Comparative Study of Intrinsic Metrics on 3D-Shapes

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Why intrinsic metrics?

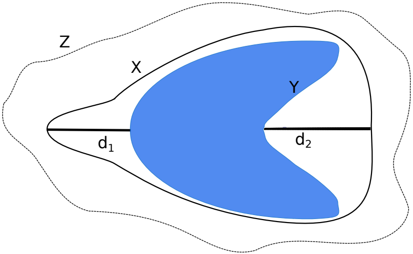
- measuring distances on surfaces is a common problem in shape analysis
- e.g. used in shape matching to find minimum distortion correspondences
- Goal: give an objective overview of the most popular intrinsic metrics

Metric Space

A set M and a metric function $d_M : M \times M \rightarrow R_+ \cup \{0, \infty\}$ are a metric space, iff for all $x, y \in M$:

- $d_M(x, y) = 0 \Leftrightarrow x = y$ (identity of indiscernibles)
- $d_M(x, y) = d_M(y, x)$ (symmetry)
- $d_M(x, y) \leq d_M(x, z) + d_M(z, y) \forall x, y, z \in M$ (triangle inequality)

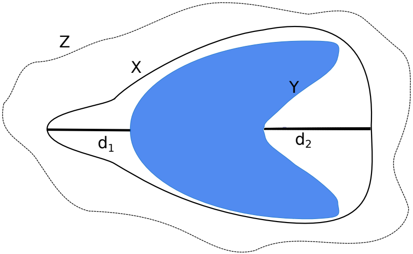
Hausdorff Distance



Gromov-Hausdorff Distance

$$d_{GH}(X, Y) = \inf_{Z,f,g} d_{GH}^{Z}(f(X), g(Y))$$

Hausdorff Distance

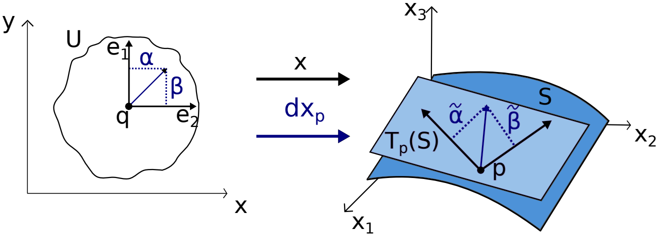


Gromov-Hausdorff Distance

$$d_{GH}(X,Y) = \inf_{Z,f,g} d_{GH}^{Z}(f(X), g(Y))$$

$$d_{GH}(X,Y) = \frac{1}{2} \inf_{R \subset X \times Y} dis\, R$$

Regular Surfaces:



First Fundamental Form:

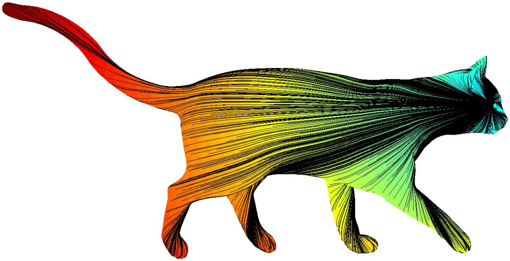
$$I_p(w) = \langle w, w \rangle_p = |w|^2 > 0$$

$$I_p(w) = \begin{pmatrix} u' & v' \end{pmatrix} \begin{pmatrix} E & F \\ F & G \end{pmatrix} \begin{pmatrix} u' \\ v' \end{pmatrix}$$

Laplace-Beltrami operator:

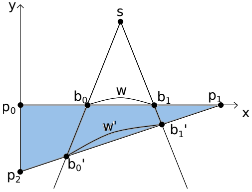
- equivalent to the Laplace operator in Euclidean space
- $\Delta f = div(\nabla f)$
- Δf is linear $\Rightarrow$ Δf has to have eigenfunctions fulfilling $\Delta f = \lambda f$

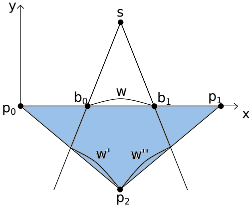
Geodesic Distance:
length of the shortest path on the surface between two points

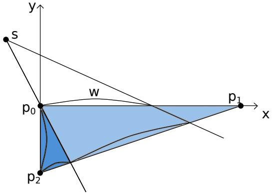


Discretisation to triangulated meshes:

- geodesics need to be straight lines within each face
- when crossing an edge, the geodesic has to correspond to a straight line if the adjacent faces are unfolded into a common plane
- distance information is propagated over the mesh in a Dijkstra-like manner by combining multiple geodesics into a single data structure called window







Diffusion Distance:

- measures the amount of heat transfered between two points after a certain time interval
- has a trade-off between local and global properties depending on a time parameter

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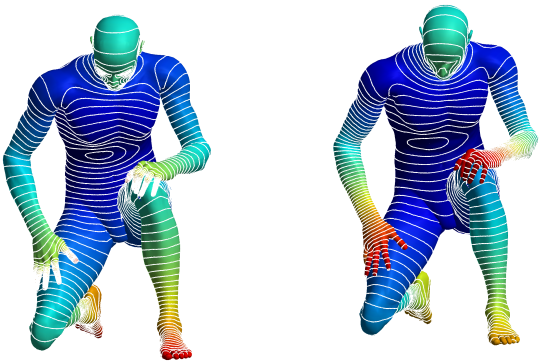
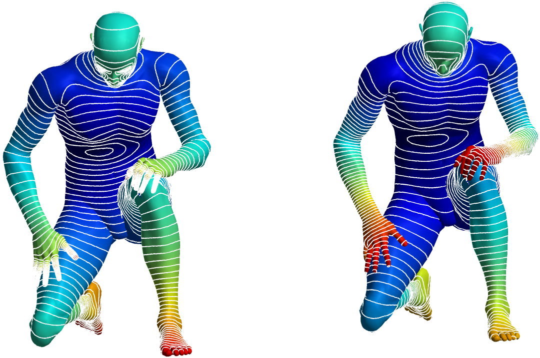
Based on the Heat equation $\Delta_S u(x,t) = -\frac{\partial u(x,t)}{\partial t}$ the formula for the diffusion distance can be obtained:

$$d_t^2(x,y) = \sum_{i=0}^{\infty} e^{-\lambda_i t}(\phi_i(x) - \phi_i(y))^2.$$





Commute-Time Distance:
Integral of the Diffusion Distance over all time steps

$$d_C^2(x, y) = \sum_{i=0}^{\infty} \frac{1}{-\lambda_i} (\phi_i(x) - \phi_i(y))^2.$$

Biharmonic Distance:

- newest metric, tries to combine the advantages of the other metrics
- based on the Green's function of the biharmonic differential equation

$$d_B^2(x,y) = \sum_{i=0}^{\infty} \frac{1}{\lambda_i^2} (\phi_i(x) - \phi_i(y))^2.$$

$$d_t^2(x,y) = \sum_{i=0}^{\infty} e^{-\lambda_i t}(\phi_i(x) - \phi_i(y))^2$$

$$d_C^2(x,y) = \sum_{i=0}^{\infty} \frac{1}{-\lambda_i}(\phi_i(x) - \phi_i(y))^2$$

$$d_B^2(x,y) = \sum_{i=0}^{\infty} \frac{1}{\lambda_i^2}(\phi_i(x) - \phi_i(y))^2$$

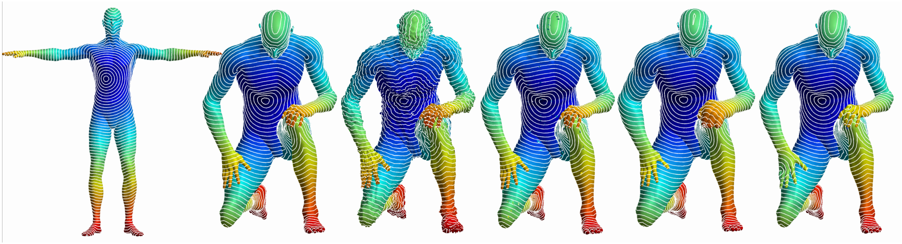
Testing goals:

- measure the computation speed
- visually compare the different metrics
- perform farthest point samplings based on different metrics
- analyse the relative error under isometries

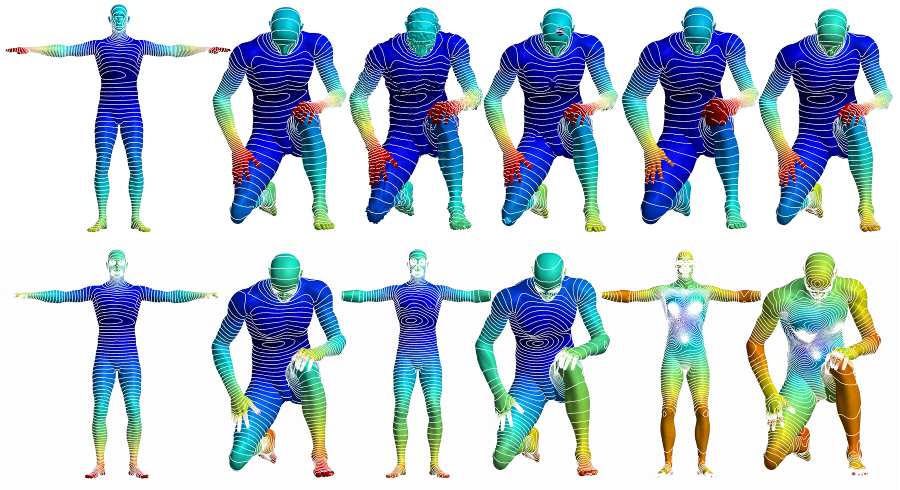
Challenges during programming:

- mesh errors for geodesic distance
- not enough isolines, so that some details were not visible first
- used the zero eigenvalue
- cut maximal values of commute-time for meshes with holes

$\lvert Vertices \rvert$	Laplacian eigenfunctions /-values	diffusion	commute time	biharmonic	geodesic exact	geodesic Dijkstra
1k	2.98	0.033	0.016	0.017	0.057	0.006
5k	12.46	0.122	0.064	0.070	0.481	0.014
10k	30.35	0.283	0.157	0.169	1.449	0.036
20k	65.08	0.487	0.268	0.288	3.632	0.063
50k	206.58	1.193	0.655	0.705	12.123	0.149



from left to right: null-shape, isometry, noise, holes, local scale, topology



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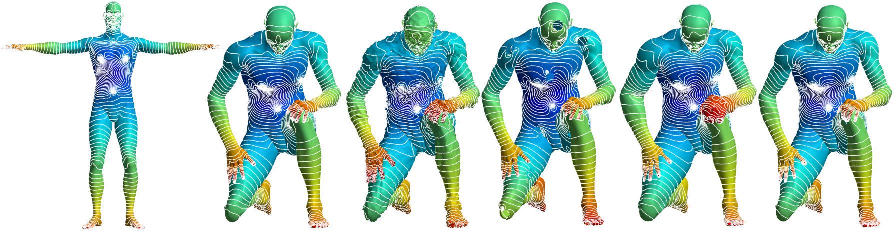

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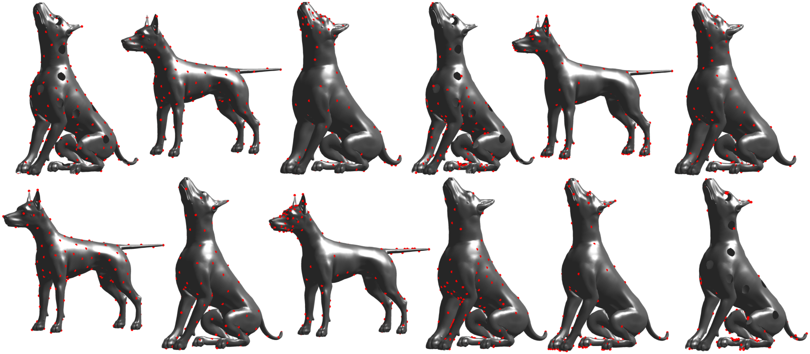

from left to right: euclidean, geodesic, diffusion(2x), commute-time, biharmonic

metric	isometry	local scale	scale	topology	noise	shot noise	micro holes	holes
geodesic	.0232	.0407	.1580	.0508	.0218	.0419	.0417	-
diffusion t=0.1	.0043	.0310	.0058	.0408	.0241	.0044	.0059	.0434
diffusion t=1	<u>.0022</u>	.0320	.0029	.0223	<u>.0180</u>	<u>.0018</u>	.0031	.0528
commute-time	.0034	<u>.0092</u>	.0037	<u>.0138</u>	.0309	.0031	.0035	<u>.0331</u>
biharmonic	.0049	.0220	<u>.0016</u>	.0420	.0625	.0024	<u>.0017</u>	.0388

Conclusions and further work:

- objective view on the properties of the metrics
- add the Earth mover's distance to the metrics
- use the FAUST dataset for testing

Thank you for your attention.
Any Questions?

Not enough isolines