Module-2

MATRICES

- **Echelon form**
- **Rank of a matrix**
- **System of linear equations**
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- **Eigenvalues and Eigenvectors**
- **Diagonalization of matrices.**
- **Conversion of an n^{th} order differential equation to a system of first order linear differential equations**
- **Solution of system of linear differential equations by diagonalization method**
- **Discuss the stability of the system.**

Introduction:

Linear algebra is the study of vectors and linear functions. It comprises of the theory and application of linear system of equations, linear transformations, Eigen values and Eigen vectors, problems.

Elementary transformations of a matrix:

The following are the elementary row transformations of a matrix. The transformations can be applied for columns.

1. The interchange of any two rows (columns).
2. The multiplication of any row (column) by a non-zero constant.
3. The addition of a constant multiple of the elements of any row (column) to the corresponding elements of any other row (column).

Ex: A= $\begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix}$

SI. No.	**Elementary row transformation**	**Notation**	**Resultant of a matrix A**
1	Interchange of first and second row	$R_1 \leftrightarrow R_2$	$\begin{bmatrix} b_1 & b_2 & b_3 \\ a_1 & a_2 & a_3 \\ c_1 & c_2 & c_3 \end{bmatrix}$
2	Multiplication of third row by a constant *k*	kR_3	$\begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ kc_1 & kc_2 & kc_3 \end{bmatrix}$
3	Addition to second row *k* times the first row	$R_2 \to kR_1 + R_2$	$\begin{bmatrix} a_1 & a_2 & a_3 \\ (ka_1 + b_1) & (ka_2 + b_2) & (ka_3 + b_3) \\ c_1 & c_2 & c_3 \end{bmatrix}$

Equivalent matrices:

Two matrices A and B of the same order are said to be *equivalent* if one matrix can be obtained from the other by a finite number of successive elementary row or column transformations. We use the notation 'A ~ B'

Echelon form of a matrix:

A non – zero matrix **A** is said to be in *row echelon form* if the following conditions prevail:

i) All the zero rows are below non zero rows.

ii) The first non zero entry in any non zero row is 1 [and the entries below 1 in the same column are zero].

$$\text{Ex : } A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \; B = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

Rank of a matrix:

The rank of a matrix A in its echelon form is equal to the number of non zero rows. It is denoted by $\rho(A)$.

Steps to find the rank of a matrix

Step 1. In order to reduce the given matrix to a row echelon form we must prefer to have the leading entry (first entry in the first row) non zero, much preferably 1.

Step 2. In case this entry is zero, we can interchange with any suitable row to meet the requirement.

Step 3. We then focus on the leading non zero entry (starting from the first row) to make all the elements in that column zero. However the transformation has to be performed for the entire row.

Step 4. Row echelon form will be achieved first and we can instantly write down the rank, which being the number of zero rows.

NOTE:

1. The **rank** of a matrix A in its **echelon form** is equal to the number of **non zero rows**.
2. Elementary transformations do not change either the order or rank of a matrix.
3. It is advisable to avoid fraction as far as possible during the process of elementary transformation.
4. Further we can say that if r is the rank of a matrix A of order $m \times n$ $(r \leq m)$, r number of rows of the matrix are linearly independent.

Problems:

1. **Find the rank of the following matrix by reducing it to the row echelon form** $A = \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix}$ **Solution:** Let $A = \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix}$ Performing $R_2 \to -3R_1 + R_2$

$$A \sim \begin{bmatrix} 1 & 2 \\ -3(1)+3 & -3(2)+6 \end{bmatrix}$$

$$A \sim \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}$$

$$\rho(A) = 1$$

2. **Find the rank of the following matrix by reducing it to the row echelon form.**

$$A = \begin{bmatrix} 0 & 2 & 3 & 4 \\ 2 & 3 & 5 & 4 \\ 4 & 8 & 13 & 12 \end{bmatrix}$$

Solution: Performing $R_1 \leftrightarrow R_2$

$$A \sim \begin{bmatrix} 2 & 3 & 5 & 4 \\ 0 & 2 & 3 & 4 \\ 4 & 8 & 13 & 12 \end{bmatrix} R_3 \to R_3 - 2R_1$$

$$A \sim \begin{bmatrix} 2 & 3 & 5 & 4 \\ 0 & 2 & 3 & 4 \\ 0 & 2 & 3 & 4 \end{bmatrix} R_3 \to R_3 - R_2$$

$$A \sim \begin{bmatrix} 2 & 3 & 5 & 4 \\ 0 & 2 & 3 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix} R_1 \to R_1/2 \quad \text{and} \quad R_2 \to R_2/2$$

$$A \sim \begin{bmatrix} 1 & 3/2 & 5/2 & 2 \\ 0 & 1 & 3/2 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\rho(A) = 2$$

Therefore A is in the row echelon form having two non zero rows.

3. Find the rank of the following matrix by reducing it to the row echelon form.

$$A = \begin{bmatrix} 2 & 3 & -1 & -1 \\ 1 & -1 & -2 & -4 \\ 3 & 1 & 3 & -2 \\ 6 & 3 & 0 & -7 \end{bmatrix}$$

Solution: Performing $R_1 \leftrightarrow R_2$

$$A=\begin{bmatrix}1 & -1 & -2 & -4\\ 2 & 3 & -1 & -1\\ 3 & 1 & 3 & -2\\ 6 & 3 & 0 & -7\end{bmatrix}$$

$R_4 \to R_4 - (R_1 + R_2 + R_3)$

$$A\sim\begin{bmatrix}1 & -1 & -2 & -4\\ 2 & 3 & -1 & -1\\ 3 & 1 & 3 & -2\\ 0 & 0 & 0 & 0\end{bmatrix}$$

$R_2 \to R_2 - 2R_1$ and $R_3 \to R_3 - 3R_1$

$$A\sim\begin{bmatrix}1 & -1 & -2 & -4\\ 0 & 5 & 3 & 7\\ 0 & 4 & 9 & 10\\ 0 & 0 & 0 & 0\end{bmatrix}$$

$R_2 \to R_2/5$ and $R_3 \to R_3/4$

$$A\sim\begin{bmatrix}1 & -1 & -2 & -4\\ 0 & 1 & 3/5 & 7/5\\ 0 & 1 & 9/4 & 5/2\\ 0 & 0 & 0 & 0\end{bmatrix}$$

$\rho(A)=3$

4. Find the rank of the following matrix by reducing it to the row echelon form.

$$A=\begin{bmatrix}-2 & -1 & -3 & -1\\ 1 & 2 & 3 & 1\\ 1 & 0 & 1 & 1\\ 0 & 1 & 1 & 1\end{bmatrix}$$

Solution: Performing $R_1 \leftrightarrow R_2$

$$A=\begin{bmatrix}1&2&3&1\\-2&-1&-3&-1\\1&0&1&1\\0&1&1&1\end{bmatrix}$$

$R_2 \to R_2 + 2R_1$ and $R_3 \to R_3 - R_1$

$$A\sim\begin{bmatrix}1&2&3&1\\0&3&3&1\\0&-2&-2&0\\0&1&1&1\end{bmatrix}$$

$R_2 \to R_4$

$$A\sim\begin{bmatrix}1&2&3&1\\0&1&1&1\\0&-2&-2&0\\0&3&3&1\end{bmatrix}$$

$R_3 \to R_3 + 2R_2$ and $R_4 \to R_4 - 3R_2$

$$A\sim\begin{bmatrix}1&2&3&1\\0&1&1&1\\0&0&0&2\\0&0&0&-2\end{bmatrix}$$

$R_4 \to R_4 + R_3$

$$A\sim\begin{bmatrix}1&2&3&1\\0&1&1&1\\0&0&0&2\\0&0&0&0\end{bmatrix}$$

$\rho(A)=3$

5. Find the rank of the following matrix by reducing it to the row echelon form.

$$A=\begin{bmatrix}1&2&3&2\\2&3&5&1\\1&3&4&5\end{bmatrix}$$

Solution:

Performing $R_2 \to R_2 - 2R_1$ and $R_3 \to R_3 - R_1$

$$A \sim \begin{bmatrix} 1 & 2 & 3 & 2 \\ 0 & -1 & -1 & -3 \\ 0 & 1 & 1 & 3 \end{bmatrix} R_3 \to R_3 + R_2$$

$$A \sim \begin{bmatrix} 1 & 2 & 3 & 2 \\ 0 & -1 & -1 & -3 \\ 0 & 0 & 0 & 0 \end{bmatrix} R_2 \to (-R_2)$$

$$A \sim \begin{bmatrix} 1 & 2 & 3 & 2 \\ 0 & 1 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$\rho(A) = 2$

6. Find the rank of the following matrix by reducing it to the row echelon form.

$$A = \begin{bmatrix} 0 & 1 & -3 & -1 \\ 1 & 0 & 1 & 1 \\ 3 & 1 & 0 & 2 \\ 1 & 1 & -2 & 0 \end{bmatrix}$$

Solution:

Performing $R_1 \leftrightarrow R_2$

$$A \sim \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & -3 & -1 \\ 3 & 1 & 0 & 2 \\ 1 & 1 & -2 & 0 \end{bmatrix}$$

$R_3 \to R_3 - 3R_1$ and $R_4 \to R_4 - R_1$

$$A \sim \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & -3 & -1 \\ 0 & 1 & -3 & -1 \\ 0 & 1 & -3 & -1 \end{bmatrix}$$

$R_3 \to R_3 - R_2$ and $R_4 \to R_4 - R_2$

$$A \sim \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & -3 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$\rho(A) = 2$

Consistency of a system of linear equations:

Consider a system of 'm' linear equations in 'n' unknowns as follows

$$a_{11}\ x_1 + a_{12}\ x_2\ + \dots\dots\dots + a_{1n}\ x_n\ = b_1$$
$$a_{21}\ x_1 + a_{22}\ x_2\ + \dots\dots\dots + a_{2n}\ x_n\ = b_2$$
$$\dots\dots\dots \qquad\qquad \dots\dots\dots$$
$$\dots\dots\dots \qquad\qquad \dots\dots\dots$$
$$a_{m1} x_1 + a_{m2}\ x_2\ + \dots\dots\dots + a_{mn}\ x_n\ = b_m$$

Where a_{ij}'s and b_i's are constants.

- If $b_1, b_2, \dots\dots\dots, b_m$ are all zero, the system is said to be homogeneous. Otherwise, it is said to be Non-homogeneous.
- The set of values $x_1, x_2, \dots, x_n$ which satisfy all the equations simultaneously is called a solution of the system of equations.
- A system of linear equations is said to be **consistent** if it possess a solution. Otherwise it is said to be **inconsistent**.

The above system of equations can be written in the matrix equation A X = B, where

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ .. & .. & \dots & .. \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}, \quad X = \begin{bmatrix} x_1 \\ x_2 \\ .. \\ x_n \end{bmatrix} \quad B = \begin{bmatrix} b_1 \\ b_2 \\ .. \\ b_m \end{bmatrix}$$

- $x_1 = x_2 = x_3 = \dots\dots\dots = x_n = 0$ is a solution of the homogeneous system of equations and is called a **trivial solution.**
- If at least one x_i, (i = 1, 2, …, n) is not equal to zero then it is called a **non trivial solution.**
- The concept of the rank of a matrix helps us to conclude
  - (i) Whether the system is consistent or not.
  - (ii) Whether the system possess unique solution or many solution.

Condition for consistency and types of solution

Consider a system of m equations in n unknowns represented in the matrix form $AX = B$

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ .. & .. & \dots & .. \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}, \quad X = \begin{bmatrix} x_1 \\ x_2 \\ .. \\ x_n \end{bmatrix} \quad B = \begin{bmatrix} b_1 \\ b_2 \\ .. \\ b_m \end{bmatrix}$$

Here A is called the coefficient matrix.

The matrix formed by appending to A an extra column consistent of the elements of B is called the **augmented matrix** denoted by **[A: B]**

That is, $$[A:B]=\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} & : & b_1 \\ a_{21} & a_{22} & \dots & a_{2n} & : & b_2 \\ .. & .. & \dots & .. & \vdots & .. \\ a_{m1} & a_{m2} & \dots & a_{mn} & : & b_m \end{bmatrix}$$

The system of equations represented by the matrix equation $AX = B$ is **consistent** if $\rho(A)=\rho(A:B)$

Suppose $\rho(A)=\rho(A:B)=r$, then the condition for two types of solution are as follows.

1. **Unique Solution:** $\rho(A)=\rho(A:B)=r=n$, (where n is the number of unknowns).
2. **Infinite Solution:** $\rho(A)=\rho(A:B)=r<n$,

In case $(n - r)$ unknowns can take arbitrary value, Obviously $\rho[A] \neq \rho[A:B]$ implies that the system in **inconsistent** (does not possess a solution).

Working procedure for problems:

Step 1: We first form the augmented matrix [A: B] and we can clearly identify the portion of the coefficient matrix A in it.

Step 2: We reduce the matrix [A: B] to an echelon form by elementary row transformations. This will enable us to immediately write down the rank of A and also [A: B], with the result we can decide the consistency aspect of the system of equations.

Step 3: The echelon form of [A: B] is converted back to the equation form and the solution will emerge easily.

Problems

1. **Test for consistency and solve**

$$x+y+z=6$$
$$x-y+2z=5$$
$$3x+y+z=8$$

Solution:

$$[A:B]=\begin{bmatrix}1 & 1 & 1: & 6\\ 1 & -1 & 2: & 5\\ 3 & 1 & 1: & 8\end{bmatrix}$$ is the augmented matrix.

Perform $R_2 \to R_2 - R_1$ and $R_3 \to R_3 - 3R_1$

$$[A:B]\sim\begin{bmatrix}1 & 1 & 1 & : & 6\\ 0 & -2 & 1 & : & -1\\ 0 & -2 & -2 & : & -10\end{bmatrix}$$

$R_3 \to R_3 - R_2$

$$[A:B]\sim\begin{bmatrix}1 & 1 & 1 & : & 6\\ 0 & -2 & 1 & : & -1\\ 0 & 0 & -3 & : & -9\end{bmatrix}$$

[Note: *We need not make the leading non zero entry in every row 1 as we can decide on the rank of the matrices A and [A: B] at this stage.***]**

Both A and [A : B] matrices have all the three rows non zero.

Therefore $\rho[A] = 3$ and $\rho[A : B] = 3$ that is, $r = 3$.

Also, the number of independent variables $n = 3$.

Since $\rho[A] = \rho[A : B] = 3$ ($r = n = 3$) the given system of equations is **consistent** and will have **unique solution**.

Let us now convert the prevailing form of [A : B] into a set of equations as follows,

$$x + y + z = 6 ....(1)$$
$$-2y + z = -1 ....(2)$$
$$-3z = -9 ....(3)$$

From (3), $z = 3$, Substitute this value in (2), we get

$y = 2$. Finally substituting these values in (1), we get $x = 1$.

Thus $x = 1, y = 2, z = 3$ is the unique solution.

2. Test for consistency and solve

$$x + 2y + 3z = 14$$
$$4x + 5y + 7z = 35$$
$$3x + 3y + 4z = 21$$

Solution:

$$[A:B]=\begin{bmatrix}1 & 2 & 3 & : & 14\\ 4 & 5 & 7 & : & 35\\ 3 & 3 & 4 & : & 21\end{bmatrix}$$ is the augmented matrix.

$R_2 \to R_2 - 4R_1$ and $R_3 \to R_3 - 3R_1$

$$[A:B]\sim\begin{bmatrix}1 & 2 & 3 & : & 14\\ 0 & -3 & -5 & : & -21\\ 0 & -3 & -5 & : & -21\end{bmatrix}$$

$R_3 \to R_3 - R_2$

$$[A:B]\sim\begin{bmatrix}1 & 2 & 3 & : & 14\\ 0 & -3 & -5 & : & -21\\ 0 & 0 & 0 & : & 0\end{bmatrix}$$

Therefore $\rho[A] = 2$ and $\rho[A : B] = 2$ that is, $r = 2$.

Also, the number of independent variables $n = 3$.

Since $\rho[A] = \rho[A : B] = 2 < 3$ ($r < n$) the given system of equations is **consistent** and will have **infinite solution**.

Here $(n - r) = 1$ and hence one of the variables can take arbitrary values.

We now have,

$$x + 2y + 3z = 14 \,.......(1)$$
$$-3y - 5z = -21 \,.......(2)$$

Let $z = k$ be arbitrary,

Therefore, from (2), $-3y - 5k = -21$ or $y = \frac{-21-5k}{3} = 7 - \frac{5k}{3}$

Substitute this value in (1), we get

$x = k/3$.

Thus $x = \frac{k}{3}, y = 7 - \frac{5k}{3}, z = k$ represents infinite solutions, since k is arbitrary.

3. Test for consistency and solve

$$x - 4y + 7z = 14$$
$$3x + 8y - 2z = 13$$
$$7x - 8y + 26z = 5$$

Solution:

$$[A:B]=\begin{bmatrix}1 & -4 & 7 & : & 14\\ 3 & 8 & -2 & : & 13\\ 7 & -8 & 26 & : & 5\end{bmatrix}$$ is the augmented matrix.

$R_2 \to R_2 - 3R_1$ and $R_3 \to R_3 - 7R_1$

$$[A:B]\sim\begin{bmatrix}1 & -4 & 7 & : & 14\\ 0 & 20 & -23 & : & -29\\ 0 & 20 & -23 & : & -93\end{bmatrix}$$

$R_3 \to R_3 - R_2$

$$[A:B]\sim\begin{bmatrix}1 & -4 & 7 & : & 14\\ 0 & 20 & -23 & : & -29\\ 0 & 0 & 0 & : & -64\end{bmatrix}$$

Therefore $\rho[A] = 2$and $\rho[A : B] = 3$.

Also, the number of independent variables $n = 3$.

Since $[A] \neq \rho[A : B]$, the given system of equations is **inconsistent.**

4. Test for consistency and solve

$$5x_1 + x_2 + 3x_3 = 20$$
$$2x_1 + 5x_2 + 2x_3 = 18$$
$$3x_1 + 2x_2 + x_3 = 14$$

Solution:

$$[A:B]=\begin{bmatrix}5 & 1 & 3 & : & 20\\ 2 & 5 & 2 & : & 18\\ 3 & 2 & 1 & : & 14\end{bmatrix}$$ is the augmented matrix.

$R_2 \to 5R_2 - 2R_1$ and $R_3 \to 5R_3 - 3R_1$

$$[A:B]\sim\begin{bmatrix}5 & 1 & 3 & : & 20\\ 0 & 23 & 4 & : & 50\\ 0 & 7 & 4 & : & 10\end{bmatrix}$$

$R_3 \to 23R_3 + 7R_2$

$$[A:B]\sim\begin{bmatrix}5 & 1 & 3 & : & 20\\ 0 & 23 & 4 & : & 50\\ 0 & 0 & -120 & : & -120\end{bmatrix}$$

Therefore $\rho[A] = 3$ and $\rho[A : B] = 3$ that is, $r = 3$.

Also, the number of independent variables $n = 3$.

Since $\rho[A] = \rho[A : B] = 3$ ($r = n = 3$) the given system of equations is **consistent** and will have **unique solution**.

Let us now convert the prevailing form of [A : B] into a set of equations as follows,

$$5x_1 + x_2 + 3x_3 = 20.......(1)$$
$$23x_2 + 4x_3 = 50.......(2)$$
$$-120x_3 = -120...........(3)$$

From (3), $x_3 = 1$, Substitute this value in (2), we get

$x_2 = 2$. Finally substituting these values in (1), we get $x_1 = 3$.

Thus $x_1 = 3,\ x_2 = 2,\ x_3 = 1$ is the unique solution.

5. Show that the following system of equation does not possess any solution

$$5x + 3y + 7z = 5$$
$$3x + 26y + 2z = 9$$
$$7x + 2y + 10z = 5$$

Solution:

$$[A:B] = \begin{bmatrix} 5 & 3 & 7 & : & 5 \\ 3 & 26 & 2 & : & 9 \\ 7 & 2 & 10 & : & 5 \end{bmatrix} \text{ is the augmented matrix.}$$

$R_2 \to 5R_2 - 3R_1$ and $R_3 \to 5R_3 - 7R_1$

$$[A:B] \sim \begin{bmatrix} 5 & 3 & 7 & : & 5 \\ 0 & 121 & -11 & : & 30 \\ 0 & -11 & 1 & : & -10 \end{bmatrix}$$

$R_3 \to R_2 + 11R_3$

$$[A:B] \sim \begin{bmatrix} 5 & 3 & 7 & : & 5 \\ 0 & 121 & -11 & : & 30 \\ 0 & 0 & 0 & : & -80 \end{bmatrix}$$

Therefore $\rho[A] = 2$and $\rho[A : B] = 3$.

Also, the number of independent variables $n = 3$.

Since $[A] \neq \rho[A : B]$, the given system of equations is **inconsistent.**

6. **Investigate the values of λ and μ such that the system of equations**

$x + y + z = 6;\quad x + 2y + 3z = 10;\quad x + 2y + \lambda z = \mu$, **may have**

a] Unique solution b] Infinite Solution c] No solution.

Solution:

$$[A:B]=\begin{bmatrix}1 & 1 & 1 & : & 6\\ 1 & 2 & 3 & : & 10\\ 1 & 2 & \lambda & : & \mu\end{bmatrix}$$ is the augmented matrix.

$R_2 \to R_2 - R_1$ and $R_3 \to R_3 - R_1$

$$[A:B]\sim\begin{bmatrix}1 & 1 & 1 & : & 6\\ 0 & 1 & 2 & : & 4\\ 0 & 1 & \lambda-1 & : & \mu-6\end{bmatrix}$$

$R_3 \to R_3 - R_2$

$$[A:B]\sim\begin{bmatrix}1 & 1 & 1 & : & 6\\ 0 & 1 & 2 & : & 4\\ 0 & 1 & \lambda-3 & : & \mu-10\end{bmatrix}$$

a] Unique solution: We must have $\rho(A)=\rho(A:B)=3$ .and $\rho(A)$ will be 3 if $\lambda-3\neq 0$

Also $\rho(A:B)$ will be 3 if $\lambda-3\neq 0$ and for any value of μ .

$\therefore$ If $\lambda\neq 3$, then the system will have unique solution.

b] Infinite solutions: We must have $\rho(A)=\rho(A:B)=r$ and $r<n$.

Since $n=3$, we must have $r=2$.

If $\lambda-3=0$ and $\mu-10=0$, then we have $\rho(A)=\rho(A:B)<n$

$\therefore$ The system have infinite solutions if $\lambda=3$ and $\mu=10$.

c] No solution: If the system is inconsistent, then we have no solution.

$\therefore$ We must have $\rho(A)\neq\rho(A:B)$.

By case **(a)**, $\rho(A)=3$, if $\lambda\neq 3$ and hence if $\lambda=3$ we obtain $\rho(A)=2$.

If we impose $\mu\neq 10$, then we have $\rho(A:B)=3$

$\therefore$ The system has no solution if $\lambda=3$ and $\mu\neq 10$.

7. Test for consistency and solve

$$x+2y+2z=1$$
$$2x+y+z=2$$
$$3x+2y+2z=3$$
$$y+z=0$$

Solution:

$$[A:B]=\begin{bmatrix}1&2&2&:&1\\2&1&1&:&2\\3&2&2&:&3\\0&1&1&:&0\end{bmatrix}$$ is the augmented matrix.

$R_2 \to R_2 - 2R_1$ and $R_3 \to R_3 - 3R_1$

$$[A:B]\sim\begin{bmatrix}1&2&2&:&1\\0&-3&-3&:&0\\0&-4&-4&:&0\\0&1&1&:&0\end{bmatrix}$$

$R_2 \leftrightarrow R_4$

$$[A:B]\sim\begin{bmatrix}1&2&2&:&1\\0&1&1&:&0\\0&-4&-4&:&0\\0&-3&-3&:&0\end{bmatrix}$$

$R_3 \to 4R_2 + R_3$ and $R_4 \to 3R_2 + R_4$

$$[A:B]\sim\begin{bmatrix}1&2&2&:&1\\0&1&1&:&0\\0&0&0&:&0\\0&0&0&:&0\end{bmatrix}$$

Therefore $\rho[A] = 2$ and $\rho[A : B] = 2$ that is, $r = 2$.

Also, the number of independent variables $n = 3$.

Since $\rho[A] = \rho[A : B] = 2 < 3$ ($r < n$) the given system of equations is **consistent** and will have **infinite solution**.

Here $(n - r) = 1$ and hence one of the variables can take arbitrary values.

We now have,

$$x + 2y + 2z = 1 ....(1)$$
$$y + z = 0 \qquad ....(2)$$

Let $z = k$ be arbitrary,

Therefore, from (2), $y + k = 0$ or $y = -k$

Substitute this value in (1), we get

$x = 1.$

Thus $x = 1, y = -k, z = k$ represents infinite solutions, since k is arbitrary.

Solution of a system of Non homogeneous equations:

Linear simultaneous Equations occur in various engineering problems. We are already familiar with the methods, Cramer's rule and matrix method for solving such equations. But these methods are tedious for large systems.

Here we study now another two methods for solving such large systems.

They are **a)** Gauss elimination method

b) Gauss –Seidel method.

Gauss – Elimination Method

In this method, the unknowns are eliminated successively and the system is reduced to an upper triangular system from which the unknowns are found by back substitution. This method is quite general and is well – adapted for computer operations.

1. Solve the following system of equations by Gauss elimination method.

$$x+y+z=9, \quad x-2y+3z=8, \quad 2x+y-z=3$$

Solution: The augmented matrix of the systems is

$$[A:B] = \begin{bmatrix} 1 & 1 & 1 & :9 \\ 1 & -2 & 3 & :8 \\ 2 & 1 & -1 & :3 \end{bmatrix}$$

$$R_2 \to R_2 - R_1 \text{ and } R_3 \to R_3 - 2R_1$$

$$[A:B] \sim \begin{bmatrix} 1 & 1 & 1 & :9 \\ 0 & -3 & 2 & :-1 \\ 0 & -1 & -3 & :-15 \end{bmatrix}$$

$$R_3 \to R_2 - 3R_3$$

$$[A:B] \sim \begin{bmatrix} 1 & 1 & 1 & :9 \\ 0 & -3 & 2 & :-1 \\ 0 & 0 & 11 & :44 \end{bmatrix}$$

Hence we have

$$x + y + z = 9$$
$$-3y + 2z = -1$$
$$11z = 44$$
$$\therefore z = 4$$

By back substitution, $y = 3, x = 2$

$\therefore$ The solution is x = 2, y = 3, z = 4

2. Solve the following system of equations by Gauss elimination method.

$$2x + y + 4z = 12, \qquad 4x + 11y - z = 33, \qquad 8x - 3y + 2z = 20$$

Solution:

The augmented matrix of the systems is

$$[A:B] \sim \begin{bmatrix} 2 & 1 & 4 & :12 \\ 4 & 11 & -1 & :33 \\ 8 & -3 & 2 & :20 \end{bmatrix}$$

$R_2 \rightarrow R_2 - 2R_1$ and $R_3 \rightarrow R_3 - 4R_1$

$$[A:B] \sim \begin{bmatrix} 2 & 1 & 4 & :12 \\ 0 & 9 & -9 & :33 \\ 0 & -7 & -28 & :20 \end{bmatrix}$$

$R_3 \rightarrow 9R_3 + 7R_2$

$$[A:B] \sim \begin{bmatrix} 2 & 1 & 4 & :12 \\ 0 & 9 & -9 & :9 \\ 0 & 0 & -189 & :-189 \end{bmatrix}$$

Hence we have

$$2x + y + 4z = 12$$
$$9y - 9z = 9$$
$$-189z = -189$$
$$\therefore z = 1$$

By back substitution, y = 2, $x = 3$.

3. Solve the following system of equations by Gauss elimination method.

$$x_1 - 2x_2 + 3x_3 = 2, \quad 3x_1 - x_2 + 4x_3 = 4, \quad 2x_1 + x_2 - 2x_3 = 5$$

Solution: The augmented matrix of the systems is

$$[A:B] = \begin{bmatrix} 1 & -2 & 3 & :2 \\ 3 & -1 & 4 & :4 \\ 2 & 1 & -2 & :5 \end{bmatrix}$$

$R_2 \to R_2 - 3R_1$ and $R_3 \to R_3 - 2R_1$

$$[A:B] = \begin{bmatrix} 1 & -2 & 3 & :2 \\ 0 & 5 & -5 & :-2 \\ 0 & 5 & -8 & :1 \end{bmatrix}$$

$R_3 \to R_3 - R_2$

$$[A:B] = \begin{bmatrix} 1 & -2 & 3 & :2 \\ 0 & 5 & -5 & :-2 \\ 0 & 0 & -3 & :3 \end{bmatrix}$$

Hence we have

$$x_1 - 2x_2 + 3x_4 = 2$$
$$5x_2 - 5x_3 = -2$$
$$-3x_3 = 3$$
$$\therefore x_3 = -1$$

By back substitution,

$x_2 = -1.4 \quad x_1 = 2.2.$

4. Solve the following system of equations by Gauss elimination method.

$$5x + y + z + w = 4$$
$$x + 7y + z + w = 12$$
$$x + y + 6z + w = -5$$
$$x + y + z + 4w = -6$$

Solution:

$$[A:B]=\begin{bmatrix} 5 & 1 & 1 & 1 & :4 \\ 1 & 7 & 1 & 1 & :12 \\ 1 & 1 & 6 & 1 & :-5 \\ 1 & 1 & 1 & 4 & :-6 \end{bmatrix}$$ is the augmented matrix.

$R_1 \leftrightarrow R_4$

$$[A:B]\sim\begin{bmatrix} 1 & 1 & 1 & 4 & :-6 \\ 1 & 7 & 1 & 1 & :12 \\ 1 & 1 & 6 & 1 & :-5 \\ 5 & 1 & 1 & 1 & :4 \end{bmatrix}$$

$R_2 \to R_2 - R_1, R_3 \to R_3 - R_1$ and $R_4 \to R_4 - 5R_1$

$$[A:B]\sim\begin{bmatrix} 1 & 1 & 1 & 4 & :-6 \\ 0 & 2 & 0 & -1 & :6 \\ 0 & 0 & 5 & -3 & :1 \\ 0 & -4 & -4 & -19 & :34 \end{bmatrix}$$

$R_4 \to R_4 + 2R_2$

$$[A:B]\sim\begin{bmatrix} 1 & 1 & 1 & 4 & :-6 \\ 0 & 2 & 0 & -1 & :6 \\ 0 & 0 & 5 & -3 & :1 \\ 0 & 0 & -4 & -21 & :46 \end{bmatrix}$$

$R_4 \to 5R_4 + 4R_3$

$$[A:B]\sim\begin{bmatrix} 1 & 1 & 1 & 4 & :-6 \\ 0 & 2 & 0 & -1 & :6 \\ 0 & 0 & 5 & -3 & :1 \\ 0 & 0 & 0 & -117 & :234 \end{bmatrix}$$

Hence we have

$x+y+z+4w=-6$

$2y-w=6$

$5z-3w=1$

$-117w=234$

$\therefore w=-2$

By back substitution,

$z=-1 \quad y=2$ and $x=1$

Gauss Seidel Method:

This is a numerical iterative method giving approximate solution. This method cannot be applied to all the system of equations.

The method is applicable when the numerically large coefficients are along the leading diagonal of the coefficient matrix .Such a system is called a diagonally dominant system.

1. **Solve the system of equations by Gauss – Seidel method**

   $10x + y + z = 12;\ x + 10y + z = 12;\ \text{and}\ x + y + 10z = 12$

   Solution: The given system of equations is diagonally dominant and the equations are put in the form

$$x = \frac{1}{10}(12 - y - z) \quad y = \frac{1}{10}(12 - x - z)\ ,\ \ z = \frac{1}{10}(12 - x - y),$$

Let us start with trial solution $x = 0, y = 0, z = 0.$

First iteration: $x^{(1)} = \frac{1}{10}(12 - 0 - 0) = 1.2$

$$y^{(1)} = \frac{1}{10}(12 - 1.2 - 0) = 1.08$$

$$z^{(1)} = \frac{1}{10}(12 - 1.2 - 1.08) = 0.972$$

Second iteration: $x^{(2)} = \frac{1}{10}(12 - 1.08 - 0.972) = 0.9948$

$$y^{(2)} = \frac{1}{10}(12 - 0.9948 - 0.972) = 1.00332$$

$$z^{(2)} = \frac{1}{10}(12 - 0.9948 - 1.00332) = 1.000188$$

Third iteration: $x^{(3)} = \frac{1}{10}(12 - 1.00332 - 1.000188) = 0.99965 \approx 1$

$$y^{(3)} = \frac{1}{10}(12 - 0.99965 - 1.000188) = 1.00002 \approx 1$$

$$z^{(3)} = \frac{1}{10}(12 - 0.99965 - 1.00002) = 1.00003 \approx 1.$$

Thus, $x = 1, y = 1, z = 1.$

2. Solve the system of equations by Gauss – Seidel method

$$20x + y - 2z = 17; \quad 3x + 20y - z = -18; \quad 2x - 3y + 20z = 25,$$

Solution: The given systems of equations are diagonally dominant and the equations are put in the form

$$x = \frac{1}{20}(17 - y + 2z), \; y = \frac{1}{20}(-18 - 3x + z), \; z = \frac{1}{20}(25 - 2x + 3y)$$

Let us start with trial solution $x = 0, y = 0, z = 0.$

First iteration: $x^{(1)} = \frac{1}{20}(17 - 0 - 0) = 0.85$

$$y^{(1)} = \frac{1}{20}(-18 - 3(0.85) - 0) = -1.0275$$

$$z^{(1)} = \frac{1}{20}(25 - 2(0.85) + 3(-1.0275)) = 1.0109$$

Second iteration: $x^{(2)} = \frac{1}{20}(17 - (-1.0275) + 2(1.0109)) = 1.0025$

$$y^{(2)} = \frac{1}{20}(-18 - 3(1.0025) + 1.0109) = -0.9998$$

$$z^{(2)} = \frac{1}{20}(25 - 2(1.0025) + 3(-0.9998)) = 0.9998$$

Third iteration: $x^{(3)} = \frac{1}{20}(17 - (-0.9998) + 2(0.9998)) = 0.99997 \approx 1$

$$y^{(3)} = \frac{1}{20}(-18 - 3(0.99997) + 0.99998) = -1.0000055 \approx -1$$

$$z^{(3)} = \frac{1}{20}(25 - 2(0.99997) + 3(-1.0000055)) = 1.0000022 \approx 1$$

Thus $x = 1, y = -1, z = 1$.

3. Solve the system of equations by Gauss–Seidel method to obtain the the final solution correct to three decimal places

$$x + y + 54z = 110; \quad 27x + 6y - z = 85; \quad 6x + 15y + 2z = 72.$$

Solution: The given system of equations is not diagonally dominant and hence we have to first rearrange the given system of equation as follows

$$27x+6y-z=85 \quad (|27|>|6|+|-1|)$$
$$6x+15y+2z=72, \quad (|15|>|6|+|2|)$$
$$x+y+54z=110; (|54|>|1|+|1|)$$

Now the given systems of equations are diagonally dominant and the equations are put in the form

$$x=\frac{1}{27}(85-6y+z), \quad y=\frac{1}{15}(72-6x-2z), \quad z=\frac{1}{54}(110-x-y),$$

Let us start with trial solution $x=0, y=0, z=0.$

First iteration: $x^{(1)}=\frac{1}{27}(85-0+0)=3.14815$

$$y^{(1)}=\frac{1}{15}(72-6(3.14815)-0)=3.54074$$
$$z^{(1)}=\frac{1}{54}(110-3.14815-3.54074)=1.91317$$

Second iteration: $x^{(2)}=\frac{1}{27}(85-6(3.54074)+1.91317)=2.43218$

$$y^{(2)}=\frac{1}{15}(72-6(2.43218)-2(1.91317))=3.57204$$
$$z^{(2)}=\frac{1}{54}(110-2.43218-3.57204)=1.92585$$

Third iteration: $x^{(3)}=\frac{1}{27}(85-6(3.57204)+1.92585)=2.42569$

$$y^{(3)}=\frac{1}{15}(72-6(2.42569)-2(1.92585))=3.57294$$
$$z^{(3)}=\frac{1}{54}(110-2.42569-3.57294)=1.92595$$

Fourth iteration: $x^{(4)}=\frac{1}{27}(85-6(3.57294)+1.92595)=2.42549$

$$y^{(4)}=\frac{1}{15}(72-6(2.42549)-2(1.92595))=3.57301$$
$$z^{(4)}=\frac{1}{54}(110-2.42549-3.57301)=1.92595$$

Thus $x=2.426, y=3.573, z=1.926.$

4. **Solve the system of equations by Gauss – Seidel method** $5x+2y+z=12; \quad x+4y+2z=15; \quad x+2y+5z=20,$ **carryout 4 iterations taking the initial approximation to the solution as (1, 0, 3).**

Solution: The given system of equations is diagonally dominant and the equations are put in the form $x=\frac{1}{5}(12-2y-z), \quad y=\frac{1}{4}(15-x-2z), z=\frac{1}{5}(20-x-2y),$

Let us start with trial solution $x=1, y=0, z=3.$

First iteration: $x^{(1)} = \frac{1}{5}(12 - 2(0) - 3) = 1.8$

$$y^{(1)} = \frac{1}{4}(15 - 1.8 - 2(3)) = 1.8$$

$$z^{(1)} = \frac{1}{5}(20 - 1.8 - 2(1.8)) = 2.92$$

Second iteration: $x^{(2)} = \frac{1}{5}(12 - 2(1.8) - 2.92) = 1.096$

$$y^{(2)} = \frac{1}{4}(15 - 1.096 - 2(2.92)) = 2.016$$

$$z^{(2)} = \frac{1}{5}(20 - 1.096 - 2(2.016)) = 2.9744$$

Third iteration: $x^{(3)} = \frac{1}{5}(12 - 2(2.016) - 2.9744) = 0.99872$

$$y^{(3)} = \frac{1}{4}(15 - 0.99872 - 2(2.9744)) = 2.01312$$

$$z^{(3)} = \frac{1}{5}(20 - 0.99872 - 2(2.01312)) = 2..995$$

Fourth iteration: $x^{(4)} = \frac{1}{5}(12 - 2(2.01312) - 2.995) = 0.995752$

$$y^{(4)} = \frac{1}{4}(15 - 0.995752 - 2(2.995)) = 2.003562$$

$$z^{(4)} = \frac{1}{5}(20 - 0.995752 - 2(2.003562)) = 2.9994248$$

Thus $x = 0.9958, \quad y = 2.0036, \quad z = 2.9994.$

Eigen values and Eigen vectors:

Let 'A' be a given square matrix of order $n \times n$. Suppose there exists a non-zero column matrix 'X' of order $1 \times n$ and a real or complex number λ such that $AX = \lambda X$ then X is called an **Eigen vector** of A and λ is called the corresponding **Eigen value** of A.

If I is the unit matrix of the same order as that of A, we have $X = IX$ and hence

$$\begin{aligned}
&AX = \lambda X \text{ can be written as,}\\
&AX = \lambda(IX)\\
&AX - \lambda IX = 0\\
&[A - I\lambda]X = [0], \quad \text{where } [0] \text{ is the null matrix.}\\
&[A - I\lambda]X = [0] \text{ represents a set of homogeneous equations.}
\end{aligned}$$

Let us consider a square matrix of order 3, represented by,

$$A = \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix} \quad \text{Also, } \lambda I = \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix}$$

$$\therefore [A - \lambda I] = \begin{bmatrix} (a_1 - \lambda) & a_2 & a_3 \\ b_1 & (b_2 - \lambda) & b_3 \\ c_1 & c_2 & (c_3 - \lambda) \end{bmatrix} \quad \text{Also, Let } X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

It can be easily seen that [A-λI][X] = [0] represents a set of homogenous equation[RHS being zero] in 3 – unknowns.

$$\text{i.e.,} \quad \begin{aligned} (a_1 - \lambda)x + a_2 y + a_3 z = 0 \\ b_1 x + (b_2 - \lambda)y + b_3 z = 0 \\ c_1 x + c_2 y + (c_3 - \lambda)z = 0 \end{aligned}$$

A nontrivial solution [at least one of $x, y, z \neq 0$] for this system exists if the determinant of the co-efficient matrix is zero.

$$\text{i.e.,} \quad \begin{vmatrix} (a_1 - \lambda) & a_2 & a_3 \\ b_1 & (b_2 - \lambda) & b_3 \\ c_1 & c_2 & (c_3 - \lambda) \end{vmatrix} = 0$$

On expanding, we get a cubic equation λ, which is called the characteristic equation of A.

The roots of this equation are Eigen values, which are also called Eigen roots or characteristic roots or latent roots.

For each value of λ, there will be an Eigen vector $X \neq 0$, which is also called a characteristic vector.

Problems

1. **Find all the Eigen values and the corresponding Eigen vectors of the matrix**

$$\begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}$$

Solution: The characteristic equation of A is $|A-\lambda I|=0$,

$$\begin{vmatrix} 8-\lambda & -6 & 2 \\ -6 & 7-\lambda & -4 \\ 2 & -4 & 3-\lambda \end{vmatrix} = 0$$

On expanding, we have $\lambda^3 - 18\lambda^2 + 45\lambda = 0$,

After solving, we get $\lambda = 0,\ 3,\ 15$ are the Eigen values.

Now the system of equations is
$$\begin{aligned} (8-\lambda)x - 6y + 2z &= 0 \\ -6x + (7-\lambda)y - 4z &= 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots(1) \\ 2x - 4y + (3-\lambda)z &= 0 \end{aligned}$$

Case (i): Let $\lambda = 0$ and the system of equations becomes,

$$\begin{aligned} 8x - 6y + 2z &= 0 \quad -\ (i) \\ -6x + 7y - 4z &= 0 \quad -\ (ii) \\ 2x - 4y + 3z &= 0 \quad -\ (iii) \end{aligned}$$

Applying the rule of cross multiplication for (i) and (ii)

$$\frac{x}{\begin{vmatrix} -6 & 2 \\ 7 & -4 \end{vmatrix}} = \frac{-y}{\begin{vmatrix} 8 & 2 \\ -6 & -4 \end{vmatrix}} = \frac{z}{\begin{vmatrix} 8 & -6 \\ -6 & 7 \end{vmatrix}}$$

i. e., $\frac{x}{10} = \frac{-y}{-20} = \frac{z}{20}$ $\quad or \quad$ $\frac{x}{1} = \frac{y}{2} = \frac{z}{2}$

$\therefore$ (x, y, z) are proportional to $(1,\ 2,\ 2)$ and we can write

$x = k,\ \ y = 2k,\ \ z = 2k$ where k is arbitrary

$\therefore$ the Eigen vector X_1 for $\lambda = 0$ is $\begin{bmatrix} k \\ 2k \\ 2k \end{bmatrix}$

[The same will be written as row vector in the form
X_1 = [2, 1, -2]]

Case (ii): Let $\lambda = 3$ and the system of equations from (1) becomes,

$$\begin{aligned} 5x - 6y + 2z &= 0 \quad -\ (iv) \\ -6x + 4y - 4z &= 0 \quad -\ (v) \\ 2x - 4y + 0z &= 0 \quad -\ (vi) \end{aligned}$$

Applying the rule of cross multiplication for (iv) and (v)

$$\frac{x}{\begin{vmatrix} -6 & 2 \\ 4 & -4 \end{vmatrix}} = \frac{-y}{\begin{vmatrix} 5 & 2 \\ -6 & -4 \end{vmatrix}} = \frac{z}{\begin{vmatrix} 5 & -6 \\ -6 & 4 \end{vmatrix}}$$

i. e., $\frac{x}{16} = \frac{-y}{-8} = \frac{z}{-16}$ *or* $\frac{x}{2} = \frac{y}{1} = \frac{z}{-2}$

$\therefore$ the Eigen vector X_2 for $\lambda = 3$ *is* $[2 \quad 1 \quad -2]$

Case (iii): Let $\lambda = 15$ and the system of equations from (1) becomes,

$$-7x - 6y + 2z = 0 \quad - (vii)$$
$$-6x - 8y - 4z = 0 \quad - (viii)$$
$$2x - 4y - 12z = 0 \quad - (ix)$$

Applying the rule of cross multiplication for (vii) and (viii)

$$\frac{x}{\begin{vmatrix} -6 & 2 \\ -8 & -4 \end{vmatrix}} = \frac{-y}{\begin{vmatrix} -7 & 2 \\ -6 & -4 \end{vmatrix}} = \frac{z}{\begin{vmatrix} -7 & -6 \\ -6 & -8 \end{vmatrix}}$$

i. e., $\frac{x}{40} = \frac{-y}{40} = \frac{z}{20}$ *or* $\frac{x}{2} = \frac{-y}{2} = \frac{z}{1}$

$\therefore$ the Eigen vector X_3 for $\lambda = 15$ *is* $[2 \quad -2 \quad 1]$

2. **Find all the Eigen values and the corresponding Eigen vectors of the matrix**

$$\begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}$$

Solution: The characteristic equation of A is $|A - \lambda I| = 0$,

$$\begin{vmatrix} 6-\lambda & -2 & 2 \\ -2 & 3-\lambda & -1 \\ 2 & -1 & 3-\lambda \end{vmatrix} = 0$$

On expanding, we have, $\lambda^3 - 12\lambda^2 + 36\lambda - 32 = 0$

After solving, we get $\lambda = 2,\ 2,\ 8$ are the Eigen values.

Now the system of equations is
$$(6-\lambda)x - 2y + 2z = 0$$
$$-2x + (3-\lambda)y - z = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots(1)$$
$$2x - y + (3-\lambda)z = 0$$

Case (i): Let $\lambda = 2$ and the system of equations becomes,

$$4x - 2y + 2z = 0 \quad - (i)$$
$$-2x + y - z = 0 \quad - (ii)$$
$$2x - y + z = 0 \quad - (iii)$$

The above sets of equations are all same as we have only one independent equation $2x - y + z = 0$ and hence we can choose two variables arbitrarily.

Let $z = k_1$ and $y = k_2$, $x = \frac{(k_2 - k_1)}{2}$

$X_1 = \left(\frac{(k_2 - k_1)}{2}, k_2, k_1 \right)$ is the eigen vector corresponding to $\lambda = 2$.

Case (ii): Let $\lambda = 8$ and the system of equations from (1) becomes,

$$-2x - 2y + 2z = 0 \quad - (iv)$$
$$-2x - 5y - z = 0 \quad - (v)$$
$$2x - y + -5z = 0 \quad - (vi)$$

Applying the rule of cross multiplication for (iv) and (v)

i. e., $\frac{x}{12} = \frac{-y}{-6} = \frac{z}{6} \quad or \quad \frac{x}{2} = \frac{y}{-1} = \frac{z}{1}$

$\therefore$ the Eigen vector X_2 for $\lambda = 8\, is$ $[2 \quad -1 \quad 1]$

3. **Find all the Eigen values and the corresponding Eigen vectors of the matrix**

$$\begin{bmatrix} -3 & -7 & -5 \\ 2 & 4 & 3 \\ 1 & 2 & 2 \end{bmatrix}$$

Solution: The characteristic equation of A is $|A - \lambda I| = 0$

$$\begin{vmatrix} -3-\lambda & -7 & -5 \\ 2 & 4-\lambda & 3 \\ 1 & 2 & 2-\lambda \end{vmatrix} = 0$$

On expanding, we obtain , $(\lambda - 1)^3 = 0$

by solving, we get $\lambda = 1, 1, 1$. All the Eigen values are equal, we now form the system of equations,

Now the system of equations is
$$\begin{aligned} &(-3-\lambda)x - 7y - 5z = 0 \\ &2x + (4-\lambda)y + 3z = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots(1) \\ &x + 2y + (2-\lambda)z = 0 \end{aligned}$$

Case (i): Let $\lambda = 1$ and the system of equations becomes,

$$-4x - 7y - 5z = 0 \quad - (i)$$
$$2x + 3y + 3z = 0 \quad - (ii)$$
$$x + 2y + z = 0 \quad - (iii)$$

Applying the rule of cross multiplication for (i) and (ii)

$$\frac{x}{\begin{vmatrix} -7 & -5 \\ 3 & 3 \end{vmatrix}} = \frac{-y}{\begin{vmatrix} -4 & -5 \\ 2 & 3 \end{vmatrix}} = \frac{z}{\begin{vmatrix} -4 & -7 \\ 2 & 3 \end{vmatrix}}$$

i. e., $\frac{x}{-6} = \frac{-y}{-2} = \frac{z}{2} \quad or \quad \frac{x}{-3} = \frac{y}{1} = \frac{z}{1}$

$\therefore$ (x, y, z) are proportional to $(-3,\ 1,\ 1)$

$\therefore$ The Eigen vector X_1 for $\lambda = 1\, is$ $[-3 \quad 1 \quad 1]$

4. **Find all the Eigen values and the corresponding Eigen vectors of the matrix**

$$\begin{bmatrix} 1 & 1 & 3 \\ 1 & 5 & 1 \\ 3 & 1 & 1 \end{bmatrix}$$

Solution: The characteristic equation of A is

$$\begin{vmatrix} 1-\lambda & 1 & 3 \\ 1 & 5-\lambda & 1 \\ 3 & 1 & 1-\lambda \end{vmatrix} = 0$$

On expanding, we obtain , $\lambda^3 - 7\lambda^2 + 36 = 0$
by solving, we get $\lambda = -2,\ 6,\ 3$.

Now the system of equations is
$$\begin{aligned} (1-\lambda)x + y + 3z &= 0 \\ x + (5-\lambda)y + z &= 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots(1) \\ 3x + y + (1-\lambda)z &= 0 \end{aligned}$$

Case (i): Let $\lambda = -2$ and corresponding equations are

$$3x + y + 3z = 0 - \ (i)$$
$$x + 7y + z = 0 - \ (ii)$$
$$3x + y + 3z = 0 - (iii)$$

Applying the rule of cross multiplication for (i) and (ii)

i. e., $\dfrac{x}{-20} = \dfrac{-y}{0} = \dfrac{z}{20}$ $\quad or \quad$ $\dfrac{x}{-1} = \dfrac{y}{0} = \dfrac{z}{1}$

$\therefore$ (x, y, z) are proportional to $(-1,\ 0,\ 1)$

$\therefore$ The Eigen vector X_1 for $\lambda = -2\, is$ $[-1 \quad 0 \quad 1]$

Case (ii): Let $\lambda = 3$ and corresponding equations are

$$-2x + y + 3z = 0 - \ (iv)$$
$$x + 2y + z = 0 \quad - \ (v)$$
$$3x + y - 2z = 0 \quad - (vi)$$

Applying the rule of cross multiplication for (iv) and (v)

i. e., $\dfrac{x}{-5} = \dfrac{-y}{-5} = \dfrac{z}{-5}$ $\quad or \quad$ $\dfrac{x}{-1} = \dfrac{y}{1} = \dfrac{z}{-1}$

$\therefore$ (x, y, z) are proportional to $(-1,\ 1,\ -1)$

$\therefore$ The Eigen vector X_2 for $\lambda = 3\, is$ $[-1 \quad 1 \quad -1]$

Case (iii): Let $\lambda = 6$ and corresponding equations are
$$\begin{aligned} -5x + y + 3z &= 0 - \ (vii) \\ x - y + z &= 0 \quad - \ (viii) \\ 3x + y - 5z &= 0 \quad - (ix) \end{aligned}$$

Applying the rule of cross multiplication for (vii) and (viii)

i. e., $\dfrac{x}{4} = \dfrac{-y}{-8} = \dfrac{z}{4}$ $\quad or \quad$ $\dfrac{x}{1} = \dfrac{y}{2} = \dfrac{z}{1}$

$\therefore (x, y, z)$ are proportional to $(1, \ 2, \ 1)$

$\therefore$ The Eigen vector X_3 for $\lambda = 6 \, is \ [1 \quad 2 \quad 1]$

5. **Find all the Eigen values and the corresponding Eigen vectors of the matrix**

$$\begin{bmatrix} 7 & -2 & 0 \\ -2 & 6 & -2 \\ 0 & -2 & 5 \end{bmatrix}$$

Solution: The characteristic equation of A is

$$\begin{vmatrix} 7-\lambda & -2 & 0 \\ -2 & 6-\lambda & -2 \\ 0 & -2 & 5-\lambda \end{vmatrix} = 0$$

On expanding, we obtain , $\lambda^3 - 18\lambda^2 + 99\lambda - 162 = 0$

by solving, we get $\lambda = 3, \ 6, \ 9$.

Now the system of equations is
$$\begin{aligned} &(7-\lambda)x - 2y + 0z = 0 \\ &-2x + (6-\lambda)y - 2z = 0 \ \ldots\ldots\ldots\ldots\ldots\ldots(1) \\ &0x - 2y + (5-\lambda)z = 0 \end{aligned}$$

Case (i): Let $\lambda = 3$ and corresponding equations are

$$\begin{aligned} &4x - 2y + 0z = 0 \quad - \ (i) \\ &-2x + 3y - 2z = 0 \ - \ (ii) \\ &0x - 2y + 2z = 0 \quad - (iii) \end{aligned}$$

Applying the rule of cross multiplication for (i) and (ii)

i. e., $\frac{x}{4} = \frac{-y}{8} = \frac{z}{8} \quad or \quad \frac{x}{1} = \frac{y}{2} = \frac{z}{2}$

$\therefore (x, y, z)$ are proportional to $(1, \ 2, \ 2)$

$\therefore$ The Eigen vector X_1 for $\lambda = 3 \ is \ [1 \quad 2 \quad 2]$

Case (ii): Let $\lambda = 6$ and corresponding equations are

$$\begin{aligned} &x - 2y + 0z = 0 - \ (iv) \\ &-2x - 0y - 2z = 0 \ - \ (v) \\ &0x - 2y - z = 0 \quad - (vi) \end{aligned}$$

Applying the rule of cross multiplication for (iv) and (v)

i. e., $\frac{x}{4} = \frac{-y}{-2} = \frac{z}{-4} \quad or \quad \frac{x}{2} = \frac{y}{1} = \frac{z}{-2}$

$\therefore (x, y, z)$ are proportional to $(2, \ 1, \ -2)$

$\therefore$ The Eigen vector X_2 for $\lambda = 6 \ is \ [2 \quad 1 \quad -2]$

Case (iii): Let $\lambda = 9$ and corresponding equations are
$$\begin{aligned} &-2x - 2y + 0z = 0 - \ (vii) \\ &-2x - 3y - 2z = 0 \quad - \ (viii) \\ &0x - 2y - 4z = 0 \quad - (ix) \end{aligned}$$

Applying the rule of cross multiplication for (vii) and (viii)

i. e., $\frac{x}{4}=\frac{-y}{4}=\frac{z}{2}$ $\quad or \quad$ $\frac{x}{2}=\frac{y}{-2}=\frac{z}{1}$

$\therefore$ (x,y,z) are proportional to $(2,\ -2,\ 1)$

$\therefore$ The Eigen vector X_3 for $\lambda=9\ is$ $[2 \quad -2 \quad 1]$

Properties of Eigen values:

I. Any square matrix A and its transpose A' have the same Eigen values.

Proof:-

We have

$$(A-\lambda I)' = A' - \lambda I' = A' - \lambda I$$

$$\therefore \left|(A-\lambda I)'\right| = \left|A' - \lambda I\right|$$

$$|A-\lambda I| = 0 \Leftrightarrow |A' - \lambda I| = 0$$

$\Rightarrow \lambda$ is an Eigen value of $A \Leftrightarrow \lambda$ is an Eigen value of A^1.

II. The Eigen values of a triangular matrix are just the diagonal elements of the matrix.

Proof: Let $A=\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ 0 & a_{22} & \dots & a_{2n} \\ .. & .. & & .. \\ .. & .. & & .. \\ 0 & 0 & & a_{nn} \end{bmatrix}$ be a triangular matrix of order n.

Then $|A-\lambda I| = (a_{11}-\lambda)(a_{22}-\lambda)\dots(a_{nn}-\lambda)$.

$\therefore$ Roots of $|A-\lambda I|=0$ are $\lambda=a_{11}, \lambda=a_{22},\dots\dots,\lambda=a_{nn}$.

$\therefore$ The Eigen values of A are the diagonal elements of A.

III. The Eigen values of an idempotent matrix are either zero or unity.

Proof:- Let A be an idempotent matrix.

$\therefore \quad A^2 = A$.

Let λ be an Eigen value of A.

Then there exists a non-zero vector x, such that $AX=\lambda X$ …….(1)

$\therefore A(AX) = A(\lambda X)$

$\Rightarrow A^2X = \lambda(AX)$

$\Rightarrow AX = \lambda(AX)$. $\quad [\because A^2 = A \quad \text{and} \quad AX=\lambda X]$

$\Rightarrow AX = \lambda^2 X$ …………(2)

From (1), (2)

$\Rightarrow \quad \lambda^2 X = \lambda X \qquad \Rightarrow \left(\lambda^2 - \lambda\right) X = 0$

$\Rightarrow \lambda^2 - \lambda = 0 \Rightarrow \lambda = 0$ or 1.

IV. The sum of the Eigen values of a matrix is the sum of the elements of the principal diagonal.

Proof: Consider the square matrix $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$

Then $|A - \lambda I| = \begin{vmatrix} a_{11}-\lambda & a_{12} & a_{13} \\ a_{21} & a_{22}-\lambda & a_{23} \\ a_{31} & a_{32} & a_{33}-\lambda \end{vmatrix}$

$$|A-\lambda I| = -\lambda^3 + \lambda^2[a_{11}+a_{22}+a_{33}] - \lambda[a_{22}a_{33}+a_{11}a_{22}+a_{11}a_{33}-a_{23}a_{32}-a_{21}a_{12}+a_{13}a_{31}]$$
$$+[a_{11}a_{22}a_{33}-a_{12}a_{21}a_{33}+a_{23}a_{31}a_{12}+a_{13}a_{21}a_{32}-a_{13}a_{31}a_{22}-a_{11}a_{23}a_{32}]................(1)$$

If $\lambda_1, \lambda_2, \lambda_3$ are the Eigen values of $|A - \lambda I|$, then

$$|A-\lambda I| = (-1)^3(\lambda-\lambda_1)(\lambda-\lambda_2)(\lambda-\lambda_3)$$
$$= -\lambda^3 + \lambda^2[\lambda_1+\lambda_2+\lambda_3] - \lambda[\lambda_2\lambda_3+\lambda_1\lambda_3+\lambda_1\lambda_2] + \lambda_1\lambda_2\lambda_3.......(2)$$

Equating the RHS of (1) and (2) and comparing the coefficients of λ^2,

we get $\lambda_1 + \lambda_2 + \lambda_3 = a_{11} + a_{22} + a_{33}$.

V. The product of the Eigen values of a matrix A is equal to its determinant. (Try yourself).

Proof: Consider the square matrix $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$

$|A - \lambda I| = \begin{vmatrix} a_{11}-\lambda & a_{12} & a_{13} \\ a_{21} & a_{22}-\lambda & a_{23} \\ a_{31} & a_{32} & a_{33}-\lambda \end{vmatrix}$

$$|A-\lambda I| = -\lambda^3 + \lambda^2[a_{11}+a_{22}+a_{33}] - \lambda[a_{22}a_{33}+a_{11}a_{22}+a_{11}a_{33}-a_{23}a_{32}-a_{21}a_{12}+a_{13}a_{31}]$$
$$+[a_{11}a_{22}a_{33}-a_{12}a_{21}a_{33}+a_{23}a_{31}a_{12}+a_{13}a_{21}a_{32}-a_{13}a_{31}a_{22}-a_{11}a_{23}a_{32}]................(1)$$

If $\lambda_1, \lambda_2, \lambda_3$ are the Eigen values of $|A - \lambda I|$, then

$$|A-\lambda I| = (-1)^3(\lambda-\lambda_1)(\lambda-\lambda_2)(\lambda-\lambda_3)$$
$$= -\lambda^3 + \lambda^2[\lambda_1+\lambda_2+\lambda_3] - \lambda[\lambda_2\lambda_3+\lambda_1\lambda_3+\lambda_1\lambda_2] + \lambda_1\lambda_2\lambda_3.......(2)$$

Equating the RHS of (1) and (2),
we get

$$\lambda_1\lambda_2\lambda_3 = a_{11}a_{22}a_{33} - a_{12}a_{21}a_{33} + a_{23}a_{31}a_{12} + a_{13}a_{21}a_{32} - a_{13}a_{31}a_{22} - a_{11}a_{23}a_{32}$$

VI. If λ is an Eigen value of a matrix A, then $1/\lambda$ is the Eigen value of A^{-1}.

Proof: Let X be the Eigen vector corresponding to λ.
Then $AX = \lambda X$..........(1)

Pre multiply both sides by A^{-1}, we get

$$A^{-1}AX = A^{-1}\lambda X$$
$$IX = \lambda A^{-1}X$$
$$X = \lambda\left(A^{-1}X\right)$$
$$\therefore A^{-1}X = \left(\frac{1}{\lambda}\right)X........(2)$$

Comparing (1) and (2), we have $\frac{1}{\lambda}$ is the Eigen value of the inverse matrix A^{-1}.

VII. If λ is an Eigen value of an orthogonal matrix then $1/\lambda$ is also its Eigen value. (Try yourself).

Proof: We know that if $\lambda$is an eigen value of a matrix A, then $1/\lambda$ is an eigen value of A^{-1}[because from previous property]**.** Since A is an orthogonal matrix, $A^{-1}$is same as its transpose A'**.**

Therefore $1/\lambda$ is an eigen value of A'. But the matrices A and A' have the same eigen values, since the determinants $|A-\lambda I|$ and $|A'-\lambda I|$ are the same

Hence $1/\lambda$ is also an eigen value of A.

VIII. If $\lambda_1, \lambda_2, ...,\lambda_n$ are the Eigen values of a matrix A, then A^m has the Eigen values $\lambda_1^m, \lambda_2^m, ...\lambda_n^m$. Where m is a positive integer.

Proof: Let λ_1 be the Eigen value of A and X_1 be the corresponding Eigen vector.

Then $AXi = \lambda_i X_i$

Now $A^2X_i = A\left(AX_i\right)$

$$= A\left(\lambda_i X_i\right) = \lambda_i\left(AX_i\right) = \lambda_i\left(\lambda_i X_i\right)$$

$$A^2X_i = \lambda_i^2 X_i$$

Similarly, $A^3X_i = \lambda_i^3X_i$

In general, $A^mX_i = \lambda_i^mX_i$.

Hence, λ_i^m is an Eigen value of A^m.

Similarity of Matrices and Diagonalization of Matrices:

Two square matrices A and B of the same order are said to be similar if there exists a non-singular matrix P, such that $B = P^{-1}AP$. Here B is said to be similar to A.

Diagonalization of a square matrix:

Property: If A is a square matrix A of order n has n linearly independent eigen vectors then there exists an n^{th} order square matrix P such that $P^{-1}AP$ is a diagonal matrix. Here "P" is called as the *modal matrix*.

We shall establish this result by considering a square matrix A of order 3, to make an important and interesting observation.

- We find eigen values $\lambda_1, \lambda_2, \lambda_3$ and the corresponding eigen vectors,

$$X_1 = \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix}, \quad X_2 = \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix}, \quad X_3 = \begin{bmatrix} x_3 \\ y_3 \\ z_3 \end{bmatrix}$$ for the square matrix of order 3.

- We form the **Modal matrix P,**

Let the square matrix P be equal to $[X_1\ X_2\ X_3]$, i.e.,

$$P = \begin{bmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ z_1 & z_2 & z_3 \end{bmatrix}$$

Now

$$AP = A[X_1\ \ X_2\ \ X_3] = [AX_1\ \ AX_2\ \ AX_3] = [\lambda_1 X_1\ \ \lambda_2 X_2\ \ \lambda_3 X_3]$$

Or

$$AP = \begin{bmatrix} \lambda_1 x_1 & \lambda_2 x_2 & \lambda_3 x_3 \\ \lambda_1 y_1 & \lambda_2 y_2 & \lambda_3 y_3 \\ \lambda_1 z_1 & \lambda_2 z_2 & \lambda_3 z_3 \end{bmatrix} = \begin{bmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ z_1 & z_2 & z_3 \end{bmatrix} \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}$$

i.e., $AP = PD$ where D is the diagonal matrix represented by

$$D = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}$$

Consider $AP = PD$

Pre multiplying by P^{-1} we have

$P^{-1}AP = P^{-1}PD = (P^{-1}P)D = ID = D$

$P^{-1}AP = D$

It is important that $P^{-1}AP$ is a diagonal matrix having the eigen values of $A(\lambda_1 \quad \lambda_2 \quad \lambda_3)$ in its principal diagonal. We say that the matrix P diagonalizes A, where P is constituted by the eigen vectors of A.

Computation of powers of square matrix:

Diagonalization of a square matrix A also helps us to find the powers of $A: A^2 \quad A^3 \quad A^4$,........etc.

We have $D = P^{-1}AP$

$\therefore D^2 = (P^{-1}AP)(P^{-1}AP) = P^{-1}A(PP^{-1})AP = P^{-1}A[AP] = P^{-1}A^2P$.

i.e., $D^2 = P^{-1}A^2P$

Pre multiplying by P and post multiplying by P^{-1} we have

$PD^2P^{-1} = (PP^{-1})A^2(PP^{-1}) = IA^2I = A^2$

i.e., $A^2 = PD^2P^{-1}$.

Thus in general, $A^n = PD^nP^{-1}$ where $D^n = \begin{bmatrix} \lambda_1^n & 0 & 0 \\ 0 & \lambda_2^n & 0 \\ 0 & 0 & \lambda_3^n \end{bmatrix}$

Working procedure for diagonalization of a square matrix A of order 3:

- We find Eigen values $\lambda_1 \quad \lambda_2 \quad \lambda_3$.
- We find the Eigen vectors $X_1 \quad X_2 \quad X_3$ corresponding to the eigen values $\lambda_1 \quad \lambda_2 \quad \lambda_3$.
- We form the modal matrix $P = [X_1 \quad X_2 \quad X_3] = \begin{bmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ z_1 & z_2 & z_3 \end{bmatrix}$
- We compute $P^{-1} = \dfrac{1}{|P|}(AdjP)$.

The diagonalization of A is given by $D = P^{-1}AP$ where $D = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}$.

Problems:

1. **Reduce the matrix $A = \begin{bmatrix} -1 & 3 \\ -2 & 4 \end{bmatrix}$ to the diagonal form and hence find A^4.**

Solution: The characteristic equation of A is $|A - \lambda I| = 0$

$$\begin{vmatrix} (-1-\lambda) & 3 \\ -2 & 4-\lambda \end{vmatrix} = 0$$

$$(-1-\lambda)(4-\lambda) + 6 = 0$$

$\lambda^2 - 3\lambda + 2 = 0$

$\therefore \lambda = 1,\ 2$ are the eigen values of A .

Now consider $[A - \lambda I][X] = [0]$

$$\begin{bmatrix} (-1-\lambda) & 3 \\ -2 & (4-\lambda) \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$(-1-\lambda)x + 3y = 0$

$-2x + (4-\lambda)y = 0$

Case-(i): Let $\lambda = 1$, We get $-2x + 3y = 0$ or $2x = 3y$ or $\frac{x}{3} = \frac{y}{2}$.

$\therefore X_1 = (3 \quad 2)'$ is the eigen vector corresponding to $\lambda = 1$.

Case-(ii): Let $\lambda = 2$, We get $-3x + 3y = 0$ or $x = y$ or $\frac{x}{1} = \frac{y}{1}$

$\therefore X_2 = (1 \quad 1)'$ is the eigen vector corresponding to $\lambda = 2$.

Modal matrix: $P = [X_1 \quad X_2] = \begin{bmatrix} 3 & 1 \\ 2 & 1 \end{bmatrix}$

We have $|P| = 1$ and $P^{-1} = \frac{1}{|P|}(AdjP)$

$$P^{-1} = \begin{bmatrix} 1 & -1 \\ -2 & 3 \end{bmatrix}$$

$$Now\ P^{-1}AP = \begin{bmatrix} 1 & -1 \\ -2 & 3 \end{bmatrix}\begin{bmatrix} -1 & 3 \\ -2 & 4 \end{bmatrix}\begin{bmatrix} 3 & 1 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ -4 & 6 \end{bmatrix}\begin{bmatrix} 3 & 1 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$$

Thus $P^{-1}AP = D$

$= \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$ is the diagonal matrix.

or $P^{-1}AP = Diag(1 \quad 2)$.

Now to find out A^4, consider the following we have $A^n = PD^nP^{-1}$

$A^4 = PD^4P^{-1}$ where $D^4 = \begin{bmatrix} 1^4 & 0 \\ 0 & 2^4 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 16 \end{bmatrix}$

$$A^4 = \begin{bmatrix} 3 & 1 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 16 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ -2 & 3 \end{bmatrix} = \begin{bmatrix} -29 & 45 \\ -30 & 46 \end{bmatrix}$$

Thus $A^4 = \begin{bmatrix} -29 & 45 \\ -30 & 46 \end{bmatrix}$.

2. **Reduce the matrix** $A=\begin{bmatrix}11 & -4 & -7\\ 7 & -2 & -5\\ 10 & -4 & -6\end{bmatrix}$ **into a diagonal matrix. Also find** A^5 **.**

Solution: The characteristic equation of A is $|A-\lambda I|=0$

$$\begin{bmatrix}11-\lambda & -4 & -7\\ 7 & (-2-\lambda) & -5\\ 10 & -4 & (-6-\lambda)\end{bmatrix}=0$$

$$\Rightarrow (11-\lambda)\,[(-2-\lambda)(-6-\lambda)-20]+4[7(-6-\lambda)+50]-7[-28-10(-2-\lambda)]=0$$

$$\Rightarrow (11-\lambda)[\lambda^2+8\lambda-8]+4[8-7\lambda]-7[10\lambda-8]=0$$

$$\Rightarrow 11\lambda^2+88\lambda-88-\lambda^3-8\lambda^2+8\lambda+32-28\lambda-70\lambda+56=0$$

$$\Rightarrow \lambda^3-3\lambda^2-2\lambda=0$$

$$\lambda=0,\quad 1,\quad 2.$$

Now consider $[A-\lambda I][X]=[0]$.

$$(11-\lambda)x-4y-7z=0$$

$$7x+(-2-\lambda)y-5z=0$$

$$10x-4y+(-6-\lambda)z=0$$

Case (i): Let $\lambda=0$ and the corresponding equations are

$$11x-4y-7z=0$$

$$7x-2y-5z=0$$

$$10x-4y-6z=0$$

$$\frac{x}{6}=\frac{-y}{-6}=\frac{z}{6}\text{ or }\frac{x}{1}=\frac{y}{1}=\frac{z}{1}$$

$X_1=(1\quad 1\quad 1)'$ is the eigen vector corresponding to $\lambda=0$.

Case (ii): Let $\lambda=1$ and the corresponding equations are

$$10x-4y-7z=0$$

$$7x-3y-5z=0$$

$$10x-4y-7z=0$$

$$\frac{x}{-1}=\frac{-y}{-1}=\frac{z}{-2}\text{ or }\frac{x}{1}=\frac{y}{-1}=\frac{z}{2}$$

$X_2=(1\quad -1\quad 2)'$ is the eigen vector corresponding to $\lambda=1$.

Case (iii): Let $\lambda=2$ and the corresponding equations are

$$9x-4y-7z=0$$

$$7x-4y-5z=0$$

$$10x-4y-8z=0$$

$$\frac{x}{-8}=\frac{-y}{4}=\frac{z}{-8} \text{ or } \frac{x}{2}=\frac{y}{1}=\frac{z}{2}$$

$X_3=(2 \quad 1 \quad 2)'$ is the Eigen vector corresponding to $\lambda = 2$.

Hence the modal matrix $P=[X_1 \quad X_2 \quad X_3]=\begin{bmatrix} 1 & 1 & 2 \\ 1 & -1 & 1 \\ 1 & 2 & 2 \end{bmatrix}$

We have $|P| = 1(-2-2)-1(2-1)+2(2+1)=1$

$$AdjP=\begin{bmatrix} -4 & 2 & 3 \\ -1 & 0 & 1 \\ 3 & -1 & -2 \end{bmatrix}$$

$$P^{-1}=\frac{1}{|P|}(AdjP)=\begin{bmatrix} -4 & 2 & 3 \\ -1 & 0 & 1 \\ 3 & -1 & -2 \end{bmatrix}$$

<u>Diagonalization of A is given by $P^{-1}AP$:</u>

Now $P^{-1}AP=\begin{bmatrix} -4 & 2 & 3 \\ -1 & 0 & 1 \\ 3 & -1 & -2 \end{bmatrix}\begin{bmatrix} 11 & -4 & -7 \\ 7 & -2 & -5 \\ 10 & -4 & -6 \end{bmatrix}\begin{bmatrix} 1 & 1 & 2 \\ 1 & -1 & 1 \\ 1 & 2 & 2 \end{bmatrix}$

$$=\begin{bmatrix} 0 & 0 & 0 \\ -1 & 0 & 1 \\ 6 & -2 & -4 \end{bmatrix}\begin{bmatrix} 1 & 1 & 2 \\ 1 & -1 & 1 \\ 1 & 2 & 2 \end{bmatrix}$$

$$=\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}=D$$

$P^{-1}AP=D=Diag(0 \quad 1 \quad 2)$.

Now to find out A^5, consider the following

we have $A^n=PD^nP^{-1}$.

$A^5=PD^5P^{-1}$ and $D^5=Diag(0^5 \quad 1^5 \quad 2^5)=Diag(0 \quad 1 \quad 32)$.

Hence $A^5=\begin{bmatrix} 1 & 1 & 2 \\ 1 & -1 & 1 \\ 1 & 2 & 2 \end{bmatrix}\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 32 \end{bmatrix}\begin{bmatrix} -4 & 2 & 3 \\ -1 & 0 & 1 \\ 3 & -1 & -2 \end{bmatrix}$

$$=\begin{bmatrix} 0 & 1 & 64 \\ 0 & -1 & 32 \\ 0 & 2 & 64 \end{bmatrix}\begin{bmatrix} -4 & 2 & 3 \\ -1 & 0 & 1 \\ 3 & -1 & -2 \end{bmatrix}$$

$$=\begin{bmatrix} 191 & -64 & -127 \\ 97 & -32 & -65 \\ 190 & -64 & -126 \end{bmatrix}.$$

3. **Determine the Eigen values and the corresponding Eigen values of the matrix**

$$A=\begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix}$$

Solution: The Characteristic equation of A is $|A-\lambda I|=0$

$$\begin{vmatrix} (-2-\lambda) & 2 & -3 \\ 2 & (1-\lambda) & -6 \\ -1 & -2 & (-\lambda) \end{vmatrix}=0$$

$$\Rightarrow (-2-\lambda)[-\lambda(1-\lambda)-12]-2[-2\lambda-6]-3[-4+1-\lambda]=0$$

$$\Rightarrow (-2-\lambda)[-\lambda+\lambda^2-12)+(4\lambda+12)+(9+3\lambda)=0$$

$$\Rightarrow (-2-\lambda)(\lambda+3)(\lambda-4)+4(\lambda+3)+3(\lambda+3)=0$$

$$\Rightarrow (\lambda+3)\left[(-2-\lambda)(\lambda-4)+4+3\right]=0$$

$$\Rightarrow (\lambda+3)(-\lambda^2+2\lambda+15)=0$$

$$\Rightarrow (\lambda+3)(\lambda+3)(\lambda-5)=0$$

$\lambda=-3 \quad -3 \quad 5.$

We now form the system of equations.

$$(-2-\lambda)+2y-3z=0$$

$$2x+(1-\lambda)y-6z=0$$

$$-1x-2y-\lambda z=0.$$

Case (i): Let $\lambda=-3$ and the corresponding equations are

$$x+2y-3z=0$$

$$2x+4y-6z=0$$

$$-x-2y+3z=0.$$

It should be observed that the equations are all same and we have only one independent equation $x+2y-3z=0$

(In case the rule of cross multiplication is applied, we get $x=y=z=0$ which is a trivial solution.)

Two variables can be arbitrary.

Let $z=k_1$, $y=k_2$ $\therefore$ $x=3k_1-2k_2$

The eigen vector corresponding to the coincident eigen value $\lambda=-3$ be denoted by

$X_{1,2}$ and we have $X_{1,2} = (3k_1 - 2K_2 \quad k_2 \quad k_1)'$ where k_1, k_2 are arbitray. We chooseconvenient values for k_1 and k_2 to obtain two distinct eigen vectors.

(i) Let $k_1 = 1,\ k_2 = 1 \therefore\ X_1 = (1 \quad 1 \quad 1)'$

(ii) Let $k_1 = 1,\ k_2 = 0 \therefore\ X_2 = (3 \quad 0 \quad 1)'$

Case (ii): Let $\lambda = 5$ and the corresponding equations are

$$-7x + 2y - 3z = 0 \quad \ldots\ldots (1)$$
$$2x - 4y - 6z = 0 \quad \ldots\ldots. (2)$$
$$-x - 2y - 5z = 0$$

Solving (1) and (2) , $\dfrac{x}{-12-12} = \dfrac{-y}{42+6} = \dfrac{z}{28-4}$

$$\frac{x}{-24} = \frac{-y}{48} = \frac{z}{24} \text{ or } \frac{x}{1} = \frac{y}{2} = \frac{z}{-1}$$

$X_3 = (1 \quad 2 \quad -1)'$ is the eigen vector corresponding to $\lambda = 5$.

<u>We have modal matrix</u>

$$P = [X_1 \quad X_2 \quad X_3] = \begin{bmatrix} 1 & 3 & 1 \\ 1 & 0 & 2 \\ 1 & 1 & -1 \end{bmatrix}$$

$$|P| = 1(-2) - 3(-3) + 1(1) = 8$$

$$AdjP = \begin{bmatrix} +(0-2) & -(-3-1) & +(6-0) \\ -(-1-2) & +(-1-1) & -(2-1) \\ +(1-0) & -(1-3) & +(0-3) \end{bmatrix} = \begin{bmatrix} -2 & 4 & 6 \\ 3 & -2 & -1 \\ 1 & 2 & -3 \end{bmatrix}$$

Diagonalization of A is given by $P^{-1}AP$,

$$= \frac{1}{8}\begin{bmatrix} -2 & 4 & 6 \\ 3 & -2 & -1 \\ 1 & 2 & -3 \end{bmatrix}\begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix}\begin{bmatrix} 1 & 3 & 1 \\ 1 & 0 & 2 \\ 1 & 1 & -1 \end{bmatrix}$$

$$= \frac{1}{8}\begin{bmatrix} 6 & -12 & -18 \\ -9 & 6 & 3 \\ 5 & 10 & -15 \end{bmatrix}\begin{bmatrix} 1 & 3 & 1 \\ 1 & 0 & 2 \\ 1 & 1 & -1 \end{bmatrix}$$

$$=\frac{1}{8}\begin{bmatrix} -24 & -0 & 0 \\ 0 & -24 & 0 \\ 0 & 0 & 40 \end{bmatrix}$$

$$=\begin{bmatrix} -3 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 5 \end{bmatrix} = D$$

Thus $P^{-1}AP = D = Diag(-3 \quad -3 \quad 5)$.

4. **Show that the following matrix is not diagnosable** $A = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix}$

Solution: The characteristic equation of A is $|A - \lambda I| = 0$

$$\begin{bmatrix} 2-\lambda & 1 & 0 \\ 0 & 2-\lambda & 1 \\ 0 & 0 & 2-\lambda \end{bmatrix} = 0$$

$$(2-\lambda)^3 = 0$$

$$\lambda = 2, \quad 2, \quad 2..$$

The eigen vector corresponding to $\lambda = 2$ has to be obtained by solving the system of equations.

$$(2-2)x + 1y + 0Z = 0$$

$$0x + (2-2)y + 1z = 0$$

$$0x + 0y + (2-2)z = 0$$

$y = 0$, $z = 0$; x can be arbitrary.

$\therefore x = k$, $y = 0$, $z = 0$ is the eigen vector corresponding to the coincident eigen value $\lambda = 2$.

It is evident that we cannot obtain three linearly independent Eigen vectors. Thus we conclude that the matrix A is not diagnozable.

Conversion of nth order differential equation into a system of first order differential equations:

Consider nth order D.E. $y^{(n)} = F\left(t, y, y', y'', \ldots\ldots\ldots, y^{(n-1)}\right)$……………………….. (1)

can always be reduced to a system of 'n' first order D.E.'s simply by setting

$$\left.\begin{array}{l} y = x_1 \\ y' = x_2 \\ y'' = x_3 \\ y''' = x_4 \\ \vdots \quad \vdots \\ y^{(n-1)} = x_n \end{array}\right\} \text{by differentiating these we have} \Rightarrow \left.\begin{array}{l} x_1' = y' = x_2 \\ x_2' = y'' = x_3 \\ x_3' = y''' = x_4 \\ \vdots \quad \vdots \\ x_n' = y^{n} = F(t, y, y', y'', \ldots, y^{(n-1)}) \\ \Rightarrow x_n' = y^{n} = F(t, x_1, x_2 x_3, \ldots, x_n \end{array}\right\} \ldots\ldots(2)$$

Examples:

1. **Write the following 3rd order differential equation as a system of first order, linear differential equations:** $\dfrac{d^3y}{dt^3} + a_2\dfrac{d^2y}{dt^2} + a_1\dfrac{dy}{dt} + a_0 y = 0$

   Solution: Let us assume that we have a higher order differential equation

$$\frac{d^3y}{dt^3} + a_2\frac{d^2y}{dt^2} + a_1\frac{dy}{dt} + a_0 y = 0$$

Our goal is to convert 3rd order differential equation into system of first order differential equations and can be written in matrix form. Replace all the terms of lower than the order with different variable. Since this is the 3rd order differential equation, we will replace 2nd, 1st, 0th term with other variables as shown below.

Now,

$$\left.\begin{array}{l} y = x_1 \\ y' = x_2 \\ y'' = x_3 \end{array}\right\} \text{by differentiating these we have} \Rightarrow \left.\begin{array}{l} x_1' = y' = x_2 \\ x_2' = y'' = x_3 \\ x_3' = y''' = -a_0 y - a_1 y' - a_2 y'' \\ x_3' = -a_0 x_1 - a_1 x_2 - a_2 x_3 \end{array}\right\}.$$

Now collect all first order differential equations,

$$\left.\begin{array}{l} x_1' = x_2 \\ x_2' = x_3 \\ x_3' = -a_0 x_1 - a_1 x_2 - a_2 x_3 \end{array}\right\} \Rightarrow \left.\begin{array}{l} x_1' = 0x_1 + 1x_2 + 0x_3 \\ x_2' = 0x_1 + 0x_2 + 1x_3 \\ x_3' = -a_0 x_1 + (-a_1)x_2 + (-a_2)x_3 \end{array}\right\} \Rightarrow \begin{bmatrix} x_1' \\ x_2' \\ x_3' \end{bmatrix} = \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ -a_0 & -a_1 & -a_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$\Rightarrow X^{I} = AX$$

This represents homogenous differential equations.

2. **Write the following 3rd order differential equation as a system of first order, linear differential equations:** $\dfrac{d^3y}{dt^3} + a_2\dfrac{d^2y}{dt^2} + a_1\dfrac{dy}{dt} + a_0 y = u(t)$

   Solution: Let us consider, that we have a 3rd order differential equation

$$\frac{d^3y}{dt^3} + a_2\frac{d^2y}{dt^2} + a_1\frac{dy}{dt} + a_0 y = u(t)$$

Our goal is to convert 3rd order differential equation into system of first order differential equations and can be written in matrix form. Replace all the terms of lower than the order with different variable. Since this is the 3rd order differential equation, we will replace 2nd, 1st, 0th term with other variables as shown below.

$$\left.\begin{aligned} y &= x_1 \\ y' &= x_2 \\ y'' &= x_3 \end{aligned}\right\}\text{by differentiating these we have} \Rightarrow \left.\begin{aligned} x_1' &= y' = x_2 \\ x_2' &= y'' = x_3 \\ x_3' &= y''' = -a_0 y - a_1 y' - a_2 y'' + u(t) \\ x_3' &= -a_0 x_1 - a_1 x_2 - a_2 x_3 + u(t) \end{aligned}\right\}.$$

Now collect all first order differential equations,

$$\left.\begin{aligned} x_1' &= x_2 \\ x_2' &= x_3 \\ x_3' &= -a_0 x_1 - a_1 x_2 - a_2 x_3 + u(t) \end{aligned}\right\} \Rightarrow \left.\begin{aligned} x_1' &= 0x_1 + 1x_2 + 0x_3 + 0u(t) \\ x_2' &= 0x_1 + 0x_2 + 1x_3 + 0u(t) \\ x_3' &= -a_0 x_1 - a_1 x_2 - a_2 x_3 + 1u(t) \end{aligned}\right\}$$

$$\Rightarrow \begin{bmatrix} x_1' \\ x_2' \\ x_3' \end{bmatrix} = \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ -a_0 & -a_1 & -a_2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u(t)$$

$$\Rightarrow X^I = AX + Bu(t)$$

This represents the non homogenous system of linear differential equations.

3. **Write the following 4th order differential equation as a system of first order linear differential equations:**

$$\frac{d^4y}{dt^4} + 3\frac{d^2y}{dt^2} - \sin t\frac{dy}{dt} + 8y = t^2; y(0) = 1; y'(0) = 2; y''(0) = 3; y'''(0) = 4;$$

Solution: Let us consider, that we have a 4th order differential equation

$$\frac{d^4y}{dt^4} + 3\frac{d^2y}{dt^2} - \sin t\frac{dy}{dt} + 8y = t^2$$

Put

$$\left.\begin{aligned} y &= x_1 \\ y' &= x_2 \\ y'' &= x_3 \\ y''' &= x_4 \end{aligned}\right\}\text{by differentiating these we have} \Rightarrow \left.\begin{aligned} x_1' &= y' = x_2 \\ x_2' &= y'' = x_3 \\ x_3' &= y''' = x_4 \\ x_4' &= y^{4} = -3y'' + \text{sint } y' - 8y + t^2 \\ x_4' &= -3x_3 + \text{sint } x_2 - 8x_1 + t^2 \end{aligned}\right\}.$$

Now consider all first order differential equations,

$$\left.\begin{aligned} x_1' &= x_2 \\ x_2' &= x_3 \\ x_3' &= x_4 \\ x_4' &= -8x_1 + \sin t\, x_2 - 3x_3 + t^2 \end{aligned}\right\} \Rightarrow \left.\begin{aligned} y(0) = 1 &\Rightarrow x_1(0) = 1 \\ y'(0) = 1 &\Rightarrow x_2(0) = 2 \\ y''(0) = 3 &\Rightarrow x_3(0) = 3 \\ y'''(0) = 4 &\Rightarrow x_4(0) = 4 \end{aligned}\right\}\text{Initial conditions.}$$

This represents the non homogenous system of linear differential equations.

Solve the system of linear differential equations by Diagonalization method and stability of the system of linear differential equations:

Solution to the system of homogenous differential equation by Diagonalization method:

Let us consider the linear first order homogenous differential equation of the form.

$$X' = AX$$

$$\begin{pmatrix} x_1' \\ x_2' \\ .. \\ x_n' \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ .. \\ x_n \end{pmatrix} \quad \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \ (1)$$

In which each x_i' is expressed as a linear combination of $x_1, x_2, x_3, \dots\dots\dots, x_n$

Let $X = PY$ …….(2), be the solution for the equation (1)

Put equation (2) in (1), we have

$PY' = APY \Rightarrow Y' = P^{-1}APY \Rightarrow Y' = DY$ where D is a diagonal matrix.

$$\begin{pmatrix} y_1' \\ y_2' \\ .. \\ y_n' \end{pmatrix} = \begin{pmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & \lambda_2 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & \lambda_n \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ .. \\ y_n \end{pmatrix}$$

Therefore each differential equation in the system is of the form

$$y_i' = \lambda_i y_i, \ i = 1,2,\dots\dots\dots, n.$$

The solution of each of these linear equations is $y_i = c_i e^{\lambda_i t}$, $i = 1,2,\dots\dots\dots, n$.

Hence the general solution is

$$Y = \begin{pmatrix} c_1 e^{\lambda_1 t} \\ c_2 e^{\lambda_2 t} \\ \\ c_n e^{\lambda_n t} \end{pmatrix}$$

Since the matrix P can be constructed from the eigen vectors of A.The general solution of the original system $X' = AX$ is obtained from $X = PY$.

Solution to the system of non-homogenous differential equation by Diagonalization method:

Let us consider non-homogenous differential equation $X^I = AX + F(t)$

$$\text{i.e.,} \begin{bmatrix} x_1' \\ x_2' \\ \vdots \\ x_n' \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} + \begin{bmatrix} F_1(t) \\ F_{12}(t) \\ \vdots \\ F_n(t) \end{bmatrix} \dots\dots\dots\dots(1)$$

Let $X = PY$. ….(2), be the solution for the non-homogenous equation

Substitute equation (2) in (1), we have

$(PY)' = APY + F$

$PY' = APY + F$

$Y' = (P^{-1}AP)Y + P^{-1}F$

$Y' = DY + P^{-1}F$....................(3)

Now solve equation (3) for $Y = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}$

Hence the solution for the non - homogenous differential equation $X^1 = AX + F$ is $X = PY$,

Where P = modal matrix

$$Y = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}$$

Eigen value and Stability of the system:

The table below gives a complete overview of the stability corresponding to each type of Eigen value.

Eigen values	Stability
All real & positive	Unstable
All real & negative	Stable
Mixed positive & negative real	Unstable
a+bi	Unstable
-a+bi	Stable
0+bi	Unstable

<u>Problems:</u>

1. **Solve** $X' = \begin{pmatrix} -2 & -1 & 8 \\ 0 & -3 & 8 \\ 0 & -4 & 9 \end{pmatrix} X$ **by diagonalization.**

Solution: Let $X=PY$, be the solution for above equation, so we need to find P and Y

$$|A - \lambda I| = 0 \Rightarrow -(\lambda + 2)(\lambda - 1)(\lambda - 5) = 0 \Rightarrow \lambda = -2, 1, 5 .$$

Now construct the homogenous differential equation, i.e.,

$$(-2 - \lambda)x - y + 8z = 0$$

$$0x + (-3 - \lambda)y + 8z = 0$$

$$0x - 4y + (9 - \lambda)z = 0$$

Since eigen values are distinct then eigen vectors are linearly independent.

Solving $|A-\lambda_i I|=0, \quad \text{for } i=1,2,3$

$$X_1=\begin{bmatrix}1\\0\\0\end{bmatrix}, X_2=\begin{bmatrix}2\\2\\1\end{bmatrix}, X_3=\begin{bmatrix}1\\1\\1\end{bmatrix}$$

Thus a modal matrix P, that Diagonalizes the coefficient matrix is $P=\begin{pmatrix}1 & 2 & 1\\0 & 2 & 1\\0 & 1 & 1\end{pmatrix}$

The entries on the main diagonal of D are the eigen values of A.

$$D=\begin{pmatrix}-2 & 0 & 0\\0 & 1 & 0\\0 & 0 & 5\end{pmatrix}$$

$$\text{and} \quad y=\begin{bmatrix}y_1\\y_2\\y_3\end{bmatrix}=\begin{bmatrix}c_1e^{\lambda_1 t}\\c_2e^{\lambda_2 t}\\c_3e^{\lambda_3 t}\end{bmatrix}=\begin{bmatrix}c_1e^{-2t}\\c_2e^{t}\\c_3e^{5t}\end{bmatrix}$$

Hence the solution of the given system is $X' = AX$ is,

$$X=PY=\begin{pmatrix}1 & 2 & 1\\0 & 2 & 1\\0 & 1 & 1\end{pmatrix}\begin{pmatrix}c_1e^{-2t}\\c_2e^{t}\\c_3e^{5t}\end{pmatrix}=\begin{pmatrix}c_1e^{-2t}+2c_2e^{t}+c_3e^{5t}\\2c_2e^{t}+c_3e^{5t}\\c_2e^{t}+c_3e^{5t}\end{pmatrix}$$

This can also be written in usual manner as

$$X=c_1\begin{pmatrix}1\\0\\0\end{pmatrix}e^{-2t}+c_2\begin{pmatrix}2\\2\\1\end{pmatrix}e^{t}+c_3\begin{pmatrix}1\\1\\1\end{pmatrix}e^{5t}$$

Stability: Since at least one eigen value is real & positive, hence the system is unstable.

Note: Solution by Diagonalization will always work provided we can find 'n' linearly independent Eigen vectors of $m \times n$ matrix A, the eigen values of A could be real & distinct, complex or repeated. The method fails when A has repeated Eigen values &'n' linearly independent eigen vectors cannot be found. In this situation A is not diagonalizable.

2. **Solve** $X'=\begin{pmatrix}4 & 2\\2 & 1\end{pmatrix}X+\begin{pmatrix}3e^{t}\\e^{t}\end{pmatrix}$ **by diagonalization and hence discuss the stability of the system.**

Solution: Let $X = PY$ be the solution of above non-homogenous differential equation.

$$A = \begin{bmatrix} 4 & 2 \\ 2 & 1 \end{bmatrix} \qquad F(t) = \begin{bmatrix} 3e^t \\ e^t \end{bmatrix}$$

Here

Now, $|A - \lambda I| = 0$

$$\begin{vmatrix} 4-\lambda & 2 \\ 2 & 1-\lambda \end{vmatrix} = 0$$

$\Rightarrow \lambda_1 = 0 \quad \text{and} \quad \lambda_2 = 5$

- The Eigen vector corresponding to Eigen value $\lambda_1 = 0$ is $X_1 = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$
- The Eigen vector corresponding to Eigen value $\lambda_2 = 5$ is $X_2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

The modal matrix

$$P = \begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix} \quad \text{and} \quad P^{-1} = \frac{1}{|P|} adj(P) = \frac{1}{5}\begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix}$$

$$\text{and} \quad P^{-1}F = \begin{bmatrix} 1/5 & -2/5 \\ 2/5 & 1/5 \end{bmatrix}\begin{bmatrix} 3e^t \\ e^t \end{bmatrix} = \begin{bmatrix} \frac{1}{5}e^t \\ \frac{7}{5}e^t \end{bmatrix}$$

$$Y' = DY + P^{-1}F$$

Now to compute Y: consider $Y' = \begin{bmatrix} 0 & 0 \\ 0 & 5 \end{bmatrix} Y + \begin{bmatrix} \frac{1}{5}e^t \\ \frac{7}{5}e^t \end{bmatrix}$

$$\begin{bmatrix} y_1' \\ y_2' \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 5 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} + \begin{bmatrix} \frac{1}{5}e^t \\ \frac{7}{5}e^t \end{bmatrix}$$

i.e., $y_1' = \frac{1}{5}e^t$ & $y_2' = 5y_2 + \frac{7}{5}e^t$

Integrating with respect to t, $y_1 = \frac{e^t}{5} + c_1 \; and \; y_2' - 5y_2 = \frac{7}{5}e^t$

This is of the form

$\frac{dy}{dx}+P(x)y=Q(x)$ is called leibnitz linear equation and $ye^{\int pdx}=\int Qe^{\int Pdx}$

Which is the required solution.

Here, $P(t)=-5$ and $Q(t)=\frac{7}{5}e^{t}$

$$e^{\int pdx}=e^{\int -5dx}=e^{-5t}$$

$$y_2e^{-5t}=\int \frac{7}{5}e^{t}e^{-5t}dt+c$$

$$y_2e^{-5t}=\frac{7}{5}\int e^{-4t}dt+c$$

$$y_2e^{-5t}=\frac{7}{5}\frac{e^{-4t}}{-4}\frac{1}{e^{-5t}}+ce^{5t}=-\frac{7}{20}e^{t}+ce^{5t}$$

Hence the solution of the original system is

$$X=PY=\begin{pmatrix}1 & 2\\ -2 & 1\end{pmatrix}\begin{pmatrix}\frac{1}{5}e^{t}+c_1\\ -\frac{7}{20}e^{t}+c_2e^{5t}\end{pmatrix}=\begin{pmatrix}-\frac{1}{2}e^{t}+c_1+2c_2e^{5t}\\ -\frac{3}{4}e^{t}-2c_1+c_2e^{5t}\end{pmatrix}$$

Writing in usual manner

$$X=c_1\begin{pmatrix}1\\ -2\end{pmatrix}+c_2\begin{pmatrix}2\\ 1\end{pmatrix}e^{5t}-\begin{pmatrix}1/2\\ 3/4\end{pmatrix}e^{t}$$

Stability: Since all the Eigen values are real, distinct & positive, hence the system is unstable.

---END---------------------------------------