
Final Portfolio: Week 8
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Question 2

(a) If $a_1, a_2, \ldots, a_n, \ldots$ is a sequence satisfying $a_n = \sum_{k=n-m}^{n} c_k a_{n-k}$, find a matrix A sending the vector $[a_{n-k}, ..., a_n]^T$ to $[a_{n-k+1}, ..., a_{n+1}]^T$.

Consider the matrix $A = \begin{bmatrix} 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 1 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & 1 \\ c_k & c_{k-1} & c_{k-2} & \ldots & c_1 \end{bmatrix}$.

This matrix will send $\begin{bmatrix} a_{n-k} \\ \vdots \\ a_k \end{bmatrix}$ to $\begin{bmatrix} a_{n-1+1} \\ \vdots \\ a_{n+1} \end{bmatrix}$.

(b) Find the 2×2 matrix which represents the Fibonacci sequence, $a_n = a_{n-1} + a_{n-2}$. Then find a formula for the nth term of the Fibonacci sequence using diagonalization.

The 2×2 matrix which represents the Fibonacci sequence is $A = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$ because the coefficients of the recurrence relation are 1.

In order to find the nth term of the Fibonacci sequence, we find $A^n \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. We can diagonalize $A = PDP^{-1}$ in order to make this an easier computation.

First, we find

$$\begin{aligned} det(A - \lambda I) &= (-\lambda)(1 - \lambda) - 1 \\ &= \lambda^2 - \lambda - 1 \end{aligned}$$

which means $\lambda_1 = \phi$ and $\lambda_2 = -\frac{1}{\phi}$ where $\phi = \frac{1+\sqrt{5}}{2}$ is the golden ratio. This gives us our diagonal matrix $D = \begin{bmatrix} \phi & 0 \\ 0 & -\frac{1}{\phi} \end{bmatrix}$.

We then use those eigenvalues to compute eigenvectors which create a basis for P. We find:

$$\begin{bmatrix}0 & 1\\1 & 1\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix} = \begin{bmatrix}\phi x\\\phi y\end{bmatrix}$$

Solving this equation for x and y we find our first eigenvector is $v_1 = \begin{bmatrix}\frac{1}{\phi}\\1\end{bmatrix}$.

We do the same process again to find v_2, but with $\lambda_2 = -\frac{1}{\phi}$ instead. We find that the corresponding eigenvector is $v_2 = \begin{bmatrix}-\phi\\1\end{bmatrix}$.

This means $P = \begin{bmatrix}\frac{1}{\phi} & -\phi\\1 & 1\end{bmatrix}$

We then use python to solve for P^{-1} and we simplify using the definition of ϕ. This results in $P^{-1} = \begin{bmatrix}\frac{\phi}{1+\phi^2} & \frac{\phi^2}{1+\phi^2}\\-\frac{\phi}{1+\phi^2} & \frac{1}{1+\phi^2}\end{bmatrix}$.

From here, we can see $A^n = PD^nP^{-1}$. This means if we want to solve for the nth term in the Fibonacci sequence we can find

$$A^n\begin{bmatrix}0\\1\end{bmatrix} = \begin{bmatrix}\frac{1}{\phi} & -\phi\\1 & 1\end{bmatrix}\begin{bmatrix}\phi^n & 0\\0 & -\frac{1}{\phi}^n\end{bmatrix}\begin{bmatrix}\frac{\phi}{1+\phi^2} & \frac{\phi^2}{1+\phi^2}\\-\frac{\phi}{1+\phi^2} & \frac{1}{1+\phi^2}\end{bmatrix}\begin{bmatrix}0\\1\end{bmatrix}$$

(c) Find the generalized eigenvalues and eigenvectors of the matrix A corresponding to the

$$a_{n+2} = 3a_{n+1} - 6a_n$$

and write A with respect to the basis given by generalized eigenvectors.
Note: In class, Prof. Masden changed our equation from $a_{n+2} = 3a_{n+1} - 6a_n$ to $a_{n+2} = -9a_{n+1} - 6a_n$.

From part (b), we know $A = \begin{bmatrix}0 & 1\\-9 & -6\end{bmatrix}$.

Solving for the eigenvalues we can find

$$\begin{aligned}\det(A - \lambda I) &= -\lambda(-6 - \lambda) + 9\\&= \lambda^2 + 6\lambda + 9\\&= (\lambda + 3)^2\end{aligned}$$

This means $\lambda = -3$. which is also the generalized eigenvalue. We then use this eigenvalue to find the generalized eigenvectors. We do this by solving $(A - 3I)v = 0$ for v. We can see $\begin{bmatrix} 3 & 1 \\ -9 & -3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$. We can then find $3x + y = 0$ So, $y = -3x$ and thus one of our eigenvectors is $v = \begin{bmatrix} -1 \\ 3 \end{bmatrix}$.

We then solve for v again and find $(A - 3I)^2 v = \begin{bmatrix} -1 \\ 3 \end{bmatrix}$. We can then see $\begin{bmatrix} 3 & 1 \\ -9 & -3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -1 \\ 3 \end{bmatrix}$ which we can then use to see, $3x + y = -1$. We can rearrange this into $3x = -1 - y$ which means $v = \begin{bmatrix} 1 \\ -4 \end{bmatrix}$. Thus, $G(-3, 2) = \begin{bmatrix} -1 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ -4 \end{bmatrix}$.

(d) Explain how the previous problem is connected to Jordan canonical form! Then find a formula for the nth power of your matrix above.

From our previous work, we know that the matrix P contains our generalized eigenvectors and the matrix J is the Jordan form of A that corresponds to our eigenvalue $\lambda = -3$.
So, $P = \begin{bmatrix} -1 & 1 \\ 3 & 4 \end{bmatrix}$ and $J = \begin{bmatrix} -3 & 1 \\ 0 & -3 \end{bmatrix}$ which follows immediately from our work in part (c). We can then calculate $P^{-1} = \begin{bmatrix} -\frac{4}{7} & \frac{1}{7} \\ \frac{3}{7} & -\frac{1}{7} \end{bmatrix}$

So we can see $A^n = PJ^nP^{-1} = \begin{bmatrix} -1 & 1 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} -3 & 1 \\ 0 & -3 \end{bmatrix}^n \begin{bmatrix} -\frac{4}{7} & \frac{1}{7} \\ \frac{3}{7} & -\frac{1}{7} \end{bmatrix} =$
$\begin{bmatrix} -1 & 1 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} (-3)^n - \binom{n}{2}3^{n-1} & 1 \\ 0 & (-3)^n - \binom{n}{2}3^{n-1} \end{bmatrix}^n \begin{bmatrix} -\frac{4}{7} & \frac{1}{7} \\ \frac{3}{7} & -\frac{1}{7} \end{bmatrix}$

Jordan Canonical form generalizes diagonalization. Our matrix from part (b) $A = \begin{bmatrix} 0 & 1 \\ -9 & -6 \end{bmatrix}$, is not diagonalizable which is why we use Jordan Canonical Form. Here, J is our Jordan block and P is our matrix of generalized eigenvectors. Our Jordan block corresponds to our only eigenvalue $\lambda = -3$, and we then compute A^n. We can still compute powers of A because Jordan canonical form allows us to compute A^n just like we would in the diagonalizable case, through our generalized eigenvectors and Jordan blocks.