Final Portfolio: Week 2
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Question 3.

Take it as given that the derivative operator, $D : C^{\infty}(\mathbb{R}) \to C^{\infty}(\mathbb{R})$ (i) has a one-dimensional kernel, and (ii) is surjective.

(a) Explain why D (the derivative) is a linear transformation $C^{\infty}(R) \to C^{\infty}(R)$.

We know D is a transformation because it maps a space onto itself. We must then show D is a *linear* transformation by showing D is closed under both scalar multiplication and addition. From the fourth edition of *Calculus: Early Transcendentals* by Jon Rogawski, Colin Adams and Robert Franzosa in Section 3.2, on page 136 Theorem 2 shows that $\frac{d}{dx}cf(x) = c\frac{d}{dx}f(x)$ and $\frac{d}{dx}f(x)+g(x) = \frac{d}{dx}f(x)+\frac{d}{dx}g(x)$. Because both of these equations are true, this means the derivative is a linear transformation. Therefore D is a linear transformation.

(b) Explain what the kernel of D is with respect to functions from $C^{\infty}(R) \to C^{\infty}(R)$, and how you know it is one-dimensional (refer to calculus).

Since D maps itself onto itself, we know from calculus that if $f(x) = c$ then $Df(x) = 0$. This is true because the only function whose derivative is zero is the constant function which we know is true from *Calculus: Early Transcendentals*, on page 315. So, we know the kernel is 1-dimensional since the only functions that get sent to zero after taking the derivative once are constants. Then a basis of the kernel could be 1 since 1 can generate any element of the real numbers.

(c) Find the kernel of $D^2 + 9I$ and demonstrate that it is 2-dimensional using tools from differential equations (in other words, I don't expect you to prove linear independence, but explain how it connects to a 2-d subspace).

This is a homogeneous second order differential equation because it is of the form $x'' + 9 = 0$. This means we can solve using a specific method. First we find the characteristic polynomial of this equation by treating it as a quadratic equation where $x'' = x^2$. We can then see $x^2 + 9 = 0$ will result in two values of λ, $\lambda_1 = 3i$ and $\lambda_2 = -3i$. We know that means a general solution to our original equation is $c_1 e^{3i} + c_2 e^{-3i}$. We can then use Euler's identity, $e^{a+bi} = e^a \cos(b) + ie^a \sin(b)$ to rewrite this equation as $c_1(\cos(3x) + i\sin(3x)) + c_2(\cos(3x) - \sin(3x)) = (c_1 + c_2)\cos(3x) + i(c_1 - c_2)\sin(3x)$. Since c_1 and c_2 are arbitrary, we can redefine $k_1 = c_1 + c_2$ and $k_2 = c_1 - c_2$. Then we can finally see the solution to $D^2 + 9I = 0$ is $k_1 \cos(3x) + k_2 sin(3x)$. Since this will send anything in $D^2 + 9I$ to zero, that means this is the kernel of $D^2 + 9I$, and because $k_1 \cos(3x) + k_2 sin(3x)$ lives in the complex plane, it is two dimensional.

(d) (Ideally, after you do everything else) $x'' + 9x = e^{3x}$ has a solution to a homogeneous and a heterogeneous part, connect this to the sum of (something in the kernel of $D^2 + 9I$) + (a unique element in the kernel of $D - 3I$).

We know $x'' + 9x$ has a solution of $c_1 \cos(3x) + c_2 \sin(3x)$ from part (c). We also know that we can find another particular solution using the method of undetermined coefficients. Using this method, we can find a third part to our solution from part (c) that is $\frac{1}{18}e^{3x}$. This gives us a solution of $c_1 cos(3x) + c_2 sin(3x) + \frac{1}{18}e^{3x}$. The method of undetermined coefficients shows us that the third part of our solution, $\frac{1}{18}e^{3x}$, is a unique element in the kernel of $D - 3I$. We can check this by seeing $\frac{d}{dx}\frac{1}{18}e^{3x} - 3(\frac{1}{18}e^{3x}) = 0$. Similarly, from part (c) we know the first part of our solution is in the kernel of $D^2 + 9I$. Therefore, our solution to $x'' + 9x = e^{3x}$ is the direct sum of an element in the kernel of $D^2 + 9I$ and a unique element in the kernel of $D - 3I$.