Final Portfolio: Week 4
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Question 1

The inner product is often thought of as a generalization of the notion of angles, and correspondingly any norm is a generalization of distance. Why? These examples explain why. You should show they work for *any* inner product, not just the standard one.

(a) A parallelogram consists of four corner vectors: 0, u, v, and $u+v$. The diagonals are $u+v$ and $u-v$. Show that the sum of the squares of the lengths of the diagonals of a parallelogram equals the sum of the squares of the lengths of the four sides.

First, we want to show the sum of the squares of the lengths of the diagonals of a parallelogram equals the sum of the squares of the lengths of the four sides. We can set this up geometrically as follows:

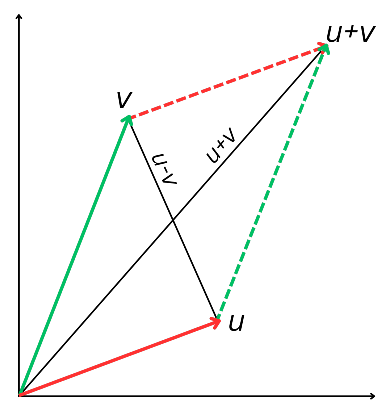

Figure 1: Parallelogram with Vertices $0, v, w$, and $v + w$

In order to prove that the sum of the squares of the lengths of the diagonal equal the sum of the squares of the lengths of the sides, we start with the sum of the squares of the lengths of the diagonal, $\|u + v\|^2$ and

$\|u-v\|^2$. We can then see,

$$\begin{aligned}\|u+v\|^2+\|u-v\|^2 &= \langle u+v,u+v\rangle+\langle u-v,u-v\rangle \\ &= \langle u,u\rangle+\langle u,v\rangle+\langle v,u\rangle+\langle v,v\rangle+\langle u,u\rangle-\langle u,v\rangle-\langle v,u\rangle+\langle v,v\rangle \\ &\text{The previous step is from the additive property in the first and second slots.} \\ &= \langle u,u\rangle+\langle v,v\rangle+\langle u,u\rangle+\langle v,v\rangle \\ &= \|u\|^2+\|v\|^2+\|u\|^2+\|v\|^2\end{aligned}$$

This is clearly the sum of the squares of the lengths of the four sides. So we can see $\|u+v\|^2+\|u-v\|^2=\|u\|^2+\|v\|^2+\|u\|^2+\|v\|^2$.

(b) Show that the diagonals of a rhombus are perpendicular to each other.

We can then look at a rhombus like in the following figure:

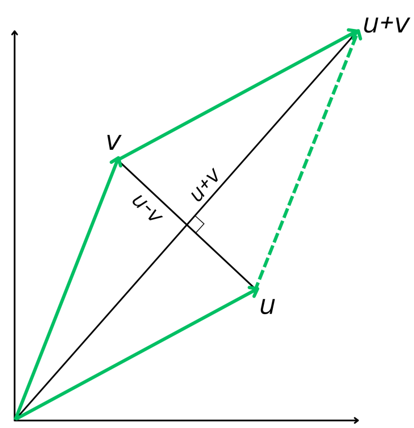

Figure 2: Rhombus

In order to show the two diagonals are perpendicular, we want to show that their inner product is 0. We can start with the inner product $\langle u+v,u-v\rangle$ and then see that:

$$\begin{aligned}\langle u+v,u-v\rangle &= \langle u,u-v\rangle+\langle v,u-v\rangle \\ &= \langle u,u\rangle-\langle u,v\rangle+\langle v,u\rangle-\langle v,v\rangle\end{aligned}$$

Since $\|u\| = \|v\|$, then $\langle u, u\rangle = \langle v, v\rangle$. Similarly, since we are in a real space, $\langle u, v\rangle = \langle \overline{v, u}\rangle = \langle v, u\rangle$. Therefore,

$$\langle u, u\rangle - \langle u, v\rangle + \langle v, u\rangle - \langle v, v\rangle = 0$$

Thus, since their inner product is zero, the diagonals of a rhombus are perpendicular.

(c) The angle between two nonzero vectors in any real inner product space is defined as

$$arccos\frac{\langle x, y\rangle}{\|x\|\|y\|}$$

Explain why the Cauchy-Schwarz Inequality is needed to show that this definition makes sense.

To apply arccos, our value must be between -1 and 1. From the Cauchy-Schwartz Inequality, we know $-\|u\|\|v\| \leq \langle u, v\rangle \leq \|u\|\|v\|$. We can then divide everything by $\|u\|\|v\|$ and see that $-1 < \frac{\langle u,v\rangle}{\|u\|\|v\|} < 1$ is true for any vectors u, v. Therefore, we are always able to apply the arccos of $\frac{\langle u,v\rangle}{\|u\|\|v\|}$.

(d) Show the above corresponds to the angle between two vectors in $\mathbb{R}^2$

Assume vectors $v, u \in \mathbb{R}$ start at the origin and let θ be the angle between the two. We can see this in the following figure:

We know $\|v\| = \sqrt{\langle v, v\rangle}$ and $\|u\| = \sqrt{\langle u, u\rangle}$. Using the law of cosines, we know $\|v - w\|^2 = \|v - u\|^2 + \|w - u\|^2 - 2\|v - u\|\|w - u\|cos(\theta)$. Since $u = 0$, we can simplify to $\|v - w\|^2 = \|v\|^2 + \|w\|^2 - 2\|v\|\|w\|cos(\theta)$. We can then rearrange to see $cos(\theta) = \frac{\|u\|+\|v\|-\|v-u\|}{2\|v\|\|u\|}$. We can then use the

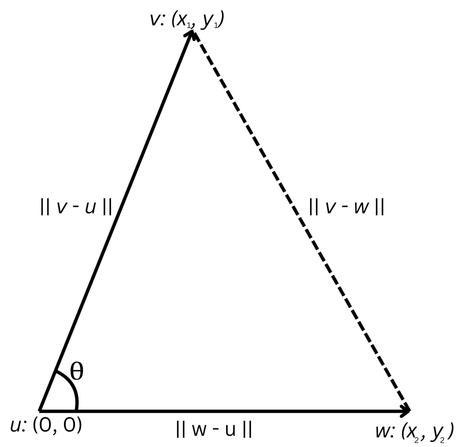

Figure 3: Diagram of our vectors and the angle between them

definition of the norm to expand this out into:

$$= \frac{\langle u, u\rangle + \langle v, v\rangle - \langle v - u, v - u\rangle}{2\|v\|\|u\|}$$

$$= \frac{\langle u, u\rangle + \langle v, v\rangle - \langle v, v\rangle + \langle v, u\rangle + \langle u, v\rangle - \langle u, u\rangle}{2\|v\|\|u\|}$$

Expanding due to additivity in 1st and 2nd slots.

$$= \frac{2\langle v, u\rangle}{2\|v\|\|u\|} \text{ Simplifying.}$$

Also, $\langle u, v\rangle = \langle v, u\rangle$ because we're in the real numbers.

$$= \frac{\langle v, u\rangle}{\|v\|\|u\|}$$

From this we can see $\cos(\theta) = \frac{\|u\|+\|v\|-\|v-u\|}{2\|v\|\|u\|}$ using the law of cosines.