$\Sigma \frac{79}{90} + \Sigma \frac{20}{30}$

作业一

姓名 范玥瑶 学号 PB18000341 日期 2020 年 10 月 28 日

第一题

(a)Proof. 由题

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$$l'(x) = \sum_{l=0}^{n} \prod_{k \neq l} (x - x_k) \tag{1}$$

$$l'(x_j) = \prod_{k \neq j} (x_j - x_k) \tag{2}$$

因此

$$\ell_j(x) = \frac{\prod_{k \neq j} (x - x_k)}{\prod_{k \neq j} (x_j - x_k)} = \frac{\prod_{k=0}^{n} \quad (x - x_k)}{(x - x_j) \prod_{k \neq j} (x_j - x_k)} = \frac{l(x)}{l'(x_j)(x - x_j)} \tag{3}$$

#

(b)Proof. 下证 $\sum_{j=0}^{n} l_j(x) = 1$

考虑 $f(x) \equiv 1$ 的 n 次 Lagrange 插值 $(n \geq 1)$

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$$p(x) = \sum_{j=0}^{n} l_j(x) \tag{4}$$

$\because n \geq 1, f^{(n-1)}(x) \equiv 0$

$\therefore$ 误差 $R_n(x) = \frac{f^{(n-1)}(\xi)}{(n+1)!} \prod_{j=0}^{n} (x - x_j) \equiv 0$, i.e.$p(x) - f(x) \equiv 0, \sum\limits_{j=0}^{n} l_j(x) = 1$

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第二问即证明

$$l(x) \sum_{l=0}^{n} \frac{\lambda_j}{x - x_j} = 1 \tag{5}$$

$$\because \sum_{j=0}^{n} l_j(x) = \sum_{j=0}^{n} \frac{l(x)}{l'(x_j)(x - x_j)} = l(x) \sum_{j=0}^{n} \frac{\lambda_j}{(x - x_j)} \tag{6}$$

$$\therefore l(x) \sum_{l=0}^{n} \frac{\lambda_j}{x - x_j} = 1 \tag{7}$$

得证题设成立

#

(c)Proof. 对于 $j = 0, 1, \ldots\ldots n$,

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$$\lambda_j = \frac{1}{l'(x_j)} = \frac{1}{\prod_{l \neq j} (x_j - x_l)} = \frac{1}{\prod_{l \neq j} (-2 sin \frac{j+l}{2n} \pi sin \frac{j-l}{2n} \pi)} \tag{8}$$

$$\therefore \frac{1}{\lambda_j}=\prod_{l\neq j}(-2sin\frac{j+l}{2n}\pi sin\frac{j-l}{2n}\pi) \tag{9}$$

当 j=0

$$\frac{1}{\lambda_0}=\prod_{k=1}^{n}2sin^2\frac{k\pi}{2n}=2^n\prod_{k=1}^{n}sin^2\frac{k\pi}{2n}=2^n\prod_{k=1}^{n-1}sin^2\frac{k\pi}{2n} \tag{10}$$

$$=2^n\prod_{k=1}^{n-1}sin\frac{k\pi}{2n}cos\frac{k\pi}{2n}=2\prod_{k=1}^{n-1}2sin\frac{k\pi}{2n}cos\frac{k\pi}{2n} \tag{11}$$

$$=2\prod_{k=1}^{n-1}sin\frac{k\pi}{n} \tag{12}$$

考虑关于 $z\in C$ 的方程 $(1+z)^n-1=0$ 的非零解 $z_k=-1+e^{\frac{i2k\pi}{n}}(k=1,2,......,n-1)$
当 $z\neq 0, \frac{(1+z)^n-1}{z}=0$, 由二项式定理

$$\sum_{k=1}^{n}C_n^k z^{k-1}=C_n^n\prod_{k=1}^{n-1}(z-z_k)=0 \tag{13}$$

$\because \{x^k\}_{k=0}^{\infty}$ 线性无关，$\therefore$ 对比常数项可得

$$n=\prod_{k=1}^{n-1}[(1-cos\frac{2k\pi}{n})-isin\frac{2k\pi}{n}]$$

$$=\prod_{k=1}^{n-1}-2isin\frac{k\pi}{n}e^{\frac{k\pi i}{n}}$$

$$=(-2i)^{n-1}e^{(n-1)\frac{\pi i}{2}}\prod_{k=1}^{n-1}sin\frac{k\pi}{n}$$

$$=2^{n-1}\prod_{k=1}^{n-1}sin\frac{k\pi}{n} \tag{14}$$

$$\therefore \lambda_0=\frac{2^{n-1}}{n}*\frac{1}{2}=\frac{2^{n-2}}{n} \tag{15}$$

当 $1\leq j\leq n-1$

$$
\begin{aligned}
\frac{1}{\lambda_j} &= \prod_{k\neq j}(x_j - x_k) = (-2)^n \prod_{k\neq j} sin\frac{j+k}{2n}\pi sin\frac{j-k}{2n}\pi \\
&= (-2)^n \prod_{k=0}^{j-1} sin\frac{j+k}{2n}\pi \prod_{k=j+1}^{n} sin\frac{j+k}{2n}\pi \prod_{k=0}^{j-1} sin\frac{j-k}{2n}\pi \prod_{k=j+1}^{n} sin\frac{j-k}{2n}\pi \\
&= (-2)^n(-1)^{n-j} \prod_{k=j}^{2j-1} sin\frac{k}{2n}\pi \prod_{k=2j+1}^{j+n} sin\frac{k}{2n}\pi \prod_{k=1}^{j} sin\frac{k}{2n}\pi \prod_{k=1}^{n-j} sin\frac{k}{2n}\pi \\
&= 2^n(-1)^j \frac{sin\frac{j\pi}{2n}}{sin\frac{j\pi}{n}} \prod_{k=1}^{j+n} sin\frac{k}{2n}\pi \prod_{k=1}^{n-j} sin\frac{k}{2n}\pi \\
&= 2^{n-1}(-1)^j \frac{1}{cos\frac{j\pi}{2n}} (\prod_{k=1}^{n} sin\frac{k}{2n})^2 \frac{\prod_{k=n+1}^{j+n} sin\frac{k}{2n}}{\prod_{k=n-j+1}^{n} sin\frac{k}{2n}\pi} \\
&= (-1)^j \frac{1}{2\lambda_0} \frac{1}{cos\frac{j\pi}{2n}} \frac{\prod_{k=1}^{j} cos\frac{k\pi}{2n}}{\prod_{k=0}^{j-1} cos\frac{k}{2n}\pi} \\
&= (-1)^j \frac{1}{2\lambda_0} \frac{1}{cos\frac{j\pi}{2n}} \frac{cos\frac{j\pi}{2n}}{cos 0} \\
&= \frac{(-1)^j}{2\lambda_0}
\end{aligned} \tag{16}
$$

$$
\therefore \lambda_j = \frac{2^{n-1}}{n}(-1)^j \tag{17}
$$

当 $j = n$

$$
\begin{aligned}
\frac{1}{\lambda_n} &= \prod_{k=0}^{n-1} -2sin\frac{n+k\pi}{2n} sin\frac{n-k\pi}{2n} \\
&= \prod_{k=1}^{n-1} -2cos^2\frac{k\pi}{2n} \\
&= (-1)^n \prod_{k=1}^{n-1} 2sin^2\frac{k\pi}{2n} \\
&= (-1)^n \frac{1}{\lambda_0} \\
&= \frac{2^{n-2}}{n}(-1)^n
\end{aligned} \tag{18}
$$

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(d) 将 (c) 中关于 $\lambda_j(j = 0, 1, \ldots\ldots n)$ 的结论代入重心插值公式第二形式即可证明题公式（6）.

MATLAB 程序显示如下：

```
clear, clc, clf
LW = 'Linewidth';lw=2;
sinh =@(x) (exp(x)-exp(-x))/2;
cosh =@(x) (exp(x)+exp(-x))/2;
tanh =@(x) sinh(x)./cosh(x);
n=5000;m = 10001;
a = linspace(0,pi,n+1);
x = cos(a);%x是Chebyshev点向量
xx = linspace(-1,1,m)';%xx是作图的点

F = @(x) tanh(20.*sin(12.*x))+0.02.*exp(3.*x).*sin(300.*x);
y = F(x);%y是插值点函数值向量

c = 1:(n-1);
d = (-1).^c;
e = [0.5 ;d'; 0.5*(-1)^n];

A = 1./(xx - x(1));%第一列

for j = 2:(n+1)
    A = [ A 1./(xx - x(j))];
end

p1 = A * (y' .* e);%分子
p2 = A*e;%分母
p = p1./p2;

figure(1)
plot(xx, F(xx), 'k', LW, lw), hold on
plot(xx, p, LW, lw)
legend('exact', 'interpolant', 'location', 'nw')

figure(2)
plot(2)
semilogy(xx, abs(F(xx) - p), 'k', LW, lw), hold on
```

```
legend('error', 'error bound', 'location', 'se')
```

得到图像如图 Fig.1&Fig.2.

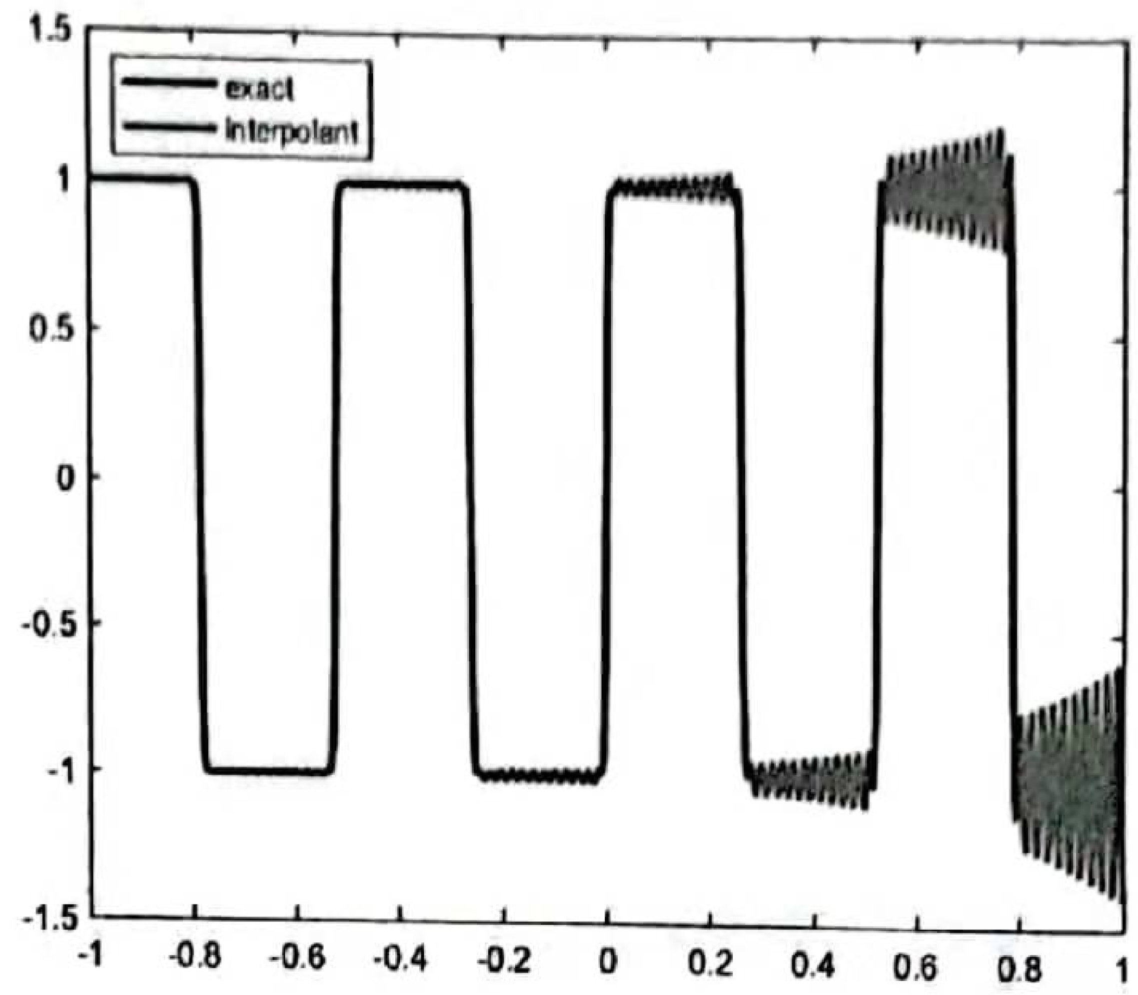

图 1: 被插值函数和原函数

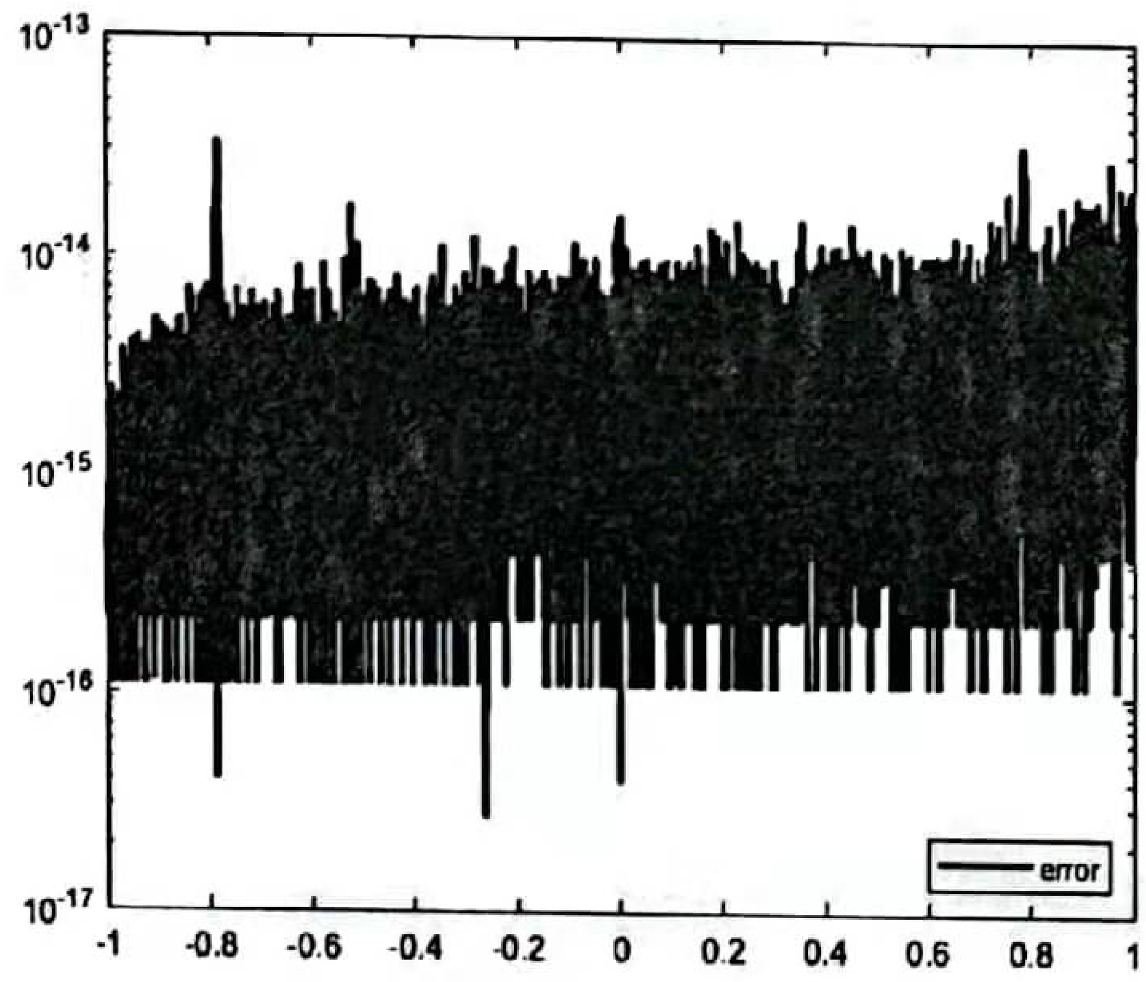

图 2: 误差

第二题 (a) $f''(x) = -3\pi^2 e^{3cos(\pi x)}(cos(\pi x) - 3sin^2(\pi x))$

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$M_0 = M_n = 3\pi^2 e^{-3}$

MATLAB 程序显示如下:

```
clear, clc, clf
LW = 'linewidth'; lw = 2;
M0 = 3*pi^2*exp(-3);Mn = 3*pi^2*exp(-3);%边界
r0 = zeros(1,7);%存放最大误差
k0 = 6 : 12;
```

```
for k = 6 : 12
  n = 2^k;
  f =@(x) exp(3 .*cos(pi .*x));
  x = linspace(-1,1,n + 1);%x=(x0,……,xn)
  y = f(x);x = linspace(-1,1,n + 1);%y=(y0,……,yn)
  h = diff(x);%h=(h0,……hn-1)
  h1 = h; h1(:,1)=[];
  h2 = h; h2(:,n)=[];
  D = diff(y);%D=(y1-y0,……,yn-y_n-1)
  D1 = D; D1(:,1)=[];
  D2 = D; D2(:,n)=[];

  l = h1 ./(h1 + h2);
  m = 1 - l;
  d = 6 ./ (h1 + h2) .*( D1 ./ h1 - D2 ./h2);
  l1 = l; l1(:,n-1)=[];
  m1 = m; m1(:,1)=[];
  A = 2.*eye(n-1) + diag(l1,1) + diag(m1,-1);
  d(1) = d(1) - m(1) * M0;
  d(n-1) = d(n-1) - l(n-1)*Mn;
  M = inv(A) *d';%得到M向量M=(M1,M2,……, Mn-1)'

  xx = linspace(x(1),x(2),6);
  xx = xx(:,2:5);
  S = ((x(2)-xx).^3.*M0+(xx-x(1)).^3.*M(1))./(6*h(1))...
  +((x(2)-xx).*y(1)+(xx-x(1)).*y(2))./h(1)...
  -1/6*h(1).*((x(2)-xx).*M0+(xx-x(1)).*M(1));
  R = abs(f(xx) - S);
  R = sort(R);
  r = R(4);%r是(x0,x1)4个插值点上最大误差

  for i = 1: n-2
      j = i + 1;% x(j)=xi, y(j)=yi,h(j)=hi.
      xx = linspace(x(j),x(j+1),6);
      xx = xx(:,2:5);%抽样点
```

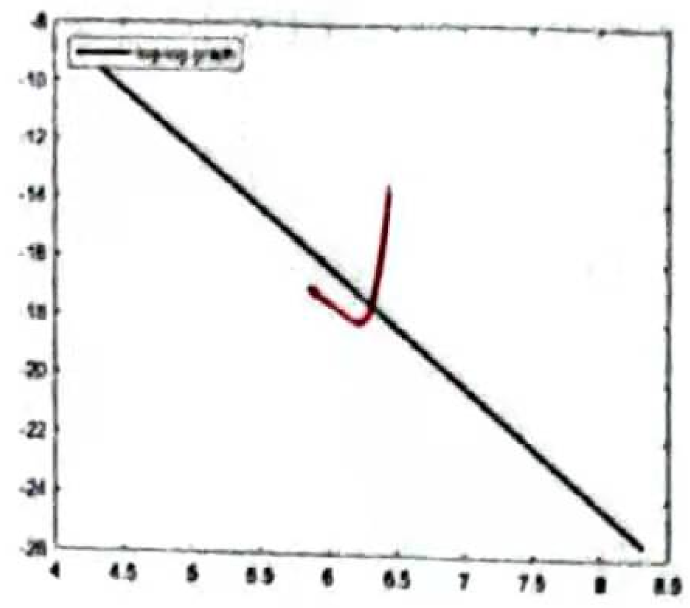
```
        S = ((x(j+1)-xx).^3.*M(i)+(xx-x(j)).^3.*M(i+1))./...
        (6*h(j))+((x(j+1)-xx).*y(j)+(xx-x(j)).*y(j+1))./...
        h(j)-1/6*h(j).*((x(j+1)-xx).*M(i)+(xx-x(j)).*M(i+1));
        R = abs(f(xx) - S);R = sort(R);
        if r <= R(4)
            r = R(4);
        end
    end

    xx = linspace(x(n),x(n+1),6);
    xx = xx(:,2:5);
    S =((x(n+1)-xx).^3.*M(n-1)+(xx-x(n)).^3.*Mn)./(6*h(n))...
    +((x(n+1)-xx).*y(n)+(xx-x(n)).*y(n+1))./h(n)...
    -1/6*h(n).*((x(n+1)-xx).*M(n-1)+(xx-x(n)).*Mn);
    R = abs(f(xx) - S);
    R = sort(R);
    if r <= R(4)%r是4n个点的最大的误差
            r = R(4);
    end

      r0(k-5) = r;
end

figure(1)
plot(k0*log(2), log(r0), 'k', LW, lw), hold on
legend('log-log graph', 'interpolant', 'location', 'nw')
```

得到图像如图 Fig.3.



log.log

图 3: 第一类边界条件 log-log 图

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(b)log-log 图的斜率 $\alpha \approx \frac{-26-(-8)}{8.5-4} = -4$, 代表误差 $r \propto n^{-4} = 2^{-4k}$，即 k 每增加 1，r 变为原来的 $\frac{1}{16}$

4
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(c) 选取第二类边值条件时:

$f'(x) = -3\pi sin(\pi x)e^{3cos(\pi x)}$

$f'(x_0) = f'(-1) = 0, f'(x_n) = f'(1) = 0$

选取第三类边值条件时, 考虑 d 插值点等距:

$f[x_0, x_1] - \frac{1}{3}h_0M_0 - \frac{1}{6}h_0M_1 = f[x_n, x_{n-1}] + \frac{1}{3}h_{n-1}M_n + \frac{1}{6}h_{n-1}M_1$

$M_0 = M_n$

$\therefore 2M_0 + \frac{1}{2}M_1 + \frac{1}{2}M_{n-1} = 3\frac{(y_1-y_0)-(y_n-y_{n-1})}{h^2}$

MATLAB 程序显示如下:

```
clear, clc, clf
LW = 'linewidth'; lw = 2;
M0 = 3*pi^2*exp(-3);Mn = 3*pi^2*exp(-3);
r0 = zeros(1,7);
r1 = zeros(1,7);
r2 = zeros(1,7);
k0 = 6 : 12;

for k = 6 : 12
  %第一类边界条件
  n = 2^k;
  f =@(x) exp(3 .*cos(pi .*x));
  x = linspace(-1,1,n + 1);
  y = f(x);x = linspace(-1,1,n + 1);
  h = diff(x);
  h1 = h; h1(:,1)=[];
  h2 = h; h2(:,n)=[];
```

```
D = diff(y);
D1 = D; D1(:,1)=[];
D2 = D; D2(:,n)=[];

l = h1 ./(h1 + h2);
m = 1 - l;
dd = 6 ./ (h1 + h2) .*( D1 ./ h1 - D2 ./h2);
d = dd;
l1 = l; l1(:,n-1)=[];
m1 = m; m1(:,1)=[];
A = 2.*eye(n-1) + diag(l1,1) + diag(m1,-1);
d(1) = d(1) - m(1) * M0;
d(n-1) = d(n-1) - l(n-1)*Mn;
M1 = inv(A) *d';

%Type2(变量没有改，因为可以覆盖)
d0 = 6*(D(1))/( h(1)^2 );
dn = - 6*(D(n))/( h(n)^2 );
d = [d0 dd dn];

l1 = [1 l];
m1 = [m 1];
A = 2.*eye(n+1) + diag(l1,1) + diag(m1,-1);
M2 = inv(A) *d';

%Type3
d = [3*(D(1)-D(n)/(h(1)^2)) dd];
l1 = [0.5, l]; l1(:,n)=[];
A =  2.*eye(n) + diag(l1,1) + diag(m,-1);
A(1,n)=0.5;  A(n,1)=l(n-1);
M3 = inv(A) *d';
M3 = [M3;M3(1)];%Mn=M0;

%检验
%1
```

```
M = M1;
xx = linspace(x(1),x(2),6);
xx = xx(:,2:5);
S = ((x(2)-xx).^3.*M0+(xx-x(1)).^3.*M(1))./(6*h(1))...
+((x(2)-xx).*y(1)+(xx-x(1)).*y(2))./h(1)...
-1/6*h(1).*((x(2)-xx).*M0+(xx-x(1)).*M(1));
R = abs(f(xx) - S);
R = sort(R);
r = R(4);
for i = 1: n-2
    j = i + 1;
    xx = linspace(x(j),x(j+1),6);
    xx = xx(:,2:5);
    S = ((x(j+1)-xx).^3.*M(i)+(xx-x(j)).^3.*M(i+1))./...
    (6*h(j))+((x(j+1)-xx).*y(j)+(xx-x(j)).*y(j+1))./...
    h(j)-1/6*h(j).*((x(j+1)-xx).*M(i)+(xx-x(j)).*M(i+1));
    R = abs(f(xx) - S);R = sort(R);
    if r <= R(4)
       r = R(4);
    end
end
xx = linspace(x(n),x(n+1),6);
xx = xx(:,2:5);
S =((x(n+1)-xx).^3.*M(n-1)+(xx-x(n)).^3.*Mn)./(6*h(n))...
+((x(n+1)-xx).*y(n)+(xx-x(n)).*y(n+1))./h(n)...
-1/6*h(n).*((x(n+1)-xx).*M(n-1)+(xx-x(n)).*Mn);
R = abs(f(xx) - S);
R = sort(R);
if r <= R(4)
       r = R(4);
end

  r0(k-5) = r;
%2
M = M2;
```

```
for i = 0 : n-1
    j = i+1;
    xx = linspace(x(j),x(j+1),6);
    xx = xx(:,2:5);
    S = ((x(j+1)-xx).^3.*M(j)+(xx-x(j)).^3.*M(j+1))./...
    (6*h(j))+((x(j+1)-xx).*y(j)+(xx-x(j)).*y(j+1))./...
    h(j)-1/6*h(j).*((x(j+1)-xx).*M(j)+(xx-x(j)).*M(j+1));
    R = abs(f(xx) - S);R = sort(R);
    if i == 0
        r = R(4);
        continue;
    end
    if r <= R(4)
       r = R(4);
    end
end

  r1(k-5) = r;
 %3
 M = M3;
for i = 0 : n-1
    j = i+1;
    xx = linspace(x(j),x(j+1),6);
    xx = xx(:,2:5);
    S = ((x(j+1)-xx).^3.*M(j)+(xx-x(j)).^3.*M(j+1))./...
    (6*h(j))+((x(j+1)-xx).*y(j)+(xx-x(j)).*y(j+1))./...
    h(j)-1/6*h(j).*((x(j+1)-xx).*M(j)+(xx-x(j)).*M(j+1));
    R = abs(f(xx) - S);R = sort(R);
    if i == 0
        r = R(4);
        continue;
    end
    if r <= R(4)
       r = R(4);
    end
```

```
  end

    r2(k-5) = r;
end

figure(1)
plot(k0*log(2), log(r0), 'k', LW, lw), hold on
plot(k0*log(2), log(r1), 'r', LW, lw), hold on
plot(k0*log(2), log(r2), 'y', LW, lw), hold on
legend('Type1', 'Type2','Type3', 'location', 'nw')
```

得到图像如图 Fig.4.

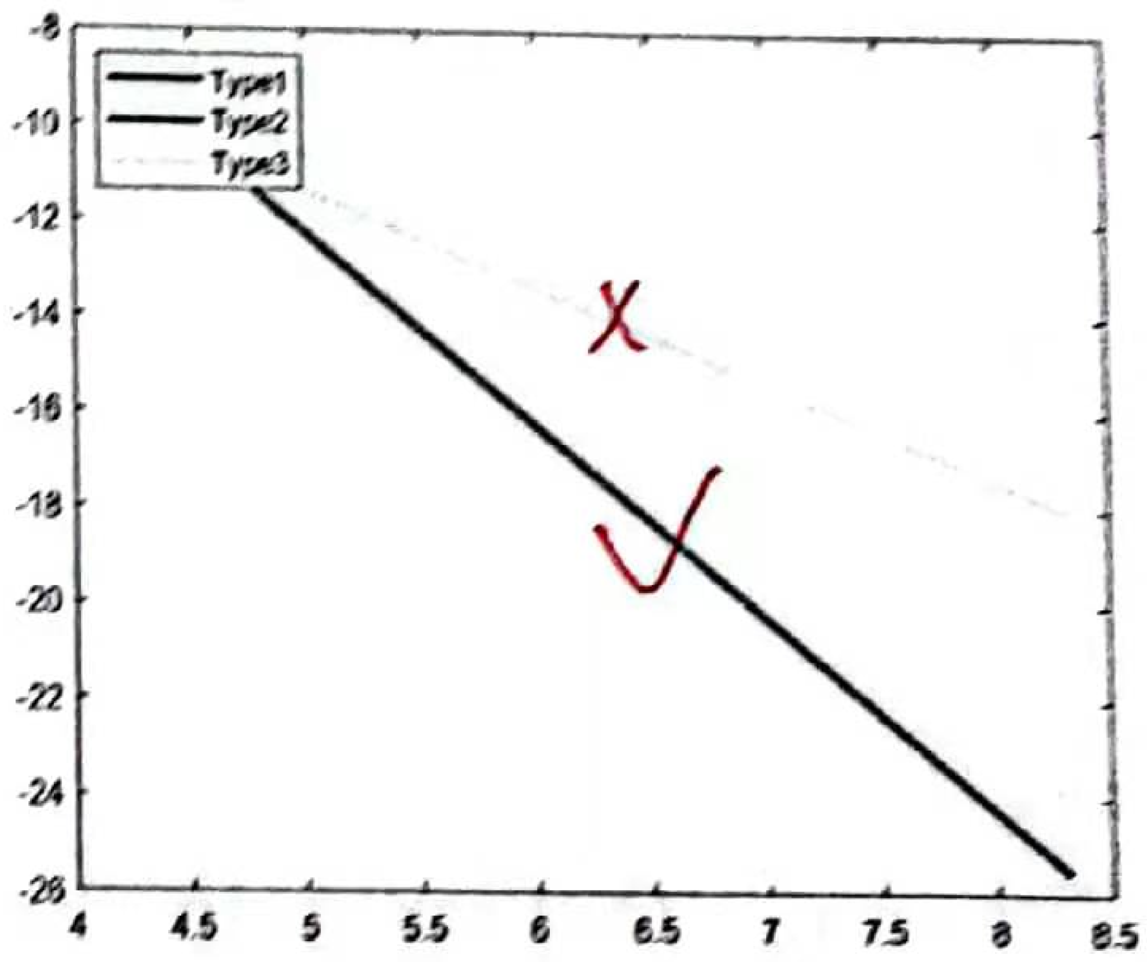

图 4: 三类边界条件 log-log 图

图上第一第二类边界条件的图像基本重合。通过对比 log-log 图，对比三次样条插值在三种不同边值条件下所得到的插值精度，可得第三类边界条件的误差明显比第一、第二类大。直观上可以认定，边界条件没有给出样条插值函数在边界上一、二阶导数的具体取值，所以精度会比另外两类边界条件的三次样条插值低。

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第三题 解：

$$\because y = ae^{bx} \tag{19}$$

$$\therefore lny = lna + bx \tag{20}$$

MATLAB 程序显示如下：

```
clear, clc, clf
LW = 'linewidth'; lw = 2;
x = [-0.70 -0.50 0.25 0.75];
y = [0.99 1.21 2.57 4.23];

Y = log(y)';

A = [ones(4,1),x'];
alpha = inv(A'*A)*A'*Y;%法方程
alpha %输出系数向量
r = A*alpha - Y;
N = r' * r;%二范数
N %输出二范数

f =@(x)exp(alpha(2)*x+alpha(1))

X= -1 : 0.001 :1;
figure(1)
plot(x,y,'o',LW, lw), hold on
plot(X, f(X), 'k', LW, lw)
legend('exact', 'LSA', 'location', 'nw')
```

则经验公式 $y = e^{1.0020x+0.6918}$ "a=?" "形如 $y=ae^{bx}$" -2.

二范数 $N = 4.0155 * 10^{-6}$ 平方. -3

拟合曲线和拟合数据点图像如图 Fig.5.

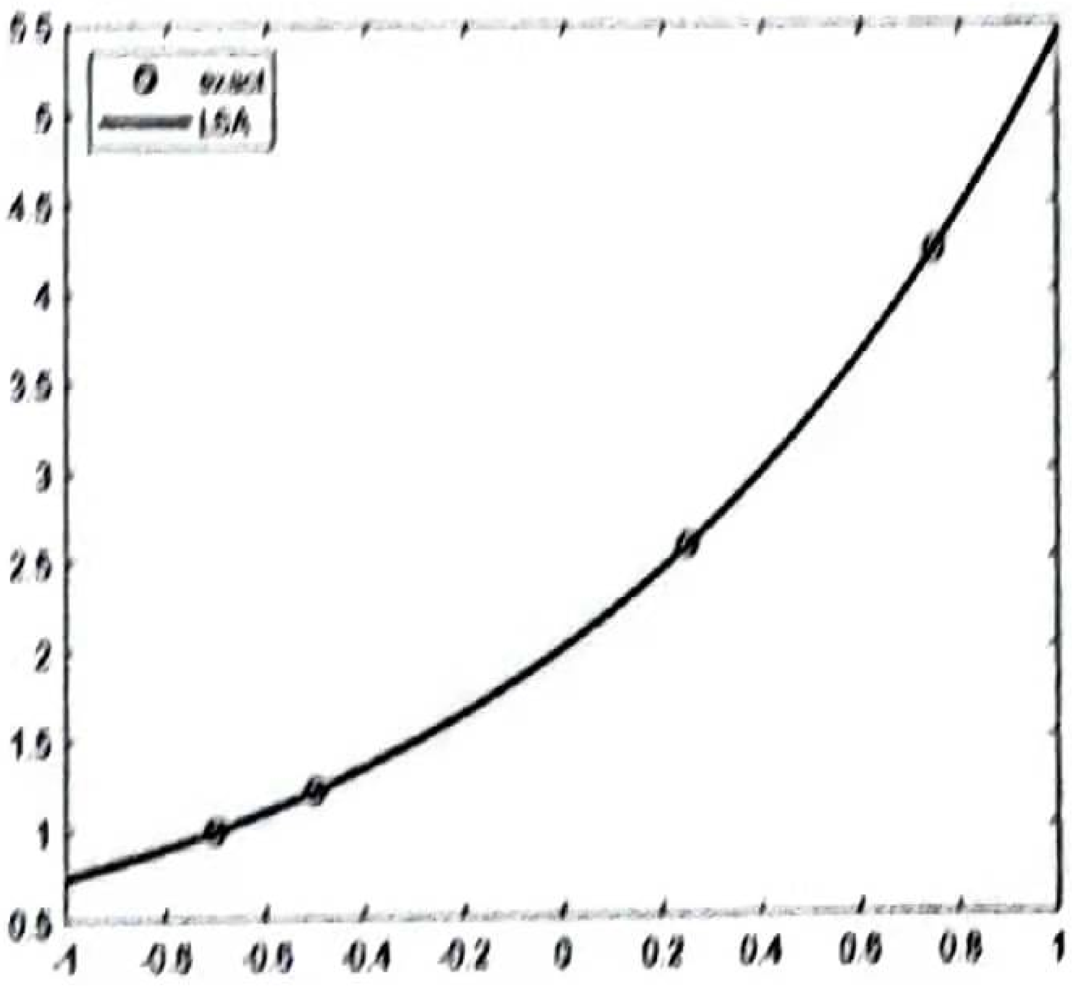

图 5: 拟合曲线和拟合原始数据点图