2021. 10. 19 Discrete-time models

$N(t+1) = F(N(t))$ — exponential model, logistic growth model, stability analysis

mapping

recursive equations

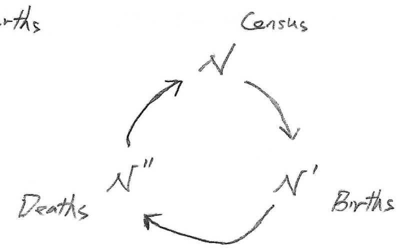
◎ Exponential model (geometric growth)

$N(t+1) = R \cdot N(t)$

Assumptions
1. closed population
2. identical individuals, no internal structure
3. no density dependence
4. time independent parameters
5. Discrete events & population census ⇒ specify order of events (synchronized)

fraction of ind. that give births

$N'(t) = N(t) + \beta N(t) = (1+\beta) N(t)$



$N''(t) = N'(t) - \delta N'(t) = (1-\delta) N'(t)$

fraction of ind. that die

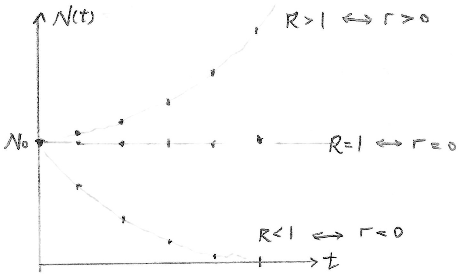
$\Rightarrow N''(t) = N(t+1) = (1-\delta)(1+\beta) N(t)$

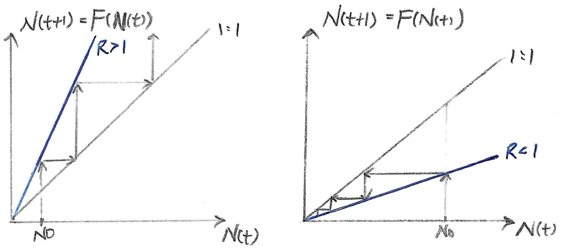
$\Rightarrow N(t+1) = R\, N(t)$ … ①

Reproductive factor, constant independent to density ; $R = \dfrac{N(t+1)}{N(t)}$
finite rate of increase (per unit time)

$\Rightarrow N(t) = R^t N_0$ … ②

- Dynamics: resemble that of continuous model





« Return map / Cobweb plots »

- Connections w/ continuous model

1. Instead of $t \sim t+1$, think of small Δt and let $\Delta t \to 0$

   In small Δt, we'd expect the per capita number of birth/death to be proportionately smaller (i.e., $\beta\Delta t$ & $\delta\Delta t$), there from $t \sim t+\Delta t$

   $$N(t+\Delta t) = (1-\delta\Delta t)(1+\beta\Delta t)N(t)$$

   $$\frac{dN}{dt} = \lim_{\Delta t\to 0} \frac{N(t+\Delta t)-N(t)}{\Delta t}$$

   $$= \lim_{\Delta t\to 0}\left[\frac{N(t)+\beta\Delta t N(t) - \delta\Delta t N(t) - \delta\beta\Delta t^2 N(t) - N(t)}{\Delta t}\right]$$

   $$= \lim_{\Delta t\to 0}\left[(\beta-\delta-\underbrace{\beta\delta\Delta t}_{0})N(t)\right] = (\beta-\delta)N$$

2. Compare with $\frac{dN}{dt} = rN$ at specific time points $t = hk$

   hk: time interval between iteration, usually $h=1$, but here more general

$t=0$	$t=h$	$t=2h$
$k=0$	$k=1$	$k=2$

   $$N(t) = N_0 e^{rt} \Rightarrow \text{at } t=hk,\ N(t) = N(hk) = N_0 e^{rhk}$$

   $\updownarrow$ compare

   $$N_0 \cdot R^k$$

   $$\Rightarrow R = e^{rh} \Rightarrow r = \frac{\ln R}{h}$$

3. Compound interest $\Rightarrow R = (1+R_0) = (1+\frac{R_0}{h})^h$, with $h\to\infty$, $R_0 \to r$, $(1+\frac{R_0}{h})^h \to e^r$

   $(1+x)^{\frac{R_0}{x}}$

⊚ Discrete Logistic growth

Can derive by discretizing the continuous model

$$\frac{dN}{dt} = r_0 N \left(1 - \frac{N}{K}\right)$$

$$\Rightarrow \frac{\Delta N}{\Delta t} = r_0 N \left(1 - \frac{N}{K}\right)$$

set $\Delta t = 1$ for discrete, non-overlapping generations

$$\Rightarrow \Delta N = N(t+1) - N(t) = r_0 N(t) \left(1 - \frac{N(t)}{K}\right) \Delta t$$

$$\Rightarrow N(t+1) = N(t) + r_0 N(t) \left(1 - \frac{N(t)}{K}\right) \quad \cdots ④ \quad = N(t)\left[1 + r_0\left(1 - \frac{N(t)}{K}\right)\right] \equiv F(N(t)) \quad \cdots ⑤$$

⊚ Equilibrium for Discrete models $N(t+1) = F(N(t))$

For continuous models, zero growth $\Rightarrow \frac{dN}{dt} = 0 = f(N^*)$ ← equilibrium

For discrete models, zero growth $\Rightarrow N_{t+1} = N_t = N^*$ ⋯ ⑥

For discrete logistic growth: $N^* = N^*\left[1 + r_0\left(1 - \frac{N^*}{K}\right)\right] \Rightarrow N^* = 0 \vee K$

($N^* = 0$: trivial equilibrium)

$\div N^*$

$$1 = 1 + r_0\left(1 - \frac{N^*}{K}\right)$$

⊚ Stability analysis for discrete models $N(t+1) = F(N(t))$

Again, think about displacement and its dynamics near the equilibrium

$$\varepsilon(t) = N(t) - N^*$$

Recall: Taylor expansion

$$f(a + \Delta x) = f(a) + \Delta x f'(a) + \frac{\Delta x^2}{2!} f''(a) + \dots$$

at next time step

$$\varepsilon(t+1) = N(t+1) - N^* = F(N(t)) - N^* = F(\underbrace{N^*}_{a} + \underbrace{\varepsilon(t)}_{\Delta x}) - N^*$$

$$\approx \left[\underbrace{F(N^*)}_{N^*} + \varepsilon(t) \cdot \frac{dF}{dN}\bigg|_{N=N^*}\right] - N^*$$

ignore higher order terms $O(\varepsilon^2)$

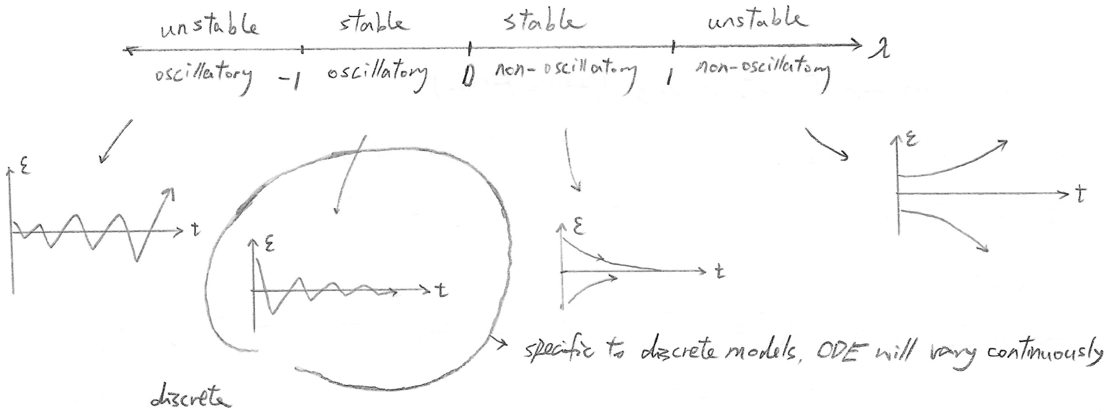
$$\Rightarrow \varepsilon(t+1) = \left[\frac{dF}{dN}\Big|_{N=N^*} \right] \varepsilon(t) \quad \ldots ⑦$$

define $\lambda = F'(N^*)$

$$\Rightarrow \varepsilon(t) = \lambda^t \varepsilon(0), \text{ with } \lambda = \frac{dF}{dN}\Big|_{N=N^*} \quad \ldots ⑧$$

1. if $|\lambda| < 1$, displacements shrink, N^* is stable
2. if $|\lambda| > 1$, displacements amplify, N^* is unstable

(3. if $\lambda = 0$, non-hyperbolic equilibrium, local stability analysis gives no answer)



specific to discrete models, ODE will vary continuously

For logistic model (discrete): $N(t+1) = F(t) = N(t)\left[1 + r_0\left(1 - \frac{N(t)}{K}\right)\right]$

$$\Rightarrow \frac{dF}{dN} = \left[1 + r_0\left(1 - \frac{N(t)}{K}\right)\right] + N(t)\left(-\frac{r_0}{K}\right) = F'(N) \quad \ldots ⑨$$

① for $N^* = 0$, $\frac{dF}{dN}\Big|_{N=0} = 1 + r_0 \Rightarrow$ stable if $|1 + r_0| < 1$

$\Rightarrow$ stable if $-2 < r_0 < 0$, not possible, 0 is unstable

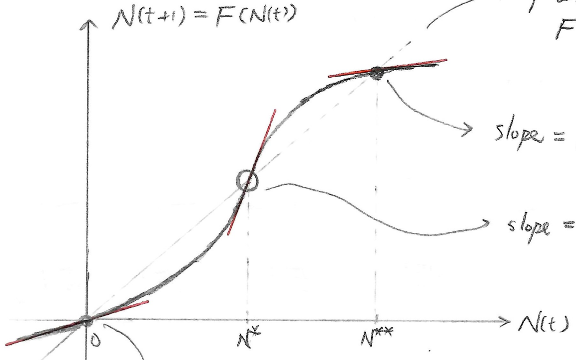
② for $N^* = K$, $\frac{dF}{dN}\Big|_{N=K} = 1 - r_0 \Rightarrow$ stable if $|1 - r_0| < 1$

$\Rightarrow$ stable if $0 < r_0 < 2$

unstable if $r_0 > 2$ ... ⑩

K is not always stable

⓪ Links w/ graphical approach



potential equilibrium cross @ $F(N(t))$ & the 1:1 diagonal line

slope $= \left.\frac{dF}{dN}\right|_{N^{**}} < 1 \Rightarrow$ stable

slope $= \left.\frac{dF}{dN}\right|_{N^{*}} > 1 \Rightarrow$ unstable

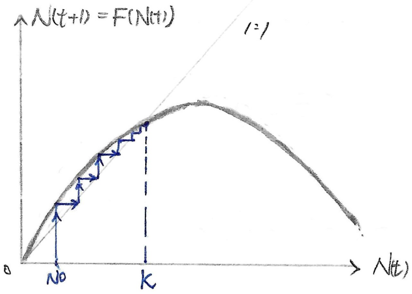
slope $= \left.\frac{dF}{dN}\right|_{0} < 1 \Rightarrow$ stable

$\left(\frac{1+r_0}{2r}\right)K$ 大於 K when $r_0 < 1$

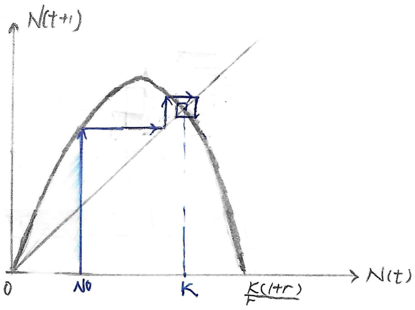
交点：0, $\frac{K}{r_0}(1+r_0)$, decreases w/ r

顶点：$\frac{1}{4}\frac{K}{r_0}(1+r_0)^2$

⓪ Cobweb plots for discrete logistic model



≪ $0 < r_0 < 1$ ≫ stable @ K
smoothly approach K



≪ $1 < r_0 < 2$ ≫ stable @ K
oscillate towards K

What happens when $R > 2$, K becomes unstable, BUT HOW? Simulate!

2-iteration $F^{(2)}(N(t)) = F(F(N(t)))$ has root : 0, K, N_1^*, N_2^* → equilibrium of $F^{(2)}(N(t))$

$$N_t\left[1+r_0\left(1-\frac{N_t}{K}\right)\right]\left\{1+r_0\left[1-\frac{1}{K}\cdot N_t\left[1+r_0\left(1-\frac{N_t}{K}\right)\right]\right]\right\} = N_{t+2}$$

$$\Rightarrow N^*(N^*-K)\left[r_0^3 N^{*2} + (-2r_0^2K - r_0^3K)N^* + (2r_0K^2 + r_0^2K^2)\right] = 0$$

$$\Rightarrow N_1^* \ \& \ N_2^* = \frac{(r_0+2)K \pm K\sqrt{(r_0+2)(r_0-2)}}{2r_0} \quad \cdots ①$$

only exist when $r_0 > 2$

from p-1 to p-2 is at $r_0 = 2$

At bifurcation boundary from p-2 to p-4 (p-2 becomes unstable)

becomes unstable

$$\left.\frac{dF^{(2)}}{dN}\right|_{N_1^*, N_2^*} = \left.\frac{dF}{dN}\right|_{N_1^*} \times \left.\frac{dF}{dN}\right|_{N_2^*} = -1$$

$$\frac{dF^{(2)}}{dN} = \frac{dF(y)}{dN} = \frac{dF(y)}{dy} \cdot \frac{dy}{dN} = \left.\frac{dF}{dN}\right|_{F(N)} \times \left.\frac{dF}{dN}\right|_{N}$$

$$= \left.\frac{dF}{dN}\right|_{F(N_1^*)} \times \left.\frac{dF}{dN}\right|_{N_1^*} = \left.\frac{dF}{dN}\right|_{N_2^*} \times \left.\frac{dF}{dN}\right|_{N_1^*}$$

$$\Rightarrow \left(1 + r_0 - \frac{2r_0}{K} N_1^*\right)\left(1 + r_0 - \frac{2r_0}{K} N_2^*\right) = -1$$

$$\Rightarrow \frac{4r_0^2}{K^2} N_1^* N_2^* - \left(\frac{2r_0}{K} + \frac{2r_0^2}{K}\right)(N_1^* + N_2^*) + (2r_0 + r_0^2 + 1) + 1 = 0 \quad \cdots ⑫$$

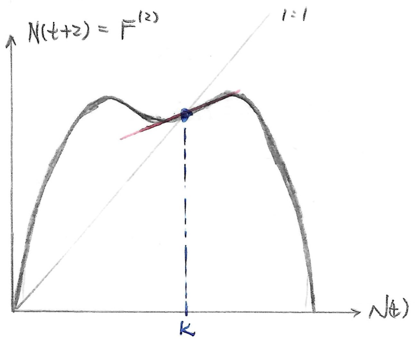
from ①

$$\begin{cases} N_1^* \times N_2^* = \dfrac{1}{r_0^3}\left[2r_0 K^2 + r_0^2 K^2\right] = K^2\left(\dfrac{2}{r_0^2} + \dfrac{1}{r_0}\right) \\ N_1^* + N_2^* = \dfrac{1}{r_0^3}\left[2r_0^2 K + r_0^3 K\right] = K\left(\dfrac{2}{r_0} + 1\right) \end{cases}$$

$$\Rightarrow 4r_0^2\left[\frac{2}{r_0^2} + \frac{1}{r_0}\right] - \left[2r_0 + 2r_0^2\right]\left[\frac{2}{r_0} + 1\right] + 2r_0 + r_0^2 + 2 = 0$$

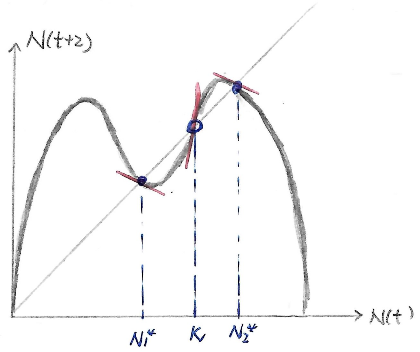
$$\Rightarrow r_0^2 - 6 = 0 \quad (負不合) \quad \cdots ⑬$$

$\Rightarrow$ p-2 becomes unstable and give rise to p-4 at $r_0 = \sqrt{6} \doteqdot 2.44949..$



« $1 < r_0 < 2$ » K is stable on $F^{(2)}$

p-1 orbit



« $2 < r_0 < \sqrt{6}$ » K becomes unstable

newly generated N_1^*, N_2^* is stable

p-2 orbit