2021. 10. 5 Exponential Population Growth

$$\frac{dN}{dt} = (b-d)N$$

- model assumptions & derivation
- related concepts
- model variation

◎ Model derivation

$N(t)$: Population size at time t

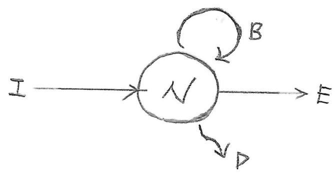
want to know $N(t)$
focus on dN/dt & related processes

$\frac{dN}{dt}$ = change in population size (population growth rate)

= Birth − Death + Immigration − emigration

$= \boxed{B - D + I - E \quad \dots ①}$

death "rate" (# ind./time)



◎ Assumptions for exponential growth

1. Closed population $(I = E = 0)$
2. All individuals are identical (no genetic, age, size structure)
3. Continuous population growth w/ no time lag
4. Birth & death rate per individual are constant (not time & density dependent)

unlimited resource

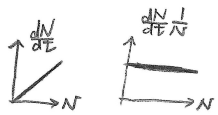
$= b(N,t)N - d(N,t)N + I - E \quad \dots ②$

per capita birth rate (# ind / ind. × time) ; assuming an average individual behavior

$= \boxed{(b-d)N} \quad ③ \rightarrow rN$, w/ r = intrinsic growth rate

$$\frac{dN}{dt} = (b-d)N$$

separation of variables

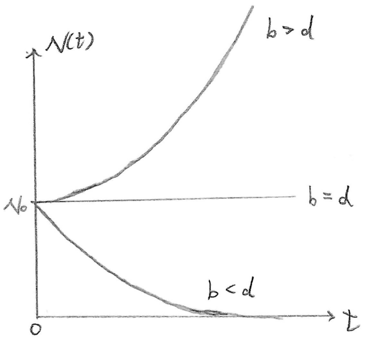


$$\Rightarrow \int \frac{1}{N}\, dN = \int (b-d)\, dt$$

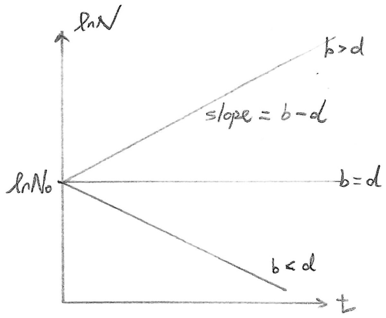
$$\Rightarrow \ln N = (b-d)t + C_1 \Rightarrow N(t) = e^{(b-d)t} \cdot C_2 \quad (C_2 = e^{C_1})$$

with initial condition N_0 at $t=0 \Rightarrow N(0) = C_2 \cdot e^0 = N_0 \Rightarrow \boxed{N(t) = N_0 e^{(b-d)t} \quad \dots ④}$

[Thomas Malthus 1798]



$$\ln N = \ln N_0 + (b-d)t \Rightarrow$$



◎ Related concepts

1. Doubling time (t_{double}) : $N(t_1 + t_{double}) = 2N(t_1)$ … ⑤

substitute ④ into ⑤ ⇒

$$N_0 e^{(b-d)(t_1+t_{double})} = N_0 e^{(b-d)t_1} \cdot e^{(b-d)t_{double}}$$
$$= 2N_0 e^{(b-d)t_1}$$

$$\Rightarrow \boxed{t_{double} = \frac{\ln 2}{r}} \quad \dots ⑥$$

doubling time is independent to time & N_0

if $b-d<0$, this is the half-life time

2. Average (expected) lifetime

concept: $\int x\, p(x)\, dx = E(X)$ — probability density function

$$\frac{dN}{dt} = -\delta N \Rightarrow N(t) = N_0 e^{-\delta t}$$

Prob that an individual dies within $t_1 \sim t_1 + \Delta t$

$$= \frac{N(t_1) - N(t_1+\Delta t)}{N_0} = \frac{N_0 e^{-\delta t_1} - N_0 e^{-\delta(t_1+\Delta t)}}{N_0}$$

$$= e^{-\delta t_1} - \underbrace{e^{-\delta(t_1+\Delta t)}}_{e^{-\delta t_1} + \Delta t \cdot (-\delta e^{-\delta t_1}) + O(\Delta t^2)}$$

※ Recall from Taylor expansion

$$f(a+\Delta x) = f(a) + \Delta x \cdot f'(a) + \frac{\Delta x^2}{2!} f''(a) \dots$$

$$= \delta e^{-\delta t_1} \Delta t + O(\Delta t^2) \Rightarrow$$ with very small Δt (dt)

Prob that an individual dies at t_1

$$= \boxed{\delta e^{-\delta t_1}}\, dt \quad \dots ⑦$$

probability density function, $P(t)$

$$\text{Expected lifetime} = \int_0^\infty t \cdot \overset{\text{eqn. ⑦}}{p(t)}\, dt = \int_0^\infty \underset{u}{t} \cdot \underbrace{\delta e^{-\delta t} dt}_{dv}$$

※ Recall: Integral by parts

$$\int u\,dv = uv - \int v\,du$$

$$\int u(x) v'(x) dx = u(x)v(x) - \int v(x) u'(x) dx$$

with $u = t,\ v = -e^{-\delta t}$

$du = dt,\ dv = +\delta e^{-\delta t} dt$

$$= t \cdot (-e^{-\delta t}) \Big|_0^\infty - \int_0^\infty (-e^{-\delta t}) dt$$

$$= -(te^{-\delta t}) \Big|_0^\infty - \left(\frac{1}{\delta} e^{-\delta t} \right) \Big|_0^\infty$$

$$= -(0-0) - (0 - \frac{1}{\delta}) = \boxed{\frac{1}{\delta}} \quad \cdots ⑧$$

◎ Relaxing assumptions (model variations)

1. Relax assumption 4. Let per capita growth become a function of time

$$\frac{dN}{dt} = [b(t) - d(t)] N = r(t) N$$

$$\Rightarrow \int_0^t \frac{1}{N} dN = \int_0^t r(t) dt \Rightarrow \ln N_{(t)} - \ln N_0 = \int_0^t r(s) ds$$

$$\Rightarrow \boxed{N_{(t)} = N_0 \cdot \exp\left[\int_0^t r(s) ds \right]} \quad \cdots ⑨$$

e.g. $r(t)$ is a fluctuating function $r(t) = \bar{r} + \frac{\sigma}{2} \sin(wt + \phi)$

$$\int_0^t r(s) ds = \bar{r} t + \frac{\sigma}{2} \left(\frac{-1}{w} \right) \cos(wt + \phi) \Big|_0^t$$

$$= \bar{r} t - \frac{\sigma}{2w} [\cos(wt+\phi) - \cos\phi]$$

$$\Rightarrow N_{(t)} = N_0 \cdot e^{\bar{r} t} \cdot \exp\left\{ -\frac{\sigma}{2w} [\cos(wt+\phi) - \cos\phi] \right\} \quad \cdots ⑩$$

2. Relax assumption 1, let there be net immigration / emigration

$$\frac{dN}{dt} = rN + I \quad \cdots (11)$$, solve $N(t)$? YOUR ASSIGNMENT

※ Recall: General solution for first order linear ODE

$$\frac{dy}{dt} + P(t)\,y = Q(t)$$

1. find integration factor: $I(t) = \exp\left[\int P(t)dt\right]$
2. Multiply both sides by $I(t)$:

$$\frac{dy}{dt}\cdot \exp\left[\int P(t)dt\right] + y\cdot P(t)\cdot \exp\left[\int P(t)dt\right] = Q(t)\cdot \exp\left[\int P(t)dt\right] \quad \cdots (12)$$

3. Simplify left hand side:

$$\frac{d}{dt}\left[y\cdot \exp\left[\int P(t)dt\right]\right] = Q(t)\cdot \exp\left[\int P(t)dt\right] \quad \cdots (13)$$

4. Integrate both sides of the equation:

$$y\cdot \exp\left[\int P(t)dt\right] = \int Q(t)\exp\left[\int P(t)dt\right]dt + C \quad \cdots (14)$$

5. Solve for $y(t)$:

$$y = e^{-\int P(t)dt}\left[\int Q(t)\, e^{\int P(t)dt}\, dt + C\right] \quad \cdots (15)$$

for this specific example: $P(t) = -r$, $Q(t) = I$, $I(t) = e^{-rt}$

$$\boxed{N(t) = N_0 e^{rt} + (e^{rt}-1)\frac{I}{r}} \quad \cdots (16)$$