2021. 11. 30 Predator – prey interactions

$N(t)$: prey population density
$P(t)$: predator population density

- model derivation
- graphical & stability analysis
- Lyapunov function

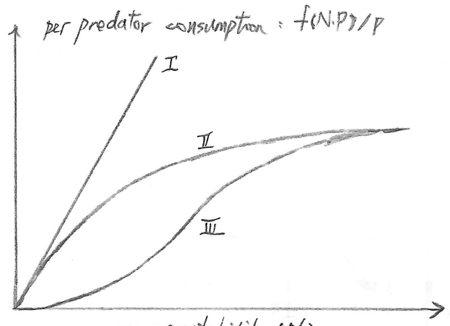
$$\begin{cases} \dfrac{dN}{dt} = \overbrace{g(N)}^{\text{growth of prey}} - \overbrace{f(N,P)}^{\text{consumption}} \\ \dfrac{dP}{dt} = \underbrace{e}_{\text{assimilation}} \cdot f(N,P) - d(P) \end{cases}$$

$d(P)$: death of predator when no food, usually exponential decay

◎ Model derivation

1. $g(N) = rN\left(1 - \dfrac{N}{K}\right)$
   - if $K \to \infty$: exponential growth
   - if $K > 0$: logistic growth

2. $f(N,P)$: often assume per predator consumption rate depends only on prey availability
   → « functional response »



- Type I : aN
- Type II : $\dfrac{aN}{1 + ahN} \Rightarrow \dfrac{\mu N}{K_m + N}$ (h: handling time, μ: max. consumption, K_m: half saturation const.)
- Type III : $\dfrac{aN^2}{1 + ahN^2} \Rightarrow \dfrac{\mu N^2}{Z + N^2}$

⇒ General form discussed in this class :

$$\begin{cases} \dfrac{dN}{dt} = rN\left(1 - \dfrac{N}{K}\right) - \left(\dfrac{aN}{1 + ahN}\right)P \\ \dfrac{dP}{dt} = e\left(\dfrac{aN}{1 + ahN}\right)P - dP \end{cases}$$

→ consider four model variates

1. exponential growth + Type I ($K \to \infty$, $h = 0$)
2. logistic growth + Type I ($K > 0$, $h = 0$)
3. exponential growth + Type II ($K \to \infty$, $h > 0$)
4. logistic growth + Type II ($K > 0$, $h > 0$)

◎ Model 1 : Exponential growth + Type I functional response

$$\begin{cases} \dfrac{dN}{dt} = rN - aNP = f_N(N,P) = N(r - aP) \\ \dfrac{dP}{dt} = eaNP - dP = f_P(N,P) = P(eaN - d) \end{cases}$$

≪ Lotka-Volterra predator-prey model ≫

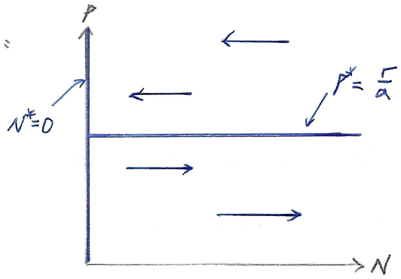
related to traits of the other species

- Find equilibrium by setting equation to zero & find ZNGI :

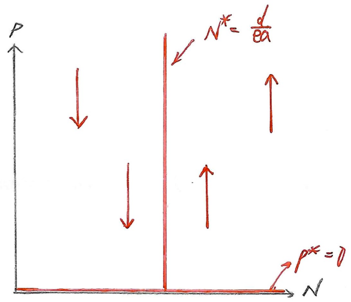
$$\Rightarrow \begin{cases} N^* = 0 \;\vee\; r - aP^* = 0 \\ P^* = 0 \;\vee\; eaN^* - d = 0 \end{cases} \Rightarrow E_0 \begin{cases} N^* = 0 \\ P^* = 0 \end{cases} \;\vee\; E_{NP} \begin{cases} N^* = \dfrac{d}{ea} \\ P^* = \dfrac{r}{a} \end{cases}$$

(2 equilibrium)

- Graphical analysis :



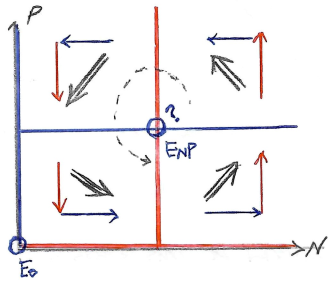
≪ For Prey, N ≫



≪ For Predator, P ≫

put together

Difficult to evaluate stability based on graphical analysis (converging ? diverging ? cycle ?)



- Local stability analysis ※ Recall : compute Jacobian → evaluate at equilibrium → eigenvalue negative real part

$$\begin{cases} \dfrac{\partial f_N}{\partial N} = r - aP \\ \dfrac{\partial f_N}{\partial P} = -aN \\ \dfrac{\partial f_P}{\partial N} = eaP \\ \dfrac{\partial f_P}{\partial P} = eaN - d \end{cases}$$

→ Basic structure of predator-prey Jacobian evaluated at $N^*>0, P^*>0$

$$\begin{bmatrix} J_{11} & - \\ + & J_{22} \end{bmatrix}$$

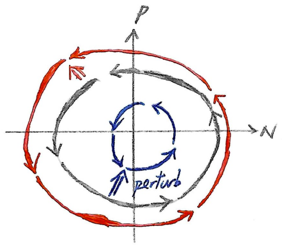
$$J_0 = \begin{bmatrix} r & 0 \\ 0 & -d \end{bmatrix} \Rightarrow \text{eigenvalues} = r,\ -d$$

unstable because $r > 0$ while we need all $Re(\lambda_i) < 0$

$$J_{NP} = \begin{bmatrix} 0 & -aN^* \\ eaP^* & 0 \end{bmatrix} \Rightarrow \lambda^2 + ea^2N^*P^* = 0$$

$$\Rightarrow \lambda^2 = -rd$$

$$\Rightarrow \lambda = \pm\sqrt{rd}\, i$$



eigenvalue has ZERO real parts → Neutral cycles

linearization predicts a center, which are structurally unstable

linear stability analysis fails to give concrete predictions

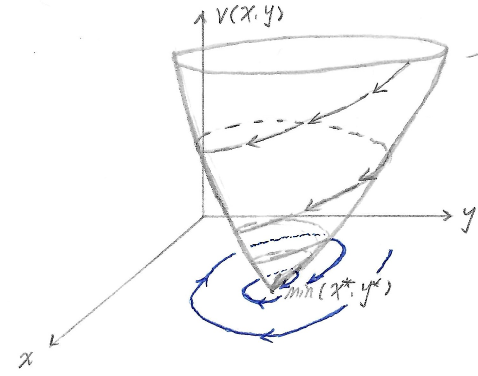
need to consider higher order terms (NOT EASY)

◎ How to judge when $Re(\lambda) = 0$? ⇝ Lyapunov function

If for a dynamic system we can find a function $V(x, y)$ such that

1. $V(x, y) > 0$ for all x, y AND $V(x^*, y^*) = 0$ (minimum value at said equilibrium)
2. $\frac{dV}{dt} < 0$ for all x, y except the equilibrium (all trajectories move downhill towards equilibrium)

⇒ Then, the existence of such function is a sufficient condition for (x^*, y^*) global stability



⟶ Energy-like functions

if all trajectories move downhill, globally stable

Unfortunately, there is no way to systematically find Lyaponov functions.

but for many cases smart mathematicians have done the hard work

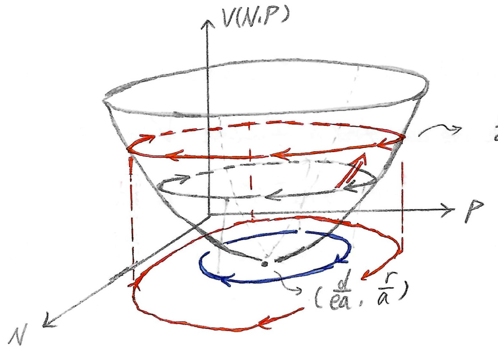
For Lotka-Volterra predator-prey system, a potential Lyapunov function is

$$V(N,P) = eaN - d\ln N + aP - r\ln P + \phi$$ ← some constant to adjust to zero

- $\dfrac{\partial V}{\partial N} = ea - d\cdot\dfrac{1}{N}$, $\dfrac{\partial V}{\partial P} = a - r\cdot\dfrac{1}{P}$ → set as zero, find minimum at $\left(N^* = \dfrac{d}{ea},\ P^* = \dfrac{r}{a}\right)$

- $$\begin{aligned}\frac{dV}{dt} &= \frac{dV}{dN}\cdot\frac{dN}{dt} + \frac{dV}{dP}\frac{dP}{dt} \\ &= \left(ea - \frac{d}{N}\right)\cdot(r - aP)N + \left(a - \frac{r}{P}\right)(eaN - d)P \\ &= (eaN - d)(r - aP) + (aP - r)(eaN - d) = 0\end{aligned}$$

$\Rightarrow \dfrac{dV}{dt} = 0$



$\dfrac{dV}{dt} = 0$ → Energy is conserved on the trajectory → nonlinear global stability analysis also predicts a center (nonlinear center)

<u>Summary</u>: Lotka-Volterra predator-prey model generates neutral cycles that are sensitivity to initial conditions and disturbance. However, the cycle will not crash as the origin is unstable