2021. 10. 12 Density-dependence & Logistic population growth

$$\frac{dN}{dt} = [b(N) - d(N)]N$$

- model assumptions & derivation
- equilibrium & graphical analysis
- Local stability analysis

@ Model derivation

$$\frac{dN}{dt} = B - D + I - E$$

@ Assumptions

1. closed population $I = E = 0$. ($\lim_{N \to 0} \frac{dN}{dt} = 0$)
2. Identical individuals w/o internal structure
3. Continuous population growth w/o time lag
4. time independent BUT density dependent per capita rates

$$= \underbrace{[b(N) - d(N)]N}_{r(N)\cdot N} \quad \cdots ①$$

5. Limited resource, so negative effect of density on growth rate ($\frac{dr(N)}{dN} < 0$)
6. Linear density dependence : $b(N) = b_0 - b_N \cdot N$
   $d(N) = d_0 + d_N \cdot N$

time independence

$$= [b_0 - b_N \cdot N - d_0 - d_N \cdot N]N$$

$$= [\underbrace{(b_0 - d_0)}_{r_0} - \underbrace{(b_N + d_N)}_{\alpha} N]N$$

$$\Rightarrow \frac{dN}{dt} = [r_0 - \alpha \cdot N]N \quad \cdots ②$$

logistic population growth model

intrinsic growth rate ($r_0 > 0$ when $b_0 > d_0$)

strength of density dependence ($\alpha > 0$)

[Pierre Francois Verhulst 1838]

[Raymond Pearl & Lowell Read 1920]

$$\Rightarrow \frac{dN}{dt} = r_0 N \left(1 - \frac{N}{K}\right) \quad \dots ③$$

carrying capacity, maximum pop. size that can be supported

$K = r_0/\alpha$

To understand the dynamics

@ Integrate the logistic equation

$$\frac{dN}{dt} = r_0 \cdot N \cdot \left(\frac{K-N}{K}\right)$$

$$\Rightarrow \frac{1}{N} \times \frac{K}{K-N}\, dN = r_0\, dt$$

$$\Rightarrow \int \left[\frac{1}{N} - \left(\frac{-1}{K-N}\right)\right] dN = \int r_0\, dt$$

$$\Rightarrow \int \frac{1}{N} dN - \int \frac{1}{N-K} dN = \int r_0\, dt$$

$$\Rightarrow \ln N - \ln(N-K) = \ln\left(\frac{N}{N-K}\right) = r_0 t + C_1$$

$$\Rightarrow \frac{N}{N-K} = C_2 \cdot e^{r_0 t}$$

$$\Rightarrow N(t) = \frac{K \cdot C_2 e^{r_0 t}}{C_2 e^{r_0 t} - 1} = \frac{K}{1 - \left(\frac{1}{C_2} e^{-r_0 t}\right)}$$

at $t=0$, $N(t=0) = N_0 = \frac{K C_2}{C_2 - 1} \rightsquigarrow C_2 = \frac{N_0}{N_0 - K}$

$$\Rightarrow N(t) = \frac{K}{1 - \left(\frac{N_0 - K}{N_0}\right) e^{-r_0 t}} \quad \dots ④$$

$$\lim_{t \to \infty} N(t) = K \Rightarrow \text{Final fate} \quad \dots ⑤$$

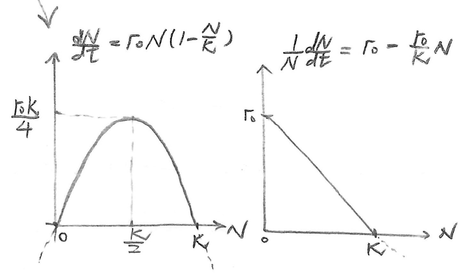
⟶ Ways to understand final fate / longterm dynamics

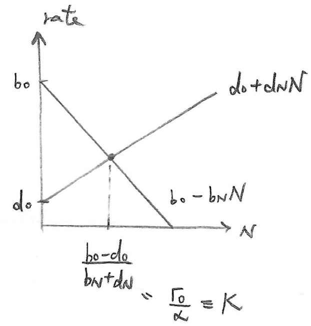
1. Integrate dN/dt
2. Graphical approach
3. Stability analysis

※ Recall: Separation of variables

$$\frac{dN}{dt} = f(N) \cdot g(t)$$

$$\int \frac{1}{f(N)} dN = \int g(t)\, dt$$



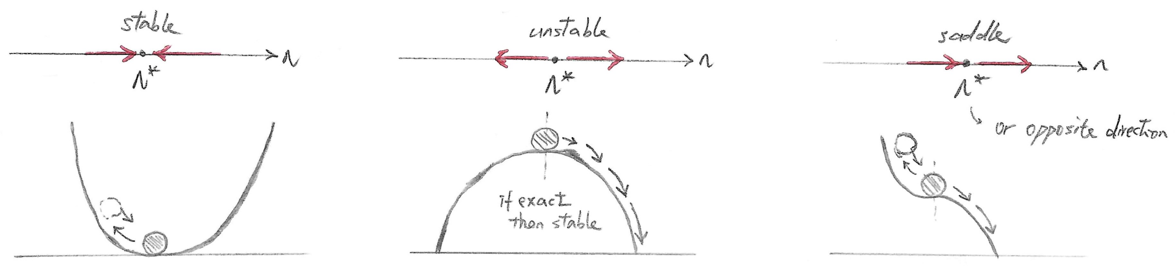


@ Equilibrium (fixed point)

- A point where the variable does not further change
  these values are good candidates where the variable will end up

- consider $\frac{dN}{dt} = f(N)$, a point N^* is an equilibrium of $\frac{dN}{dt} = f(N)$ — full model, population growth. not per capita growth
  if $f(N^*) = 0$. that is $\frac{dN}{dt}\Big|_{N=N^*} = f(N^*) = 0$ — do logistic example here

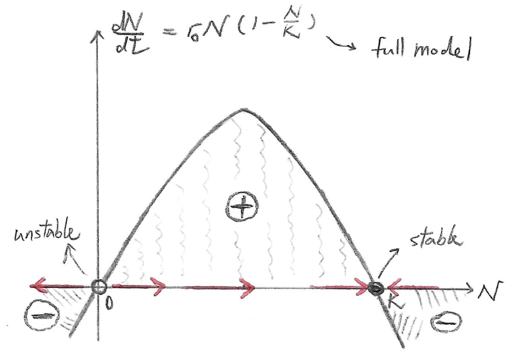
- Equilibrium (equilibria) can be classified based on their stability
  1. Attracting (stable) equilibrium, attracts nearby trajectories
  2. Repelling (unstable) equilibrium, repels nearby trajectories
  3. Saddle equilibrium, attract trajectories on one side but repels on the other



Stable (attracting) equilibrium   Unstable (repeling) equilibrium   Saddle

@ Graphical approach   plot $\frac{dN}{dt}$ versus N

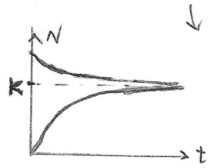
$$\frac{dN}{dt} = r_0 N\left(1 - \frac{N}{K}\right) \Rightarrow \overset{\text{find equilibrium}}{r_0 N^*\left(1 - \frac{N^*}{K}\right) = 0} \Rightarrow \text{equilibrium } N^* = 0 \vee K$$



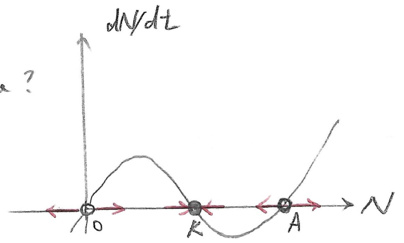
from graphical method

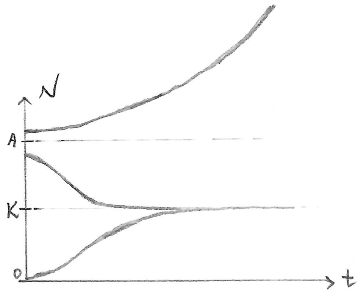
$$\begin{cases} N^* = 0 \text{ is unstable} & (\text{if } N_0 = 0 \text{ exact. } N(t) = 0) \\ N^* = K \text{ is stable equilibrium} \end{cases}$$

→ if $N_0 > 0$, $N(t) \to K$



Practice ?





◎ Local stability analysis

near the equilibrium N^*, do taylor expansion of $\frac{dN}{dt} = f(N)$ about N^*

think about "displacements" from N^*, $\varepsilon = N - N^*$

think how ε changes through time (study behavior of nearby points)

$$\frac{d\varepsilon}{dt} = \frac{d(N-N^*)}{dt} = \frac{dN}{dt} - \underbrace{\frac{dN^*}{dt}}_{0} = f(N)$$

not function of ε

$$= f(N^* + \varepsilon)$$

Recall: Taylor expansion

$$f(a+\Delta x) = f(a) + \Delta x f'(a) + \frac{\Delta x^2}{2!} f''(a) \dots$$

under small ε, near neighbors

$$= \underbrace{f(N^*)}_{0} + \varepsilon \cdot \frac{df}{dN}\Big|_{N=N^*} + O(\varepsilon^2)$$

$$\Rightarrow \frac{d\varepsilon}{dt} = \left[\frac{df}{dN}\Big|_{N=N^*}\right]\varepsilon \quad \dots ⑥$$

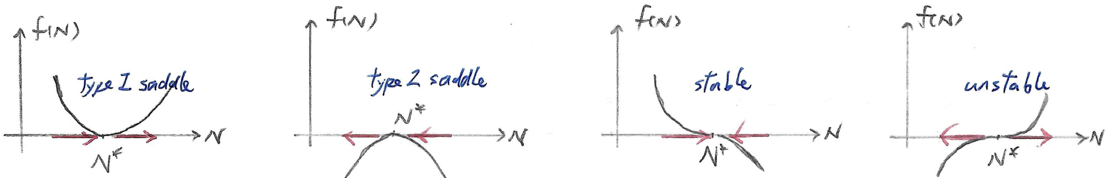
define as λ (constant)

$$\Rightarrow \varepsilon(t) = \varepsilon_0 \cdot e^{\lambda t}, \quad \text{with } \lambda = \frac{df}{dN}\Big|_{N=N^*} \quad \dots ⑦$$

1. if $\lambda > 0$, displacements amplify, N^* is unstable
2. if $\lambda < 0$, displacements shrink, N^* is stable

(3. if $\lambda = 0$, linear term is insufficient & higher order terms are required.)

nonhyperbolic equilibrium



$\left(\frac{d^2f}{dN^2} > 0\right)$ $\left(\frac{d^2f}{dN^2} < 0\right)$ $\left(\frac{d^2f}{dN^2} = 0, \frac{d^3f}{dN^3} < 0\right)$ $\left(\frac{d^2f}{dN^2} = 0, \frac{d^3f}{dN^3} > 0\right)$

◎ Link between graphical approach & stability analysis

for logistic growth: $\frac{dN}{dt} = r_0 N\left(1 - \frac{N}{K}\right) = f(N)$

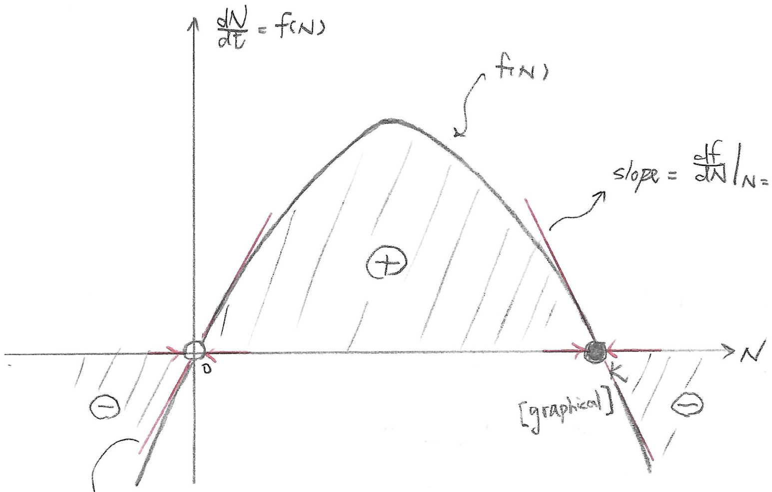
$$\frac{df}{dN} = r_0 - \frac{2r_0}{K} \cdot N = f'(N) \quad \cdots ⑧$$

Trick: don't expand everything

$$\frac{df}{dN} = r_0\left(1 - \frac{N}{K}\right) + r_0 N\left(-\frac{1}{K}\right)$$

① for $N^* = 0$, $\left.\frac{df}{dN}\right|_{N=0} = r_0 > 0 \rightsquigarrow$ unstable

② for $N^* = K$, $\left.\frac{df}{dN}\right|_{N=K} = -r_0 < 0 \rightsquigarrow$ stable



[stability analysis]

slope = $\left.\frac{df}{dN}\right|_{N=K} = -r_0 < 0$

< stable >

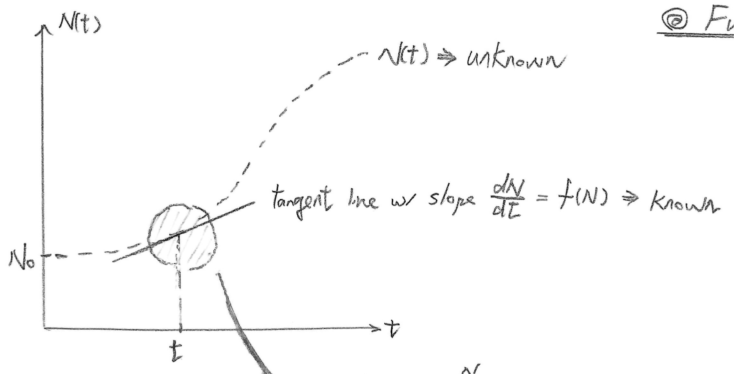
slope = $\left.\frac{df}{dN}\right|_{N=0} = r_0 > 0$

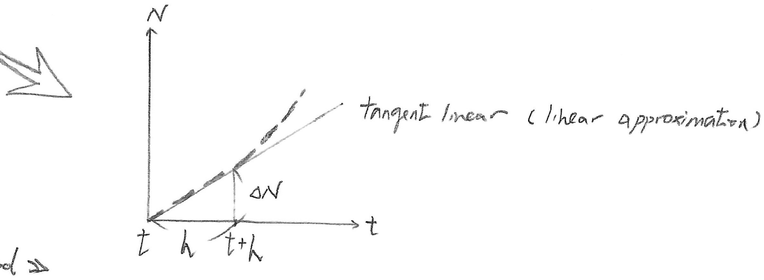
< unstable >

◎ Final Note: Parameter constraint is sometimes important

2021.10.12. Basic Intro to Numerical Methods

@ Fundamentals





« Forward Euler Method »

$$N(t+h) = N(t) + h \cdot \text{slope of tangent line at time point } t$$
$$= N(t) + h \cdot \underline{f(N(t))}$$

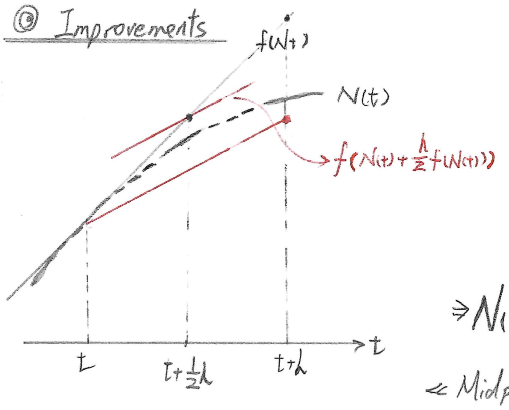
→ your ODE, provide this to deSolve

Recall: Taylor expansion

$$N(t+h) = N(t) + h \cdot \frac{dN}{dt}\Big|_{N=N_t} + \mathcal{O}(h^2)$$

only works when h is small enough (but this is costly)

@ Improvements



$$N(t+h) = N(t) + h \cdot f(N(t+\tfrac{h}{2}))$$

$$N(t+\tfrac{h}{2}) \approx N(t) + \frac{h}{2}\frac{dN}{dt}\Big|_t \approx N(t) + \frac{h}{2} f(N(t))$$

$$\Rightarrow N(t+h) = N(t) + h \cdot f(N(t) + \tfrac{h}{2} f(N(t)))$$

« Midpoint method »

weighted average of slopes

Commonly used algorithm: 4th order Runge-Kutta methods...

Multistep methods, implicit methods, adaptive timesteps.