Introduction to Theoretical Ecology Assignment 7

Linear Stability Analysis of Lotka-Volterra Competition Model

Continuing on the assignment last week, in this assignment you will analyze the same Lotka-Volterra competition model using linear stability analysis:

$$\frac{dN_1}{dt} = r_1 N_1 \left(1 - \frac{N_1 + \alpha N_2}{K_1}\right)$$

$$\frac{dN_2}{dt} = r_2 N_2 \left(1 - \frac{N_2 + \beta N_1}{K_2}\right)$$

, where r_1 and r_2 are the intrinsic population growth rates; K_1 and K_2 are the carrying capacities; α is the effect of N_1 on the population growth of N_2; β is the effect of N_2 on the population growth of N_1.

1. Perform linear stability analysis for all four equilibrium points you got in the previous assignment. Your answer should include (1) the Jacobian matrix evaluated at the equilibrium point, (2) the two eigenvalues of the Jacobian matrix (in the case of two species coexistence, simply show the characteristic equation), and (3) the stability criteria. (10 pts; 2.5 pts for each equilibrium point)

Solution:

(1) Equilibrium point (0, 0)

- Jacobian matrix: $\begin{bmatrix} r_1 & 0 \\ 0 & r_2 \end{bmatrix}$
- Eigenvalues: r_1 and r_2
- Stability criteria: always unstable since both r_1 and $r_2 > 0$

(2) Equilibrium point (K_1, 0)

- Jacobian matrix: $\begin{bmatrix} -r_1 & -r_1\alpha \\ 0 & r_2(1-\beta\frac{K_1}{K_2}) \end{bmatrix}$
- Eigenvalues: $-r_1$ and $r_2(1-\beta\frac{K_1}{K_2})$
- Stability criteria: stable if $\frac{K_1}{K_2} > \frac{1}{\beta}$

(3) Equilibrium point (0, K_2)

- Jacobian matrix: $\begin{bmatrix} r_1(1-\alpha\frac{K_2}{K_1}) & 0 \\ -r_2\beta & -r_2 \end{bmatrix}$
- Eigenvalues: $r_1(1-\alpha\frac{K_2}{K_1})$ and $-r_2$
- Stability criteria: stable if $\frac{K_2}{K_1} > \frac{1}{\alpha}$

(4) Equilibrium point $(\frac{K_1-\alpha K_2}{1-\alpha\beta}, \frac{K_2-\beta K_1}{1-\alpha\beta})$

- Jacobian matrix: $\begin{bmatrix} r_1 N_1^*(-\frac{1}{K_1}) & r_1 N_1^*(-\frac{\alpha}{K_1}) \\ r_2 N_2^*(-\frac{\beta}{K_2}) & r_2 N_2^*(-\frac{1}{K_2}) \end{bmatrix}$

  (No need to plug in the actual equilibrium values N_1* and N_2* in this step)

- Characteristic equation:

  $$\left(r_1 N_1^*\left(-\frac{1}{K_1}\right) - \lambda\right)\left(r_2 N_2^*\left(-\frac{1}{K_2}\right) - \lambda\right) - r_1 r_2 N_1^* N_2^*\left(\frac{\alpha\beta}{K_1 K_2}\right) = 0$$

  $$\longrightarrow \lambda^2 + \left(\frac{r_1 N_1^*}{K_1} + \frac{r_2 N_2^*}{K_2}\right)\lambda + \frac{r_1 r_2 N_1^* N_2^*}{K_1 K_2}(1 - \alpha\beta) = 0$$

- Stability criteria:

  $$-\frac{b}{a} = \lambda_1 + \lambda_2 < 0 \rightarrow \frac{r_1 N_1^*}{K_1} + \frac{r_2 N_2^*}{K_2} > 0$$

  $$\frac{c}{a} = \lambda_1 \lambda_2 > 0 \rightarrow \frac{r_1 r_2 N_1^* N_2^*}{K_1 K_2}(1 - \alpha\beta) > 0$$

→ stable if N_1* and N_2* > 0 (feasibility) & $\alpha\beta < 1$ (stabilization)