2021. 12. 28 Disease dynamics

Compartmental models : assign the population to different compartments and model the transition between compartments (SIR model of epidemiology)

Classify individuals as : $S(t)$ ⇒ density of susceptible individuals

$I(t)$ ⇒ density of infected individuals

$R(t)$ ⇒ density of resistant (recovered) individuals

→ e.g. vaccinated , exposed (latent)

@ SI model without demography

$$\begin{cases} \frac{dS}{dt} = - \text{Infection rate per infected individual} \times I \\ \frac{dI}{dt} = + \text{Infection rate per infected individual} \times I \end{cases}$$

$S \xrightarrow{\beta} I$

Infection rate per infected individual → rate of contacting other ind. × prob. of susceptible ind. × prob. of transmission

$$\begin{cases} \text{density-based transmission} = c \times N \times \frac{S}{N} \times a = \beta S \\ \text{frequency-based transmission} = c \times \frac{S}{N} \times a = \beta \times \frac{S}{N} \end{cases}$$

$c \times N$ → scales with N, e.g. cold, COVID

c → contact determined by social constraints (sexual disease)

$$\Rightarrow \begin{cases} \frac{dS}{dt} = - \beta S I \\ \frac{dI}{dt} = + \beta S I \end{cases}$$

→ define $N = S + I$

$$\frac{dN}{dt} = \frac{dS}{dt} + \frac{dI}{dt} = 0$$

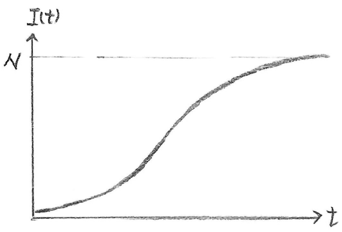
imply a closed population w/ $N = \text{const.} = S(0) + I(0)$

The system can be reduced to : $\frac{dI}{dt} = \beta (N - I) I$ ← 可積分 ! 如 logistic growth !

$$\Rightarrow I(t) = \frac{N}{1 + \left(\frac{N - I_0}{I_0}\right) e^{-N\beta t}}$$

↑ $t \to \infty$, $I(t) \to N$

all individuals become infected



◎ SIR model without demography

$S \xrightarrow{\beta} I \xrightarrow{\rho} R$

$$\begin{cases} \frac{dS}{dt} = -\beta S I \\ \frac{dI}{dt} = \beta S I - \rho I \\ \frac{dR}{dt} = \rho I \end{cases}$$

ρ ← recovery rate

※ observe that $N = S + I + R = \text{const.}$

since $\frac{dN}{dt} = \frac{dS}{dt} + \frac{dI}{dt} + \frac{dR}{dt} = 0$

* Basic Reproduction number (R_0) → R-naught

≡ the number of secondary infections caused by 1 infected ind. in a fully susceptible population

= (per capita infection in a fully susceptible population) × (infectious period)

$$= \beta \cdot S_0 \cdot \frac{1}{\rho} = \frac{\beta}{\rho} S_0$$

1. Can the disease spread in a fully susceptible population, S_0 ?

$$\lim_{I \to 0} \frac{dI}{dt}\frac{1}{I} = \beta \cdot S_0 - \rho : \begin{cases} \text{can spread if } \beta S_0 - \rho > 0 \Rightarrow R_0 > 1 \\ \text{cannot spread if } \beta S_0 - \rho < 0 \Rightarrow R_0 < 1 \end{cases}$$

2. What happens after initial spread ?

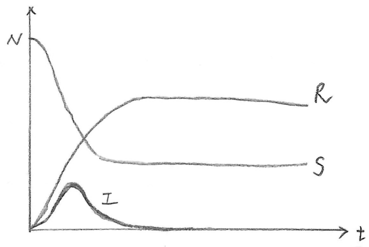
⇒ Observe $I^* = 0$ will stop the whole system (end on epidemic)

eventually disease will burnout despite initial spread, but will it infect everyone ?

⇒ How does I change with respect to S, the fuel of the epidemic

$$\frac{dI}{dS} = \frac{\beta S I - \rho I}{-\beta S I} = -1 + \frac{\rho}{\beta S}$$

← solve by separation of variables



$$\Rightarrow \int 1 \, dI = \int \left(-1 + \frac{\rho}{\beta S}\right) dS$$

$$\Rightarrow -S + \frac{\rho}{\beta} \ln S = I + C$$

$$\Rightarrow \frac{\rho}{\beta} \ln S = \frac{S_{0}}{R_0} \ln S = I + S + C = (N - R) + C$$

$$\Rightarrow S_{(t)} = \exp\left[\frac{R_0}{S_{(0)}}(N - R_{(t)})\right] \times C'$$

Note: $I_{(0)}$ is very small

with $t=0$, $S_{(t)} = S_{(0)} = \exp\left[\frac{R_0}{S_{(0)}}(N - R_{(0)})\right] \times C'$

$C' = S_{(0)} \times \exp\left[-\frac{R_0}{S_{(0)}}(N - R_{(0)})\right]$

$$\Rightarrow S_{(t)} = S_{(0)} \exp\left[\frac{-R_0}{S_{(0)}}(R_{(t)} - R_{(0)})\right]$$

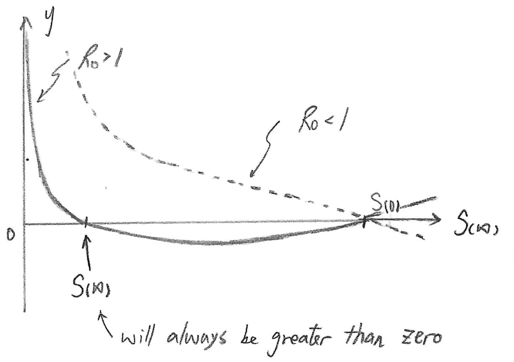
as $t \to \infty$, $I_{(t)} \to 0$, $R_{(t)} = R_{(\infty)} = N - S_{(\infty)}$

$$\Rightarrow S_{(\infty)} = S_{(0)} \exp\left[\frac{-R_0}{S_{(0)}}(N - S_{(\infty)} - R_{(0)})\right]$$

$$\Rightarrow \text{let } y = f(S_{(\infty)}) = \ln S_{(0)} - \frac{R_0}{S_{(0)}}(N - S_{(\infty)} - R_{(0)}) - \ln S_{(\infty)}$$

Use computer to look for $S_{(\infty)}$ that lets $y = 0$

when $R_0 > 1$ & $I_{(0)}$ is small (invasion scenario), $S_{(0)}$ is a solution, generally looks like...



without new susceptible individuals disease will self-extinguish

Summary: following initial spread, the epidemic eventually dies off and NOT all individuals will become infected. This is because eventually recovery outpace infection before all remaining susceptibles are infected (i.e. a fraction of ind. escape the disease)

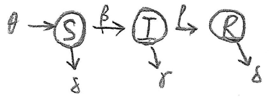
exact trajectory & final state depend on initial states (non-unique equilibrium with $I^* = 0$)

◎ SIR model with demography

slower disease dynamics so that the population is no longer closed

const arrival of new individuals

$$\begin{cases} \frac{dS}{dt} = \theta - \beta S I - \delta S = f_S \\ \frac{dI}{dt} = \beta S I - \rho I - \gamma I = f_I \\ \frac{dR}{dt} = \rho I - \delta R = f_R \end{cases}$$



infected individuals suffer higher mortality ($\gamma > \delta$)

mortality rate

※ Some notes on parameterization that can make model more traceable:

1. if infected individuals do not suffer higher mortality: $\gamma = \delta$
   then $\frac{dN}{dt} = \frac{dS}{dt} + \frac{dI}{dt} + \frac{dR}{dt} = \theta - \delta(S+I+R) = \theta - \delta N$

2. since $\frac{dR}{dt}$ is separated from the other two equations, the dynamics can be understood by analyzing $\frac{dS}{dt}$ & $\frac{dI}{dt}$, since $R(t) = N(t) - S(t) - I(t)$

$$N(t) = \frac{\theta}{\delta} + \left(N_0 - \frac{\theta}{\delta}\right) e^{-\delta t}$$

3. the S-I system can be analyzed via ZNGIs: $I = 0$, $S = \frac{\rho+\gamma}{\beta}$, $I = \frac{\theta}{\beta S} - \frac{\delta}{\beta}$

※ R_0 = (per capita infection power in a fully susceptible population) × (infectious period)

$$= \beta \times \left(\frac{\theta}{\delta}\right) \times \frac{1}{\rho + \gamma} = \left(\frac{\beta}{\rho+\gamma}\right) \times \underbrace{\frac{\theta}{\delta}}_{S^*_{DF}}$$

1. Find equilibrium: from $f_I = I(\beta S - \rho - \gamma)$; from f_R: $R^* = \frac{\rho}{\delta} I^*$

disease free equilibrium

$$E_{DF} \begin{cases} S^* = \frac{\theta}{\delta} \\ I^* = 0 \\ R^* = 0 \end{cases}$$

(2) For E_E : $J_E = \begin{bmatrix} -\beta I^* - \delta & -\beta S^* & 0 \\ \beta I^* & 0 & 0 \\ 0 & \rho & -\delta \end{bmatrix}$

$$\Rightarrow -\lambda(\delta+\lambda)(\beta I^* + \delta + \lambda) - (\delta+\lambda)\beta S^* \cdot \beta I^* = 0$$

$$\Rightarrow \underbrace{(\delta+\lambda)}_{\lambda = -\delta} \Big[\underbrace{\lambda^2 + (\beta I^* + \delta)\lambda + \beta^2 S^* I^*}_{\text{gives two roots w/ } Re(\lambda_i) < 0 \text{ if } I^* \text{ is feasible}} \Big] = 0$$

Summary: if $R_0 > 1$, disease can invade E_{DF}, E_E is feasible and stable

if $R_0 < 1$, disease cannot invade E_{DF}. E_E is unfeasible while E_{DF} is stable