Introduction to Theoretical Ecology (EEB5096)

Midterm exam (2021/11/24)

1. The dynamics of a population, $N(t)$, with self-limitation can be described using the logistic growth equation: $\frac{dN}{dt} = rN\left(1 - \frac{N}{K}\right)$, where r and K are the species' intrinsic rate of increase and carrying capacity, respectively. Use the technique of "separation of variables" to find the solution of this ordinary differential equation. In particular, show that the solution can be expressed as the following when the initial condition is N_0 at t = 0:

$$N(t) = \frac{K}{1 - \left(\frac{N_0 - K}{N_0}\right)e^{-rt}}$$

Based on the above solution, explain why K is the stable final state of the dynamics. [15 pts]

2. Levin's metapopulation model describes a species' occupancy on a landscape and models its dynamics as: $\frac{dP}{dt} = c(P)(1-P) - e(P)P$. Here, $P(t)$ is the species' occupancy, $c(P)$ and $e(P)$ represent the local colonization and extinction processes, both as a function of current occupancy. This general form of the

model can incorporate both internal colonization and rescue effect. The model will then take the form as:

$$\frac{dP}{dt} = cP(1-P) - e(1-P)P$$

(1) Find all equilibrium points of this model. [5 pts]

(2) Use graphical analysis to determine the stability criteria for the equilibrium points. [5 pts]

(3) Derive the stability criteria for the equilibrium points using local stability analysis. [5 pts]

3. In the 1940s, some researchers have proposed the following modified Lotka-Volterra competition model to describe the dynamics of two species engaged in mutualism:

$$\frac{dN_1}{dt} = r_1 N_1 \left(\frac{K_1 - N_1 + \alpha N_2}{K_1} \right)$$

$$\frac{dN_2}{dt} = r_2 N_2 \left(\frac{K_2 - N_2 + \beta N_1}{K_2} \right)$$

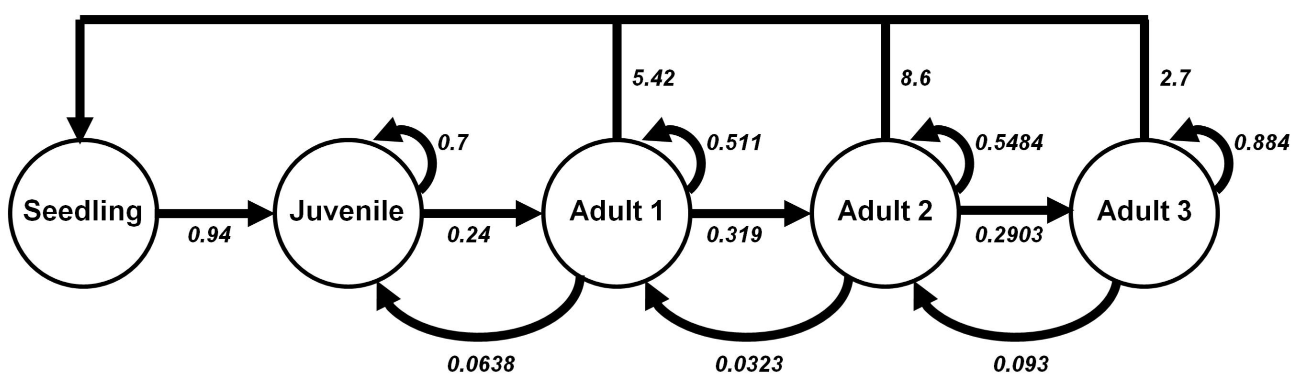
Here, the two species densities are represented by $N_1(t)$ and $N_2(t)$, with intrinsic growth rates r_1 and r_2 as well as carrying capacity K_1 and K_2, respectively. Heterospecific facilitation effect of N_2 on N_1 is captured by the parameter α and that of N_1 on N_2 by β (all parameters strictly positive).

(1) Find all equilibrium points of this model. [5 pts]

(2) Draw the ZNGI and movement arrows on the *s*tate space for N_1 and N_2 separately. [5 pts]

(3) Combine the two ZNGIs and discuss the stability of the equilibrium points (and dynamics of the system) under different parameter scenarios using graphical analysis. [10 pts]

4. The life history diagram of *Mammillaria dixanthocentron* (a species of cactus) can



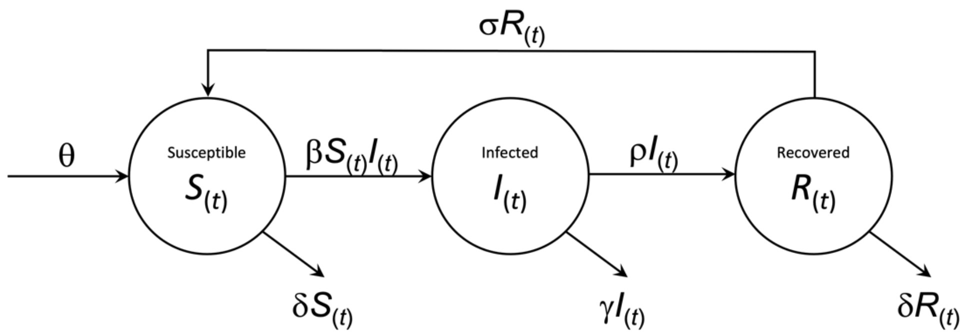
be represented by the following diagram.

(1) Based on the above diagram, write out its transition matrix. [5 pts]

(2) Report its long-term finite rate of increase to the fourth decimal place. Would the population be growing or dying? [2.5 pts]

(3) Report the steady stage distribution, rounded to the fourth decimal place. [2.5 pts]

5. The spread of a disease within a population can be described by the classic SIR model, which stands for susceptible ($S_{(t)}$), infected ($I_{(t)}$), and recovered ($R_{(t)}$) individuals within the population. The model assumes a constant arrival of new susceptible individuals (θ), a transmission process following the mass-action assumption with efficiency β, recovery with rate ρ, and decay of immunity with rate σ. Uninfected individuals have mortality rate δ while infected individuals suffer a higher mortality rate γ. The model can be represented by the following diagram:



(1) Write down the set of ordinary differential equations for the system. [10 pts]

(2) With initial population size $S_{(0)}$ = 100, $I_{(0)}$ = 2, and $R_{(0)}$ = 0, run the system to equilibrium using the following parameter values: θ = 0.1, β = 0.01, ρ = 0.05, σ = 0.0, δ = 0.01, and γ = 0.02. Report the equilibrium population size (for all three state variables) rounded to the fourth decimal place. [5 pts]

(3) Sketch the dynamics of the three state variables during the first 100 time steps. [5 pts]

(4) Under certain parameters, the disease will not spread and the system will end up at the "disease-free equilibrium" with I^* = R^* = 0. With your model, derive the analytical form of S^* at this disease-free equilibrium. [5 pts]

(5) By changing only one parameter in (2) at a time, provide two different parameter sets that will cause the system to end up at the disease-free equilibrium. Explain the ecological meaning of your parameter alternation. [5 pts]

(6) Consider a new population state, $V_{(t)}$, representing vaccinated individuals. Assume that the country's vaccination rate is μ = 0.005 (relatively low), write a new set of equations and simulate its long-term dynamics again with the parameters provided in (2). Report the new equilibrium for all four state variables rounded to the fourth decimal place. [10 pts]