2021. 9. 28 Mathematical Background

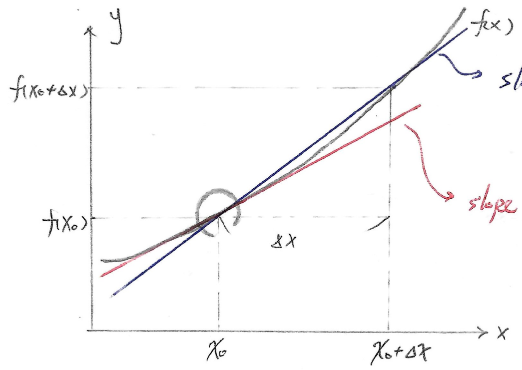
◎ Derivatives

1 Consider a function of an independent variable x, $f(x)$

its derivative is defined as : $\dfrac{df(x)}{dt} = \lim_{\Delta x \to 0} \dfrac{f(x+\Delta x) - f(x)}{\Delta x} = f'(x)$

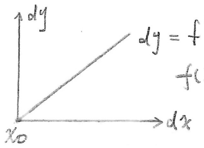
change in function value

change in input value



slope $= \dfrac{f(x_0+\Delta x) - f(x_0)}{\Delta x}$

slope $= \lim_{\Delta x \to 0} \dfrac{f(x_0+\Delta x) - f(x_0)}{\Delta x} = \left.\dfrac{df(x)}{dt}\right|_{x_0}$ → tangent line



$dy = f'(x_0)\,dx$

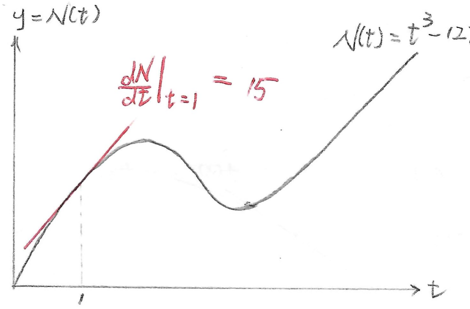
$f(x_0+dx) = f(x_0) + f'(x_0)\,dx$

e.g.
rate of population size change

↓

$\dfrac{dN(t)}{dt} = 3t^2 - 24t + 36 = N'(t)$

$\Rightarrow N'(1) = 15$ ※



$N(t) = t^3 - 12t^2 + 36t$

$\left.\dfrac{dN}{dt}\right|_{t=1} = 15$

derivatives of power, exponential, log, trig functions

2. Some rules

$$\frac{d(f(x)g(x))}{dx} = \frac{df(x)}{dx}\cdot g(x) + f(x)\cdot\frac{dg(x)}{dx}$$ « product rule »

$$\frac{d\left(\frac{f(x)}{g(x)}\right)}{dx} = \frac{g(x)\cdot\frac{df(x)}{dx} - f(x)\,\frac{dg(x)}{dx}}{g(x)^2}$$ « quotient rule »

$$\frac{d(f(g(x)))}{dx} = \frac{df}{dg}\cdot\frac{dg}{dx} = f'(g(x))\cdot g'(x)$$ « chain rule »

e.g. $f(x) = x^n e^{ax}$, $f'(x) = nx^{n-1}e^{ax} + x^n e^{ax}\cdot a$ ※

Integral:

1. If $F'(x) = f(x)$, then $F(x)$ is the antiderivative of $f(x)$

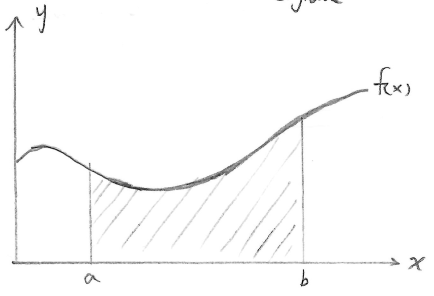
$f(x)$ gives the rate of change of $F(x)$

$$\int f(x)dx = F(x) + C$$

C: integration constant

« indefinite integral »

2. Integrals represent the signed area under the curve $f(x)$



« definite integral »

$$\int_a^b f(x)dx = F(b) - F(a) = F(x)\Big|_a^b$$

e.g. $f(x) = x^2$

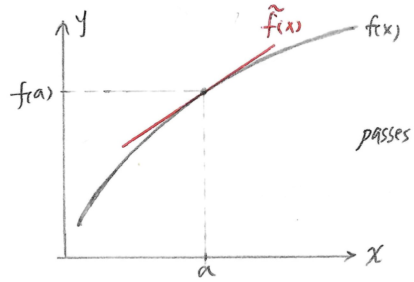
$$\int_0^1 x^2 dx = \frac{1}{3}x^3\Big|_0^1 = \frac{1}{3}$$

Linear approximations

for smooth & continuous functions

We want to approximate a function when the variable lies near a particular value

⇒ Any curve looks like a line when we look close enough (i.e. $y = mx + c$)



$$f(a) = \left(\frac{df}{dx}\Big|_a\right)a + c \rightarrow c = f(a) - \left(\frac{df}{dx}\Big|_a\right)a$$

passes through point

$$\Rightarrow \tilde{f}(x) = \left(\frac{df}{dx}\Big|_a\right)x + f(a) - \left(\frac{df}{dx}\Big|_a\right)a$$

$$\Rightarrow \tilde{f}(x) = f(a) + \frac{df}{dx}\Big|_a (x-a)$$

e.g. $f(x) = e^x \Rightarrow \tilde{f}(x) = 1 + e^0 x = 1 + x$ (around 0)

Taylar Series

Most functions $f(x)$ can be approximated by a power series around the point a:

$$f(x) = \sum_0^\infty \frac{f^{(n)}(a)}{n!}(x-a)^n = f(a) + \frac{df}{dx}\Big|_a (x-a) + \frac{1}{2!}\frac{d^2f}{dx^2}\Big|_a (x-a)^2 + \frac{1}{3!}\frac{d^3f}{dx^3}\Big|_a (x-a)^3 \cdots$$

$$= f(a) + f'(a)(x-a) + \frac{1}{2!}f''(a)(x-a)^2 + \frac{1}{3!}f'''(a)(x-a)^3 \cdots$$

e.g. $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$ (around 0)