The "Difficulties" in Fermat's Original Discourse on the Indecomposability of Powers Greater Than a Square: A Retrospect

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Abstract
A possible version of the original proof of the decomposability of whole degrees above the square that Pierre Fermat spoke of has been identified. This reconstructed evidence is discussed with some extra conclusions drawn from it.

Keywords: number theory, Fermat's Big/Last Theorem, Hisitory of Mathematics, Algebra, Proof

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Dr Dedenko's research interests range from nuclear science and engineering to pure mathematics, history and philosophy.

1 Introduction

The present work is the result of an attempted reconstruction of Fermat's original discourse along with an explanation of why he might have not written it down. The author had performed it within a two-time period of time– between 1990 and 1993 – trying proving the theorem and the final revision in 2017-2025.[1] When completed, it did look like a proof of Fermat's epoch, as it only involved the knowledge and techniques available and utilised by Fermat's contemporary and pre-Fermat mathematical world.

Not to overburden this text with details of a real historical study, let us briefly recall the history of the conjecture. Around 1637, Fermat wrote his Last Theorem in the margin of his copy of the Arithmetica next to Diophantus' sum-of-squares problem [7]:

Cubum autem in duos cubos, aut quadratoquadratum in duos quadratoquadra-tos & generaliter nullam in infinitum ultra quadratum potestatem in duos eiusdem nominis fas est dividere cuius rei demonstrationem mirabilem sane detexi. Hanc marginis exiguitas non caperet.

Attempts to prove this conjecture employed a diverse range of methods. Early attempts, including Fermat's own proof for the case $n = 4$, utilized the **method of infinite descent**. A significant step forward was Leonhard Euler's proof for the case $n = 3$, which involved the use of **Gaussian integers** [13]. Subsequent efforts included approaches based on the work of Sophie Germain and other techniques, before Andrew Wiles finally presented a complete proof in 1995, drawing upon deep results from **the theory of modular forms and elliptic curves** [1–6].

Modern methods, such as the theory of modular forms, deal with **the transformation properties of specific curves over particular types of spaces** (e.g., rational numbers), highlighting the **stability of elliptic curves with respect to modular transformations**. While the contemporary formalizations of algebraic curves, spaces, transformations, and groups used in these methods were not present in Fermat's time, mathematicians of that era possessed their own approaches and intuitive understandings for studying the properties of natural numbers (and primes). (More on this can be found in the Conclusions section of this paper.)

Theorem 1 (Conditional on Dedenko's Ansatz)**.** *Assume the following Ansatz: for every $n > 2$ and every integer solution $x^n + y^n = z^n$ there exists an integer $o > 1$ such that $o^n = 2 \cdot n$. Then the Fermat equation $x^n + y^n = z^n$ has no solutions in $\mathbb{N}$ for all $n > 2$.*

[1] *The work was mainly performed on the scholarship of the NRNU MEPhI student in* $1990-93$, *final revision in* $2017-2025$.

Proof sketch. The Ansatz forces $o^n = 2 \cdot n$. Elementary growth estimates imply $o = 2$ and $n \in \{1, 2\}$ only; hence for $n > 2$ a contradiction. See the accompanying Coq file `FLT.v` for a machine-checked proof.

2 Possible Proof

The statement of the theorem is rather straightforward and as follows:
Neither a cube for two cubes, nor a biquadrate or two biquadrates, and generally no power greater than two can be decomposed into two powers of the same grade. In other words, the equation

$$x^n + y^n = z^n$$

has no solutions in natural numbers if n is an integer greater than 2.
Therefore, first

1). Let's write down the theorem

$$x^n + y^n = z^n \tag{2.1}$$

2). Let's rewrite (2.1)

$$z^n - x^n = y^n \tag{2.2}$$

3). Let's perform the transformation of (2.2) to

$$\big(m^n + p^n\big)^n - \big(m^n - p^n\big)^n = y^n, \tag{2.3}$$

where $n \in \mathbb{N}$, m, p are <u>arbitrary numbers</u> (not necessarily integers; signs arbitrary), and $z, x \in \mathbb{N}$ satisfy

$$\begin{cases} m^n + p^n = z, \\ m^n - p^n = x. \end{cases} \tag{2.4}$$

Raising both identities to the n-th power yields

$$\begin{cases} (m^n + p^n)^n = z^n, \\ (m^n - p^n)^n = x^n. \end{cases} \tag{2.5}$$

Thus (2.4) implies (2.5).

Solving (2.4). Adding and subtracting the equations gives

$$2m^n = z + x, \qquad 2p^n = z - x,$$

so formally

$$m = \sqrt[n]{\frac{z+x}{2}}, \qquad p = \sqrt[n]{\frac{z-x}{2}}.$$

<u>Domain note.</u> Over $\mathbb{R}$: for odd n the roots are unique; for even n we need $(z \pm x)/2 \geq 0$ and there is a sign choice $m = \pm\big((z+x)/2\big)^{1/n}$, $p = \pm\big((z-x)/2\big)^{1/n}$. Over $\mathbb{C}$: there are n branches for the n-th root; once a branch is fixed, the reconstruction is consistent.

Equivalence of (2.4)–(2.5). Forward (2.4) $\Rightarrow$ (2.5)**.** Immediate by raising to the n-th power.

Reverse (2.5) $\Rightarrow$ (2.4)**.** Assume $z, x \in \mathbb{N}$ and

$$z^n = (m^n + p^n)^n, \qquad x^n = (m^n - p^n)^n.$$

Taking n-th roots in the same domain as $m^n \pm p^n$ (principal root over $\mathbb{R}_{\geq 0}$, or a fixed branch over $\mathbb{C}$) gives

$$z = m^n + p^n, \qquad x = m^n - p^n,$$

i.e. (2.4). For even n one fixes consistent signs as above. Hence, (2.4) and (2.5) are equivalent <u>in the chosen number domain</u> once the root/branch convention is fixed.

Proposition 1 (Integer case: necessary & sufficient conditions)**.** *If, in addition, one requires $m, p \in \mathbb{Z}$, then necessarily*

$$z \pm x \text{ are even}, \qquad \frac{z+x}{2} = m^n, \qquad \frac{z-x}{2} = p^n.$$

Conversely, if $z \pm x$ are even and both halves are perfect n-th powers in $\mathbb{Z}$, then the reconstructed m, p are integers (up to signs for even n).

Remark 1. *This separates the general real/complex reconstruction (no parity obstruction) from the integer reconstruction, where parity and perfect-power constraints are essential.*

Example (real domain). For $n = 2$, $m = 3$, $p = 2$:

$$z = 3^2 + 2^2 = 13, \qquad x = 3^2 - 2^2 = 5,$$

hence

$$z^2 = 169, \qquad x^2 = 25,$$

and reversing via $(z \pm x)/2$ returns $m = 3$, $p = 2$.

Counterexample (integer case). For $n = 3$, $z = 2$, $x = 1$ we have $(z + x)/2 = 3/2$, which is not a perfect cube in $\mathbb{Z}$; hence no integer m, p satisfy (2.4), although real m, p exist.

4). Let's ask ourselves the question: what is the number y? Is it positive integer or not? If it is not positive integer, then under what conditions will it be natural number? Does its naturalness depend on the degree of n?

From (2.3) we have the difference:

$$y^n = z^n - x^n = (m^n + p^n)^n - (m^n - p^n)^n = \tag{2.6}$$

that can be expanded or decomposed into a sum according to Newton's binomial [8, 9]:

$$\begin{aligned}
&= [(m^n)^n + C_n^1(m^n)^{n-1}p^n + C_n^2(m^n)^{n-2}(p^n)^2 + \cdots + C_n^{n-1}m^n(p^n)^{n-1} + (p^n)^n] - \\
&-[(m^n)^n - C_n^1(m^n)^{n-1}p^n + C_n^2(m^n)^{n-2}(p^n)^2 \pm \cdots \pm C_n^{n-1}m^n(p^n)^{n-1} \pm (p^n)^n] = \\
&= 2C_n^1(m^n)^{n-1}p^n + 2C_n^3(m^n)^{n-3}(p^n)^3 + \cdots + 2C_n^k(m^n)^{n-k}(p^n)^k + \{2C_n^n(p^n)^n\} = \\
&= 2\textstyle\sum_{i=0}^k C_n^{(2i+1)}(m^n)^{n-(2i+1)}(p^n)^{(2i+1)}
\end{aligned}$$

with $k = (n-1)/2$ if n is odd and $k = (n-2)/2$ if n is even.

Rewrite then (2.6) as

$$z^n = x^n + y^n,$$

$$\text{where } x,\ y,\ z \text{ are} \begin{cases} z = m^n + p^n \\ x = m^n - p^n \\ y = \sqrt[n]{2}\left[\sum_{i=0}^k C_n^{(2i+1)}(m^n)^{n-(2i+1)}(p^n)^{(2i+1)}\right]^{1/n} \end{cases}$$

with $k = (n-1)/2$ if n is odd and $k = (n-2)/2$ if n is even;

we know that in common case for $n = 2$ we have

$$\text{where } x,\ y,\ z \text{ are given by} \begin{cases} z = r \cdot (m^2 + p^2), \\ x = r \cdot (m^2 - p^2), \\ y = r \cdot 2mp. \end{cases} \tag{2.7}$$

but in our case, we omit the factor r – is some positive ($r > 0$) integer constant since it is reduced in our calculations. Therefore, we made the transformation (2.3) in order to take into account this case (n=2), known since the time of Pythagoras, since the general case should include a particular solution as a subset.

5). scrutinise now the y:

$$y = \sqrt[n]{2}\left[\sum_{i=0}^{k} C_n^{(2i+1)}(m^n)^{n-(2i+1)}(p^n)^{(2i+1)}\right]^{1/n}. \tag{2.8}$$

In order for the y to maybe a positive integer, $\sqrt[n]{2}$ must leave, since for $n > 1$ $\sqrt[n]{2}$ is an irrational number (see Appendix A). It is thus necessary that the expression

$$\left[\sum_{i=0}^{k} C_n^{(2i+1)}(m^n)^{n-(2i+1)}(p^n)^{(2i+1)}\right]^{1/n} \tag{2.9}$$

contain some common factor that destroys the radical expression $\sqrt[n]{2}$, let's find out what it is. Otherwise, y is not a positive integer due to the presence of $\sqrt[n]{2}$. Consider now what largest divisor this sum may contain and what it is equal to:

$$\begin{aligned}
&\left[\textstyle\sum_{i=0}^{k} C_n^{(2i+1)}(m^n)^{n-(2i+1)}(p^n)^{(2i+1)}\right] = \\
&= C_n^1(m^n)^{n-1}p^n + C_n^3(m^n)^{n-3}(p^n)^3 + \cdots + C_n^k(m^n)^{n-k}(p^n)^k + \{C_n^n(p^n)^n\} = \\
&= n(m^n)^{n-1}p^n + \tfrac{n(n-1)(n-2)}{3!}(m^n)^{n-3}p^3 + \cdots + \\
&+ \tfrac{n(n-1)\ldots(n-k+1)}{k!}(m^n)^{n-k}(p^n)^k + \left\{\tfrac{n(n-1)\ldots 2\cdot 1}{n!}(p^n)^n\right\} = \\
&= n\cdot\left[(m^n)^{n-1}p^n + \tfrac{(n-1)(n-2)}{3!}(m^n)^{n-3}p^3 + \ldots + \right. \\
&\left. + \tfrac{(n-1)\ldots(n-k+1)}{k!}(m^n)^{n-k}(p^n)^k + \left\{\tfrac{(n-1)\ldots 2\cdot 1}{n!}(p^n)^n\right\}\right] = n\cdot l^n
\end{aligned} \tag{2.10}$$

with $k = (n-1)/2$ if n is odd and $k = (n-2)/2$ if n is even,

Hence the conclusion follows: n is a common divisor, and where l is some constant about which we know nothing (it is maybe real in common case or maybe integer), through which we denoted the rest of the radical after the allocation of the common set n.

Since all the terms in the sum (2.10) contain the factor n, the expression is divisible by n.

To show the uniqueness of the decomposition of expression (2.10), we can use the following considerations:

(a) Degree n is a fixed positive integer.

(b) Exponentiation is an unambiguous operation; for any number, there is a unique value for the degree.

(c) Addition and subtraction of real numbers are commutative operations, and the result does not depend on the order of actions.

Based on these properties, it can be argued that for fixed m, p and n there is a single result of calculating the expression (2.10). Rearranging the members will not affect the final answer.

We can see that from (2.8 - 2.10) (where l is some constant (which will be it for fixed m, p, n), which we don't know anything about yet)

$$y = \sqrt[n]{2\cdot n}\cdot l \tag{2.11}$$

Hence, we see from (2.6-2.11) that

$$z^n - x^n = (m^n + p^n)^n - (m^n - p^n)^n = y^n$$

$$(m^n + p^n)^n - (m^n - p^n)^n = 2\cdot n\cdot l^n$$

Let's put $m = j\cdot p$, where $j > 1$ is any number.

$$((j\cdot p)^n + p^n)^n - ((j\cdot p)^n - p^n)^n = 2\cdot n\cdot l^n$$

$$(p^n \cdot (j^n+1))^n - (p^n \cdot (j^n-1))^n = 2 \cdot n \cdot l^n$$
$$(p^n)^n \cdot ((j^n+1)^n - (j^n-1)^n) = 2 \cdot n \cdot l^n$$

Consider the difference $(j^n+1)^n - (j^n-1)^n$, decompose it according to the Newtonian Binomial

$$(j^n+1)^n - (j^n-1)^n = \sum_{k=0}^{n} \binom{n}{k} (j^n)^{n-k} - \sum_{k=0}^{n} \binom{n}{k} (j^n)^{n-k} (-1)^k$$

Combining the amounts

$$\sum_{k=0}^{n} \binom{n}{k} (j^n)^{n-k} \left[1-(-1)^k\right]$$

Simplify the amount. Since $\left[1-(-1)^k\right]$ it is 2 when k is odd and 0 when k is even, the sum will only include the odd values of k.

Thus, we have:

$$(p^n)^n \cdot 2 \cdot \sum_{k \quad odd} \binom{n}{k} (j^n)^{n-k} = 2 \cdot n \cdot l^n$$

Let's denote

$$q^n = 2 \cdot \sum_{k \quad odd} \binom{n}{k} (j^n)^{n-k}$$

Therefore

$$(p^n)^n \cdot q^n = 2 \cdot n \cdot l^n$$
$$\frac{(p^n)^n \cdot q^n}{l^n} = 2 \cdot n \tag{2.12}$$

Considering the expressions for q and l in equation (2.12), we have an infinite number of solutions.

Why infinity?

From (2.8-2.11), we see that

$$l^n = \frac{1}{n} \sum_{k \text{ odd}} \binom{n}{k} m^{n(n-k)} p^{nk}$$

When we accept $m = j \cdot p$, we see that

$$l^n = (p^n)^n \frac{1}{n} \sum_{k \quad odd} \binom{n}{k} (j^n)^{n-k}$$

Therefore, substituting the obtained expressions for l and q in (2.12), we see an infinite number of solutions; in fact, we have obtained an identity.

$$\frac{(p^n)^n \cdot 2 \cdot \sum_{k \quad odd} \binom{n}{k} (j^n)^{n-k}}{(p^n)^n \frac{1}{n} \sum_{k \quad odd} \binom{n}{k} (j^n)^{n-k}} = 2 \cdot n$$

Consequently, the only possible way to solve equation (2.12) is to assume, by imposing an additional constraint on it, that there exists a real number $o > 1$ satisfying the condition:

$$o^n = 2 \cdot n \tag{2.13}$$

Where o is

$$o = \left(\frac{p^n \cdot q}{l}\right)$$

Statement 1: In equation (2.13), the lift side o^n exhibits exponential growth as the argument n increases continuously, while the right side $2 \cdot n$ exhibits linear growth. Therefore, for $o > 2$, this equation has no solutions. Based on this condition and the additional condition $o > 1$, we observe that the only possible value for o that could lead to solutions is $o = 2$, which is a positive real number. Analyzing the right side of the equation, we see that this value of o is an integer because n is an integer.

Let us prove the **Statement 1**

For this a closer look at the expression (2.13)

Equation for a Fixed n

Consider the equation

$$\left(\frac{p^n \cdot q}{l}\right)^n = 2 \cdot n. \tag{2.14}$$

Rewrite (2.14) as

$$\frac{p^n \cdot q}{l} = \sqrt[n]{2 \cdot n}. \tag{2.15}$$

Then modify (2.15):

$$\frac{p^n \cdot q}{l} - \sqrt[n]{2 \cdot n} = 0. \tag{2.16}$$

Applying the ansatz method to (2.16) (see Appendix F) and introducing a new variable o, we get:

$$o - \sqrt[n]{2 \cdot n} = 0. \tag{2.17}$$

Hence,

$$o = \sqrt[n]{2 \cdot n}. \tag{2.18}$$

This allows us to reduce the analysis of the original equation to analyzing the properties of the function $f(n) = \sqrt[n]{2 \cdot n}$.

For a specific n, this equation determines a unique value of o. Here, there is no implication that n "changes" or "tends to something": n is a fixed parameter of the problem.

Behavior of $f(n) = \sqrt[n]{2 \cdot n}$ for Various n

Although in each specific formulation n is fixed, it is useful to consider the function

$$f(n) = \sqrt[n]{2 \cdot n}$$

for different integer values $n \geq 1$ in order to understand its behavior in a family of equations (one equation for each n).

- For $n = 1$, we have $f(1) = 2$.
- For $n = 2$, we have $f(2) = 2$.

When $n = 1$ and $n = 2$, the function attains the value 2. If we consider all n starting from 1, we can also prove that

$$f(n) \leq 2 \quad \text{for all } n \geq 1,$$

with the maximum equal to 2, reached only at $n = 1$ and $n = 2$. Moreover, for $n > 2$, $f(n)$ decreases monotonically (see Appendix C). Following this decreasing trend all the way to large n, we get (see Appendix B)

$$\sqrt[n]{2 \cdot n} \to 1 \quad \text{as} \quad n \to \infty.$$

However, this decrease toward 1 applies to the family of equations as we go from n to $n+1$, etc.; in a problem with a fixed n, such a transition is not used.

Maximum Number of Integer Solutions When Choosing o

Returning to the equation

$$o = \sqrt[n]{2 \cdot n},$$

for a family of values n, we want to find for which o this equation has the largest number of integer solutions for n. From the behavior of $f(n)$ described above, we see:

$$f(n) \leq 2, \quad \text{and } f(n) = 2 \text{ only when } n = 1 \text{ and } n = 2.$$

Therefore, if $o = 2$, we get two integer solutions ($n = 1$ and $n = 2$). If $o > 2$, there are no solutions; if $1 < o < 2$, then (due to the decreasing nature of $f(n)$ for $n > 2$) there is at most one solution. For $o \leq 1$, there are no solutions because $f(n)$ does not drop below 1.

Thus, choosing $o = 2$ yields the greatest number of integer solutions for n, which, in the context of the original formulation, leads to the final equality

$$2 = \sqrt[n]{2 \cdot n}. \tag{2.19}$$

The choice of the number 2 is driven by our desire to investigate the behavior of the original equation in the region most "saturated" with integer solutions. As shown, the value $o = 2$ corresponds to the maximum of the function $f(n) = \sqrt[n]{2 \cdot n}$ and thus ensures the presence of two integer solutions for n. Focusing on this value allows us to analyze the situation where the relationship between the parameters of the original equation permits the greatest number of integer variants, which is important for understanding its integer properties. Therefore, analyzing the case $o = 2$ enables us to gain the most comprehensive understanding of the possible integer interrelationships in the original equation, while identifying all potentially significant parameter combinations (see Appendix D, E).

Conclusion:

- **Fixed n in a single equation.** When analyzing a specific equation, the parameter n is unchanging. In this context, the limit $\lim_{n\to\infty} \sqrt[n]{2 \cdot n}$ is not considered, because n does not "move."
- **Behavior for different n.** If we look at a family of equations (one for each n), then analyzing the function $f(n) = \sqrt[n]{2 \cdot n}$ reveals its global properties:
  - $\max f(n) = 2$, achieved at $n = 1$ and $n = 2$.
  - As n increases (starting from 3), $f(n)$ strictly decreases and asymptotically approaches 1.

Thus, from (2.14) to (2.19), we finally arrive at

$$2^n = 2 \cdot n \tag{2.20}$$

Therefore, from (2.13) we have (2.20) and the **Statement 1** is proved.

Rewrite (2.20)

$$n = 2^{n-1} \tag{2.21}$$

This equation has only two roots in the positive integers *1* and *2*, let's prove it.

Consider the function $f(n) = n - 2^{n-1}$:

(a) For $n = 1$ we get:

$$f(1) = 1 - 2^{1-1} = 1 - 1 = 0.$$

That is, when $n = 1$, the function equals 0; hence, $n = 1$ is a root of the equation.

(b) For $n = 2$ we get:

$$f(2) = 2 - 2^{2-1} = 2 - 2 = 0.$$

Thus, when $n = 2$, the function also equals 0, and so $n = 2$ is another root.

(c) For any other integer $n > 2$, the function satisfies

$$f(n) < 0,$$

since 2^{n-1} grows exponentially while n grows linearly.

That is, for $n > 2$, the equation of roots cannot have.

Thus, it is proved that this equation (2.21) has exactly two root s in positive integers - for $n = 1$ and $n = 2$.

Therefore the equation $o = \sqrt[n]{2 \cdot n}$ (from $o^n = 2 \cdot n$) requires $\sqrt[n]{2 \cdot n}$ to be integer for integer o. By Appendix A, $\sqrt[n]{2 \cdot n}$ is irrational for $n > 2$. Thus, no integer o exists for $n > 2$, proving the impossibility of $y \in \mathbb{N}$ in $x^n + y^n = z^n$.

Conclusion: maybe the number y is likely to be a positive integer at $n = 1$ or $n = 2$. **Hence eventually comes Fermat's conclusion:** *this equation is not positive integer numbers is no longer there, if and only if* $n = 1$ *or* $n = 2$, *i.e. maybe the number* y *is likely to be a positive integer at* $n = 1$ *or* $n = 2$.

6.) Verification (without considering the total multiplier r from (2.7), which has been reduced)

(a) Consider the case $n = 1$

$$\begin{aligned}
&z = m + p \\
&x = m - p \\
&y = 2 \cdot [C_1^1 (m^1)^{1-(2 \cdot 0+1)} (p^1)^{(2 \cdot 0+1)} = 2[1 \cdot 1 \cdot p] = 2p
\end{aligned}$$

that is, for the case $n = 1$, we have a solution in natural numbers x, y, z, if m and p - are positive integer numbers and $m > p$.

(b) Consider the case $n = 2$

$$\begin{aligned}
&z = m^2 + p^2 \\
&x = m^2 - p^2 \\
&y = \sqrt{2} \cdot [C_1^2 (m^2)^{2-(2 \cdot 0+1)} (p^2)^{(2 \cdot 0+1)}]^{1/2} = \sqrt{2} \cdot [2m^2p^2]^{1/2} = 2mp
\end{aligned}$$

that is, for the case $n = 2$, we also have a solution in positive integer numbers x, y, z, if m and p are positive integer numbers and $m > p$, or according to [10], if $m > p$, and m, p are such non-integer numbers that combinations with them will give the positive integer numbers x, y, z, for example: $m = 3/\sqrt{2}$, $p = 1/\sqrt{2}$, in this case, we also get the classical Pythagorean triple:

$$\begin{aligned}
&x = m^2 - p^2 = \left(\tfrac{3}{\sqrt{2}}\right)^2 - \left(\tfrac{1}{\sqrt{2}}\right)^2 = \tfrac{9}{2} - \tfrac{1}{2} = \tfrac{8}{2} = 4 \\
&y = 2 \cdot m \cdot p = 2 \cdot \tfrac{3}{\sqrt{2}} \cdot \tfrac{1}{\sqrt{2}} = 3 \\
&z = m^2 + p^2 = \left(\tfrac{3}{\sqrt{2}}\right)^2 + \left(\tfrac{1}{\sqrt{2}}\right)^2 = \tfrac{9}{2} + \tfrac{1}{2} = 5
\end{aligned}$$

We have obtained the classical Pythagorean triple and can see that this formula works.

7). Let us now consider the more general case where r is not reduced to proceed to final formulas like (2.7). If r is a positive integer number, then all the above applies. But according to [10], r – can be a rational number. For example, let's say $r = 0.5$:

(a) Consider the case of $n = 1$, $m = 4$, and $p = 2$

$x = 0.5 \cdot (m - p) = 0.5 \cdot (4 - 2) = 1$

$z = 0.5 \cdot (m + p) = 0.5 \cdot (4 + 2) = 3$

$y = 0.5 \cdot (2 \cdot p) = 0.5 \cdot (2 \cdot 2) = 2$

We see positive integer numbers. That is,, m, p must be such that $7(a)$ is performed similarly. In this case, we can see that the formula $6(a)$ works.

(b) Consider the case *of* $n = 2$, $m = 2 \cdot \sqrt{2}$, $p = \sqrt{2}$,
$x = 0.5 \cdot (m^2 - p^2) = 0.5 \cdot \left(\left(2 \cdot \sqrt{2}\right)^2 - \left(\sqrt{2}\right)^2 \right) = 3$
$z = 0.5 \cdot (m^2 + p^2) = 0.5 \cdot \left(\left(2 \cdot \sqrt{2}\right)^2 + \left(\sqrt{2}\right)^2 \right) = 5$
$y = 0.5 \cdot (2 \cdot m \cdot p) = 0.5 \cdot \left(2 \cdot \left(2 \cdot \sqrt{2}\right) \cdot \left(\sqrt{2}\right)\right) = 4$
We see positive integer numbers. That is,, m, p must be such that 7*(b)* holds similarly. In this case, we can see that the formula 6*(b)* works.

This preserves the generality of solutions for $n = 2$, as seen in the Pythagorean triple example.

8). Therefore, the cases $n = 1$ and $n = 2$ **exhaust all possible integer solutions**, which is consistent with the theorem.
The test showed that for *n=1* or for *n=2*, in all the cases considered, we have solutions of the equation $x^n + y^n = z^n$in the positive integer numbers *x, y, z*

9). **the equation $x^n + y^n = z^n$ has roots in the positive integer numbers *x, y, z* only for $n = 1$ and for $n = 2$**

Q.E.D.

3 Remark and corollaries

REMARK. *Note that the expression (2.8) can be simplified, namely*

$$\begin{aligned} y &= \sqrt[n]{2}\left[\sum_{i=0}^{k} C_n^{(2i+1)} (m^n)^{n-(2i+1)} (p^n)^{(2i+1)}\right]^{1/n} = \\ &= \sqrt[n]{2}\left[\sum_{i=0}^{k} C_n^{(2i+1)} \frac{(m^n)^n}{(m^n)^{2i} m^n} (p^n)^{2i} p^n\right]^{1/n} = \\ &= \sqrt[n]{2} m^n \frac{1}{m} p \left[\sum_{i=0}^{k} C_n^{(2i+1)} \left(\frac{p}{m}\right)^{n2i}\right]^{1/n} = \\ &= \sqrt[n]{2} m^{n-1} p \left[\sum_{i=0}^{k} C_n^{(2i+1)} \left(\frac{p}{m}\right)^{2in}\right]^{1/n} \end{aligned} \tag{3.1}$$

with $k = (n-1)/2$ if n is odd and $k = (n-2)/2$ if n is even.
COROLLARY 1. *Consider the case $m = p$, then from the expression (3.1) it can be derived that*

$x = 0$
$z = m^n + m^n = 2m^n$
$y = \sqrt[n]{2} m^{n-1} m \left[\sum_{i=0}^{k} C_n^{(2i+1)} \left(\frac{m}{m}\right)^{2in}\right]^{1/n} = \sqrt[n]{2} m^n \left[\sum_{i=0}^{k} C_n^{(2i+1)}\right]^{1/n}$

with $k = (n-1)/2$ if n is odd and $k = (n-2)/2$ if n is even.
It is obvious that $y = z$

$$\begin{aligned} \sqrt[n]{2} m^n \left[\sum_{i=0}^{k} C_n^{(2i+1)}\right]^{1/n} &= 2m^n \\ \sqrt[n]{2} \left[\sum_{i=0}^{k} C_n^{(2i+1)}\right]^{1/n} &= 2 \end{aligned},$$

whence

$$\sum_{i=0}^{k} C_n^{(2i+1)} = 2^{n-1} \tag{3.2}$$

with $k = (n-1)/2$ if n is odd and $k = (n-2)/2$ if n is even.

COROLLARY 2. *Based on (3.2), the sum of even combinations can be calculated.*
Consider Pascal's triangle [7]:

1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1

Similarly to the above, it is concluded that

$$\sum_{j=0}^{s} C_n^{2i} = 2^{n-1} \tag{3.3}$$

with $k = (n-1)/2$ if n is odd and $k = (n-2)/2$ if n is even.

PROOF.
Expand

$$(m^n + p^n)^n + (m^n - p^n)^n =$$

into a binomial [8, 9]:

$$\begin{aligned}
&= [(m^n)^n + C_n^1 (m^n)^{n-1} p^n + C_n^2 (m^n)^{n-2} (p^n)^2 + \cdots + C_n^{n-1} m^n (p^n)^{n-1} + (p^n)^n] + \\
&+ [(m^n)^n - C_n^1 (m^n)^{n-1} p^n + C_n^2 (m^n)^{n-2} (p^n)^2 \pm \cdots \pm C_n^{n-1} m^n (p^n)^{n-1} \pm (p^n)^n] = \\
&= 2C_n^0 (m^n)^n + 2C_n^2 (m^n)^{n-2} (p^n)^2 + \ldots + 2C_n^k (m^n)^{n-k} (p^n)^k + \{2C_n^n (p^n)^n\} = \\
&= 2 \textstyle\sum_{j=0}^{s} C_n^{2j} (m^n)^{n-2j} (p^n)^{2j}
\end{aligned}$$

with $s = (n-1)/2$ if n is odd and $s = n/2$ if n is even.
If $m = p$

$$(p^n + p^n)^n + (p^n - p^n)^n = 2 \sum_{j=0}^{s} C_n^{2j} (p^n)^{n-2j} (p^n)^{2j}$$

$$(2p^n)^n = 2 \sum_{j=0}^{s} C_n^{2j} \frac{(p^n)^n}{(p^n)^{2j}} (p^n)^{2j}$$

$$2^n (p^n)^n = 2(p^n)^n \sum_{j=0}^{s} C_n^{2j}$$

$$2^{n-1} = \sum_{j=0}^{s} C_n^{2j}$$

with $s = (n-1)/2$ if n is odd and $s = n/2$ if n is even.
Corollary 2 is proved.

COROLLARY 3. *Analysing (3.2) and (3.3), it can be concluded that*

$$\sum_{i=0}^{k} C_n^{(2i+1)} = \sum_{j=0}^{s} C_n^{2j} \tag{3.4}$$

with $k = (n-1)/2$, $s = (n-1)/2$ if n is odd and $k = (n-2)/2$, $s = n/2$ if n is even.
Why such borders?

Type of number	Condition	Last constant in equation
Odd	$k = 2((n-1)/2)+1 = n$	C_n^n
	$s = 2((n-1)/2) = n-1$	C_n^{n-1}
Even	$k = 2((n-2)/2)+1 = n-1$	C_n^{n-1}
	$s = 2(n/2) = n$	C_n^n

Table 1: Boundaries of binomial coefficients

Conclusion: the sum of even coefficients is equal to the sum of odd ones and is equal to 2^{n-1}, therefore from (3.4) we have

$$\sum_{r=0}^{n} C_n^r = \sum_{i=0}^{k} C_n^{(2i+1)} + \sum_{j=0}^{s} C_n^{2j} = 2 \cdot 2^{n-1} = 2^n \tag{3.5}$$

with $k = (n-1)/2$, $s = (n-1)/2$ if n is odd and $k = (n-2)/2$, $s = n/2$ if n is even.

4 Conclusion

The “difficulties” were for Fermat the lengthiness of the run of his deductions ***put in writing***, as in the first half of the seventeenth century the mathematical notations had been way far from their present concise and diverse shape***,*** many actions had to be written down ***in words. Besides, a purely mathematical challenge was that he had to operate the then entirely new notions of binomials and logarithms***, both having just appeared for use and to be learnt “on the fly”.

As mentioned in the introduction, the mathematical methods from Pierre Fermat’s era used in this article’s proof are accessible to any first-year physics and mathematics student. This contrasts favorably with Andrew Wiles’s proof, which is quite complex for the average mathematician due to its use of advanced and intricate modern mathematical tools.

Fermat was obviously “playing” with the new notions, decomposing powers of differences into sums of powers and suddenly found out that as one confines oneself with positive integers in the power, the logarithmic equation yields immediately that $x^n + y^n = z^n$(which is a difference rewritten as a sum) is correct for whole x, y, z only and if only $n = 1$ or 2.

He (would have) had first to introduce the two new notions so as to fully explain his finding. One can imagine ***how much room it would take to put down all the deliberations*** that had led him to his discovery ***on the margins of a book solely without the proper symbolic notations that a contemporary mathematician avails***.

Why Pierre Fermat did not write down all those ideas in a dedicated document is the dedicated question of a dedicated research endeavour. It can come out that he had authored such a separate document indeed, which afterwards was somehow lost or – alternatively – has survived to this day, hidden in an archive or a library or in somebody’s unrealised custody.

The author requests the mathematical society to look critically at the deliberations set forth above and to return their assessment.

This paper has been published as a preprint on the ResearchGate platform and on the OSF platform at the following links [11, 12].

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A Appendix A. Proof of the Irrationality of $\sqrt[n]{2}$ for $n \geq 2$

A.1 Theorem: The number $\sqrt[n]{2}$ is irrational for any integer $n \geq 2$.

A.2 Proof:

We will prove by contradiction. Assume that $\sqrt[n]{2}$ is a rational number. This means it can be represented as an irreducible fraction $\frac{p}{q}$, where p and q are integers, $q \neq 0$, and p and q have no common divisors other than 1.

Thus, we have:

$$\sqrt[n]{2} = \frac{p}{q}$$

Raise both sides of the equation to the power of n:

$$(\sqrt[n]{2})^n = \left(\frac{p}{q}\right)^n$$

$$2 = \frac{p^n}{q^n}$$

Multiply both sides by q^n:

$$2q^n = p^n$$

From this equation, we see that p^n is an even number, since it is equal to $2q^n$. If p^n is even, then p must also be even (if p were odd, then p^n would also be odd).

Since p is even, we can write it as $p = 2k$, where k is some integer. Substitute this expression for p into the equation $2q^n = p^n$:

$$2q^n = (2k)^n$$

$$2q^n = 2^n k^n$$

Divide both sides of the equation by 2 (which is permissible since $n \geq 2$, and therefore 2^n is divisible by 2):

$$q^n = 2^{n-1} k^n$$

Since $n \geq 2$, then $n - 1 \geq 1$. From the last equation, we see that q^n is an even number, since it is equal to $2^{n-1}k^n$, where 2^{n-1} is an even factor. If q^n is even, then q must also be even.

Thus, we have reached the conclusion that both p and q are even numbers. This means they have a common divisor of 2. However, at the beginning of the proof, we assumed that the fraction $\frac{p}{q}$ was irreducible, meaning p and q have no common divisors other than 1.

The resulting contradiction indicates that our initial assumption about the rationality of $\sqrt[n]{2}$ is incorrect.

A.3 Conclusion:

Therefore, the number $\sqrt[n]{2}$ is irrational for any integer $n \geq 2$.

B Appendix B. Proof that the limit of the function $f(n) = \sqrt[n]{2 \cdot n}$ is 1

B.1 Problem Statement

Prove that the limit of the function $f(n) = \sqrt[n]{2 \cdot n}$ as n approaches infinity is equal to 1.

B.2 Solution

Consider the function $f(n) = \sqrt[n]{2 \cdot n}$. Our goal is to find the limit of this function as $n \to \infty$. Let's rewrite the function in the form of a power:

$$f(n) = (2 \cdot n)^{\frac{1}{n}}$$

To find the limit of this function, let's consider the natural logarithm of $f(n)$:

$$\ln(f(n)) = \ln\left((2 \cdot n)^{\frac{1}{n}}\right)$$

Using the properties of logarithms, we get:

$$\ln(f(n)) = \frac{1}{n}\ln(2 \cdot n)$$

Separate the logarithm of the product into the sum of logarithms:

$$\ln(f(n)) = \frac{\ln(2) + \ln(n)}{n}$$

Now, let's find the limit of $\ln(f(n))$ as $n \to \infty$:

$$\lim_{n\to\infty} \ln(f(n)) = \lim_{n\to\infty} \frac{\ln(2) + \ln(n)}{n}$$

This limit has the indeterminate form $\frac{\infty}{\infty}$, so we can apply L'Hôpital's Rule. Take the derivatives of the numerator and denominator with respect to n:

$$\frac{d}{dn}(\ln(2) + \ln(n)) = \frac{1}{n}$$

$$\frac{d}{dn}(n) = 1$$

Thus, the limit becomes:

$$\lim_{n\to\infty} \frac{\frac{1}{n}}{1} = \lim_{n\to\infty} \frac{1}{n} = 0$$

So, we have found that the limit of the natural logarithm of the function is 0:

$$\lim_{n\to\infty} \ln(f(n)) = 0$$

Now, to find the limit of the original function $f(n)$, we use the continuity of the exponential function:

$$\lim_{n\to\infty} f(n) = \lim_{n\to\infty} e^{\ln(f(n))} = e^{\lim_{n\to\infty} \ln(f(n))} = e^0 = 1$$

B.3 Conclusion

Therefore, we have proven that the limit of the function $f(n) = \sqrt[n]{2 \cdot n}$ as n approaches infinity is equal to 1.

C Appendix C. A short proof of monotonicity

It is well known that $\sqrt[n]{2 \cdot n} \to 1$ as $n \to \infty$ (see Appendix B). However, in this appendix we focus on showing that $(2 \cdot n)^{1/n}$ is strictly decreasing for $n \geq 3$. Below is one way to prove this, using the logarithmic derivative:

C.1 Define the function.

Let

$$f(n) \;=\; (2 \cdot n)^{\frac{1}{n}}.$$

It is convenient to work with its natural logarithm:

$$\ln\bigl(f(n)\bigr) \;=\; \ln\bigl((2 \cdot n)^{\frac{1}{n}}\bigr) \;=\; \frac{1}{n}\,\ln(2 \cdot n).$$

Denote

$$g(n) \;=\; \ln\bigl(f(n)\bigr) \;=\; \frac{\ln(2 \cdot n)}{n} \;=\; \frac{\ln(2) + \ln(n)}{n}.$$

C.2 Compute the derivative.

Treating n as a real variable $n > 0$, we have

$$g'(n) \;=\; \frac{d}{dn}\Bigl(\tfrac{\ln(2 \cdot n)}{n}\Bigr).$$

Using the quotient rule,

$$g'(n) = \frac{\frac{d}{dn}[\ln(2 \cdot n)] \cdot n \;-\; \ln(2 \cdot n) \cdot 1}{n^2} = \frac{\frac{1}{n} \cdot n \;-\; \ln(2 \cdot n)}{n^2} = \frac{1 - \ln(2 \cdot n)}{n^2}.$$

(Note that $\frac{d}{dn}[\ln(2 \cdot n)] = \frac{1}{2 \cdot n} \cdot 2 = \frac{1}{n}$.)

C.3 Sign of the derivative.

We want to see where $g'(n) < 0$:

$$g'(n) < 0 \iff 1 - \ln(2 \cdot n) < 0 \iff \ln(2 \cdot n) > 1 \iff 2 \cdot n > e \iff n > \frac{e}{2}.$$

Since $e \approx 2.718$, the inequality $n > e/2$ is certainly true for all integer $n \geq 2$, and hence strictly for $n \geq 3$. Therefore, for $n \geq 3$, $g(n)$ is strictly decreasing.

C.4 Implication for $f(n)$.

Since $f(n) = \exp(g(n))$, and $\exp(\cdot)$ is a strictly increasing function in its argument, $f(n)$ decreases whenever $g(n)$ decreases. Hence, $f(n)$ is indeed strictly decreasing for $n \geq 3$.

Thus, only for $n = 1$ and $n = 2$ does $f(n)$ attain the maximum value 2, and for all $n \geq 3$, it strictly decreases (which is consistent with the limit $\lim_{n\to\infty}(2 \cdot n)^{1/n} = 1$).

C.5 Conclusion

Hence, we have demonstrated that $f(n)$ is strictly decreasing for $n \geq 3$, either by analyzing the derivative of $\ln(f(n))$ (as shown above) or by comparing $f(n+1)$ and $f(n)$. This completes the proof of the monotonicity of $f(n)$.

D Appendix D: Analysis of Functions and Verification of Critical Points

D.1 Introduction

The goal of this analysis is to study the behavior of the function:

$$f(p, q) = q - \sqrt[p]{qp},$$

and to identify its critical points where the second derivative $f''(p)$ equals zero. This study aims to support the conclusions presented in the main article, particularly regarding the unique role of $o = 2$ in the context of Fermat's Last Theorem:

$$x^n + y^n = z^n.$$

In this article, the number o represents a key parameter associated with the symmetry of the equation. In our calculations, this corresponds to $q = o$. The uniqueness of $q = 2$ (or $o = 2$) is supported both geometrically and algebraically.

D.2 Results of the Analysis

D.2.1 Combined Graph of Functions

The following graph (Fig. 1) illustrates three functions, each with its corresponding discrete values:

- $1.5 - \sqrt[p]{1.5p}$ (blue line) and $1.5 - \sqrt[n]{1.5n}$ (blue squares),
- $2 - \sqrt[p]{2p}$ (green line) and $2 - \sqrt[n]{2 \cdot n}$ (green squares),
- $3 - \sqrt[p]{3p}$ (orange line) and $3 - \sqrt[n]{3n}$ (orange squares).

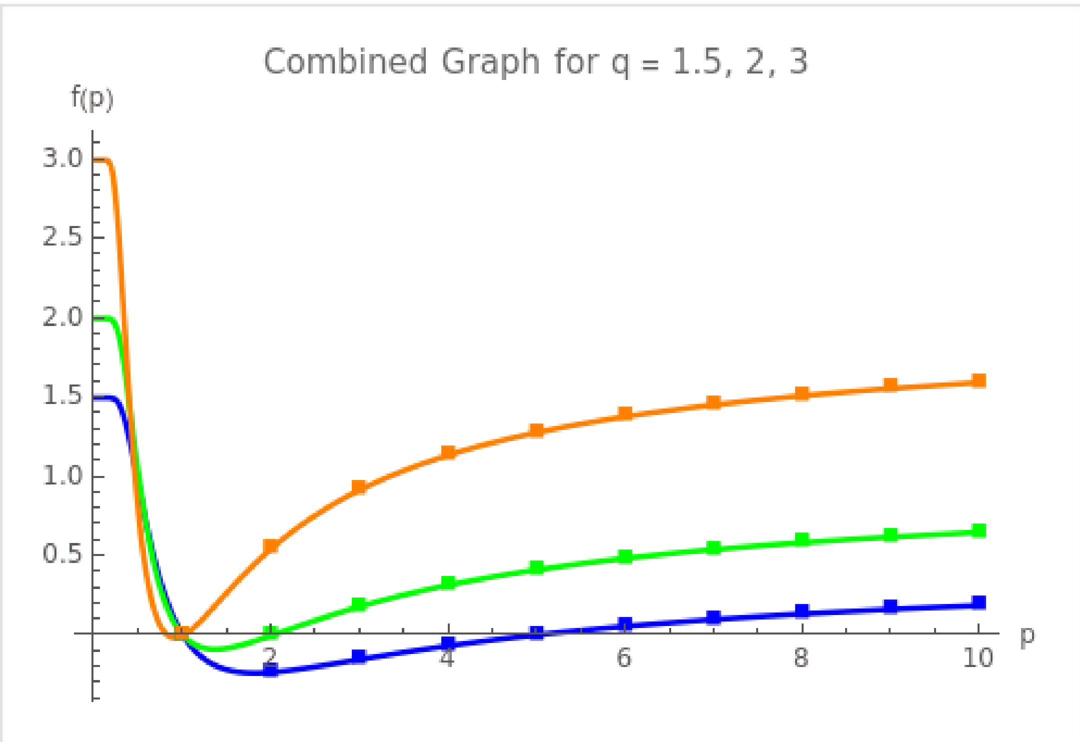


Figure 1: Combined graph of $q - \sqrt[p]{qp}$ for $q = 1.5, 2, 3$. Lines represent continuous p, and squares represent discrete n.

D.2.2 Surface Plot of $f(p, q)$

The three-dimensional surface plot below (Fig. 2) illustrates the behavior of $f(p, q) = q - \sqrt[p]{qp}$ for continuous values of $p > 0$ and $q > 0$. The perspective highlights the change in curvature as both parameters vary.

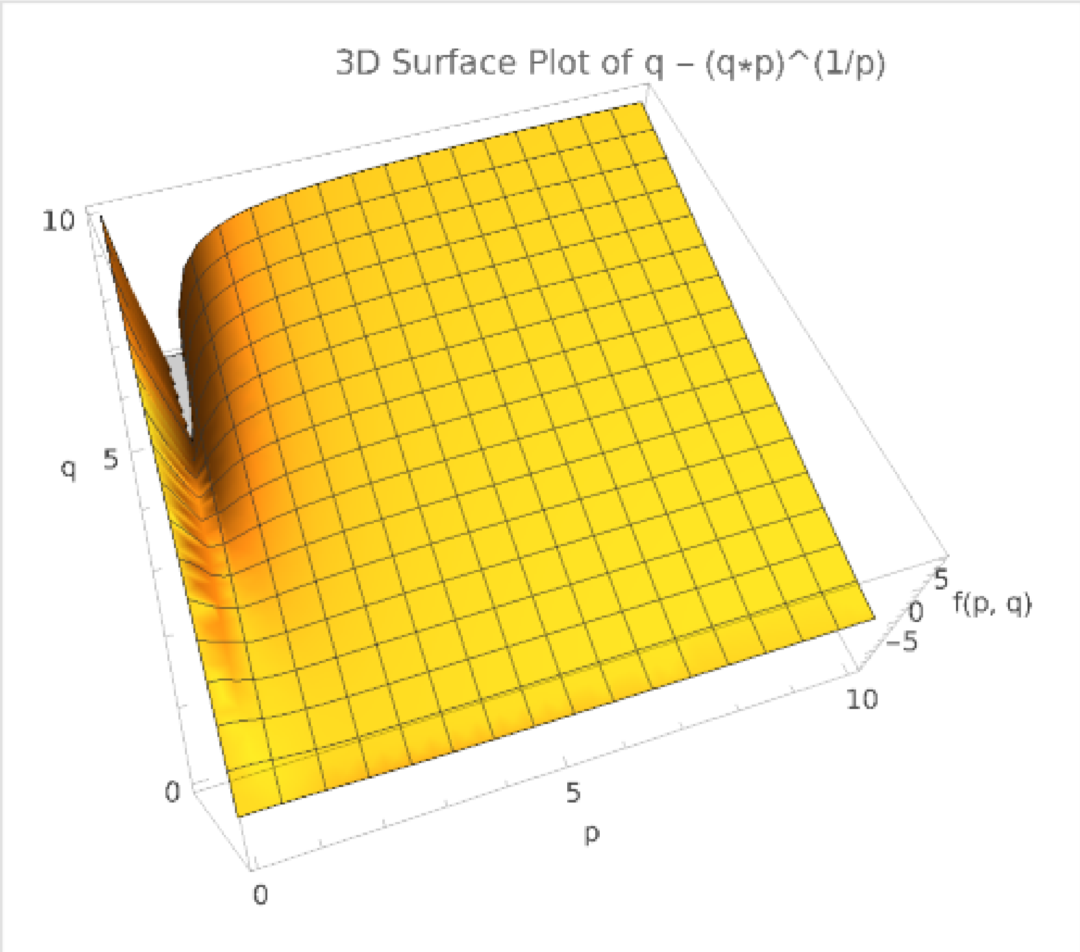


Figure 2: 3D surface plot of $f(p, q) = q - \sqrt[p]{qp}$ for $p > 0$ and $q > 0$.

D.3 Analysis and Conclusions

- The critical point at $p = 2$ exists uniquely for $q = 2$, where the second derivative $f''(p)$ equals zero. This highlights the special role of $q = 2$ (or $o = 2$).
- For other values of q, critical points exist, but they occur at values $p \neq 2$, showing that $q = 2$ is unique in its symmetry and simplicity.

These results confirm the dual nature of this article solution: both **geometric** and **algebraic**. The parameter $o = 2$ serves as a unifying concept in the analysis of Fermat's Last Theorem (See Appendix E for more details).

E Appendix E: Geometric Verification of Fermat's Last Theorem through the Analysis of the Function $f(p, q)$

In this appendix, we investigate the connection between the function

$$f(p, q) = q - \sqrt[p]{qp}$$

and Fermat's equation

$$x^n + y^n = z^n.$$

It is shown that for $n = 2$ there exists an inflection point of the second derivative, which corresponds to the existence of Pythagorean triples. For $n > 2$, this inflection point shifts toward lower values of p, indicating a change in the mathematical properties of the equation and the impossibility of integer solutions.

E.1 Analysis of the Second Derivative

The function

$$f(p, q) = q - \sqrt[p]{qp}$$

allows us to study the behavior of inflection points, which are determined by the condition

$$f_p''(p, q) = \frac{\partial^2}{\partial p^2}\left(q - \sqrt[p]{qp}\right) = 0.$$

Inflection points are important because they reveal specific mathematical patterns in the equation.

E.2 Additional Analysis for the Variable q

Investigations were also conducted on the variable q. At the point $q = 2$ (with $p = 2$), both the first and second partial derivatives with respect to q are equal to 0.5. This result indicates a fundamental property of Fermat's equation at that point. However, in our case the variable q is fixed at the value 2, as determined by the binomial computations, although the very fact of isolating the value of q is interesting in itself.

E.3 Numerical Results

Below are the numerical values of the second derivative $f_p''(p, q)$ for various values of p and q:

p	$q = 1.5$	$q = 2.0$	$q = 2.5$	$q = 3.0$	$q = 4.0$
1	2.75	0.77	2.90	0.96	1.10
2	0.17	0.00	-0.11	-0.34	-0.55
3	-0.0056	-0.08	-0.099	-0.21	-0.31
4	-0.018	-0.11	-0.056	-0.12	-0.17
5	-0.014	-0.10	-0.033	-0.067	-0.093

Table 2: Numerical values of the second derivative $f_p''(p, q)$ for various p and q.

E.4 Conclusions from the Data

- For $p = 2,\ q = 2$ we have $f''_p(p,q) = 0$, which corresponds to the existence of Pythagorean triples.
- For $q > 2$, the inflection point shifts toward lower values of p, indicating a change in the properties of the equation and the impossibility of integer solutions.
- Figure 3 illustrates the surface of $f''_p(p,q)$ with the marked inflection points.

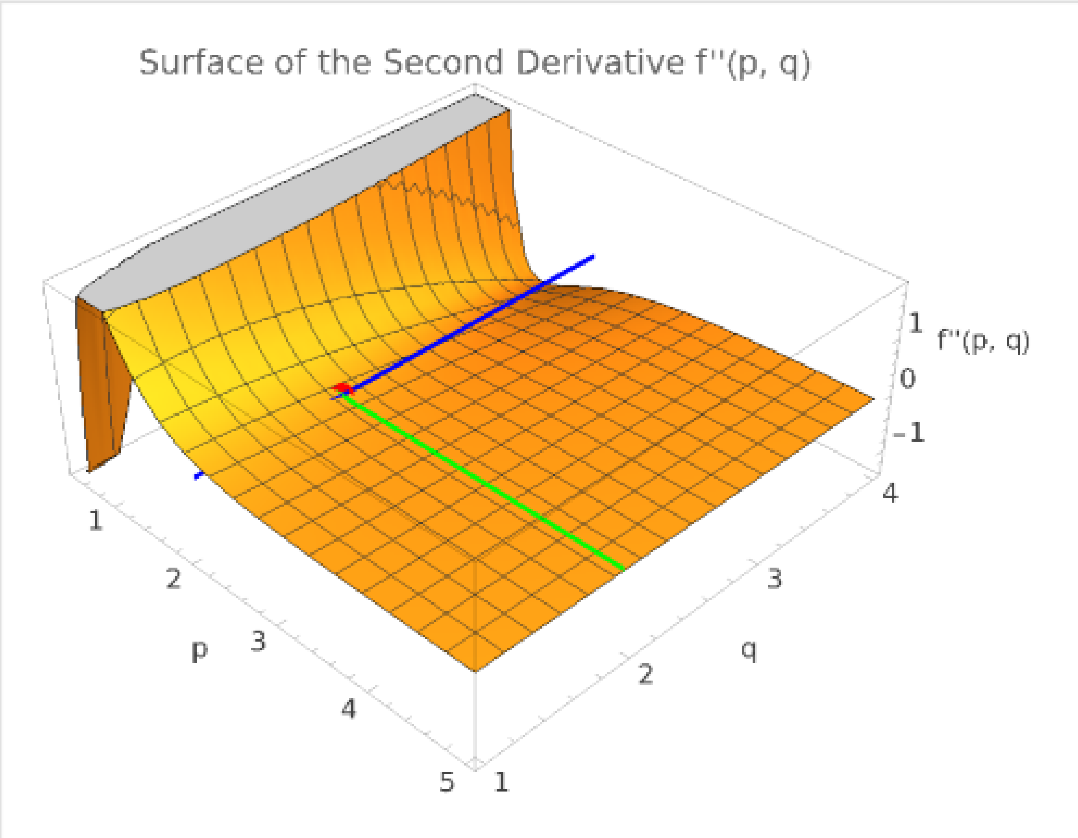


Figure 3: Surface of the second derivative $f''_p(p,q)$, demonstrating the shift of the inflection point.

E.5 Verification of Fermat's Last Theorem

E.5.1 Relation to the Inflection Points

From the data, it follows that:

1. If the inflection point occurs at $p = n$, then integer solutions are possible.
2. If the inflection point shifts to lower values of p, then integer solutions are impossible.

Thus, **only for $n = 2$ are integer solutions possible**, while for $n > 2$ the properties of the equation change, excluding such solutions.

E.5.2 Final Conclusion

Based on our calculations:

- **For $q = 2$, the inflection point corresponds to $p = 2$** $\Rightarrow$ Fermat's equation has integer solutions.

- **For $q > 2$, the inflection point shifts to lower values of p** $\Rightarrow$ the mathematical properties of the equation change, and integer solutions become impossible.
- This confirms that the Fermat equation

$$x^n + y^n = z^n, \quad n > 2$$

has no solutions in the integers.

Thus, the function $f(p, q)$ clearly demonstrates that when $n > 2$ the inflection point shifts and the properties of Fermat's equation change, confirming the impossibility of integer solutions.

F Appendix F: Axiomatic Formulation of the Ansatz and Derivation of FLT

F.1 Introduction

In this appendix, the Dedenko ansatz is presented in the form of an independent axiom. This emphasizes that it is not a consequence of standard arithmetic but is introduced as an additional postulate. In such an axiomatic style, it can be rigorously shown that upon accepting the ansatz, Fermat's Last Theorem (FLT) immediately follows, whereas upon its rejection, the proof does not work. Thus, the formalization separates the hypothesis from the conclusion derived from it.

Axiom F.1 (Dedenko's Ansatz). *For any $n, x, y, z \in \mathbb{N}$, the following holds:*

$$n > 2 \ \wedge \ x^n + y^n = z^n \quad \Longrightarrow \quad \exists o \in \mathbb{N}, \ o > 1 \ \wedge \ o^n = 2 \cdot n.$$

Theorem 2 (On the solutions of the equation $o^n = 2 \cdot n$). *If $o > 1$ and $o^n = 2 \cdot n$, then*

$$(o, n) \in \{(2, 1), (2, 2)\}.$$

Proof F.1. Let $o = 2$. Then $2^n = 2 \cdot n$. This equation has solutions only for $n = 1$ and $n = 2$, since for $n \geq 3$, the inequality $2^n > 2 \cdot n$ holds.

Let $o \geq 3$. Then $o^n \geq 3^n > 2 \cdot n$ for any $n \geq 1$, which is impossible if $o^n = 2 \cdot n$.

Therefore, there are no other solutions. □

Theorem 3 (Fermat's Last Theorem under the Ansatz).

$$\forall n > 2, \ \forall x, y, z \in \mathbb{N}, \quad x^n + y^n \neq z^n.$$

Proof F.2. Assume, contrary to the statement, that there exist $n > 2$ and $x, y, z \in \mathbb{N}$ such that $x^n + y^n = z^n$. According to the ansatz, there exists an $o > 1$ such that $o^n = 2 \cdot n$. But by the preceding theorem, this is only possible for $n = 1$ or $n = 2$. This is a contradiction, since $n > 2$. □

F.2 Clarification on Non-integer Values of o

A frequent objection is that in equation $o^n = 2 \cdot n$, the parameter o might take non-integer values. Several clarifications are in order:

- **Restriction to integers.** The reconstruction is carried out entirely within the framework of natural numbers, consistent with Fermat's original problem. Formal verification in Coq confirms that only the integer pairs $(o, n) = (2, 1)$ and $(2, 2)$ satisfy the equation.
- **Heuristic observation.** Considering $f(n) = (2 \cdot n)^{1/n}$, one sees that $o = 2$ is the only value producing more than one integer solution in n (namely $n = 1, 2$). This makes $o = 2$ a center of integer stability.
- **Ansatz as a razor.** While a single solution with non-integer o cannot be excluded by pure algebra, the ansatz deliberately postulates that only the structurally perfect integer case $o = 2$ is relevant. All other cases are cut off by this postulate.
- **Analytic uniqueness.** As shown in Appendices D and E, the case $o = 2, n = 2$ corresponds to an inflection point of the associated function (the second derivative vanishes). This analytical property reinforces the special role of $o = 2$: at $n = 2$ integer solutions exist, but for $n > 2$ the system passes beyond this critical point and integer solutions disappear.

Thus, the ansatz is not a denial of possible non-integer o, but a structural principle: if integer solutions (x, y, z) exist, they must correspond to the unique integer-centered case $o = 2$, which in turn leads to contradiction for $n > 2$.

F.3 Conclusion

The axiomatic presentation emphasizes:

- the ansatz is an independent postulate, not a provable fact;
- its acceptance immediately implies FLT for all $n > 2$;
- in terms of formal logic, this reasoning has the structure

$$\text{Arithmetic} + \text{Ansatz} \implies \text{FLT}.$$

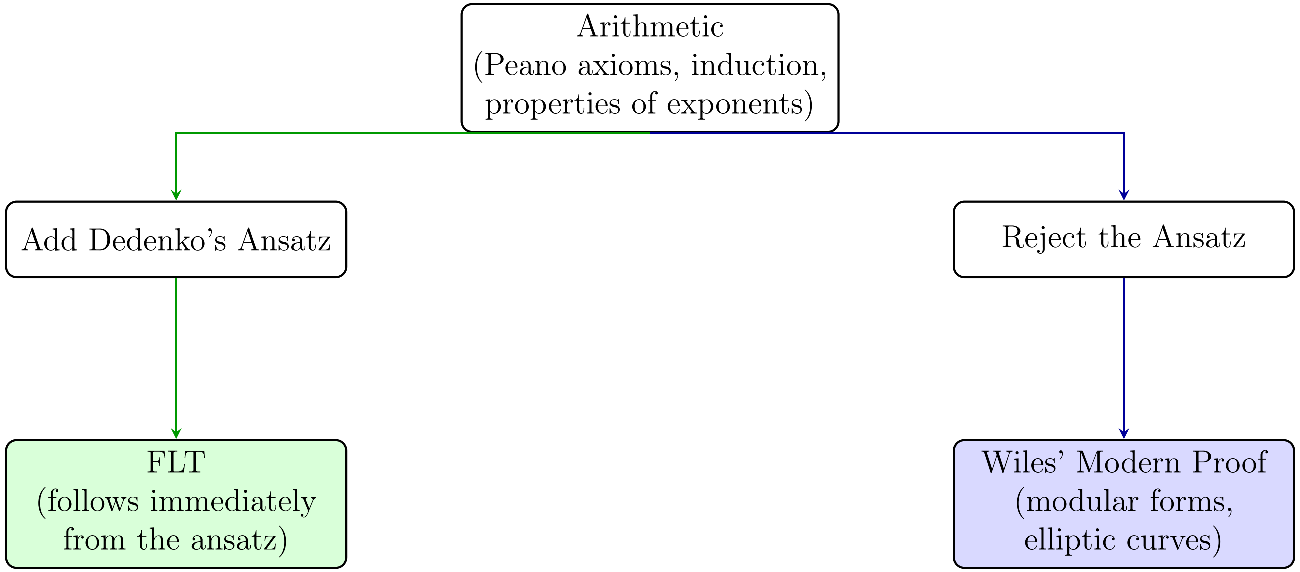
F.4 Methodological Analogy

The situation with the Dedenko ansatz is reminiscent of the history of Euclid's fifth postulate. For centuries, attempts were made to derive the parallel postulate from the other axioms of geometry. However, it turned out to be an independent assumption: one can accept it—and get Euclidean geometry, or replace it with an alternative—and get Lobachevskian or Riemannian geometry.

Similarly, the Dedenko ansatz is not derivable from standard arithmetic. If it is accepted, Fermat's Last Theorem becomes an immediate consequence. If it is rejected, what remains is Wiles' modern proof using the theory of elliptic curves and modular forms.

Remark 2 (On non-integer values of o)**.** *It is important to emphasize that the exclusion of non-integer values of o does not follow from algebraic proof but is part of the ansatz itself. The ansatz acts as a structural postulate: if integer solutions of Fermat's equation were to exist, they would necessarily correspond to the integer-centered case $o = 2$. By this assumption, fractional or irrational o are not considered relevant, and the reasoning proceeds entirely within the integer domain.*

F.5 Reasoning Diagram



Legend:
→ Green branch: the "fast track" if the ansatz is accepted (immediately implies FLT).
→ Blue branch: the historical path—without the ansatz, the proof is only possible via Wiles' methods (1995).