



# Analytical Geometry and Linear Algebra II, Lab 1

The geometry of linear equation

Gaussian elimination

Matrix notation and matrix multiplication

# Task 1

*Warm up*



Rewrite the following system in the matrix form:

$$\begin{cases} 3x + 4y - 2z = 1 \\ 3y - 2z + x = -2 \\ 5x - 7z - 2y = 3 \end{cases}$$

## Task 2



Describe geometrically (line, plane, or whole space) all linear combinations of:

1.  $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$  and  $\begin{bmatrix} 3 \\ 6 \\ 9 \end{bmatrix}$

2.  $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$  and  $\begin{bmatrix} 0 \\ 2 \\ 3 \end{bmatrix}$

3.  $\begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}$ ,  $\begin{bmatrix} 0 \\ 2 \\ 2 \end{bmatrix}$  and  $\begin{bmatrix} 2 \\ 2 \\ 3 \end{bmatrix}$

## Task 2



Describe geometrically (line, plane, or whole space) all linear combinations of:

1.  $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$  and  $\begin{bmatrix} 3 \\ 6 \\ 9 \end{bmatrix}$  *Ans: line*

2.  $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$  and  $\begin{bmatrix} 0 \\ 2 \\ 3 \end{bmatrix}$

3.  $\begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}$ ,  $\begin{bmatrix} 0 \\ 2 \\ 2 \end{bmatrix}$  and  $\begin{bmatrix} 2 \\ 2 \\ 3 \end{bmatrix}$

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3.  $\begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}$ ,  $\begin{bmatrix} 0 \\ 2 \\ 2 \end{bmatrix}$  and  $\begin{bmatrix} 2 \\ 2 \\ 3 \end{bmatrix}$  Ans: whole space

## Task 3



Draw  $v = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$  and  $w = \begin{bmatrix} -2 \\ 2 \end{bmatrix}$  and  $v + w, v - w$  in a single  $xy$  plane

## Task 4



Explain, why the system is singular?

$$\begin{cases} u + v + w = 2 \\ u + 2v + 3w = 1 \\ v + 2w = 0 \end{cases}$$

What value should replace that last zero on the right side, to allow the equations to have solutions, and what is one of the solutions?



# What does it mean, singular?

*Lec 1, page 11*



**Example 2.** Singular (incurable)

$$\begin{cases} u + v + w = \\ 2u + 2v + 5w = \\ 4u + 4v + 8w = \end{cases} \Rightarrow \begin{cases} u + v + w = \\ 3w = \\ 4w = \end{cases}$$

There is no exchange of equations that can avoid zero in the second pivot position. The equations themselves may be solvable or unsolvable. If the last two equations are  $3w=6$  and  $4w=7$ , there is no solution. If those two equations happen to be consistent as in  $3w=6$  and  $4w=8$  then this singular case has an infinity of solutions. We know that  $w=2$ , but the first equation cannot decide both  $u$  and  $v$ .

# What does it mean, pivot point?



## Definition

**The pivot or pivot element** is the element of a matrix, or an array, which is selected first on a particular step by an algorithm (e.g. Gaussian elimination, simplex algorithm, etc.), to do certain calculations.

## Task 4



Explain, why the system is singular?

$$\begin{cases} u + v + w = 2 \\ u + 2v + 3w = 1 \\ v + 2w = 0 \end{cases}$$

What value should replace that last zero on the right side, to allow the equations to have solutions, and what is one of the solutions?

## Task 5



1. Choose a coefficient  $b$  that makes this system singular.
2. Then choose a right-hand side  $g$  that makes it solvable.
3. Find two solutions in that singular case.

$$\begin{cases} 2x + by = 16 \\ 4x + 8y = g \end{cases}$$



## Task 6

Give 3x3 examples (not just the zero matrix):

1. a diagonal matrix:  $a_{ij} = 0$ , if  $i \neq j$ ;
2. a symmetric matrix:  $a_{ij} = a_{ji}$  for all  $i$  and  $j$ ;
3. an upper triangular matrix:  $a_{ij} = 0$ , if  $i > j$ ;
4. a skew-symmetric matrix:  $a_{ij} = -a_{ji}$  for all  $i$  and  $j$ .

## Task 7



Obtain a Reduced Row Echelon Form (rref) of

$$\begin{cases} 2u + 3v + 0w = 0 \\ 4u + 5v + w = 3 \\ 2u - 1v - 3w = 5 \end{cases}$$



# The difference between REF and RREF

## Gaussian elimination vs Gauss-Jordan elimination

**Gaussian elimination** refers to the process until it has reached its upper triangular.

**Gauss-Jordan elimination** is the algorithm for converting a matrix into RREF.

The **Row Echelon Form** is *not unique*

$$\begin{bmatrix} 1 & 3 & -1 \\ 0 & 1 & 7 \end{bmatrix} \xrightarrow{\text{add row 2 to row 1}} \begin{bmatrix} 1 & 4 & 6 \\ 0 & 1 & 7 \end{bmatrix}.$$

Every matrix has a *unique* **Reduced Row Echelon Form**

$$\begin{bmatrix} 1 & 3 & -1 \\ 0 & 1 & 7 \end{bmatrix} \xrightarrow{\text{subtract } 3 \times (\text{row 2}) \text{ from row 1}} \begin{bmatrix} 1 & 0 & -22 \\ 0 & 1 & 7 \end{bmatrix}.$$

## Task 7



Obtain a Reduced Row Echelon Form (rref) of

$$\begin{cases} 2u + 3v + 0w = 0 \\ 4u + 5v + w = 3 \\ 2u - 1v - 3w = 5 \end{cases}$$

$$\text{Ans: } \left[ \begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 1 \end{array} \right]$$



## Reference material



- Lectures 1 – 3

# Deserve "A" grade!

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🏢 Room 105 (Underground robotics lab)