Analytical Geometry and Linear Algebra II, Lab 2

Gaussian elimination recap

A=LU, A=LDV, PA=LU factorization

Innopolis University

Gaussian elimination

Algorithm

$$\begin{bmatrix} 2 & 1 & 1 \\ 4 & -6 & 0 \\ -2 & 7 & 2 \end{bmatrix}$$

Gaussian elimination

Algorithm

$$\begin{bmatrix} 2 & 1 & 1 \\ 4 & -6 & 0 \\ -2 & 7 & 2 \end{bmatrix} \to \begin{bmatrix} 2 & 1 & 1 \\ 0 & -8 & -2 \\ 0 & 0 & 1 \end{bmatrix}$$

A=LU factorization

Example of A=LU

$$\underbrace{\begin{bmatrix} 2 & 1 & 1 \\ 4 & -6 & 0 \\ -2 & 7 & 2 \end{bmatrix}}_{A}$$

A=LU factorization

Example of A=LU

$$\underbrace{\begin{bmatrix} 2 & 1 & 1 \\ 4 & -6 & 0 \\ -2 & 7 & 2 \end{bmatrix}}_{A} \rightarrow \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & -1 & 1 \end{bmatrix}}_{L} \underbrace{\begin{bmatrix} 2 & 1 & 1 \\ 0 & -8 & -2 \\ 0 & 0 & 1 \end{bmatrix}}_{U}$$

A=LU factorization

Example of A=LDV

$$\underbrace{\begin{bmatrix} 2 & 1 & 1 \\ 4 & -6 & 0 \\ -2 & 7 & 2 \end{bmatrix}}_{A} \rightarrow \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & -1 & 1 \end{bmatrix}}_{L} \underbrace{\begin{bmatrix} 2 & 0 & 0 \\ 0 & -8 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{D} \underbrace{\begin{bmatrix} 1 & 1/2 & 1/2 \\ 0 & 1 & 1/4 \\ 0 & 0 & 1 \end{bmatrix}}_{V \text{ or } U}$$

A=LU factorization

How to convert LU to LDV

Split U into $\begin{bmatrix} d_1 & & & \\ & d_2 & & \\ & & \ddots & \\ & & & d_n \end{bmatrix} \begin{bmatrix} 1 & u_{12}/d_1 & u_{13}/d_1 & \cdot \\ & 1 & u_{23}/d_2 & \cdot \\ & & \ddots & \vdots \\ & & & 1 \end{bmatrix}$

A=LU factorization

Formal Idea

$$A = LU = \begin{pmatrix} \text{lower triangular } L \\ \text{1's on the diagonal} \end{pmatrix} \begin{pmatrix} \text{upper triangluar } U \\ \text{pivots on the diagonal} \end{pmatrix}$$

Requirements: No row exchanges as Gaussian elimination reduces square A to U

$$A = LDU = \begin{pmatrix} \text{lower triangular } L \\ \text{1's on the diagonal} \end{pmatrix} \begin{pmatrix} \text{pivot matrix} \\ D \text{ is diagonal} \end{pmatrix} \begin{pmatrix} \text{upper triangluar } U \\ \text{1's on the diagonal} \end{pmatrix}$$

Requirements: No row exchanges. The pivots in D are divided out to leave 1's on the diagonal of U. If A is symmetric, then U is L^T and $A = LDL^T$.

A=LU factorization

Application: Theory

$$Ax = b \longrightarrow$$

A=LU factorization

Application: Theory

$$Ax = b \to \underbrace{LU}_{A} x = b$$

A=LU factorization

Application: Theory

$$Ax = b \rightarrow \underbrace{LU}_{A} x = b \rightarrow L \underbrace{c}_{Ux} = b$$

A=LU factorization

Application: Theory

$$Ax = b \to \underbrace{LU}_{A} x = b \to L \underbrace{c}_{Ux} = b \to Ux = c \to \text{Ans: } x$$

There are two steps:

1. **Factor** (from A find its factors L and U)
2. **Solve** (from L and U and b find the solution x)

A=LU factorization

Application: Case Study

Description

Try to solve such systems using:

- Gauss-Jordan elimination
- LU factorization

Compare time consumption

$$AX = B, \text{ where } A = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 3 & 3 \\ -3 & 10 & 2 \end{bmatrix}, B = \begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix}$$

$$B_{j+1} = B_j + X_j, \text{ for } j = 1, 2, \text{ where } B_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

A=LU factorization

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$$A = \overbrace{\underbrace{\begin{bmatrix} \frac{-1}{3} & \frac{16}{29} & 1 \\ \frac{-2}{3} & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}}_{L} \underbrace{\begin{bmatrix} -3 & 10 & 2 \\ 0 & \frac{29}{3} & \frac{13}{3} \\ 0 & 0 & \frac{-21}{2} \end{bmatrix}}_{U}}^{Matlab} = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -3 & -16 & 1 \end{bmatrix}}_{L} \underbrace{\begin{bmatrix} 1 & 2 & 1 \\ 0 & -1 & 1 \\ 0 & 0 & 21 \end{bmatrix}}_{U}, B_2 = \begin{bmatrix} \frac{5}{7} \\ \frac{3}{7} \\ \frac{-4}{7} \end{bmatrix}, B_3 = \begin{bmatrix} \frac{73}{49} \\ \frac{109}{147} \\ \frac{-185}{147} \end{bmatrix}$$

Task 0

1. Solve the system
2. Make $A = LU$ factorization of the matrix
3. Solve the system, using LU.

$$\begin{cases} 2x_1 + 3x_2 - x_3 = 9 \\ x_1 - 2x_2 + x_3 = 3 \\ x_1 + 2x_3 = 2 \end{cases}$$

Task 0

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$$\begin{cases} 2x_1 + 3x_2 - x_3 = 9 \\ x_1 - 2x_2 + x_3 = 3 \\ x_1 + 2x_3 = 2 \end{cases}$$

$$\text{Ans: } A = \begin{bmatrix} 2 & 3 & -1 \\ 1 & -2 & 1 \\ 1 & 0 & 2 \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{2} & 1 & 0 \\ \frac{1}{2} & \frac{3}{7} & 1 \end{bmatrix}}_{L} \underbrace{\begin{bmatrix} 2 & 3 & -1 \\ 0 & \frac{-7}{2} & \frac{3}{2} \\ 0 & 0 & \frac{13}{7} \end{bmatrix}}_{U}; \ X = \begin{cases} x_1 = 4 \\ x_2 = 0 \\ x_3 = -1 \end{cases}$$

Task 1

Which number c leads to zero in the second pivot position? A row exchange is needed and $A = LU$ will not be possible. Which c produces zero in the third pivot position? Then a row exchange can't help and elimination fails:

$$A = \begin{bmatrix} 1 & c & 0 \\ 2 & 4 & 1 \\ 3 & 5 & 1 \end{bmatrix}.$$

Task 1

Which number c leads to zero in the second pivot position? A row exchange is needed and $A = LU$ will not be possible. Which c produces zero in the third pivot position? Then a row exchange can't help and elimination fails:

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Answer

$c = 2$ leads to zero in the second pivot position: exchange rows and not singular.
$c = 1$ leads to zero in the third pivot position. In this case the matrix is *singular*.

Task 2

This nonsymmetric matrix will have the same L as in Problem **13**:

Find L and U for

$$A = \begin{bmatrix} a & r & r & r \\ a & b & s & s \\ a & b & c & t \\ a & b & c & d \end{bmatrix}.$$

Find the four conditions on a, b, c, d, r, s, t to get $A = LU$ with four pivots.

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Answer

$$\begin{bmatrix} a & r & r & r \\ a & b & s & s \\ a & b & c & t \\ a & b & c & d \end{bmatrix} = \begin{bmatrix} 1 & & & \\ 1 & 1 & & \\ 1 & 1 & 1 & \\ 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} a & r & r & r \\ & b-r & s-r & s-r \\ & & c-s & t-s \\ & & & d-t \end{bmatrix}. \text{ Need } \begin{matrix} a \neq 0 \\ b \neq r \\ c \neq s \\ d \neq t \end{matrix}$$

Task 3

If A and B have nonzeros in the positions marked by x, which zeros (marked by 0) *stay zero* in their factors L and U?

$$A = \begin{bmatrix} x & x & x & x \\ x & x & x & 0 \\ 0 & x & x & x \\ 0 & 0 & x & x \end{bmatrix} \qquad B = \begin{bmatrix} x & x & x & 0 \\ x & x & 0 & x \\ x & 0 & x & x \\ 0 & x & x & x \end{bmatrix}.$$

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$$A = \begin{bmatrix} x & x & x & x \\ x & x & x & 0 \\ 0 & x & x & x \\ 0 & 0 & x & x \end{bmatrix} \qquad B = \begin{bmatrix} x & x & x & 0 \\ x & x & 0 & x \\ x & 0 & x & x \\ 0 & x & x & x \end{bmatrix}.$$

Answer

For the first matrix A, L keeps the 3 lower zeros at the start of rows. But U may not have the upper zero where $A_{24} = 0$. For the second matrix B, L keeps the bottom left zero at the start of row 4. U keeps the upper right zero at the start of column 4. One zero in A and two zeros in B are filled in.

Reference material

- Lecture 4 MIT course
- *"Linear Algebra and Applications", pdf pages 46–86*
- *"Introduction to Linear Algebra", pdf pages 42–134*

  2.2 – The Idea of Elimination, 2.6 – Elimination = Factorization: $A = LU$
- This lab video, 2022 year

Preparation material for the next class

1. Lecture 6, Gilbert Strang

   Goal is to understand the basics of spaces and how Null Space appeared.

2. Khan Academy: Null space

   It contains a good case study how to calculate Null space.

3. Matrix Algebra for Engineers: Null Space

   Another nice example how to find $N(A)$.

4. *"Linear Algebra and Applications", pdf pages 96–106*

   What does partial and full solutions means

5. The Big Picture of Linear Algebra

   Extra for now If you want to get the global view of four subspaces

Deserve “A” grade!

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Room 105 (Underground robotics lab)