Long solutions

Solution Problem Set 5 – ACSE-2 – 2020

(1)

(a) $\mathrm{tr}(\underline{\underline{S}}) = -1+7+6 = 12$

(b) $\underline{\underline{S}} : \underline{\underline{S}} = S_{ij} S_{ij} = (-1)^2 + 0^2 + 5^2 + 3^2 + 7^2 + 4^2 + 9^2 + 8^2 + 6^2$

$= 1 + 25 + 9 + 49 + 16 + 81 + 64 + 36 = 281$

(c) $\underline{\underline{S}} : \underline{\underline{S}}^T = S_{ij} S_{ji} = \begin{bmatrix} -1 & 0 & 5 \\ 3 & 7 & 4 \\ 9 & 8 & 6 \end{bmatrix} : \begin{bmatrix} -1 & 3 & 9 \\ 0 & 7 & 8 \\ 5 & 4 & 6 \end{bmatrix}$

$= -1^2 + 0 \cdot 3 + 5 \cdot 9 + 3 \cdot 0 + 7 \cdot 7 + 8 \cdot 4$

$+ 9 \cdot 5 + 4 \cdot 8 + 6 \cdot 6 =$

$1 + 49 + 36 + 2 \cdot (0 + 45 + 32) = 240$

(d) $\vec{a} \cdot \underline{\underline{S}} = \begin{pmatrix} -2 - 3 + 36 \\ 0 - 7 + 32 \\ 10 - 4 + 24 \end{pmatrix} = \begin{pmatrix} 31 \\ 25 \\ 30 \end{pmatrix}$

$a_i S_{ij}$

(e) $\underline{\underline{S}} \cdot \vec{a}$

$S_{ij} a_j = \begin{pmatrix} -2 + 0 + 20 \\ 6 - 7 + 16 \\ 18 - 8 + 24 \end{pmatrix} = \begin{pmatrix} 18 \\ 15 \\ 34 \end{pmatrix}$

$\Big\} \; \vec{a} \cdot \underline{\underline{S}} \neq \underline{\underline{S}} \cdot \vec{a}$

(f) $\underline{\underline{S}}^T \cdot \vec{a} = \begin{bmatrix} -1 & 3 & 9 \\ 0 & 7 & 8 \\ 5 & 4 & 6 \end{bmatrix} \begin{pmatrix} 2 \\ -1 \\ 4 \end{pmatrix} = \begin{pmatrix} -2 - 3 + 36 \\ 0 - 7 + 32 \\ 10 - 4 + 24 \end{pmatrix} = \vec{a} \cdot \underline{\underline{S}}$

(g) $\vec{a}\vec{b} = \begin{bmatrix} 2 \cdot 8 & 2 \cdot 5 & 2 \cdot -7 \\ -1 \cdot 8 & -1 \cdot 5 & -1 \cdot -7 \\ 4 \cdot 8 & 4 \cdot 5 & 4 \cdot -7 \end{bmatrix} = \begin{bmatrix} 16 & 10 & -14 \\ -8 & -5 & 7 \\ 32 & 20 & -28 \end{bmatrix}$

$a_i b_j$

(h) $\vec{b}\vec{a} = \begin{bmatrix} 8 \cdot 2 & 8 \cdot -1 & 8 \cdot 4 \\ 5 \cdot 2 & 5 \cdot -1 & 5 \cdot 4 \\ -7 \cdot 2 & -7 \cdot -1 & -7 \cdot 4 \end{bmatrix} = \begin{bmatrix} 16 & -8 & 32 \\ 10 & -5 & 20 \\ -14 & 7 & -28 \end{bmatrix} = (\vec{a}\vec{b})^T$

(2)

Orthonormal base transformation

$\hat{e}_1, \hat{e}_2, \hat{e}_3$ to $\hat{e}'_1 = \frac{1}{3}(2\hat{e}_1 + 2\hat{e}_2 + \hat{e}_3)$

$\hat{e}'_2 = \frac{1}{\sqrt{2}}(\hat{e}_1 - \hat{e}_2)$

$\hat{e}'_3 = \frac{1}{3\sqrt{2}}(\hat{e}_1 + \hat{e}_2 - 4\hat{e}_3)$

$$\hat{e}'_1 \cdot \hat{e}'_2 = \frac{1}{3} \cdot \frac{1}{\sqrt{2}} \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix} \cdot (1, -1, 0) = \frac{1}{3\sqrt{2}} (2 - 2 + 0) = 0$$

$$\hat{e}'_1 \times \hat{e}'_2 = \frac{1}{3\sqrt{2}} \begin{vmatrix} \hat{e}_1 & \hat{e}_2 & \hat{e}_3 \\ 2 & 2 & 1 \\ 1 & -1 & 0 \end{vmatrix} = \frac{1}{3\sqrt{2}} \left[\hat{e}_1 (0+1) + \hat{e}_2 (1-0) + \hat{e}_3 (-2-2) \right] = \frac{1}{3\sqrt{2}} (\hat{e}_1 + \hat{e}_2 - 4\hat{e}_3)$$

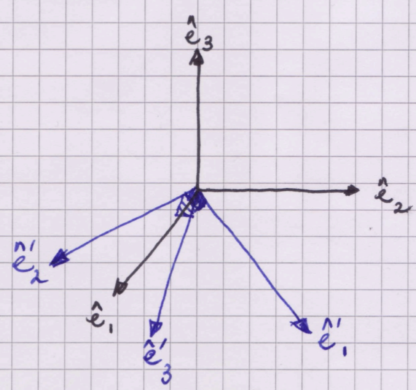
direction cosines:

$l_{11} = \hat{e}'_1 \cdot \hat{e}_1 = \frac{2}{3}$, $l_{12} = \hat{e}'_1 \cdot \hat{e}_2 = \frac{2}{3}$, $l_{13} = \hat{e}'_1 \cdot \hat{e}_3 = \frac{1}{3}$

$l_{21} = \hat{e}'_2 \cdot \hat{e}_1 = \frac{1}{\sqrt{2}}$, $l_{22} = \hat{e}'_2 \cdot \hat{e}_2 = -\frac{1}{\sqrt{2}}$, $l_{23} = \hat{e}'_2 \cdot \hat{e}_3 = 0$

$l_{31} = \hat{e}'_3 \cdot \hat{e}_1 = \frac{1}{3\sqrt{2}}$, $l_{32} = \hat{e}'_3 \cdot \hat{e}_2 = \frac{1}{3\sqrt{2}}$, $l_{33} = \hat{e}'_3 \cdot \hat{e}_3 = \frac{-4}{3\sqrt{2}}$

transformation matrix $L = \frac{1}{3\sqrt{2}} \begin{bmatrix} 2\sqrt{2} & 2\sqrt{2} & \sqrt{2} \\ 3 & -3 & 0 \\ 1 & 1 & -4 \end{bmatrix}$



Indeed $\hat{e}'_1, \hat{e}'_2, \hat{e}'_3$ are an orthonormal basis:

$\hat{e}'_1 \cdot \hat{e}'_1 = 1 \; (= \frac{4}{9} + \frac{4}{9} + \frac{1}{9})$

$\hat{e}'_2 \cdot \hat{e}'_2 = 1 \; (\frac{1}{2} + \frac{1}{2})$

$\hat{e}'_3 \cdot \hat{e}'_3 = 1 \; (= \frac{1}{18} + \frac{1}{18} + \frac{16}{18})$

and $\hat{e}'_i \cdot \hat{e}'_j = 0$ if $i \neq j$

(3)

(a) $\delta_{ij}\delta_{jk}\delta_{kp}\delta_{pi} = ?$ assuming $n=3$

$\delta_{kp}\delta_{pi}$ → for any given k, i

$= \delta_{k1}\delta_{1i} + \delta_{k2}\delta_{2i} + \delta_{k3}\delta_{3i}$

→ one of these terms is non-zero if $k=i$

$= \delta_{ki}$

$\Rightarrow \delta_{ij}\delta_{jk}\delta_{kp}\delta_{pi} = \delta_{ij}\delta_{jk}\delta_{ki} = \delta_{ij}\delta_{ji} = \delta_{ii} = 3$

(b) $\varepsilon_{ijk}\varepsilon_{lmk} \stackrel{?}{=} \delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}$ assume $n=3$

For any given i, j, l, m

$\varepsilon_{ijk}\varepsilon_{lmk} = \varepsilon_{ij1}\varepsilon_{lm1} + \varepsilon_{ij2}\varepsilon_{lm2} + \varepsilon_{ij3}\varepsilon_{lm3}$

one of these terms will be non-zero

if $i \neq j$ and either $i=l$ and $j=m$ → +1

or $i=m$ and $j=l$ → −1

then for one term $i \neq j \neq k$

the other two will be $=0$

The case. $i=l, j=m \Rightarrow \delta_{il}\delta_{jm}$

$i=m, j=l \Rightarrow -\delta_{im}\delta_{jl}$

(c) $\vec{a} \times (\vec{b} \times \vec{c}) = (\vec{a} \cdot \vec{c})\vec{b} - (\vec{a} \cdot \vec{b})\vec{c}$

call $\vec{v} = (\vec{b} \times \vec{c})$

$\vec{a} \times \vec{v} = \varepsilon_{ijk} a_i v_j \hat{e}_k$

$v_j = \varepsilon_{lmj} b_l c_m$

$\vec{a} \times (\vec{b} \times \vec{c}) = \varepsilon_{ijk} \varepsilon_{lmj} a_i b_l c_m$ $\qquad \varepsilon_{ijk} = \varepsilon_{kij}$

$= \varepsilon_{kij} \varepsilon_{lmj} a_i b_l c_m$

use b. $\quad \varepsilon_{kij} \varepsilon_{lmj} = \delta_{kl} \delta_{im} - \delta_{km} \delta_{il}$

$\Rightarrow \vec{a} \times (\vec{b} \times \vec{c}) = (\delta_{kl} \delta_{im} - \delta_{km} \delta_{il}) a_i b_l c_m$

$= a_i c_i b_k - a_l b_l c_k$

$= (\vec{a} \cdot \vec{c})\vec{b} - (\vec{a} \cdot \vec{b})\vec{c}$

pay attention to free indices (should be same) and dummy indices (different as separate sums)

Can use similar derivation for (4) b to show

that $\nabla \times (\nabla \times \vec{b}) = \nabla(\nabla \cdot b) - (\nabla \cdot \nabla)\vec{b}$

$= \frac{\partial}{\partial x_k}\left(\frac{\partial}{\partial x_i} b_i\right) - \frac{\partial}{\partial x_j}\frac{\partial}{\partial x_j} b_k$

$= \nabla(\nabla \cdot b) - \nabla^2 b$

or $\nabla^2 b = \nabla(\nabla \cdot b) - \nabla \times (\nabla \times \vec{b})$

Laplacian in terms of divergence and curl of $\vec{b}$

(4) a

$$\nabla \times (\nabla F) = \left(\hat{e}_l \frac{\partial}{\partial x_l}\right) \times \left(\hat{e}_j \frac{\partial F}{\partial x_j}\right)$$

$$= \varepsilon_{ijk}\, \hat{e}_k \frac{\partial^2 F}{\partial x_i \partial x_j}$$

Note that $\dfrac{\partial^2 F}{\partial x_i \partial x_j}$ is symmetric $= \dfrac{\partial^2 F}{\partial x_j \partial x_i}$

$\Rightarrow \varepsilon_{ijk} \dfrac{\partial^2 F}{\partial x_i \partial x_j} = -\varepsilon_{jik} \dfrac{\partial^2 F}{\partial x_i \partial x_j}$ → i, j interchanged

$= -\varepsilon_{jik} \dfrac{\partial^2 F}{\partial x_j \partial x_i}$ → using symmetry

$= -\varepsilon_{ijk} \dfrac{\partial^2 F}{\partial x_i \partial x_j}$ → renaming dummy indices on right-hand side $i \to j$, $j \to i$

⇒ expression equal to its own negative ⇒ must be $= 0$

Or can write out all components

$$\nabla \times (\nabla F) = \begin{vmatrix} \hat{e}_1 & \hat{e}_2 & \hat{e}_3 \\ \frac{\partial}{\partial x_1} & \frac{\partial}{\partial x_2} & \frac{\partial}{\partial x_3} \\ \frac{\partial F}{\partial x_1} & \frac{\partial F}{\partial x_2} & \frac{\partial F}{\partial x_3} \end{vmatrix} = \hat{e}_1 \left(\frac{\partial^2 F}{\partial x_2 \partial x_3} - \frac{\partial^2 F}{\partial x_3 \partial x_2}\right) + \hat{e}_2 \left(\frac{\partial^2 F}{\partial x_3 \partial x_1} - \frac{\partial^2 F}{\partial x_1 \partial x_3}\right) + \hat{e}_3 \left(\frac{\partial^2 F}{\partial x_1 \partial x_2} - \frac{\partial^2 F}{\partial x_2 \partial x_1}\right)$$

due to symmetry $\dfrac{\partial^2 F}{\partial x_i \partial x_j} = \dfrac{\partial^2 F}{\partial x_j \partial x_i}$

each of these components is equal to zero

$\Rightarrow \nabla \times (\nabla F) = 0$