Learning Objectives

- Understand meaning of different components of 3D Cauchy stress tensor, and know how to determine state of stress on given plane
- Be able to transform rank 2 tensor to a new basis.
- Be able to decompose a rank 2 tensor into symmetric and anti-symmetric components
- Be able to find principal stresses and stress invariants and know what they represent
- Be able to balance body forces and stresses

Tensor symmetry

A tensor can be symmetric in 1 or more indices

In 2-D:

$S_{ij} = S_{ji} \Rightarrow \mathbf{S}=\mathbf{S}^T$ symmetric

$S_{ij} = -S_{ji} \Rightarrow \mathbf{S}=-\mathbf{S}^T$ antisymmetric

Higher rank:

e.g., $S_{ijk} = S_{jik}$ for all i,j,k => symmetric in i,j

antisymmetric **T** of rank 2

Write out general antisymmetric **T** *rank 2, n=3 => how many independent components?*

symmetric **T** of rank 2

has n(n+1)/2 independent components

Any **T** of rank 2 can be decomposed in symm. and antisymm. part:

$$T_{ij} = (T_{ij}+T_{ji})/2 + (T_{ij}-T_{ji})/2$$

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Higher rank:

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antisymmetric $\mathbf{T}$ of rank 2

$\Rightarrow T_{ij} = 0$ for i=j, trace($\mathbf{T}$)=0

has n(n-1)/2 independent components

$$\begin{bmatrix} 0 & T_{12} & T_{13} \\ -T_{12} & 0 & T_{23} \\ -T_{13} & -T_{23} & 0 \end{bmatrix}$$

symmetric $\mathbf{T}$ of rank 2

has n(n+1)/2 independent components

$$\begin{bmatrix} T_{11} & T_{12} & T_{13} \\ T_{12} & T_{22} & T_{23} \\ T_{13} & T_{23} & T_{33} \end{bmatrix}$$

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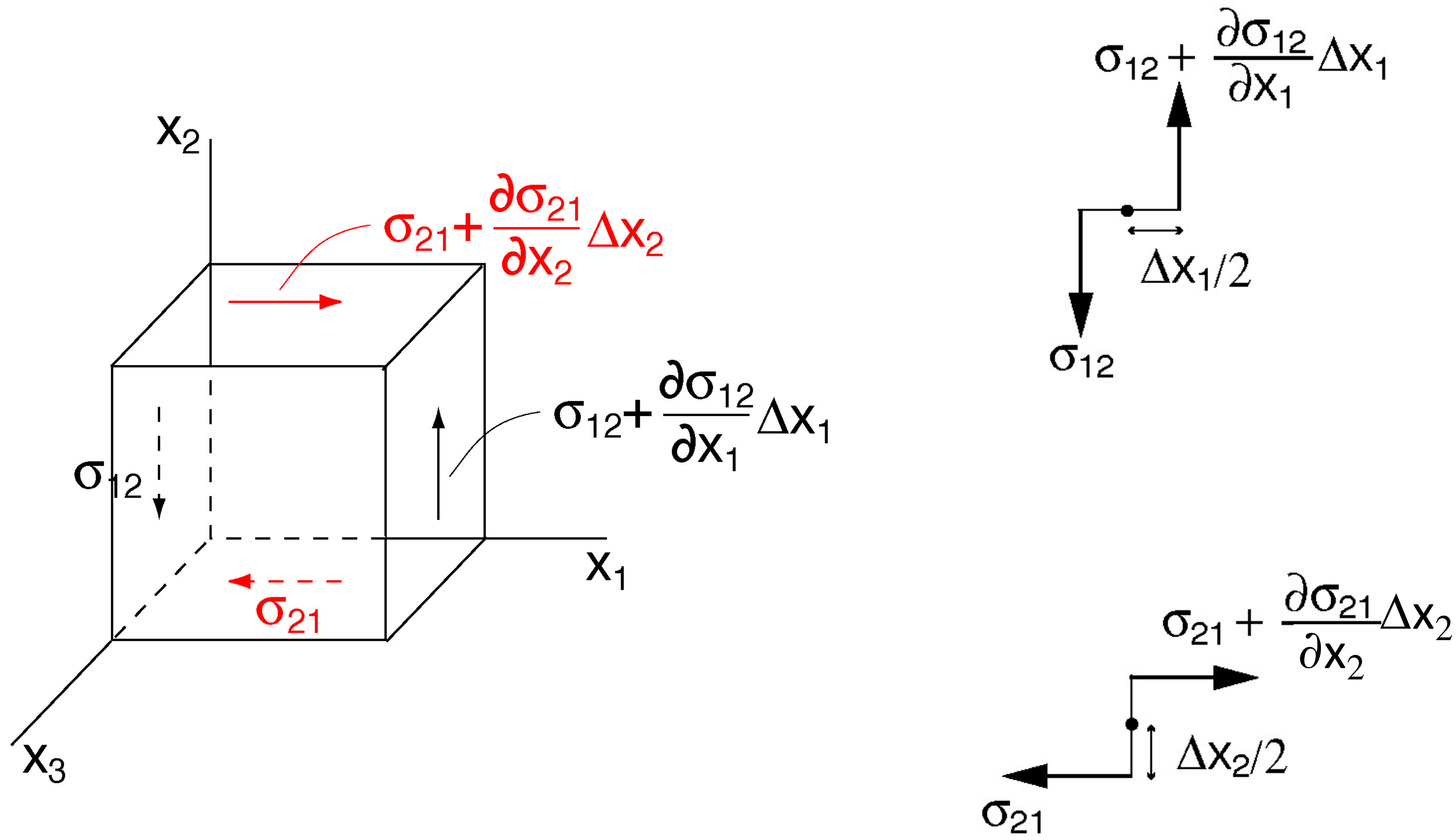
Symmetry of the stress tensor



Try writing out the balance of moments in x_3 direction,

assuming static equilibrium

A balance of moments in x_3 direction:

$$m_3 = [\qquad\qquad\qquad\qquad\qquad\qquad] \qquad \cdot \Delta x_1 / 2$$

$$-[\qquad\qquad\qquad\qquad\qquad\qquad] \qquad \cdot \Delta x_2 / 2 = 0$$

$$\Rightarrow [2\sigma_{12} + \Delta x_1 \frac{\partial \sigma_{12}}{\partial x_1})] - [2\sigma_{21} + \Delta x_2 \frac{\partial \sigma_{21}}{\partial x_2})] = 0$$

$$\lim_{\Delta x_1, \Delta x_2} \to 0 \Rightarrow \boxed{\sigma_{12} = \sigma_{21}}$$

Note: if body force induced rotation:

$$I_{33} \frac{\partial \omega}{\partial t} = O(\Delta x^2)$$

Balancing m_1 and m_2: $\boxed{\sigma_{23} = \sigma_{32}}$ *and* $\boxed{\sigma_{13} = \sigma_{31}}$

*thus, the stress tensor is **symmetric***

$$\mathbf{t} = \boldsymbol{\sigma}^{\mathrm{T}} \cdot \hat{\mathbf{n}} \Rightarrow \mathbf{t} = \boldsymbol{\sigma} \cdot \hat{\mathbf{n}}$$

A balance of moments in x_3 direction:

$$m_3 = [\sigma_{12} + (\sigma_{12} + \Delta x_1 \frac{\partial \sigma_{12}}{\partial x_1})]\Delta x_2 \cancel{\Delta x_3} \cdot \Delta x_1 / 2$$

$$-[\sigma_{21} + (\sigma_{21} + \Delta x_2 \frac{\partial \sigma_{21}}{\partial x_2})]\cancel{\Delta x_1 \Delta x_3} \cdot \Delta x_2 / 2 = 0$$

$$\Rightarrow [2\sigma_{12} + \Delta x_1 \frac{\partial \sigma_{12}}{\partial x_1})] - [2\sigma_{21} + \Delta x_2 \frac{\partial \sigma_{21}}{\partial x_2})] = 0$$

$$\lim_{\Delta x_1, \Delta x_2} \to 0 \Rightarrow \boxed{\sigma_{12} = \sigma_{21}}$$

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Diagonalizing

Real-valued, symmetric rank 2 tensors (square, symmetric matrices) can be diagonalized, i.e. a coordinate frame can be found, such that only the diagonal elements (normal stresses) remain.

For stress tensor, these elements, $\sigma_1, \sigma_2, \sigma_3$ are called the principal stresses

$$\begin{bmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & \sigma_3 \end{bmatrix}$$

Such a transformation can be cast as:

$\mathbf{T}\cdot\mathbf{x} = \lambda\mathbf{x}$

where $\mathbf{x}_i$ are eigenvectors or characteristic vectors

and λ_i are the eigenvalues, characteristic or principal values

$\Rightarrow(\mathbf{T}-\lambda\delta)\cdot\mathbf{x} = \mathbf{0}$

Non-trivial solution only if $\det(\mathbf{T}-\lambda\delta) = 0$

Determinant

For 2-dimensional rank 2 tensor

$$\det(\mathbf{T}) = \begin{vmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{vmatrix} = T_{11}T_{22} - T_{12}T_{21}$$

$\det(\mathbf{T}) \neq 0$
columns of **T**
are linearly
independent,
and $\mathbf{T}^{-1}$ exists

$$\det(\mathbf{a},\mathbf{b}) = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = a_1b_2 - a_2b_1 = \mathbf{a} \times \mathbf{b}$$

signed area

For 3-dimensional rank 2 tensor $\mathbf{T}=\{\mathbf{a},\mathbf{b},\mathbf{c}\}$

$$\mathbf{T}\cdot\hat{\mathbf{e}}_1 = \mathbf{a}$$
$$\mathbf{T}\cdot\hat{\mathbf{e}}_2 = \mathbf{b}$$
$$\mathbf{T}\cdot\hat{\mathbf{e}}_3 = \mathbf{c}$$

$$\det(\mathrm{T}) = \det(\mathbf{a},\mathbf{b},\mathbf{c}) = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = a_1b_2c_3 + a_2b_3c_1 + a_3b_1c_2 - a_1b_3c_2 - a_2b_1c_3 - a_3b_2c_1$$

$$= \varepsilon_{ijk}a_ib_jc_k = (\mathbf{a}\times\mathbf{b})\cdot\mathbf{c}$$

signed volume

Determinant
and cross product

Can write cross product as a determinant

$$\mathbf{a} \times \mathbf{b} = \varepsilon_{ijk} a_i b_j \hat{\mathbf{e}}_k = \begin{vmatrix} \hat{\mathbf{e}}_1 & \hat{\mathbf{e}}_2 & \hat{\mathbf{e}}_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

$$\begin{vmatrix} \hat{\mathbf{e}}_1 & \hat{\mathbf{e}}_2 & \hat{\mathbf{e}}_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} - \begin{vmatrix} \hat{\mathbf{e}}_1 & \hat{\mathbf{e}}_2 & \hat{\mathbf{e}}_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

$$\hat{\mathbf{e}}_1 \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} + \hat{\mathbf{e}}_2 \begin{vmatrix} a_3 & a_1 \\ b_3 & b_1 \end{vmatrix} + \hat{\mathbf{e}}_3 \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}$$