Recap Fluid - Solid

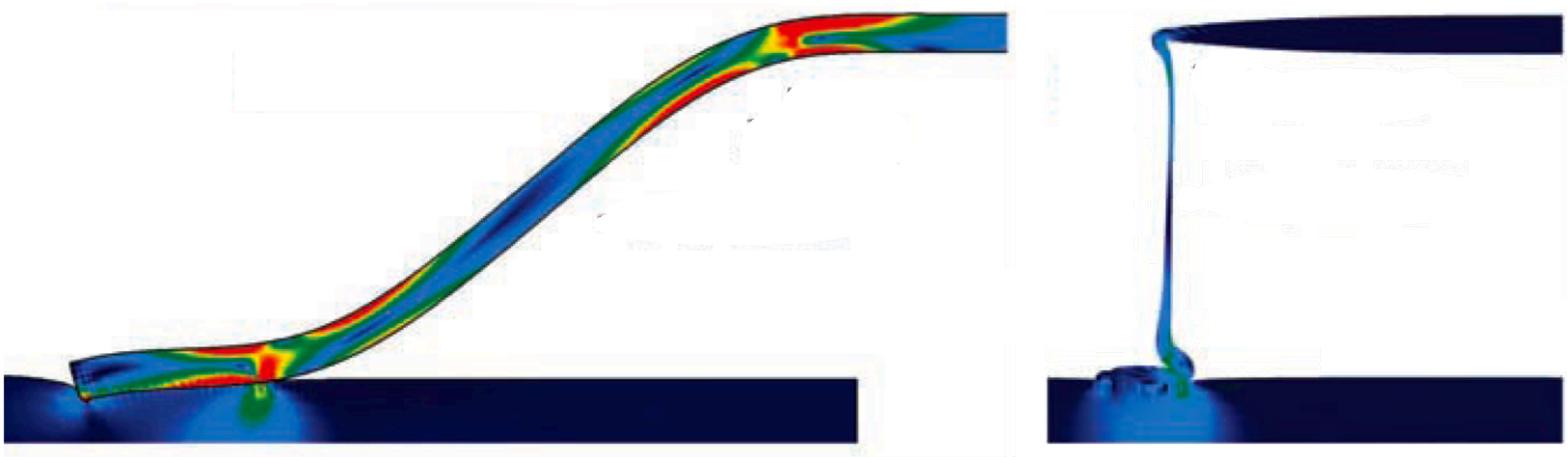
- What is a solid?

  A solid acquires finite deformation under stress

  stress σ ~ *strain* ε

- What is a fluid?

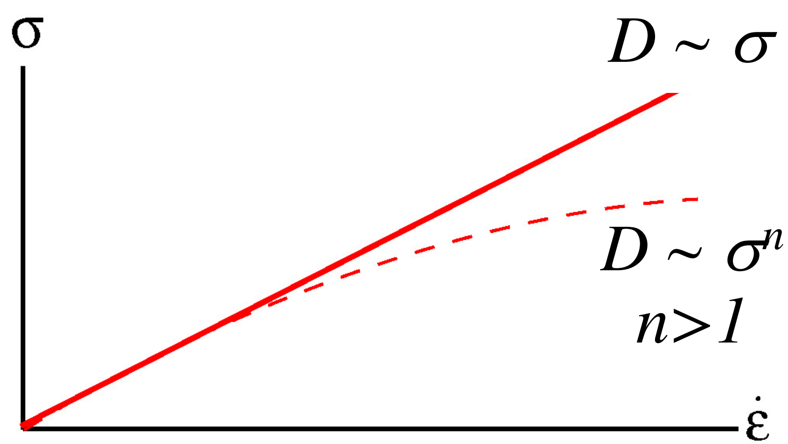
  A material that flows in response to applied stress *stress* σ ~ *strain rate* Dε/Dt



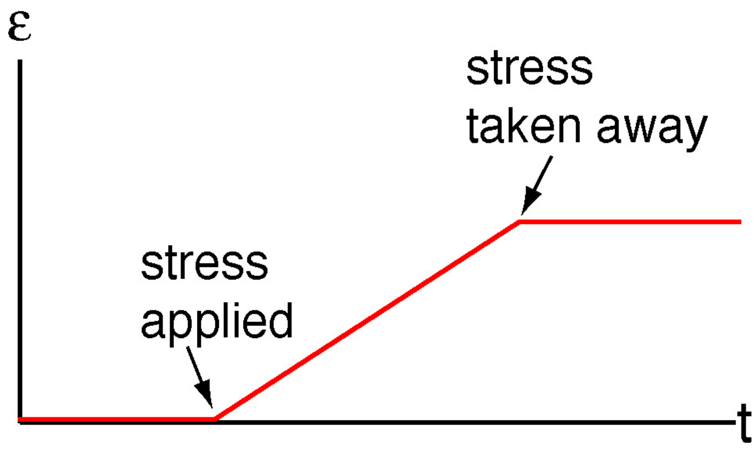
Figures from Funiciello et al. (2003a)

Viscous Flow

- steady state flow at constant stress
- permanent deformation
- linear (Newtonian) or non-linear (e.g., Powerlaw) relation between strain rate and stress
- isotropic stress does not cause flow
- *on timescales > years base tectonic plates and mantle deform predominantly viscously -> plate motions, postseismic deformation, but also glaciers, magmas*



$$\dot{\varepsilon} = \frac{\partial \varepsilon}{\partial t}$$



Hydrostatics

Fluids can not support shear stresses

i.e. if in rest/rigid body motion: $\boldsymbol{\sigma} \cdot \hat{\mathbf{n}} = \lambda \hat{\mathbf{n}}$

and this normal stress is the same on any plane: $\boldsymbol{\sigma} = -p\mathbf{I}$

p is *hydrostatic pressure*

In force balance:

$$\nabla \cdot \boldsymbol{\sigma} + \mathbf{f} = 0$$

$$-\nabla p = -\mathbf{f}$$

In gravity field $\frac{\partial p}{\partial z} = \rho g \quad \Rightarrow \quad p_2 - p_1 = \rho g h$

Newtonian Fluids

In general motion: $$\boldsymbol{\sigma} = -p\mathbf{I} + \boldsymbol{\sigma}'$$

In Newtonian fluids,
deviatoric stress varies ***linearly*** with ***strain rate***, $\mathbf{D}$
$D_{ij} = (\partial v_i/\partial x_j + \partial v_j/\partial x_i)/2$

For *isotropic*, Newtonian fluids, 2 *material parameters*:

Viscous stress tensor $\sigma'_{ij} = -\zeta D_{kk}\delta_{ij} + 2\eta D_{ij}$

where ζ is ***bulk viscosity*** and η **(*shear*) *viscosity***, $\Delta = D_{kk} = \nabla \cdot \mathbf{v}$

$$\boldsymbol{\sigma} = \left(-p + \varsigma\Delta\right)\mathbf{I} + 2\eta\mathbf{D}$$

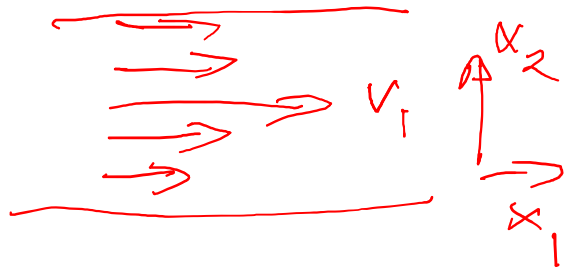
p not always mean normal stress: $\sigma_{kk} = -3p + (3\zeta + 2\eta)D_{kk}$

Consider a Newtonian shear flow with velocity field $v_1(x_2)$, $v_2=v_3=0$

What is $\mathbf{D}$? What is $\boldsymbol{\sigma}$?

Consider a Newtonian shear flow with velocity field $v_1(x_2)$, $v_2=v_3=0$

What is **D**? What is **σ**?



$$D_{11}=D_{22}=D_{33}=0$$

$$D_{12}=D_{21}=\frac{1}{2}\frac{\partial v_1}{dx_2}$$

$$D_{13}=D_{31}=D_{23}=D_{32}=0$$

$$\sigma_{11}=\sigma_{22}=\sigma_{33}=-\mathrm{p}$$

$$\sigma_{12}=\sigma_{21}=\eta\frac{\partial v_1}{dx_2}$$

$$\sigma_{13}=\sigma_{31}=\sigma_{23}=\sigma_{32}=0$$

Illustrates that η represents resistance to shearing

Navier-Stokes for incompressible Newtonian Flow

For incompressible fluids Δ=0, so that: $\boldsymbol{\sigma} = -p\mathbf{I} + 2\eta\mathbf{D}$

Force balance: $\nabla \cdot \underline{\underline{\sigma}} + \mathbf{f} = \rho \dfrac{D\mathbf{v}}{Dt}$

$\nabla \cdot \vec{v} = 0$

Show that: $\dfrac{\partial \sigma_{ij}}{\partial x_j} = -\dfrac{\partial p}{\partial x_i} + \eta \dfrac{\partial^2 v_i}{\partial x_j \partial x_j}$

Assuming constant η

$$\nabla \cdot \underline{\underline{\sigma}} = -\nabla p + \eta \nabla^2 \mathbf{v}$$

Navier-Stokes for incompressible Newtonian Flow

For incompressible fluids Δ=0, so that: $\boldsymbol{\sigma} = -p\mathbf{I} + 2\eta\mathbf{D}$

Force balance: $\nabla \cdot \underline{\underline{\sigma}} + \mathbf{f} = \rho \dfrac{D\mathbf{v}}{Dt}$

Show that: $$\frac{\partial \sigma_{ij}}{\partial x_j} = -\frac{\partial p}{\partial x_i} + \eta \frac{\partial^2 v_i}{\partial x_j \partial x_j}$$

Assuming constant η

$$\sigma_{ij} = -p\delta_{ij} + \eta\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right)$$

$$\frac{\partial \sigma_{ij}}{\partial x_j} = -\frac{\partial p}{\partial x_i} + \eta\left(\frac{\partial^2 v_i}{\partial x_j \partial x_j} + \cancel{\frac{\partial^2 v_j}{\partial x_i \partial x_j}}\right)$$

Because $\dfrac{\partial v_j}{\partial x_j} = \Delta = 0$

$$\nabla \cdot \underline{\underline{\sigma}} = -\nabla p + \eta \nabla^2 \mathbf{v}$$

Navier-Stokes for incompressible Newtonian Flow

For incompressible fluids Δ=0, so that: $\boldsymbol{\sigma} = -p\mathbf{I} + 2\eta\mathbf{D}$

Force balance: $\nabla \cdot \underline{\underline{\sigma}} + \mathbf{f} = \rho \dfrac{D\mathbf{v}}{Dt}$

Navier Stokes equation of motion:

$$-\nabla p + \eta \nabla^2 \mathbf{v} + \mathbf{f} = \rho \left[\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{v} \right]$$

Assuming constant η

Together with continuity, 4 equations, 4 unknowns (p, v_x, v_y, v_z)

$$\nabla \cdot \mathbf{v} = 0$$

Navier-Stokes for compressible Newtonian Flow

$$\boldsymbol{\sigma} = \left(-p + \varsigma\Delta\right)\mathbf{I} + 2\eta\mathbf{D}$$

$$\nabla \cdot \underline{\underline{\sigma}} + \mathbf{f} = \rho \frac{D\mathbf{v}}{Dt}$$

Navier Stokes equation of motion:

$$-\nabla p + \left(\zeta + \eta\right)\nabla\Delta + \eta\nabla^2\mathbf{v} + \mathbf{f} = \rho\left[\frac{\partial\mathbf{v}}{\partial t} + \mathbf{v}\cdot\nabla\mathbf{v}\right]$$

Assuming constant ζ, η

\+ Conservation of mass:

$$\frac{D\rho}{Dt} + \rho\nabla\cdot\mathbf{v} = 0$$

6 equations
6 unknowns
(p, v_x, v_y, v_z, ρ, T)

\+ Energy equation

\+ Equation of state for $\rho(T,p)$

Continuum Mechanics Equations

General:

1.Kinematics – describing deformation and velocity without considering forces

2.Dynamics – equations that describe force balance, conservation of linear and angular momentum

3.Thermodynamics – relations temperature, heatflux, stress, entropy

Material-specific

4.Constitutive equations – relations describing how material properties vary as a function of T,P, stress,…. Such material properties govern dynamics (e.g., ***density***), response to stress (***viscosity***, ***elastic parameters***), heat transport (***thermal conductivity***, ***heat capacity***)

Learning Objectives

- Learn main conservation equations used in continuum mechanics modelling and understand what different terms in these equations represent
- Be able to solve conservation equations for basic analytical solutions given boundary/initial conditions.
- Understand basic properties of elastic and viscous rheology and understand how the choice of rheology leads to different forms of the momentum conservation equation
- Using tensor analysis to obtain relations between the main isotropic elastic parameters

Outline

- Conservation equations
- Energy equation
- Rheology
- Elasticity and Wave Equation
- Newtonian Viscosity and Navier Stokes

More reading on the topics covered in this lecture can be found in, for example: Lai et al. Ch 4.14-4-16, 6.18, Ch 5.1-5.6, Ch 6.1-6.7; Reddy parts of Ch 5 & Ch 6

Friday Coursework

- Released at 9:00, due at 11:00 with 30 minutes to upload (until 11:30)
- Similar style to the class exercises from lectures 5 through 8
- Understand material covered in lectures and slides, practise class exercises (with answers), plus old exam questions (no answers released)

- Good luck!

Outline of course

- **1.** Mathematical essentials – *Matthew Piggott*
- **2.** Linear Algebra I – *Matthew Piggott*
- **3.** Linear Algebra II, ODEs– *Matthew Piggott*
- **4.** Verifying models– *Matthew Piggott*
- **5.** Vector and Tensor Calculus - *Saskia Goes*
- **6.** Stress principles - *Saskia Goes*
- **7.** Kinematics and strain - *Saskia Goes*
- **8.** Rheology and conservation equations - *Saskia Goes*
- **9.** Dimensional analysis - *Stephen Neethling*
- **10.** Potential flow - *Stephen Neethling*
- **11.** Fluid flow I - *Stephen Neethling*
- **12.** Fluid flow & turbulence – *Stephen Neethling*