# Long Solutions Problem Set 7 - ACSE-2 2020

① Motion of a material particle initially at position $\vec{\xi}$ is

$$x'_1 = \xi_1 + kt\xi_2 \,,\quad x'_2 = (1+kt)\xi_2 \,,\quad x'_3 = \xi_3$$

Temperature field in spatial description

$$T(\vec{x},t) = \alpha\,(x_1 + x_2)$$

(a) $T$ in material description:

At time $t$ the particle $\xi$ is in position $\vec{x}'$

$T$ at that position is

$$T(\vec{\xi},t) = \alpha\left[\xi_1 + kt\xi_2 + (1+kt)\xi_2\right]$$

(b) velocity of the particle is $\dfrac{\partial \vec{x}'}{\partial t}$ or $\dfrac{D\vec{x}}{Dt}$

$$\vec{v}(\vec{\xi},t) = \frac{\partial \vec{x}'}{\partial t} = (k\xi_2, k\xi_2, 0)$$

in spatial description → same but express $\xi_2$ in $x'_i$

$$\xi_2 = \frac{x'_2}{(1+kt)}$$

$$\Rightarrow \vec{v}(\vec{x},t) = \left(\frac{kx_2}{(1+kt)}, \frac{kx_2}{(1+kt)}, 0\right)$$

(c) $\dfrac{DT}{Dt}$ in material description:

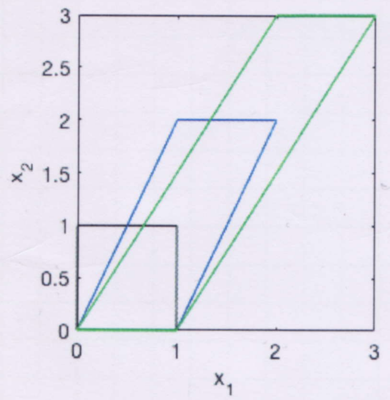
$$\frac{DT}{Dt}(\vec{\xi},t) = \alpha k\xi_2 + \alpha k\xi_2 = 2\alpha k\xi_2$$

same in spatial description

- substitute $\xi_2 = \dfrac{x_2}{1+kt} \Rightarrow \dfrac{DT}{Dt}(\vec{x},t) = \dfrac{2\alpha k x_2}{1+kt}$
- or use $\dfrac{DT}{Dt}(\vec{x},t) = \dfrac{\partial T}{\partial t}(\vec{x},t) + \vec{v}\cdot\nabla T(\vec{x},t)$

$$= 0 + \frac{kx_2}{1+kt}(1,1,0)\cdot\begin{pmatrix}\alpha\\ \alpha\\ 0\end{pmatrix} = \frac{2\alpha k x_2}{1+kt}$$

(d)




*black at t=0*
*blue at t=1*
*green at t=2*

*Describes shearing and stretching*

(2) $\underline{\underline{E}} = \begin{bmatrix} 5 & 3 & 0 \\ 3 & 4 & -1 \\ 0 & -1 & 2 \end{bmatrix} \cdot 10^{-4}$  note: $\underline{\underline{E}} \cdot \hat{x} \to \vec{x}'$ deformed $\hat{x}$ by strain $\underline{\underline{E}}$

$\hat{x} \cdot \underline{\underline{E}} \cdot \hat{x} \longrightarrow$ component of deformed $\vec{x}'$ in direction $\hat{x}$

(a) unit elongation in direction $2\hat{e}_1 + 2\hat{e}_2 + \hat{e}_3$?

$\Rightarrow$ vector $\hat{s} = \frac{1}{3}(2,2,1)$

$\underline{\text{elongation}} = \hat{s} \cdot \underline{\underline{E}} \cdot \hat{s} = \frac{1}{9}(2,2,1) \cdot \underline{\underline{E}} \cdot \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix}$

$= \frac{1}{9}(2,2,1) \cdot \begin{pmatrix} 16 \\ 13 \\ 0 \end{pmatrix} \cdot 10^{-4} = \frac{58}{9} \cdot 10^{-4}$

(b) new angle of vectors originally: $\hat{s} = (2\hat{e}_1 + 2\hat{e}_2 + \hat{e}_3)_{\text{unit length}}$
$= \frac{1}{3}(2,2,1)$
$\hat{p} = (3\hat{e}_1 - 6\hat{e}_3)_{\text{unit length}}$
$= \frac{1}{\sqrt{45}}(3,0,-6)$

$\downarrow$

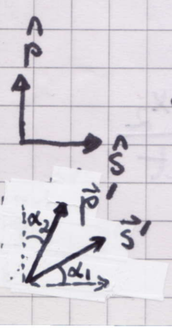
originally $90^\circ$ $\hat{s} \cdot \hat{p} = 0$

$\underline{\text{Change in angle}}$

$2 \times \hat{p} \cdot \underline{\underline{E}} \cdot \hat{s} = \frac{2}{3\sqrt{45}}(3,0,-6)\begin{pmatrix} 16 \\ 13 \\ 0 \end{pmatrix} \cdot 10^{-4}$

$= \frac{2 \times 48}{3\sqrt{45}} \cdot 10^{-4} = \frac{32}{\sqrt{45}} \cdot 10^{-4}$ in radians $= \alpha_1 + \alpha_2$

$= 2 \times \hat{s} \cdot \underline{\underline{E}} \cdot \hat{p}$




before

after $\underline{\underline{E}}$

(3) Displacement field

$u_1 = k\xi_1^2$ ; $u_2 = k\xi_2\xi_3$ ; $u_3 = k(2\xi_1\xi_3 + \xi_1^2)$ $\qquad k = 10^{-6}$

$\downarrow$ small strain

maximum unit elongation

$\Rightarrow$ equals maximum eigenvalue of strain tensor

$$\underline{\underline{E}} = \frac{1}{2}(\nabla u + \nabla u^T)$$

$$\nabla u = \begin{bmatrix} 2k\xi_1 & 0 & 0 \\ 0 & k\xi_3 & k\xi_2 \\ 2k(\xi_3+\xi_1) & 0 & 2k\xi_1 \end{bmatrix}$$

$$\underline{\underline{E}} = \begin{bmatrix} 2k\xi_1 & 0 & k(\xi_1+\xi_3) \\ 0 & k\xi_3 & \frac{1}{2}k\xi_2 \\ k(\xi_1+\xi_3) & \frac{1}{2}k\xi_2 & 2k\xi_1 \end{bmatrix}$$

in point $(1,0,0)$ $\rightarrow$ $\underline{\underline{E}} = \begin{bmatrix} 2k & 0 & k \\ 0 & 0 & 0 \\ k & 0 & 2k \end{bmatrix}$

Find eigenvalues

$$-\lambda^3 + tr(\underline{\underline{E}})\lambda^2 - minor(\underline{\underline{E}})\lambda + \det \underline{\underline{E}} = 0$$

$tr(\underline{\underline{E}}) = 4k$

$minor(\underline{\underline{E}}) = 4k^2 - k^2 = 3k^2$

$\det(\underline{\underline{E}}) = 0$

$-\lambda(\lambda^2 - 4k\lambda + 3k^2) = 0$

$-\lambda(\lambda - 3k)(\lambda - k) = 0 \Rightarrow \lambda_1 = 0, \lambda_2 = 3k, \lambda_3 = k$

$\underbrace{\phantom{\lambda_2 = 3k}}_{\text{max. elongation}}$

Can also find direction $\vec{x}$

Solve $\underline{\underline{E}} \cdot \vec{x} = 3k\vec{x} \Rightarrow \begin{pmatrix} 2kx_1 + kx_3 \\ 0 \\ kx_1 + 2kx_3 \end{pmatrix} = \begin{pmatrix} 3kx_1 \\ 3kx_2 \\ 3kx_3 \end{pmatrix}$

$\Rightarrow x_1 = x_3$ and $x_2 = 0$

$\Rightarrow$ <u>direction</u> $\vec{x} = (1,0,1)$ gives max. elongation

④ Given displacement: $u_1 = k\xi_2^2$, $u_2 = u_3 = 0$, $k = 10^{-4}$

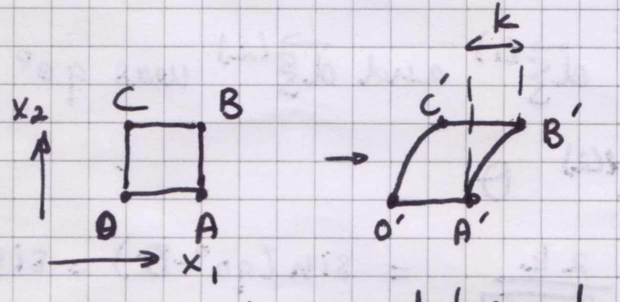
(a) Sketch deformation of a unit square OABC

$$O: \vec{\xi} = (0,0,0) \Rightarrow \vec{x}' = \vec{\xi} + \vec{u} = \vec{\xi} + (0,0,0) = (0,0,0)$$

$$A: \vec{\xi} = (1,0,0) \Rightarrow \vec{x}' = (1,0,0) + (0,0,0) = (1,0,0)$$

$$B: \vec{\xi} = (1,1,0) \Rightarrow \vec{x}' = (1,1,0) + (k,0,0) = (1+k,1,0)$$

$$C: \vec{\xi} = (0,1,0) \Rightarrow \vec{x}' = (0,1,0) + (k,0,0) = (k,1,0)$$




original deformed.

deformation exaggerated because actually $k$ is only $10^{-4}$

C′ and B′ displaced by same amount in $x_1$ direction

lines O′-C′ and A′-B′ take on parabolic shape

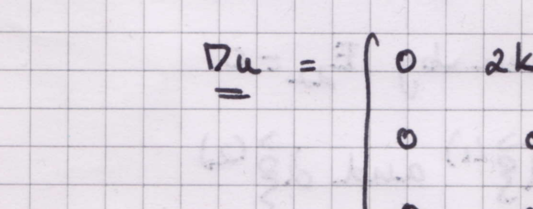
(b) Find deformed vectors $\vec{dx}'^{(1)}$ and $\vec{dx}'^{(2)}$ for elements originally $\vec{d\xi}^{(1)} = d\xi_1 \hat{e}_1$, $\vec{d\xi}^{(2)} = d\xi_2 \hat{e}_2$ at point $C(0,1,0)$

$$\vec{dx}'^{(1)} = \vec{d\xi}^{(1)} + \underline{\underline{\nabla u}} \cdot \vec{d\xi}^{(1)}$$

$$\underline{\underline{\nabla u}} = \begin{bmatrix} 0 & 2k\xi_2 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad \underline{\underline{\nabla u}} \cdot \vec{d\xi}^{(1)} = (0,0,0)$$

$$\Rightarrow \vec{dx}'^{(1)} = \vec{d\xi}^{(1)} = (d\xi_1, 0, 0)$$

$$\vec{dx}'^{(2)} = \vec{d\xi}^{(2)} + \underline{\underline{\nabla u}} \cdot \vec{d\xi}^{(2)} \qquad \underline{\underline{\nabla u}} \cdot \vec{d\xi}^{(2)} = \begin{pmatrix} 2k\xi_2 \, d\xi_2 \\ 0 \\ 0 \end{pmatrix}$$




$$\text{At } C(0,1,0) = \begin{pmatrix} 2k\,d\xi_2 \\ 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow \vec{dx}'^{(2)} = \vec{d\xi}^{(2)} + (2k,0,0)$$

$$= (2k\,d\xi_2, d\xi_2, 0) = d\xi_2(2k, 1, 0)$$

(c) difference between original and deformed lengths:

For $\vec{d\xi}^{(1)} \rightarrow$ length $d\xi_1$

For $\vec{dx}'^{(1)} \rightarrow$ length $d\xi_1$

$$\left| \vec{dx}'^{(1)} \right| - \left| \vec{d\xi}^{(1)} \right| = 0$$

For $\vec{d\xi}^{(2)} \rightarrow$ length $d\xi_2$

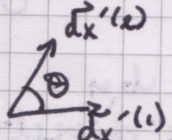
For $\vec{dx}'^{(2)} \rightarrow$ length $\sqrt{4k^2+1} \cdot d\xi_2$

$$\left| \vec{dx}'^{(2)} \right| - \left| \vec{d\xi}^{(2)} \right| = d\xi_2 \left( \sqrt{4k^2+1} - 1 \right)$$

(d) change in angle between the two elements

original angle between $\vec{d\xi}^{(1)}$ and $\vec{d\xi}^{(2)}$ was $90°$

new angle $\vec{dx}'^{(1)}$ and $\vec{dx}'^{(2)}$ $\theta$




$$\cos\theta = \frac{2k\, d\xi_2}{\sqrt{4k^2+1}\, d\xi_2} = \frac{2k}{\sqrt{4k^2+1}} = \sin(90° - \theta) = \sin(d\psi)$$

i.e. sin(change in angle)

(e) infinitesimal strain tensor

$$\underline{\underline{E}} = \frac{1}{2}(\nabla u + \nabla u^T)$$

$$= \frac{1}{2}\begin{bmatrix} 0 & 2k\xi_2 & 0 \\ 2k\xi_2 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & k\xi_2 & 0 \\ k\xi_2 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

(f) $\vec{d\xi}^{(1)}$ – change in length given by $E_{11} = 0$

$\vec{d\xi}^{(2)}$ – change in length given by $E_{22} = 0$

(g) change in angle $\overset{d\psi}{}$ between $\vec{d\xi}^{(1)}$ and $\vec{d\xi}^{(2)}$

given by $2E_{12} = 2k\xi_2 = 2k$ (radians)

(h) compare result (c) and (f)

Actual change in length $\vec{d\xi}^{(1)} = E_{11}$

Actual change in length $\vec{d\xi}^{(2)} = \sqrt{4k^2+1} - 1$

$\approx 0 = E_{22}$ if $k$ small

compare (d) and (g)

$\sin(d\psi) \approx d\psi$ if $d\psi$ = small and $\frac{2k}{\sqrt{4k^2+1}} \approx 2k$ for small $k$

## (5) Conservation of mass

Given $v_i = \frac{kx_i}{1+kt}$

Density of a material particle through time, i.e. assume $\rho(t)$ (not fion of $\vec{x} \Rightarrow \nabla\rho = 0$)

$$\frac{D\rho}{Dt} + \rho\nabla\cdot\vec{v} = 0$$

$$\rho(t) \Rightarrow \quad \frac{\partial\rho}{\partial t} + \rho\nabla\cdot\vec{v} = 0$$

$$\nabla\cdot\vec{v} = \frac{k}{1+kt}\left(\frac{\partial x_i}{\partial x_i}\right) = \frac{3k}{1+kt}$$

$$\Rightarrow \frac{\partial\rho}{\partial t} = \frac{-3\rho k}{1+kt}$$

integrate to get $\rho(t)$

$$\int_{\rho_0}^{\rho} \frac{d\rho'}{\rho'} = \int_0^t \frac{-3k\,dt'}{1+kt'}$$

$$\ln\rho - \ln\rho_0 = -3\ln(1+kt)$$

$$\ln\left(\frac{\rho}{\rho_0}\right) = \ln(1+kt)^{-3}$$

$$\Rightarrow \rho = \frac{\rho_0}{(1+kt)^3}$$