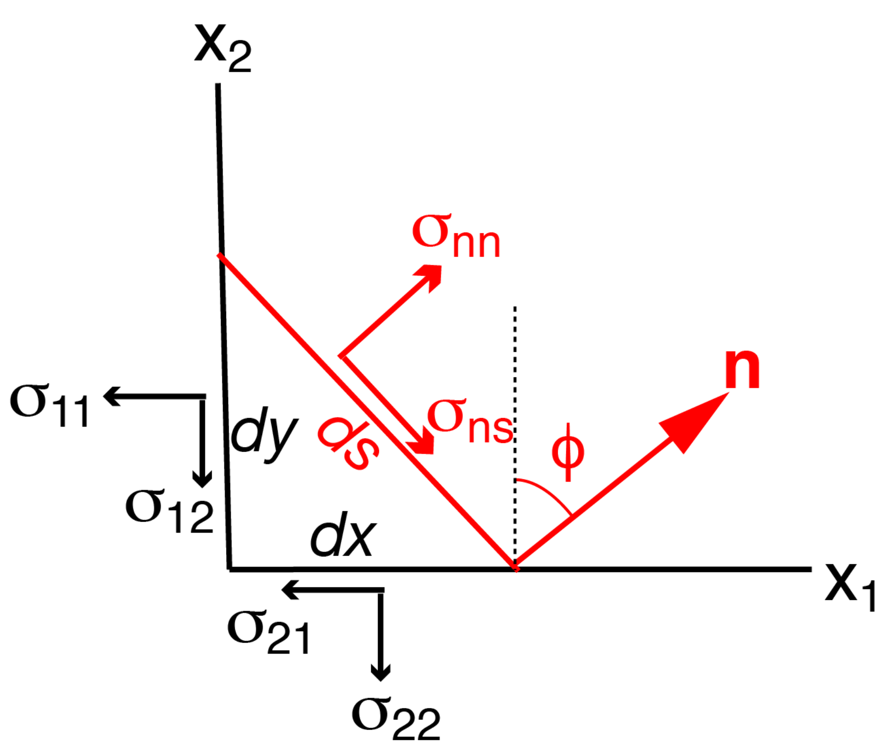
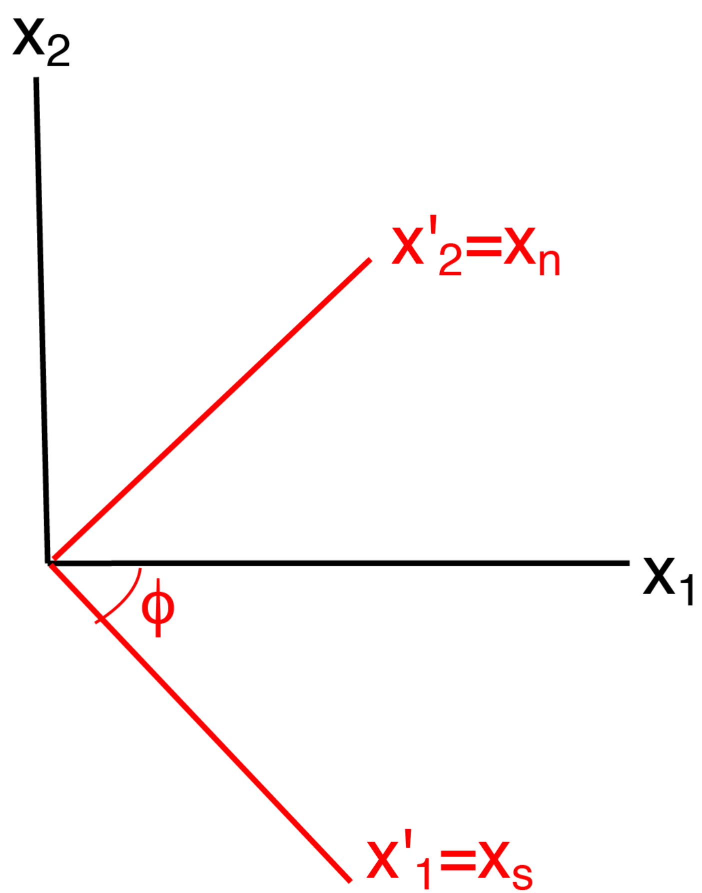
Learning Objectives

- Understand meaning of different components of 3D Cauchy stress tensor, and know how to determine state of stress on given plane
- Be able to transform rank 2 tensor to a new basis.
- Be able to decompose a rank 2 tensor into symmetric and anti-symmetric components
- Be able to find principal stresses and stress invariants and know what they represent
- Be able to balance body forces and stresses

Transforming the 2-D stress tensor

(determining normal and shear stress on a plane)





$dx = \cos\phi \, ds$

$dy = \sin\phi \, ds$

Try writing force balance in x_1 direction

Force balance

in x_1 direction: (1)

Try writing force balance in x_1 direction

in x_2 direction: (2)

$$\sigma_{12} dy + \sigma_{22} dx = \sigma_{nn} \cos\phi ds - \sigma_{ns} \sin\phi ds$$

$$\sigma_{12} \sin\phi + \sigma_{22} \cos\phi = \sigma_{nn} \cos\phi - \sigma_{ns} \sin\phi$$

(1)·sinϕ + (2)·cosϕ: *verify yourself*

$$\sigma_{nn} = \sigma_{11} \sin^2\phi + \sigma_{21} \cos\phi \sin\phi + \sigma_{12} \cos\phi \sin\phi + \sigma_{22} \cos^2\phi$$

(1)·cosϕ – (2)·sinϕ:

$$\sigma_{ns} = \sigma_{11} \cos\phi \sin\phi + \sigma_{21} \cos^2\phi - \sigma_{12} \sin^2\phi - \sigma_{22} \cos\phi \sin\phi$$

This is equivalent to the **tensor transformation** $\sigma'_{qp}=\alpha_{pi}\alpha_{qj}\sigma_{ji}$

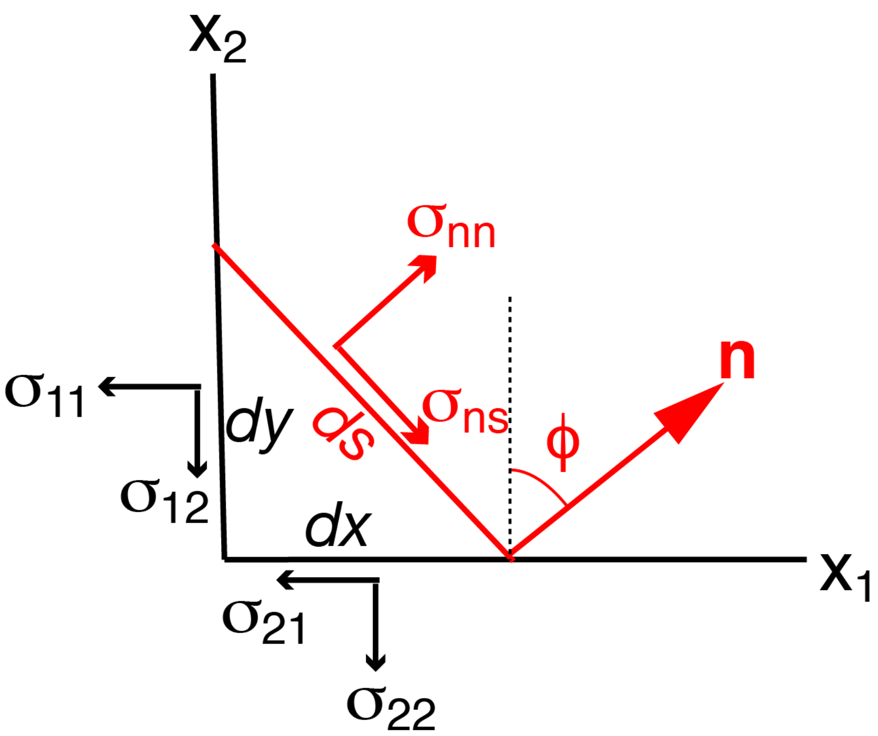
$$\sigma'_{nn}=\alpha_{ni}\alpha_{nj}\sigma_{ji}$$

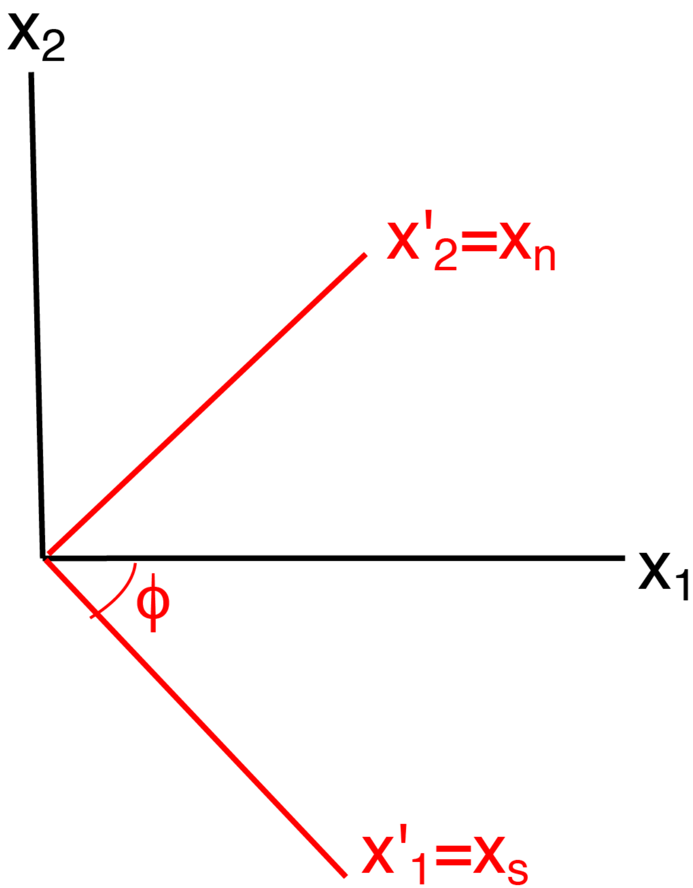
$$\sigma'_{ns}=\alpha_{si}\alpha_{nj}\sigma_{ji}$$

With $\alpha_{n1} = \sin\phi$, $\alpha_{n2} = \cos\phi$, $\alpha_{s1} = \cos\phi$, $\alpha_{s2} = -\sin\phi$

Transforming the 2-D stress tensor

(determining normal and shear stress on a plane)





$$dx = \cos\phi \, ds$$
$$dy = \sin\phi \, ds$$

$$\sigma_{11} dy + \sigma_{21} dx = \sigma_{nn} \sin\phi \, ds + \sigma_{ns} \cos\phi \, ds$$

Force balance

in x_1 direction: (1) $$\sigma_{11}dy+\sigma_{21}dx=\sigma_{nn}\sin\phi ds+\sigma_{ns}\cos\phi ds$$
$$\sigma_{11}\sin\phi+\sigma_{21}\cos\phi=\sigma_{nn}\sin\phi+\sigma_{ns}\cos\phi$$

in x_2 direction: (2) $$\sigma_{12}dy+\sigma_{22}dx=\sigma_{nn}\cos\phi ds-\sigma_{ns}\sin\phi ds$$
$$\sigma_{12}\sin\phi+\sigma_{22}\cos\phi=\sigma_{nn}\cos\phi-\sigma_{ns}\sin\phi$$

(1)·sinϕ + (2)·cosϕ: *verify yourself*
$$\sigma_{nn}=\sigma_{11}\sin^2\phi+\sigma_{21}\cos\phi\sin\phi+\sigma_{12}\cos\phi\sin\phi+\sigma_{22}\cos^2\phi$$

(1)·cosϕ – (2)·sinϕ:
$$\sigma_{ns}=\sigma_{11}\cos\phi\sin\phi+\sigma_{21}\cos^2\phi-\sigma_{12}\sin^2\phi-\sigma_{22}\cos\phi\sin\phi$$

This is equivalent to the **tensor transformation** $\sigma'_{qp}=\alpha_{pi}\alpha_{qj}\sigma_{ji}$

$\sigma'_{nn}=\alpha_{ni}\alpha_{nj}\sigma_{ji}$

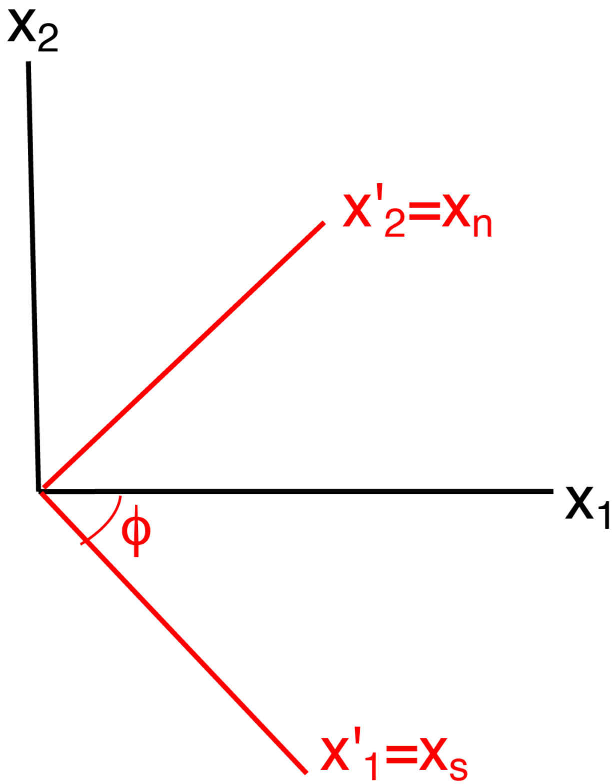
$\sigma'_{ns}=\alpha_{si}\alpha_{nj}\sigma_{ji}$

With $\alpha_{n1}=\sin\phi$, $\alpha_{n2}=\cos\phi$, $\alpha_{s1}=\cos\phi$, $\alpha_{s2}=-\sin\phi$

$\boxed{\begin{array}{l} x_1' = x_s \\ x_2' = x_n \end{array}}$

Write out transformation

$\sigma'_{qp} = \alpha_{pi}\alpha_{qj}\sigma_{ji}$



$$\alpha_{ij} = \hat{\mathbf{e}}'_i \cdot \hat{\mathbf{e}}_j$$

$$\alpha_{s1} = \hat{\mathbf{e}}_s \cdot \hat{\mathbf{e}}_1 = \cos\phi$$

$$\alpha_{s2} = \hat{\mathbf{e}}_s \cdot \hat{\mathbf{e}}_2 = -\sin\phi$$

$$\alpha_{n2} = \hat{\mathbf{e}}_n \cdot \hat{\mathbf{e}}_2 = \cos\phi$$

$$\alpha_{n1} = \hat{\mathbf{e}}_n \cdot \hat{\mathbf{e}}_1 = \sin\phi$$

In tensor notation:

$$\boldsymbol{\sigma}'^{T} = \mathbf{A}\cdot\boldsymbol{\sigma}^{T}\cdot\mathbf{A}^{T}$$

In matrix notation:

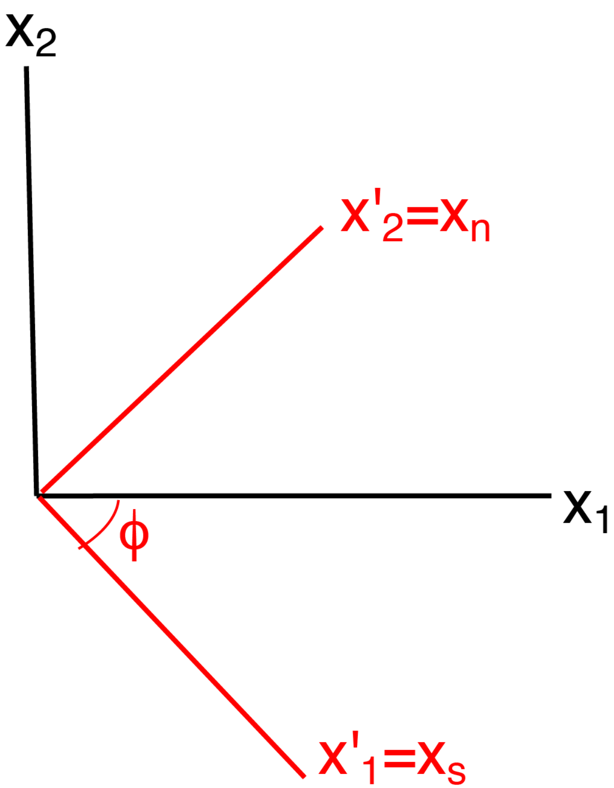
$$\begin{bmatrix} \sigma_{ss} & \sigma_{ns} \\ \sigma_{sn} & \sigma_{nn} \end{bmatrix} = \begin{bmatrix} \quad & \quad \\ \quad & \quad \end{bmatrix} \begin{bmatrix} \sigma_{11} & \sigma_{21} \\ \sigma_{12} & \sigma_{22} \end{bmatrix} \begin{bmatrix} \quad & \quad \\ \quad & \quad \end{bmatrix}$$

Write out matrices A and A^T

Check that the expressions for σ_{nn}, σ_{ns} of previous slide obtained

Write out transformation

$\sigma'_{qp}=\alpha_{pi}\alpha_{qj}\sigma_{ji}$



$$\alpha_{ij}=\hat{\mathbf{e}}'_i\cdot\hat{\mathbf{e}}_j$$

$$\alpha_{s1}=\hat{\mathbf{e}}_s\cdot\hat{\mathbf{e}}_1=\cos\phi$$

$$\alpha_{s2}=\hat{\mathbf{e}}_s\cdot\hat{\mathbf{e}}_2=-\sin\phi$$

$$\alpha_{n2}=\hat{\mathbf{e}}_n\cdot\hat{\mathbf{e}}_2=\cos\phi$$

$$\alpha_{n1}=\hat{\mathbf{e}}_n\cdot\hat{\mathbf{e}}_1=\sin\phi$$

In tensor notation:

$\boldsymbol{\sigma}'^{T}=\mathbf{A}\cdot\boldsymbol{\sigma}^{T}\cdot\mathbf{A}^{T}$

In matrix notation:

$$\begin{bmatrix}\sigma_{ss} & \sigma_{ns}\\ \sigma_{sn} & \sigma_{nn}\end{bmatrix}=\begin{bmatrix}\alpha_{s1} & \alpha_{s2}\\ \alpha_{n1} & \alpha_{n2}\end{bmatrix}\begin{bmatrix}\sigma_{11} & \sigma_{21}\\ \sigma_{12} & \sigma_{22}\end{bmatrix}\begin{bmatrix}\alpha_{s1} & \alpha_{n1}\\ \alpha_{s2} & \alpha_{n2}\end{bmatrix}$$

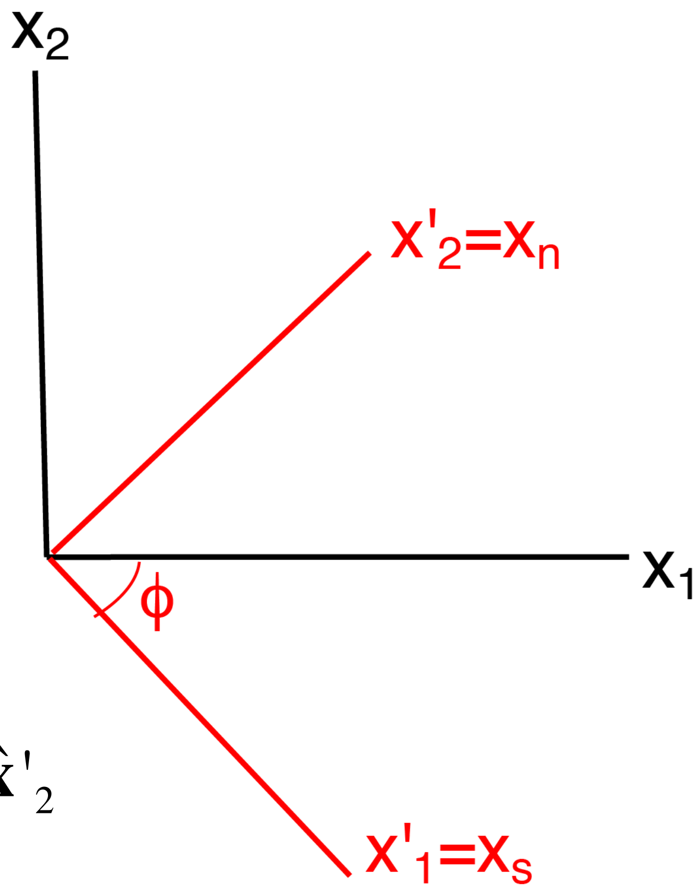
$$\begin{bmatrix}\sigma_{ss} & \sigma_{ns}\\ \sigma_{sn} & \sigma_{nn}\end{bmatrix}=\begin{bmatrix}\cos\phi & -\sin\phi\\ \sin\phi & \cos\phi\end{bmatrix}\begin{bmatrix}\sigma_{11} & \sigma_{21}\\ \sigma_{12} & \sigma_{22}\end{bmatrix}\begin{bmatrix}\cos\phi & \sin\phi\\ -\sin\phi & \cos\phi\end{bmatrix}$$

For $\hat{\mathbf{x}}_1=(1,0)$, $\hat{\mathbf{x}}_2=(0,1)$,
first row of $\mathbf{A}$ consists of $\hat{\mathbf{x}}'_1$, second of $\hat{\mathbf{x}}'_2$

$$\hat{\mathbf{x}}'_1 = (\cos\phi, -\sin\phi)$$

$$\hat{\mathbf{x}}'_2 = (\sin\phi, \cos\phi)$$

$$\mathbf{A} = \begin{bmatrix} \mathbf{x}'_1 \cdot \mathbf{x}_1 & \mathbf{x}'_1 \cdot \mathbf{x}_2 \\ \mathbf{x}'_2 \cdot \mathbf{x}_1 & \mathbf{x}'_2 \cdot \mathbf{x}_2 \end{bmatrix} = \begin{bmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{bmatrix}$$



You may recognise $\mathbf{A}$ *as a matrix that describes a rigid-body rotation over and angle* $-\phi$

A^T *describes a rotation over angle* ϕ

First column of $\mathbf{A}^T$ consists of $\hat{\mathbf{x}}'_1$, second of $\hat{\mathbf{x}}'_2$

$$\mathbf{A}^T = \begin{bmatrix} \mathbf{x}_1 \cdot \mathbf{x}'_1 & \mathbf{x}_1 \cdot \mathbf{x}'_2 \\ \mathbf{x}_2 \cdot \mathbf{x}'_1 & \mathbf{x}_2 \cdot \mathbf{x}'_2 \end{bmatrix} = \begin{bmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{bmatrix}$$

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