Short solutions
Problem Set 5
ACSE-2 2020

(1) (a) $tr(\underline{\underline{S}}) = 12$

(b) $\underline{\underline{S}} : \underline{\underline{S}} = 281$

(c) $\underline{\underline{S}} : \underline{\underline{S}}^T = 240$

(d) $\vec{a} \cdot \underline{\underline{S}} = \begin{pmatrix} 31 \\ 25 \\ 30 \end{pmatrix}$

(e) $\underline{\underline{S}} \cdot \vec{a} = \begin{pmatrix} 18 \\ 15 \\ 34 \end{pmatrix}$

(f) $\underline{\underline{S}}^T \cdot \vec{a} = \begin{pmatrix} 31 \\ 25 \\ 30 \end{pmatrix} = \vec{a} \cdot \underline{\underline{S}}$

(g) $\vec{a}\vec{b} = \begin{bmatrix} 16 & 10 & -14 \\ -8 & -5 & 7 \\ 32 & 20 & -28 \end{bmatrix}$

(h) $\vec{b}\vec{a} = \begin{bmatrix} 16 & -8 & 32 \\ 10 & -5 & 20 \\ -14 & 7 & -28 \end{bmatrix} = (\vec{a}\vec{b})^T$

(2) orthonormal base transformation

check $\hat{e}'_i \cdot \hat{e}'_j = \delta_{ij}$

transformation matrix $L = \frac{1}{3\sqrt{2}} \begin{bmatrix} 2\sqrt{2} & 2\sqrt{2} & \sqrt{2} \\ 3 & -3 & 0 \\ 1 & 1 & -4 \end{bmatrix}$

(3) (a) simplify $\delta_{ij}\delta_{jk}\delta_{kp}\delta_{pi}$

$\delta_{kp}\delta_{pi} \xrightarrow[\text{sum}]{} \delta_{ki}$

similarly for $\delta_{jk}\delta_{ki}$

$= \delta_{ij}\delta_{ji} = 3$

for dimension=3

(b) $\varepsilon_{ijk}\varepsilon_{lmk} = \delta_{il}\delta_{jm} - \delta_{lm}\delta_{jl}$

For any given i, j, l, m

$\varepsilon_{ijk}\varepsilon_{lmk} = \sum_{k=1}^{3} \varepsilon_{ijk}\varepsilon_{lmk}$

only one non-zero term if $i \neq j$ and either $i = l$ and $j = m \Rightarrow +1$ or $i = m$ and $j = l \Rightarrow -1$

then for one term $i \neq j \neq k$ other two will be $= 0$

The case $i = l, j = m \Rightarrow \delta_{il}\delta_{jm}$, case $i = m, j = l \Rightarrow -\delta_{im}\delta_{jl}$

(c) $\vec{a} \times (\vec{b} \times \vec{c}) = (\vec{a}\cdot\vec{c})\vec{b} - (\vec{a}\cdot\vec{b})\vec{c}$

call $\vec{v} = (\vec{b} \times \vec{c})$

④ (b)

can use similar derivation as 3c

$$\vec{a} \times \vec{v} = \varepsilon_{ijk}\, a_i\, v_j\, \hat{e}_k$$

$$v_j = \varepsilon_{lmj}\, b_l\, c_m$$

$$[\vec{a} \times (\vec{b} \times \vec{c})]_k = \varepsilon_{ijk}\,\varepsilon_{lmj}\, a_i\, b_l\, c_m$$

$$= \varepsilon_{kij}\,\varepsilon_{lmj}\, a_i\, b_l\, c_m$$

$\varepsilon_{ijk} = \varepsilon_{kij}$

use (b)

$$\rightarrow [\vec{a} \times (\vec{b} \times \vec{c})]_k = (\delta_{kl}\,\delta_{im} - \delta_{km}\,\delta_{il})\, a_i\, b_l\, c_m$$

$$= (a_i\, c_i\, b_k) - (a_l\, b_l\, c_k)$$

$$= (\vec{a}\cdot\vec{c})\, b_k - (\vec{a}\cdot\vec{b})\, c_k$$

(d) in similar way can use index notation to prove other triple product identities

④ (a) $\nabla \times (\nabla F) = 0$

- either write out all components + realise $\dfrac{\partial^2 F}{\partial x_i \partial x_j} = \dfrac{\partial^2 F}{\partial x_j \partial x_i}$ (*) $\Rightarrow$ all components $= 0$

- or use index notation

$$= \varepsilon_{ijk}\, \hat{e}_k\, \frac{\partial^2 F}{\partial x_i \partial x_j} \qquad (i)$$

$$= -\varepsilon_{jik}\, \hat{e}_k\, \frac{\partial^2 F}{\partial x_j \partial x_i} \qquad (ii)$$

however from symmetry (*) + renaming dummy indices $i \to j$, $j \to i$

$$\rightarrow \; = -\varepsilon_{ijk}\, \hat{e}_k\, \frac{\partial^2 F}{\partial x_i \partial x_j} \qquad (iii)$$

$\Rightarrow$ (i) = (iii) expression equals own negative $\Rightarrow = 0$