Fluid Flow (part 1)

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Outline of Lecture

- The Navier-Stokes equation
- Solid boundaries
- Fluid flow in simple geometries
  - Solved using Navier-Stokes equation
  - Soled using a momentum/force balance
- Fluid flow in a rheometer

Learning Objectives (Next 2 lectures)

- Learn how to calculate velocity profiles in simple geometries
  - Directly from the Navier-Stokes equation
  - …or using a force/momentum balance
- Understand the mechanisms behind the onset of turbulence
  - Pipe flow
  - Modelling turbulence in CFD
- Understanding non-Newtonian rheologies
  - Solve non-Newtonian flows in simple geometries

The Navier-Stokes Equation

- Momentum balance:

$$\rho \frac{D\mathbf{u}}{Dt} = -\nabla P + \nabla \cdot \boldsymbol{\tau} + \rho\, \mathbf{g}$$

- Mass balance (continuity equation):

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0$$

Material derivative:

$$\frac{D\mathbf{u}}{Dt} = \frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u}$$

or equivalently

$$\frac{D\mathbf{u}}{Dt} = \frac{\partial \mathbf{u}}{\partial t} + \nabla \cdot \mathbf{u}\mathbf{u}$$

Assumptions in this lecture

- We will only be considering incompressible flow – constant density:

$$\nabla \cdot \mathbf{u} = 0$$

- We will start by considering Newtonian Flow:
  - Shear stress is proportional to the strain rate

$$\boldsymbol{\tau} = 2\mu\,\mathbf{S} \qquad\qquad \mathbf{S} = \frac{1}{2}(\nabla \boldsymbol{u} + \nabla \boldsymbol{u}^T)$$

- This assumes viscosity is isotropic and also ignores the bulk viscosity, which is associated with changes in the volume/density of the fluid (we are making an incompressible assumption here).
  - Bulk viscosity can be important in, for instance shocks

Incompressible Newtonian Fluid

- In an incompressible Newtonian fluid the gradient of viscous stress can be further simplified:

$$\nabla \cdot \boldsymbol{\tau} = \mu \, \nabla \cdot (\nabla \boldsymbol{u} + \nabla \boldsymbol{u}^T) = \mu \nabla^{\mathbf{2}} \boldsymbol{u}$$

- This requires the implication of the incompressible assumption:

$$\nabla \cdot \mathbf{u} = 0$$

- Navier-Stokes equation for incompressible Newtonian fluids:

$$\rho \frac{D\mathbf{u}}{Dt} = -\nabla P + \mu \nabla^{\mathbf{2}} \boldsymbol{u} + \rho \, \mathbf{g}$$

Exercise

- Given $\nabla \cdot \boldsymbol{\tau} = \mu \, \nabla \cdot (\nabla \boldsymbol{u} + \nabla \boldsymbol{u}^T)$ and $\nabla \cdot \mathbf{u} = 0$
- Show that

$$\nabla \cdot \boldsymbol{\tau} = \mu \nabla^{2} \boldsymbol{u}$$

Calculations

Side note: Dynamic and Kinematic Viscosity

Be careful when talking about viscosity as to what is meant

Dynamic viscosity (sometimes called absolute viscosity)

- μ (units: Pa.s)
- Proportionality between shear stress and strain rate

Kinematic viscosity

- $\upsilon = \frac{\mu}{\rho}$ (units: m^2/s)
- Equivalent to a diffusivity for momentum

Solid Boundaries in Fluid Dynamics

No Slip Boundary Condition

- A useful assumption for most solid boundaries
  - Together with no flux through the boundary this implies a zero velocity vector at the boundary
- Not completely true at the very smallest scales
  - Some molecular slip may need to be included for micro or nano scale fluid flows
  - Good approximation for macroscopic flows

- We will be using no slip conditions in all the examples in this lecture

Solid Boundaries in Fluid Dynamics

Velocity Boundary Layers

- Some flows will have velocity boundary layers against solid walls
  - No slip at the wall, but rapid increase towards free stream velocity away from the wall
  - Boundary layers often thin compared to the scale of the system
- Especially true for turbulent flows
  - Can also occur in entry regions for laminar flows
  - More on turbulent flow in the next lecture

Solid Boundaries in Fluid Dynamics

Modelling Systems with Boundary Layers

- Could simply have enough resolution near the wall to resolve the wall
  - Can be computationally very expensive especially in large scale simulations
- Simplest approximation is that there is full slip at the boundary
  - Not a bad approximation in very turbulent flows, but no stresses on the wall
- Can use a sub-resolution model to impose a stress at the boundary that is a function of the velocity at the boundary
  - More complex models will include predictions for the velocity profile in this boundary region and a more complex inter-relationship between the stress at the wall and the velocity profile in the resolved near wall region

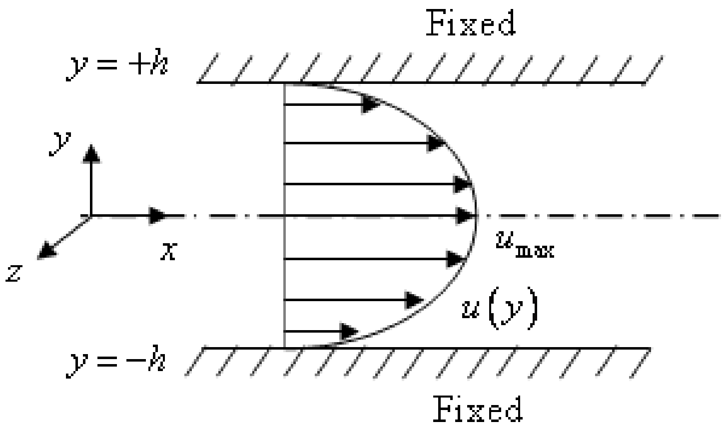
Steady Flows in Simple Geometries

- The Navier-Stokes equation is non-linear and can have complex interactions between mass and momentum conservation, especially in transient flows
  - Most fluid flow problems don’t have analytical solutions
  - This is why you will be learning how to solve problems like this numerically
- It is useful, though, to have some analytical solutions
  - Can give insights into the behaviour of more complex systems
  - Useful for validating numerical codes
- Typically only possible for steady flows in very simple geometries
  - How to solve for these types of flows is what the rest of this lecture is about

Pressure Driven Steady Flow between Parallel Plates

$$\rho \frac{D\mathbf{u}}{Dt} = -\nabla P + \mu \nabla^2 \boldsymbol{u} + \rho\, \mathbf{g}$$

- Can be derived easily either from the Navier-Stoke equation or by considering a force balance on a unit of fluid:
  - No time dependency
  - Only non-zero velocity components are in the x direction
  - Ignore body forces (e.g. gravity)



$$\frac{dP}{dx} = -\frac{\Delta P}{L} = \mu \frac{d^2 u_x}{dy^2}$$

- Integrate and remember no slip at boundaries ($u_x = 0$ at $y = h$ and $y = -h$):

$$u_x = \frac{\Delta P}{2\mu L}(h^2 - y^2)$$

Pressure Driven Steady Flow in a pipe

- Similar derivation as before, but done in cylindrical coordinates:
  - Navier Stokes momentum equation in cylindrical coordinates:
- r direction:

$$\rho\left(\frac{\partial u_r}{\partial t}+u_r\frac{\partial u_r}{\partial r}+\frac{u_\theta}{r}\frac{\partial u_r}{\partial \theta}+u_z\frac{\partial u_r}{\partial z}\right)=$$

$$-\frac{\partial P}{\partial r}+\mu\left[\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}(ru_r)\right)+\frac{1}{r^2}\frac{\partial^2 u_r}{\partial \theta^2}-\frac{2}{r^2}\frac{\partial u_\theta}{\partial \theta}+\frac{\partial^2 u_r}{\partial z^2}\right]+\rho g_r$$

- θ direction:

$$\rho\left(\frac{\partial u_\theta}{\partial t}+u_r\frac{\partial u_\theta}{\partial r}+\frac{u_\theta}{r}\frac{\partial u_\theta}{\partial \theta}+\frac{u_\theta u_r}{r}+u_z\frac{\partial u_\theta}{\partial z}\right)=$$

$$-\frac{1}{r}\frac{\partial P}{\partial \theta}+\mu\left[\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}(ru_\theta)\right)+\frac{1}{r^2}\frac{\partial^2 u_\theta}{\partial \theta^2}+\frac{2}{r^2}\frac{\partial u_r}{\partial \theta}+\frac{\partial^2 u_\theta}{\partial z^2}\right]+\rho g_\theta$$

- z direction:

$$\rho\left(\frac{\partial u_z}{\partial t}+u_r\frac{\partial u_z}{\partial r}+\frac{u_\theta}{r}\frac{\partial u_z}{\partial \theta}+u_z\frac{\partial u_z}{\partial z}\right)=$$

$$-\frac{\partial P}{\partial z}+\mu\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u_z}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2 u_r}{\partial \theta^2}+\frac{\partial^2 u_r}{\partial z^2}\right]+\rho g_z$$

Pressure Driven Steady Flow in a pipe

- Need only consider momentum balance in z direction
  - z direction only non-zero velocity and no time dependency:

$$\frac{dP}{dz} = -\frac{\Delta P}{L} = \mu \frac{1}{r} \frac{d}{dr}\left(r \frac{du_z}{dr}\right)$$

- Integrate once:

$$\frac{du_z}{dr} = -\frac{\Delta P}{2 \mu L} r$$

Constant of integration is zero as symmetry means that there is a zero velocity gradient at the centre of the pipe

- Integrate again and note that velocity is zero at the wall ($r = R$):

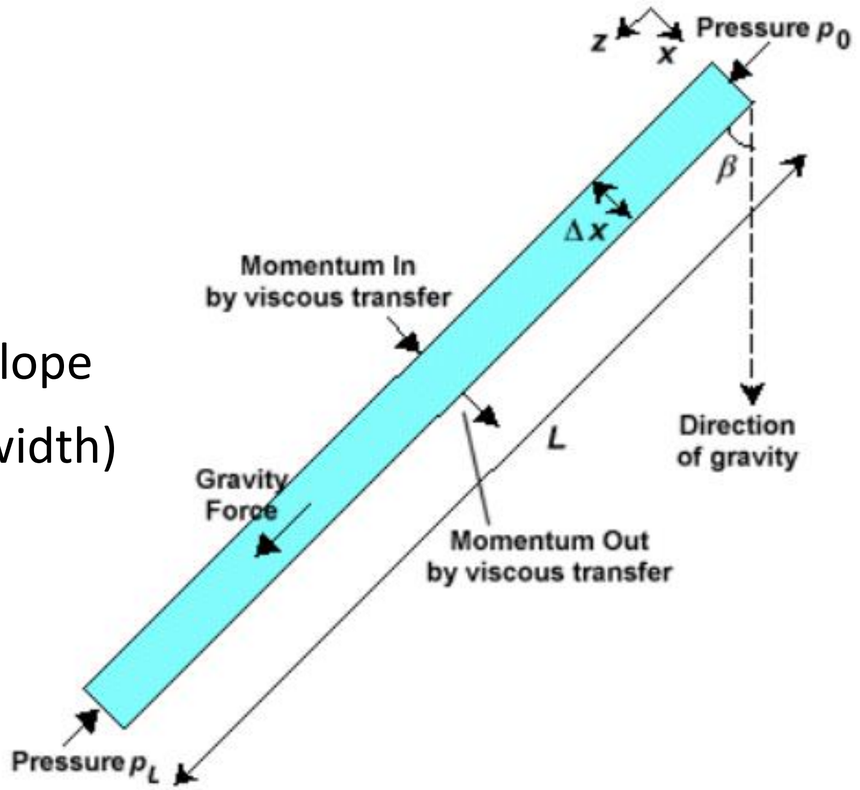
$$u_z = \frac{\Delta P}{4 \mu L} (R^2 - r^2)$$

Calculating flow and pressure

- For simple geometries either simplify the Navier-Stokes equation or do your own momentum balance

- Free surface flow on an inclined plane using a momentum balance
  - For a given slope angle, volumetric fluid flow-rate, fluid density and fluid viscosity, how thick will the fluid layer be (assuming a large enough plane that the thickness is constant)?

Calculations:



- Assume x is distance from free surface and z is distance down slope
- Force balance on control in direction parallel to slope (for unit width)

$$p_0 \Delta x - p_L \Delta x + \rho g \cos(\beta)\, \Delta x\, L - L\, \tau_{xz}(x) + L\, \tau_{xz}(x + \Delta x) = 0$$

Gravity force down the slope

Pressure force on ends of control volume

Shear force on sides of control volume (note that τ_{xz} refers to the stress on the x direction face acting in the z direction)

- As the system is at steady state, there are no viscous stresses exerting a force in the direction normal to the slope
  - Pressure with depth below free surface purely hydrostatic

    – no pressure gradient down the slope ($p_0 = p_L$)
- Rearranging the equation:

$$-\rho g \cos(\beta) = \frac{\tau_{xz}(x + \Delta x) - \tau_{xz}(x)}{\Delta x}$$

Taking the limit as Δx goes to zero:

$$\frac{d\tau_{xz}}{dx} = -\rho g \cos(\beta)$$

Calculations continued:

$$\frac{d\tau_{xz}}{dx} = \rho g \cos(\beta)$$

- Integrate to get shear stress as a function of the depth below the surface:

$$\tau_{xz} = -\rho g \cos(\beta)\, x + A$$

- As the air above the free surface is much lower viscosity than the liquid we can assume that it exerts negligible viscous stress:

$$\tau_{xz} = 0 \text{ at } x = 0 \text{ therefore } A = 0$$

- If we assume a Newtonian fluid then the shear stress is proportional to the strain rate, with viscosity being the proportionality:
    - Note that the sign depends on whether you consider the stress to be the force exerted by the fluid or on the fluid – Either can be used, but it must be used consistently

$$\tau_{xz} = \mu \frac{dv_z}{dx}$$

- Substituting and integrating again yields:

$$v_z = -\frac{\rho g \cos(\beta)}{2\mu} x^2 + B$$

Calculations continued:

$$v_z = -\frac{\rho g \cos(\beta)}{2\mu} x^2 + B$$

- Constant of integration obtain from bottom boundary condition:
- If h is the thickness of the slide, then $v_z = 0$ at $x = h$ (noting that we have defined x as being the distance below the free surface)

$$B = \frac{\rho g \cos(\beta)}{2\mu} h^2$$

- The velocity profile is therefore parabolic with the following shape

$$v_z = \frac{\rho g \cos(\beta)}{2\mu} (h^2 - x^2)$$

- Note that this is the same velocity profile as would be obtained for similar flow between 2 parallel plates $2\ h$ apart
  - Planes of symmetry and free surfaces are physically very similar as they involve no shear stress

Example – A slightly more complex flow

- Two fluids are flowing between parallel horizontal plates
  - Assume steady flow with no instabilities
- The top fluid is less dense and has a lower viscosity
  - μ_u, μ_l and ρ_u, ρ_l are the viscosity and density of the upper and lower fluids
- Each fluid occupies half of the vertical gap between the plates
- If the fluids are driven be a pressure gradient of $\frac{\Delta P}{L}$, what is the velocity profile between the plates as a function of the viscosity of the two fluids?
- Note:
  - There can be assumed to be no slip at the walls
  - Where the two fluids meet the velocity in each fluid will be the same.
  - The shear stress will be continuous over the interface, but there will be a discontinuity in the velocity gradient and the flow won’t be symmetric about the centre line

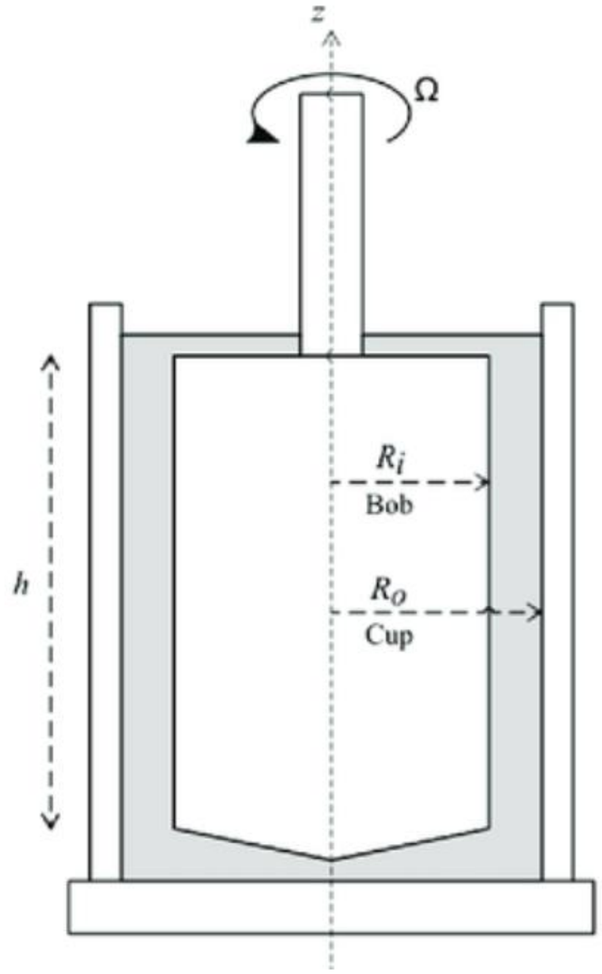
Calculations

- Do this example using a force/momentum balance

Calculating flows in circular geometries

- A common way of measuring the rheology of a fluid is using a Couette-flow rheometer
  - Spinning inner cylinder
  - Fixed outer wall
  - Measure the torque required to spin the cylinder at a given speed

- We can do the calculations for this relationship if we assume:
  - Virtually all viscous losses occur in the gap on the sides
  - There are negligible end effects
  - There is a uniform and steady velocity distribution in the gap



Calculate the velocity distribution in the gap

- Given the angular velocity of the cylinder, Ω, the radius of the cup, r_O and the radius of the bob, r_O, calculate the velocity profile as a function of radius
  - Assume no slip at the walls

- Either use a force balance on a cylindrical region of the fluid or use the Navier-Stokes equation in cylindrical coordinates

Calculations

Torque on the cylinder

- If the height of the cylinder is h, what is the relationship between the torque, the angular velocity and the viscosity of the fluid?
  - Note that this relationship will depend strongly on the radius of the inner and outer cylinders
- How does this relationship simplify if the gap between the two cylinders is small compared to their radii?

Calculations

These simple balances are not the end of the story even for flows in simple geometries – Why?

- An explicit assumption in these derivations is that the flow is steady (no time dependency)
- What happens if there is a perturbation in the flow?
  - If this perturbation grows, it means that the flow can not be steady and that this assumption is invalid
- Perturbation will be damped if viscous forces are stronger than the inertial force associated with the perturbation

How does this effect the modelling and simulation of these systems?

- Tomorrow' lecture topic