

Vector transformation

- Vector magnitude and direction do not depend on basis
- When defined on orthonormal basis, like rectangular Cartesian, the transformation to other orthonormal bases is simple, with real coefficients.

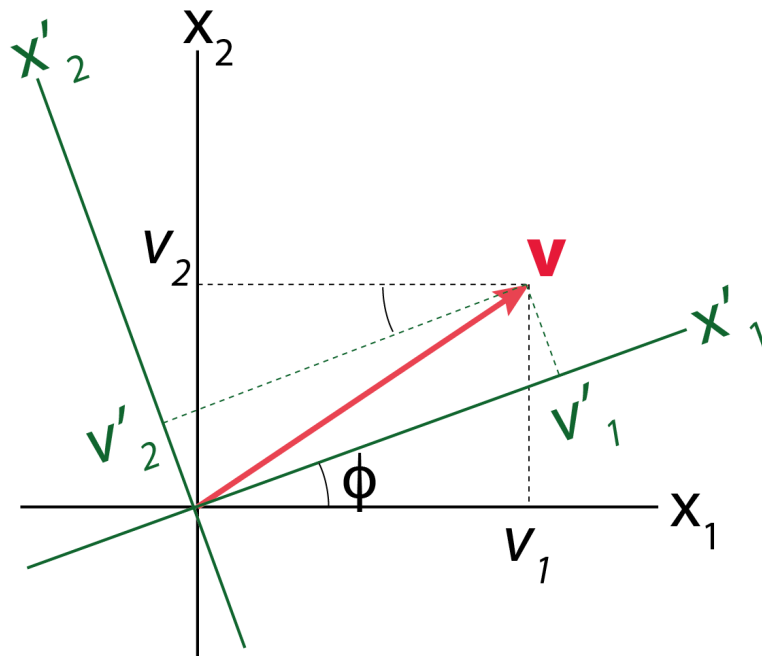
$$\hat{\mathbf{e}}_i \cdot \hat{\mathbf{e}}_j = 0 \quad \text{if } i \neq j$$

$$\hat{\mathbf{e}}_i \cdot \hat{\mathbf{e}}_i = |\hat{\mathbf{e}}_i|^2 = 1$$

check out Khan Academy lectures on orthonormal bases

Vector transformation

physical parameters should not depend on coordinate frame



for vectors on **orthonormal** basis,
transformed vector \mathbf{v}' depends on \mathbf{v} :

$$v'_1 = \alpha_{11}v_1 + \alpha_{12}v_2$$

$$v'_2 = \alpha_{21}v_1 + \alpha_{22}v_2$$

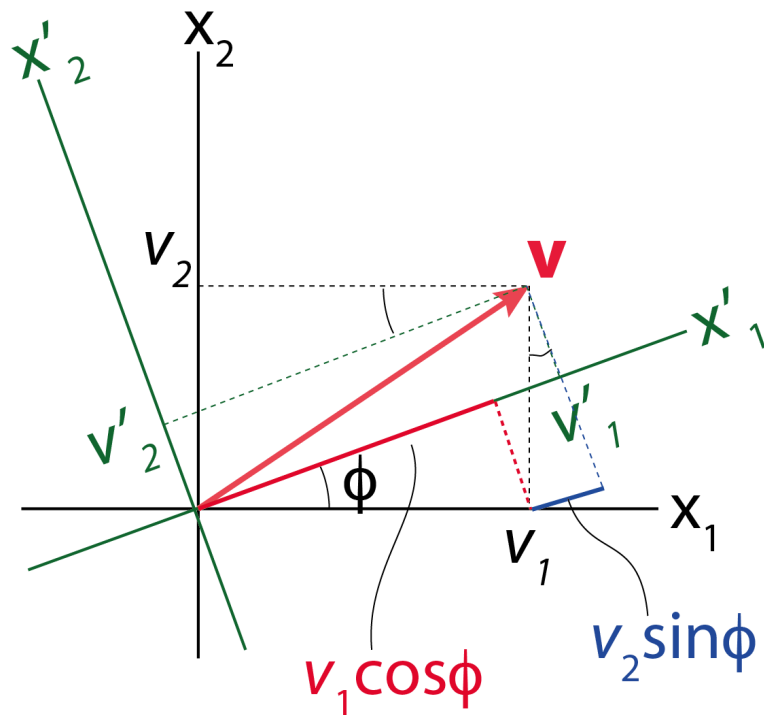
$$v'_1 = \cos\phi v_1 + \sin\phi v_2$$

$$v'_2 = -\sin\phi v_1 + \cos\phi v_2$$

coefficients α_{ij} depend on angle ϕ between x_1 and x'_1 (or x_2 and x'_2)

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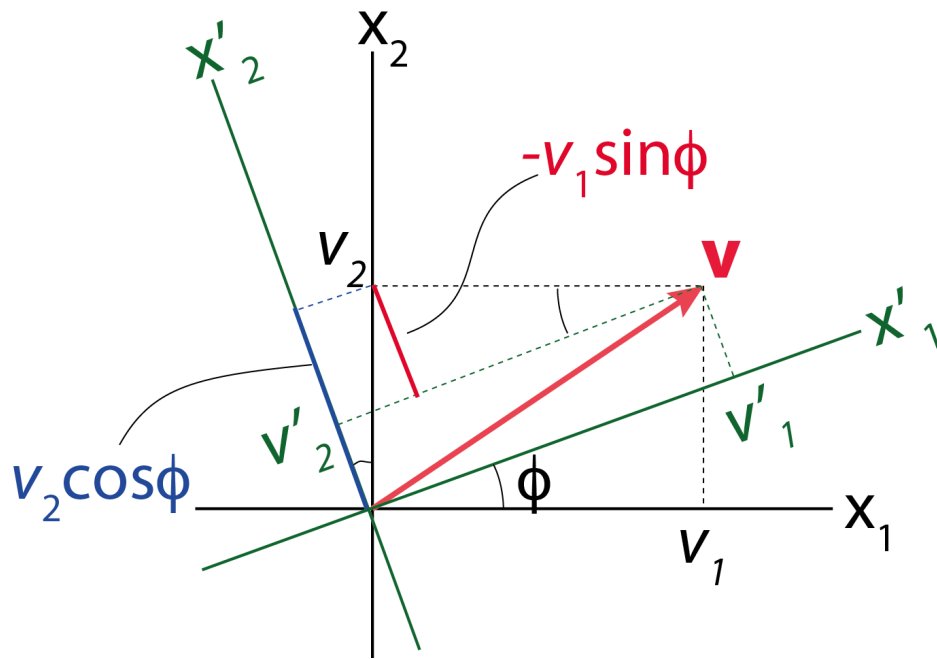
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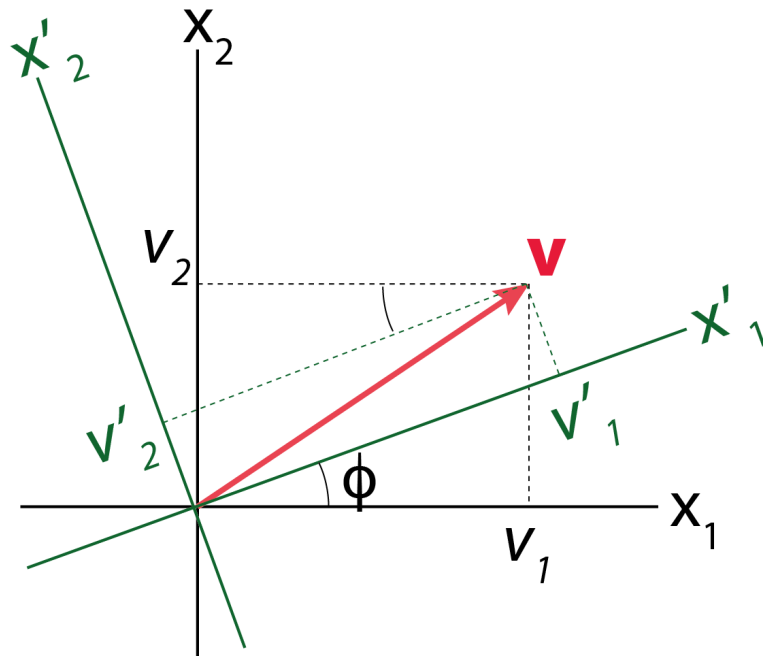
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$$v'_1 = \alpha_{11}v_1 + \alpha_{12}v_2$$

$$v'_2 = \alpha_{21}v_1 + \alpha_{22}v_2$$

$$\Rightarrow \mathbf{v}' = \begin{bmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{bmatrix} \mathbf{v}$$

coefficients α_{ij} depend on angle ϕ between x_1 and x'_1 (or x_2 and x'_2)

$$\mathbf{v}' = \begin{bmatrix} \cos \phi & \sin \phi \\ -\sin \phi & \cos \phi \end{bmatrix} \mathbf{v}$$

$$\alpha_{11} = \hat{\mathbf{e}}'_1 \cdot \hat{\mathbf{e}}_1 \quad \alpha_{12} = \hat{\mathbf{e}}'_1 \cdot \hat{\mathbf{e}}_2$$

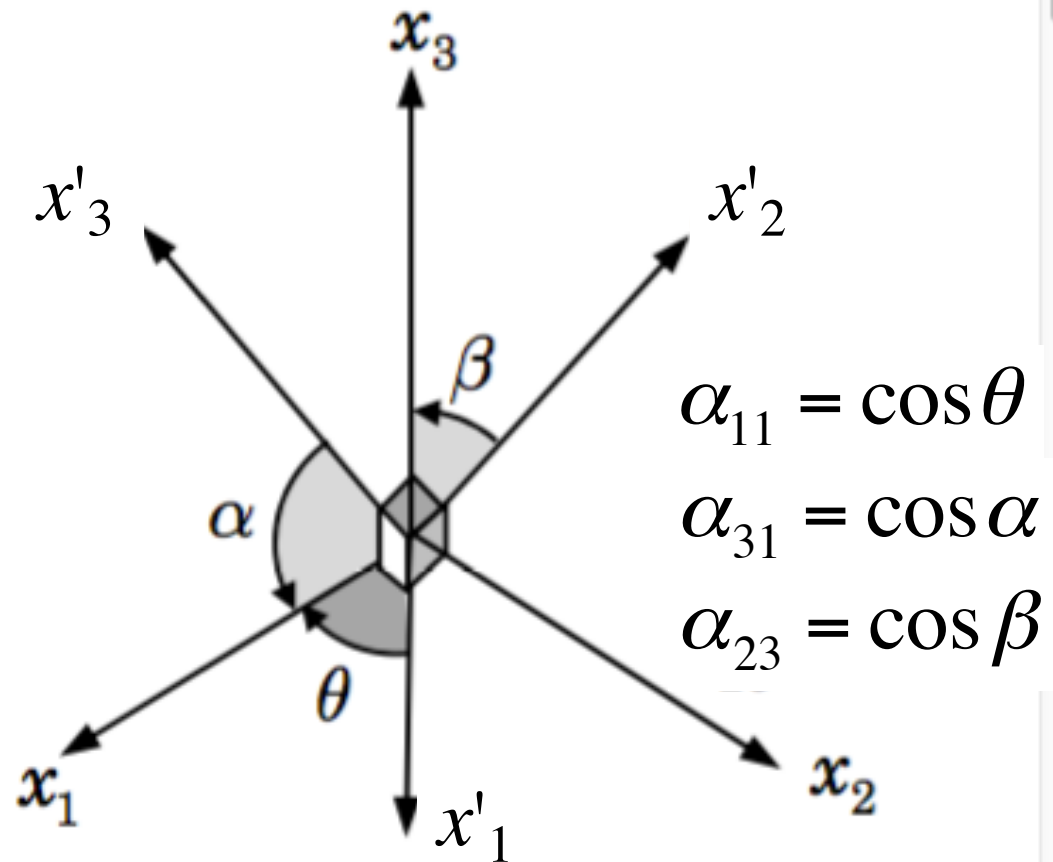
$$\alpha_{21} = \hat{\mathbf{e}}'_2 \cdot \hat{\mathbf{e}}_1 \quad \alpha_{22} = \hat{\mathbf{e}}'_2 \cdot \hat{\mathbf{e}}_2$$

Transformation orthonormal bases

$$\hat{\mathbf{e}}'_i = \sum_{j=1,n} \alpha_{ij} \hat{\mathbf{e}}_j$$

In other words:

$$\alpha_{ij} = \hat{\mathbf{e}}'_i \cdot \hat{\mathbf{e}}_j$$

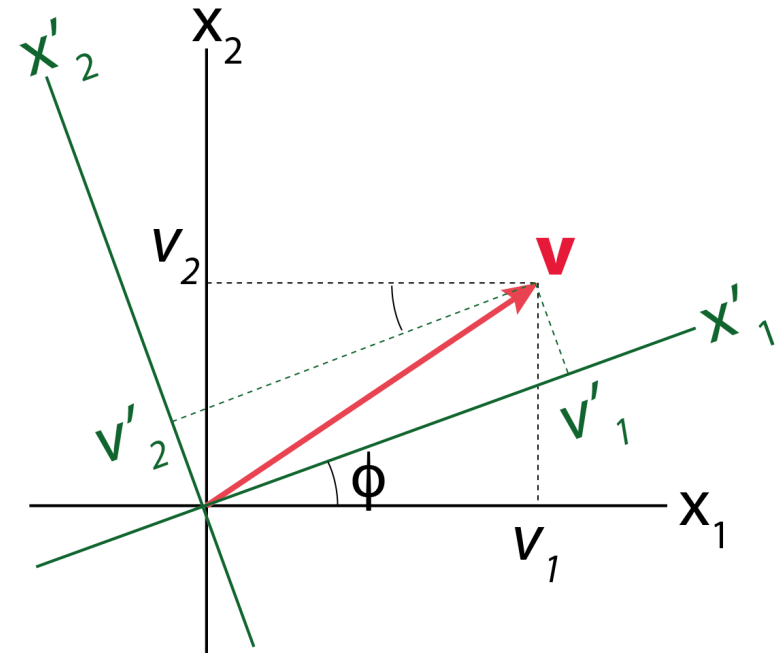


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angle x'_1 and x_2

$$\mathbf{v}' = \begin{bmatrix} \cos \phi & \sin \phi \\ -\sin \phi & \cos \phi \end{bmatrix} \mathbf{v} \quad \mathbf{v}' = \begin{bmatrix} \cos \phi & \cos(90 - \phi) \\ \cos(90 + \phi) & \cos \phi \end{bmatrix} \mathbf{v}$$

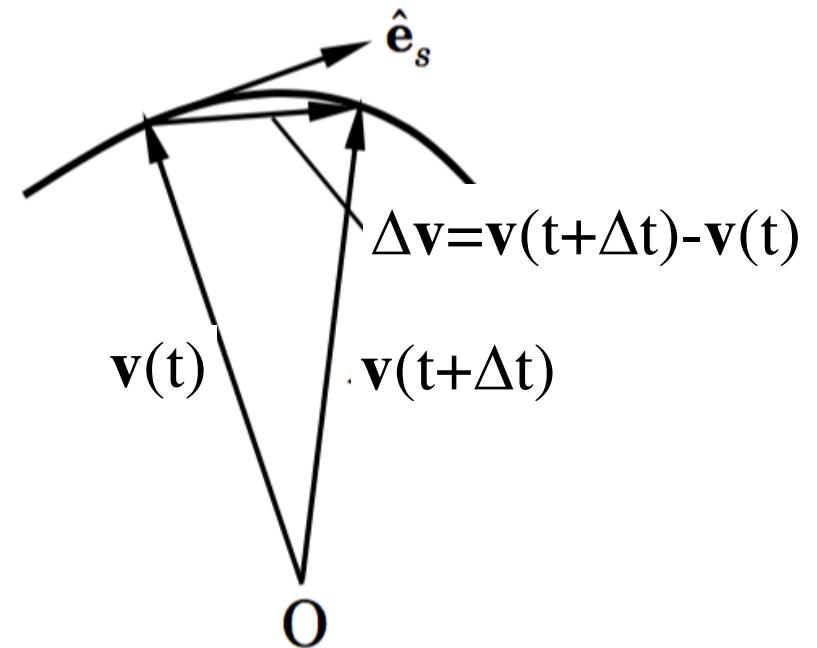
angle x'_2 and x_1 angle x'_1 and x_2

Vector derivatives

Scalar: e.g. time

$$\frac{d\mathbf{v}}{dt} = \begin{pmatrix} \frac{dv_1}{dt} & \frac{dv_2}{dt} & \frac{dv_3}{dt} \end{pmatrix}$$

$$\frac{d\mathbf{v}}{dt} = \lim_{\Delta t \rightarrow 0} \frac{\mathbf{v}(t + \Delta t) - \mathbf{v}(t)}{\Delta t}$$



usually has a different direction than \mathbf{v}

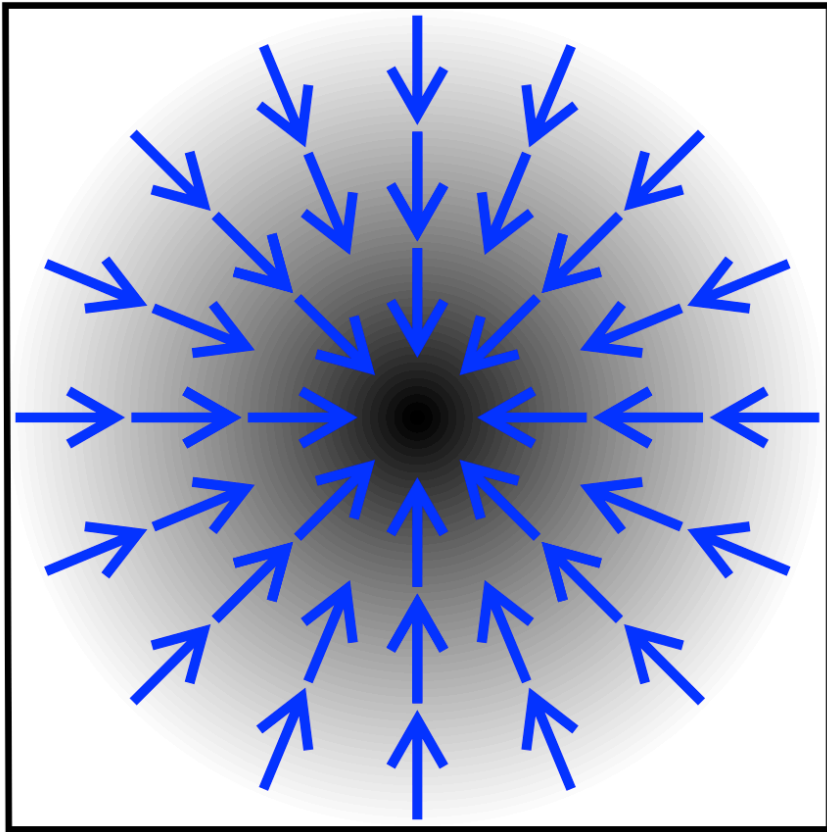
remember: $\mathbf{v} = v_1 \hat{\mathbf{e}}_1 + v_2 \hat{\mathbf{e}}_2 + v_3 \hat{\mathbf{e}}_3$

$$\frac{d\mathbf{v}}{dt} = \frac{dv_1}{dt} \hat{\mathbf{e}}_1 + \frac{dv_2}{dt} \hat{\mathbf{e}}_2 + \frac{dv_3}{dt} \hat{\mathbf{e}}_3 + v_1 \frac{d\hat{\mathbf{e}}_1}{dt} + v_2 \frac{d\hat{\mathbf{e}}_2}{dt} + v_3 \frac{d\hat{\mathbf{e}}_3}{dt}$$

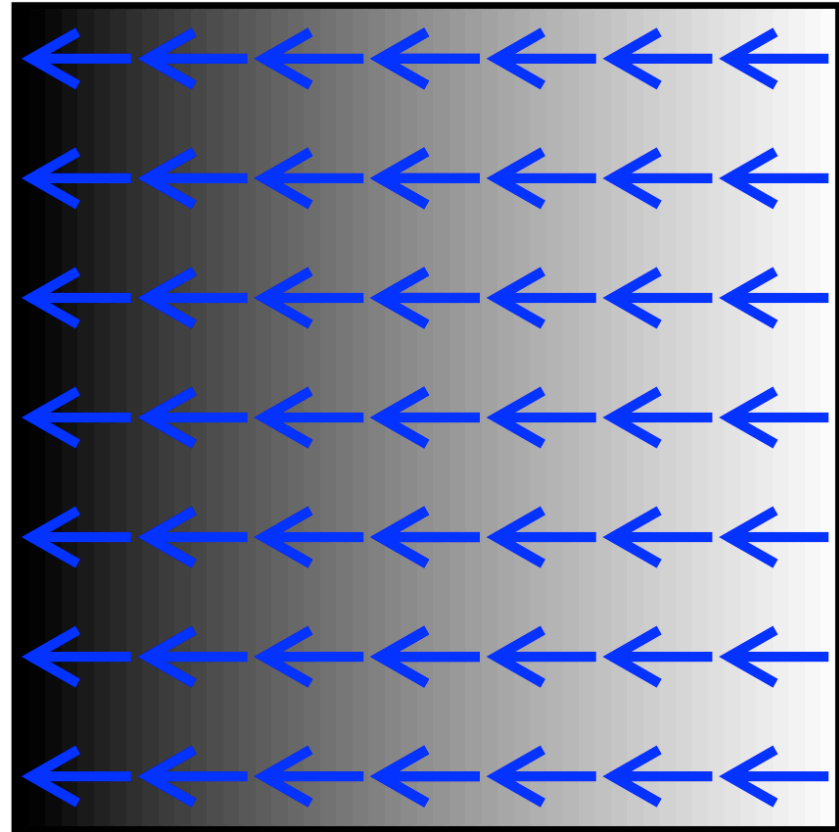
for Cartesian systems $= 0$

Vector derivatives

directional derivative: space



ϕ - high



ϕ - low

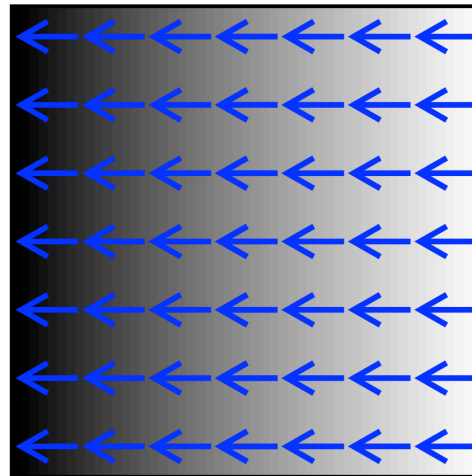
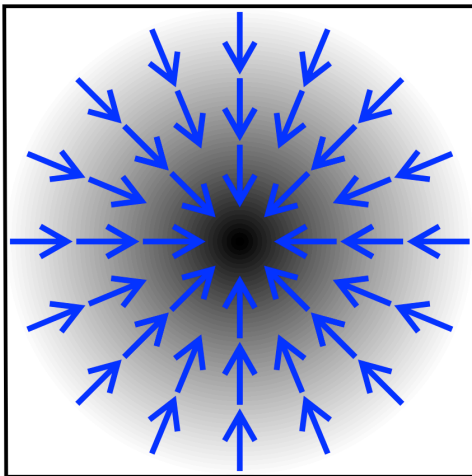
Vector derivatives

directional derivative: space

∂ - partial
derivative

$$d\phi = \frac{\partial \phi}{\partial x_1} dx_1 + \frac{\partial \phi}{\partial x_2} dx_2 + \frac{\partial \phi}{\partial x_3} dx_3$$

$$d\phi = \begin{pmatrix} \frac{\partial \phi}{\partial x_1} & \frac{\partial \phi}{\partial x_2} & \frac{\partial \phi}{\partial x_3} \end{pmatrix} \cdot \begin{pmatrix} dx_1 \\ dx_2 \\ dx_3 \end{pmatrix} = \nabla \phi \cdot \mathbf{dx}$$



Gradient $\nabla \phi$:
vector that is a
measure of change
in scalar field with
direction

Try yourself later

- *Define a scalar field, e.g. $\phi(x,y)=x+2y$*
- *Determine values of ϕ on a regular grid*
- *Make a contour plot of the field*
- *Numerically determine the gradient $\nabla\phi$*
- *Plot the gradient components as contours and compare with the analytical solution.*
- *Plot the gradient as vectors*

Del operator

$$\nabla = \hat{\mathbf{e}}_1 \frac{\partial}{\partial x_1} + \hat{\mathbf{e}}_2 \frac{\partial}{\partial x_2} + \hat{\mathbf{e}}_3 \frac{\partial}{\partial x_3} = \begin{pmatrix} \frac{\partial}{\partial x_1} & \frac{\partial}{\partial x_2} & \frac{\partial}{\partial x_3} \end{pmatrix}$$

Has some properties of a vector, but not all

$$\mathbf{v} \cdot \nabla \phi \neq (\nabla \cdot \mathbf{v}) \phi$$

Vector products with derivatives

divergence, curl

- Divergence of a vector: $\nabla \cdot \mathbf{v} = \sum_{i=1,3} \frac{\partial v_i}{\partial x_i} = \frac{\partial v_1}{\partial x_1} + \frac{\partial v_2}{\partial x_2} + \frac{\partial v_3}{\partial x_3}$

- Curl of a vector: $\nabla \times \mathbf{v} = \begin{pmatrix} \frac{\partial v_3}{\partial x_2} - \frac{\partial v_2}{\partial x_3} \\ \frac{\partial v_1}{\partial x_3} - \frac{\partial v_3}{\partial x_1} \\ \frac{\partial v_2}{\partial x_1} - \frac{\partial v_1}{\partial x_2} \end{pmatrix}$

Useful calculus theorems

Gauss or divergence theorem: $\int_V \nabla \cdot \mathbf{v} d\mathbf{x} = \oint_S \mathbf{v} \cdot \hat{\mathbf{n}} ds$

Stokes or curl theorem: $\int_V \nabla \times \mathbf{v} d\mathbf{x} = \oint_S \mathbf{v} \cdot \hat{\mathbf{t}} ds$

Take volume V enclosed by a closed surface S
within a vector field \mathbf{v} with continuous partial derivatives

Flow perpendicular to the boundary: $\mathbf{v} \cdot \hat{\mathbf{n}}$

Flow parallel to the boundary: $\mathbf{v} \cdot \hat{\mathbf{t}}$

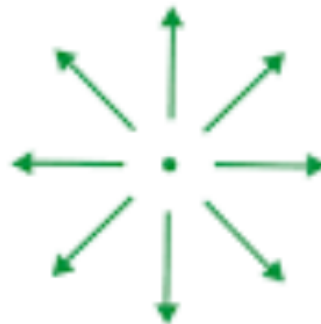
These can be used to simplify integration over volumes or closed surfaces as well as to gain understanding of the meaning of div and curl

Divergence of a vector field

$$\nabla \cdot \vec{v} < 0$$



$$\nabla \cdot \vec{v} > 0$$



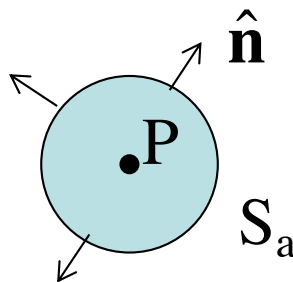
$$\nabla \cdot \vec{v} = 0$$



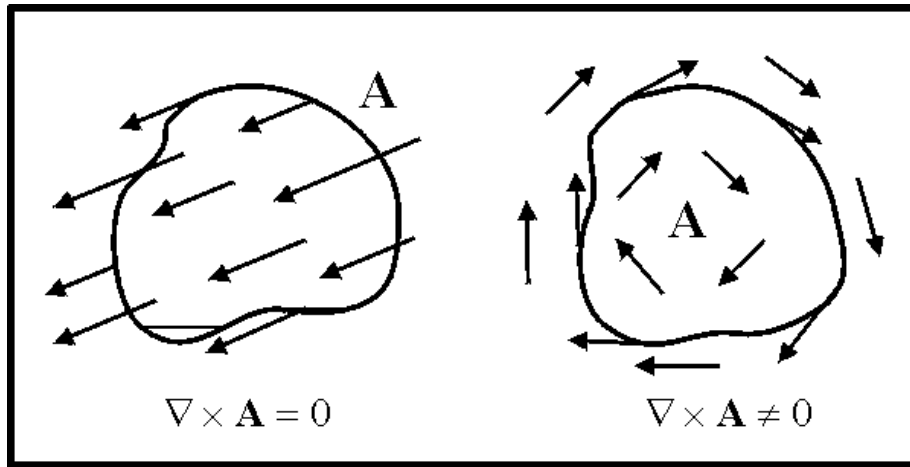
$$\int_V \nabla \cdot \mathbf{v} d\mathbf{x} = \oint_S \mathbf{v} \cdot \hat{\mathbf{n}} ds$$

Imagine a very small sphere,
radius a , with boundary S_a
around a point P

$$(\nabla \cdot \mathbf{v})_P \frac{4}{3} \pi a^3 = \oint_{S_a} \mathbf{v} \cdot \hat{\mathbf{n}} ds$$



Divergence of a
vector field
represents the net
outward flux per
unit volume, i.e.
measure of
source/sink of flow



Curl of a vector field

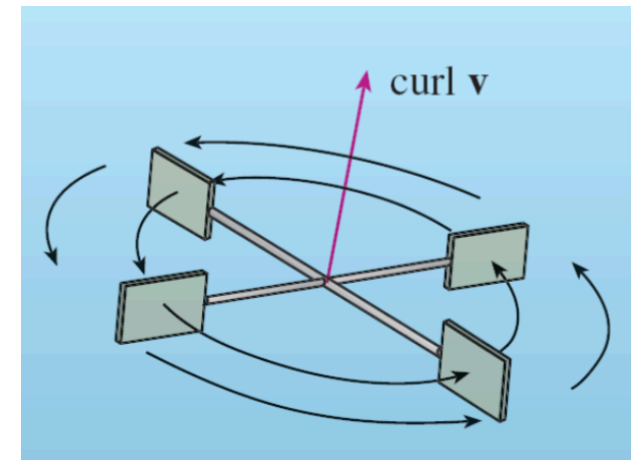
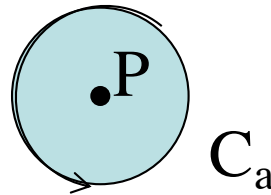
Amount of turn or spin, vorticity, in a vector field

Curl theorem:
$$\int_V \nabla \times \mathbf{v} d\mathbf{x} = \oint_S \mathbf{v} \cdot \hat{\mathbf{t}} ds$$

Right-hand side larger if velocities more parallel to the boundary, spinning in consistent direction, circulation around the boundary

Imagine a very small disk, radius a , with boundary C_a around a point P

$$(\nabla \times \mathbf{v})_P \pi a^2 = \oint_{C_a} \mathbf{v} \cdot \hat{\mathbf{t}} ds$$



Try yourself later

e.g. check out Khan Academy lectures on divergence, curl

- Take a 2D vector field, e.g. $\mathbf{v} = \begin{pmatrix} xy \\ y^2 - x^2 \end{pmatrix}$
- Plot the field as vectors on a regular grid
- Calculate $\nabla \cdot \mathbf{v}$ and plot as a contour plot
- Calculate $\nabla \times \mathbf{v}$ and plot as contour plot
- What do the different values of divergence and curl mean?

Covered so far

- Be able to perform vector/tensor operations (addition, multiplication) on Cartesian orthonormal bases
- Be able to do basic vector/tensor calculus (time and space derivatives, divergence, curl of a vector field) on these bases.
- Perform transformation of a vector from one to another Cartesian basis.
- Understand differences/commonalities tensor and vector
- Use index notation and Einstein convention
- Be able to use the special tensors δ_{ij} and ε_{ijk}