Learning Objectives

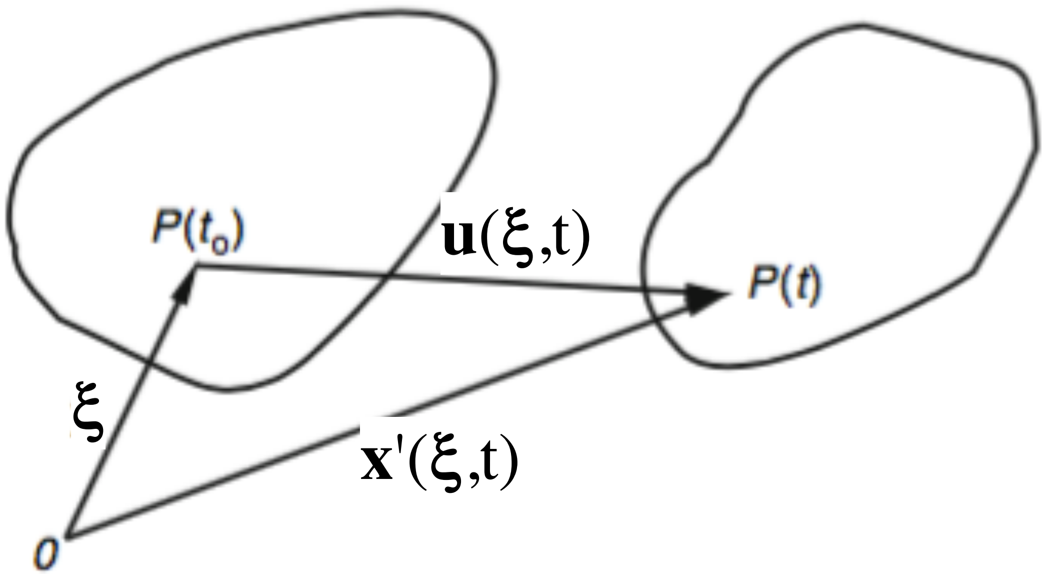
- Be able to use material and spatial descriptions of variables and their time derivatives.
- Be able to compute infinitesimal strain (strain rate) tensor given a displacement (velocity) field.
- Know meaning of the different components of the infinitesimal strain (rate) tensor
- Be able to find principal strain(rate)s and strain (rate) invariants and know what they represent
- Understand difference between infinitesimal and finite strain
- Be able to use the conservation of mass equation

Displacement

Can result in

(a) Rigid body motion

(b) Deformation of the body



Rigid body motion

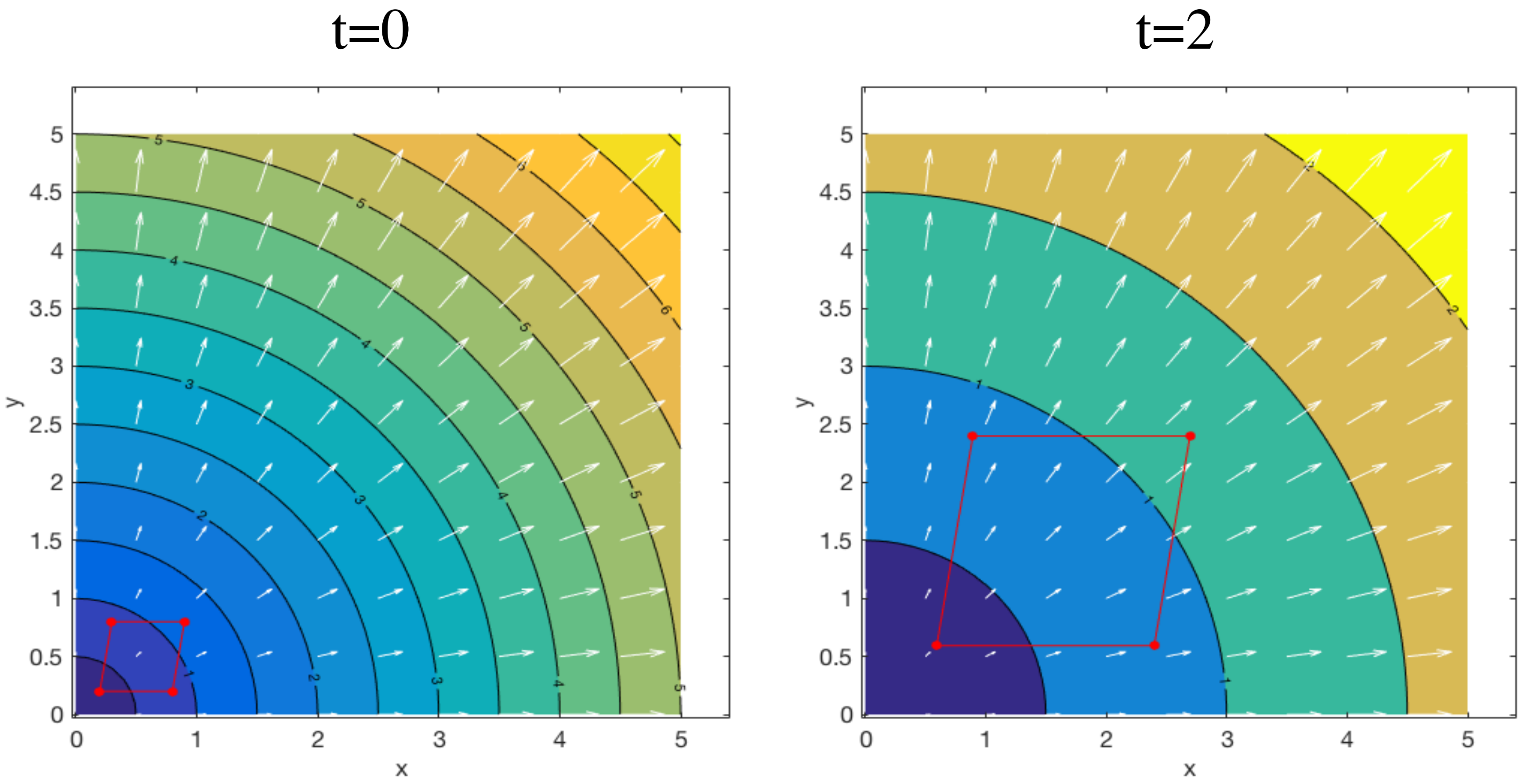
- Translation: $\mathbf{x}'=\boldsymbol{\xi}+\mathbf{c}(t)$, with $\mathbf{c}(0)=\mathbf{0}$

  $\Rightarrow$ $\mathbf{u}=\mathbf{x}'-\boldsymbol{\xi}$, each point same $\mathbf{u}(t)=\mathbf{c}(t)$

- Rotation: $\mathbf{x}'-\mathbf{b}=\mathbf{R}(t)(\boldsymbol{\xi}-\mathbf{b})$, where $\mathbf{R}(t)$ is rotation tensor, with $\mathbf{R}(0)=\mathbf{I}$, $\mathbf{b}$ is the point of rotation. $\mathbf{R}(t)$ is an orthogonal transformation (preserves lengths and angles, $\mathbf{R}^T\mathbf{R}=\mathbf{I}$, $\det(\mathbf{R})=1$)

If $\mathbf{u}$ depends on $\mathbf{x}$ and t, then internal deformation

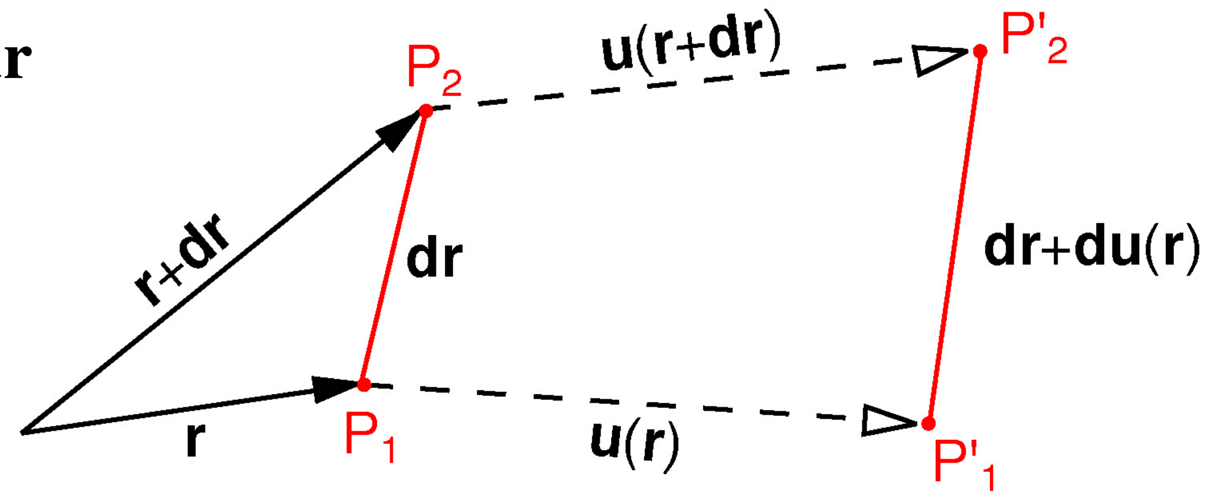
Displacement



translation &
deformation

Deformation tensor

For small **dr**



P_1 at $\mathbf{r}$ -> P'_1 at $\mathbf{r}+\mathbf{u}(\mathbf{r})$, P_2 at $\mathbf{r}+\mathbf{dr}$ -> P'_2 at $\mathbf{r}+\mathbf{dr}+\mathbf{u}(\mathbf{r}+\mathbf{dr})$.

$\mathbf{dr}' = P'_2 - P'_1 = \mathbf{dr} + [\mathbf{u}(\mathbf{r}+\mathbf{dr}) - \mathbf{u}(\mathbf{r})] = \mathbf{dr} + \nabla\mathbf{u}(\mathbf{r})\cdot\mathbf{dr} = \mathbf{dr}+\mathbf{du}(\mathbf{r})$

deformation of P_2-P_1 described by: $$du_i = \frac{\partial u_i}{\partial x_j} dx_j$$

$\mathbf{du} = \nabla\mathbf{u} \cdot \mathbf{dr} = \mathbf{dr} \cdot \nabla^T\mathbf{u}$

$$du_i = \frac{\partial u_i}{\partial x_j} dx_j \qquad : \qquad \begin{pmatrix} du_1 \\ du_2 \\ du_3 \end{pmatrix} = \begin{bmatrix} \frac{\partial u_1}{\partial x_1} & \frac{\partial u_1}{\partial x_2} & \frac{\partial u_1}{\partial x_3} \\ \frac{\partial u_2}{\partial x_1} & \frac{\partial u_2}{\partial x_2} & \frac{\partial u_2}{\partial x_3} \\ \frac{\partial u_3}{\partial x_1} & \frac{\partial u_3}{\partial x_2} & \frac{\partial u_3}{\partial x_3} \end{bmatrix} \begin{pmatrix} dx_1 \\ dx_2 \\ dx_3 \end{pmatrix}$$

$$\frac{\partial u_i}{\partial x_j} = \underbrace{\frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)}_{\varepsilon_{ij}} + \underbrace{\frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i}\right)}_{\omega_{ij}}$$

Total deformation is:

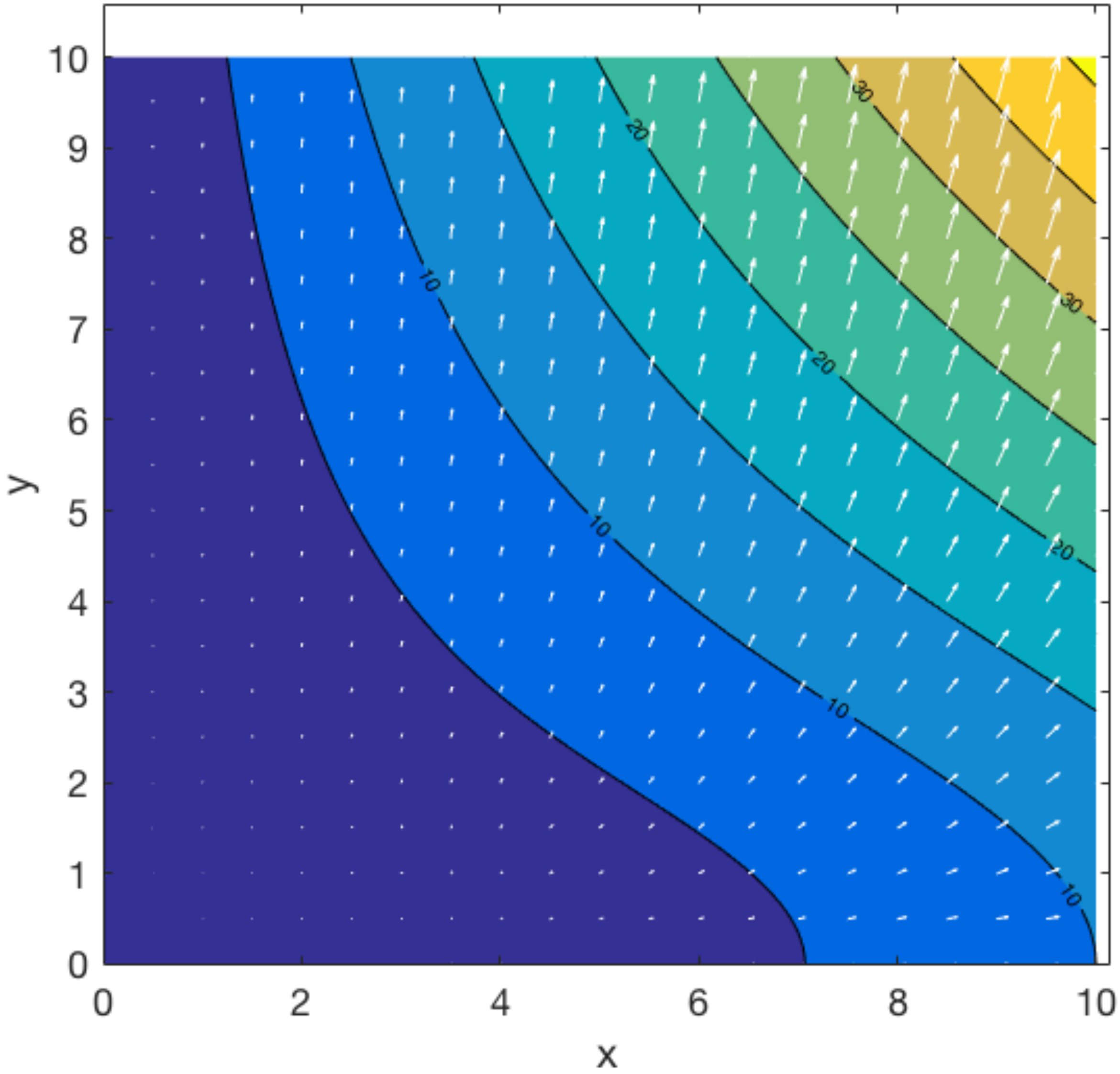
- rigid body translation - $\mathbf{u}(\mathbf{r})$
- rigid body rotation - $\boldsymbol{\omega}\cdot\mathbf{dr}$
- internal deformation, strain - $\boldsymbol{\varepsilon}\cdot\mathbf{dr}$ - result of stresses

Infinitesimal strain and rotation tensors

$$\varepsilon = \begin{bmatrix} \frac{\partial u_1}{\partial x_1} & \frac{1}{2}\left(\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1}\right) & \frac{1}{2}\left(\frac{\partial u_1}{\partial x_3} + \frac{\partial u_3}{\partial x_1}\right) \\ \frac{1}{2}\left(\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1}\right) & \frac{\partial u_2}{\partial x_2} & \frac{1}{2}\left(\frac{\partial u_2}{\partial x_3} + \frac{\partial u_3}{\partial x_2}\right) \\ \frac{1}{2}\left(\frac{\partial u_1}{\partial x_3} + \frac{\partial u_3}{\partial x_1}\right) & \frac{1}{2}\left(\frac{\partial u_2}{\partial x_3} + \frac{\partial u_3}{\partial x_2}\right) & \frac{\partial u_3}{\partial x_3} \end{bmatrix}$$

$$\omega = \begin{bmatrix} 0 & \frac{1}{2}\left(\frac{\partial u_1}{\partial x_2} - \frac{\partial u_2}{\partial x_1}\right) & \frac{1}{2}\left(\frac{\partial u_1}{\partial x_3} - \frac{\partial u_3}{\partial x_1}\right) \\ \frac{1}{2}\left(\frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2}\right) & 0 & \frac{1}{2}\left(\frac{\partial u_2}{\partial x_3} - \frac{\partial u_3}{\partial x_2}\right) \\ \frac{1}{2}\left(\frac{\partial u_3}{\partial x_1} - \frac{\partial u_1}{\partial x_3}\right) & \frac{1}{2}\left(\frac{\partial u_3}{\partial x_2} - \frac{\partial u_2}{\partial x_3}\right) & 0 \end{bmatrix}$$

Example displacement – infinitesimal strain



displacement in time interval =1

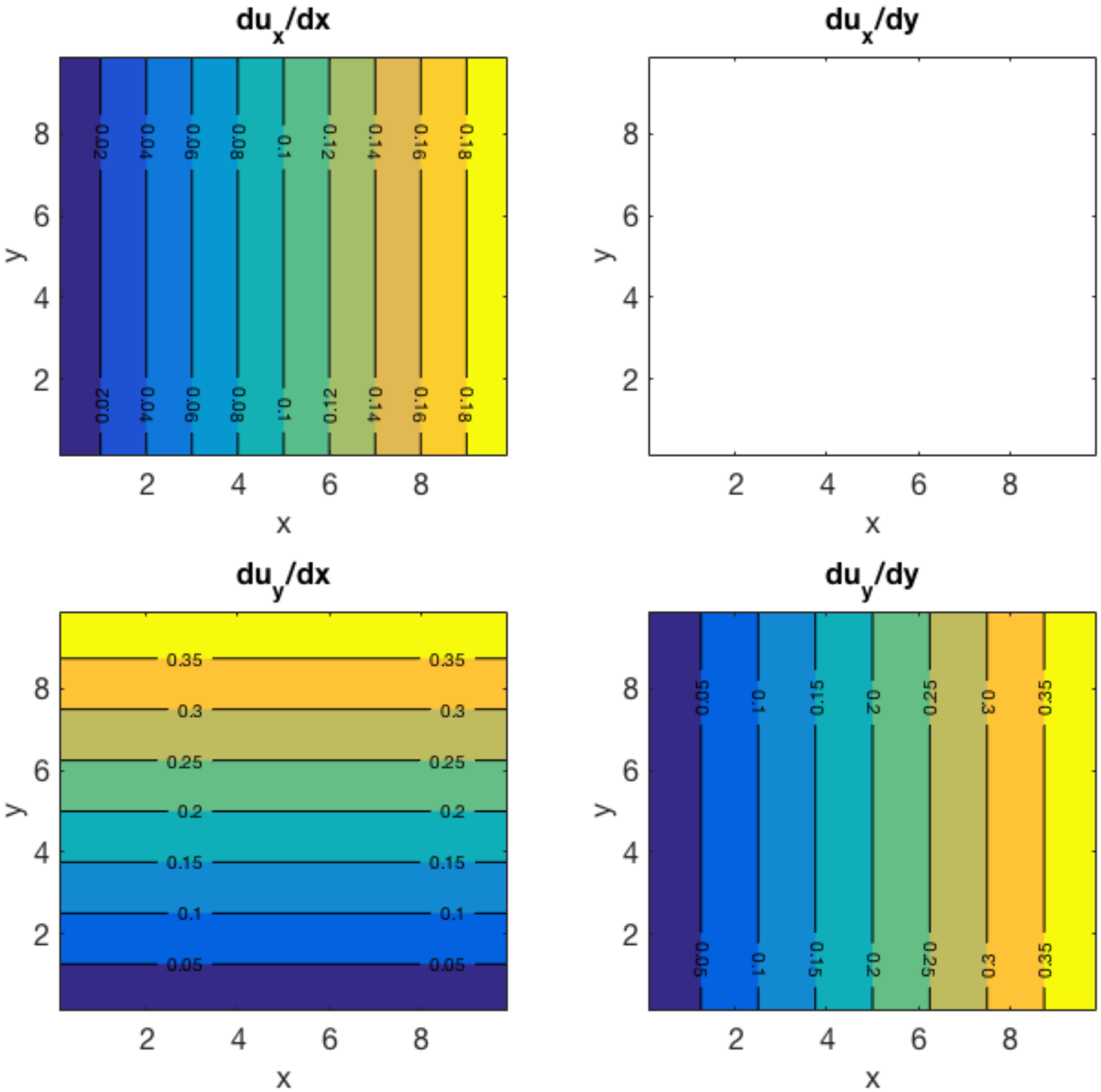
$$u_x = 0.1x^2$$

$$u_y = 0.4xy$$

Example displacement – infinitesimal strain

$$\frac{\partial u_x}{\partial x} = 0.2x$$

$$\frac{\partial u_x}{\partial y} = 0$$



for small δt

(=0.05)

$$u_x = 0.1x^2$$

$$u_y = 0.4xy$$

$$\frac{\partial u_y}{\partial x} = 0.4y$$

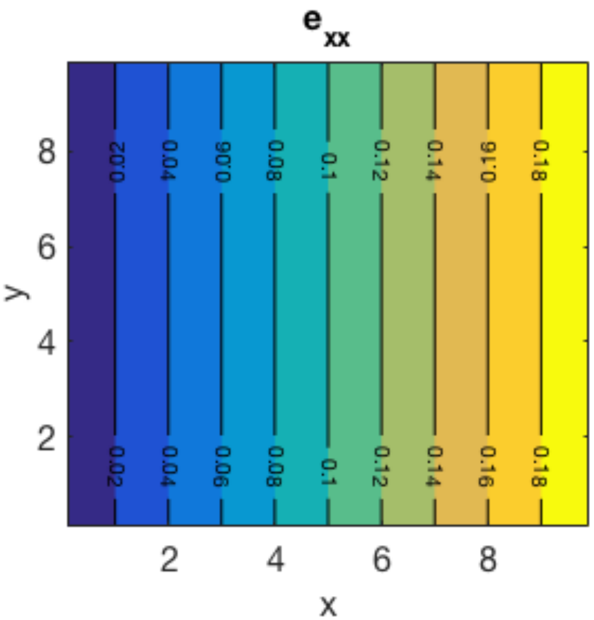
$$\frac{\partial u_y}{\partial y} = 0.4x$$

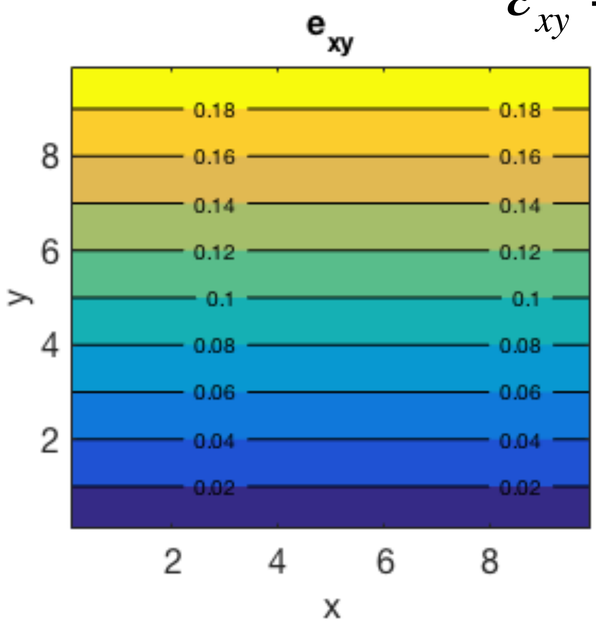
Example displacement – infinitesimal strain

$$\varepsilon_{xy} = \frac{1}{2}\left[\frac{\partial u_y}{\partial x} + \frac{\partial u_x}{\partial y}\right]$$

$$\frac{\partial u_x}{\partial x} = 0.2x$$

$$\varepsilon_{xx} = \frac{\partial u_x}{\partial x}$$





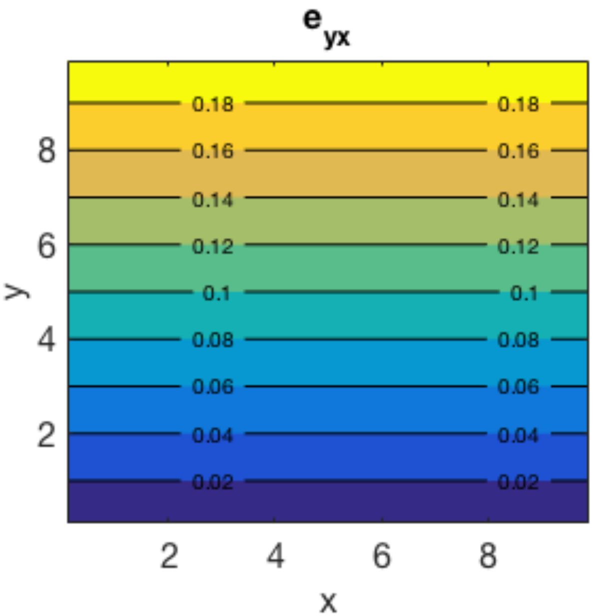
$$\frac{\partial u_x}{\partial y} = 0$$

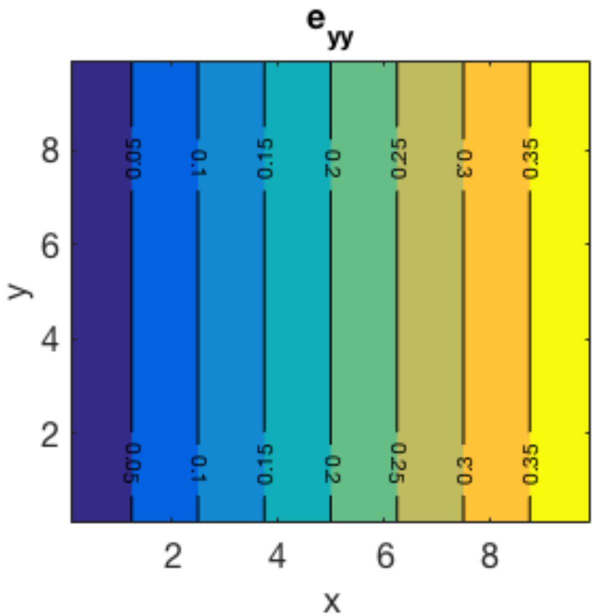
$$u_x = 0.1x^2$$

$$u_y = 0.4xy$$

$$\frac{\partial u_y}{\partial x} = 0.4y$$

$$\varepsilon_{yx} = \varepsilon_{xy}$$





$$\frac{\partial u_y}{\partial y} = 0.4x$$

$$\varepsilon_{yy} = \frac{\partial u_y}{\partial y}$$

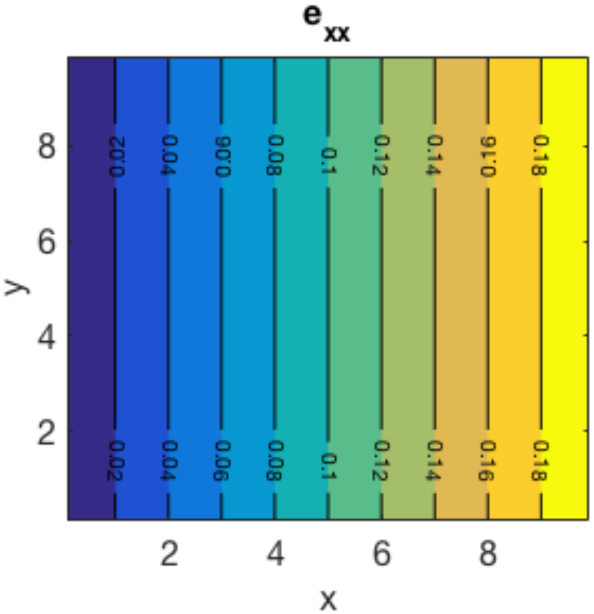
How about ω_{xx}, ω_{yy}, ω_{xy}, ω_{yx}?

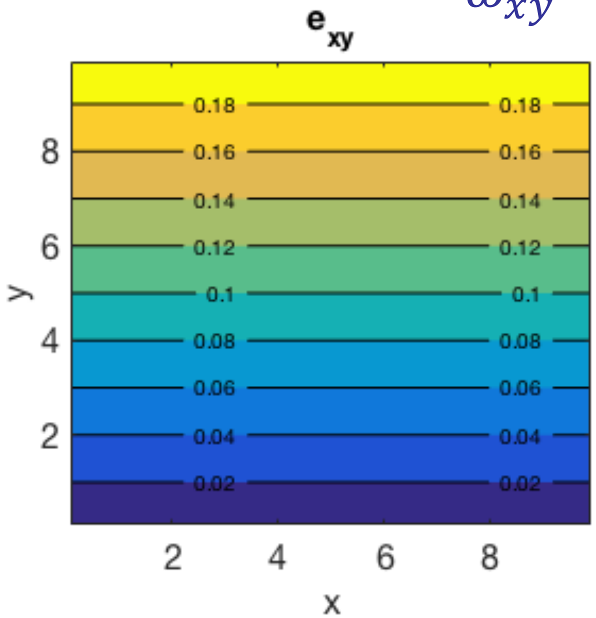
Example displacement – infinitesimal rotation

$$\omega_{xy} = \left[\frac{\partial u_x}{\partial y} - \frac{\partial u_y}{\partial x}\right]$$

$$\varepsilon_{xx} = \frac{\partial u_x}{\partial x}$$

$$\omega_{xx} = 0$$

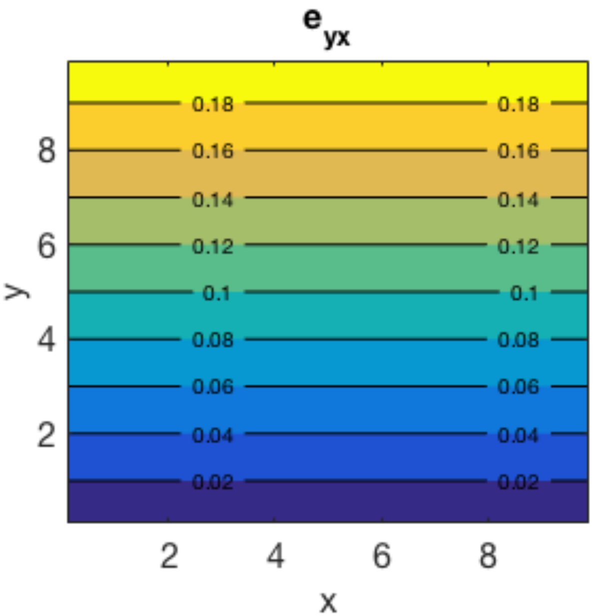


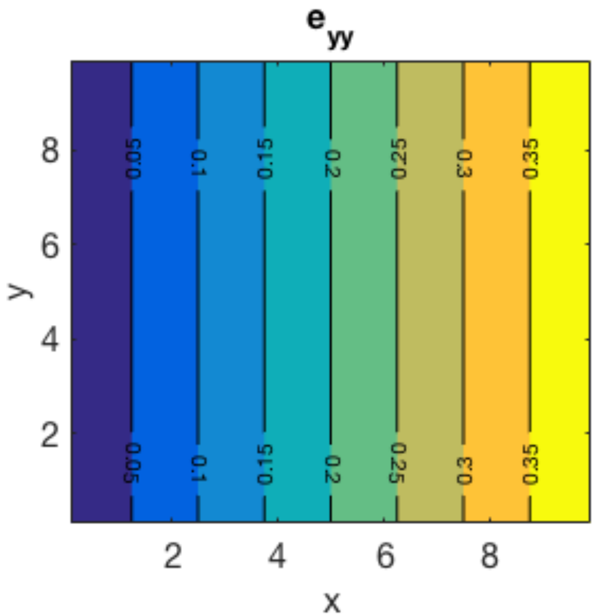


$$\varepsilon_{xy} = -\omega_{xy}$$

$$u_x = 0.1x^2$$

$$u_y = 0.4xy$$





$$\varepsilon_{yx} = \varepsilon_{xy}$$

$$\varepsilon_{yx} = \omega_{yx}$$

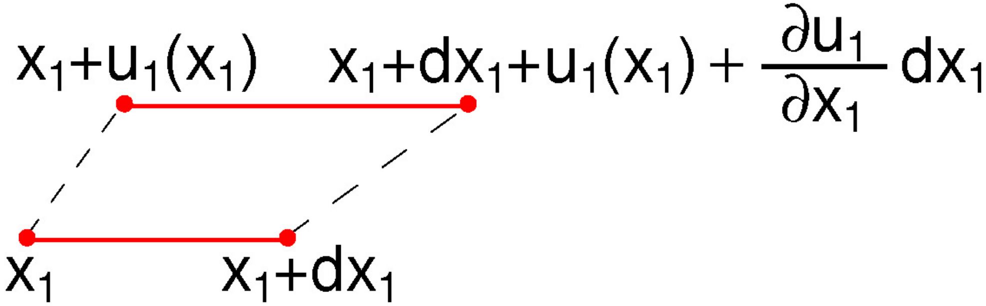
$$\varepsilon_{yy} = \frac{\partial u_y}{\partial y}$$

$$\omega_{yy} = 0$$

How about ω_{xx}, ω_{yy}, ω_{xy}, ω_{yx}?

<u>diagonal</u> infinitesimal strain tensor elements

For a line segment
dr = $(dx_1,0,0)$ deforming in displacement field **u**=$(u_1,0,0)$:



For small dx_1

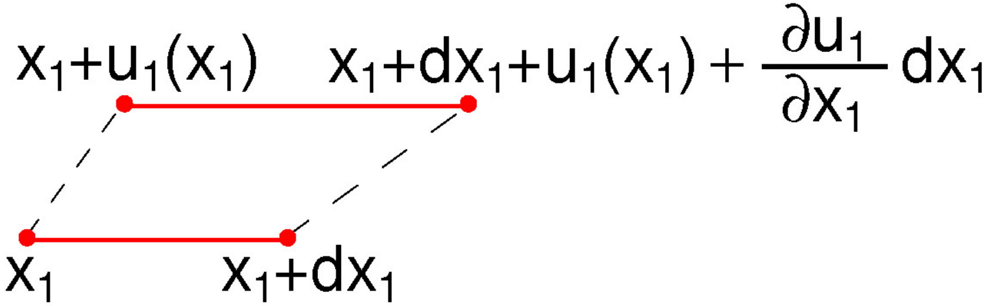
the new length $dx'_1 \approx dx_1 + (\partial u_1/\partial x_1)dx_1 = (1+\varepsilon_{11})dx_1$

$\Rightarrow$ $\varepsilon_{11}=[dx'_1 - dx_1]/dx_1$ = the relative change in length of a line element, originally in x_1 direction.

The relative change in volume $(V'-V)/V$ of a cube $V=dx_1dx_2dx_3 \approx$?

<u>diagonal</u> infinitesimal strain tensor elements

For a line segment
dr = (dx_1,0,0) deforming
in velocity field **u**=(u_1,0,0):



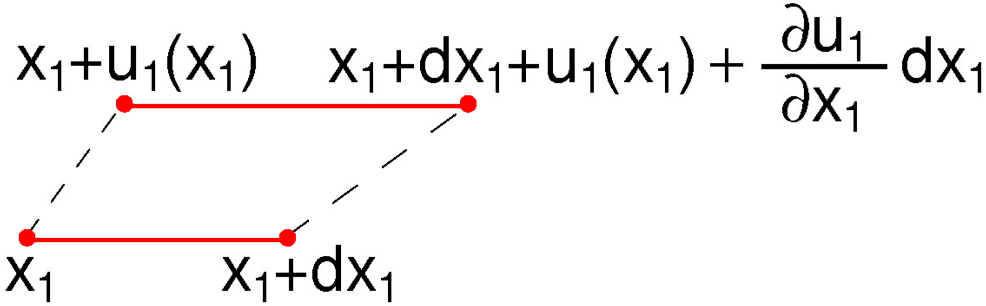
the new length $dx'_1 \approx dx_1 + (\partial u_1/\partial x_1)dx_1 = (1+\varepsilon_{11})dx_1$

$\Rightarrow$ $\varepsilon_{11}=[dx'_1 - dx_1]/dx_1$ = the relative change in length of a line element, originally in x_1 direction.

The relative change in volume (V'-V)/V of a cube $V=dx_1dx_2dx_3$
$\approx \varepsilon_{11}+\varepsilon_{22}+\varepsilon_{33} = \varepsilon_{ii} = \mathrm{tr}(\boldsymbol{\varepsilon}) = \nabla\cdot\mathbf{u}$.

<u>**diagonal**</u> **infinitesimal strain** tensor elements

For a line segment
dr = (dx_1,0,0) deforming
in velocity field **u**=(u_1,0,0):

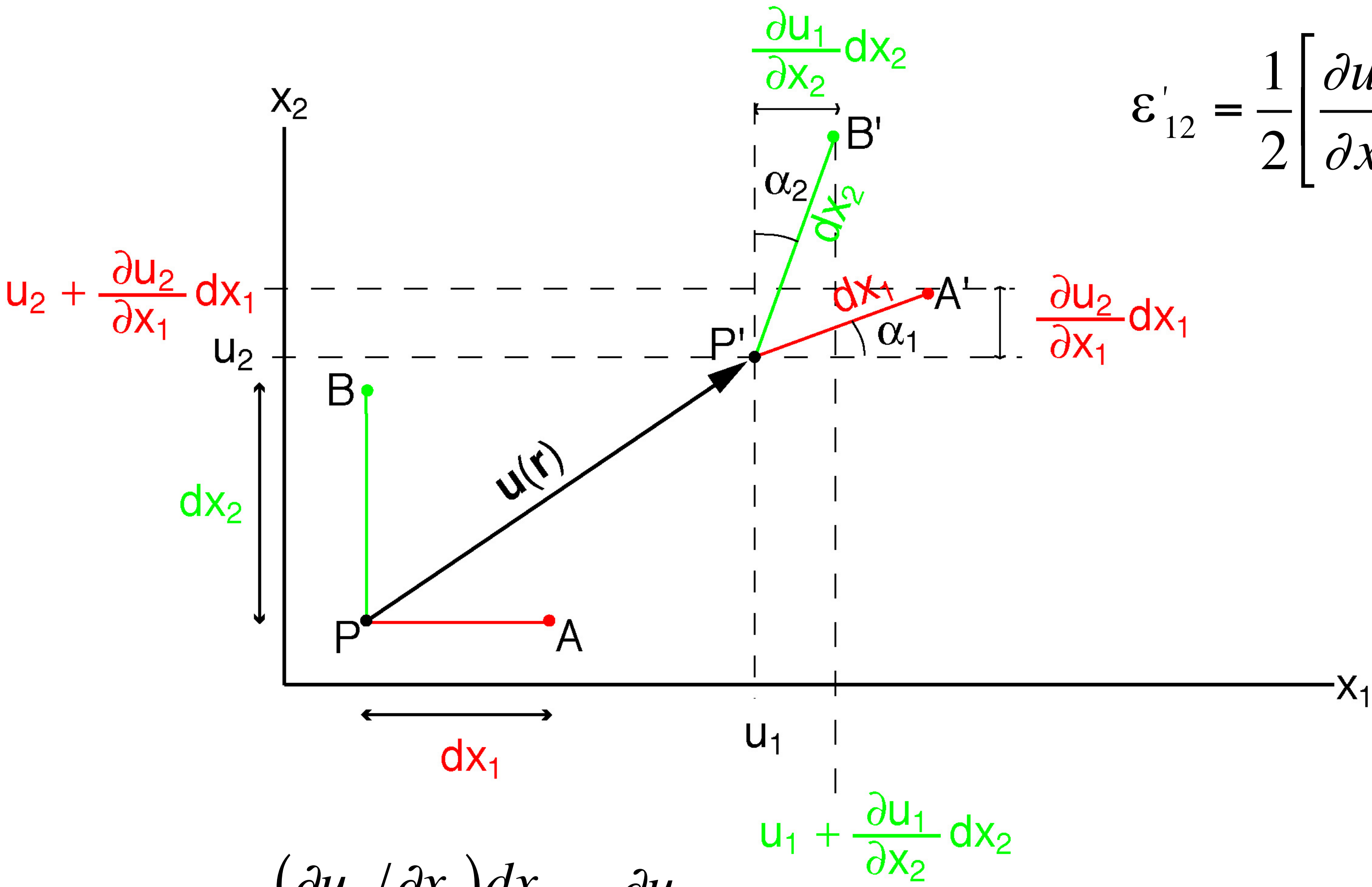


the new length $dx'_1 \approx dx_1 + (\partial u_1/\partial x_1)dx_1 = (1+\varepsilon_{11})dx_1$

The relative change in volume (V'-V)/V of a cube $V=dx_1dx_2dx_3$
$\approx \varepsilon_{11}+\varepsilon_{22}+\varepsilon_{33} = \varepsilon_{ii} = \mathrm{tr}(\boldsymbol{\varepsilon}) = \nabla\cdot\mathbf{u}$.

off-diagonal infinitesimal strain tensor elements

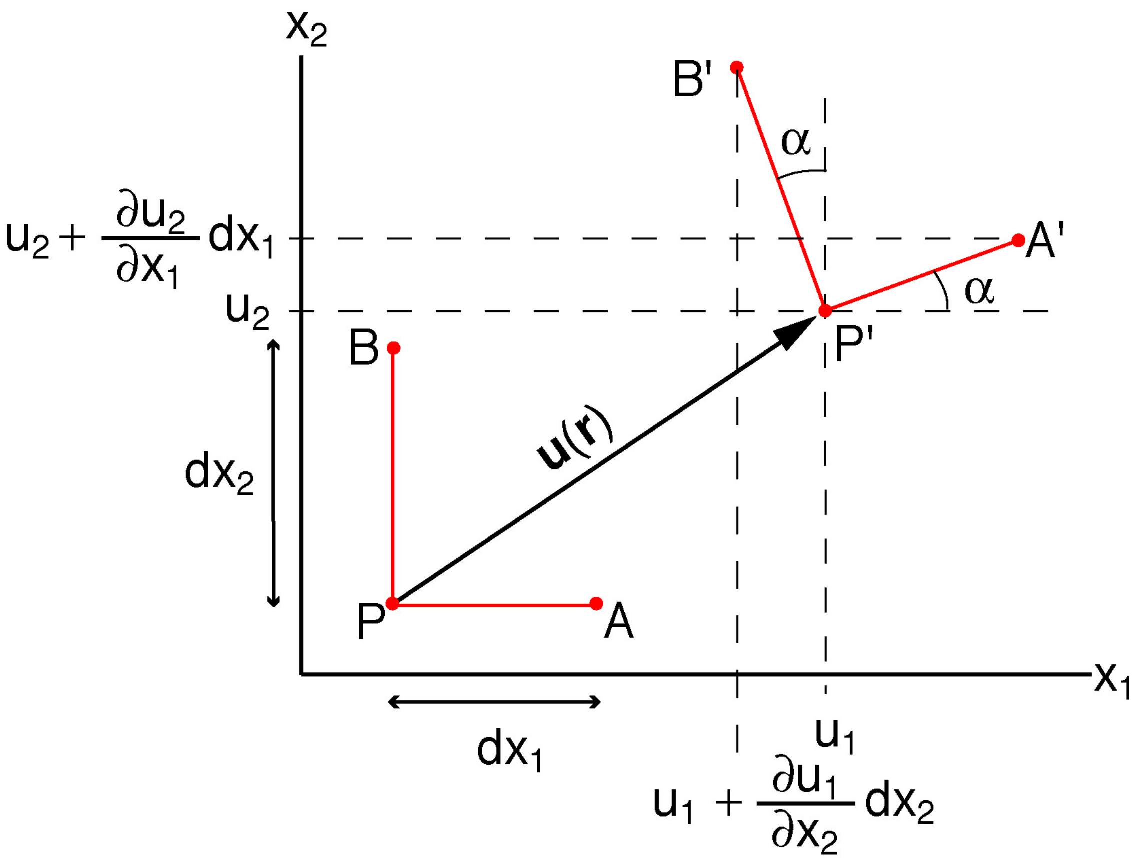
$$\varepsilon'_{12} = \frac{1}{2}\left[\frac{\partial u_2}{\partial x_1} + \frac{\partial u_1}{\partial x_2}\right]$$



$$\left.\begin{aligned} \alpha_1 &\approx \tan\alpha_1 = \frac{(\partial u_2/\partial x_1)dx_1}{dx_1} = \frac{\partial u_2}{\partial x_1} \\ \alpha_2 &\approx \frac{(\partial u_1/\partial x_2)dx_2}{dx_2} = \frac{\partial u_1}{\partial x_2} \end{aligned}\right\} \quad \boxed{\varepsilon_{12}=\varepsilon_{21}=(\alpha_1+\alpha_2)/2}$$

$2\varepsilon_{12}$ is the change in angle of an originally 90^o angle between dx_1 and dx_2

infinitesimal rotation tensor elements



$$\omega_{12}=-\omega_{21}= [(\partial u_2/\partial x_1)-(\partial u_1/\partial x_2)]/2=(\alpha_1-\alpha_2)/2$$

ω_{12} is common rigid rotation angle of vectors in the dx_1 - dx_2 plane (around x_3)