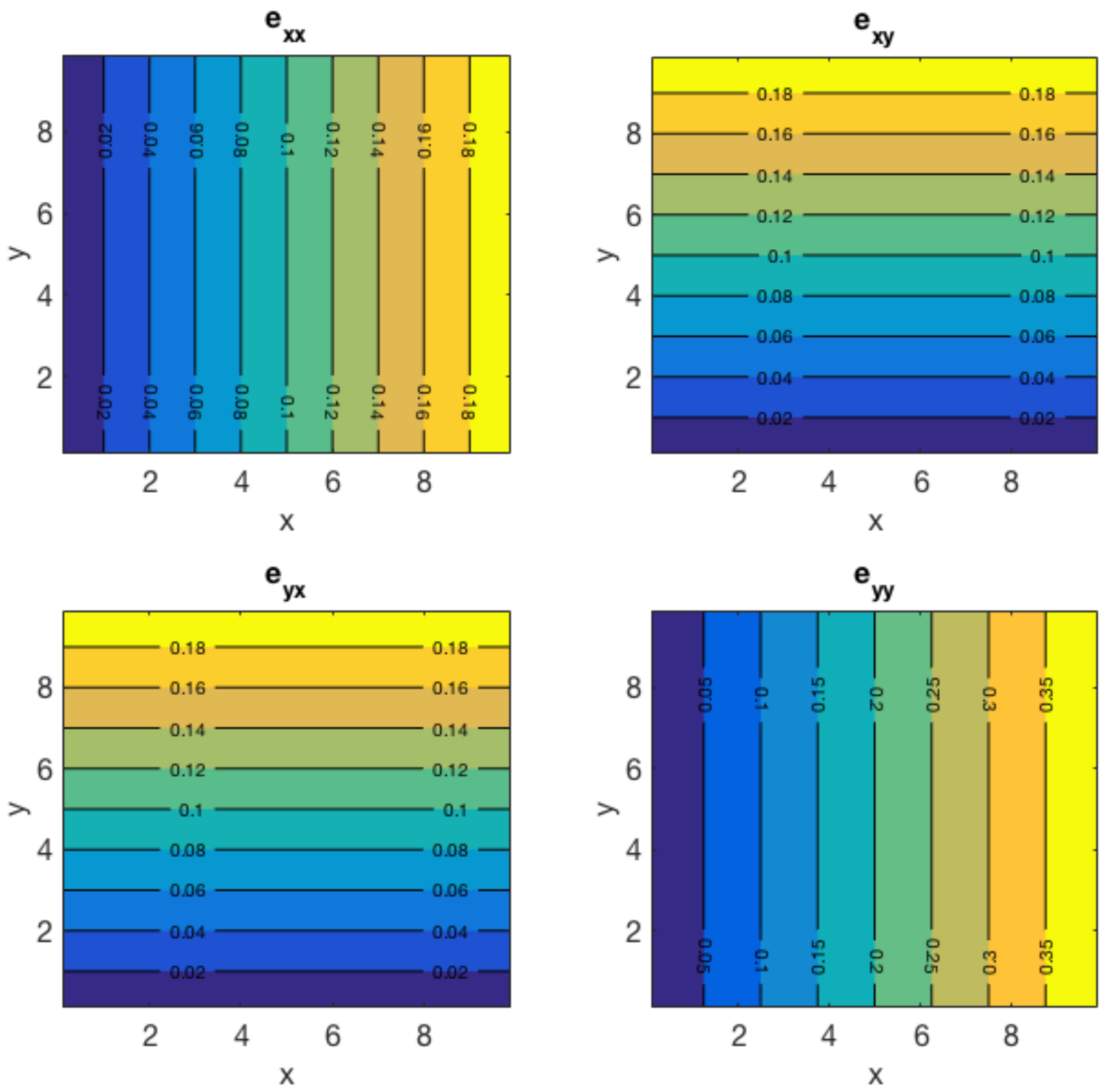
Learning Objectives

- Be able to use material and spatial descriptions of variables and their time derivatives.
- Be able to compute infinitesimal strain (strain rate) tensor given a displacement (velocity) field.
- Know meaning of the different components of the infinitesimal strain (rate) tensor
- Be able to find principal strain(rate)s and strain (rate) invariants and know what they represent
- Understand difference between infinitesimal and finite strain
- Be able to use the conservation of mass equation

Infinitesimal strain

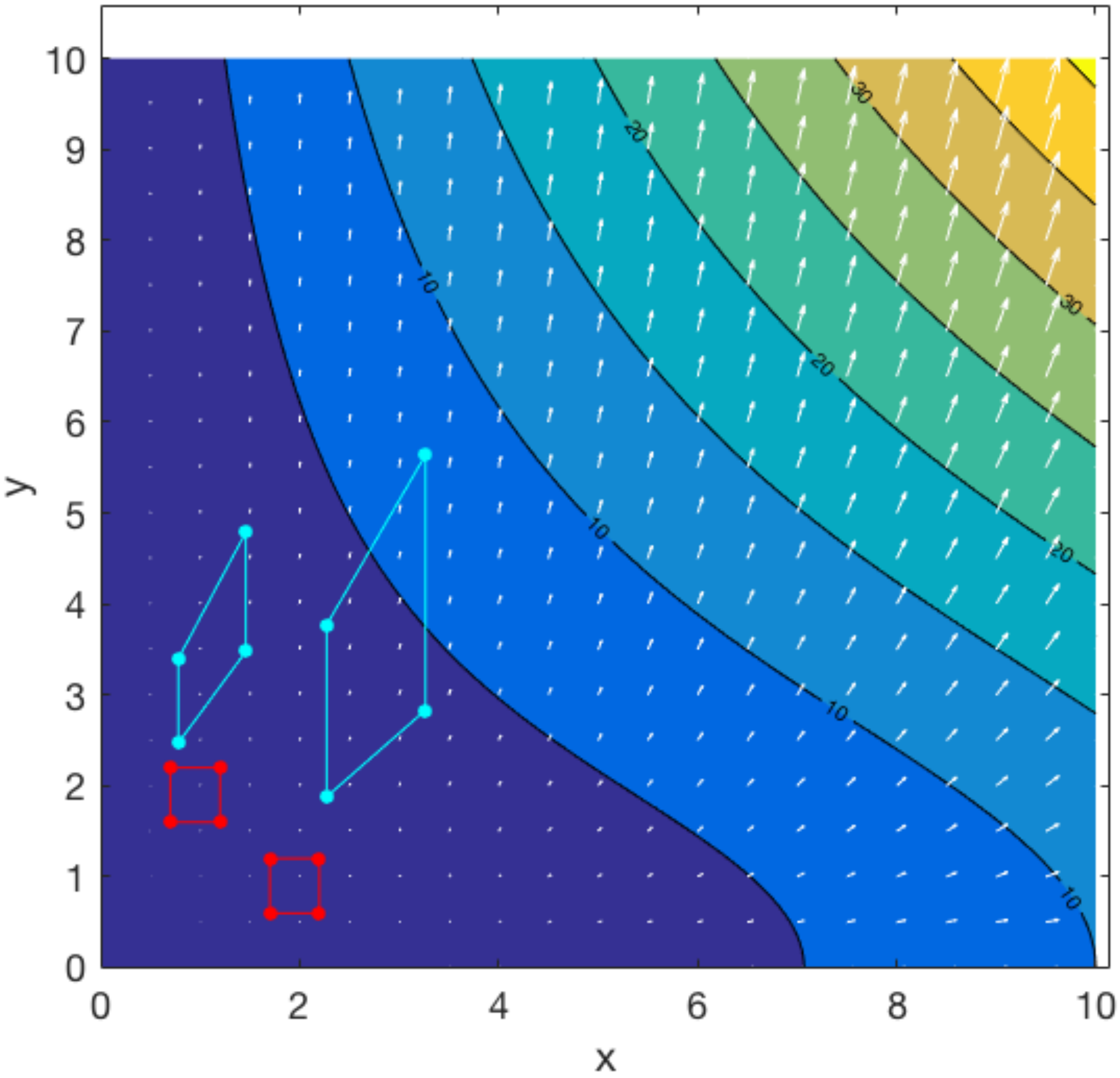
small time step, can assume constant displacement gradient encountered



Deformation after finite strain

original shape

shape at time=1.5



displacement in time interval =1

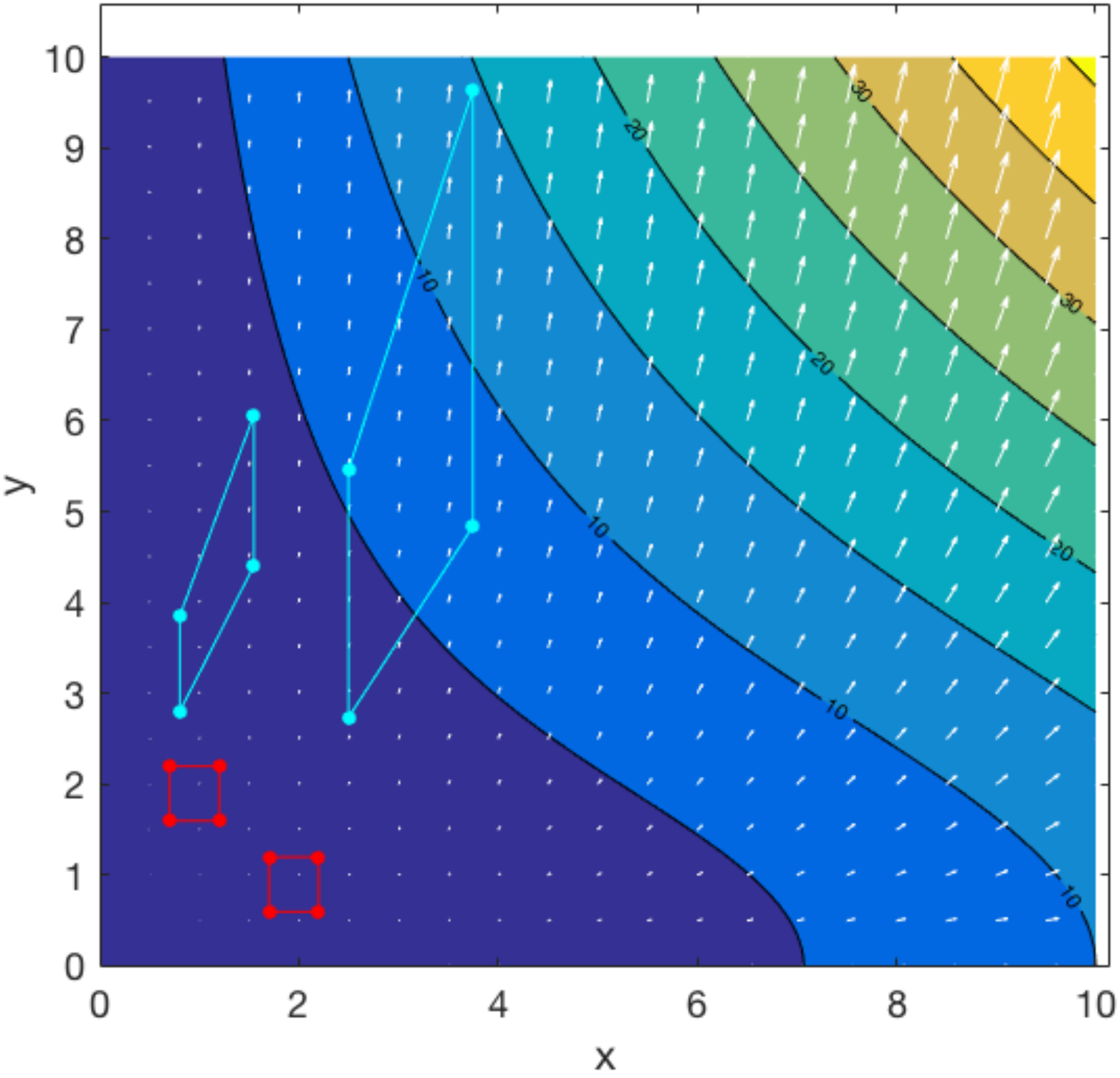
$$u_x = 0.1x^2$$

$$u_y = 0.4xy$$

Deformation after finite strain

original shape

shape at time=1.9



displacement in time interval =1

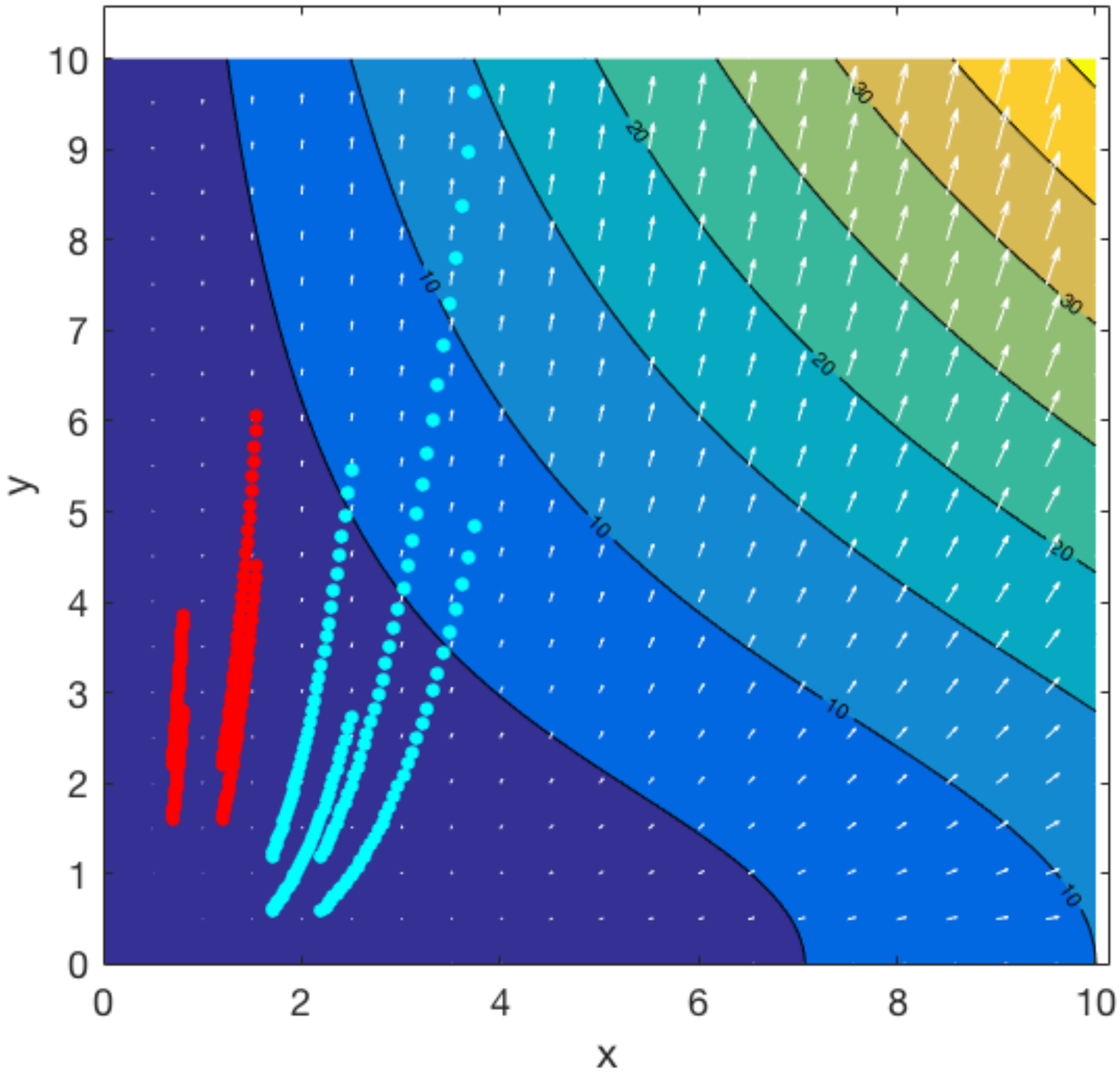
$$u_x = 0.1x^2$$

$$u_y = 0.4xy$$

Deformation after finite strain

points shape 1

points shape 2

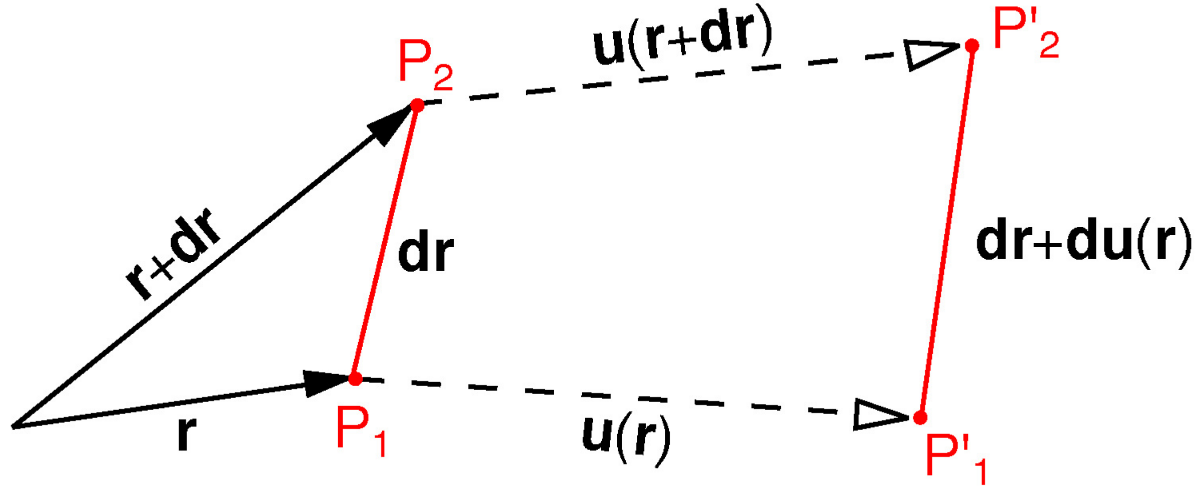


displacement in time interval =1

$$u_x = 0.1x^2$$

$$u_y = 0.4xy$$

Finite Strain



$\mathbf{dr}' = P'_2 - P'_1 = \mathbf{dr} + \nabla\mathbf{u}(\mathbf{r})\cdot\mathbf{dr} = [\mathbf{I} + \nabla\mathbf{u}(\mathbf{r})]\cdot\mathbf{dr} = \mathbf{F}\cdot\mathbf{dr}$

new length of segment P'_2-P'_1:

-$\mathbf{dr}'\cdot\mathbf{dr}' = (\mathbf{F}\cdot\mathbf{dr})\cdot(\mathbf{F}\cdot\mathbf{dr}) = \mathbf{dr}\cdot(\mathbf{F}^T\cdot\mathbf{F})\cdot\mathbf{dr} = \mathbf{dr}\cdot\mathbf{C}\cdot\mathbf{dr}$

-$\mathbf{C} = \mathbf{F}^\mathbf{T}\cdot\mathbf{F} = (\mathbf{I}+\nabla\mathbf{u})^T\cdot(\mathbf{I}+\nabla\mathbf{u}) = \mathbf{I}+\nabla\mathbf{u}+(\nabla\mathbf{u})^T+(\nabla\mathbf{u})^T\cdot\nabla\mathbf{u}$

-$\mathbf{C} = \mathbf{I}+2\mathbf{E}^*$

-$\mathbf{E}^* = 1/2\ [\nabla\mathbf{u}+(\nabla\mathbf{u})^T+(\nabla\mathbf{u})^T\cdot\nabla\mathbf{u}]$

C - ***right Cauchy-Green deformation tensor***

E* - finite deformation tensor, also called ***Lagrange strain tensor***

Finite Strain

$\mathbf{dr'}\cdot\mathbf{dr'} = \mathbf{dr}\cdot\mathbf{C}\cdot\mathbf{dr}$

$\mathbf{C} = \mathbf{F}^{\mathbf{T}}\cdot\mathbf{F} = \mathbf{I}+2\mathbf{E}^*$ - ***right Cauchy-Green deformation tensor***

$\mathbf{E}^*$ - finite deformation tensor, also called ***Lagrange strain tensor***

$\mathbf{E}^*=1/2\ [\nabla\mathbf{u}+(\nabla\mathbf{u})^{\mathrm{T}}+(\nabla\mathbf{u})^{\mathrm{T}}\cdot\nabla\mathbf{u}]$ *In terms of ∂/∂x'*

The inverse: $\mathbf{dr}\cdot\mathbf{dr} = \mathbf{dr'}\cdot\mathbf{B}\cdot\mathbf{dr'}$

gives the ***left C-G deformation tensor***: $\mathbf{B}=\mathbf{F}\cdot\mathbf{F}^{\mathbf{T}}=\mathbf{I}+2\mathbf{e}^*$

where $\mathbf{e}^*$ is the ***Euler strain tensor***

$\mathbf{e}^*=1/2\ [\nabla'\mathbf{u}+(\nabla'\mathbf{u})^{\mathrm{T}}+(\nabla'\mathbf{u})^{\mathrm{T}}\cdot\nabla'\mathbf{u}]$ *In terms of ∂/∂x*

For small deformation, $\partial/\partial x' \approx \partial/\partial x$ and quadratic term in $\nabla\mathbf{u}$ negligible

$\Rightarrow$ **E*=e*=*infinitesimal strain tensor* ε**

Meaning Finite Strain Tensor Components

Similar to infinitesimal strain tensor, the components of the finite strain tensor can be related to length and angle changes:

$$\mathbf{dx}^{(1)'}\cdot\mathbf{dx}^{(2)'} = \mathbf{dx}^{(1)}\cdot(\mathbf{F}^{T}\cdot\mathbf{F})\cdot\mathbf{dx}^{(2)} = \mathbf{dx}^{(1)}\cdot\mathbf{C}\cdot\mathbf{dx}^{(2)} = \mathbf{dx}^{(1)}\cdot(\mathbf{I}+2\mathbf{E}^{*})\cdot\mathbf{dx}^{(2)}$$

$$\mathbf{dx}^{(1)'}\cdot\mathbf{dx}^{(2)'} - \mathbf{dx}^{(1)}\cdot\mathbf{dx}^{(2)} = 2\,\mathbf{dx}^{(1)}\cdot\mathbf{E}^{*}\cdot\mathbf{dx}^{(2)}$$

Take $\mathbf{dx}^{(1)\,'} = ds_1'\,\hat{\mathbf{s}}$ as the deformed vector of $\mathbf{dx}^{(1)} = ds_1\,\hat{\mathbf{e}}_1$,

So that: $(ds_1')^2 - (ds_1)^2 = 2\,\mathbf{dx}^{(1)}\cdot\mathbf{E}^{*}\cdot\mathbf{dx}^{(1)} = ?$

What is r.h.s
in terms of ds_1?

Then $E^{*}_{11} =$, and similarly for other on-diagonal E^{*}_{ij}

For small strain:

Meaning Finite Strain Tensor Components

Similar to infinitesimal strain tensor, the components of the finite strain tensor can be related to length and angle changes:

$$\mathbf{dx}^{(1)'}\cdot\mathbf{dx}^{(2)'} = \mathbf{dx}^{(1)}\cdot(\mathbf{F}^{\mathrm{T}}\cdot\mathbf{F})\cdot\mathbf{dx}^{(2)} = \mathbf{dx}^{(1)}\cdot\mathbf{C}\cdot\mathbf{dx}^{(2)} = \mathbf{dx}^{(1)}\cdot(\mathbf{I}+2\mathbf{E}^*)\cdot\mathbf{dx}^{(2)}$$

$$\mathbf{dx}^{(1)'}\cdot\mathbf{dx}^{(2)'} - \mathbf{dx}^{(1)}\cdot\mathbf{dx}^{(2)} = 2\,\mathbf{dx}^{(1)}\cdot\mathbf{E}^*\cdot\mathbf{dx}^{(2)}$$

Take $\mathbf{dx}^{(1)}{}' = ds_1'\,\hat{\mathbf{s}}$ as the deformed vector of $\mathbf{dx}^{(1)} = ds_1\,\hat{\mathbf{e}}_1$,

So that: $\left(ds_1'\right)^2 - \left(ds_1\right)^2 = 2\left(ds_1\right)^2\,\hat{\mathbf{e}}_1\cdot\mathbf{E}^*\cdot\hat{\mathbf{e}}_1$

Then $\mathrm{E}^*_{11} = \dfrac{\left(ds_1'\right)^2 - \left(ds_1\right)^2}{2\left(ds_1\right)^2}$, and similarly for other on-diagonal E^*_{ij}

For small strain:

$$\mathrm{E}^*_{11} = \frac{\left(ds_1' - ds_1\right)\left(ds_1' + ds_1\right)}{2\left(ds_1\right)^2} \approx \frac{\left(ds_1' - ds_1\right)2ds_1}{2\left(ds_1\right)^2} = \frac{\left(ds_1' - ds_1\right)}{ds_1} = \varepsilon_{11}$$

Meaning Finite Strain Tensor Components

Similar to infinitesimal strain tensor, the components of the finite strain tensor can be related to length and angle changes:

$$\mathbf{dx}^{(1)\prime}\cdot\mathbf{dx}^{(2)\prime} - \mathbf{dx}^{(1)}\cdot\mathbf{dx}^{(2)} = 2\,\mathbf{dx}^{(1)}\cdot\mathbf{E}^{*}\cdot\mathbf{dx}^{(2)}$$

Take $\mathbf{dx}^{(1)}{}' = ds_1{}'\,\hat{\mathbf{s}}$ as the deformed vector of $\mathbf{dx}^{(1)} = ds_1\,\hat{\mathbf{e}}_1$,
And $\mathbf{dx}^{(2)}{}' = ds_2{}'\,\hat{\mathbf{p}}$ as the deformed vector of $\mathbf{dx}^{(2)} = ds_2\,\hat{\mathbf{e}}_2$

So that: $ds_1{}'ds_2{}'\cos(\hat{\mathbf{s}},\hat{\mathbf{p}}) - 0 =$

Then $2\mathrm{E}^{*}_{12} =$, and similar for other off-diagonal E^{*}_{ij}

For small strain:

$$\cos(\hat{\mathbf{s}},\hat{\mathbf{p}}) = \sin(90^{\circ} - (\hat{\mathbf{s}},\hat{\mathbf{p}})) \approx 90^{\circ} - (\hat{\mathbf{s}},\hat{\mathbf{p}}) \qquad \frac{ds_i{}'}{ds_i} \approx 1$$

$$\mathrm{E}^{*}_{12} \approx \varepsilon_{12}$$

Meaning Finite Strain Tensor Components

Similar to infinitesimal strain tensor, the components of the finite strain tensor can be related to length and angle changes:

$$\mathbf{dx}^{(1)'}\cdot\mathbf{dx}^{(2)'} - \mathbf{dx}^{(1)}\cdot\mathbf{dx}^{(2)} = 2\,\mathbf{dx}^{(1)}\cdot\mathbf{E}^{*}\cdot\mathbf{dx}^{(2)}$$

Take $\mathbf{dx}^{(1)}{}' = ds_1'\,\hat{\mathbf{s}}$ as the deformed vector of $\mathbf{dx}^{(1)} = ds_1\,\hat{\mathbf{e}}_1$,
And $\mathbf{dx}^{(2)}{}' = ds_2'\,\hat{\mathbf{p}}$ as the deformed vector of $\mathbf{dx}^{(2)} = ds_2\,\hat{\mathbf{e}}_2$

So that: $ds_1' ds_2' \cos(\hat{\mathbf{s}},\hat{\mathbf{p}}) - 0 = 2 ds_1 ds_2 \hat{\mathbf{e}}_1 \cdot \mathbf{E}^{*} \cdot \hat{\mathbf{e}}_2$

Then $2\mathrm{E}^{*}_{12} = \dfrac{ds_1'}{ds_1}\,\dfrac{ds_2'}{ds_2}\cos(\hat{\mathbf{s}},\hat{\mathbf{p}})$, and similar for other off-diagonal $\mathrm{E}^{*}{}_{ij}$

For small strain:

$$\cos(\hat{\mathbf{s}},\hat{\mathbf{p}}) = \sin(90^{\circ} - (\hat{\mathbf{s}},\hat{\mathbf{p}})) \approx 90^{\circ} - (\hat{\mathbf{s}},\hat{\mathbf{p}}) \qquad \frac{ds_i'}{ds_i} \approx 1$$

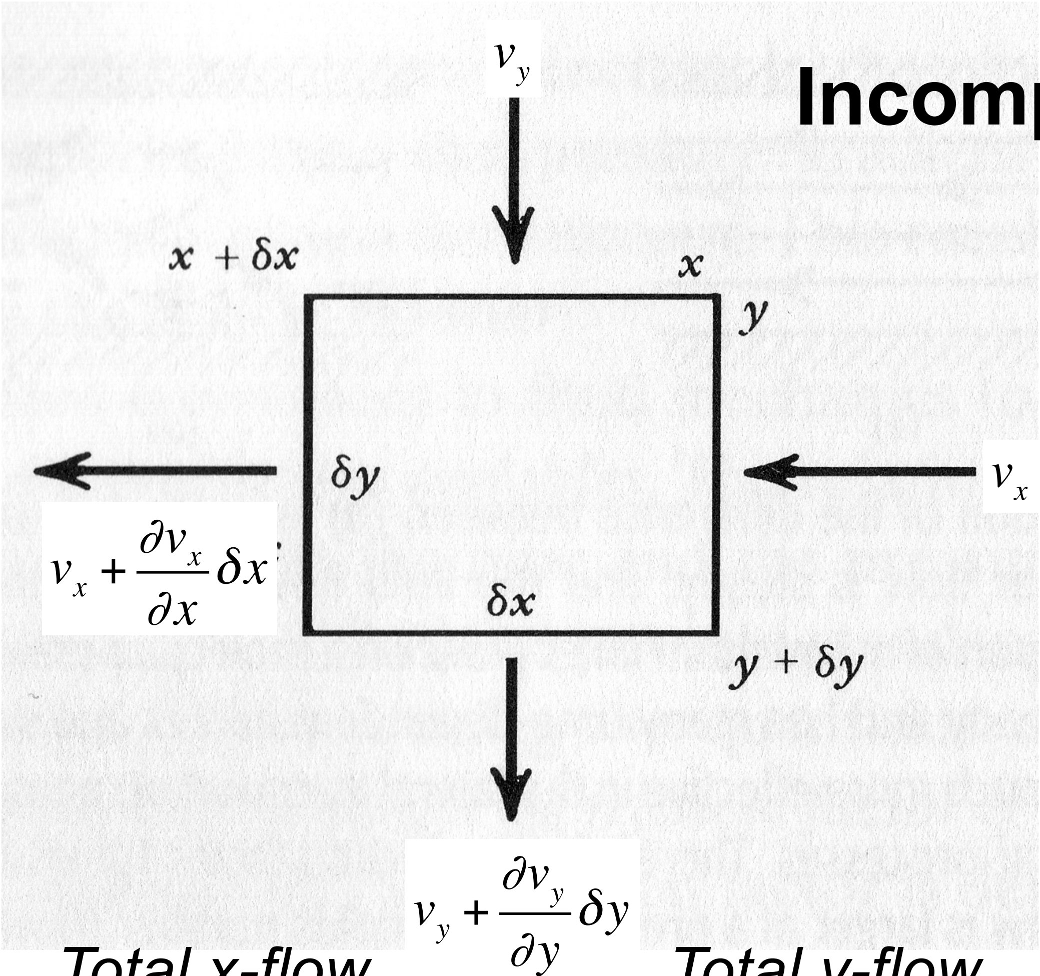
$$\mathrm{E}^{*}{}_{12} \approx \varepsilon_{12}$$

Compatibility equations

Computing strain (rate) field from a displacement (velocity) field is straightforward.

The inverse is only defined if the strain rate field satisfies a set of ***compatibility equations*** to ensure that the 6 strain components uniquely relate to a continuous field of 3 displacement components.

2-D Conservation of Mass



Incompressible

Continuity Equation

x-flow in: $v_x \delta y$

x-flow out: $(v_x + \frac{\partial v_x}{\partial x}\delta x)\delta y$

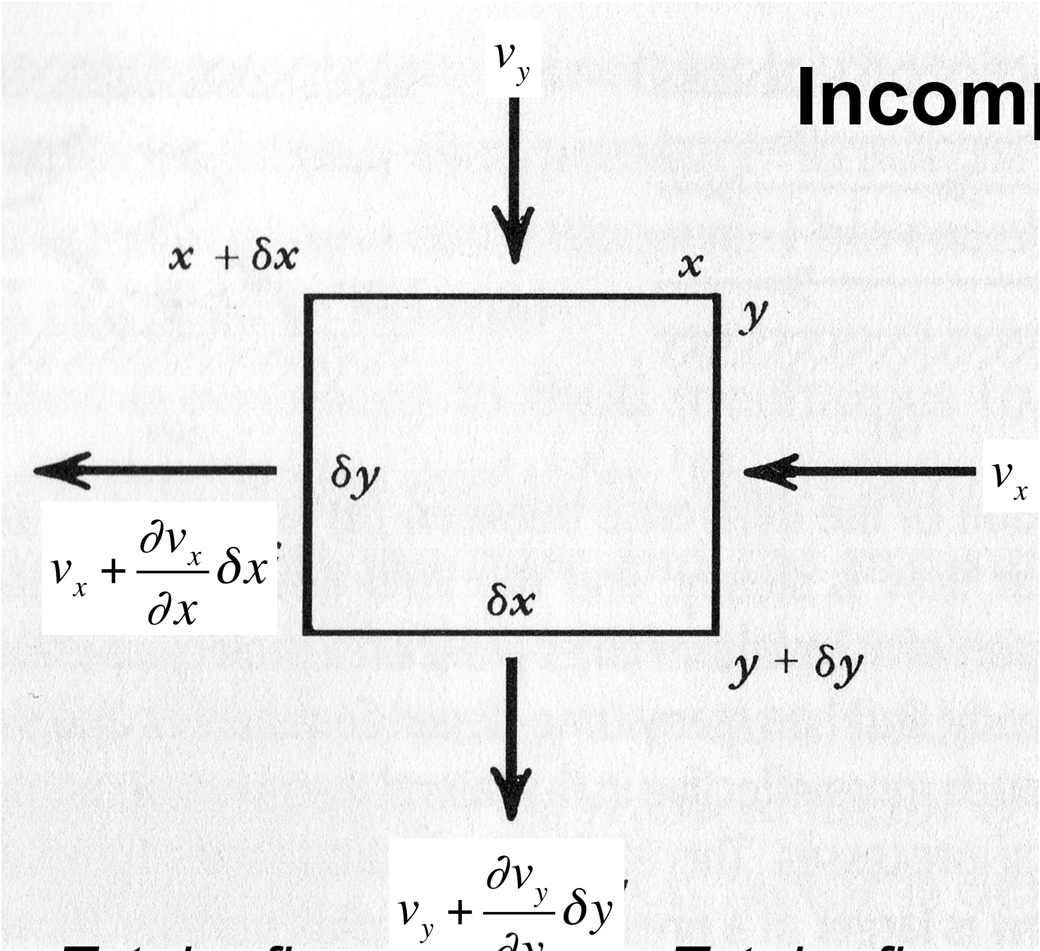
Total x-flow *Total y-flow* *Per unit area*

$$\left\{ \qquad \right\}\delta y + \left\{ \qquad \right\}\delta x = 0 \Rightarrow$$

i.e., which also applies in 3-D

No volume changes!

2-D Conservation of Mass



Incompressible

Continuity Equation

x-flow in: $v_x \delta y$

x-flow out: $(v_x + \frac{\partial v_x}{\partial x}\delta x)\delta y$

<u>*Total x-flow*</u> <u>*Total y-flow*</u> *Per unit area*

$$\left\{(v_x + \frac{\partial v_x}{\partial x}\delta x) - v_x\right\}\delta y + \left\{(v_y + \frac{\partial v_y}{\partial y}\delta y) - v_y\right\}\delta x = 0 \Rightarrow \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} = 0$$

i.e., $\boxed{\nabla \cdot \mathbf{v} = 0}$ which also applies in 3-D

No volume changes!

Conservation of Mass

Full expression: compressible

$$\frac{D\rho dV}{Dt} = 0$$

ρ – density
dV – infinitesimal volume

density changes

$$\frac{D\rho}{Dt} dV + \rho \frac{DdV}{Dt} = 0$$

volume changes

$$\boxed{\frac{D\rho}{Dt} + \rho \nabla \cdot \mathbf{v} = 0}$$

In spatial description: $\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0$, where $\frac{D\rho}{Dt} = \frac{\partial \rho}{\partial t} + \mathbf{v} \cdot \nabla \rho$

ρ(time) advected

$\frac{\partial \rho}{\partial t} \cancel{+} v \cdot \nabla \rho + \rho \nabla \cdot v = 0$

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Outline Lecture 7

- Material vs. spatial descriptions
- Time derivatives
- Displacement
- Infinitesimal Deformation
- Finite Deformation
- Conservation of Mass

Further reading on the topics in the lecture can be done in for example: Lai, Rubin, Kremple (2010): Ch. 3-1 through 3-15 and we covered some of the basics discussed in 3-20 to 3-26