ACSE-2

Lecture 6

Stress and Tensors

Outline

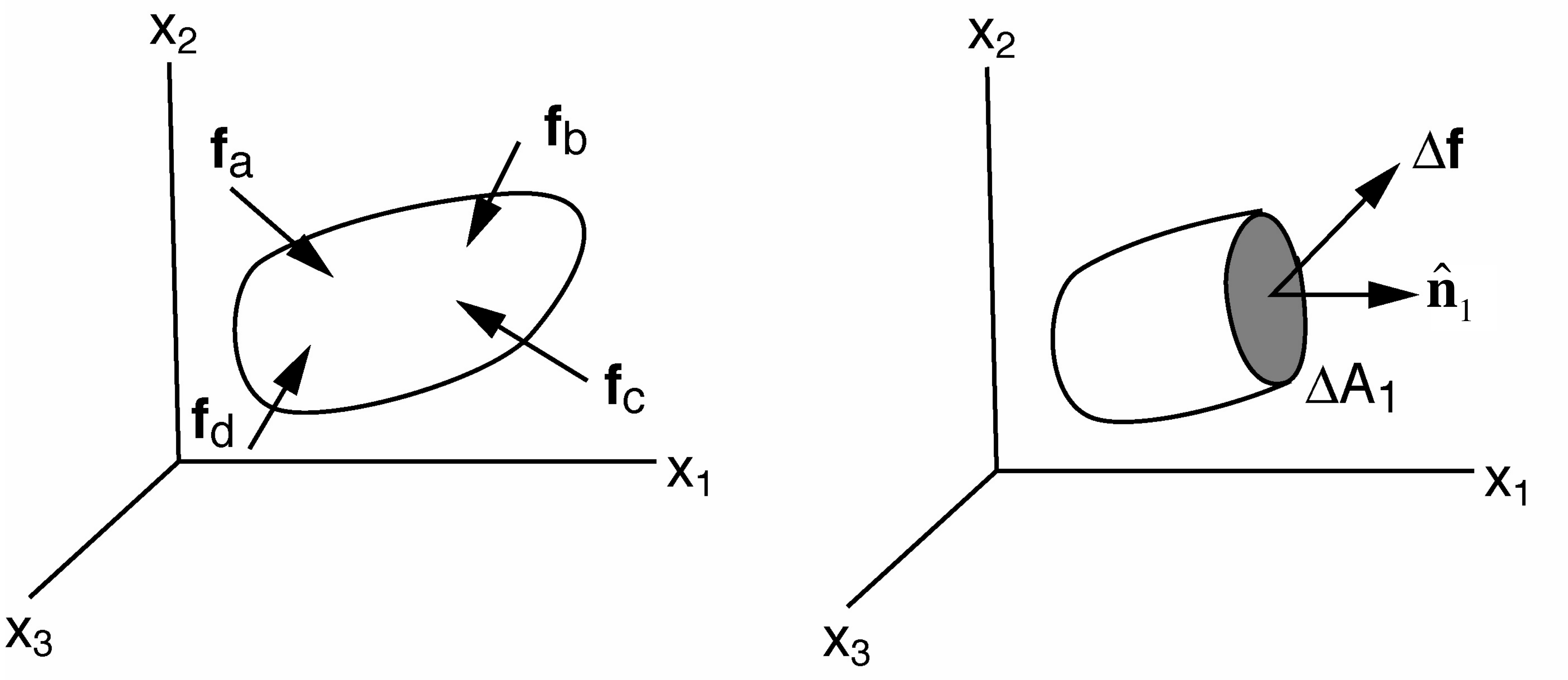
- Cauchy stress tensor recap
- Coordinate transformation (stress) tensors
- (Stress) tensor symmetry
- Tensor invariants
- Diagonalising, eigenvalues, eigenvectors
- Special stress states
- Equation of motion

Learning Objectives

- Understand meaning of different components of 3D Cauchy stress tensor, and know how to determine state of stress on given plane
- Be able to transform rank 2 tensor to a new basis.
- Be able to decompose a rank 2 tensor into symmetric and anti-symmetric components
- Be able to find principal stresses and stress invariants and know what they represent
- Be able to balance body forces and stresses

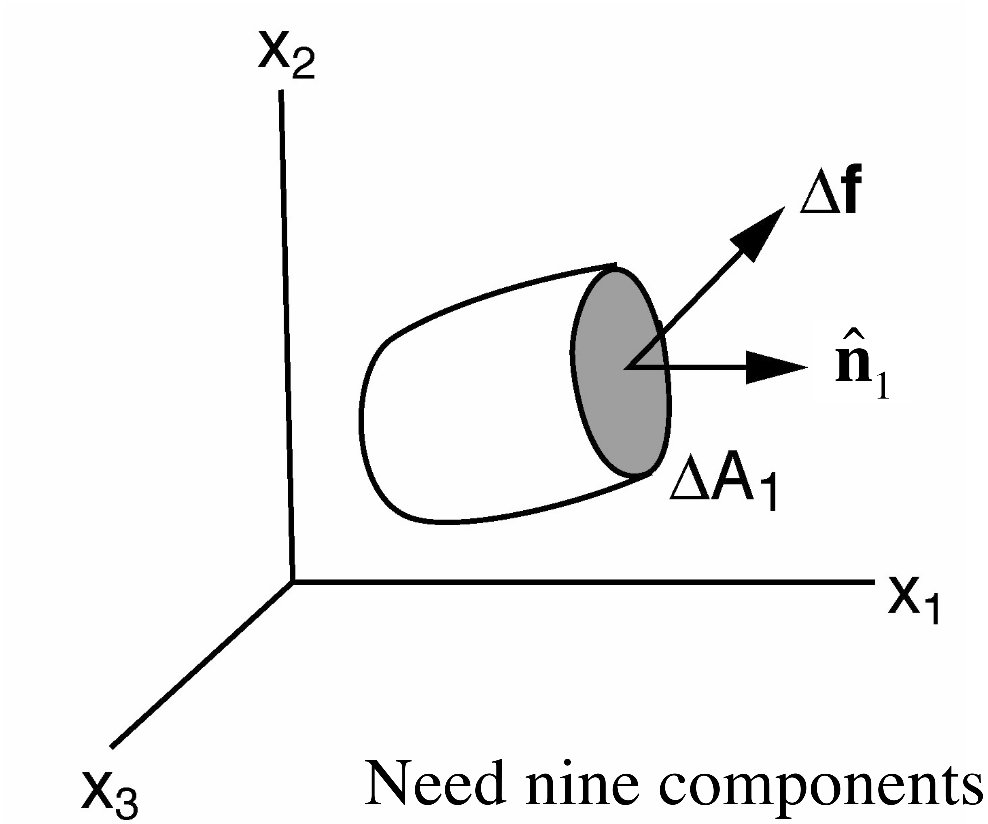
Cauchy Stress

Stress in a point, measured in medium as deformed by the stress experienced.



forces introduce a state of stress in a body

(Other stress measures, e.g., Piola-Kirchhoff tensor, used in Lagrangian formulations)



traction, stress vector

$$\mathbf{t_1} = \mathbf{t}(\hat{\mathbf{n}}_1) = \lim_{\Delta A \to 0} \Delta\mathbf{f}/\Delta A_1$$

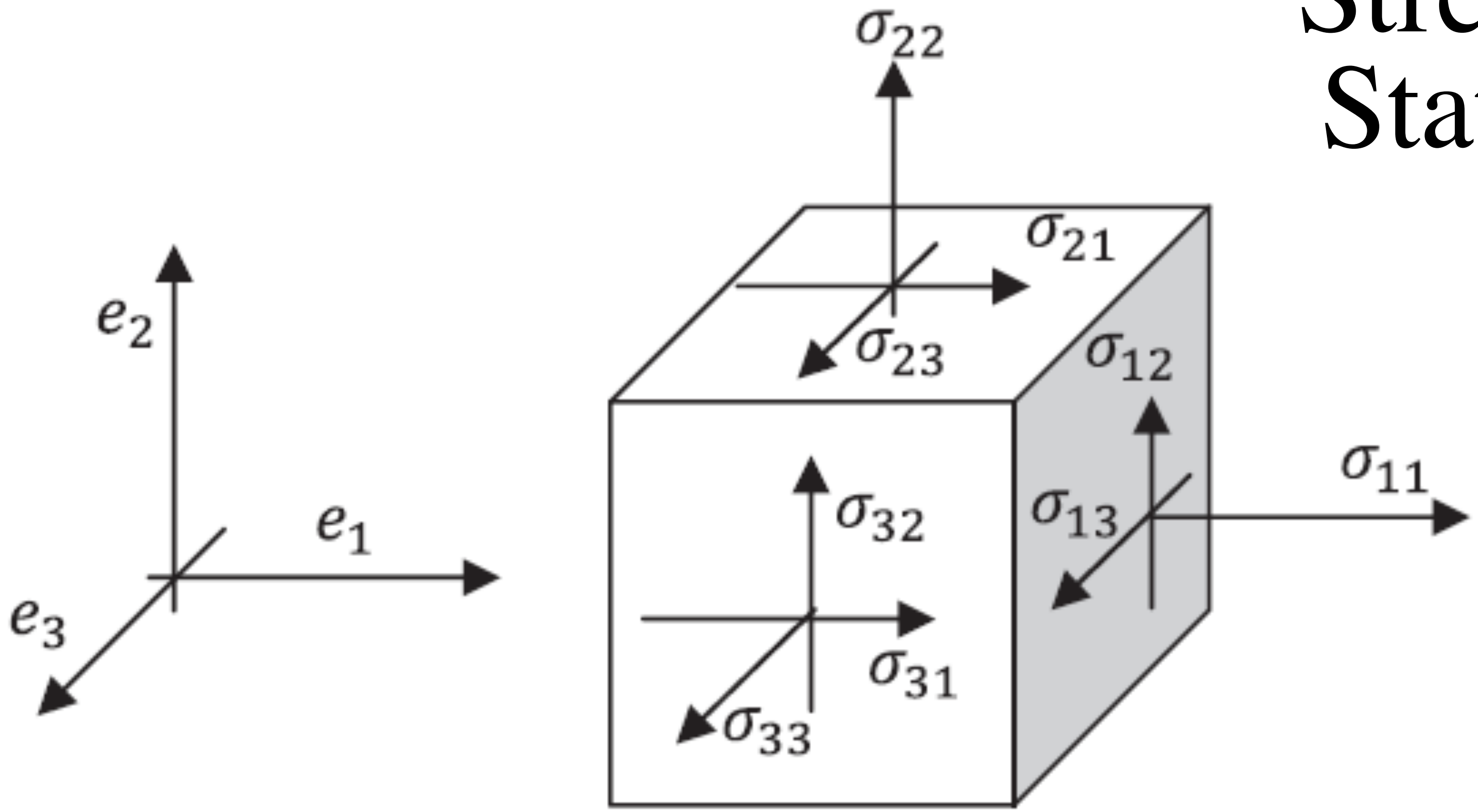
$$\mathbf{t_1} = (\sigma_{11}, \sigma_{12}, \sigma_{13})$$

Need nine components to fully describe the stress

$\sigma_{11}, \sigma_{12}, \sigma_{13}$ for ΔA_1
$\sigma_{22}, \sigma_{21}, \sigma_{23}$ for ΔA_2
$\sigma_{33}, \sigma_{31}, \sigma_{32}$ for ΔA_3

first index = orientation of plane
second index = orientation of force

3-D Stress State



first index = orientation of plane
second index = orientation of force

Positive if force in direction of normal (as shown)

$t_i = \sigma_{ji} n_j$ $\quad$ $\mathbf{t} = \boldsymbol{\sigma}^T \cdot \hat{\mathbf{n}}$ $\quad$ Transpose: $\sigma_{ji} = \sigma^T_{ij}$

Note: unusual index order

in matrix notation: $$\mathbf{t} = \begin{bmatrix} \sigma_{11} & \sigma_{21} & \sigma_{31} \\ \sigma_{12} & \sigma_{22} & \sigma_{32} \\ \sigma_{13} & \sigma_{23} & \sigma_{33} \end{bmatrix} \cdot \hat{\mathbf{n}}$$

$\mathbf{t}$ and $\hat{\mathbf{n}}$ - tensors of rank 1 (vectors) in 3-D

$\underline{\boldsymbol{\sigma}}$ - tensor of rank 2 in 3-D

compression - negative

tension - positive

σ_{ji} where $i=j$ - normal stresses

σ_{ji} where $i \neq j$ - shear stresses

2nd order tensors can be written as square matrices and have algebraic properties similar to some of those of matrices.

Example to try

Assume state of stress in a point described by stress tensor

$$\boldsymbol{\sigma} = -p\mathbf{I}$$

How could you show that there is no shearing stress on any plane containing this point?

$$\begin{bmatrix} -p & & \emptyset \\ & -p & \\ \emptyset & & -p \end{bmatrix}$$

Example to try

Assume state of stress in a point described by stress tensor

$$\boldsymbol{\sigma} = -p\mathbf{I}$$

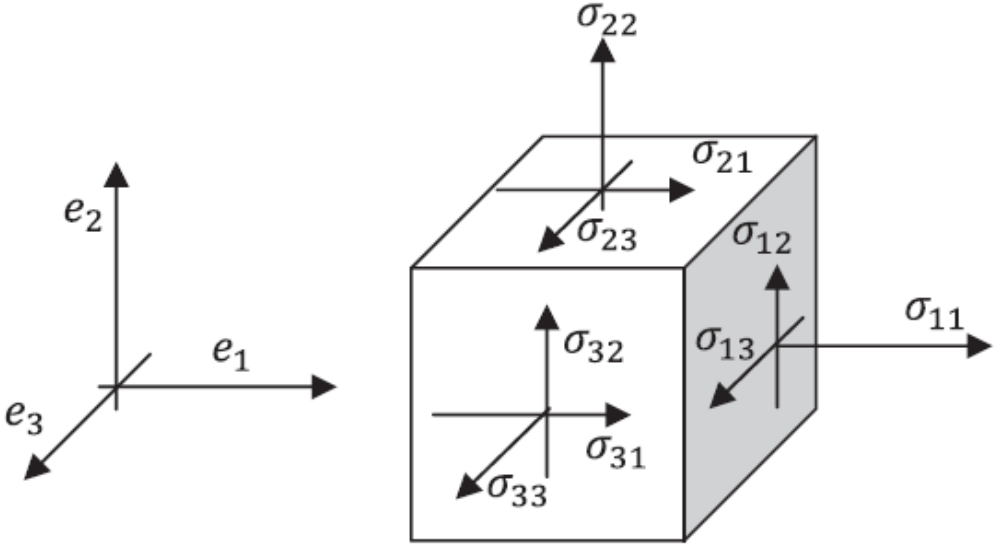
How could you show that there is no shearing stress on any plane containing this point?

By showing that traction vector on any plane with normal $\hat{\mathbf{n}}$

$$\mathbf{t} = \boldsymbol{\sigma}^T \cdot \hat{\mathbf{n}} = -p\hat{\mathbf{n}}$$

i.e., normal stress, no matter which orientation of a plane

Stress components



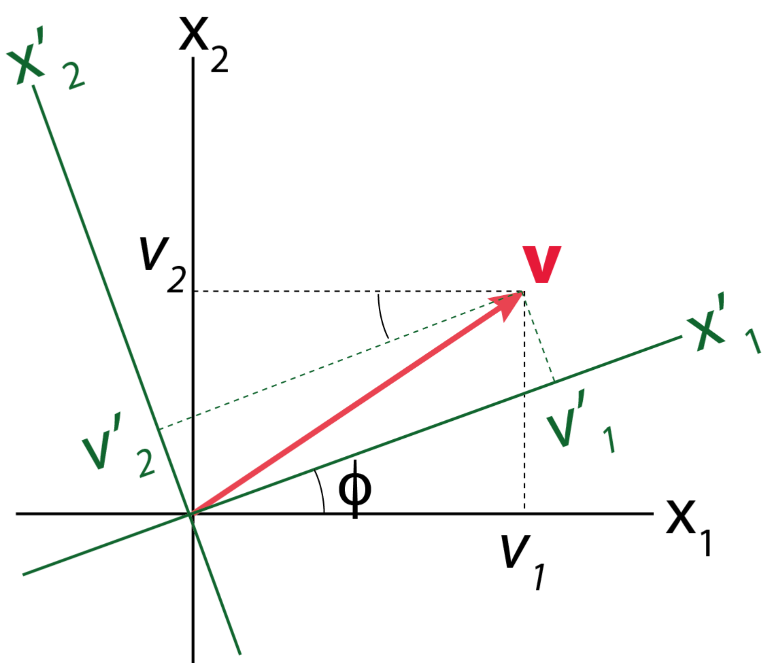
traction on a plane $$\mathbf{t} = \begin{bmatrix} \sigma_{11} & \sigma_{21} & \sigma_{31} \\ \sigma_{12} & \sigma_{22} & \sigma_{32} \\ \sigma_{13} & \sigma_{23} & \sigma_{33} \end{bmatrix} \cdot \hat{\mathbf{n}}$$

what is (1) $\hat{\mathbf{e}}_1 \cdot \mathbf{t} = \hat{\mathbf{e}}_1 \cdot \boldsymbol{\sigma}^T \cdot \hat{\mathbf{n}}$?

what is (2) $\hat{\mathbf{e}}_1 \cdot \boldsymbol{\sigma}^T \cdot \hat{\mathbf{e}}_1$? what is (3) $\hat{\mathbf{e}}_1 \cdot \boldsymbol{\sigma}^T \cdot \hat{\mathbf{e}}_2$?

physical parameters should not depend on coordinate frame

$\Rightarrow$ **tensors follow linear transformation laws**



for vectors on orthonormal basis:

$v'_1 = \alpha_{11} v_1 + \alpha_{12} v_2$

$v'_2 = \alpha_{21} v_1 + \alpha_{22} v_2$

$$\Rightarrow \quad \mathbf{v}' = \begin{bmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{bmatrix} \mathbf{v}$$

coefficients α_{ij} depend on angle ϕ between x_1 and x'_1 (or x_2 and x'_2)

$$\mathbf{v}' = \begin{bmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{bmatrix} \mathbf{v} = \begin{bmatrix} \cos\phi & \cos(90-\phi) \\ \cos(90+\phi) & \cos\phi \end{bmatrix} \mathbf{v}$$

$$\boxed{\alpha_{ij} = \hat{\mathbf{e}}'_i \cdot \hat{\mathbf{e}}_j}$$

Inverse transform: $v_j = \alpha_{ji}\, v'_i \qquad \alpha_{ji} = \hat{e}_j \cdot \hat{e}'_i$

In a new coordinate system:

traction $t'_i=\alpha_{ik}t_k$

normal $n'_j=\alpha_{jl}n_l$

$t_k=\sigma^T{}_{kl}n_l$

$t'_i=\sigma'^T{}_{ij}n'_j$

Relation $\boldsymbol{\sigma}'$ to $\boldsymbol{\sigma}$?

$\Rightarrow$ *transformation for stress tensor*

$$t'_i=\alpha_{ik}\sigma^T{}_{kl}n_l$$
$$=\alpha_{ik}\sigma^T{}_{kl}\alpha^{-1}{}_{lj}n'_j$$
$$=\alpha_{ik}\sigma^T{}_{kl}\alpha_{jl}n'_j$$

- transformation matrices are orthogonal $\alpha^{-1}{}_{jl}=\alpha_{lj}$ $(\mathbf{A}^{-1}=\mathbf{A}^T)$
- *remember* $\alpha_{ij}=\hat{\mathbf{e}}'_i\cdot\hat{\mathbf{e}}_j$
  $\alpha_{ij}^{-1}=\hat{\mathbf{e}}_i\cdot\hat{\mathbf{e}}'_j=\alpha_{ji}=\alpha_{ij}^T$

$\Rightarrow$ $\sigma'^T{}_{ij}=\alpha_{ik}\sigma^T{}_{kl}\alpha_{jl}=\alpha_{ik}\alpha_{jl}\sigma^T{}_{kl}$

$\boldsymbol{\sigma}'^T=\mathbf{A}\boldsymbol{\sigma}^T\mathbf{A}^T$

$\Rightarrow$ each dependence on direction transforms as a vector, requiring two transformations

An *n-dimensional* tensor of *rank r* consists of n^r components

This tensor $T_{i1,i2,\ldots,in}$ is defined relative to a basis of the real, linear n-dimensional space S_n

and under a coordinate transformation **T** transforms as:

$$T'_{ij\ldots n} = \alpha_{ip}\alpha_{jq}\ldots\alpha_{nt}\, T_{pq\ldots t}$$

For *orthonormal* bases the matrices α_{ik} are *orthogonal* transformations, i.e. $\alpha_{ik}^{-1} = \alpha_{ki}$. (columns and rows are orthogonal and have length =1, i.e., perpendicular unit vectors are transformed to unit vectors)
If the basis is *Cartesian*, α_{ik} are *real*.

Difference tensor and its matrix

Tensor – physical quantity which is independent of coordinate system used

Matrix of a tensor – contains components of that tensor in a particular coordinate frame

Could test that indeed tensor addition and multiplication satisfy transformation laws