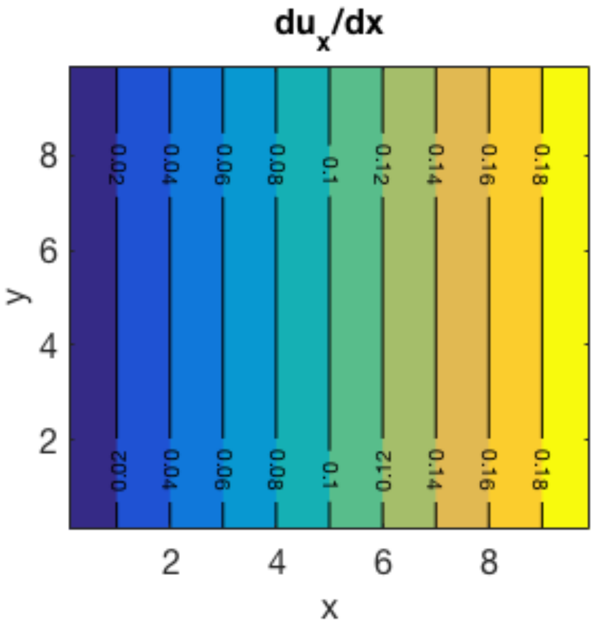
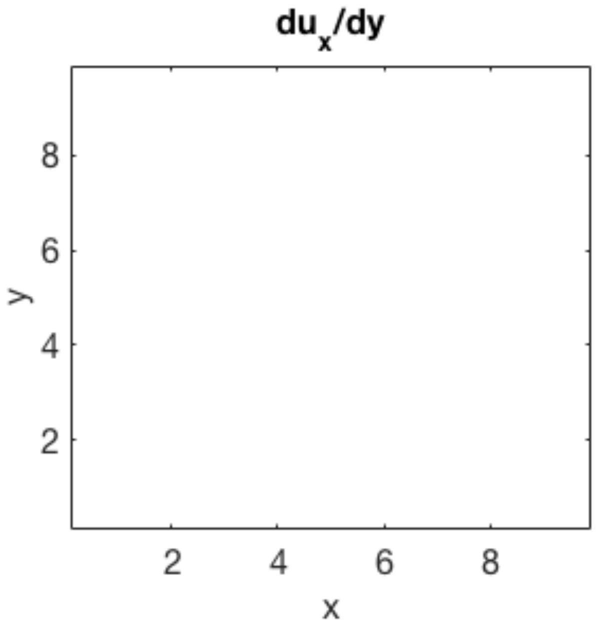
Learning Objectives

- Be able to use material and spatial descriptions of variables and their time derivatives.
- Be able to compute infinitesimal strain (strain rate) tensor given a displacement (velocity) field.
- Know meaning of the different components of the infinitesimal strain (rate) tensor
- Be able to find principal strain(rate)s and strain (rate) invariants and know what they represent
- Understand difference between infinitesimal and finite strain
- Be able to use the conservation of mass equation

Example displacement – infinitesimal strain

$$\frac{\partial u_x}{\partial x} = 0.2x$$





$$\frac{\partial u_x}{\partial y} = 0$$

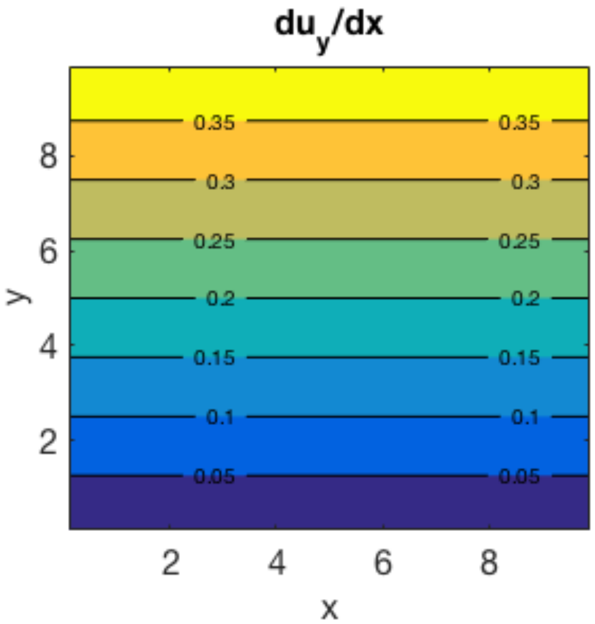
for small δt

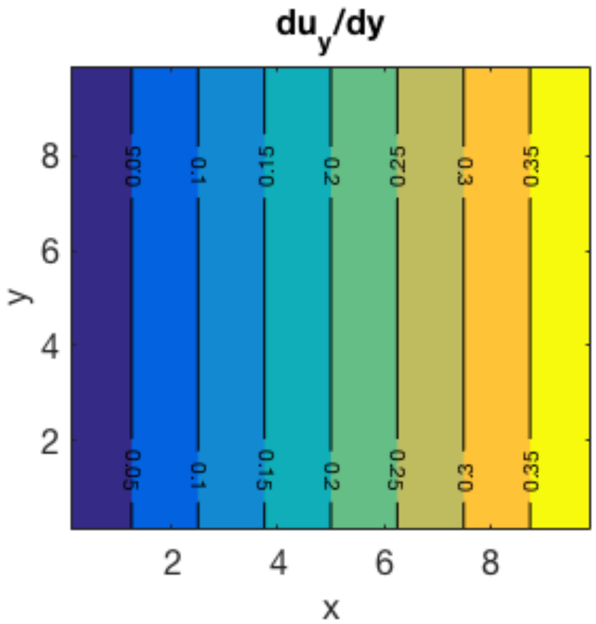
(=0.05)

$$u_x = 0.1x^2$$

$$u_y = 0.4xy$$

$$\frac{\partial u_y}{\partial x} = 0.4y$$





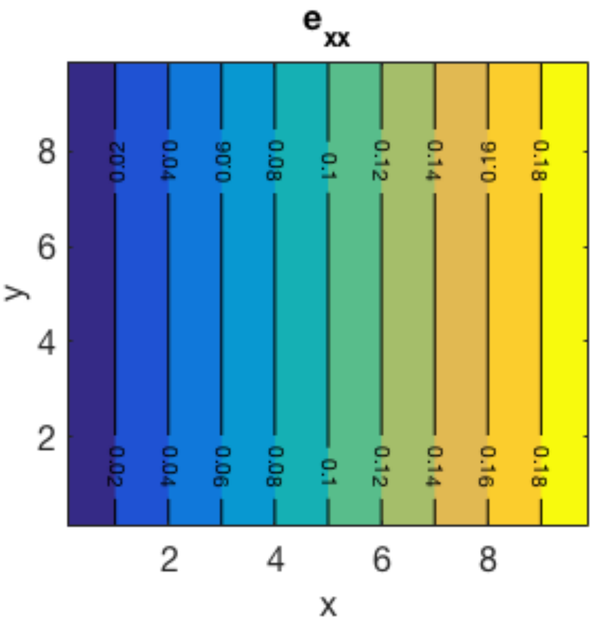
$$\frac{\partial u_y}{\partial y} = 0.4x$$

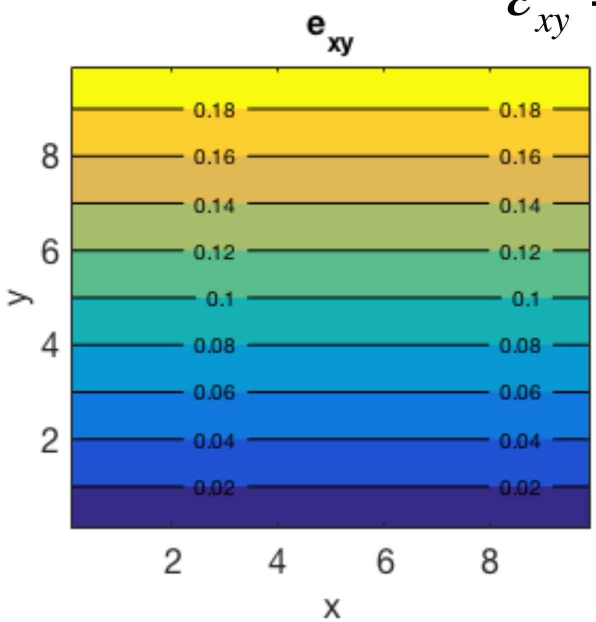
Example displacement – infinitesimal strain

$$\varepsilon_{xy} = \frac{1}{2}\left[\frac{\partial u_y}{\partial x} + \frac{\partial u_x}{\partial y}\right]$$

$$\frac{\partial u_x}{\partial x} = 0.2x$$

$$\varepsilon_{xx} = \frac{\partial u_x}{\partial x}$$





$$\frac{\partial u_x}{\partial y} = 0$$

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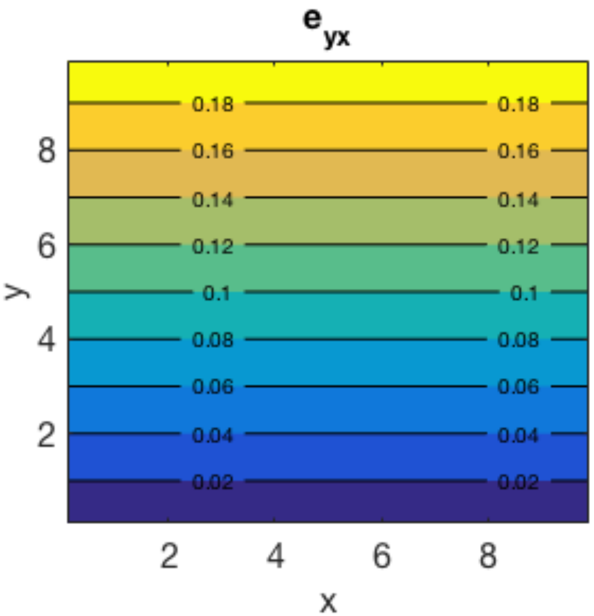
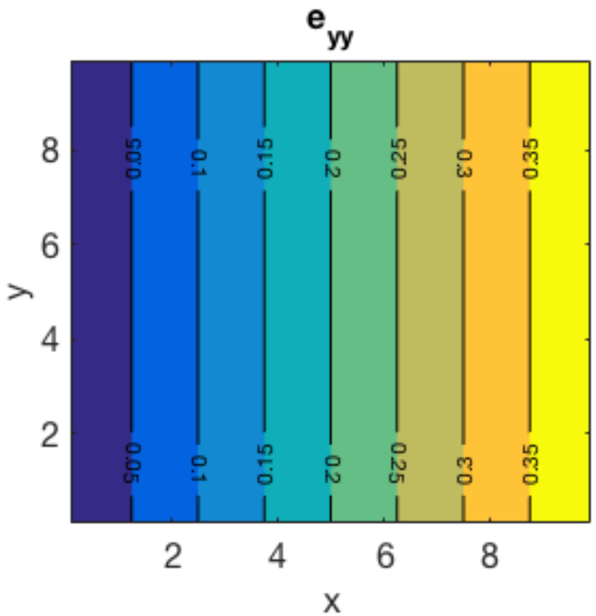
$$\frac{\partial u_y}{\partial x} = 0.4y$$

$$\varepsilon_{yx} = \varepsilon_{xy}$$

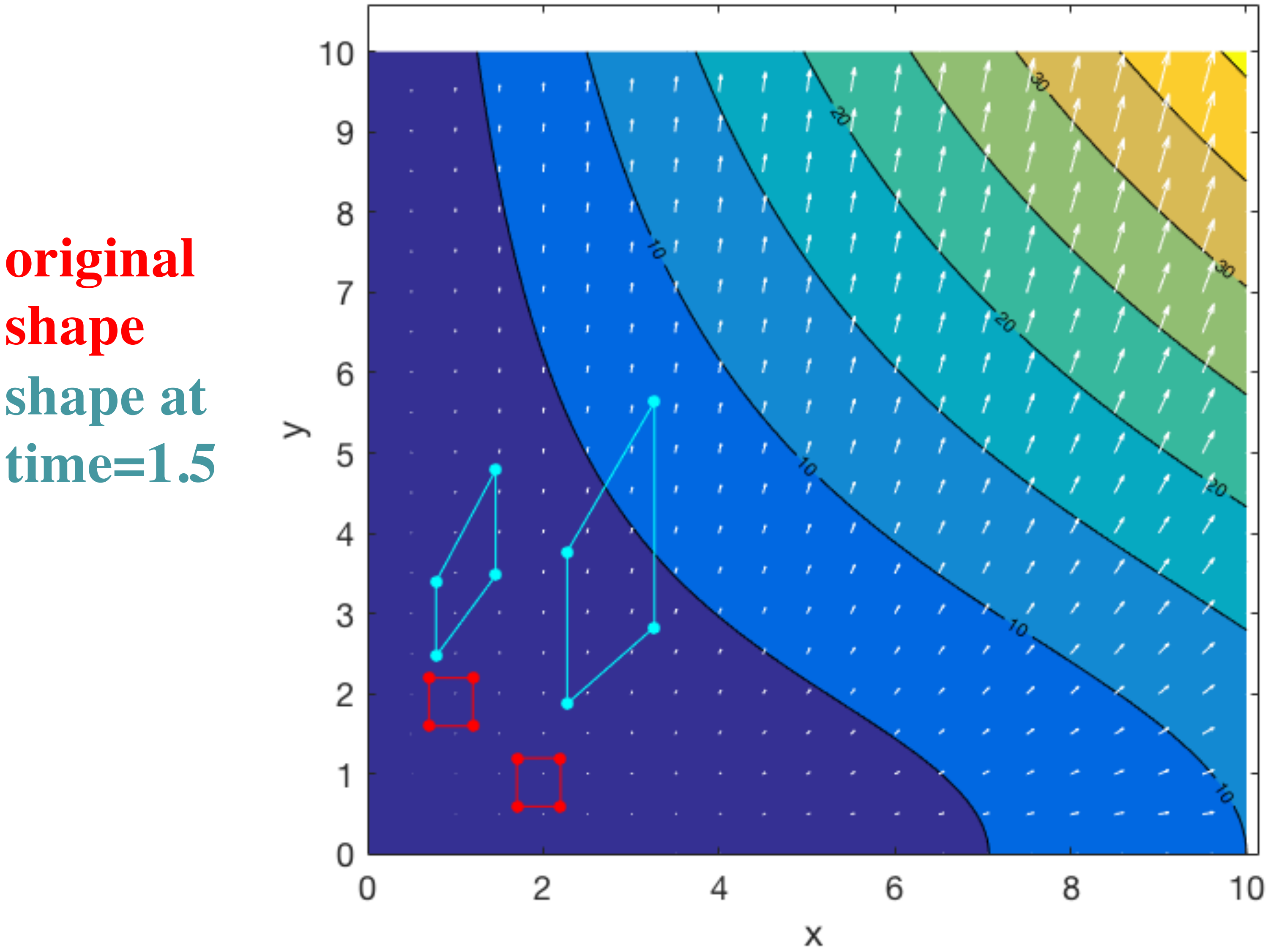




$$\frac{\partial u_y}{\partial y} = 0.4x$$

$$\varepsilon_{yy} = \frac{\partial u_y}{\partial y}$$

Deformation after finite strain

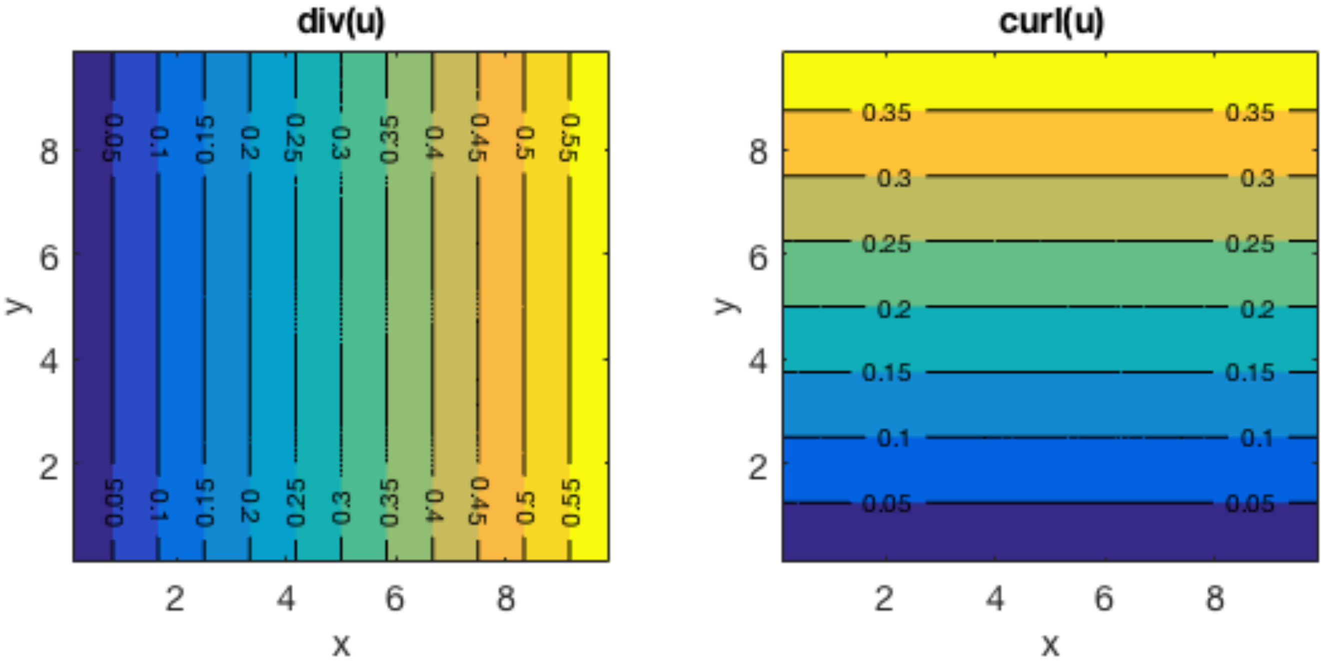


displacement in time interval =1

$$u_x = 0.1x^2$$

$$u_y = 0.4xy$$

Example displacement – infinitesimal strain



$$\nabla \cdot \mathbf{u} = \frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} \qquad \nabla \times \mathbf{u} = \left(\frac{\partial u_y}{\partial x} - \frac{\partial u_x}{\partial y} \right) \hat{\mathbf{e}}_z$$

Try later: squarestrain.ipynb

Rotation tensor and rotation vector

For any antisymmetric tensor $\mathbf{W}$, a corresponding ***dual*** or ***axial vector*** $\mathbf{w}$ can be found so that

$$\mathbf{W} \cdot \mathbf{a} = \mathbf{w} \times \mathbf{a}$$

How does vector $\mathbf{w}$ relate to the components of $\mathbf{W}$?

$$\mathbf{w} = \qquad \hat{\mathbf{e}}_1 \qquad \hat{\mathbf{e}}_2 \qquad \hat{\mathbf{e}}_3$$

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$$\mathbf{w} = \qquad \hat{\mathbf{e}}_1 \qquad \hat{\mathbf{e}}_2 \qquad \hat{\mathbf{e}}_3$$

$$\underline{\underline{W}} = \begin{bmatrix} 0 & W_{12} & W_{13} \\ -W_{12} & 0 & W_{23} \\ -W_{13} & -W_{23} & 0 \end{bmatrix} \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} = \begin{pmatrix} a_2 W_{12} + a_3 W_{13} \\ \vdots \\ \vdots \end{pmatrix} \Rightarrow \begin{matrix} w_2 = W_{12} \\ w_3 = -W_{13} \end{matrix}$$

$$\vec{w} \times \vec{a} = (w_2 a_3 - w_3 a_2, \; \cdot \; , \; \cdot \;)$$

Rotation tensor and rotation vector

For any antisymmetric tensor $\mathbf{W}$, a corresponding ***dual*** or ***axial vector*** $\mathbf{w}$ can be found so that

$$\mathbf{W} \cdot \mathbf{a} = \mathbf{w} \times \mathbf{a}$$

Vector $\mathbf{w}$ relates to the components of $\mathbf{W}$ as:

$$\mathbf{w} = -W_{23}\hat{\mathbf{e}}_1 + W_{13}\hat{\mathbf{e}}_2 - W_{12}\hat{\mathbf{e}}_3$$

For the rotation tensor, an equivalent rotation vector exists:

$$\boldsymbol{\omega} \cdot \mathbf{dx} = r_\omega \times \mathbf{dx} \qquad \text{where:} \qquad r_\omega = \frac{1}{2} \nabla \times \mathbf{u}$$

Note that $\boldsymbol{\omega}$ only describes the overall rigid body rotation, not the total rotation of each individual segment $\mathbf{dx}$, which is also influenced by $\boldsymbol{\varepsilon}$

infinitesimal strain tensor properties

transform to fault plane *coordinate frame*:

$\varepsilon_{nn} \quad = \varepsilon_{11}\cos^2\phi + \varepsilon_{21}\sin\phi\cos\phi + \varepsilon_{12}\sin\phi\cos\phi + \varepsilon_{22}\sin^2\phi$

$\varepsilon_{ns} \quad = \varepsilon_{11}\sin\phi\cos\phi + \varepsilon_{21}\sin^2\phi - \varepsilon_{12}\cos^2\phi - \varepsilon_{22}\sin\phi\cos\phi$

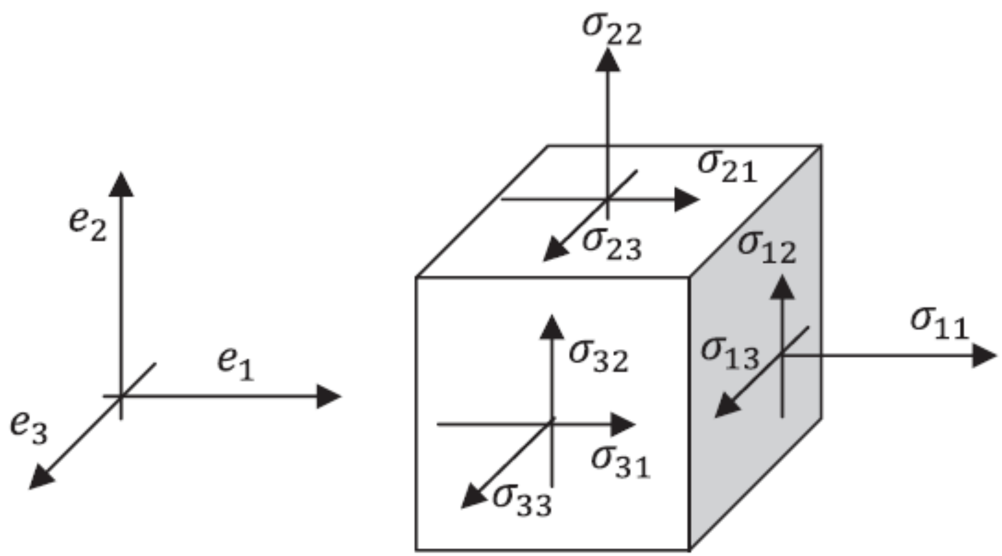
$\varepsilon_1, \varepsilon_2, \varepsilon_3$ - *principal strains* : minimum, maximum and intermediate fractional length changes

isotropic, *deviatoric* strain: $\varepsilon_{ij} = (\theta/3)\delta_{ij} + \varepsilon'_{ij}$

- $\mathrm{tr}(\boldsymbol{\varepsilon}) = \theta$ = sum of normal strains=volume change
- ε'_{ij} is deviatoric strain, change in shape, involves no change in volume
- $\mathrm{tr}(\boldsymbol{\varepsilon}') = 0$, does not imply $\varepsilon'_{ij} = 0$ for i=j
- $\varepsilon_{ij} = 0$ for $i \neq j$ does not ensure no volume change

Stress components

Reminder



traction on a plane $$\mathbf{t} = \begin{bmatrix} \sigma_{11} & \sigma_{21} & \sigma_{31} \\ \sigma_{12} & \sigma_{22} & \sigma_{32} \\ \sigma_{13} & \sigma_{23} & \sigma_{33} \end{bmatrix} \cdot \hat{\mathbf{n}}$$

what is $\hat{\mathbf{e}}_1 \cdot \mathbf{t} = \hat{\mathbf{e}}_1 \cdot \boldsymbol{\sigma}^T \cdot \hat{\mathbf{n}}$?

t_1 on plane with normal $\hat{\mathbf{n}}$

what is $\hat{\mathbf{e}}_1 \cdot \boldsymbol{\sigma}^T \cdot \hat{\mathbf{e}}_1$? σ_{11}

what is $\hat{\mathbf{e}}_1 \cdot \boldsymbol{\sigma}^T \cdot \hat{\mathbf{e}}_2$? σ_{21}

Strain components

$$\boldsymbol{\varepsilon} = \begin{bmatrix} \varepsilon_{11} & \varepsilon_{12} & \varepsilon_{13} \\ \varepsilon_{21} & \varepsilon_{22} & \varepsilon_{23} \\ \varepsilon_{31} & \varepsilon_{32} & \varepsilon_{33} \end{bmatrix}$$

$$\hat{\mathbf{e}}_1 \cdot \boldsymbol{\varepsilon} \cdot \hat{\mathbf{e}}_1 = \varepsilon_{11}$$

$$\hat{\mathbf{e}}_1 \cdot \boldsymbol{\varepsilon} \cdot \hat{\mathbf{e}}_2 = \varepsilon_{12}$$

$\boldsymbol{\varepsilon} \cdot \hat{\mathbf{p}} = \mathbf{p}'$ the unit vector $\hat{\mathbf{p}}$ after deformation by $\boldsymbol{\varepsilon}$

$\hat{\mathbf{p}} \cdot \boldsymbol{\varepsilon} \cdot \hat{\mathbf{p}} =$ elongation by $\boldsymbol{\varepsilon}$ of unit vector $\hat{\mathbf{p}}$ in direction $\hat{\mathbf{p}}$

$= \hat{\mathbf{p}} \cdot \mathbf{p}' = |\mathbf{p}'| \cos\alpha$

Strain Rate Tensor

$$\frac{\mathrm{D}}{\mathrm{D}t}\mathbf{dr} = \nabla\mathbf{v}$$

In similar way as strain tensor, a tensor that describes the rate of change of deformation can be defined from **velocity gradient**:

$$\nabla\mathbf{v} = \begin{bmatrix} \frac{\partial \mathrm{v}_1}{\partial \mathrm{x}_1} & \frac{\partial \mathrm{v}_1}{\partial \mathrm{x}_2} & \frac{\partial \mathrm{v}_1}{\partial \mathrm{x}_3} \\ \frac{\partial \mathrm{v}_2}{\partial \mathrm{x}_1} & \frac{\partial \mathrm{v}_2}{\partial \mathrm{x}_2} & \frac{\partial \mathrm{v}_2}{\partial \mathrm{x}_3} \\ \frac{\partial \mathrm{v}_3}{\partial \mathrm{x}_1} & \frac{\partial \mathrm{v}_3}{\partial \mathrm{x}_2} & \frac{\partial \mathrm{v}_3}{\partial \mathrm{x}_3} \end{bmatrix}$$

$$\nabla\mathbf{v} = \frac{1}{2}\left(\nabla\mathbf{v} + \nabla\mathbf{v}^T\right) + \frac{1}{2}\left(\nabla\mathbf{v} - \nabla\mathbf{v}^T\right)$$

$$\nabla\mathbf{v} = \mathbf{D} + \mathbf{W}$$

Velocity gradient tensor is the sum of **strain rate** and **vorticity** tensors