(1) • Given stress tensor $\underline{\underline{\sigma}} = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 0 \end{bmatrix}$ MPa

On planes w/ normal $\hat{n}_1 = (1,0,0)$ and $\hat{n}_3 = (0,0,1)$

determine (a) <u>normal stress</u>

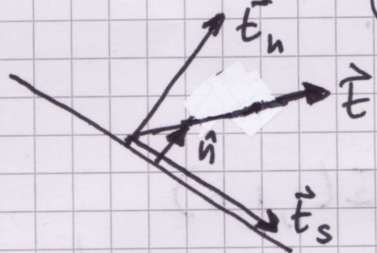
traction on plane 1 $= \vec{t}_1 = \underline{\underline{\sigma}} \cdot \hat{n}_1 = (1,2,3)$ MPa

normal stress $= \hat{n}_1 \cdot \vec{t}_1 = 1$ MPa $= \sigma_{11}$

on plane 3 $\vec{t}_3 = \underline{\underline{\sigma}} \cdot \hat{n}_3 = (3,5,0)$ MPa

normal stress $= \hat{n}_3 \cdot \vec{t}_3 = 0$ MPa $= \sigma_{33}$

(b) total shearing stress



$\Rightarrow |\vec{t}|^2 = |\vec{t}_n|^2 + |\vec{t}_s|^2$

or $\vec{t}_s = \vec{t} - \vec{t}_n$

length total shearing stress

plane 1 $\Rightarrow |t_s|^2 = (1+4+9) - 1 \Rightarrow |t_s| = \sqrt{4+9} \approx 3.61$ MPa

plane 3 $\Rightarrow |t_s|^2 = |\vec{t}_3|^2 \Rightarrow |t_s| = \sqrt{9+25} \approx 5.83$ MPa.

• (c) traction on a plane w/ a normal in the direction $2\hat{e}_1 + 2\hat{e}_2 + \hat{e}_3$, i.e. $\vec{n} = (2,2,1)$

and normalised $\hat{n} = \frac{1}{3}(2,2,1)$

$\vec{t} = \underline{\underline{\sigma}} \cdot \hat{n} = \frac{1}{3} \begin{pmatrix} 9 \\ 17 \\ 16 \end{pmatrix}$ MPa

(2) $\underline{\underline{\sigma}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

(a) only plane on which $\vec{t} = 0$?

$$\vec{t} = \underline{\underline{\sigma}} \cdot \hat{n} = \begin{pmatrix} n_1 \\ -n_2 \\ 0 \end{pmatrix} \overset{?}{=} \vec{0}$$

This is true if $n_1 = n_2 = 0$, n_3 is free $\Big\}$ $\hat{n} = (0,0,1)$

so indeed plane where $\hat{n} = \hat{e}_3$

(b) planes with no normal stress?

$$\vec{t} \cdot \hat{n} = n_1^2 - n_2^2 \overset{?}{=} 0$$

$\Rightarrow$ $n_1 = n_2$ $\longrightarrow$ $\hat{n} = \frac{1}{2}\sqrt{2}(1,1,0)$

or

$n_1 = -n_2$ $\qquad \hat{n} = \frac{1}{2}\sqrt{2}(1,-1,0)$ $\Big\}$ three planes

or

$n_1 = n_2 = 0$ (as in a) $\qquad \hat{n} = (0,0,1)$

(3) $\underline{\underline{\sigma}} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ MPa

change coordinate system such that
$\hat{e}_1' = \hat{e}_2$; $\hat{e}_2' = -\hat{e}_1$, $\hat{e}_3' = \hat{e}_3$

to obtain $\sigma_{11}', \sigma_{21}', \sigma_{33}'$

either use $\sigma_{ij}' = (\hat{e}_i')^T \cdot \underline{\underline{\sigma}} \cdot \hat{e}_j'$

$\uparrow$ single component

$\Rightarrow \sigma_{11}' = (0,1,0) \cdot \underline{\underline{\sigma}} \cdot \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$
$= 2$ MPa

or use the transformation matrices A and A^T

$\underline{\underline{\sigma}}' = A \cdot \underline{\underline{\sigma}} \cdot A^T$ where $A^T = \{ \hat{e}_1', \hat{e}_2', \hat{e}_3' \}$

$$= \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \hat{e}_i \cdot \hat{e}_j'$$

new basis vectors as columns

$$\Rightarrow \underline{\underline{\sigma}}' = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \text{ MPa}$$

(4) Stress state where only nonvanishing stress components are set of shearing stresses is called simple shear

Take $\sigma_{12} = \sigma_{21} = \tau$, all other $\sigma_{ij} = 0$

$$\Rightarrow \quad \underline{\underline{\sigma}} = \begin{bmatrix} 0 & \tau & 0 \\ \tau & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

(a) <u>principal values</u> and directions

Solve $-\lambda^3 + tr(\underline{\underline{\sigma}})\lambda^2 - minor(\underline{\underline{\sigma}})\lambda + det(\underline{\underline{\sigma}}) = 0$

$tr(\underline{\underline{\sigma}}) = 0 = \sigma_{ii}$

$$minor(\underline{\underline{\sigma}}) = \begin{vmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{vmatrix} + \begin{vmatrix} \sigma_{11} & \sigma_{13} \\ \sigma_{31} & \sigma_{33} \end{vmatrix} + \begin{vmatrix} \sigma_{22} & \sigma_{23} \\ \sigma_{32} & \sigma_{33} \end{vmatrix}$$

$$= -\tau^2 + 0 + 0 = -\tau^2$$

$det(\underline{\underline{\sigma}}) = 0$

$$\Rightarrow -\lambda^3 + \tau^2\lambda = 0$$

$$-\lambda(\lambda^2 - \tau^2) = 0$$

$$-\lambda(\lambda+\tau)(\lambda-\tau) = 0 \Rightarrow \lambda_1 = -\tau, \ \lambda_2 = \tau, \ \lambda_3 = 0$$

$\Rightarrow$ diagonalised stress tensor $\begin{bmatrix} -\tau & & \phi \\ & \tau & \\ \phi & & 0 \end{bmatrix}$ $\to$ can be arranged in preferred order as long as accompanying eigenvectors similarly ordered

<u>eigenvectors</u> $x^{(1)}, x^{(2)}, x^{(3)}$

- $\underline{\underline{\sigma}} \cdot \vec{x}^{(1)} = -\tau \vec{x}^{(1)}$

$$\begin{pmatrix} \tau x_2^{(1)} \\ \tau x_1^{(1)} \\ 0 \end{pmatrix} = \begin{pmatrix} -\tau x_1^{(1)} \\ -\tau x_2^{(1)} \\ -\tau x_3^{(1)} \end{pmatrix} \Rightarrow \begin{aligned} \tau x_2^{(1)} &= -\tau x_1^{(1)} \Rightarrow x_2^{(1)} = -x_1^{(1)} \\ \tau x_1^{(1)} &= -\tau x_2^{(1)} \\ 0 &= -\tau x_3^{(1)} \Rightarrow x_3^{(1)} = 0 \end{aligned}$$

$\Rightarrow \vec{x}^{(1)}$ can be $\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$

normalised $\hat{x}^{(1)} = \frac{1}{2}\sqrt{2}\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$

- $\underline{\underline{\sigma}} \cdot \vec{x}^{(2)} = \tau \vec{x}^{(2)}$

$$\left.\begin{aligned} \tau x_2^{(2)} &= \tau x_1^{(2)} \\ \tau x_1^{(2)} &= \tau x_2^{(2)} \end{aligned}\right\} x_2^{(2)} = x_1^{(2)} \quad \Bigg\} \quad \vec{x}^{(2)} \text{ can be } \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$$

$$0 = \tau x_3^{(2)} \Rightarrow x_3^{(2)} = 0 \qquad \hat{x}^{(2)} = \frac{1}{2}\sqrt{2}\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$$

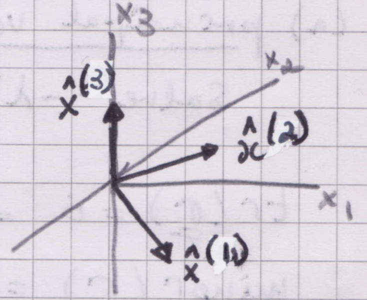
Last eigenvector

$$\underline{\underline{\sigma}} \cdot \vec{x}^{(3)} = 0 \cdot \vec{x}^{(3)}$$

$$\left.\begin{array}{l} \Rightarrow \quad \vec{x}^{(3)}_2 = 0 \\ \qquad \vec{x}^{(3)}_1 = 0 \\ \qquad 0 = 0 \cdot \vec{x}^{(3)}_3 \rightarrow \text{satisfied by any } \vec{x}^{(3)}_3 \end{array}\right\} \quad \hat{x}^{(3)} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

Can check that $\hat{x}^{(1)} \times \hat{x}^{(2)} = \hat{x}^{(3)}$

$\Rightarrow$ also a righthanded system



(b) Max. shearing stress?
Stresses in original coordinate system

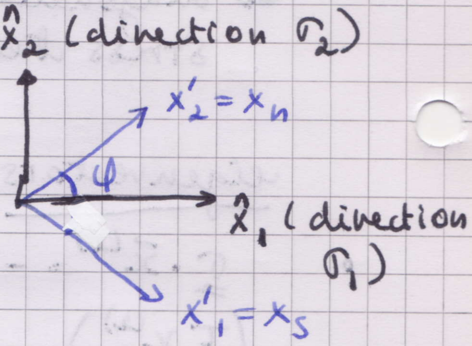
directions $\hat{x}^{(1)}$ and $\hat{x}^{(2)}$ at angle of $45°$ relative to original coordinate system
$\Rightarrow$ max. shear makes angle $45°$ with principal stress directions $\hat{x}^{(1)}, \hat{x}^{(2)}$

Can also show as follows:
Shear stress on a plane in 2D stress state $\begin{bmatrix} -\tau & 0 \\ 0 & \tau \end{bmatrix} = \begin{bmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \end{bmatrix}$

$$\sigma'_{ns} = \sigma_{11} \cos\varphi \sin\varphi + \sigma_{21} \sin^2\varphi - \sigma_{12} \cos^2\varphi - \sigma_{22} \cos\varphi \sin\varphi$$

$$= (\sigma_1 - \sigma_2) \cos\varphi \sin\varphi$$

max where $\dfrac{\partial \sigma'_{ns}}{\partial \varphi} = 0$



$$\frac{\partial \sigma'_{ns}}{\partial \varphi} = (\sigma_1 - \sigma_2)\left[\cos^2\varphi - \sin^2\varphi\right]$$

$$= (\sigma_1 - \sigma_2) \cos 2\varphi$$

$$= 0 \text{ if } \cos 2\varphi = 0 \Rightarrow \varphi = 45° \text{ or } \varphi = 135°$$

i.e. indeed original coordinate axes were those in the direction of maximum shear stress
$\Rightarrow$ max. shear stresses on planes w/ normal $(1,0,0)$ and normal $(0,1,0)$

values of these shear stresses $= \tau$

(5) Given the stress: $\underline{\underline{\sigma}} = \begin{bmatrix} x_1 + x_2 & \sigma_{12}(x_1, x_2) & 0 \\ \sigma_{12}(x_1, x_2) & x_1 - 2x_2 & 0 \\ 0 & 0 & x_2 \end{bmatrix}$

Find σ_{12} if equilibrium w/ zero body force, $\vec{f} = 0$

Also on plane $x_1 = 1$, $\vec{t} = (1+x_2)\hat{e}_1 + (5 - x_2)\hat{e}_2$

- Plane $x_1 = 1$ has as normal $\hat{n} = (1, 0, 0)$

Stress on such a plane

$$\vec{t} = \underline{\underline{\sigma}} \cdot \hat{n} = \begin{pmatrix} x_1 + x_2 \\ \sigma_{12} \\ 0 \end{pmatrix} \quad \text{at } x_1 = 1 \text{ this becomes } \begin{pmatrix} 1 + x_2 \\ \sigma_{12} \\ 0 \end{pmatrix}$$

$$\vec{t} = \begin{pmatrix} 1 + x_2 \\ 5 - x_2 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 + x_2 \\ \sigma_{12} \\ 0 \end{pmatrix} \quad \Rightarrow \sigma_{12}(x_1 = 1) = 5 - x_2$$

assuming linear in x_1

$\sigma_{12} = a \cdot x_1 + c - x_2$

where $a + c = 5$

- For equilibrium $\rightarrow$ no acceleration, no body force

$$\frac{\partial \sigma_{ji}}{\partial x_j} = 0$$

in x_1 direction: $\dfrac{\partial \sigma_{11}}{\partial x_1} + \dfrac{\partial \sigma_{21}}{\partial x_2} + \dfrac{\partial \sigma_{31}}{\partial x_3} \overset{?}{=} 0$

$1 \quad -1 \quad +0 \quad = 0$ ✓ satisfied

in x_2 direction: $\dfrac{\partial \sigma_{12}}{\partial x_1} + \dfrac{\partial \sigma_{22}}{\partial x_2} + \dfrac{\partial \sigma_{32}}{\partial x_3} \overset{?}{=} 0$

$a \quad -2 \quad +0 \quad \overset{?}{=} 0$

$\Rightarrow a = 2 \quad \Rightarrow c = 3$

In x_3 direction $\dfrac{\partial \sigma_{13}}{\partial x_1} + \dfrac{\partial \sigma_{23}}{\partial x_2} + \dfrac{\partial \sigma_{33}}{\partial x_3} = 0$

$0 \quad 0 \quad 0 \quad = 0$ ✓ satisfied

$$\Rightarrow \boxed{\sigma_{12}(x_1, x_2) = 2x_1 - x_2 + 3}$$

(6) $\underline{\underline{\sigma}}' = \underline{\underline{\sigma}} - \frac{\sigma_{kk}}{3}\underline{\underline{I}} = \begin{bmatrix} \sigma_{11} - \frac{\sigma_{kk}}{3} & \sigma_{12} & \sigma_{13} \\ \sigma_{12} & \sigma_{22} - \frac{\sigma_{kk}}{3} & \sigma_{23} \\ \sigma_{13} & \sigma_{23} & \sigma_{33} - \frac{\sigma_{kk}}{3} \end{bmatrix}$

(a) $tr(\underline{\underline{\sigma}}') = \sigma_{11} + \sigma_{22} + \sigma_{33} - 3\left(\frac{\sigma_{kk}}{3}\right)$

$= \sigma_{kk} - \sigma_{kk} = 0$

(b) $tr(\underline{\underline{\sigma}}) = 6+3+9 = 18 \Rightarrow \frac{\sigma_{kk}}{3} = 6$

$\Rightarrow \underline{\underline{\sigma}}' = \begin{bmatrix} 0 & 5 & -2 \\ 5 & -3 & 4 \\ -2 & 4 & 3 \end{bmatrix} \Rightarrow$ diagonal elements of $\underline{\underline{\sigma}}$ modified while for $i \neq j$ $\sigma'_{ij} = \sigma_{ij}$

(c) principal stress directions of $\underline{\underline{\sigma}}'$?

remember that tensor transformation is linear and diagonalising of $\underline{\underline{\sigma}}$ comes down to transforming $\underline{\underline{\sigma}}$ into its principal stress direction coordinate fram

$\Rightarrow \begin{bmatrix} \lambda_1 & & \emptyset \\ & \lambda_2 & \\ \emptyset & & \lambda_3 \end{bmatrix} = \mathbb{Q}^T \cdot \underline{\underline{\sigma}} \cdot \mathbb{Q}$ where $\mathbb{Q} = \{\hat{x}_1, \hat{x}_2, \hat{x}_3\}$ with $\hat{x}_i$ the principal stress directions

$\underline{\underline{\sigma}} = \underline{\underline{\sigma}}' + \frac{\sigma_{kk}}{3}\underline{\underline{I}}$ so

$\begin{bmatrix} \lambda_1 & & \emptyset \\ & \lambda_2 & \\ \emptyset & & \lambda_3 \end{bmatrix} = \mathbb{Q}^T \cdot \left[\underline{\underline{\sigma}}' + \frac{\sigma_{kk}}{3}\underline{\underline{I}}\right] \cdot \mathbb{Q}$

The unit tensor $\underline{\underline{I}}$ is invariant to coordinate transformation, so

$$\begin{bmatrix} \lambda_1 & & \emptyset \\ & \lambda_2 & \\ \emptyset & & \lambda_3 \end{bmatrix} = \underbrace{\mathbb{Q}^T \cdot \underline{\underline{\sigma}}' \cdot \mathbb{Q}}_{\text{must also be a diagonal tensor}} + \begin{bmatrix} \frac{\sigma_{kk}}{3} & & \emptyset \\ & \frac{\sigma_{kk}}{3} & \\ \emptyset & & \frac{\sigma_{kk}}{3} \end{bmatrix}$$

must also be a diagonal tensor, i.e. same transformation that diagonalises $\underline{\underline{\sigma}}$ also does so for $\underline{\underline{\sigma}}'$