Short Solutions
Problem Set 6
ACSE-2
2020

1

a) $\vec{n}_1 = \hat{e}_1 = (1,0,0)$

$\vec{t}_{1,n} = \hat{e}_1 \cdot \underline{\underline{\sigma}}^T \cdot \hat{e}_1 = 1\ \mathrm{MPa}$

$\hat{n}_2 = \hat{e}_3 = (0,0,1)$

$\vec{t}_{2,n} = \hat{e}_3 \cdot \underline{\underline{\sigma}}^T \cdot \hat{e}_3 = 0\ \mathrm{MPa}$

b) $|\vec{t}_s|^2 = |\vec{t}|^2 - |\vec{t}_n|^2$

(draction considered as normal + shear component)

$|\vec{t}_{1,s}|^2 = (1+4+9) - 1 \quad \rightarrow \quad |t_{1,s}| = 3.61\ \mathrm{MPa}$

$|t_{2,s}| = 5.83\ \mathrm{MPa}$

c) normal $= \frac{1}{3}(2,2,1)$ unit length

$\vec{t} = \underline{\underline{\sigma}}^T \cdot \hat{n} = \frac{1}{3}\begin{pmatrix} 9 \\ 17 \\ 16 \end{pmatrix}\ \mathrm{MPa}$

2

a) for which $\hat{n}$ is $\vec{t} = \underline{\underline{\sigma}}^T \cdot \hat{n} = \vec{0}$?

$(n_1, -n_2, 0) = (0,0,0)$?

$\Rightarrow$ requires $n_1 = n_2 = 0$, n_3 free

$\Rightarrow \hat{n} = (0,0,1)$

b) no normal stress

$\Rightarrow \vec{t} \cdot \hat{n} = (n_1^2 - n_2^2) \overset{?}{=} 0$

$\Rightarrow n_1 = n_2$ or $n_1 = -n_2$ or $n_1 = n_2 = 0$

$n_1 = n_2 \Rightarrow \hat{n} = \frac{1}{2}\sqrt{2}(1,1,0)$

$n_1 = -n_2 \Rightarrow \hat{n} = \frac{1}{2}\sqrt{2}(1,-1,0)$

$n_1 = n_2 = 0 \Rightarrow \hat{n} = (0,0,1)$

③ $\hat{e}'_1 = \hat{e}_2$; $\hat{e}'_2 = -\hat{e}_1$; $\hat{e}'_3 = \hat{e}_3$

transformation matrix $A^T = \{\hat{e}'_1, \hat{e}'_2, \hat{e}'_3\}$

$\underline{\underline{\sigma}}' = A \cdot \underline{\underline{\sigma}} \cdot A^T$

$\Rightarrow \sigma'_{11} = 2$; $\sigma'_{21} = -1$; $\sigma'_{33} = 1$ all in MPa.

④

(a) principal stress = eigenvalues

$-\lambda^3 + \tau^2 \lambda = 0 \Rightarrow \lambda_1 = -\tau$; $\lambda_2 = \tau$, $\lambda_3 = 0$

eigenvectors = principal stress directions

$\underline{\underline{\sigma}} \cdot \vec{x}^{(1)} = -\tau \vec{x}^{(1)}$

$\Rightarrow x_2^{(1)} = -x_1^{(1)}$; $x_3^{(1)} = 0$

normalised: $\hat{x}^{(1)} = \frac{1}{2}\sqrt{2} \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$

$\hat{x}^{(2)} = \frac{1}{2}\sqrt{2} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$

$\hat{x}^{(3)} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ $\Rightarrow$ check that right-handed system

(b) max. shear stress $\rightarrow = \tau$

$\rightarrow$ original coordinate system which is rotated $45°$ relative to principal stress directions $\hat{x}^{(1)}$ and $\hat{x}^{(2)}$ (around $\hat{x}^{(3)}$)

or

$$\sigma'_{ns} = \sigma_{11} \cos\varphi \sin\varphi + \sigma_{21} \sin^2\varphi - \sigma_{12} \cos^2\varphi - \sigma_{22} \cos\varphi \sin\varphi$$

$$= (\sigma_1 - \sigma_2) \cos\varphi \sin\varphi$$

$\hookrightarrow$ in principal stress system

maximum when $\frac{\partial \sigma'_{ns}}{\partial \varphi} = 0 \Rightarrow \cos 2\varphi = 0$

$\varphi = 45°$ or $135°$

(5) plane $x_1 = 1$ $\rightarrow$ normal $(1,0,0)$

$\vec{t}$ on this plane $= \begin{pmatrix} x_1 + x_2 \\ \sigma_{12} \\ 0 \end{pmatrix} \overset{x_1=1}{=} \begin{pmatrix} 1 + x_2 \\ \sigma_{12} \\ 0 \end{pmatrix}$

$\Rightarrow \quad \sigma_{12}(x_1 = 1) = 5 - x_2$

take linear in x_1 $\Rightarrow$ $\sigma_{12} = \underbrace{a x_1 + c}_{=5 \text{ for } x_1 = 1} - x_2$

equilibrium: $\dfrac{\partial \sigma_{ji}}{\partial x_j} = 0$

in x_2 direction $\Rightarrow$ $\dfrac{\partial \sigma_{12}}{\partial x_1} + \dfrac{\partial \sigma_{22}}{\partial x_2} + \dfrac{\partial \sigma_{32}}{\partial x_3} = 0$

$a - 2 + 0 = 0$

$\Rightarrow a = 2, \; c = 3$

$\Rightarrow \sigma_{12}(x_1, x_2) = 2x_1 + 3 - x_2$

(6) (a) write out $\underline{\underline{\sigma}}'$ and sum diagonal elements $\Rightarrow$ $tr(\underline{\underline{\sigma}}') = 0$

(b) $\underline{\underline{\sigma}}' = 100 \begin{bmatrix} 0 & 5 & -2 \\ 5 & -3 & 4 \\ -2 & 4 & 3 \end{bmatrix}$ kPa

(c) $\underline{\underline{\sigma}} = \underline{\underline{\sigma}}' + \dfrac{\sigma_{kk}}{3} \underline{\underline{I}}$

if A transformation that diagonalises $\underline{\underline{\sigma}}$

$$\begin{bmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & \sigma_3 \end{bmatrix} = A \cdot \left[\underline{\underline{\sigma}}' + \frac{\sigma_{kk}}{3} \underline{\underline{I}} \right] \cdot A^T$$

$$= \underbrace{A \cdot \underline{\underline{\sigma}}' \cdot A^T}_{\text{A also diagonalises } \underline{\underline{\sigma}}'} + \frac{\sigma_{kk}}{3} \underline{\underline{I}}$$