Question 1

Using *x* as the distance along the plates and *y* as the distance between the plates, the useful momentum balance for the NS equation is that in the *x* direction. Leaving only the non-zero terms:

$$-\frac{dP}{dx}+\frac{d\tau_{yx}}{dy}+g_x\rho=0$$

If it assumed that the gravity points in the negative direction and that the pressure drop is in the positive direction:

$$\frac{d\tau_{yx}}{dy}=\rho g\sin(\theta)-\frac{\Delta P}{L}$$

Therefore

$$\tau_{yx}=\left(\rho g\sin(\theta)-\frac{\Delta P}{L}\right)y+A$$

As it is a Newtonian flow:

$$\tau_{yx}=\mu\frac{dv_x}{dy}$$

Integrating:

$$v_x=\frac{\left(\rho g\sin(\theta)-\frac{\Delta P}{L}\right)}{2\mu}y^2+Ay+B$$

Boundary conditions:

$v_x=0$ at $y=0$ and $y=H$

Therefore $B=0$

And $A=-\frac{\left(\rho g\cos(\theta)-\frac{\Delta P}{L}\right)}{2\mu}H$

$$v_x=\frac{\left(\rho g\sin(\theta)-\frac{\Delta P}{L}\right)}{2\mu}(y^2-Hy)$$

The volumetric flowrate (per unit in the *z* direction) is:

$$Q=\int_{y=0}^{y=H}v_x\,dy=\frac{\left(\frac{\Delta P}{L}-\rho g\sin(\theta)\right)H^3}{12\mu}$$

Flow stops when

$$\sin(\theta)=\frac{\Delta P}{L\,\rho\,g}$$

Question 2

a) Flow in the anulus requires the *z* direction momentum balance from the NS equation. Ignoring all the terms that are zero gives:

$$0 = -\frac{dP}{dz} + \mu\left[\frac{1}{r}\frac{d}{dr}\left(r\frac{du_z}{dr}\right)\right]$$

The pressure gradient is given:

$$\frac{dP}{dz} = \frac{\Delta P}{L}$$

Rearranging:

$$\frac{d}{dr}\left(r\frac{du_z}{dr}\right) = -\frac{\Delta P\, r}{L\mu}$$

Integrating once and rearranging:

$$\frac{du_z}{dr} = -\frac{\Delta P\, r}{2\,L\mu} + \frac{A}{r}$$

Integrating again:

$$u_z = -\frac{\Delta P\, r^2}{4\,L\mu} + A ln(r) + B$$

Boundary conditions

$v_z = 0$ at $r = R_i$ and $r = R_o$:

$$0 = -\frac{\Delta P\, {R_i}^2}{4\,L\mu} + A ln(R_i) + B$$

$$0 = -\frac{\Delta P\, {R_O}^2}{4\,L\mu} + A ln(R_O) + B$$

$$A = \frac{-\frac{\Delta P}{4\,L\mu}\left({R_o}^2 - {R_i}^2\right)}{ln\left(\frac{R_i}{R_o}\right)}$$

$$B = \frac{\Delta P\, {R_i}^2}{4\,L\mu} + \frac{\frac{\Delta P}{4\,L\mu}\left({R_o}^2 - {R_i}^2\right)}{ln\left(\frac{R_i}{R_o}\right)} ln(R_i)$$

$$u_z = -\frac{\Delta P}{4\,L\mu}\left(r^2 - {R_i}^2 + \frac{\left({R_o}^2 - {R_i}^2\right)}{ln\left(\frac{R_i}{R_o}\right)} ln\left(\frac{r}{R_i}\right)\right)$$