Continuum Mechanics Equations

General:

1.<u>Kinematics</u> – describing deformation and velocity without considering forces

2.<u>Dynamics</u> – equations that describe force balance, conservation of linear and angular momentum

3.<u>Thermodynamics</u> – relations temperature, heatflux, stress, entropy

Material-specific

4.<u>Constitutive equations</u> – relations describing how material properties vary as a function of T,P, stress,…. Such material properties govern dynamics (e.g., ***density***), response to stress (***viscosity***, ***elastic parameters***), heat transport (***thermal conductivity***, ***heat capacity***)

Rheology

$$\boxed{\text{deformation } (\boldsymbol{\varepsilon}) = \textit{rheology} \cdot \text{stress } (\boldsymbol{\sigma})}$$

material response to stress, depends on material, P,T, time, deformation history, environment (volatiles, water)

- *elastic*
- *viscous*
- *brittle*
- *plastic*

• experiments under simple stress conditions
$\Rightarrow$ strain evolution under constant stress, stress-strain rate diagrams

• thermodynamics + experimental parameters

• ab-initio calculations

Recap Fluid - Solid

- What is a solid?

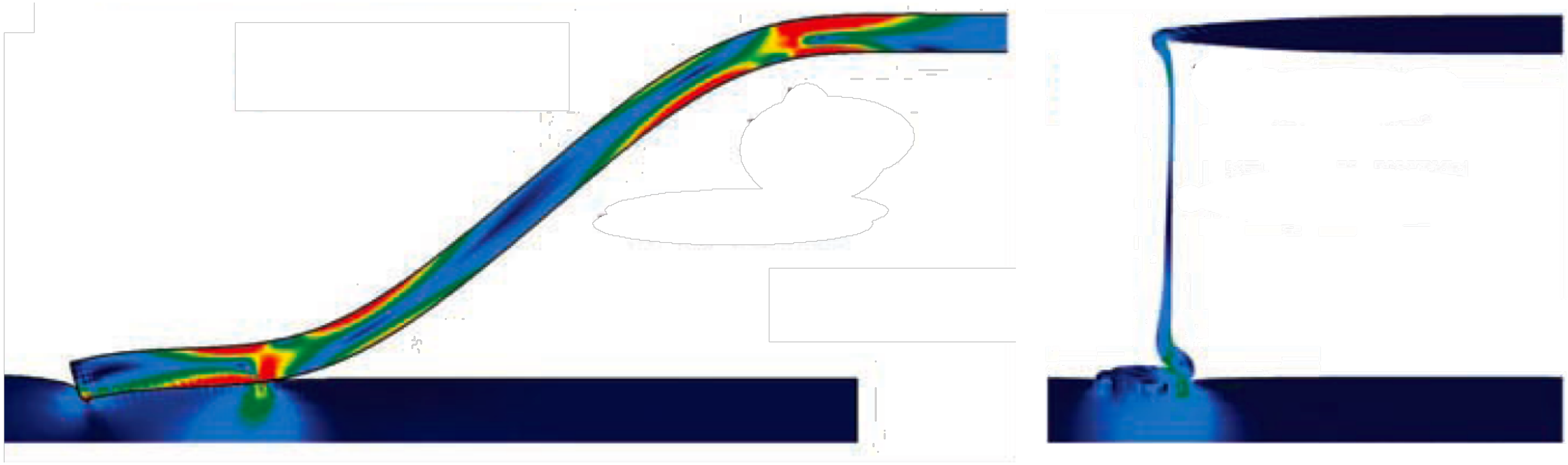
  A solid acquires finite deformation under stress

  stress σ ~ *strain* ε

- What is a fluid?

  A material that flows in response to applied stress

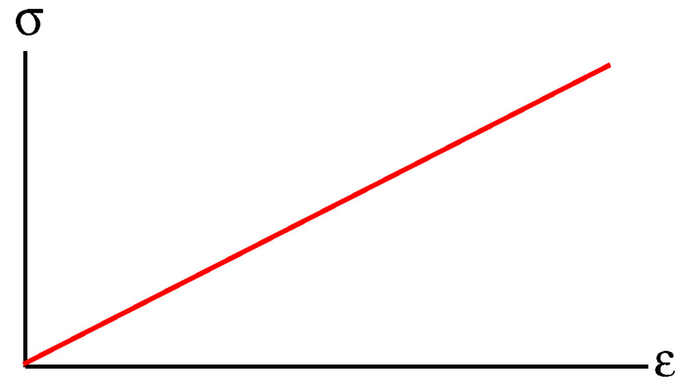
  stress σ ~ *strain rate* $D\varepsilon/Dt$

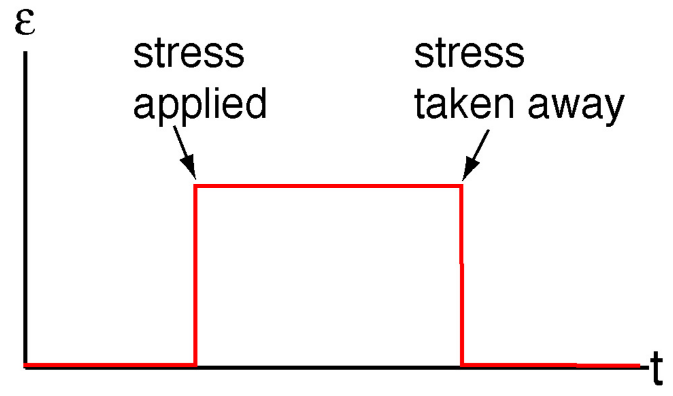


Figures from Funiciello et al. (2003a)

Elasticity

- linear response to load applied
- instantaneous
- completely recoverable
- below threshold (yielding) stress
- *dominates behaviour of coldest part of tectonic plates on time scales of up to 100 m.y. -> fault loading*
- *on time scale of seismic waves the whole Earth is elastic*





- $\sigma_{ij} = C_{ijkl}\varepsilon_{kl}$ - Hooke's law
  C_{ijkl} - rank 4 elasticity tensor
  3^4 elements, up to 21 independent

Elasticity tensor

C_{ijkl} $3^4=81$ elements (for n=3)

- symmetry of σ_{ij} and ε_{kl}

  $\Rightarrow$ only 36 independent elements

  Why 36?

Elasticity tensor

C_{ijkl} $3^4=81$ elements (for n=3)

- symmetry of σ_{ij} and ε_{kl}

  $\Rightarrow$ only 36 independent elements

$P=\boldsymbol{\sigma}:\mathbf{D}\approx \boldsymbol{\sigma}:\mathrm{D}\varepsilon/\mathrm{D}t=\mathrm{D}U/\mathrm{D}t$

- conservation of elastic energy $U=\boldsymbol{\sigma}:\varepsilon=\mathbf{C}:\varepsilon:\varepsilon \geq 0$

$\Rightarrow C_{ijkl}=C_{klij}$

$\Rightarrow$ only 21 independent elements - most general form of **C**

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$$\left.\begin{matrix} C_{1111} & C_{2222} & C_{3333} \\ C_{1212} & C_{1313} & C_{23}\,C_{23} \end{matrix}\right\} \text{6 symmetric}$$

$(36-6)/2=15$

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$\Rightarrow$ only 21 independent elements - most general form of **C**

- other symmetries further reduce the number of independent elements

Elasticity tensor

- for example for *isotropic* media

  Only 2 independent elements (λ, μ):

What is isotropic?

$$\sigma_{ij} = \lambda\delta_{ij}\delta_{kl}\varepsilon_{kl}+\alpha\delta_{ik}\delta_{jl}\varepsilon_{kl}+\beta\delta_{il}\delta_{jk}\varepsilon_{kl}$$
$$= \lambda\delta_{ij}\varepsilon_{kk}+\alpha\varepsilon_{ij}+\beta\varepsilon_{ji}$$
$$= \lambda\delta_{ij}\theta+(\alpha+\beta)\varepsilon_{ij}$$

3 isotropic rank 4 tensors:

$\delta_{ij}\delta_{kl}$, $\delta_{ik}\delta_{jl}$, $\delta_{il}\delta_{jk}$

$$\Rightarrow \sigma_{ij}= \lambda\theta\delta_{ij}+2\mu\varepsilon_{ij}$$

Hooke's law for isotropic material: 2 independent coefficients

Lamé constants

λ and **μ** : $\sigma_{ij} = \lambda\varepsilon_{kk}\delta_{ij} + 2\mu\varepsilon_{ij}$

Bulk and shear modulus

K and **μ=G** : $\sigma_{ij} = -p\delta_{ij} + \sigma'_{ij}$

where: $-p = \dfrac{\sigma_{kk}}{3} = \left(\lambda + \dfrac{2}{3}\mu\right)\theta = K\theta$ *hydrostatic*

$\sigma'_{ij} = \cancel{\lambda\theta\delta_{ij}} + 2\mu\varepsilon_{ij} + p\delta_{ij} = 2\mu\varepsilon'_{ij}$ *deviatoric*

$\downarrow$

$-(\cancel{\lambda} + \frac{2}{3}\mu)\theta\,\delta_{ij}$

$2\mu\underbrace{\left(\varepsilon_{ij} - \frac{\theta}{3}\delta_{ij}\right)}_{\varepsilon'_{ij}}$

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Young's modulus and Poisson's ratio

E and **ν** : $E = \sigma_{11}/\varepsilon_{11}$, $\nu=-\varepsilon_{33}/\varepsilon_{11}$ (uniaxial stress)

Determine in problem set

Wave equation

For infinitesimal deformation:
spatial coordinates ≈ material coordinates

v_i (spatial) ≈ $\partial u_\mathrm{i}/\partial t$
a_i (spatial) ≈ $\partial v_\mathrm{i}/\partial t = \partial^2 u_\mathrm{i}/\partial t^2$

Equation of motion: $\mathrm{f_i} + \partial\sigma_{\mathrm{ji}}/\partial \mathrm{x_j} = \rho\partial^2 \mathrm{u_i}/\partial \mathrm{t}^2$ (1)

Elastic rheology: $\sigma_{\mathrm{ij}} = \lambda\varepsilon_{\mathrm{kk}}\delta_{\mathrm{ij}} + 2\mu\varepsilon_{\mathrm{ij}}$ (2)

Substitute (2) in (1) if (infinitesimal) deformation is consequence of force balance

Wave equation

Equation of motion: $f_i + \partial\sigma_{ji}/\partial x_j = \rho\partial^2 u_i/\partial t^2$

Elastic rheology: $\sigma_{ij} = \lambda\varepsilon_{kk}\delta_{ij} + 2\mu\varepsilon_{ij}$

$$\partial\sigma_{ji}/\partial x_j = \lambda\partial\varepsilon_{kk}/\partial x_i + \mu\partial(\partial u_i/\partial x_j + \partial u_j/\partial x_i)/\partial x_j$$

$$= \lambda\partial(\partial u_k/\partial x_k)/\partial x_i + \mu\partial^2 u_i/\partial^2 x_j + \mu\partial(\partial u_j/\partial x_j)/\partial x_i$$

$\nabla \cdot \sigma$ = Write vector equation

Using: $\nabla^2 \mathbf{u} = \nabla(\nabla \cdot \mathbf{u}) - \nabla \times \nabla \times \mathbf{u}$

=> $\boxed{\rho\partial^2 \mathbf{u}/\partial t^2 = \mathbf{f} + (\lambda + 2\mu)\nabla(\nabla \cdot \mathbf{u}) - \mu\nabla \times \nabla \times \mathbf{u}}$

what type of deformation do the two terms represent?

Wave equation

Equation of motion: $f_i + \partial\sigma_{ji}/\partial x_j = \rho\partial^2 u_i/\partial t^2$

Elastic rheology: $\sigma_{ij} = \lambda\varepsilon_{kk}\delta_{ij} + 2\mu\varepsilon_{ij}$

$$\partial\sigma_{ji}/\partial x_j = \lambda\partial\varepsilon_{kk}/\partial x_i + \mu\partial(\partial u_i/\partial x_j + \partial u_j/\partial x_i)/\partial x_j$$

$$= \lambda\partial(\partial u_k/\partial x_k)/\partial x_i + \mu\partial^2 u_i/\partial^2 x_j + \mu\partial(\partial u_j/\partial x_j)/\partial x_i$$

$$\nabla\cdot\sigma = (\lambda+\mu)\nabla(\nabla\cdot\mathbf{u}) + \mu\nabla^2\mathbf{u}$$

Using: $\nabla^2\mathbf{u} = \nabla(\nabla\cdot\mathbf{u}) - \nabla\times\nabla\times\mathbf{u}$

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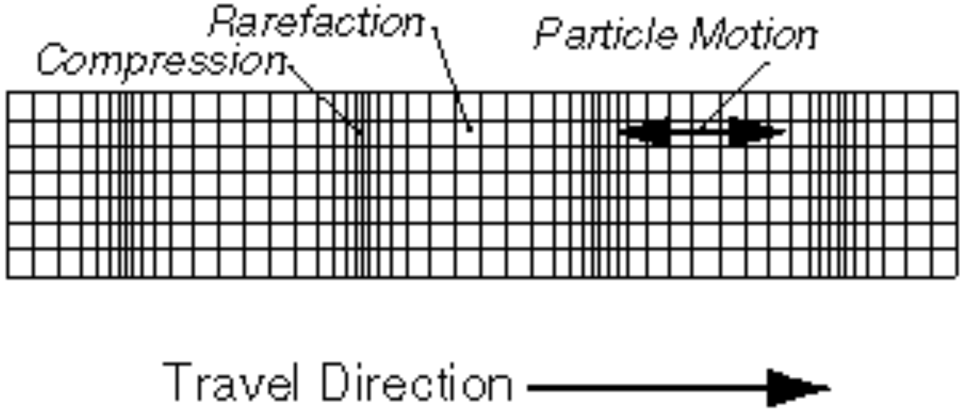
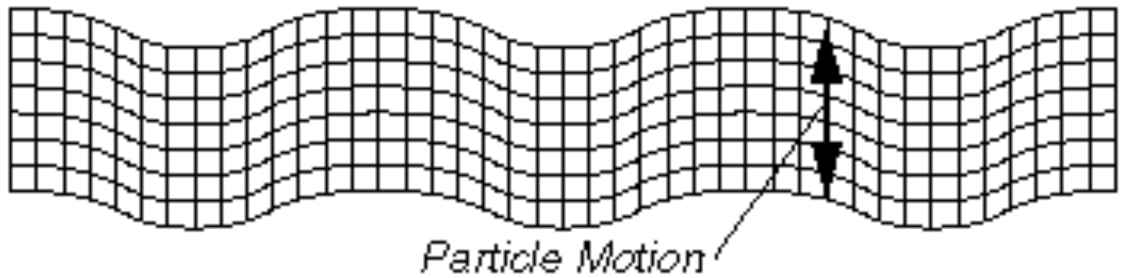
P wave


Rarefaction
Compression
Particle Motion
Travel Direction

S wave


Particle Motion