Diagonalizing

Real-valued, symmetric rank 2 tensors (square, symmetric matrices) can be diagonalized, i.e. a coordinate frame can be found, such that only the diagonal elements (normal stresses) remain.

For stress tensor, these elements, $\sigma_1, \sigma_2, \sigma_3$ are called the principal stresses

$$\begin{bmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & \sigma_3 \end{bmatrix}$$

Such a transformation can be cast as:

$\mathbf{T}\cdot\mathbf{x} = \lambda\mathbf{x}$

where $\mathbf{x}_i$ are eigenvectors or characteristic vectors

and λ_i are the eigenvalues, characteristic or principal values

$\Rightarrow(\mathbf{T}-\lambda\delta)\cdot\mathbf{x} = \mathbf{0}$

Non-trivial solution only if $\det(\mathbf{T}-\lambda\delta) = 0$

Diagonalising

Write out characteristic equation for n=2

$\det(\mathbf{T}-\lambda\boldsymbol{\delta}) = 0$ => eigenvalues λ_i $\quad$ i=1,n

$\det(\mathbf{T}-\lambda\boldsymbol{\delta}) = -\lambda^3+\mathrm{tr}(\mathbf{T})\lambda^2-\mathrm{minor}(\mathbf{T})\lambda+\det(\mathbf{T})=0$ $\quad$ for n=3

characteristic equation + coefficients are *tensor invariants*

$$I_1 = \mathrm{tr}(\mathbf{T}) = T_{11} + T_{22} + T_{33}$$

$$I_2 = \mathrm{minor}(\mathbf{T}) = \begin{vmatrix} T_{11} & T_{21} \\ T_{21} & T_{22} \end{vmatrix} + \begin{vmatrix} T_{11} & T_{31} \\ T_{31} & T_{33} \end{vmatrix} + \begin{vmatrix} T_{22} & T_{32} \\ T_{32} & T_{33} \end{vmatrix}$$

$$=$$

$$I_3 = \det(\mathbf{T}) = \begin{vmatrix} T_{11} & T_{21} & T_{31} \\ T_{21} & T_{22} & T_{32} \\ T_{31} & T_{32} & T_{33} \end{vmatrix} =$$

Diagonalising

Write out characteristic equation for n=2

$\det(\mathbf{T}-\lambda\boldsymbol{\delta}) = 0 \Rightarrow$ eigenvalues λ_i $\quad i=1,n$

$\det(\mathbf{T}-\lambda\boldsymbol{\delta}) = -\lambda^3+\mathrm{tr}(\mathbf{T})\lambda^2-\mathrm{minor}(\mathbf{T})\lambda+\det(\mathbf{T})=0$ $\quad$ for n=3

characteristic equation + coefficients are *tensor invariants*

$$I_1 = \mathrm{tr}(\mathbf{T}) = T_{11} + T_{22} + T_{33}$$

$$I_2 = \mathrm{minor}(\mathbf{T}) = \begin{vmatrix} T_{11} & T_{21} \\ T_{21} & T_{22} \end{vmatrix} + \begin{vmatrix} T_{11} & T_{31} \\ T_{31} & T_{33} \end{vmatrix} + \begin{vmatrix} T_{22} & T_{32} \\ T_{32} & T_{33} \end{vmatrix}$$

$$= T_{11}T_{22} + T_{22}T_{33} + T_{11}T_{33} - T_{21}^2 - T_{32}^2 - T_{31}^2$$

$$I_3 = \det(\mathbf{T}) = \begin{vmatrix} T_{11} & T_{21} & T_{31} \\ T_{21} & T_{22} & T_{32} \\ T_{31} & T_{32} & T_{33} \end{vmatrix} = T_{11}T_{22}T_{33} + 2T_{21}T_{32}T_{31} - T_{11}T_{32}^2 - T_{22}T_{31}^2 - T_{33}T_{21}^2$$

Eigenvalues, eigenvectors

For real-valued, symmetric rank 2 order *n* tensors

$\vec{x}^T \cdot \underline{\underline{S}} \cdot \vec{x} > 0$

$\vec{x} \neq 0$

- All eigenvalues are real
- If S is positive definite, then eigenvalues are positive
- Eigenvectors for two distinct λ are orthogonal.
- There are *n* linearly independent eigenvectors

$$\mathbf{T}\cdot\mathbf{x}_1=\lambda_1\mathbf{x}_1 \qquad \text{where } \lambda_1\neq\lambda_2$$
$$\mathbf{T}\cdot\mathbf{x}_2=\lambda_2\mathbf{x}_2$$

$$\mathbf{x}_2\cdot\mathbf{T}\cdot\mathbf{x}_1=\lambda_1\mathbf{x}_2\cdot\mathbf{x}_1 \qquad \mathbf{x}_1\cdot\mathbf{T}\cdot\mathbf{x}_2=\lambda_2\mathbf{x}_1\cdot\mathbf{x}_2=\lambda_2\mathbf{x}_2\cdot\mathbf{x}_1$$

$$\mathbf{x}_2\cdot\mathbf{T}\cdot\mathbf{x}_1=\mathbf{x}_1\cdot\mathbf{T}^T\cdot\mathbf{x}_2 \text{ with symmetry } =\mathbf{x}_1\cdot\mathbf{T}\cdot\mathbf{x}_2$$

$$\mathbf{x}_2\cdot\mathbf{T}\cdot\mathbf{x}_1-\mathbf{x}_1\cdot\mathbf{T}\cdot\mathbf{x}_2=(\lambda_1-\lambda_2)\mathbf{x}_2\cdot\mathbf{x}_1=0$$

$$\Rightarrow\mathbf{x}_2\cdot\mathbf{x}_1=0$$

Eigenvectors

- If x is an eigenvector with eigenvalue λ, then any multiple $\alpha\mathbf{x}$ is also an eigenvector: $\mathbf{T}\cdot\alpha\mathbf{x} = \alpha\lambda\mathbf{x}$

  $\Rightarrow$ Eigenvectors often scaled to unit vectors

- For repeated λ, infinite range of possible $\mathbf{x}$, usually set of orthonormal vectors chosen

Example: $$\mathbf{T} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

Write out the characteristic equation. What are λ_i?

Try finding eigenvectors so that $\mathbf{T}\cdot\mathbf{x}_i=\lambda_i\mathbf{x}_i$

Eigenvectors

- If x is an eigenvector with eigenvalue λ, then any multiple $\alpha\mathbf{x}$ is also an eigenvector: $\mathbf{T}\cdot\alpha\mathbf{x} = \alpha\lambda\mathbf{x}$

  $\Rightarrow$ Eigenvectors often scaled to unit vectors

- For repeated λ, infinite range of possible $\mathbf{x}$, usually set of orthonormal vectors chosen

Example: $$\mathbf{T} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

$$\det(\underline{\underline{T}} - \lambda \underline{\underline{S}}) = \begin{vmatrix} 2-\lambda & & \emptyset \\ & 2-\lambda & \\ \emptyset & & 3-\lambda \end{vmatrix}$$

Characteristic equation: $(2-\lambda)^2(3-\lambda)=0$

$\Rightarrow \lambda=2$ (twice), $\lambda=3$

Easy to verify that: $\mathbf{T}\cdot\hat{\mathbf{e}}_1=2\hat{\mathbf{e}}_1$, $\mathbf{T}\cdot\hat{\mathbf{e}}_2=2\hat{\mathbf{e}}_2$, $\mathbf{T}\cdot\hat{\mathbf{e}}_3=3\hat{\mathbf{e}}_3$

$\Rightarrow$ $\hat{\mathbf{e}}_1$ and $\hat{\mathbf{e}}_2$ eigenvectors, but so are any $a\hat{\mathbf{e}}_1 + b\hat{\mathbf{e}}_2$

Try yourself

- Program finding eigenvalues for a 2-dimensional, rank 2 tensor with components input by the user. *What would you like to check for before starting calculations? What needs to be done to find eigenvalues?*
- Find eigenvectors for the eigenvalues. Bear in mind that because $|\sigma-\lambda I|=0$, the two linear equations for a single λ will be multiples of each other. *What additional requirement do you need to impose to obtain unique vectors? What different cases are there?*
- *What would you need to find the eigenvalues for a 3-dimensional, rank-2 tensor?*
- *How would you deal with finding eigenvectors for repeated eigenvalues?*

Try yourself

- Program finding eigenvalues for a 2-dimensional, rank 2 tensor with components input by the user. *What would you like to check for before starting calculations? Symmetry. What needs to be done to find eigenvalues? Solve quadratic equation (quadratic formula).*
- Find eigenvectors for the eigenvalues. Bear in mind that because $|\sigma-\lambda I|=0$, the two linear equations for a single λ will be multiples of each other. *What additional requirement do you need to impose to obtain unique vectors? What different cases are there? If $T_{21} = 0$, then already diagonal. Otherwise, choose value for x_1, solve for x_2, then normalise to unit length.*
- *What would you need to find the eigenvalues for a 3-dimensional, rank-2 tensor? Root finder to solve cubic equation*
- *How would you deal with finding eigenvectors for repeated eigenvalues? Find eigenvector for unique λ, others perpendicular. In 2-D, repeated eigenvalues => isotropic stress*

Invariants

$I_1 = \mathrm{tr}(\mathbf{T}) \qquad = T_{ii}$

$I_2 = \mathrm{minor}(\mathbf{T}) = T_{ii}T_{jj} - T_{ij}T_{ji}$

$I_3 = \det(\mathbf{T}) \qquad = \varepsilon_{ijk}T_{i1}T_{j2}T_{k3}$

In terms of eigenvalues, invariants simplify to:

$I_1 = \mathrm{tr}(\mathbf{T}) \qquad = \lambda_1 + \lambda_2 + \lambda_3$

$I_2 = \mathrm{minor}(\mathbf{T}) = \lambda_1\lambda_2 + \lambda_2\lambda_3 + \lambda_1\lambda_3$

$I_3 = \det(\mathbf{T}) \qquad = \lambda_1\lambda_2\lambda_3$

Check yourself

Stress components

Diagonalizing
=> principal stress coordinate frame

$$\begin{bmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & \sigma_3 \end{bmatrix}$$

(σ_1 to σ_3 usually ordered from largest to smallest)

$\sigma_{ij} = -p\delta_{ij} + \sigma'_{ij}$

$tr(\boldsymbol{\sigma})$ = sum of normal stresses
$tr(\boldsymbol{\sigma})/3$ = - pressure p = average normal stress = ***hydrostatic stress***
$\Rightarrow$ volume change

$\boldsymbol{\sigma}'_{ij}$ is ***deviatoric stress*** $= \sigma_{ij}+p\delta_{ij}$
$\Rightarrow$ shape change

$tr(\boldsymbol{\sigma}') = ?$

Second invariant deviatoric stress

σ'_{ij} is deviatoric stress = $\sigma_{ij}+p\delta_{ij}$

$\mathrm{minor}(\boldsymbol{\sigma}') = \sigma'_{11}\sigma'_{22} + \sigma'_{22}\sigma'_{33} + \sigma'_{11}\sigma'_{33} - \sigma'^2_{21} - \sigma'^2_{32} - \sigma'^2_{31}$ (1)

=

(2)

$- \sigma'^2_{21} - \sigma'^2_{32} - \sigma'^2_{31}$

Rewrite first three terms using expression for tr(σ') i.e.,
$\sigma'_{22} = -\sigma'_{11} - \sigma'_{33}$

= ½ [(1)+(2)]

$= - \frac{1}{2} [\sigma'^2_{11} + \sigma'^2_{22} + \sigma'^2_{33} + \sigma'^2_{21} + \sigma'^2_{32} + \sigma'^2_{31}]$

$\mathrm{minor}(\boldsymbol{\sigma})=\frac{1}{2}[\mathrm{tr}(\boldsymbol{\sigma}^2)-(\mathrm{tr}\boldsymbol{\sigma})^2]$, $\mathrm{minor}(\boldsymbol{\sigma}')=\frac{1}{2}\mathrm{tr}(\boldsymbol{\sigma}'^2)$

measure of stress magnitude, important in flow and plastic yielding

Second invariant deviatoric stress

σ'_{ij} is deviatoric stress = $\sigma_{ij}+p\delta_{ij}$

$$\text{minor}(\boldsymbol{\sigma'}) = \sigma'_{11}\sigma'_{22} + \sigma'_{22}\sigma'_{33} + \sigma'_{11}\sigma'_{33} - \sigma'^{2}_{21} - \sigma'^{2}_{32} - \sigma'^{2}_{31} \quad (1)$$

$$= - \sigma'^{2}_{11} - \sigma'^{2}_{22} - \sigma'^{2}_{33} - \sigma'_{11}\sigma'_{33} - \sigma'_{11}\sigma'_{22} - \sigma'_{22}\sigma'_{33} - \sigma'^{2}_{21} - \sigma'^{2}_{32} - \sigma'^{2}_{31} \quad (2)$$

Using that:
$\mathrm{tr}(\sigma')=\sigma'_{11}+\sigma'_{22}+\sigma'_{33}$
$=0$

$$= \tfrac{1}{2}\,[(1)+(2)]$$

$$= - \tfrac{1}{2}\,[\sigma'^{2}_{11} + \sigma'^{2}_{22} + \sigma'^{2}_{33} + \sigma'^{2}_{21} + \sigma'^{2}_{32} + \sigma'^{2}_{31}]$$

$\text{minor}(\boldsymbol{\sigma})=\tfrac{1}{2}[\mathrm{tr}(\boldsymbol{\sigma}^2)-(\mathrm{tr}\boldsymbol{\sigma})^2]$, $\text{minor}(\boldsymbol{\sigma'})=\tfrac{1}{2}\mathrm{tr}(\boldsymbol{\sigma}'^2)$

measure of stress magnitude, important in flow and plastic yielding

Maximum shear stress

Principal stresses include largest and smallest normal stresses in given stress system (*see proof in Lai et al*.)

If σ_1 is largest and σ_3 smallest principal stress, then maximum shear stress

$$\left|\sigma_s^{\max}\right| = \frac{\sigma_1 - \sigma_3}{2}$$

- Show this using case of 2-D stress in σ_1, σ_3 coordinate frame,
- Determine the orientation of the corresponding direction relative to the σ_1, σ_3 coordinate frame

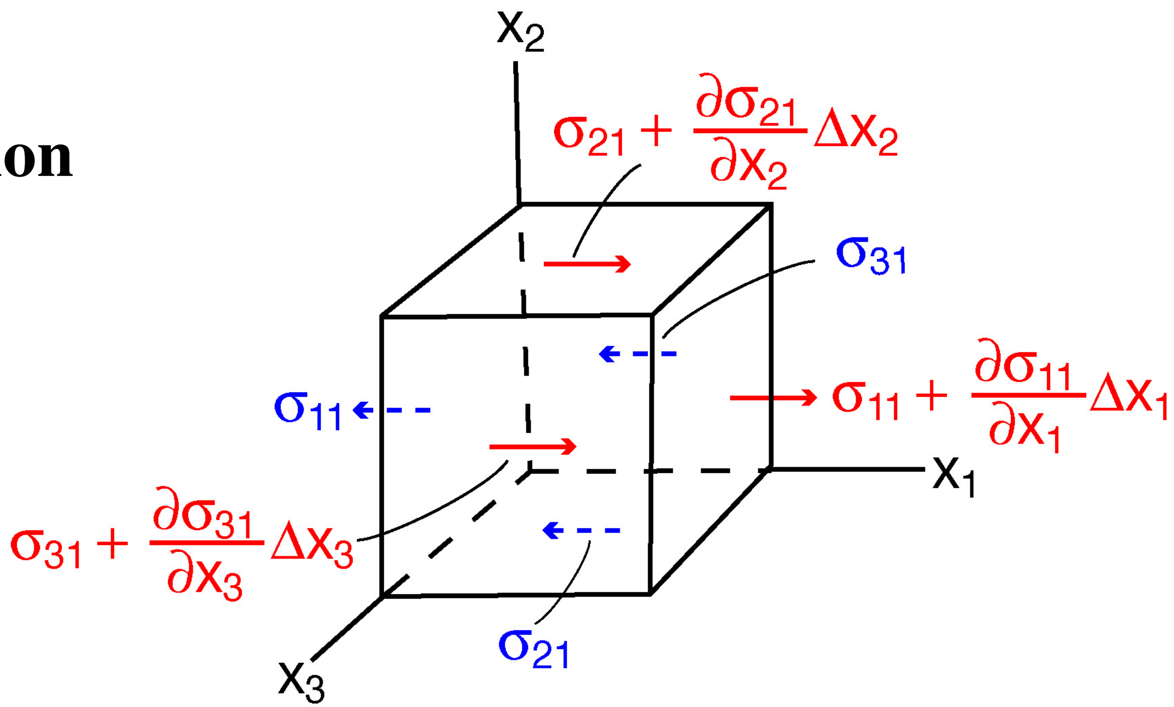
Maximum shear stress important for yield criteria

Equation of motion

Force balance:

$\mathbf{F}_{body}+\mathbf{F}_{stress}=m\mathbf{a}$

In x_1- direction:



+

+

+

$= \rho \Delta x_1 \Delta x_2 \Delta x_3 \partial^2 u_1/\partial t^2$

$\Rightarrow \; f_1+\partial\sigma_{11}/\partial x_1 + \partial\sigma_{21}/\partial x_2 + \partial\sigma_{31}/\partial x_3 = \rho\partial^2 u_1/\partial t^2$

$\Rightarrow \; f_i + \partial\sigma_{ji}/\partial x_j = \rho\partial^2 u_i/\partial t^2$

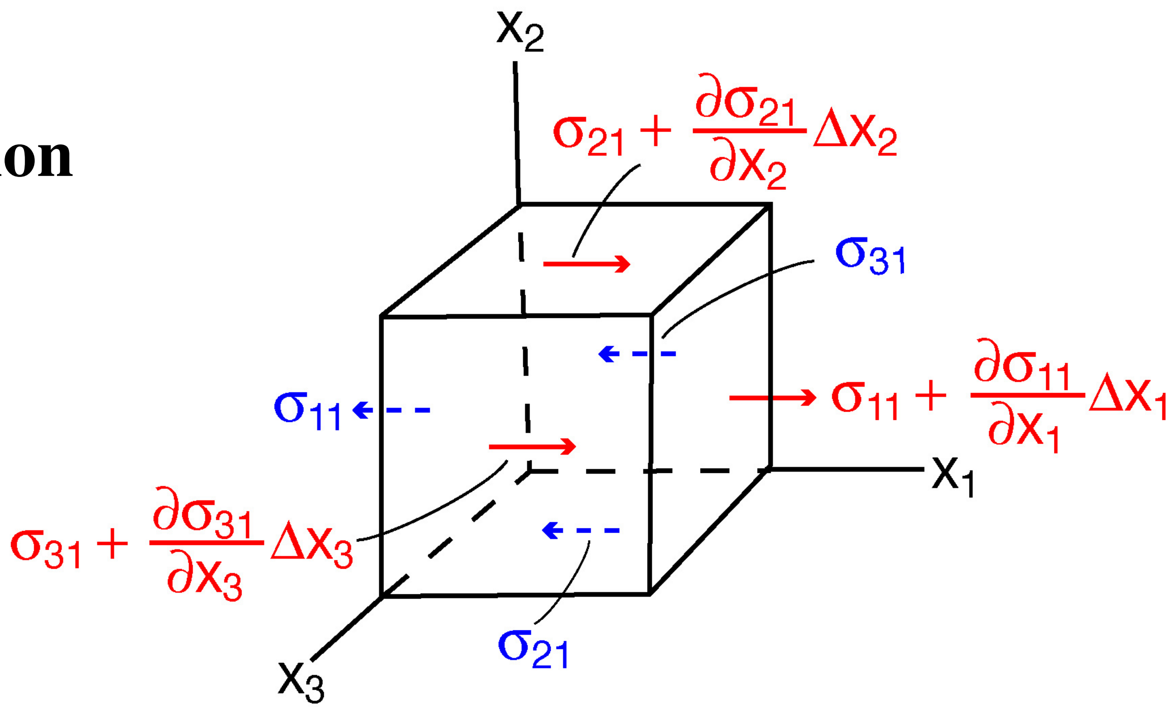
$\Rightarrow \; \mathbf{f} + \nabla\cdot\underline{\boldsymbol{\sigma}} = \rho\partial^2\mathbf{u}/\partial t^2$

Equation of motion

Force balance:

$\mathbf{F}_{body}+\mathbf{F}_{stress}=m\mathbf{a}$

In x_1- direction:



$f_1\Delta x_1\Delta x_2\Delta x_3+$
$(\sigma_{11}+\Delta x_1\partial\sigma_{11}/\partial x_1 - \sigma_{11})\Delta x_2\Delta x_3 +$
$(\sigma_{21}+\Delta x_2\partial\sigma_{21}/\partial x_2 - \sigma_{21})\Delta x_1\Delta x_3 +$
$(\sigma_{31}+\Delta x_3\partial\sigma_{31}/\partial x_3 - \sigma_{31})\Delta x_1\Delta x_2 = \rho\Delta x_1\Delta x_2\Delta x_3\partial^2 u_1/\partial t^2$

$\Rightarrow\ f_1+\partial\sigma_{11}/\partial x_1 + \partial\sigma_{21}/\partial x_2 + \partial\sigma_{31}/\partial x_3 = \rho\partial^2 u_1/\partial t^2$

$\Rightarrow\ f_i + \partial\sigma_{ji}/\partial x_j = \rho\partial^2 u_i/\partial t^2$
$\Rightarrow\ \mathbf{f} + \nabla\cdot\underline{\boldsymbol{\sigma}} = \rho\partial^2\mathbf{u}/\partial t^2$

Learning Objectives

- Understand meaning of different components of 3D Cauchy stress tensor, and know how to determine state of stress on given plane
- Be able to transform rank 2 tensor to a new basis.
- Be able to decompose a rank 2 tensor into symmetric and anti-symmetric components
- Be able to find principal stresses and stress invariants and know what they represent
- Be able to balance body forces and stresses

Summary
Stress Tensors

- Cauchy stress tensor
- Tensor coordinate transformation
- (Stress) tensor symmetry
- Tensor invariants
- Diagonalizing, eigenvalues, eigenvectors
- Special stress states
- Equation of motion

Further reading on the topics in the lecture can be done in for example: Lai, Rubin, Kremple (2010): Ch. 2.18 through 2.25, 4.4 through 4.7