Dimensional Analysis Worksheet Solutions

Question 1

a) There are 5 variables (d, γ, g, ρ_h and ρ_l) and 3 dimensions (M, L, T) and so **TWO** dimensionless groups are required.

b) Doing it the proper Buckingham Pi way:

$$\begin{array}{c} \\ M \\ L \\ T \end{array} \begin{array}{c} \begin{array}{ccccc} \gamma & g & \rho_h & \rho_l & d \end{array} \\ \begin{pmatrix} 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & -3 & -3 & 1 \\ -2 & -2 & 0 & 0 & 0 \end{pmatrix} \end{array}$$

Setting $x_\gamma = 1$ and $x_{\rho_l} = 0$ results in the following dimensionless group.

The resulting dimensionless group is:

$$\boldsymbol{N = \frac{\gamma}{\rho_h g d^2}}$$

(This is actually the inverse of the Bond number).

c) This dimensionless group represents the ratio of the capillary to gravity force.

Question 2

a) There are 7 variables (A, Q, C_p, h, ρ, ΔT and V) and 4 dimensions (M, L, T and θ) and so **THREE** dimensionless groups are required.

b) Note that Energy has the following dimensions: $E = \frac{ML^2}{T^2}$

$$\begin{array}{c} \\ M \\ L \\ T \\ \theta \end{array} \begin{array}{c} \begin{array}{ccccccc} V & \rho & C_P & h & A & \Delta T & Q \end{array} \\ \begin{pmatrix} 0 & 1 & 0 & 1 & 0 & 0 & 1 \\ 3 & -3 & 2 & 0 & 2 & 0 & 2 \\ -1 & 0 & -2 & -3 & 0 & 0 & -3 \\ 0 & 0 & -1 & -1 & 0 & 1 & 0 \end{pmatrix} \end{array}$$

This results in the following vectors:

$$\boldsymbol{x_1} = \begin{pmatrix} -1 \\ -1 \\ -1 \\ 1 \\ 1 \\ 0 \\ 0 \end{pmatrix} \quad \boldsymbol{x_2} = \begin{pmatrix} 0 \\ 2 \\ 3 \\ -2 \\ 0 \\ 1 \\ 0 \end{pmatrix} \quad \boldsymbol{x_3} = \begin{pmatrix} -1 \\ 1 \\ 2 \\ -2 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

$N_1 = \frac{h\,A}{V\rho\,C_P}$ $N_2 = \frac{\Delta T \rho^2 {C_P}^3}{h^2}$ $N_3 = \frac{Q\,\rho\,{C_P}^2}{V\,h^2}$

A nicer set of dimensionless groups would be to replace N_3 with N_3/N_2 and to replace N_2 with $N_2\,N_1$ (note that this is a matter of choice and not a "right" or "wrong" answer. I just fell that the resultant groups look nicer, mainly because of the smaller exponents involved):

$N_1 = \frac{h\,A}{V\rho\,C_P}$ $N_2 = \frac{A\,\Delta T \rho {C_P}^2}{h}$ $N_3 = \frac{Q}{\Delta T \rho\,C_P\,V}$

Question 3

a)

$$\frac{\partial \phi}{\partial t} = \frac{2k_1}{\lambda}\phi\frac{\partial \phi}{\partial z} + \frac{k_2}{2\sqrt{\lambda}\sqrt{\phi}}\left(\frac{\partial \phi}{\partial z}\right)^2 + \frac{k_2\sqrt{\phi}}{\sqrt{\lambda}}\frac{\partial^2 \phi}{\partial z^2}$$

where $k_1 = \frac{\rho g}{3\mu C_{PB}}$ and $k_2 = \frac{\gamma\sqrt{3-\frac{\pi}{2}}}{6\mu C_{PB}}$

b) $z^* = z\sqrt{\frac{\rho g}{\gamma}}$ and

$$t^* = t\frac{\sqrt{\gamma\rho g}}{\mu}$$

(other ways of non-dimensionalising the problem are possible).

c)

$$\frac{\partial \phi}{\partial t^*} = Bo\frac{2}{3k_\lambda C_{PB}}\phi\frac{\partial \phi}{\partial z^*} + \frac{\sqrt{3-\frac{\pi}{2}}}{6C_{PB}\sqrt{k_\lambda}}\sqrt{Bo}\left(\frac{1}{2\sqrt{\phi}}\left(\frac{\partial \phi}{\partial z^*}\right)^2 + \sqrt{\phi}\frac{\partial^2 \phi}{\partial {z^*}^2}\right)$$

Where $Bo = \frac{\rho g {d_b}^2}{\gamma}$

Optional bit

d) Simply replace the derivatives with their approximations and also note that ϕ should be replaced with $\phi_{i,j}$:

$$\phi_{i,j+1} = \phi_{i,j} + \Delta t^* \left(Bo \frac{2}{3k_\lambda C_{PB}} \phi_{i,j} \frac{\phi_{i+1,j} - \phi_{i-1,j}}{2\Delta z^*} + \frac{\sqrt{3 - \frac{\pi}{2}}}{6C_{PB}\sqrt{k_\lambda}} \sqrt{Bo} \left(\frac{1}{2\sqrt{\phi_{i,j}}} \left(\frac{\phi_{i+1,j} - \phi_{i-1,j}}{2\Delta z^*} \right)^2 + \sqrt{\phi_{i,j}} \frac{\phi_{i+1,j} + \phi_{i-1,j} - 2\phi_{i,j}}{\Delta {z^*}^2} \right) \right)$$

For finding the time step:

$$D_{max} = \frac{\sqrt{3 - \frac{\pi}{2}}}{6C_{PB}\sqrt{k_\lambda}} \sqrt{Bo} \sqrt{\phi_{max}}$$

$$v_{max} = \max \left(Bo \frac{2}{3k_\lambda C_{PB}} \phi_{max}, \frac{\sqrt{3 - \frac{\pi}{2}}}{6C_{PB}\sqrt{k_\lambda}} \sqrt{Bo} \frac{1}{2\sqrt{\phi_{min}}} \frac{\partial \phi}{\partial {z^*}_{max}} \right)$$

The maximum gradient occurs on the first timestep, which given the resolution, is $\frac{\partial \phi}{\partial {z^*}_{max}} = \frac{\phi_{max} - \phi_{min}}{\Delta z^*}$. Ignoring this gradient condition won't actually cause proper divergence, but can add wiggles in the formation of the capillary boundary layer.

$\phi_{max} = 0.3$ and $\phi_{min} = 0.01$ as the liquid contents cannot go outside these limits if the code is working correctly.