ACSE-2

Lecture 7
Kinematics of Continua

Description of deformation, motion of a continuum

Outline Lecture 7

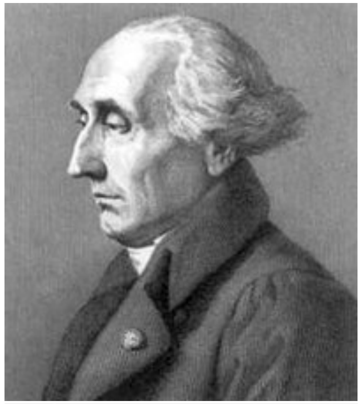
- Material vs. spatial descriptions
- Time derivatives
- Displacement
- Infinitesimal Deformation
- Finite Deformation
- Conservation of Mass

Learning Objectives

- Be able to use material and spatial descriptions of variables and their time derivatives.
- Be able to compute infinitesimal strain (strain rate) tensor given a displacement (velocity) field.
- Know meaning of the different components of the infinitesimal strain (rate) tensor
- Be able to find principal strain(rate)s and strain (rate) invariants and know what they represent
- Understand difference between infinitesimal and finite strain
- Be able to use the conservation of mass equation

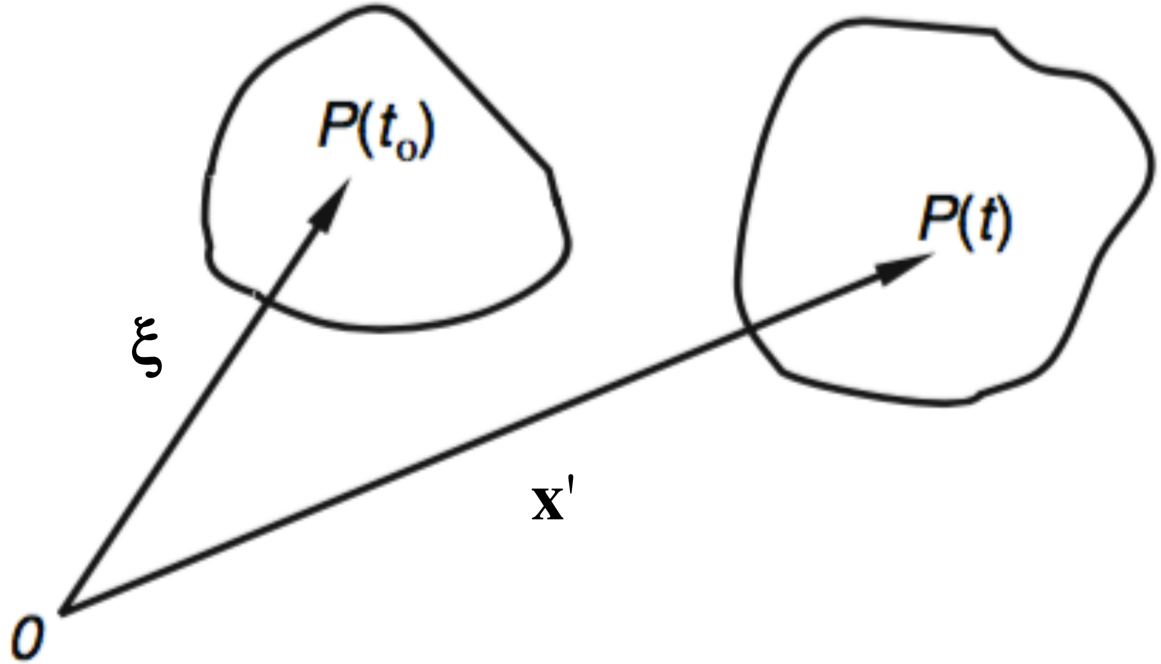
Two ways to describe motion

- Material (Lagrangian)
  - following a "particle"
- Spatial (Eulerian)
  - from a fixed observation point



Preferred description depends on application

Material description



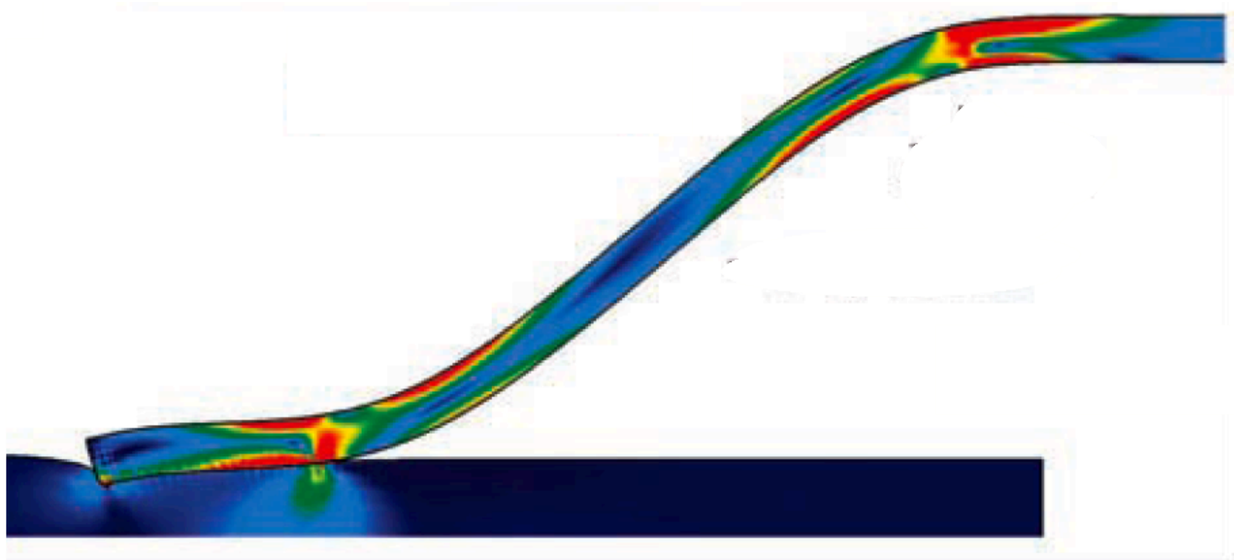
Position vector $\boldsymbol{\xi}=(\xi_1,\xi_2,\xi_3)$

"Particle" at point $\boldsymbol{\xi}$ at a reference time t_0, moves to point $\mathbf{x}'$ at a later time t

Field P described as function of $\boldsymbol{\xi}$ and t

Often the preferred description for solids

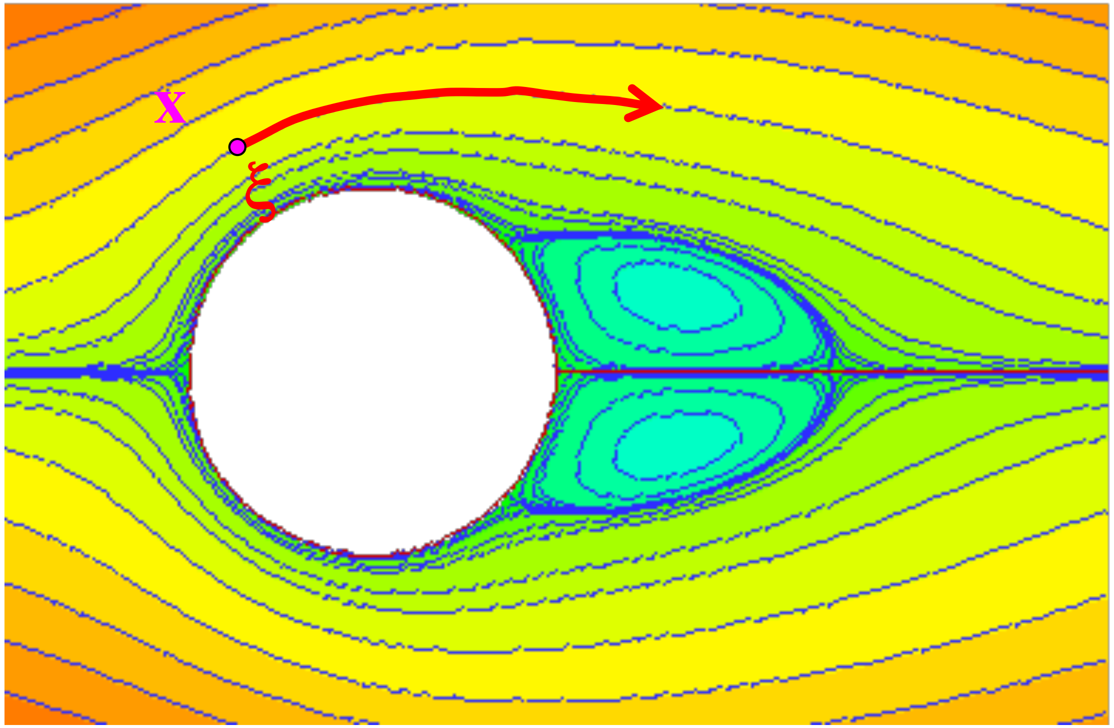
Material description



"Particle" at point ξ at a reference time t_0,
moves to point $\mathbf{x}'$ at a later time t
Field P described as function of ξ and t

Often the preferred description for solids

Spatial description



Field P described as function of a given position $\mathbf{x}$ and t

In the example flow, velocity in point $\mathbf{x}$ does not change with time, but velocity that a particle originally in same position ξ experiences with time does change

Often the preferred description for fluids

Material Derivative

- Rate of change (with time) of a quantity (e.g., T,$\mathbf{v}$,$\boldsymbol{\sigma}$) for a material particle
- In <u>material description</u>, time derivative of P: $$\frac{DP}{Dt} = \left(\frac{\partial P}{\partial t}\right)_{\xi}$$

Note: here $P(\boldsymbol{\xi},t)$

- In <u>spatial description</u>, $$\frac{DP}{Dt} = \left(\frac{\partial P}{\partial t}\right)_{\xi} = \left(\frac{\partial P}{\partial t}\right)_{\mathbf{x}} + \frac{\partial P}{\partial x_i}\left(\frac{\partial x'_i}{\partial t}\right)_{\xi}$$

where $\left(\frac{\partial \mathbf{x}'}{\partial t}\right)_{\xi} = \frac{D\mathbf{x}}{Dt}$ velocity of particle $\boldsymbol{\xi}$

Note: here $P(\mathbf{x},t)$

material *spatial*

$$\frac{DP}{Dt} = \frac{\partial P}{\partial t} + \mathbf{v} \cdot \nabla P$$

This definition works in any coordinate frame

Acceleration

- In spatial description: $\mathbf{a} = \frac{D\mathbf{v}}{Dt} = \frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{v}$

Try yourself:

Determine component a_1 of the acceleration of a particle

in a spatial velocity field: $v_i = \frac{kx_i}{1+kt}$

Could start with single component a_1
And then for general case of a_i

Acceleration

- In spatial description: $\mathbf{a} = \dfrac{D\mathbf{v}}{Dt} = \dfrac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{v}$

Try yourself:

Determine component a_1 of the acceleration of a particle

in a spatial velocity field: $v_i = \dfrac{kx_i}{1+kt}$

$$a_i = \frac{Dv_i}{Dt} = \frac{\partial v_i}{\partial t} + v_j \frac{\partial v_i}{\partial x_j} = -\frac{k^2 x_i}{(1+kt)^2} + \frac{kx_j}{1+kt} \frac{k\delta_{ij}}{1+kt} = 0$$

Acceleration

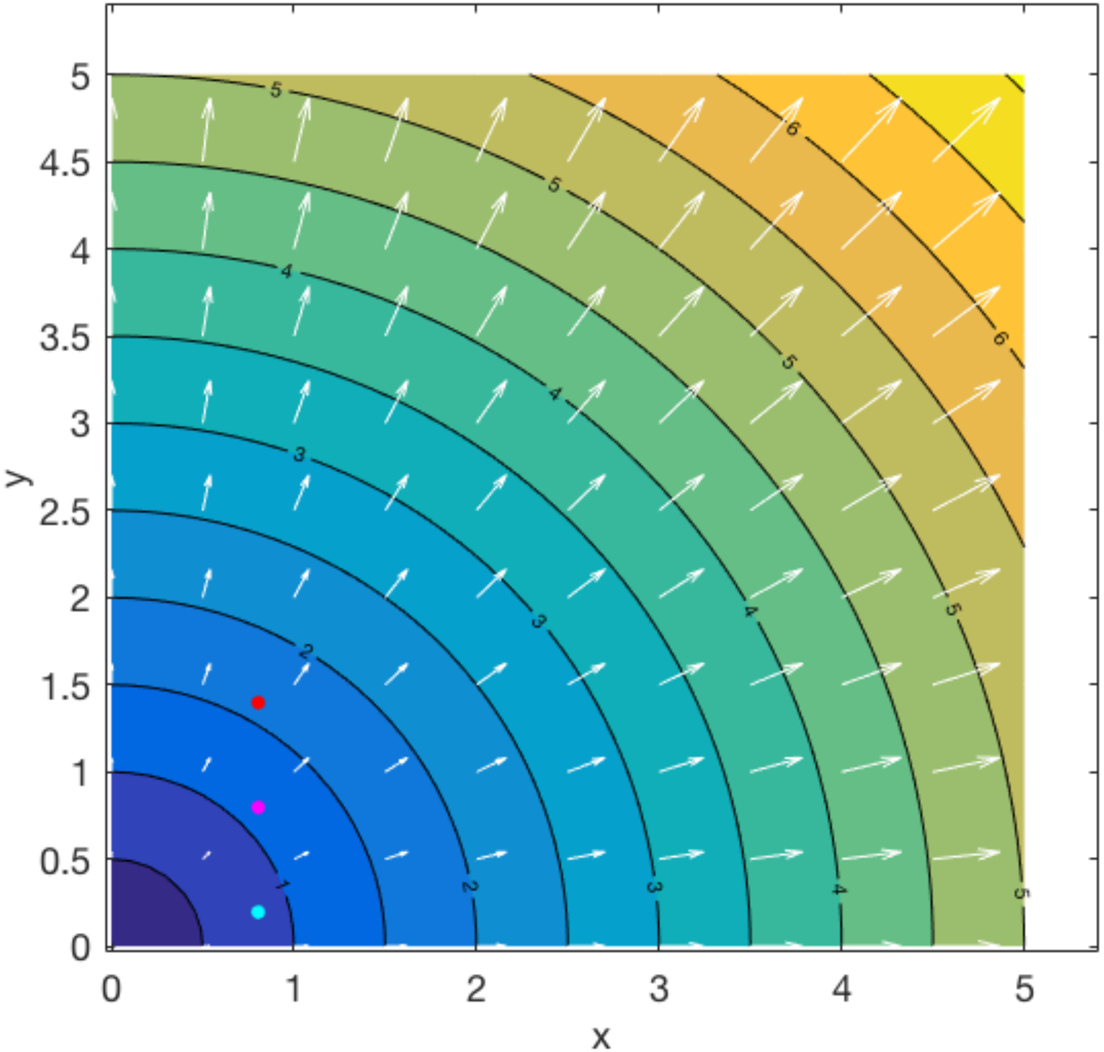
Spatial velocity field:

$$v_i = \frac{kx_i}{1+kt}$$

Acceleration:

$$a_i = \frac{Dv_i}{Dt} = \frac{\partial v_i}{\partial t} + v_j \frac{\partial v_i}{\partial x_j} = 0$$

velocity field at t=0 (k=1)



contours for magnitude, arrows direction and size

Acceleration

Spatial velocity field:

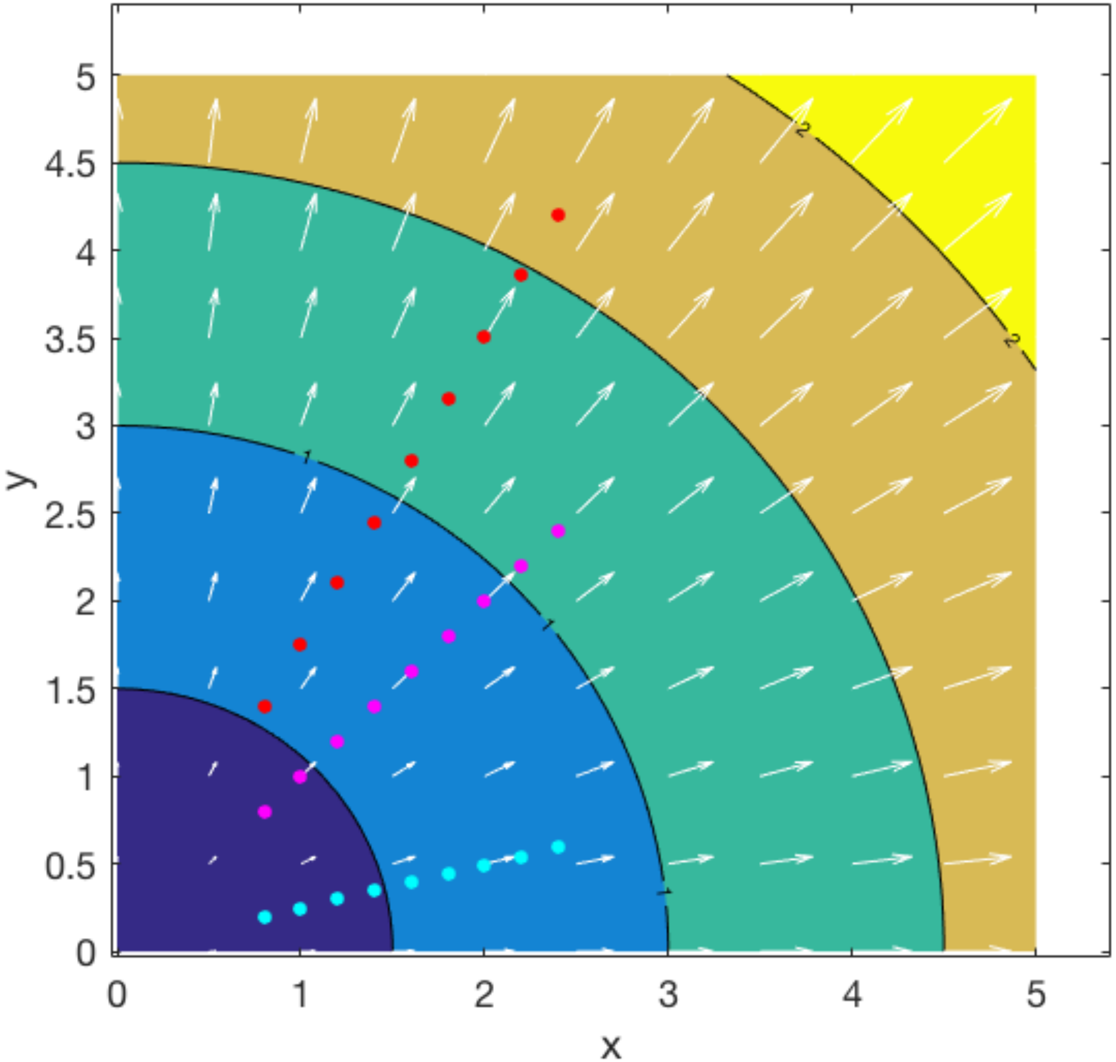
$$v_i = \frac{kx_i}{1+kt}$$

Acceleration:

$$a_i = \frac{Dv_i}{Dt} = \frac{\partial v_i}{\partial t} + v_j \frac{\partial v_i}{\partial x_j} = 0$$

*How can you see that **a** = 0?*

velocity field at t=2 (k=1)



marker positions at constant time intervals between [0:2]

Acceleration

- In spatial description: $\mathbf{a} = \dfrac{D\mathbf{v}}{Dt} = \dfrac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{v}$

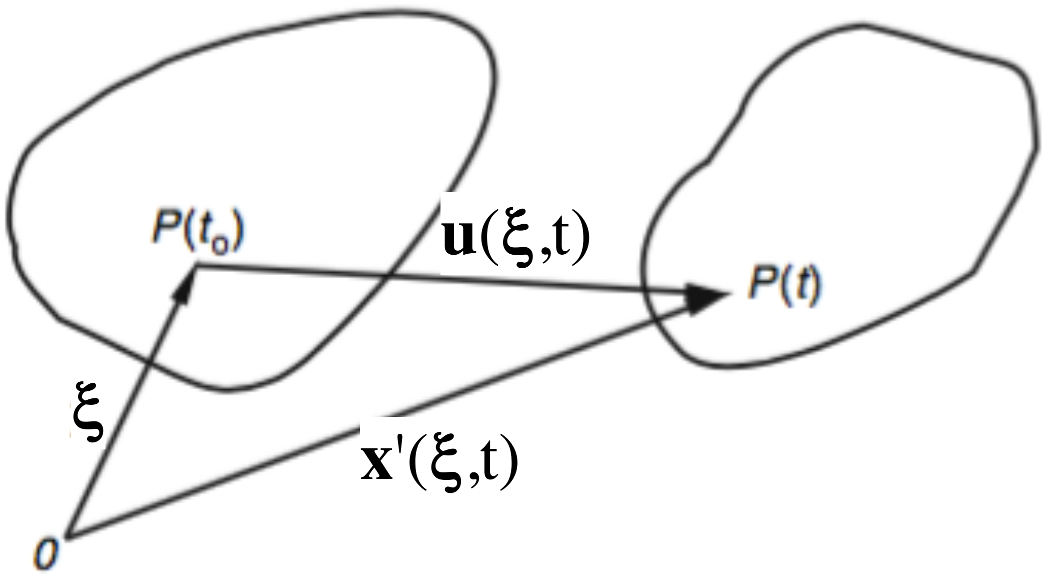
Equation of motion then becomes:

$$\rho \mathbf{a} = \nabla \cdot \underline{\underline{\sigma}} + \mathbf{f} = \rho \frac{D\mathbf{v}}{Dt} = \rho \left(\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{v} \right)$$

Displacement

Motion of a continuum can be described by:

- path lines $\mathbf{x}'=\mathbf{x}'(\xi,t)$
- displacement field $\mathbf{u}(\xi,t)=\mathbf{x}'(\xi,t)-\xi$



Pathlines

Try yourself:

Determine the pathline for the x'_1 component of the particle's position for the spatial velocity field of the acceleration example

$$v_i = \frac{kx_i}{1+kt}$$

Realise that

$$v_i = \frac{\partial x'_i}{\partial t} = \frac{kx_i}{1+kt}$$

Pathlines

Try yourself:

Determine the pathline for the x'_1 component of the particle's position for the spatial velocity field of the acceleration example

$$v_i = \frac{kx_i}{1+kt}$$

Realise that

$$v_i = \frac{\partial x'_i}{\partial t} = \frac{kx_i}{1+kt}$$

$$\int_{\xi_i}^{x'_i} \frac{dx_i}{kx_i} = \int_0^t \frac{dt}{1+kt}$$

$$\frac{1}{k}\left[\ln x'_i - \ln \xi_i\right] = \frac{1}{k}\left[\ln(1+kt) - \ln(1)\right]$$

$$x'_i(\xi, t) = (1+kt)\xi_i$$

Pathlines

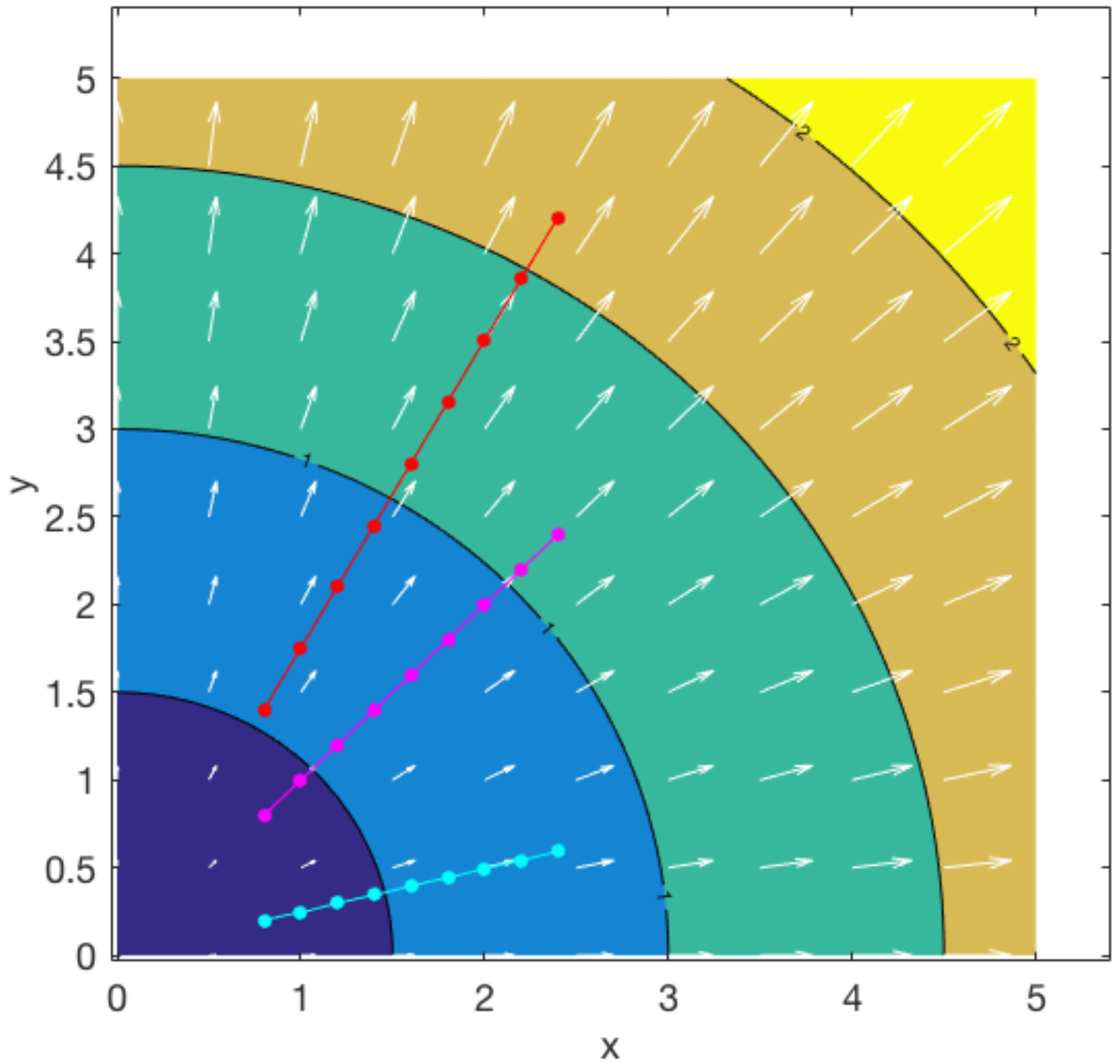
Determine the pathline for the x'_1 component of the particle's position for the spatial velocity field of the acceleration example

$$v_i = \frac{kx_i}{1+kt}$$

$$x'_i(\xi,t) = (1+kt)\xi_i$$

Material velocity field:

$$v'_i = v_i = k\xi_i$$



Try later: acceleration.ipynb