

Solutions Problem Set 8 - ACSE 2 - 2020

(1) (a) Energy equation

1-D, steady state, no flow, no strain



$$\nabla \cdot \underline{D} = 0$$

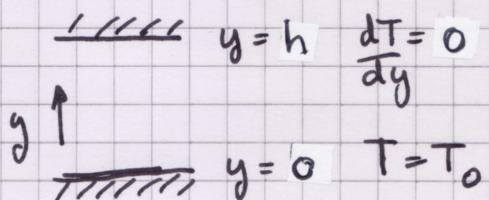
$$\frac{\partial}{\partial t} = 0$$

no advection

$$\Rightarrow 0 = \nabla \cdot k \nabla T + A$$

in 1-D $0 = \frac{d}{dy} \left(k \frac{dT}{dy} \right) + A$

(b) k constant



$$\frac{d^2 T}{dy^2} = -\frac{A}{k}$$

integrate once $\frac{dT}{dy} = -\frac{Ay}{k} + B$

use $\frac{dT}{dy}(y=h) = 0 \Rightarrow B = \frac{Ah}{K}$

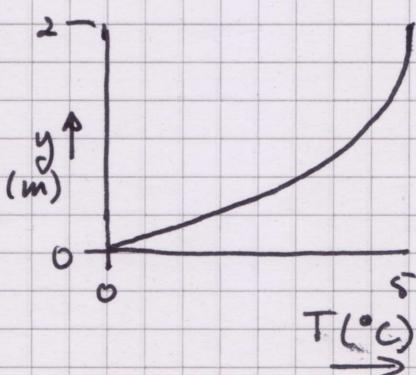
integrate again $T(y) = -\frac{A}{2k} y^2 + By + C$

use $T(y=0) = T_0 \Rightarrow C = T_0$

solution $T(y) = -\frac{A}{2k} y^2 + \frac{Ah}{K} y + T_0$

(c) $k = 80 \text{ W/mK}$; $A = 200 \text{ W/m}^3$, $h = 2 \text{ m}$, $T_0 = 0$

$$T(y) = \frac{A}{2k} (2h - y) y$$



(2) (a) principal stress and strain axes same for Hookean solid?

stress $\underline{\underline{\sigma}}$, strain $\underline{\underline{\epsilon}}$

Take \vec{x}_1 as eigenvector of $\underline{\underline{\epsilon}}$
with eigenvalue c_1 ,

$$\underline{\underline{\epsilon}} \cdot \vec{x}_1 = c_1 \vec{x}_1$$

$$\Rightarrow \underline{\underline{\sigma}} \cdot \vec{x}_1 = \lambda \Theta \underline{\underline{I}} \cdot \vec{x}_1 + 2\mu \underline{\underline{\epsilon}} \cdot \vec{x}_1$$

where $\Theta = \text{tr}(\underline{\underline{\epsilon}})$, i.e. constant

$$\begin{aligned}\underline{\underline{\sigma}} \cdot \vec{x}_1 &= \lambda \Theta \vec{x}_1 + 2\mu c_1 \vec{x}_1 \\ &= (\lambda \Theta + 2\mu c_1) \vec{x}_1\end{aligned}$$

$\Rightarrow \vec{x}_1$ is also an eigenvector of $\underline{\underline{\sigma}}$
with eigenvalue $\lambda \Theta + 2\mu c_1$

(b) Relation between principal stress and strain components

$$\begin{bmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & \sigma_3 \end{bmatrix} = \lambda(\epsilon_1 + \epsilon_2 + \epsilon_3) \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + 2\mu \begin{bmatrix} \epsilon_1 & 0 & 0 \\ 0 & \epsilon_2 & 0 \\ 0 & 0 & \epsilon_3 \end{bmatrix}$$

$$\sigma_1 = \lambda \Theta + 2\mu \epsilon_1$$

$$\sigma_2 = \lambda \Theta + 2\mu \epsilon_2$$

$$\sigma_3 = \lambda \Theta + 2\mu \epsilon_3$$

$$\left. \begin{array}{l} \sigma_i = \lambda \Theta + 2\mu \epsilon_i \end{array} \right\}$$

(3) Young's modulus E and Poisson's ratio v
in terms of Lamé parameters λ, μ

Defined for uniaxial stress

$$\sigma_1 = \sigma, \quad \sigma_2 = \sigma_3 = 0$$

Strain for this case:

$$\epsilon_1 = \epsilon, \quad \epsilon_2 = \epsilon_3$$

$$\text{Young's modulus} \quad E = \frac{\sigma}{\epsilon}$$

$$\text{Poisson's ratio} \quad v = -\frac{\epsilon_3}{\epsilon}$$

$$\sigma_1 = \lambda(\epsilon_1 + 2\epsilon_3) + 2\mu\epsilon_1$$

$$\sigma_3 = \lambda(\epsilon_1 + 2\epsilon_3) + 2\mu\epsilon_3 = 0$$

$$\Rightarrow 2(\lambda + \mu)\epsilon_3 = -\lambda\epsilon_1 \Rightarrow 2\epsilon_3 = -\frac{\lambda}{\lambda + \mu}\epsilon_1$$

$$\Rightarrow v = \left[\frac{\lambda}{2\lambda + \mu} \right]$$

$$\Rightarrow \sigma = \left(\frac{\lambda + \mu}{\lambda + \mu} - \frac{\lambda}{\lambda + \mu} \right) \epsilon_1 = \frac{\mu}{\lambda + \mu} \epsilon_1$$

$$\sigma_1 = \left(\frac{\lambda\mu}{\lambda + \mu} + 2\mu \right) \epsilon_1$$

$$= \left(\frac{\lambda\mu + 2\mu\lambda + 2\mu^2}{\lambda + \mu} \right) \epsilon_1$$

$$= (3\lambda + 2\mu) \frac{\mu}{\lambda + \mu} \epsilon_1$$

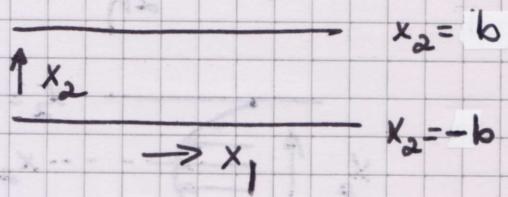
$$\boxed{E = \frac{\sigma}{\epsilon_1} = \frac{(3\lambda + 2\mu)\mu}{\lambda + \mu}}$$

(q) Steady unidirectional flow

constant viscosity η
incompressible

$$v_1 = v(x_2), v_2 = v_3 = 0$$

$$v(-b) = v(b) = 0$$



$$\text{Steady} \Rightarrow \frac{\partial}{\partial t} = 0$$

no external force $\Rightarrow \vec{f} = 0$

$$(a) \Rightarrow -\nabla p + \eta \nabla^2 \vec{v} = 0$$

$$\text{only } \frac{\partial^2 v_1}{\partial x_2^2} \neq 0$$

$$-\frac{\partial p}{\partial x_i} + \eta \frac{\partial^2}{\partial x_j^2} v_i = 0$$

$$(b) -\frac{\partial p}{\partial x_1} + \eta \frac{\partial^2 v}{\partial x_2^2} = 0, \quad -\frac{\partial p}{\partial x_2} = -\frac{\partial p}{\partial x_3} = 0$$

\Rightarrow pressure does not vary in x_2 and x_3 direction

$$\frac{\partial p}{\partial x_1} = \eta \frac{\partial^2 v}{\partial x_2^2}$$

$$\frac{\partial^2 p}{\partial x_1^2} = 0 \quad \text{because } v \text{ only depends on } x_2 \\ \text{so } \frac{\partial^2 v}{\partial x_2^2} \text{ also does not vary with } x_1$$

$$\Rightarrow \frac{\partial p}{\partial x_1} = \text{constant} = -c \quad \rightarrow \text{set to } -c \text{ realising flow will move in the direction of decreasing } p$$

$$(c) \eta \frac{\partial^2 v}{\partial x_2^2} = -c$$

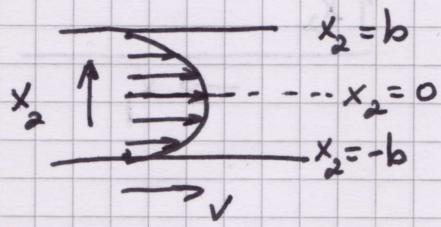
Integrate twice as for T solution of problem (l)

$$\Rightarrow v(x_2) = -\frac{c}{2\eta} x_2^2 + B_1 x + B_2$$

$$v(b) = v(-b) = 0$$

$$\begin{aligned} -\frac{c}{2\eta} b^2 + B_1 b + B_2 &= 0 \\ -\frac{c}{2\eta} b^2 - B_1 b + B_2 &= 0 \end{aligned} \quad \left. \begin{array}{l} \Rightarrow B_1 = 0 \\ B_2 = +\frac{c}{2\eta} b^2 \end{array} \right.$$

$$\Rightarrow v(x_2) = \frac{c}{2\eta} (b^2 - x_2^2)$$



parabolic flow profile