Page. 1.

(a)

$(a)\ \hat{e_1'} = T\cdot\hat{e_1} = \begin{bmatrix} \sqrt{3} & 0 & -\frac{1}{2} \\ 0 & 1 & 0 \\ 1 & 0 & \frac{1}{2}\sqrt{3} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} \sqrt{3} \\ 0 \\ 1 \end{bmatrix}.$

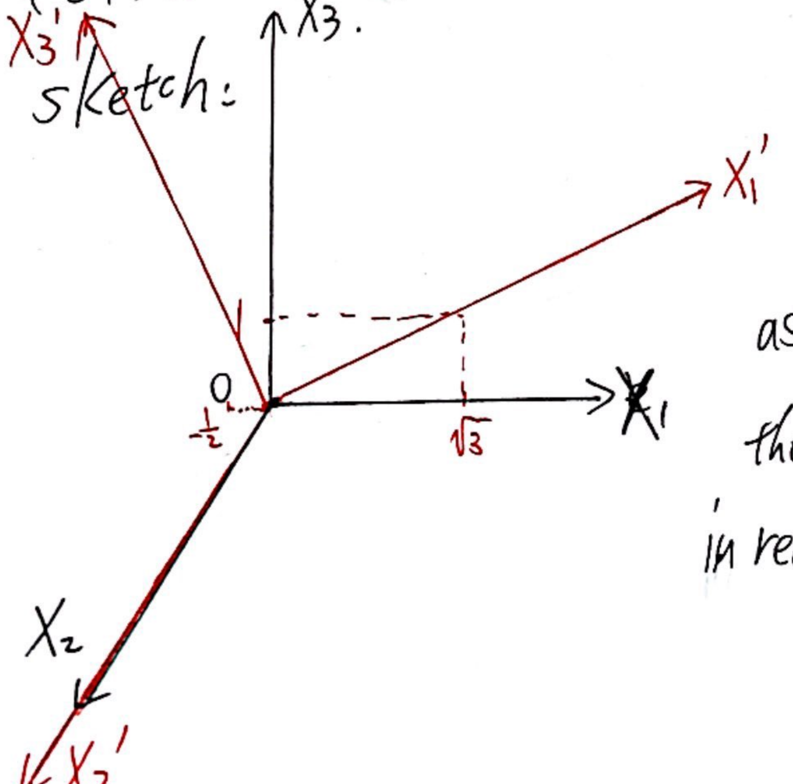
$\cancel{T\hat{e_2} =}\ \hat{e_2'} = T\cdot\hat{e_2} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \hat{e_2}.$

$\hat{e_3'} = T\cdot\hat{e_3} = \frac{1}{2}\begin{bmatrix} -1 \\ 0 \\ \sqrt{3} \end{bmatrix}$

so we have the transformed coordinate system.

$(\hat{e_1'}, \hat{e_2'}\ \hat{e_3'}) = (\sqrt{3}\hat{e_1} + \hat{e_3}, \hat{e_2}, -\frac{1}{2}\hat{e_1} + \frac{\sqrt{3}}{2}\hat{e_3}).$

sketch:



as the graph on the left, the coordinate (X_1', X_2', X_3') plotted in red is the modified coordinate.

~~(b)~~. Page. 2

(b) No, it's not.

if T is an orthonormal transformation,

its column vectors should all be normal.

but ~~apparent~~ obviously, let $T = [V_1, V_2, V_3]$.

$V_1 = \begin{bmatrix} \sqrt{3} \\ 0 \\ 1 \end{bmatrix}$. $\Rightarrow \|V_1\| = 2$, not normal, that is to

say, for $\hat{e_1}' = T \cdot \hat{e_1} = \begin{bmatrix} \sqrt{3} \\ 0 \\ 1 \end{bmatrix}$, $\|\hat{e_1}'\| = 2 \neq \|\hat{e_1}\|$.

so some vector changes norm after transformation,

so T is not an orthonormal transformation.

(2)

(a) let ξ_a be some point on the material filament at $t=0$.

so $\xi_a = (a, 0, 0)$, $a \in [0,1]$.

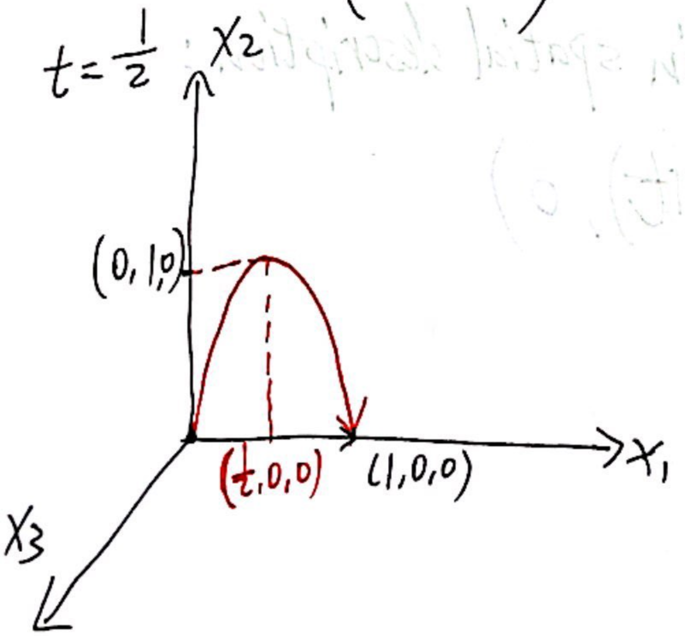
given. $X_a = \begin{pmatrix} \xi_{a_1} \\ \xi_{a_2} + \sin(\pi t)(\sin \pi \xi_{a_1}) \\ \xi_{a_3} \end{pmatrix}$.

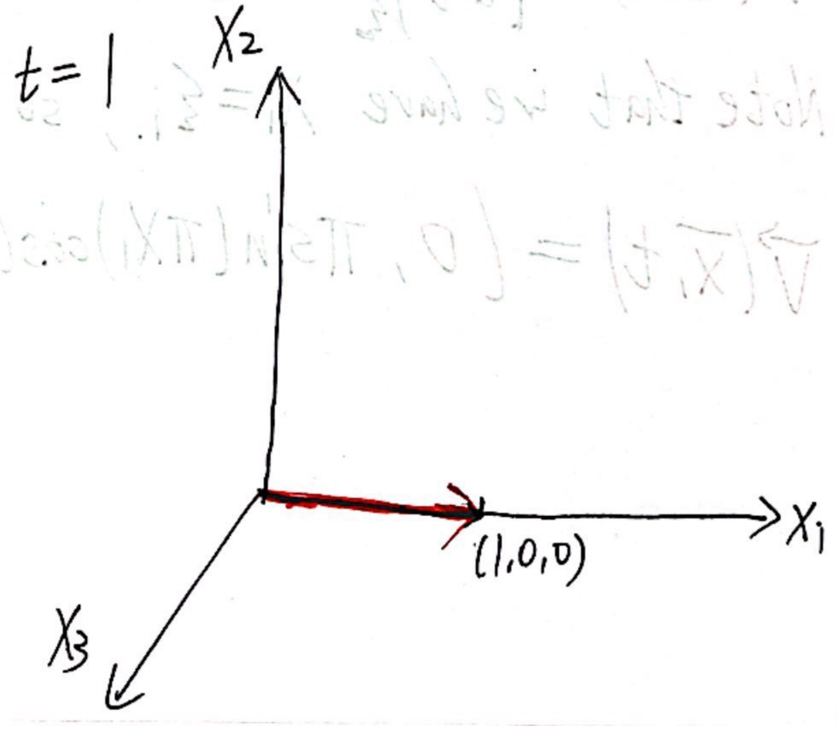
we have $X_a|_{t=\frac{1}{2}} = \begin{pmatrix} a \\ \sin(a\pi) \\ 0 \end{pmatrix}$,

$X_a|_{t=1} = \begin{pmatrix} a \\ 0 \\ 0 \end{pmatrix}$

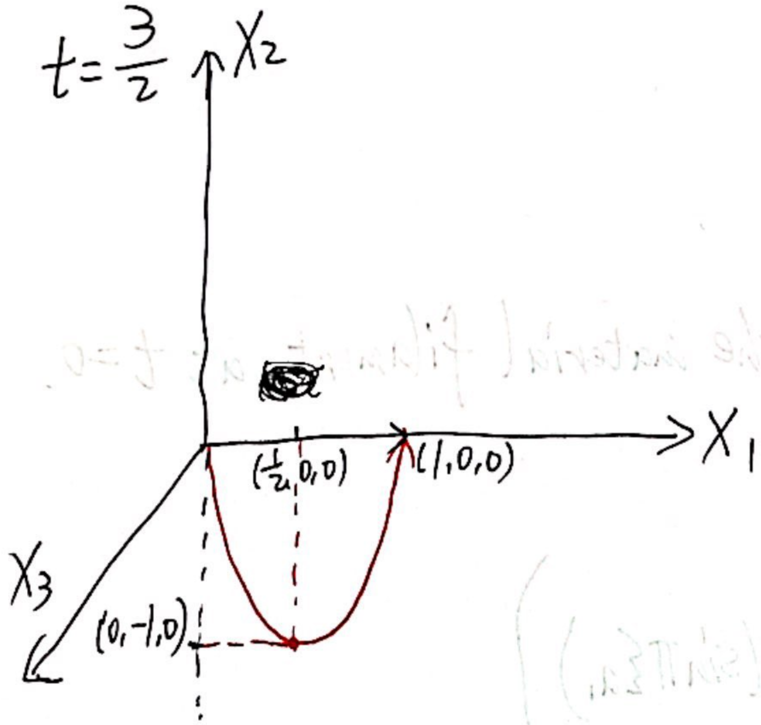
$X_a|_{t=\frac{3}{2}} = \begin{pmatrix} a \\ -\sin(a\pi) \\ 0 \end{pmatrix}$

Plots are following:
(red line is the deformed shape.)





$t=\frac{3}{2}$



To conclude, at $t=\frac{1}{2}$ and $t=\frac{3}{2}$, the filament defoms to be a trigonometric-like shape. And at $t=1$, the defomed shape ~~con~~ coincides with the initial shape.

(b) For velocity $\vec{V}$,

In material description, we have

$$\vec{V}(\bar{\xi},t)=\left(\frac{\partial \bar{X}}{\partial t}\right)_{\xi}=\left(0,\ \pi\sin(\pi\xi_1)\cos(\pi t),\ 0\right).$$

Note that we have $X_1=\xi_1$, so in spatial description:

$$\vec{V}(\bar{X},t)=\left(0,\ \pi\sin(\pi X_1)\cos(\pi t),\ 0\right)$$

For acceleration $\vec{a}$ in material description:

$$\vec{a}(\vec{\xi},t) = \left(\frac{\partial^2 \vec{x}}{\partial t^2}\right)_{\xi} = (0, -\pi^2 \sin(\pi \xi_1)\sin(\pi t), 0)$$

and still we have $x_1 = \xi_1$, so in spatial description we have:

$$\vec{a}(\vec{x},t) = (0, -\pi^2 \sin(\pi x_1)\sin(\pi t), 0)$$

(3)

(a) Displacement field.

$u_1 = k(X_1 + X_3^2)$, $u_2 = kX_1X_3$, $u_3 = k(X_1^2 - X_3)$.

compute
$$\nabla u = \begin{bmatrix} \frac{\partial u_1}{\partial X_1} & \frac{\partial u_1}{\partial X_2} & \frac{\partial u_1}{\partial X_3} \\ \frac{\partial u_2}{\partial X_1} & \frac{\partial u_2}{\partial X_2} & \frac{\partial u_2}{\partial X_3} \\ \frac{\partial u_3}{\partial X_1} & \frac{\partial u_3}{\partial X_2} & \frac{\partial u_3}{\partial X_3} \end{bmatrix} = \begin{bmatrix} k & 0 & 2kX_3 \\ kX_3 & 0 & kX_1 \\ 2kX_1 & 0 & -k \end{bmatrix}.$$

and the stress is given by

$$E = \frac{1}{2}(\nabla u + \nabla u^T).$$

$$= \begin{bmatrix} k & \frac{k}{2}X_3 & k(X_1+X_3) \\ \frac{k}{2}X_3 & 0 & \frac{k}{2}X_1 \\ k(X_1+X_3) & \frac{k}{2}X_1 & -k \end{bmatrix}$$

The components of E are the stress components.

Page. 6.

comments on the style of deformation:

write $\nabla u = \frac{1}{2}(\nabla u + \nabla u^T) + \frac{1}{2}(\nabla u - \nabla u^T)$

$E = \frac{1}{2}(\nabla u + \nabla u^T)$, $W = \frac{1}{2}(\nabla u - \nabla u^T)$.

E represents the internal deformation, also known as strain.

W represents the rigid body rotation.

to further comment on E, suppose we have our original vector $d\bar{\xi}_1 = (d\xi_1, 0, 0)$, $d\bar{\xi}_2 = (0, d\xi_2, 0)$ at $\bar{X} = (X_1, X_2, X_3)$.

the change of length of $d\bar{\xi}_1$: $d\xi_1' - d\xi_1 \approx (1 + E_{11} - 1)\, d\xi_1$.

so $\frac{D(d\bar{\xi}_1)}{Dt} = E_{11} = K$, similarly, change in length of $d\bar{\xi}_2$ is given by $E_{22} = 0$

the change in angle between $d\vec{\xi}_1$, $d\vec{\xi}_2$ is given by

$2E_{12} = KX_3$

(b) Recall the equation of motion:

$$F_{body} + F_{stress} = m\vec{a}.$$

In this question, $\begin{cases} \text{equilibrium} \Rightarrow \vec{a} = 0 \\ \text{in the absence of body forces} \Rightarrow F_{body} = 0. \end{cases}$

Thus we have:

$$F_{stress} = \nabla \cdot \sigma = 0.$$

so $\dfrac{\partial \sigma_{ji}}{\partial X_j} = 0.$

In X_1 direction:

$$\frac{\partial \sigma_{11}}{\partial X_1} + \frac{\partial \sigma_{21}}{\partial X_2} + \frac{\partial \sigma_{31}}{\partial X_3} = 0 + 0 + 0 = 0 \text{ (Already satisfied.)}$$

In X_2 direction:

$\cancel{\partial\sigma}$

$$\frac{\partial \sigma_{12}}{\partial X_1} + \frac{\partial \sigma_{22}}{\partial X_2} + \frac{\partial \sigma_{32}}{\partial X_3} = 0 + 0 + 0 = 0 \text{ (Already satisfied.)}.$$

In X_3 direction:

$$\frac{\partial \sigma_{13}}{\partial X_1} + \frac{\partial \sigma_{23}}{\partial X_2} + \frac{\partial \sigma_{33}}{\partial X_3} = 1 + C - B = 0$$

write σ as

$$\sigma = \begin{pmatrix} AX_3 & CX_3 & X_1+X_2 \\ CX_3 & X_1+X_3 & CX_2 \\ X_1+X_2 & CX_2 & CX_1-BX_3 \end{pmatrix}$$

the plane $X_1+X_3=0$ is stress free,

let $s=(a,b,c)$ on plane $X_1+X_3=0$., also $\bar{0}=(0,0,0)$ is on the plane.

wehave $a+c=0 \Rightarrow c=-a$.

let $\hat{n}$ be unit normal of $X_1+X_3=0$, suppose $\hat{n}=(n_1,n_2,n_3)$

wehave
$$\begin{cases} \|\hat{n}\|=1 \\ \hat{n}\cdot s = n_1 a + n_2 b + n_3\cdot(-a) = a(n_1-n_3)+bn_2=0, \quad \forall a,b\in\mathbb{R}. \end{cases}$$

so wehave $\hat{n}=\frac{\sqrt{2}}{2}(1,0,1)$

stress on such a plane $\bar{t}=\sigma\cdot\hat{n}$,

so $\bar{t}=\frac{\sqrt{2}}{2}(AX_3+X_1+X_2,\ C(X_2+X_3),\ CX_1-BX_3+X_1+X_2)=\bar{0}$.

also, we have $X_1+X_3=0$ on the plane.

so we have
$$\begin{cases} A-1=0 \\ C=0 \\ C+1-B=0 \end{cases} \Rightarrow \begin{cases} A=1 \\ B=1 \\ C=0 \end{cases}$$