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Q1.

(i) There are 7 variables ($\tau_0, \mu_\infty, \rho, V, M, r$ and g) and 3 dimensions (M, L, T) and so 4 dimensionless groups are required.

(ii). According to Buckingham Pi Theory,

write as

$$\begin{array}{c|ccccccc} & \tau_0 & \mu_\infty & \rho & g & V & M & r \\ \hline M & 1 & 1 & 1 & 0 & 0 & 1 & 0 \\ L & -1 & -1 & -3 & 1 & 1 & 0 & 1 \\ T & -2 & -1 & 0 & -2 & -1 & 0 & 0 \end{array}$$

so

$$A = \begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 1 & 0 \\ -1 & -1 & -3 & 1 & 1 & 0 & 1 \\ -2 & -1 & 0 & -2 & -1 & 0 & 0 \end{pmatrix} \quad \text{let } \bar{X} = (X_{\tau_0}\ X_{\mu_\infty}\ X_\rho\ X_g\ X_V\ X_M\ X_r)$$

$$A_1 = \begin{pmatrix} 1 & 1 & 1 \\ -1 & -1 & -3 \\ -2 & -1 & 0 \end{pmatrix} \quad A_2 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ -2 & -1 & 0 & 0 \end{pmatrix} \quad \bar{X}_1 = (X_{\tau_0}\ X_{\mu_\infty}\ X_\rho)$$

$$\bar{X}_2 = (X_g\ X_V\ X_M\ X_r)$$

~~we~~ then the linear equation $A\bar{X} = 0$ is equivalent to.

$$A_1\bar{X}_1 + A_2\bar{X}_2 = \bar{0} \Rightarrow A_1\bar{X}_1 = -A_2\bar{X}_2 \Rightarrow \bar{X}_1 = -A_1^{-}A_2\bar{X}_2 .$$

compute A_1^{-} by python Code, we have $A_1^{-} = \begin{pmatrix} -\frac{3}{4} & \frac{1}{4} & -1 \\ \frac{3}{2} & -\frac{1}{2} & 1 \\ \frac{1}{4} & \frac{1}{4} & 0 \end{pmatrix}$

so take $\bar{X}_2 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$ or $\begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$ or $\begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$ or $\begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$.

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we solve $\overline{X}$ to be

$$\overline{X} = \begin{pmatrix} -\frac{9}{4} \\ \frac{5}{2} \\ -\frac{1}{4} \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \text{ or } \begin{pmatrix} -\frac{5}{4} \\ \frac{3}{2} \\ -\frac{1}{4} \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \text{ or } \begin{pmatrix} \frac{3}{4} \\ -\frac{3}{2} \\ -\frac{1}{4} \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \text{ or } \begin{pmatrix} -\frac{1}{4} \\ \frac{1}{2} \\ -\frac{1}{4} \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

thus we have the following dimensionless groups:

$$N_1 = \frac{\mu_\infty^{\frac{5}{2}}\, g}{T_0^{\frac{9}{4}}\, \rho^{\frac{1}{4}}} \quad N_2 = \frac{\mu_\infty^{\frac{3}{2}}\, v}{T_0^{\frac{5}{4}}\, \rho^{\frac{1}{4}}} \quad N_3 = \frac{T_0^{\frac{3}{4}}\, M}{\mu_\infty^{\frac{3}{2}}\, \rho^{\frac{1}{4}}}$$

$$N_4 = \frac{\mu_\infty^{\frac{1}{2}}\, r}{T_0^{\frac{1}{4}}\, \rho^{\frac{1}{4}}}$$

(iii) take new $N_2 = \dfrac{N_2}{\sqrt{N_1 N_4}} = \dfrac{\dfrac{\mu_\infty^{\frac{3}{2}} v}{T_0^{\frac{5}{4}} \rho^{\frac{1}{4}}}}{\sqrt{\dfrac{\mu_\infty^{\frac{5}{2}} g}{T_0^{\frac{9}{4}} \rho^{\frac{1}{4}}} \cdot \dfrac{\mu_\infty^{\frac{1}{2}} r}{T_0^{\frac{1}{4}} \rho^{\frac{1}{4}}}}}$

$= \dfrac{v}{\sqrt{gr}}$ ~ the Fronde Number, represents the ratio of inertia to gravity, and $\underset{\text{new}}{N_3} = \dfrac{N_3}{N_4} = \dfrac{T_0 M}{\mu_\infty^2}$

new $N_1 = N_1 N_3^3 = \dfrac{g M^3}{\mu_\infty^2 \rho}$, new $N_4 = N_3^3 \cdot N_4^3 = \dfrac{M^3 \cdot r^3}{\rho}$

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Thus our new dimension less group should be:

$$N_1 = \frac{g M^3}{\mu \omega^2 \rho}$$

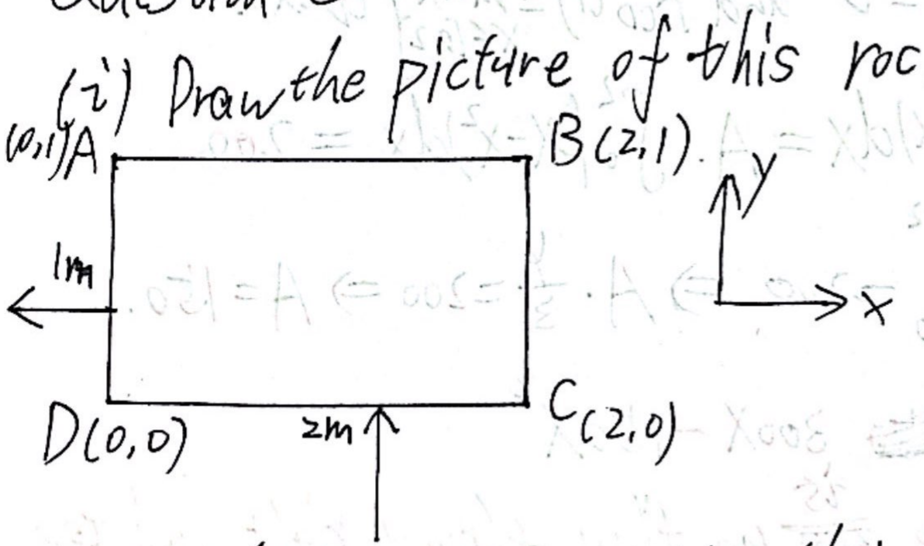
$$N_2 = \frac{v}{\sqrt{g r}} \sim \text{the Fronde Number.}$$

$$N_3 = \frac{T_0 M}{\mu \omega^2}$$

$$N_4 = \frac{M \cdot r^3}{\rho}$$

Question 2

(i) Draw the picture of this rock:



From the question, we notice that there is only inlet ~~[illegible]~~ heat flow at the bottom, and the only outlet is the left edge. No heat flows through the right and top sides.

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And also compute the total heat flow inlet

$Q_{in} = 100\, W/m \cdot 2m = 200 W = Q_{out}$.

the edges arebeing evenly cooled, thus at the left edge we have: $F_x = -\frac{200}{1} = -200$ (Note the outlet flow direction is negative-x direction.).

So $F_{AD} = -200$.

At the bottom, assuming $F_{CD}(X) = A \cdot (2-X) \cdot X$, A some constant.

we have $F_{CD}(2) = F_{CD}(0) = 0$ and $F_{CD}(1) = \max_{X \in [0,2]} F_{CD}(X)$.

Note that $Q_{in} = \int_0^2 F_{CD}(X) dX = A \cdot \int_0^2 (2X - X^2) dX = 200$.

wehave $A \cdot \left[X^2 - \frac{X^3}{3}\right]_{X=0}^{X=2} = 200 \Rightarrow A \cdot \frac{4}{3} = 200 \Rightarrow A = 150$.

thus we have $F_{CD}(x) = 300X - 150X^2$

As mentioned before, there ~~are~~ is no flow through right and top sides, thus naturally, $F_{AB} = F_{BC} = 0$.

To conclude, wehave

$$\begin{cases} F_{left} = -200 \\ F_{top} = 0 \\ F_{right} = 0 \\ F_{bottom} = 300X - 150X^2 \end{cases}$$

(iii) For the stream function ψ, we always have:

$$F_x = \frac{\partial \psi}{\partial y},\ F_y = -\frac{\partial \psi}{\partial x},\ \text{suppose } \psi(C) = 0.$$

at the bottom we have:

$$F_{DC}(x) = 300x - 150x^2 = -\frac{\partial \psi_{CD}}{\partial x} = -\frac{d\psi_{CD}}{dx}$$

$$\text{so } \psi_{CD}(x) = \int 150x^2 - 300x \, dx = 150\left(\frac{x^3}{3} - x^2\right) + C.$$

$$\psi(C) = 0 \Rightarrow \psi_{CD}(2) = 0 \Rightarrow C = 200.$$

thus we have $\psi(D) = \psi_{CD}(0) = C = 200.$

at the left side we have:

$$F_{AD} = -200 = \frac{\partial \psi_{AD}}{\partial y} = \frac{d\psi_{AD}}{dy},\ \text{so } \psi_{AD}(y) = -200y + D.$$

$$\psi(D) = 200 \Rightarrow \psi_{AD}(0) = 200 \Rightarrow D = 200.$$

also indicates $\psi(A) = \psi_{AD}(1) = -200 \cdot 1 + 200 = 0$

Note no flow through AB, BC, Thus $F_{AB} = F_{BC} = 0$.

$\psi_{AB} = \alpha_1$, $\psi_{BC} = \alpha_2$, (α_1, α_2 constants). and

$$\psi(A) = 0 \Rightarrow \alpha_1 = 0$$

$$\psi(C) = 0 \Rightarrow \alpha_2 = 0.$$

To conclude, we have the stream functions around the boundaries:

$$\begin{cases} \psi_{left} = -200y + 200 \\ \psi_{top} = 0 \\ \psi_{right} = 0 \\ \psi_{bottom} = 50x^3 - 150x^2 + 200 \end{cases}$$

Q3.

(i) the flow is steady, so $\frac{D\vec{u}}{Dt} = 0$.

body forces be ignored, so the term $\rho \vec{g}$ can be threw away.

let's write the Navier-Stokes equation in the x-direction.

we have $\cancel{0 = -\frac{DP}{DX}}$

$$0 = -\frac{dP}{dx} + \frac{d(\tau_{yx})}{dx}$$

No pressure gradient in the direction of flow implies $\frac{dP}{dx} = 0$.

thus we have $\cancel{\tau_{yx} = 0}$. $\tau_{yx} = A$, (A some constant.)

~~Also, note at the top we measured τ_{top}.~~

~~thus we have $\tau_{yx} = \tau_{top}$~~

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(ii) Note at the top we have measured I_{top}.

So we have $T_{yx} = I_{top}$.

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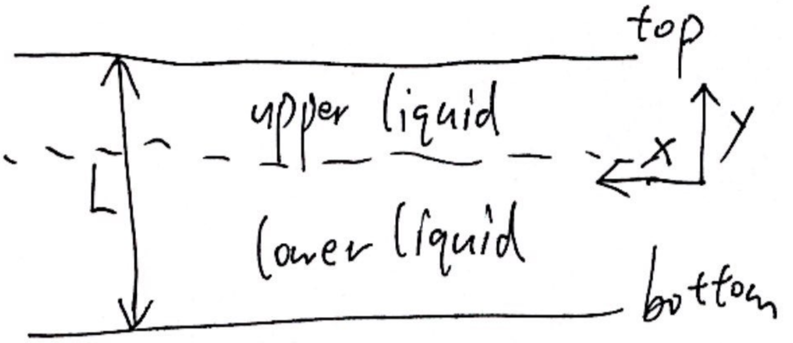
(iii) Draw the picture:



let y be the distance from the bottom, we have:

for the upper liquid:

$$\tau_{yx} = \mu_{upper} \frac{dV_x}{dy} = \tau_{top}$$

$$\Downarrow$$

$$V_x = \frac{\tau_{top}}{\mu_{upper}} y + B,$$

Given $y = L$, $V_x = V_{top}$.

we can derive $B = -\frac{\tau_{top}}{\mu_{upper}} L + V_{top}$

thus $V_x = \frac{\tau_{top}}{\mu_{upper}}(y-L) + V_{top}$., $y > \phi L$.

for the lower liquid:

$$\tau_{yx} = \mu_{lower} \frac{dV_x}{dy} = \tau_{top}.$$

thus $V_x = \frac{\tau_{top}}{\mu_{lower}} y + C$ with $V_x = 0$ at $y = 0$.

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Thus, $C = 0$.

So we have $V_x = \frac{\tau_{top}}{\mu_{lower}} y$ for $y < \phi L$