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Problem 1.

(i). as we are given $y(x) = c + mx$,

and $m = f'(x_0)$, $y(x_0) = f(x_0)$,

we ~~we~~ will have the equation:

$$y(x_0) = f(x_0) = c + f'(x_0) \cdot x_0,$$

just by taking $x = x_0$,

then we can derive: $c = f(x_0) - f'(x_0) \cdot x_0$.

~~plug~~ substitude back, we will have:

$$y(x) = c + mx = f(x_0) - f'(x_0) \cdot x_0 + f'(x_0) \cdot x$$
$$= f(x_0) + (x - x_0) \cdot f'(x_0).$$

which ended up with the same expression as given.

(ii) given $y(x) = c + mx + nx^2$.

compute its derivative: $y'(x) = 2nx + m$

second derivative: $y''(x) = 2n$.

Also, Note the conditions given by the question:

$$\begin{cases} y(x_0) = f(x_0) \\ y'(x_0) = f'(x_0) \\ y''(x_0) = f''(x_0) \end{cases} \Rightarrow \begin{cases} c + mx_0 + n \cdot x_0^2 = f(x_0) \\ 2nx_0 + m = f'(x_0) \\ 2n = f''(x_0) \end{cases}$$

we will get:

$$\begin{cases} c = f(x_0) - x_0 \cdot f'(x_0) + \frac{x_0^2}{2} \cdot f''(x_0) \\ m = f'(x_0) - f''(x_0) \cdot x_0 \\ n = \frac{f''(x_0)}{2} \end{cases}$$

Thus, $y(x) = c + mx + nx^2$

$$= f(x_0) - x_0 f'(x_0) + \frac{x_0^2}{2} \cdot f''(x_0) + [f'(x_0) - f''(x_0) \cdot x_0] \cdot x + \frac{f''(x_0)}{2} \cdot x^2$$

$$= f(x_0) + (x - x_0) \cdot f'(x_0) + \left[\frac{x^2}{2} - x_0 \cdot x + \frac{x_0^2}{2}\right] \cdot f''(x_0)$$

$$y(x) = f(x_0) + (x-x_0)\cdot f'(x_0) + (x-x_0)^2 \cdot \frac{f''(x_0)}{2}$$

Then, recall the ~~definition~~ formula for the infinite Taylor series for the source function f around x_0:

$f(x) = \sum_{n=0}^{\infty} \frac{(x-x_0)^n}{n!} \cdot f^{(n)}(x_0)$. [Here, $f^{(n)}(x_0)$ represents the value of the nth derivative of $f(x)$ taking $x = x_0$.]

Expend it, we get:

$$f(x) = f(x_0) + (x-x_0)\cdot f'(x_0) + \frac{(x-x_0)^2}{2!} f''(x_0) + \frac{(x-x_0)^3}{3!} \cdot f'''(x_0)$$

$$+ \cdots\cdots \frac{(x-x_0)^n}{n!} f^{(n)}(x_0) + \cdots\cdots$$

comparing with $y(x)$ we derived, we notice that $y(x)$ just coincides with the Taylar series of $f(x)$ expends at x_0, truncating ~~$O(x^3)$~~ $O((x-x_0)^3)$ terms.

That is: $f(x) = y(x) + O\left((x-x_0)^3\right)$

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(iii) we express the nth derivative of f as $f^{n}(X)$,

thus, $\sin^{(n)}(X) = \begin{cases} \cos(x), & n \bmod 4 \equiv 1 \\ -\sin(x), & n \bmod 4 \equiv 2 \\ -\cos(x), & n \bmod 4 \equiv 3 \\ \sin(x), & n \bmod 4 \equiv 0 \end{cases}$,

Thus, $\sin^{(n)}(0) = \begin{cases} 1, & n \bmod 4 \equiv 1 \\ 0, & n \bmod 4 \equiv 2 \\ -1, & n \bmod 4 \equiv 3 \\ 0, & n \bmod 4 \equiv 0 \end{cases}$

similarly, we derive that

$$\cos^{(n)}(0) = \begin{cases} 0, & n \bmod 4 \equiv 1 \\ -1, & n \bmod 4 \equiv 2 \\ 0, & n \bmod 4 \equiv 3 \\ 1, & n \bmod 4 \equiv 0 \end{cases}$$

Note, when expend at $X=0$, the Taylor series of $f(x)$ is simply $f(X) = f(0) + X \cdot f'(X_0) + \frac{X^2}{2!} f''(X_0) + \cdots \frac{X^n}{n!} f^{(n)}(X_0) + \cdots$

Thus, we have:

$$\sin(X) = 0 + X \cdot 1 + \frac{X^2}{2!} \cdot 0 + \frac{X^3}{3!} \cdot (-1) + \frac{X^4}{4!} \cdot 0 + \cdots$$

$$= X - \frac{X^3}{3!} + \frac{X^5}{5!} - \frac{X^7}{7!} + \frac{X^9}{9!} + \cdots (-1)^k \cdot \frac{X^{2k+1}}{(2k+1)!} \cdots$$

$$\cos(X) = 1 + X \cdot 0 + \frac{X^2}{2!} \cdot (-1) + \frac{X^3}{3!} \cdot 0 + \frac{X^4}{4!} \cdot 1 + \cdots$$

$$= 1 - \frac{X^2}{2!} + \frac{X^4}{4!} - \frac{X^6}{6!} + \frac{X^8}{8!} + \cdots (-1)^k \cdot \frac{X^{2k}}{(2k)!} \cdots, \ k \in \mathbb{N}.$$

(iii)

The simple approximation to $\sin(x)$, could be made as $y_1(x) = x - \frac{x^3}{3!}$ (or more terms, depending on the power of error required.)

we could see $\sin(x) = y_1(x) + \mathcal{O}(x^5)$,

when x is very small, or say, x close to 0,

we have $|\mathcal{O}(x^5)| \leq C \cdot x^5$ where C is a constant.

so $y_1(x)$ becomes a good approximation.

Similarly, choose $y_2(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!}$,

then we have $\cos(x) = y_2(x) + \mathcal{O}(x^6)$,

which seems to be a good approximation

when x is very small.

[Note: if we want our approximation y_1, y_2 to be more accurate on a larger interval, then we should choose a longer polynomial regarding the Taylor series for $\sin(x)$, $\cos(x)$ to be $y_1(x)$, $y_2(x)$.]

Page. 6 Problem 2

(i) Consider $X = [X_1, X_2]^T$, such that $A \cdot X = 0$.

we have ~~$A|X$~~ $\left(\begin{array}{cc|c} 1 & 0 & 0 \\ 2 & 0 & 0 \end{array}\right) \xrightarrow{R_2 \leftarrow R_2 - 2\cdot R_1} \left(\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right)$,

Thus, we have $X_1 = 0$, $X_2 = C$, for C be arbitary constant.

Thus, the null space of A $null(A) = t \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, $t \in \mathbb{R}$.

look at the column vectors for A, $\xi_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$, $\xi_2 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$.

for $a, b \in \mathbb{R}$,

if $a \cdot \xi_1 + b \cdot \xi_2 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$, we will have $\begin{cases} a = 0 \\ b = C \in \mathbb{R}. \end{cases}$

thus, ξ_1, ξ_2 is linear dependent,

also. $a\xi_1 = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow a = 0$, so $rank(A) = 1$.

$Range(A) = span\{\xi_1, \xi_2\} = span\left\{\begin{pmatrix} 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \end{pmatrix}\right\}$

$= a \cdot \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ for $a \in \mathbb{R}$.

Consider linear system $Ax=b$.

① when $b=\begin{pmatrix}4\\8\end{pmatrix}$, we have $\left(\begin{array}{cc|c}1&0&4\\2&0&8\end{array}\right) \xrightarrow{R_2 \leftarrow R_2-2\cdot R_1} \left(\begin{array}{cc|c}1&0&4\\0&0&0\end{array}\right)$,

Thus, we have ~~solution~~ non-trivial solution $X=\begin{pmatrix}4\\t\end{pmatrix}$, $t\in\mathbb{R}$.

② when $b=\begin{pmatrix}4\\9\end{pmatrix}$, we have $\left(\begin{array}{cc|c}1&0&4\\2&0&9\end{array}\right) \xrightarrow{R_2 \leftarrow R_2-2\cdot R_1} \left(\begin{array}{cc|c}1&0&4\\0&0&1\end{array}\right)$,

which indicate the system has no solutions, because

$0\cdot X_1+0\cdot X_2=0\neq 1$ for any $(X_1,X_2)\in\mathbb{R}^2$.

Given a solution X for $Ax=b$, and $X'\in \text{null}(A)$.

other valid solutions is simply $X+X'$, for any X' in null (A). Reason! $A\cdot(X+X')=AX+A\cdot X'=AX+0=AX=b$.

As for a example for this A, when $b=\begin{pmatrix}4\\8\end{pmatrix}$, and we know $X=\begin{pmatrix}4\\0\end{pmatrix}$ is a solution for $Ax=b$, then ~~any~~ other valid solution is given by $X'=\begin{pmatrix}4\\0\end{pmatrix}+t\cdot\begin{pmatrix}0\\1\end{pmatrix}$, $t\in\mathbb{R}$.

$$=\begin{pmatrix}4\\t\end{pmatrix},\ t\in\mathbb{R}.$$

(ii) for eigen values λ_1, λ_2 of A,

we have $Ax = \lambda x$. for some $x \in \mathbb{R}^2$.

thus $(A-\lambda I)\cdot x = 0 \Longrightarrow \det(A-\lambda I) = 0$.

Thus.
$$\begin{vmatrix} 2-\lambda & 4 \\ 4 & -4-\lambda \end{vmatrix} = (\lambda-2)(\lambda+4) - 16$$
$$= \lambda^2 + 2\lambda - 24$$
$$= (\lambda+6)(\lambda-4)$$
$$= 0.$$

so we have $\lambda_1 = -6$. $\lambda_2 = 4$.

Define eigenvectors v_1, v_2 for λ_1, λ_2:

we have $Av_1 = \lambda_1 v_1$.
$Av_2 = \lambda_2 v_2$.

consider P a matrix with $P = [v_1, v_2]$.

$$\begin{cases} Av_1 = \lambda_1 v_1 \\ Av_2 = \lambda_2 v_2 \end{cases} \Longrightarrow A\cdot P = [\lambda_1 v_1, \lambda_2 v_2]$$
$$= [v_1, v_2]\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$$
$$= P\cdot\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$$

take Δ the diagonal matrix.

such that $\Delta = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$.

we have $A \cdot P = P \cdot \Delta$

$$\Downarrow$$

$$A = P \cdot \Delta \cdot P^{-1},$$

so A is diagonalised.

In this particular question,

for $\lambda_1 = -6$.

we have $(A - \lambda_1 I) \cdot V_1 = 0 \Rightarrow \begin{pmatrix} 8 & 4 \\ 4 & 2 \end{pmatrix} \cdot V_1 = 0$

$\Rightarrow V_1 = \begin{pmatrix} 1 \\ -2 \end{pmatrix} \cdot t,\ t \in \mathbb{R}$; here we just take $V_1 = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$.

for $\lambda_2 = 4$,

we have $(A - \lambda_2 I) \cdot V_2 = 0 \Rightarrow \begin{pmatrix} -2 & 4 \\ 4 & -8 \end{pmatrix} \cdot V_2 = 0$

$\Rightarrow V_2 = \begin{pmatrix} 2 \\ 1 \end{pmatrix} \cdot t,\ t \in \mathbb{R}$. here we just take $V_2 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$.

so. $P = (V_1, V_2) = \begin{pmatrix} 1 & 2 \\ -2 & 1 \end{pmatrix}$, $P^{-1} = \dfrac{\begin{pmatrix} 1 & -2 \\ 2 & 1 \end{pmatrix}}{\det(P)} = \begin{pmatrix} \frac{1}{5} & -\frac{2}{5} \\ \frac{2}{5} & \frac{1}{5} \end{pmatrix}$

Thus,

$$A=\begin{pmatrix} 2 & 4 \\ 4 & -4 \end{pmatrix}=\begin{pmatrix} 1 & 2 \\ -2 & 1 \end{pmatrix}\cdot\begin{pmatrix} -6 & 0 \\ 0 & 4 \end{pmatrix}\cdot\begin{pmatrix} \frac{1}{5} & -\frac{2}{5} \\ \frac{2}{5} & \frac{1}{5} \end{pmatrix}$$

$$=P\cdot\Delta\cdot P^{-1}$$

Problem 3.

There may be some cases that it's inconvenient for us to find a exact analytical solution for a problem, but we still want to verify that our program works. In this case, we could manufacture a solution which could give a very similar equation to the one we want to save. For example, we want to solve the equation:

$y'=3y+6t^2-2t+1$,

suppose we have completed no idea of how to work out an analytical solution for an ODE, we could just writing a function of t to be our "manifactured function",

like $\hat{y}(t) = \exp(2t) + 3t^2 + 2t + 5$.

then we have $\hat{y}'(t) = 2\exp(2t) + 6t + 2$.

and our equation to be: $\hat{y}'(t) = 2\hat{y}(t) - 6t^2 + 2t - 8$.

which is quite similar to the original equation.

Then we could implement $\hat{y}' = 2\hat{y} - 6t^2 + 2t - 8$ ~~to the~~ as the argument for our program to check if it could works out the correct result.

The reason why it is such a valuable technique is that it's easy for us to monitor the errors in the program by MMS analysis, as well as how ~~them~~ they behaves when the form of inputs change.