Mathematical Proof of Correctness for Sum of Multiples of 3 or 5 Using the Inclusion-Exclusion Principle

Problem Statement

Given a positive integer N, find the sum of all natural numbers less than N that are multiples of 3 or 5.

Understanding the Inclusion-Exclusion Principle

The inclusion-exclusion principle is a fundamental concept in combinatorics used to calculate the size or sum of the union of overlapping sets. It corrects for overcounting elements that are common to multiple sets.

General Formulation

For two finite sets A and B:

$$|A \cup B| = |A| + |B| - |A \cap B|$$

- $|A|$: Number of elements in set A.
- $|B|$: Number of elements in set B.
- $|A \cap B|$: Number of elements common to both A and B.

This formula ensures that elements counted in both A and B are not double-counted in $A \cup B$.

Application to Sums

When dealing with sums of elements in sets, the principle extends as:

$$\sum_{x \in A \cup B} x = \sum_{x \in A} x + \sum_{x \in B} x - \sum_{x \in A \cap B} x$$

Why the Principle is Necessary

If we simply add $\sum_{x \in A} x$ and $\sum_{x \in B} x$, any elements that are in both A and B (i.e., $A \cap B$) will be counted twice. The inclusion-exclusion principle corrects this by subtracting $\sum_{x \in A \cap B} x$, ensuring each element is counted exactly once.

Aspect Utilized in This Context

In the context of our problem:

- A: Multiples of 3 less than N.
- B: Multiples of 5 less than N.
- $A \cap B$: Multiples of both 3 and 5 (i.e., multiples of 15) less than N.

By applying the inclusion-exclusion principle to sums, we can accurately compute the total sum of multiples of 3 or 5 below N without double-counting the multiples of 15.

Proof of Correctness

Definition of Sets

Define the sets as:

$$A = \{n \in \mathbb{N} \mid 1 \leq n < N,\ 3 \mid n\}$$

$$B = \{n \in \mathbb{N} \mid 1 \leq n < N,\ 5 \mid n\}$$

$$A \cap B = \{n \in \mathbb{N} \mid 1 \leq n < N,\ 15 \mid n\}$$

Calculating the Sums

Multiples of 3:

- Number of terms:

$$n_3 = \left\lfloor \frac{N-1}{3} \right\rfloor$$

- Sum:

$$S_3 = 3 \times \frac{n_3(n_3+1)}{2}$$

Multiples of 5:

- Number of terms:

$$n_5 = \left\lfloor \frac{N-1}{5} \right\rfloor$$

- Sum:

$$S_5 = 5 \times \frac{n_5(n_5+1)}{2}$$

Multiples of 15:

- Number of terms:

$$n_{15} = \left\lfloor \frac{N-1}{15} \right\rfloor$$

- Sum:

$$S_{15} = 15 \times \frac{n_{15}(n_{15}+1)}{2}$$

Applying the Inclusion-Exclusion Principle

Using the principle:

$$S = S_3 + S_5 - S_{15}$$

This equation ensures that the sum S includes all multiples of 3 or 5 below N exactly once.

Explanation in Context

- **Overcounting Issue:** Multiples of 15 are included in both S_3 and S_5 because they are divisible by both 3 and 5.
- **Correction:** Subtracting S_{15} removes the duplicated sums of the multiples of 15, correcting the overcounting.

Example with $N = 1000$

Calculations:

- $n_3 = 333$
- $S_3 = 166833$
- $n_5 = 199$

- $S_5 = 99500$
- $n_{15} = 66$
- $S_{15} = 33165$

Final Sum:

$$S = 166833 + 99500 - 33165 = 233168$$

Conclusion

By applying the inclusion-exclusion principle, we have accurately calculated the sum of all natural numbers less than 1000 that are multiples of 3 or 5:

$$\boxed{233168}$$

This confirms the correctness of our function `calculate_sum_of_multiples` $(N = 1000)$.