A General Approach to Computing the Sum of Multiples of Integers Below a Given Upper Limit

Overview and Problem Statement

Given a set of positive integers $D = \{d_1, d_2, \ldots, d_k\}$, our objective is to compute the sum of all natural numbers less than a positive integer N that are divisible by at least one element of the set D.

For example, if $D = \{3, 5\}$ and $N = 10$, the numbers divisible by 3 or 5 are $3, 5, 6, 9$. The sum of these numbers is $3 + 5 + 6 + 9 = 23$.

The challenge is to generalize this process for any set D of divisors and any upper limit N, ensuring no overlaps are double-counted. To accomplish this, we apply the **Principle of Inclusion-Exclusion** (PIE).

Notation and Definitions

- $\mathbb{N}$: The set of natural numbers, defined as $\mathbb{N} = \{1, 2, 3, \ldots\}$.
- $N \in \mathbb{N}$: The positive integer upper limit (exclusive), meaning we are only interested in numbers $< N$.
- $D = \{d_1, d_2, \ldots, d_k\}$: A set of positive divisors, with each $d_i \in \mathbb{N}$.
- $\text{Sum}(d_i, N)$: The sum of all positive integers less than N that are divisible by d_i.
- $\text{lcm}(d_{i_1}, \ldots, d_{i_j})$: The least common multiple of the given set of divisors.
- $\lfloor x \rfloor$: The floor function, which returns the greatest integer less than or equal to x.

Additional Notation:

- $\sum_{1 \leq i < j \leq k}$: Summation over all distinct pairs (i, j) such that $i < j$.
- $\sum_{1 \leq i < j < \ell \leq k}$: Summation over all distinct triplets (i, j, ℓ) such that $i < j < \ell$. This ensures that each combination of three distinct divisors is counted exactly once.

Theorem and Proof

Theorem: The sum of all natural numbers less than N that are divisible by at least one of the integers in the set $D = \{d_1, d_2, \ldots, d_k\}$ is given by:

$$S(N, D) = \sum_{\emptyset \neq S \subseteq D} (-1)^{|S|+1} \cdot \mathrm{Sum}(\mathrm{lcm}(S), N).$$

This formula follows from the **Principle of Inclusion-Exclusion** (PIE), which ensures that elements divisible by multiple divisors are not over-counted.

Step 1: Sum of Multiples of a Single Divisor

Let $d \in \mathbb{N}$ be a divisor. The set of all natural numbers divisible by d and strictly less than N is given by:

$$\{d, 2d, 3d, \ldots, m \cdot d\},$$

where m is the largest positive integer such that $m \cdot d < N$. Formally:

$$m = \left\lfloor \frac{N-1}{d} \right\rfloor.$$

Thus, the numbers divisible by d below N form an **arithmetic progression** with the first term d and common difference d. The general formula for the sum of the first m terms of an arithmetic progression with first term a and common difference d is:

$$\mathrm{Sum}(d, N) = d \cdot \frac{m(m+1)}{2}.$$

Step 2: Inclusion-Exclusion for Multiple Divisors

When we are dealing with multiple divisors, we need to ensure that numbers divisible by more than one divisor are not counted multiple times. For this, we use the **Principle of Inclusion-Exclusion (PIE)**.

Let $S \subseteq D$ be a non-empty subset of the divisors. The set of numbers divisible by all elements of S is precisely the set of numbers divisible by $\mathrm{lcm}(S)$. Therefore:

$$\mathrm{Sum}(\mathrm{lcm}(S), N) = \mathrm{Sum}\left(\mathrm{lcm}(d_{i_1}, \ldots, d_{i_j}), N\right),$$

where $j = |S|$ is the size of the subset.

$$S(N, D) = \sum_{i=1}^{k} \mathrm{Sum}(d_i, N)$$

$$- \sum_{1 \leq i < j \leq k} \mathrm{Sum}(\mathrm{lcm}(d_i, d_j), N)$$

$$+ \sum_{1 \leq i < j < \ell \leq k} \mathrm{Sum}(\mathrm{lcm}(d_i, d_j, d_\ell), N)$$

$$-\cdots$$

In general:

$$S(N,D)=\sum_{\emptyset\neq S\subseteq D}(-1)^{|S|+1}\cdot \mathrm{Sum}(\mathrm{lcm}(S),N).$$

Explanation of Summation Notation

Summation $\sum_{1\leq i<j\leq k}$: This summation counts over all pairs (i,j) such that $i<j$.

This ensures that each distinct pair is considered only once.

Summation $\sum_{1\leq i<j<\ell\leq k}$: This summation counts over all distinct triplets (i,j,ℓ) such that $i<j$

This ensures that each unique triplet of divisors is included exactly once in the computation.

Step 3: Example with $D=\{3,5\}$ and $N=1000$

Let $D=\{3,5\}$ and $N=1000$. We compute:

$$\mathrm{Sum}(3,1000)=3\cdot\frac{333\cdot 334}{2}=166833,$$

$$\mathrm{Sum}(5,1000)=5\cdot\frac{199\cdot 200}{2}=99500,$$

$$\mathrm{Sum}(15,1000)=15\cdot\frac{66\cdot 67}{2}=33165.$$

Applying PIE:

$$S(1000,\{3,5\})=166833+99500-33165=233168.$$

Conclusion

The general formula for the sum of multiples of any combination of positive integers below a given upper limit has been rigorously derived using the principle of inclusion-exclusion. The use of ordered summations ensures correct counting of intersections without redundancy, and the methodology can be applied to any set of divisors D and any upper limit N.