# Calculating the metric tensor of an egg.

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## 1 Introduction.

We know that the metric tensor of the egg is:

$$g_{ij} = \begin{bmatrix} (\cos u)^2 + \frac{1}{16}(\sin u)^2(2\cos u + 5)^2 & 0 \\ 0 & (\sin u)^2 \end{bmatrix}$$

and

$$g^{ij} = g_{ij}^{-1} = \begin{bmatrix} \frac{1}{(\cos u)^2 + \frac{1}{16}(\sin u)^2(2\cos u+5)^2} & 0 \\ 0 & \frac{1}{(\sin u)^2} \end{bmatrix}$$

The geodesic equation is:

$$\frac{d^2u^a}{dt^2} + \Gamma^a_{bc}\frac{du^b}{dt}\frac{du^c}{dt} = 0$$

where $u^1 = v$, $u^2 = v$ and $\Gamma^a_{bc} = \frac{\delta^2\vec{R}}{\delta u^b \delta u^c}\frac{\delta\vec{R}}{\delta u^d}g^{da}$

## 2 Calculating the christoffel symbols.

From before we know that:

$$\frac{\delta\vec{R}}{\delta u} = \begin{bmatrix} \cos v \cos u \\ \sin v \cos u \\ -\frac{1}{4}(\sin u)(2\cos u + 5) \end{bmatrix}$$

$$\frac{\delta\vec{R}}{\delta v}=\begin{bmatrix}-\sin v\sin u\\ \cos v\sin u\\ 0\end{bmatrix}$$

We can calculate that:

$$\frac{\delta^2\vec{R}}{\delta u^2}=\begin{bmatrix}-\cos v\sin u\\ -\sin v\sin u\\ \frac{1}{4}(-2\cos 2u-5\cos u)\end{bmatrix}$$

$$\frac{\delta^2\vec{R}}{\delta v^2}=\begin{bmatrix}-\cos v\sin u\\ -\sin v\sin u\\ 0\end{bmatrix}$$

$$\frac{\delta^2\vec{R}}{\delta v\delta u}=\frac{\delta^2\vec{R}}{\delta u\delta v}=\begin{bmatrix}-\sin v\cos u\\ \cos v\cos u\\ 0\end{bmatrix}$$

Now we can calculate the christoffel symbols

$$\Gamma^1_{bc}=\frac{\delta^2\vec{R}}{\delta u^b\delta u^c}\frac{\delta\vec{R}}{\delta u^d}g^{d1}=\frac{\delta^2\vec{R}}{\delta u^b\delta u^c}\frac{\delta\vec{R}}{\delta u^1}g^{11}+\frac{\delta^2\vec{R}}{\delta u^b\delta u^c}\frac{\delta\vec{R}}{\delta u^2}g^{21}$$

$g^{21}$ is equal to zero so

$$\Gamma^1_{bc}=\frac{\delta^2\vec{R}}{\delta u^b\delta u^c}\frac{\delta\vec{R}}{\delta u^1}g^{11}$$

$$\Gamma^1_{11}=\frac{\delta^2\vec{R}}{\delta u^1\delta u^1}\frac{\delta\vec{R}}{\delta u^1}g^{11}=\frac{\delta^2\vec{R}}{\delta u^2}\frac{\delta\vec{R}}{\delta u}g^{11}$$

In this coordinate system multiplying vectors is easier:

$$\begin{aligned}\vec{a}\cdot\vec{b}&=(a^x\vec{e}_x+a^y\vec{e}_y+a^z\vec{e}_z)\cdot(b^x\vec{e}_x+b^y\vec{e}_y+b^z\vec{e}_z)=\\ &=a^xb^x(\vec{e}_x\cdot\vec{e}_x)+a^xb^y(\vec{e}_x\cdot\vec{e}_y)+a^xb^z(\vec{e}_x\cdot\vec{e}_z)+\\ &+a^yb^x(\vec{e}_y\cdot\vec{e}_x)+a^yb^y(\vec{e}_y\cdot\vec{e}_y)+a^yb^z(\vec{e}_y\cdot\vec{e}_z)+\\ &+a^zb^x(\vec{e}_z\cdot\vec{e}_x)+a^zb^y(\vec{e}_z\cdot\vec{e}_y)+a^zb^z(\vec{e}_z\cdot\vec{e}_z)\end{aligned}$$

$(\vec{e}_a \cdot \vec{e}_b) = 0$ when $a \neq b$ and $(\vec{e}_a \cdot \vec{e}_b) = 1$ when $a = b$ so:

$$\vec{a} \cdot \vec{b} = a^x b^x + a^y b^y + a^z b^z$$

$$\frac{\delta^2 \vec{R}}{\delta u^2} \cdot \frac{\delta \vec{R}}{\delta u} = \begin{bmatrix} -\cos v \sin u \\ -\sin v \sin u \\ \frac{1}{4}(-2\cos 2u - 5\cos u) \end{bmatrix} \cdot \begin{bmatrix} \cos v \cos u \\ \sin v \cos u \\ -\frac{1}{4}(\sin u)(2\cos u + 5) \end{bmatrix} =$$

$$= -(\cos v)^2 (\cos u)(\sin u) - (\sin v)^2 (\cos u)(\sin u) -$$

$$-\frac{1}{16}(-2\cos 2u - 5\cos u)(\sin u)(2\cos u + 5) =$$

$$= -(\cos u)(\sin u) - \frac{1}{16}(-2\cos 2u - 5\cos u)(\sin u)(2\cos u + 5)$$

$$\Gamma^1_{11} = \frac{\delta^2 \vec{R}}{\delta u^2} \frac{\delta \vec{R}}{\delta u} g^{11} =$$

$$= -(\cos u)(\sin u) - \frac{1}{16}(-2\cos 2u - 5\cos u)(\sin u)(2\cos u + 5) \cdot g^{11} =$$

$$= \frac{-(\cos u)(\sin u) - \frac{1}{16}(-2\cos 2u - 5\cos u)(\sin u)(2\cos u + 5)}{(\cos u)^2 + \frac{1}{16}(\sin u)^2 (2\cos u + 5)^2}$$

$$\Gamma^1_{12} = \Gamma^1_{21} = \frac{\delta^2 \vec{R}}{\delta u \delta v} \frac{\delta \vec{R}}{\delta u} g^{11} = \frac{\delta^2 \vec{R}}{\delta v \delta u} \frac{\delta \vec{R}}{\delta v} g^{21} = 0$$

We will just skip the christoffel symbols when there's $g^{21}$ or $g^{12}$

$$\Gamma^1_{22} = \frac{\delta^2 \vec{R}}{\delta v^2} \frac{\delta \vec{R}}{\delta u} g^{11}$$

$$\begin{bmatrix} -\cos v \sin u \\ -\sin v \sin u \\ 0 \end{bmatrix} \cdot \begin{bmatrix} \cos v \cos u \\ \sin v \cos u \\ -\frac{1}{4}(\sin u)(2\cos u + 5) \end{bmatrix} =$$

$$= -(\cos v)^2 (\sin u)(\cos u) - (\sin v)^2 (\sin u)(\cos u) = -(\sin u)(\cos u)$$

$$\Gamma^1_{22} = -(\sin u)(\cos u) \cdot g^{11} = \frac{-(\sin u)(\cos u)}{(\cos u)^2 + \frac{1}{16}(\sin u)^2(2\cos u + 5)^2}$$

$$\Gamma^2_{11} = \frac{\delta^2 \vec{R}}{\delta u^2}\frac{\delta \vec{R}}{\delta v} g^{22}$$

$$\begin{bmatrix} -\cos v \sin u \\ -\sin v \sin u \\ \frac{1}{4}(-2\cos 2u - 5\cos u) \end{bmatrix} \cdot \begin{bmatrix} -\sin v \sin u \\ \cos v \sin u \\ 0 \end{bmatrix} =$$
$$= (\cos v)(\sin v)(\sin u)^2 - (\cos v)(\sin v)(\sin u)^2 = 0$$

$$\Gamma^2_{11} = 0$$

$$\Gamma^2_{12} = \Gamma^2_{21} = \frac{\delta^2 \vec{R}}{\delta u \delta v}\frac{\delta \vec{R}}{\delta v} g^{22}$$

$$\frac{\delta^2 \vec{R}}{\delta u \delta v}\frac{\delta \vec{R}}{\delta v} = \begin{bmatrix} -\sin v \cos u \\ \cos v \cos u \\ 0 \end{bmatrix} \cdot \begin{bmatrix} -\sin v \sin u \\ \cos v \sin u \\ 0 \end{bmatrix} =$$
$$= (\sin v)^2(\sin u)(\cos u) + (\cos v)^2(\sin u)(\cos u) = (\sin u)(\cos u)$$

$$\Gamma^2_{12} = \Gamma^2_{21}(\sin u)(\cos u) \cdot g^{22} = (\sin u)(\cos u)(\frac{1}{(\sin u)^2}) = \frac{\cos u}{\sin u}$$

$$\Gamma^2_{22} = \frac{\delta^2 \vec{R}}{\delta v^2}\frac{\delta \vec{R}}{\delta v} g^{22}$$

$$\frac{\delta^2 \vec{R}}{\delta v^2}\frac{\delta \vec{R}}{\delta v} = \begin{bmatrix} -\cos v \sin u \\ -\sin v \sin u \\ 0 \end{bmatrix} \cdot \begin{bmatrix} -\sin v \sin u \\ \cos v \sin u \\ 0 \end{bmatrix} =$$
$$= (\sin u)^2(\sin v)(\cos v) - (\sin u)^2(\sin v)(\cos v) = 0$$

$$\Gamma^2_{22} = 0$$

## 3 The christoffel symbols.

We found that:

$$\Gamma^1_{11} = \frac{-(\cos u)(\sin u) - \frac{1}{16}(-2\cos 2u - 5\cos u)(\sin u)(2\cos u + 5)}{(\cos u)^2 + \frac{1}{16}(\sin u)^2(2\cos u + 5)^2}$$

$$\Gamma^1_{12} = \Gamma^1_{21} = 0$$

$$\Gamma^1_{22} = \frac{-(\sin u)(\cos u)}{(\cos u)^2 + \frac{1}{16}(\sin u)^2(2\cos u + 5)^2}$$

$$\Gamma^2_{11} = 0$$

$$\Gamma^2_{12} = \Gamma^2_{21} = \frac{\cos u}{\sin u}$$

$$\Gamma^2_{22} = 0$$