9.7 Application of Taylor's Formula

Many of you may be wondering about the purpose of learning Taylor expansions, as they can be difficult to understand. However, this chapter aims to demonstrate the incredible power of the Taylor formula.

What methods can we employ to find the limits of a function? The limit operator (quadratic + composite)? Substitution? Equivalent infinitesimal? No, no, and no! In fact, once you grasp Taylor's formula, there's no need to learn any of these methods. You won't need to use them anymore! Using them could lead to mistakes, and if you make a mistake, you won't know where it went wrong!

Introduction of Taylor's formula

Limits of Polynomials

When it comes to finding limits, dealing with functions like e^x, $\sin(x)$, or $\ln(1+x)$ can often be challenging. They can seem quite perplexing. However, there's one type of function that is very friendly when it comes to limits, and that is the polynomial!

The general formula for a polynomial is:

$$a_n x^n + a_{n-1} x^{n-1} + \cdots + a_2 x^2 + a_1 x + a_0 = \sum_{i=0}^{n} a_i x^i$$

where a_i are constants, and $a_n \neq 0$.

Adding and subtracting polynomials is straightforward; you just need to add and subtract the coefficients of the corresponding terms. Multiplying and dividing polynomials is a bit more involved, but when it comes to finding the limit as $x \to \infty$, there is an easily understandable conclusion:

Consider the limit when two polynomials are divided:

$$L = \lim_{x \to 0} \frac{a_m x^m + a_{m+1} x^{m+1} + \cdots}{b_n x^n + b_{n+1} x^{n+1} + \cdots}, \quad (a_m b_n \neq 0)$$

The outcome of this limit depends on the relationship between the exponents m and n:

- If $m > n$, which means the numerator is a higher-order infinitesimal and tends to 0 faster than the denominator, the result is $L = 0$.
- If $m < n$, which means the numerator is a lower-order infinitesimal and the denominator tends to 0 faster than the numerator, the result is $L = \infty$.
- If $m = n$, which means the numerator and denominator are of the same order of infinitesimals, you can obtain:

$$L = \lim_{x \to \infty} \frac{(a_m x^m + a_{m+1} x^{m+1} + \cdots)/x^m}{(b_m x^n + b_{m+1} x^{m+1} + \cdots)/x^m} = \lim_{x \to 0} \frac{a_m x^m + a_{m+1} x + \cdots}{b_m x^n + b_{m+1} x + \cdots} = \frac{a_m}{b_m}$$

When finding the limit as $x \to 0$, polynomial factoring can be achieved by keeping only the term with the lowest order. (Note that this must be the factorization of the entire function, as explained in the example at the end of the text in the Appendix.)

Furthermore, note that when finding the limit as $x \to \infty$, it is generally advisable to perform an inverse substitution, i.e., replace all occurrences of x with $\frac{1}{x}$.

From these observations, we can conclude that the limits of polynomial expressions in quadratic form with respect to x are easy to determine as $x \to 0$.

Polynomial Expressions for General Functions

Having established the simplicity of finding the limit of a polynomial, a natural question arises: can we approximate a general function using a polynomial? After all, a polynomial can have an arbitrary number of terms, each with its own coefficients. These coefficients act as adjustable controls for the function, allowing us to manipulate them in an attempt to make the polynomial closely approximate the general function.

Let's denote the general function as $f(x)$ and assume that it can be approximated by a polynomial:

$$f(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots + a_n x^n + \cdots \quad (1)$$

The question now is how to determine the coefficients of this polynomial.

It's easy to observe that $a_0 = f(0)$, but finding the subsequent coefficients doesn't seem as straightforward. We would like the equation above to hold for all values of x, which allows us to differentiate both sides of the equation with respect to x simultaneously. Doing so, we find:

$$f'(x) = a_1 + 2a_2 x + 3a_3 x^2 + \cdots + na_n x^{n-1} + \cdots$$

From this, we can easily deduce that $a_1 = f'(0)$. Similarly, we can determine the remaining coefficients in a similar manner.

By equating the equation (1) and taking the nth derivative of x on both sides, we obtain:

$$f^{(n)}(x) = n! a_n + \cdots$$

Setting $x = 0$ yields:

$$a_n = \frac{f^{(n)}(0)}{n!}$$

Hence, we can now determine all the coefficients $a_1, a_2, \ldots, a_n$ (assuming that $f(x)$ has derivatives of order n). Consequently, we can express $f(x)$ as:

$$f(x) = \sum_{i=0}^{n} a_i x^i + R(x), \quad a_i = \frac{f^{(n)}(0)}{i!}$$

Here, $R(x)$ represents the remainder term or the error of the function $f(x)$ with respect to this polynomial approximation. Currently, we do not know the exact form of this remainder term.

Taylor's Theorem

Although we haven't determined the exact form of the remainder term, Taylor's theorem provides us with reassurance (and I won't attempt to prove it here). Let's recall the **Maclaurin's formula** for **Peano's remainder term**:

If $f(x)$ has derivatives of order n at $x = 0$, then in the vicinity of $x = 0$, we have:

$f(x) = \sum_{i=0}^{n} a_i x^i + o(x^n)$, where $a_i = \frac{f^{(n)}(0)}{i!}$.

Here, $o(x^n)$ denotes a higher order infinitesimal compared to x^n, which means that $\lim_{x \to 0} \frac{o(x^n)}{x^n} = 0$.

Taylor's formula involves substituting x in Maclaurin's formula with $x - x_0$, which represents a translation of x. I don't strictly differentiate between Taylor's formula and Maclaurin's formula. For now, we won't delve into the Lagrange remainder term.

The beauty lies in the fact that elementary functions, along with some power functions and variable limit integrals, constitute a significant portion of our common limit problems. Since these functions possess infinitely differentiable properties, we can comfortably apply Taylor's formula to them.

Commonly Used Taylor Formulas

Here are some commonly used Taylor formulas, as mentioned in the previous chapter:

1. **Exponential Function Expansion**:

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

2. **Trigonometric Function Expansion**:

- Sine function:

$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$$

- Cosine function:

$$\cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$$

- Arctangent Expansion:

$$\arctan(x) = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \ldots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1}$$

- Tangent Expansion:

$$\tan(x) = x + \frac{x^3}{3} + \frac{2x^5}{15} + \frac{17x^7}{315} + \ldots = \sum_{n=1}^{\infty} \frac{B_{2n}(-4)^n(1-4^n)x^{2n-1}}{(2n)!}$$

3. **Natural Logarithm Expansion**:

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} x^n}{n}$$

4. **Geometric Series**:

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \ldots = \sum_{n=0}^{\infty} x^n, \quad |x| < 1$$

5. **Binomial Expansion**:

The binomial expansion, based on the binomial theorem, is used to expand expressions of the form $(a+b)^n$:

$$(a+b)^n = \binom{n}{0} a^n b^0 + \binom{n}{1} a^{n-1} b^1 + \binom{n}{2} a^{n-2} b^2 + \ldots + \binom{n}{n-1} a^1 b^{n-1} + \binom{n}{n} a^0 b^n$$

6. **Error Function Expansion**:

$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \left(x - \frac{x^3}{3} + \frac{x^5}{10} - \frac{x^7}{42} + \ldots \right) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{n!(2n+1)}$$

These formulas provide useful approximations for various functions and can be utilized in calculations and problem-solving.

The Power of The Taylor Formula

The Taylor formula provides valuable insights into the limitations of substitution and the use of equivalent infinitesimals.

In the case of substitution, if we replace $x \to 0$ with $cosx$, we initially obtain the value of 1. However, it is essential to consider higher order terms such as $-\frac{1}{2!}x^2$, which emerge subsequently. Neglecting these terms can lead to incorrect limit evaluations.

Similarly, when considering equivariant infinitesimals, the infinitesimal equivalent of $sinx$ as $x \to 0$ is indeed x. However, it is crucial to acknowledge the presence of higher order terms like $-\frac{1}{3!}x^3$. Disregarding these terms can lead to inaccuracies in limit calculations.

The Taylor formula also reinforces L'Hôpital's rule:

$$L = \lim_{x \to 0} \frac{a_m x^m + a_{m+1} x^{m+1} + \cdots}{b_m x^m + b_{m+1} x^{m+1} + \cdots} \xRightarrow{\text{L'Hôpital's rule}} \lim_{x \to 0} \frac{m a_m x^{m-1} + (m+1) a_{m+1} x^m + \cdots}{m b_m x^{m-1} + (m+1) b_{m+1} x^m + \cdots} = \cdots$$

From this perspective, it becomes evident that L'Hôpital's rule involves the step-by-step reduction of the numerator and denominator, akin to a draw. In contrast, Taylor's formula allows for a comprehensive breakdown of both the numerator and denominator into polynomials, akin to an X-ray of a function, providing a holistic view at a glance.

Application of Taylor's Formula

When applying Taylor's formula to evaluate limits, it is crucial to include the Peano remainder term. This inclusion allows you to assess the accuracy of your approximation. Let's consider an example:

$$\lim_{x \to 0} \frac{e^x - 1 - x}{x^2} = \lim_{x \to 0} \frac{(1 + x + \frac{1}{2!} x^2 + o(x^2)) - 1 - x}{x^2} = \lim_{x \to 0} \frac{\frac{1}{2} x^2 + o(x^2)}{x^2} = \frac{1}{2}$$

Substitution, equivalent infinitesimals, and L'Hôpital's rule are indeed useful and concise techniques. However, it is essential to exercise caution and apply them selectively rather than indiscriminately. When unsure about their applicability, it is advisable to resort to Taylor's formula instead.

By using Taylor's formula, you can maintain a clear understanding of when these alternative methods can be employed, thereby avoiding potential inaccuracies.

Quadratic Problems

1. Calculate limits $\lim_{x \to 0} \frac{e^x - cosx - x}{x^2}$.

Since the denominator is of the second order, we expand the numerator to the second order as well:

$$\begin{aligned} L = & \lim_{x \to 0} \frac{(1 + x + \frac{1}{2} x^2 + o(x^2)) - (1 - \frac{1}{2} x^2 + o(x^2)) - x}{x^2} \\ = & \lim_{x \to 0} \frac{\frac{1}{2} x^2 + \frac{1}{2} x^2 + o(x^2)}{x^2} \\ = & \, 1 \end{aligned}$$

2. Calculate limits $\lim_{x \to 0} \frac{3sinx + x^2 cos\frac{1}{x}}{(1 + cosx)\ln(1+x)}$.

The numerator has oscillatory behavior, which presumably does not affect the limit. We will first simplify the denominator using the conclusion from before, where we factorize and keep only the lowest-order term:

$$\begin{aligned} L = & \lim_{x \to 0} \frac{3sinx + x^2 cos\frac{1}{x}}{(1 + 1 + o(x))(x + o(x))} \\ = & \lim_{x \to 0} \frac{3sinx + x^2 cos\frac{1}{x}}{2x} \end{aligned}$$

Now, we can split the limit calculation into two parts:

$$L = \lim_{x \to 0} \frac{3sinx}{2x} + \frac{x}{2} cos\frac{1}{x} = \frac{3}{2} + 0 = \frac{3}{2}$$

3. Calculate limits $L = \lim_{x \to 0} \frac{xsinxcosx - x^2}{e^x \ln(1+x) + \ln(1-x)}$

This problem involves multiplying polynomials and can be a bit tedious. If you're unsure how many terms to expand, start with expanding two terms and include the remainder. If the expansion is not accurate enough, continue expanding.

Since the numerator appears more manageable, let's expand the numerator first to determine the order of expansion:

$$\begin{aligned} xsinxcosx - x^2 &= x(x - \frac{1}{3!} x^3 + o(x^3))(1 - \frac{1}{2!} x^2 + o(x^2)) - x^2 \\ &= x^2 - \frac{1}{2} x^4 + o(x^4) - \frac{1}{6} x^4 + o(x^4) - x^2 \\ &= -\frac{2}{3} x^4 + o(x^4) \end{aligned}$$

Note that terms above x^4 in the expansion can be combined as $o(x^4)$ without the need to compute each term individually. This gives us a fourth-order infinitesimal with x in the numerator. Since the coefficient of the fourth-order term is nonzero, this expansion is sufficient. If the coefficient were zero, we would continue expanding further. Next, we need to expand the denominator to the fourth order (if the numerator and denominator have different orders, the limit is either 0 or infinity, which is easy to determine and typically not asked in this form).

For the denominator:

$$e^x \ln(1+x) + \ln(1-x) = (1+x+\frac{1}{2}x^2+o(x^2))(x-\frac{1}{2}x^2+o(x^2)) + (-x-\frac{1}{2}x^2+o(x^2))$$

It is evident that this expansion is not sufficient, as the form of $o(x^2)$ appearing is uncertain and can affect the limits. We further increase the accuracy of the expansion by:

$$\begin{aligned}
& e^x \ln(1+x) + \ln(1-x) \\
&= (1+x+\frac{1}{2}x^2+\frac{1}{6}x^3+o(x^3))(x-\frac{1}{2}x^2+\frac{1}{3}x^3-\frac{1}{4}x^4+o(x^4)) + (-x-\frac{1}{2}x^2-\frac{1}{3}x^3-\frac{1}{4}x^4+o(x^4)) \\
&= x-\frac{1}{2}x^2+\frac{1}{3}x^3-\frac{1}{4}x^4+x^2-\frac{1}{2}x^3+\frac{1}{3}x^4+\frac{1}{2}x^3-\frac{1}{4}x^4+\frac{1}{6}x^4+o(x^4)-x-\frac{1}{2}x^2-\frac{1}{3}x^3-\frac{1}{4}x^4+o(x^4) \\
&= (-\frac{1}{4}+\frac{1}{3}-\frac{1}{4}+\frac{1}{6}-\frac{1}{4})x^4+o(x^4) \\
&= -\frac{1}{4}x^4+o(x^4)
\end{aligned}$$

In summary:

$$L = \lim_{x\to 0} \frac{-\frac{2}{3}x^4+o(x^4)}{-\frac{1}{4}x^4+o(x^4)} = \frac{8}{3}$$

Compound Function Questions

When dealing with Taylor's formula, it is essential to consider the function approaching zero as a whole.

1. Calculate limits $L = \lim_{x\to 0} \frac{(1-\cos x)(x-\ln(1+\tan x))}{\sin^4 x}$.

The factorization of the denominator and numerator in the first step is well handled, while the factorization of the second term in the numerator poses some challenges. Let's consider expanding $tanx$ as a whole.

$$L = \lim_{x\to 0} \frac{\frac{1}{2}x^2(x-(tanx-\frac{1}{2}(tanx)^2+o((tanx)^2)))}{x^4}$$

Since we know that $\tan x = x + o(x)$, we can substitute $(\tan x)^2 = x^2 + o(x^2)$, resulting in:

$$L = \lim_{x\to 0} \frac{x-tanx+\frac{1}{2}x^2+o(x^2)}{2x^2} = \lim_{x\to 0} \frac{x-tanx}{2x^2} + \frac{1}{4}$$

By Section 2, we can determine that $x - \tan x$ is a third-order infinitesimal. Hence:

$$L = 0 + \frac{1}{4} = \frac{1}{4}$$

2. Calculate limits $L = \lim_{x\to 0} \frac{2\ln(cosx)+x^2}{sin^2x-x^2}$.

We can handle the denominator more effectively by the following approach:

$$\begin{aligned}
sin^2x - x^2 &= (sinx+x)(sinx-x) \\
&= 2x(-\frac{1}{3!}x^3+o(x^3)) \\
&= -\frac{1}{3}x^4+o(x^4)
\end{aligned}$$

Now, let's expand the numerator to the fourth order, taking into account the entire expression $(\cos x - 1)$:

$$
\begin{aligned}
2\ln(cosx)+x^2 &= 2\ln(1+cosx-1)+x^2 \\
&= 2((cosx-1)-\frac{1}{2}(cosx-1)^2+o((cosx-1)^2))+x^2 \\
&= 2(-\frac{1}{2}x^2+\frac{1}{4!}x^4+o(x^4)-\frac{1}{2}(-\frac{1}{2}x^2+o(x^2))^2+o(x^4))+x^2 \\
&= 2(-\frac{1}{2}x^2+\frac{1}{24}x^4-\frac{1}{8}x^4+o(x^4))+x^2 \\
&= -\frac{1}{6}x^4+o(x^4)
\end{aligned}
$$

Therefore,

$$
L=\lim_{x\to 0}\frac{-\frac{1}{6}x^4+o(x^4)}{-\frac{1}{3}x^4+o(x^4)}=\frac{1}{2}
$$

Power Finger Function Questions

When dealing with idempotent functions of the form $f(x)^{g(x)}$, it is common to take the logarithm and utilize the approximation $\ln(1+\alpha(x))=\alpha(x)+o(\alpha(x))$.

1. Calculate limits $L=\lim_{x\to 0}(\frac{x}{sinx}^{\frac{1}{x^2}})$.

To simplify the expression, we can take the natural logarithm:

$$
\begin{aligned}
\ln L = & \lim_{x\to 0}\frac{1}{x^2}\ln\frac{x}{sinx} \\
= & \lim_{x\to 0}\frac{1}{x^2}\ln(1+\frac{x}{sinx}-1) \\
= & \lim_{x\to 0}\frac{1}{x^2}(\frac{x}{sinx}-1) \\
= & \lim_{x\to 0}\frac{x-sinx}{x^2 sinx} \\
= & \lim_{x\to 0}\frac{x-(x-\frac{1}{3!}x^3+o(x^3))}{x^3} \\
= & \frac{1}{6}
\end{aligned}
$$

Taking the exponentiation of both sides, we find:

$$
L=e^{\frac{1}{6}}
$$