Power Series

Power series are a fundamental topic in calculus and mathematical analysis. They provide a way to represent functions as infinite series of powers of a variable. Let's dive into the concepts and properties of power series.

Definition of Power Series

A power series is an infinite series of the form:

$$\sum_{n=0}^{\infty} a_n (x-c)^n$$

where a_n and c are constants, and x is a variable. The constants a_n are called the coefficients of the power series.

Convergence of Power Series

The convergence of a power series depends on the value of x and the coefficients a_n. We define the interval of convergence, which is the set of all values of x for which the series converges.

To determine the interval of convergence, we use the **ratio test** or the **root test**. These tests help determine whether the series converges absolutely, diverges, or converges conditionally.

Ratio Test

The ratio test states that if

$$\lim_{n\to\infty} |\frac{a_{n+1}}{a_n}| = L$$

where $L$is a finite number, then the series converges absolutely if $L < 1$ and diverges if $L > 1$.

We still use the ratio test more often, so I will derive the ratio test in detail.

Suppose we have a power series of the form:

$$\sum_{n=0}^{\infty} a_n (x-c)^n$$

We want to determine the convergence or divergence of this series using the ratio test. The ratio test involves taking the limit of the absolute value of the ratio of consecutive terms:

$$L = \lim_{n\to\infty} |\frac{a_{n+1}}{a_n}|$$

Now, we'll consider the following cases:

1. If $L < 1$: The series converges absolutely.
2. If $L > 1$: The series diverges.
3. If $L = 1$: The test is inconclusive.

Let's derive the ratio test to determine the value of L:

First, consider the term $\frac{a_{n+1}}{a_n}$:

$$\frac{a_{n+1}}{a_n} = \frac{a_n(x-c)^{n+1}}{a_n(x-c)^n} = (x-c)$$

Now, take the absolute value of this term:

$$|\frac{a_{n+1}}{a_n}| = |x-c|$$

Since a_n and a_{n+1} are coefficients of the power series, they are constant with respect to n, so they cancel out.

Now, we have:

$$L = \lim_{n\to\infty} |x-c|$$

Let's analyze this limit:

- If $|x-c| < 1$, then $\lim_{n\to\infty} |x-c| = |x-c|$.
- If $|x-c| > 1$, then $\lim_{n\to\infty} |x-c| = |x-c|$.
- If $|x-c| = 1$, the limit does not exist since the terms alternate in sign.

Therefore, we have:

$$L = \begin{cases} |x-c| & \text{if} \quad |x-c| < 1 \\ |x-c| & \text{if} \quad |x-c| > 1 \\ \text{undefined} & \text{if} \quad |x-c| = 1 \end{cases}$$

From this, we can conclude:

Root Test

The root test states that if

$$\lim_{n\to\infty} |a_n|^{\frac{1}{n}} = L$$

where L is a finite number, then the series converges absolutely if $L < 1$ and diverges if $L > 1$.

Interval of Convergence

Once you apply either the ratio test or the root test and find the value of L, you can determine the interval of convergence. The interval will have one of the following forms:

1. The series converges only at $x = c$.
2. The series converges for all x.
3. The series converges for $c - R < x < c + R$, where R is a positive number called the radius of convergence.

Radius of Convergence

The radius of convergence, denoted by R, determines how far from the center c the power series converges. It can be found using the following formulas:

Ratio Test Formula

If the ratio test is used to find the value of L for the series, then the radius of convergence is given by:

$$R = \frac{1}{L}$$

Consider the power series:

$$a_0 + a_1 x + a_2 x^2 + \cdots$$

The theorem states the following:

1. If $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = L$ where $0 < L < \infty$, then the series has a convergence radius (R) given by $R = \frac{1}{L}$. In other words, the series converges absolutely for all (x) values within the interval $-R < x < R$.
2. If $L = 0$, then the series has an infinite convergence radius, $R = \infty$. This means that the series converges absolutely for all x values.
3. If $L = \infty$, then the series has a convergence radius of $R = 0$. In this case, the series only converges at $x = c$, where c is the center of the power series.

To summarize:

- When $0 < L < \infty$, the series converges absolutely within the interval $-R < x < R$ where $R = \frac{1}{L}$.
- When $L = 0$, the series converges absolutely for all x values.
- When $L = \infty$, the series only converges at $x = c$ (the center of the power series), and the convergence radius is $R = 0$.

Properties of Power Series

Power series possess several important properties that make them useful for approximating functions.

Sum of a Power Series

The sum of a power series is denoted by $S(x)$ and is defined as:

$$S(x) = \sum_{n=0}^{\infty} a_n (x-c)^n$$

where $S(x)$ represents the function represented by the power series.

Differentiation and Integration

Within the interval of convergence, a power series can be differentiated and integrated term by term. This means that if $S(x)$ represents a function $f(x)$, then its derivative $f'(x)$ and integral $\int f(x)$ can be found by differentiating and integrating each term of the power series.

And after integrating and differentiating term by term, the radius of convergence is the same. However, it should be noted that whether the endpoints converge or not needs to be re-determined.

Uniqueness

A power series represents a unique function within its interval of convergence. That is, if two power series have the same coefficients and the same center, they represent the same function.

Operations on Power Series

We can perform arithmetic operations on power series, such as addition, subtraction, multiplication, and division, within their respective intervals of convergence. These operations are carried out by manipulating the coefficients of the power series.