9.6 Taylor Series

The Taylor Series is a mathematical tool that allows us to represent a wide range of functions as an infinite series of terms. It provides a way to approximate complex functions using simpler polynomial expressions. The series is named after the English mathematician Brook Taylor, who introduced the concept in the 18th century.

Taylor Polynomial

At the heart of the Taylor Series lies the Taylor polynomial, which approximates a function around a specific point (often denoted as *c*) by using a polynomial of increasing degree. The Taylor polynomial of degree *n* for a function *f(x)* is given by the formula:

$$P_n(x) = f(c) + f'(c)(x-c) + \frac{f''(c)(x-c)^2}{2!} + \frac{f'''(c)(x-c)^3}{3!} + \ldots + \frac{f^n(c)(x-c)^n}{n!}$$

In this formula, *f'(x)* represents the first derivative of *f(x)*, *f''(x)* represents the second derivative, and so on, with $f^{(n)}(x)$ representing the *n*-th derivative of *f(x)*.

Taylor Series Expansion

The Taylor Series expands the Taylor polynomial to an infinite series. It provides a way to represent a function *f(x)* as the sum of an infinite number of terms. The general form of the Taylor Series expansion for a function *f(x)* centered around a point *c* is:

$$f(x) = f(c) + f'(c)(x-c) + \frac{f''(c)(x-c)^2}{2!} + \frac{f'''(c)(x-c)^3}{3!} + \ldots$$

This series continues indefinitely, with each subsequent term involving higher derivatives of the function evaluated at the center *c*.

Taylor Series: Sufficient and Necessary Conditions

The Taylor series has both sufficient and necessary conditions for its validity. Let's discuss them in detail:

Sufficient Condition: Infinitely Differentiable Function

The **sufficient condition** for the Taylor series expansion is that the function must be **infinitely differentiable** within a specific interval. In other words, all the derivatives of the function exist and are continuous in that interval.

If a function satisfies this condition, we can construct its Taylor series expansion by evaluating the derivatives of the function at a chosen center.

Necessary Condition: Convergence of Remainder Term

The **necessary condition** for the Taylor series expansion is that the **remainder term must converge to zero**. This means that as we include more terms in the Taylor series, the difference between the original function and the truncated series becomes arbitrarily small.

Mathematically, the necessary condition can be stated as:

$$\lim_{n\to\infty} R_n(x) = 0$$

where *R_n(x)* represents the remainder term of the Taylor series truncated at the *n*-th term.

The convergence of the remainder term ensures that the Taylor series provides a valid approximation of the original function within the specified interval.

Properties of Taylor Series

Here are some important properties of Taylor Series:

1. **Convergence**: The Taylor Series expansion converges to the original function *f(x)* within a certain interval around the center *c*. The size of this interval depends on the behavior of *f(x)* and the point of expansion.
2. **Approximation**: By truncating the Taylor Series at a specific term, we can obtain an approximation of the function *f(x)*. The accuracy of the approximation increases as we include more terms in the series.
3. **Remainder Term**: The remainder term represents the difference between the original function *f(x)* and the truncated Taylor Series approximation. It provides a measure of the error in the approximation and can be expressed using the Lagrange form of the remainder, Cauchy form of the remainder, or other forms depending on the situation.
4. **Uniform Convergence**: In some cases, the Taylor Series expansion may converge uniformly, meaning that the approximation is accurate across the entire interval of interest, rather than just at a specific point.
5. **Center Selection**: The choice of the center *c* affects the behavior and accuracy of the Taylor Series. Selecting a suitable center can simplify calculations and improve the approximation in specific regions.

Example

Let's take an example to illustrate the concept. Consider the function *f(x) = sin(x)*. We can find the Taylor Series expansion for *f(x)* centered at *c = 0*.

The derivatives of *f(x) = sin(x)* are:

- *f'(x) = cos(x)*
- *f''(x) = -sin(x)*
- *f'''(x) = -cos(x)*
- *f''''(x) = sin(x)*

Evaluating these derivatives at *x = 0* gives us:

- *f(0) = 0*
- *f'(0) = 1*
- *f''(0) = 0*
- *f'''(0) = -1*
- *f''''(0) = 0*

Substituting these values into the Taylor polynomial formula, we have:

$$P_n(x) = 0 + 1(x-0) + \frac{0(x-0)^2}{2!} + \frac{-1(x-0)^3}{3!} + \dots$$

Simplifying the terms, we get:

$$P_n(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots$$

This is the Taylor Series expansion of *f(x) = sin(x)* centered at *c = 0*.

Important Taylor expansions

1. **Exponential Function Expansion**:

The Taylor expansion for the exponential function is:

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

This expansion is particularly useful in various mathematical and scientific contexts, especially when dealing with exponential growth or decay phenomena.

2. **Trigonometric Function Expansion**:

The Taylor expansions for common trigonometric functions are:

- Sine function:

$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$$

- Cosine function:

$$\cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$$

- Arctangent Expansion:

$$\arctan(x) = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \dots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1}$$

- Tangent Expansion:

$$\tan(x) = x + \frac{x^3}{3} + \frac{2x^5}{15} + \frac{17x^7}{315} + \dots = \sum_{n=1}^{\infty} \frac{B_{2n}(-4)^n(1-4^n)x^{2n-1}}{(2n)!}$$

3. **Natural Logarithm Expansion**:

The Taylor expansion for the natural logarithm function is:

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} x^n}{n}$$

This expansion is particularly useful in approximating the logarithm of values close to 1.

4. **Geometric Series**:

The Taylor expansion for the geometric series is:

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \dots = \sum_{n=0}^{\infty} x^n, \quad |x| < 1$$

This expansion is widely used in mathematics and physics, especially in the study of sequences and series.

5. **Binomial Expansion**:

The binomial expansion, based on the binomial theorem, is used to expand expressions of the form $(a+b)^n$:

$$(a+b)^n = \binom{n}{0}a^n b^0 + \binom{n}{1}a^{n-1}b^1 + \binom{n}{2}a^{n-2}b^2 + \ldots + \binom{n}{n-1}a^1 b^{n-1} + \binom{n}{n}a^0 b^n$$

This expansion is applicable in combinatorics, probability theory, and algebraic manipulations.

6. **Error Function Expansion**:

The Taylor expansion for the error function is:

$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}}\left(x - \frac{x^3}{3} + \frac{x^5}{10} - \frac{x^7}{42} + \ldots\right) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{n!(2n+1)}$$

The error function is widely used in statistics and mathematical analysis, particularly in the study of normal distributions and probability.

The last two are not required, understanding is sufficient