9.4 Arbitrary Series

Alternating series

An alternating series is a type of infinite series where the terms alternate in sign, meaning that each term is positive or negative. Mathematically, an alternating series can be represented as:

$$\sum_{n=1}^{\infty}(-1)^{n+1}a_n$$

where a_n is a sequence of positive numbers. The first term of the series is $(-1)^2a_1 = a_1$, which is positive. The second term is $(-1)^3a_2 = -a_2$, which is negative. The third term is $(-1)^4a_3 = a_3$, which is positive, and so on.

The key property of alternating series is that they converge if the sequence a_n is decreasing and the limit of a_n as n approaches infinity is 0, that is, $\lim_{n\to\infty} a_n = 0$. This property is known as the Alternating Series Test.

The Alternating Series Test states that if a series $\sum_{n=1}^{\infty}(-1)^{n+1}a_n$ satisfies the conditions:

1. $a_n \geq 0$ for all n,
2. $a_{n+1} \leq a_n$ for all n, and
3. $\lim_{n\to\infty} a_n = 0$,

then the series converges.

In addition to the Alternating Series Test, there are some other important results related to alternating series. For example, the error of an alternating series approximation can be bounded by the size of the first neglected term. This result is known as the Alternating Series Estimation Theorem.

The Alternating Series Estimation Theorem states that if a series $\sum_{n=1}^{\infty}(-1)^{n+1}a_n$ satisfies the conditions of the Alternating Series Test, and S is the sum of the series, then the error $R_n = S - \sum_{k=1}^{n}(-1)^{k+1}a_k$ in approximating S by the first n terms of the series is bounded by the size of the first neglected term, that is, $|R_n| \leq a_{n+1}$.

Another important result related to alternating series is the Leibniz criterion, which provides a sufficient condition for convergence of an alternating series. The Leibniz criterion states that if a series $\sum_{n=1}^{\infty}(-1)^{n+1}a_n$ satisfies the conditions:

1. $a_n \geq 0$ for all n,
2. $a_{n+1} \leq a_n$ for all n, and
3. $\lim_{n\to\infty} a_n = 0$,

then the series converges.

Absolute Series

An absolute series is a type of infinite series where the terms are all positive, meaning that each term is greater than or equal to zero. Mathematically, an absolute series can be represented as:

$$\sum_{n=1}^{\infty} |a_n|$$

where a_n is a sequence of real numbers. The absolute value of each term $|a_n|$ ensures that the terms are all positive, regardless of the sign of a_n.

The behavior of absolute series is important in the study of infinite series because if an absolute series converges, then the original series, which may or may not alternate in sign, also converges. This is known as the Absolute Convergence Test.

The Absolute Convergence Test states that if a series $\sum_{n=1}^{\infty} |a_n|$ converges, then the series $\sum_{n=1}^{\infty} a_n$ also converges. The converse, however, is not necessarily true: if the series $\sum_{n=1}^{\infty} a_n$ converges but the series $\sum_{n=1}^{\infty} |a_n|$ diverges, then the series $\sum_{n=1}^{\infty} a_n$ is said to converge conditionally.

Theorem A

If the series $\sum_{n=1}^{\infty} |a_n|$ converges, then the series $\sum_{n=1}^{\infty} a_n$ also converges. Conversely, if the series $\sum_{n=1}^{\infty} a_n$ diverges, then the series $\sum_{n=1}^{\infty} |a_n|$ also diverges.

Explanation:

The theorem states that if the series of absolute values $\sum_{n=1}^{\infty} |a_n|$ converges, then the original series $\sum_{n=1}^{\infty} a_n$ also converges. Conversely, if the original series $\sum_{n=1}^{\infty} a_n$ diverges, then the series of absolute values $\sum_{n=1}^{\infty} |a_n|$ also diverges.

Let's understand why this theorem holds.

Proof (Forward Direction):

Suppose $\sum_{n=1}^{\infty} |a_n|$ converges. This means that the series of positive terms $\sum_{n=1}^{\infty} a'_n$, where $a'_n = |a_n|$, converges.

Since $a'_n \geq 0$ for all n, the convergence of $\sum_{n=1}^{\infty} a'_n$ implies that the series $\sum_{n=1}^{\infty} a_n$ also converges. The reason is that the terms of $\sum_{n=1}^{\infty} a'_n$ and $\sum_{n=1}^{\infty} a_n$ have the same magnitudes but potentially opposite signs.

Therefore, if the series of absolute values converges, the original series must also converge.

Proof (Converse Direction):

Now, let's prove the converse part of the theorem.

Suppose $\sum_{n=1}^{\infty} a_n$ diverges. This means that the series does not have a finite sum. Since the terms a_n can have both positive and negative values, their absolute values $|a_n|$ can be considered as a series with nonnegative terms.

If $\sum_{n=1}^{\infty} |a_n|$ were to converge, it would mean that the series of nonnegative terms has a finite sum. However, since the original series $\sum_{n=1}^{\infty} a_n$ diverges, the series of absolute values $\sum_{n=1}^{\infty} |a_n|$ must also diverge.

Hence, if the original series diverges, the series of absolute values must also diverge.

Therefore, the theorem holds in both directions.

This theorem is useful in analyzing the convergence behavior of series. If we can show that the series of absolute values converges, we can conclude that the original series converges as well. On the other hand, if the original series diverges, we can conclude that the series of absolute values also diverges.

Knowledge Point

If the series $\sum_{n=1}^{\infty} |u_n|$ converges, then the series $\sum_{n=1}^{\infty} u_n$ is said to be **absolutely convergent**. Conversely, if the series $\sum_{n=1}^{\infty} |u_n|$ diverges but the series $\sum_{n=1}^{\infty} u_n$ converges, then it is referred to as **conditionally convergent**.

Explanation:

1. **Convergence of Absolute Series**: If the series of absolute values $\sum_{n=1}^{\infty} |u_n|$ converges, it means that the series formed by the magnitudes of the terms converges. In other words, the series $\sum_{n=1}^{\infty} u'_n$, where $u'_n = |u_n|$, is convergent. As a result, the original series $\sum_{n=1}^{\infty} u_n$ is said to be absolutely convergent. The absolute convergence of a series guarantees that the series converges regardless of the signs of its terms.
2. **Conditional Convergence**: On the other hand, if the series of absolute values $\sum_{n=1}^{\infty} |u_n|$ diverges, but the series $\sum_{n=1}^{\infty} u_n$ converges, then the original series is known as conditionally convergent. This means that the convergence of the series is dependent on the alternating signs of its terms. The series may converge as a result of the cancellation between positive and negative terms.

These concepts can be better understood by considering some examples. For instance:

- The series $\sum_{n=1}^{\infty}(-1)^{n+1}/n$ is an example of a conditionally convergent series. The series of absolute values $\sum_{n=1}^{\infty} 1/n$ is known as the harmonic series, which diverges. However, the original series $\sum_{n=1}^{\infty}(-1)^{n+1}/n$ converges to the natural logarithm of 2.
- Conversely, the series $\sum_{n=1}^{\infty}(-1)^{n+1}/n^2$ is an example of an absolutely convergent series. Both the series of absolute values $\sum_{n=1}^{\infty} 1/n^2$ and the original series $\sum_{n=1}^{\infty}(-1)^{n+1}/n^2$ converge.

Understanding the distinction between absolute convergence and conditional convergence is crucial when studying the convergence behavior of series and their applications in various mathematical contexts.

Remember:

- If the series of absolute values converges, the original series is absolutely convergent.
- If the series of absolute values diverges but the original series converges, the original series is conditionally convergent.

These concepts play a significant role in series analysis and have implications in various branches of mathematics and beyond.

Theorem B

For an infinite series $\sum_{n=1}^{\infty} u_n$, if the limit $\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = L$, where L is a nonnegative real number, then:

- If $L < 1$, the series $\sum_{n=1}^{\infty} u_n$ is absolutely convergent.
- If $L > 1$ or $L = \infty$, the series $\sum_{n=1}^{\infty} u_n$ is divergent.
- If $L = 1$, the test is inconclusive, and the series may be convergent or divergent.

Explanation:

The theorem, known as the Ratio Test, provides information about the convergence or divergence of an infinite series based on the limit of the ratio of consecutive terms $\left|\frac{u_{n+1}}{u_n}\right|$.

Let's examine the different cases described by the theorem:

1. $L < 1$**: Absolute Convergence** If the limit of the ratio is less than 1 ($L < 1$), it implies that the terms of the series eventually become smaller in magnitude as n increases. In this case, the series $\sum_{n=1}^{\infty} u_n$ is said to be absolutely convergent. The convergence is guaranteed regardless of the signs of the terms. Moreover, the smaller the value of L, the faster the series converges.
2. $L > 1$ **or** $L = \infty$**: Divergence** If the limit of the ratio is greater than 1 ($L > 1$) or infinite ($L = \infty$), it indicates that the terms of the series eventually increase in magnitude as n increases. Consequently, the series $\sum_{n=1}^{\infty} u_n$ is divergent. The divergence can be understood as the series growing without bounds, rendering its sum undefined.
3. $L = 1$**: Inconclusive** If the limit of the ratio is exactly 1 ($L = 1$), the test does not provide a definitive conclusion about the convergence or divergence of the series. The behavior of the series could be either convergent or divergent, and further analysis or alternative tests are necessary to determine its convergence.

Theorem C

Rearrangement Theorem

The Rearrangement Theorem is a result in the theory of infinite series that states: If an infinite series $\sum_{n=1}^{\infty} a_n$ is conditionally convergent, meaning it converges but not absolutely, then for any real number S, it is possible to rearrange the terms of the series such that the new rearranged series converges to S.

Explanation:

The Rearrangement Theorem deals with the behavior of conditionally convergent series, which are series that converge but not absolutely. It asserts that by rearranging the terms of a conditionally convergent series, we can achieve a new series that converges to any desired real number S.

To understand the theorem, let's consider an example of a conditionally convergent series:

$$\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \cdots$$

This series, known as the alternating harmonic series, is conditionally convergent. If we sum the terms in their original order, the series converges to the natural logarithm of 2. However, the Rearrangement Theorem allows us to rearrange the terms of this series in any manner we choose, creating a new series that converges to any desired real number.

For instance, if we rearrange the terms by first taking two positive terms followed by one negative term, we get:

$$(1 - \frac{1}{2}) + (\frac{1}{3} - \frac{1}{4}) + (\frac{1}{5} - \frac{1}{6}) + \cdots$$

This rearranged series converges to a different value, specifically, $\ln(2)/2$. By employing various rearrangement patterns, we can obtain different convergent sums.

It is important to note that the Rearrangement Theorem applies to conditionally convergent series but not absolutely convergent series. For absolutely convergent series, the order of the terms does not affect the sum.

The Rearrangement Theorem showcases the intriguing behavior of conditionally convergent series. It reveals that rearranging the terms can lead to different convergence behavior and different sums. This result demonstrates the delicate nature of conditionally convergent series and highlights the importance of considering the order in which the terms are summed.