9.2 Properties of infinite series

Welcome to the fascinating world of infinite series! In this chapter, we will explore the properties of infinite series, including convergence, divergence, and the algebraic operations we can perform on series. Understanding these properties is crucial in many areas of mathematics and science, and it allows us to make accurate predictions, solve important problems, and develop new computational methods. So, let's dive in and discover the amazing properties of infinite series together!

A few important infinite series

In addition to exploring the properties of infinite series, we will also introduce and study some of the most important series in mathematics. These include the geometric series, which has many important applications in calculus, and the harmonic series, which is a classic example of a divergent series. We will also study other important series, such as the alternating series and the Taylor series, which have wide-ranging applications in many areas of mathematics and science. So, get ready to explore not only the properties of infinite series, but also some of the most important series in mathematics!

Geometric Series

A *geometric series* is a special type of infinite series where each term is a constant multiple of the previous term. The general form of a geometric series is

$$\sum_{n=0}^{\infty} ar^n = a + ar + ar^2 + ar^3 + \cdots$$

where a is the first term and r is the common ratio. Geometric series are important in mathematics and have many applications in science and engineering.

Derivation of the General Formula

To derive a formula for the sum of a geometric series, we start by considering the partial sums of the series. Let S_n be the sum of the first n terms of the series, so

$$S_n = a + ar + ar^2 + a^3 + \cdots + ar^{n-1}$$

We can multiply both sides of this equation by r to get

$$rS_n = ar + ar^2 + a^3 + \cdots + ar^n$$

Subtracting the second equation from the first, we get

$$S_n - rS_n = a - ar^n$$

which simplifies to

$$(1-r)S_n = a(1-r^n)$$

If $r \neq 1$, we can divide both sides by $(1-r)$ to get

$$S_n = \frac{a(1-r^n)}{1-r}$$

This formula gives the sum of the first n terms of the geometric series.

To find the sum of the infinite geometric series, we take the limit as n goes to infinity:

$$\sum_{n=0}^{\infty} ar^n = \lim_{n\to\infty} S_n = \lim_{n\to\infty} \frac{a(1-r^n)}{1-r}$$

If $|r| < 1$, then r^n goes to zero as n goes to infinity, so the limit simplifies to

$$\sum_{n=0}^{\infty} ar^n = \frac{a}{1-r}$$

This is the formula for the sum of an infinite geometric series when $|r| < 1$.

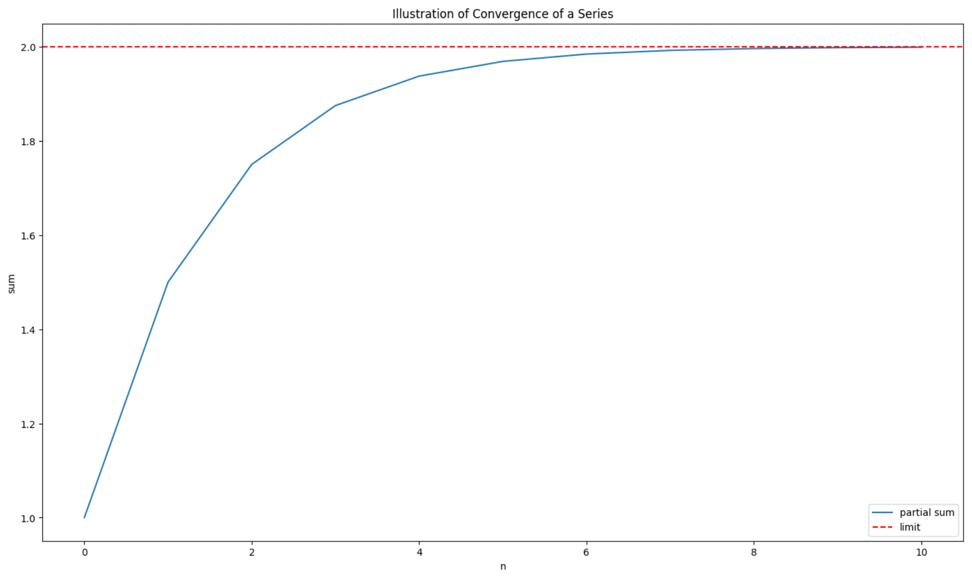
Convergence of the Geometric Series

The convergence of a geometric series depends on the value of the common ratio r. We can classify geometric series into three types:

- If $|r| < 1$, then the series converges absolutely. This means that the series converges and the sum is finite.
- If $|r| = 1$, then the series may converge or diverge. In this case, we need to look at the behavior of the terms in the series to determine convergence or divergence.
- If $|r| > 1$, then the series diverges. This means that the sum of the series is infinite.

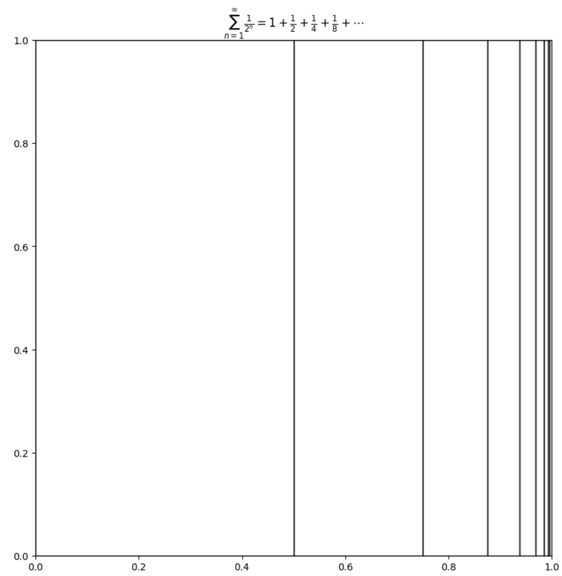
For example, consider the series

$$\sum_{n=0}^{\infty} \frac{1}{2^n} = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots$$



This is a geometric series with $a = 1$ and $r = \frac{1}{2}$. Since $|r| < 1$, the series converges absolutely, and the sum is

$$\sum_{n=0}^{\infty} \frac{1}{2^n} = \frac{1}{1-\frac{1}{2}} = 2$$



> This image shows us vividly why the sum of this geometric series is 2. Of course, for convenience we let the first term of the series start from n=1. The graph is divided into halves each time and one-half of the remaining parts are taken, that is, one-fourth, and so on, and we find that the total area of the graph is still 1 until the infinite term.

Applications of Geometric Series

Geometric series have many applications in mathematics, science, and engineering. For example, they can be used to model exponential growth or decay, such as in the growth of populations or the decay of radioactive substances. They can also be used to evaluate certain types of integrals and to approximate functions.

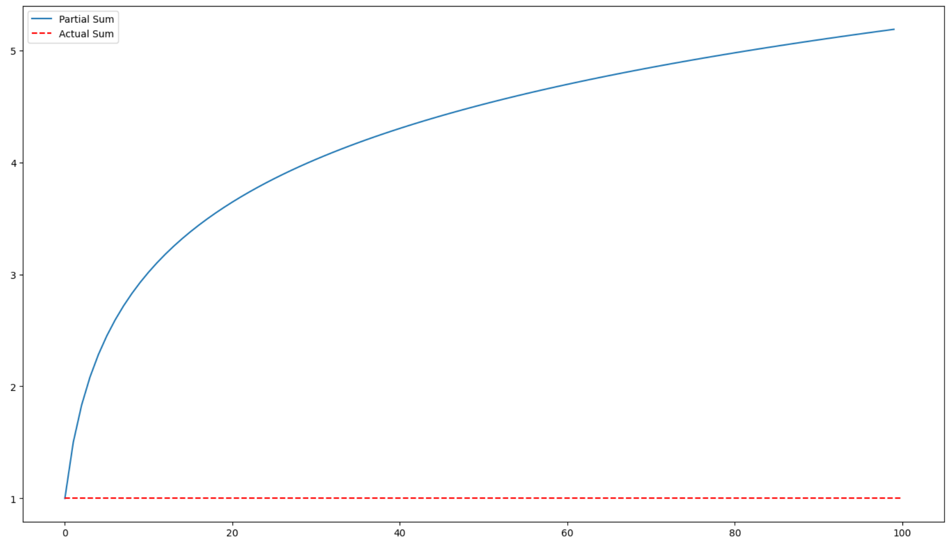
Conclusion

Geometric series are a fundamental concept in calculus and have many applications in various fields. Understanding the convergence and divergence of a geometric series is crucial for using the closed-form solution to find the sum of the series.

Harmonic Series

The harmonic series is a well-known series in mathematics that arises naturally in many different contexts, including calculus, number theory, and physics. In particular, it is a series of the form:

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots$$



The harmonic series is interesting because it is a divergent series, meaning that it does not have a finite sum. This can be seen by examining the partial sums of the series, which grow without bound as more and more terms are added.

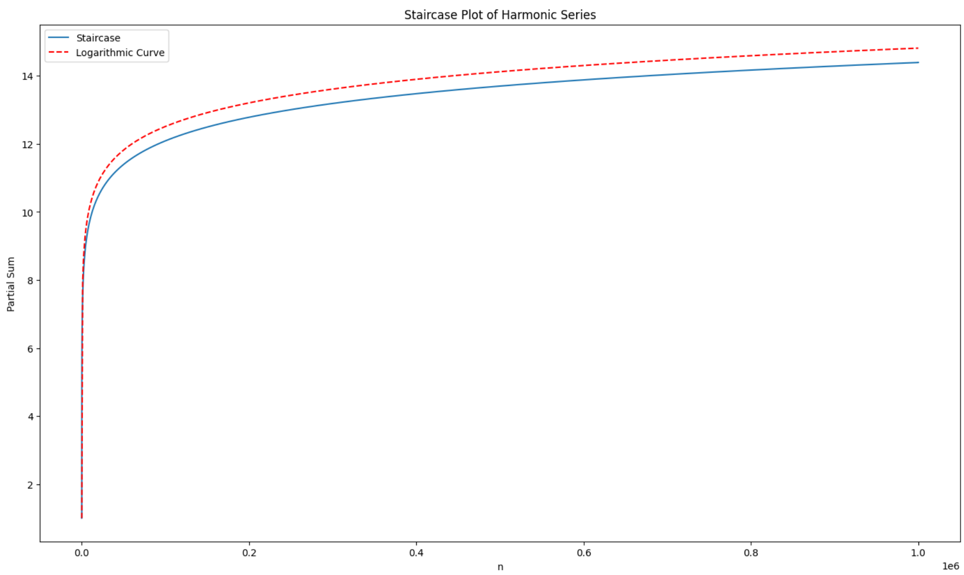
Proof of convergence

To prove that the harmonic series is divergent, we can use the integral test. The integral test states that if the function $f(x)$ is positive, decreasing, and continuous for all $x \geq 1$, and if $a_n = f(n)$ for all n, then the series $\sum_{n=1}^{\infty} a_n$ and the improper integral $\int_1^{\infty} f(x)dx$ either both converge or both diverge.

In the case of the harmonic series, we can choose $f(x) = \frac{1}{x}$, which satisfies the conditions of the integral test. The improper integral $\int_1^{\infty} \frac{1}{x}dx$ can be evaluated as:

$$\int_1^{\infty} \frac{1}{x}dx = \lim_{t \to \infty} ln(t) - ln(1) = \infty$$

Since the integral diverges, the harmonic series must also diverge.



It can be seen that the area of each rectangle is $\frac{1}{n}$, so the total area of the first n rectangles is $\sum_{i=1}^{n} \frac{1}{i}$. As n increases, the area of the rectangles gradually approaches $\ln n$, and thus the step-like graph gradually approaches the curve with slope $y = \ln x$. Eventually, when n tends to infinity, the area of the rectangle tends to ∞ and the stepped graph tends to the curve $y = \ln x$.

This visualization can help us better understand the behavior of the series and how it relates to the natural logarithm.

Of course, if you can't read it, it's okay, you just have to remember that the harmonic series is divergent.

Applications of Harmonic Series

The divergence of the harmonic series has many important applications in mathematics and science. For example:

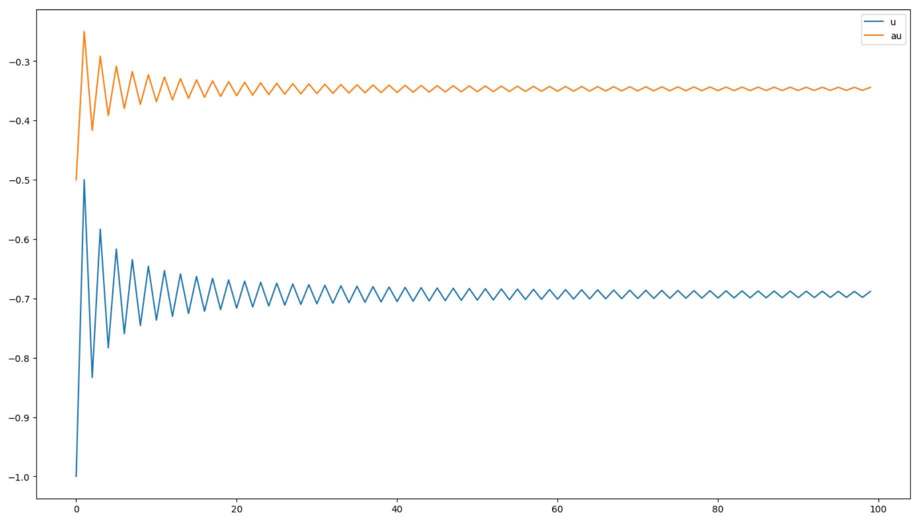
- The harmonic series is used in number theory to study the distribution of prime numbers. The divergence of the series implies that there are infinitely many primes, which is a fundamental result in number theory.
- The divergence of the harmonic series also has important implications in physics, particularly in the study of electric fields. The potential energy of a point charge in an electric field is proportional to the sum of the inverses of the distances between the charge and all other charges in the field. This sum is equivalent to the harmonic series, and its divergence implies that the potential energy of a point charge is infinite.

Conclusion

In conclusion, the harmonic series is an important and interesting series in mathematics and science. It is a divergent series, meaning that it does not have a finite sum. We can prove the divergence of the series using the integral test, and the divergence has important applications in number theory and physics.

Some properties of infinite series

1. **If a given infinite series $\sum_{n=1}^{\infty} u_n$ converges, then any series of the form $\sum_{n=1}^{\infty} au_n$, where a is a constant, will also converge. This property can be derived using the definition of the limit of a sequence.**



> We can see that $\sum_{n=1}^{\infty}(-1)^n/n$ converges and $\sum_{n=1}^{\infty} 0.5[(-1)^n/n]$ also converges and their convergence is the same as that of $\sum_{n=1}^{\infty}(-1)^n/n$.
>
> Therefore, this theorem can be used to prove that $\sum_{n=1}^{\infty} 0.5[(-1)^n/n]$ converges because it is obtained by multiplying $\sum_{n=1}^{\infty}(-1)^n/n$ by a constant 0.5.

- To prove the linearity property, let s_n be the sequence of partial sums of the series $\sum_{n=1}^{\infty} u_n$, and let t_n be the sequence of partial sums of the series $\sum_{n=1}^{\infty} au_n$. Then, we have:

$$\sum_{n=1}^{\infty} S_n = u_1 + u_2 + u_3 + \cdots + u_n + \cdots$$

and

$$\sum_{n=1}^{\infty} t_n = au_1 + au_2 + au_3 + \cdots + au_n + \cdots$$

Multiplying the first equation by a, we obtain:

$$aS_n = au_1 + au_2 + au_3 + \cdots + au_n + \cdots$$

Now, let L be the limit of the sequence s_n, i.e., $\lim_{n\to\infty} s_n = L$. Since the series $\sum_{n=1}^{\infty} u_n$ converges, we know that the limit L is finite. Therefore, by the algebraic property of limits, we have:

$$\lim_{n\to\infty} aS_n = a \lim_{n\to\infty} S_n = aL$$

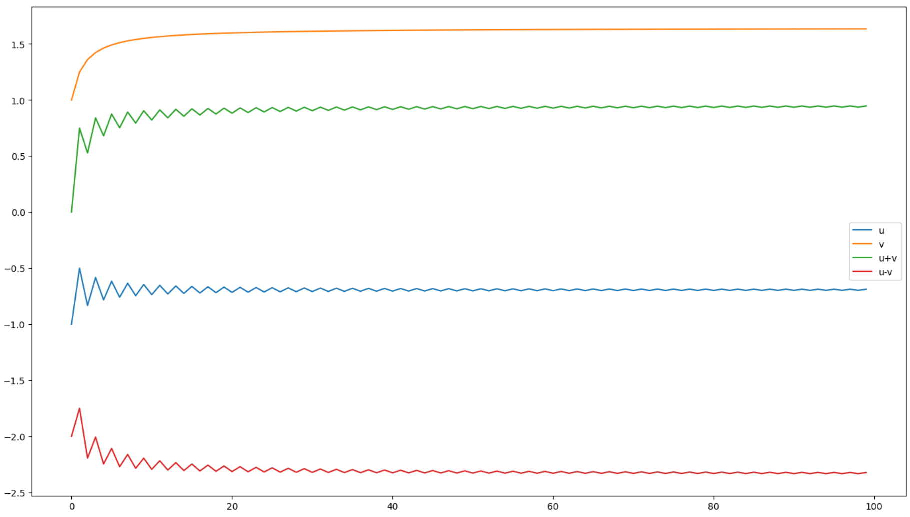
Similarly, the limit of the sequence t_n is given by:

$$\lim_{n\to\infty} t_n = \lim_{n\to\infty} (au_1 + au_2 + \cdots + au_n + \cdots) = a(u_1 + u_2 + u_3 + \cdots + u_n + \cdots) = aL$$

Therefore, the series $\sum_{n=1}^{\infty} au_n$ also converges, and its sum is given by aL.

In conclusion, the linearity property of infinite series states that if a given infinite series converges, then any series obtained by multiplying the terms of the original series by a constant will also converge, and its sum will be the product of the constant and the sum of the original series. This property can be derived using the definition of the limit of a sequence and the algebraic property of limits.

2. **If $\sum_{n=1}^{\infty} u_n$ and $\sum_{n=1}^{\infty} v_n$ converge, then $\sum_{n=1}^{\infty}(u_n \pm v_n)$ also converges.**



> We can see that both $\sum_{n=1}^{\infty}(-1)^n/n$ and $\sum_{n=1}^{\infty} 1/n^2$ converge, while $\sum_{n=1}^{\infty}[(-1)^n/n \pm 1/n^2]$ also converges, and their convergence is the same as that of $\sum_{n=1} infty(-1)^n/n$ and $\sum_{n=1}^{\infty} 1/n^2$ converge in the same way.
>
> Therefore, this theorem can be used to prove that $\sum_{n=1}^{\infty}[(-1)^n/n + 1/n^2]$ converges because it is the sum of $\sum_{n=1}^{\infty}(-1)^n/n$ and $\sum_{n=1}^{\infty} 1/n^2$.

- To prove this result, we can use the following steps:
  1. Let $s_n = u_n + v_n$. Then $\sum_{n=1}^{\infty} s_n$ is a series formed by adding corresponding terms of $\sum_{n=1}^{\infty} u_n$ and $\sum_{n=1}^{\infty} v_n$.
  2. Since both $\sum_{n=1}^{\infty} u_n$ and $\sum_{n=1}^{\infty} v_n$ converge, their sequences of partial sums (S_n) and (T_n) converge as well. That is, $\lim_{n\to\infty} S_n = S$ and $\lim_{n\to\infty} T_n = T$.
  3. We want to show that $\sum_{n=1}^{\infty} s_n = \sum_{n=1}^{\infty}(u_n + v_n)$ converges. To do so, we need to show that the sequence of partial sums (U_n) of $\sum_{n=1}^{\infty} s_n$ converges. That is, we need to show that $\lim_{n\to\infty} U_n$ exists.
  4. We can express U_n in terms of S_n and T_n as follows:

  $$U_n = S_n + T_n = (\sum_{i=1}^{n} u_i) + (\sum_{i=1}^{n} v_i)$$

  5. By the properties of limits, we have:

  $$\lim_{n\to\infty} U_n = \lim_{n\to\infty}(S_n + T_n) = \lim_{n\to\infty} S_n + \lim_{n\to\infty} T_n = S + T$$

  6. Therefore, we have shown that the sequence of partial sums (U_n) of $\sum_{n=1}^{\infty} s_n$ converges, and hence the series $\sum_{n=1}^{\infty} s_n$ converges.
  7. Since s_n was defined as $u_n + v_n$, the result holds for both the $+$ and $-$ cases. That is, if $\sum_{n=1}^{\infty} u_n$ and $\sum_{n=1}^{\infty} v_n$ converge, then $\sum_{n=1}^{\infty}(u_n + v_n)$ and $\sum_{n=1}^{\infty}(u_n - v_n)$ also converge.

3. **Convergent infinite series that still converge after arbitrary addition of parentheses.**

   If you have a Python environment, I strongly suggest you run the python file I provided for "Verifying_Property_3" and you can see the difference in the images after the different brackets.

- To see why this is true, consider an infinite series $\sum_{n=1}^{\infty} a_n$ that converges to a limit L. That is, the sequence of partial sums (S_n), where $S_n = \sum_{i=1}^{n} a_i$, converges to L as n approaches infinity.

  Now, suppose we add parentheses to the terms of the series in any way. That is, we group some terms together with parentheses, but the order of the terms is not changed. For example, we might write:

  $$a_1 + (a_2 + a_3) + (a_4 + a_5 + a_6) + (a_7 + a_8 + a_9 + a_{10}) + \cdots$$

  or

  $$(a_1 + a_2) + (a_3 + a_4 + a_5) + (a_6 + a_7 + a_8 + a_9) + \cdots$$

  or any other grouping of terms.

  Let's denote the new sequence of partial sums by (T_n), where T_n is the sum of the terms up to the n-th group. For example, in the first grouping above, we have $T_1 = a_1$, $T_2 = a_1 + a_2 + a_3$, $T_3 = a_1 + a_2 + a_3 + a_4 + a_5 + a_6$, and so on.

  Now, consider any two adjacent groups of terms in the series. Let's call the terms in the first group $a_{k_1}, a_{k_1+1}, \ldots, a_{k_2-1}$ and the terms in the second group $a_{k_2}, a_{k_2+1}, \ldots, a_{k_3-1}$. Then the sum of these two groups is:

  $$(a_{k_1} + a_{k_1+1} + a_{k_1+2} + \cdots + a_{k_2-1}) + (a_{k_2} + a_{k_2+1} + a_{k_2+2} + \cdots + a_{k_3-1})$$

  By the associative property of addition, we can rearrange this sum as:

  $$a_{k_1} + a_{k_1+1} + a_{k_1+2} + \cdots + a_{k_3-1}$$

  That is, we can group these terms together in any way we like, and the resulting sum will be the same. Therefore, the sequence of partial sums (T_n) for the new grouping of terms is the same as the original sequence of partial sums (S_n). In other words, adding parentheses to the terms of a convergent series does not change the limit of the series.

  Therefore, we have shown that a convergent infinite series is still convergent after adding any parentheses to the terms of the series.

4. **By removing, adding or changing the finite term, the convergence of the series does not change, but the sum may change.**

   - To formally prove this property, let's consider an infinite series $\sum_{n=1}^{\infty} a_n$, where a_n is the nth term of the series. We want to show that if we modify this series by adding, removing, or changing any finite number of terms, the convergence or divergence of the series will not change.

     First, let's consider the case where we add or remove finitely many terms. Let S be the original sum of the series $\sum_{n=1}^{\infty} a_n$, and let S' be the new sum of the modified series $\sum_{n=1}^{\infty} a'_n$. We can express S' as the sum of two series: the first is the original series with a finite number of terms removed or added, and the second is a finite series consisting of the terms that were removed or added. Formally, we can write:

     $$S' = (\sum_{n=1}^{k} a_n + \sum_{n=k+m+1}^{\infty} a_n) + \sum_{n=k+1}^{k+m} a_n$$

     where k is a positive integer, m is a non-negative integer, and a'_n is the nth term of the modified series.

     Since the original series converges to S, we have:

$$\lim_{n\to\infty}(\sum_{i=1}^{n} a_i) = S$$

Now, let's consider the modified series. The first part of the modified series converges to the same limit as the original series since it differs only by a finite number of terms. The second part is a finite series, and therefore, it converges to a finite sum. Thus, the modified series also converges to a sum S'.

Therefore, we have shown that if we modify an infinite series by adding or removing finitely many terms, the convergence or divergence of the series does not change.

Next, let's consider the case where we change a finite number of terms. Suppose we modify the series by changing the kth term to a'_k, where $a'_k \neq a_k$. Let S and S' be the original sum and the modified sum, respectively. Then, we can write:

$$S' = S - a_k + a'_K$$

Since the difference between S' and S is a finite number, the convergence or divergence of the series does not change. However, the actual sum of the series is different.

In conclusion, the property "By removing, adding or changing the finite term, the convergence of the series does not change, but the sum may change" is a fundamental result in the theory of infinite series. It tells us that we can modify an infinite series by adding, removing, or changing a finite number of terms without affecting its convergence or divergence. However, the actual sum of the series may be different.

5. **The necessary condition for the convergence of the series is $\lim_{n\to\infty} u_n = 0$. In other words, if $\sum_{n=1}^{\infty} u_n$ is a convergent series, then $\lim_{n\to\infty} u_n = 0$.**
   - To prove this result, suppose that $\sum_{n=1}^{\infty} u_n$ is a convergent series. By definition, this means that the sequence of partial sums $S_k = \sum_{n=1}^{k} u_n$ converges to some finite limit S.

     We can express the kth term of the series as the difference between two consecutive partial sums:

     $$u_k = S_k - S_{k-1}$$

     Taking the limit of both sides as $k \to \infty$, we get:

     $$\lim_{k\to\infty} u_k = \lim_{k\to\infty}(S_k - S_{k-1}) = S - S = 0$$

     where we have used the fact that the sequence S_k converges to S, so $\lim_{k\to\infty} S_k = S$ and $\lim_{k\to\infty} S_{k-1} = S$.

     Therefore, we have shown that if $\sum_{n=1}^{\infty} u_n$ is a convergent series, then $\lim_{n\to\infty} u_n = 0$
     .

     It's worth noting that the converse of this result is not necessarily true. That is, just because $\lim_{n\to\infty} u_n = 0$, it doesn't necessarily mean that $\sum_{n=1}^{\infty} u_n$ is convergent. For example, the harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges even though $\lim_{n\to\infty} \frac{1}{n} = 0$. So, the condition $\lim_{n\to\infty} u_n = 0$ is only a necessary condition for convergence, not a sufficient one.

Of course we ask that it is not necessary to remember or fully understand the derivation of these properties, these are just to help you understand. For the purpose of the exam, you just need to remember the bolded text, it shouldn't be that hard, right?