9.1 Infinite Series

Let's spice things up! We've got a scoop for you: sequences are ordered sets of terms, and when you add them all together, you've got yourself a series. Now, hold on to your hats because we're about to dive into the wild world of infinite series and how they relate to sequences. Get ready to learn about what it means for a series to converge or diverge – it's about to get crazy!

But wait, there's more! We can't forget about the geometric series, one of the most important types of series out there. Trust us, you'll be seeing this bad boy in action in the next chapter, where we'll be using it to write certain functions as polynomials with an infinite number of terms. This is a crucial process because it allows us to evaluate, differentiate, and integrate complicated functions using polynomials that are easier to handle.

Oh, and don't even get us started on the harmonic series – arguably the most interesting divergent series out there. Why is it so fascinating, you ask? Because it just fails to converge, and that's pretty darn cool if you ask us. So, buckle up and get ready to go on a wild ride through the infinite world of series and sequences!

Introduction to Infinite Series

Infinite series are a fundamental concept in calculus, particularly in the field of analysis. An infinite series is simply the sum of an infinite sequence of terms. For example, the infinite series

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \cdots$$

is the sum of the sequence of terms $\frac{1}{n^2}$ for $n = 1, 2, 3, \ldots$.

Infinite series can be used to approximate functions, solve differential equations, and explore the properties of functions. However, it's important to understand the convergence and divergence of an infinite series before using it in any calculations.

Explicit Formula

Explicit formula refers to a mathematical formula or expression that directly gives the value of any term in a sequence or series, without having to calculate the preceding terms. It is also called a closed form solution.

For example, the explicit formula for the nth term of a arithmetic sequence with a common difference of d and a first term of a can be written as:

$$a_n = a + (n-1)d$$

Similarly, the explicit formula for the nth term of a geometric sequence with a common ratio of r and a first term of a can be written as:

$$a_n = a \times r^{n-1}$$

Having an explicit formula can be very helpful in solving problems involving sequences and series, as it allows for quick and efficient calculations of any term in the sequence without having to compute all the preceding terms.

Recursion Formula

Recursion formula, also known as recurrence relation, is a mathematical formula that defines each term of a sequence or series in terms of one or more preceding terms. In other words, it is a formula that relates each term of a sequence to the ones that come before it.

For example, the Fibonacci sequence is defined recursively as:

$$F_n = F_n - 1 + F_n - 2$$

where $F_1 = 1$ and $F_2 = 1$. This means that each term of the sequence (starting from the third term onwards) is the sum of the two preceding terms.

Another example is the arithmetic sequence, which can be defined recursively as:

$$a_n = a_n - 1 + d$$

where a_1 is the first term of the sequence and d is the common difference.

Recursion formulas are useful in mathematics as they can often be used to generate a sequence more efficiently than explicitly computing each term, and can also provide insights into the properties of a sequence or series.

Convergence and Divergence

An infinite series is said to be *convergent* if the sum of its terms approaches a finite value as the number of terms in the series approaches infinity. On the other hand, if the sum of the terms of a series approaches infinity, the series is said to be *divergent*.

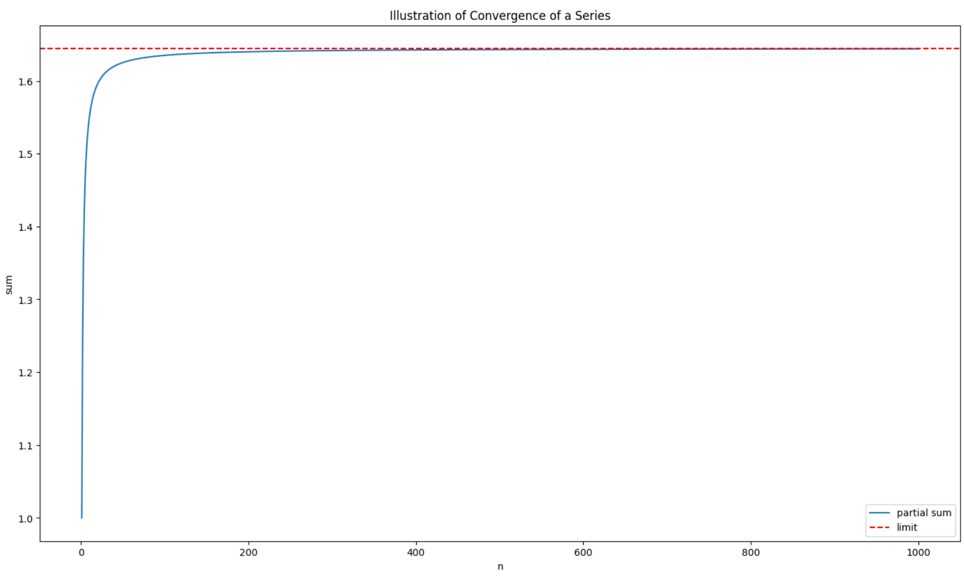
For example, the harmonic series

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots$$

is a well-known divergent series, while the geometric series

$$\sum_{n=1}^{\infty} r^n = 1 + r + r^2 + r^3 + \cdots$$

is convergent if $|r| < 1$ and divergent if $|r| \geq 1$.



> The graph illustrates the convergence of a simple infinite series, $\sum_{n=1}^{\infty} \frac{1}{n^2}$. The blue curve represents the partial sums of the series up to the first 1000 terms, while the red dashed line represents the limit of the series, which is $\frac{\pi^2}{6}$. As we can see from the graph, as the number of terms in the partial sum increases, the sum gets closer to the limit of the series, indicating that the series converges. The x-axis represents the number of terms in the partial sum, while the y-axis represents the value of the partial sum or the limit of the series.

Tests for Convergence

There are several tests that can be used to determine the convergence or divergence of an infinite series, including the comparison test, the limit comparison test, the ratio test, the root test, and the alternating series test. These tests are used to determine if the series is absolutely convergent, conditionally convergent, or divergent.

Understanding infinite series is essential for many applications in calculus, and it's important to have a good grasp of the convergence and divergence of a series before using it in any calculations.

Why Do We Care About Convergence and Divergence of Infinite Series?

Infinite series are important in many areas of mathematics and science, but not all series converge to a finite limit. Understanding whether a series converges or diverges is crucial in many applications, including physics, engineering, and computer science.

For example, consider the series:

$$\sum_{n=1}^{\infty} \frac{1}{n^2}$$

This series is known as the "Basel problem", named after the Swiss mathematician who posed the question of finding the sum of this series. Surprisingly, the sum of this series is a finite value, approximately equal to 1.644934. This value is known as the Basel constant, and it appears in many areas of mathematics, including number theory and probability theory.

On the other hand, consider the series:

$$\sum_{n=1}^{\infty} \frac{1}{n}$$

This series is known as the harmonic series, and it is an example of a divergent series. This means that the sum of the series goes to infinity as the number of terms increases. This is an important result, as it shows that even though the individual terms of the series get smaller and smaller, their sum can still be infinite.

Knowing whether a series converges or diverges is also important in numerical analysis and computational methods. For example, in numerical integration, we use methods that approximate the sum of a series to a certain degree of accuracy. If we know that a series converges, we can use these methods confidently and obtain accurate results. However, if a series diverges, these methods may fail and lead to incorrect or unreliable results.

Therefore, understanding the convergence and divergence of infinite series is crucial in many areas of mathematics and science, and it allows us to make accurate predictions, solve important problems, and develop new computational methods.

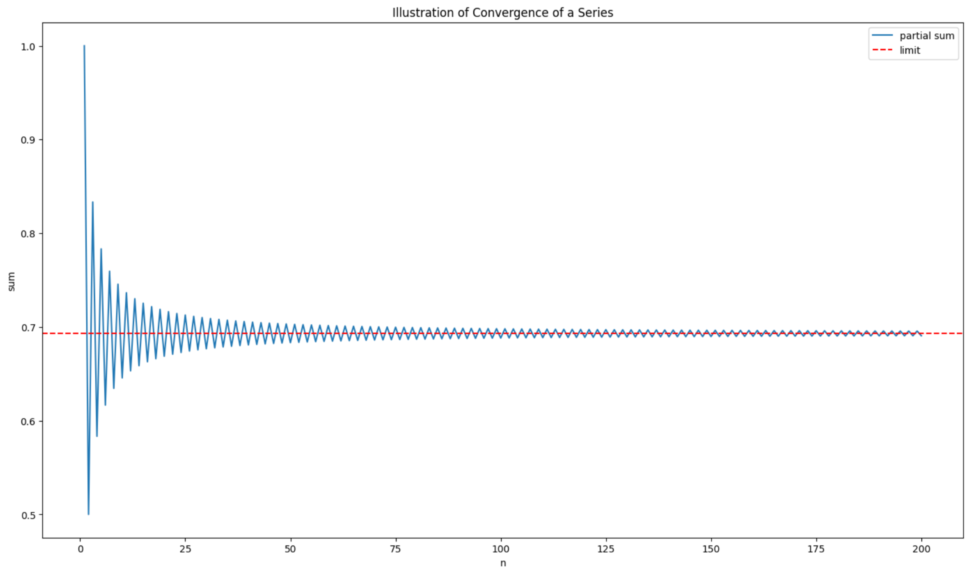
Absolute Convergence and Conditional Convergence

An infinite series is said to be *absolutely convergent* if the sum of the absolute values of its terms converges. If the sum of the absolute values of the terms diverges, but the original series still converges, the series is said to be *conditionally convergent*.

For example, the alternating harmonic series

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$$

is conditionally convergent, since the sum of the absolute values of its terms (the regular harmonic series) diverges, but the original series converges.



Power Series

A *power series* is an infinite series of the form

$$\sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \cdots$$

where x is a variable and c_n are constants. Power series are a useful tool in calculus, particularly for representing functions as infinite series.

Conclusion

Infinite series are a fascinating and important topic in calculus. Understanding the convergence and divergence of a series is crucial for many applications, and there are several tests that can be used to determine the convergence of a series. Absolute and conditional convergence are also important concepts to understand. Finally, power series are a powerful tool for representing functions as infinite series.

This section mainly introduces what concepts we will cover in this chapter on infinite series, and it doesn't matter if you don't understand some of them, because I will focus on them in detail later in the lecture.