9.3 Positive Series

This is a very important lesson as we will be starting to learn about the positive series, so please concentrate on that.

Definition of a Positive Series

A positive series is an infinite series whose terms are all positive real numbers. It has the following form:

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \cdots$$

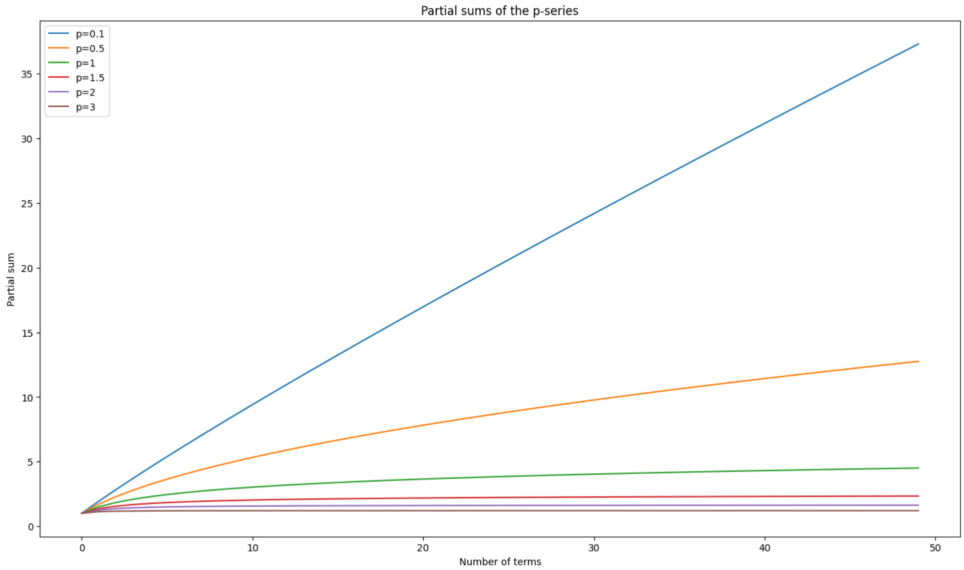
where a_n is the nth term of the series.

p-Series

A **p-series** is a series of the form $\sum_{n=1}^{\infty} \frac{1}{n^p}$, where p is a positive real number. The series is named after the exponent p in the denominator.

The p-series has the following properties:

- If $p > 1$, then the p-series converges.
- If $p \leq 1$, then the p-series diverges.



The proof of this result is based on the integral test. To see why this is true, we can consider the function $f(x) = \frac{1}{x^p}$. This function is continuous, positive, and decreasing on the interval $[1, \infty)$, so we can apply the integral test to the series to obtain:

$$\sum_{n=1}^{\infty} \frac{1}{n^p} \quad \text{if and only if} \quad \int_1^{\infty} \frac{1}{x^p} dx < \infty$$

Evaluating the integral, we get:

$$\int_1^\infty \frac{1}{x^p} dx \begin{cases} \frac{1}{1-p} & \text{if} \quad p > 1 \\ \infty & \text{if} \quad p \leq 1 \end{cases}$$

Thus, the p-series converges if and only if $p > 1$, and diverges if $p \leq 1$.

The sum of the p-series for $p > 1$ can be expressed in terms of the Riemann zeta function $\zeta(p)$, which is defined by:

$$\zeta(p) = \sum_{n=1}^{\infty} \frac{1}{n^p}$$

The Riemann zeta function has many interesting properties and connections to other areas of mathematics, including number theory and complex analysis.

Convergence tests

There are several convergence tests that can be used to determine whether a positive series converges or diverges. Here are a few of the most commonly used tests:

1. Comparison Test

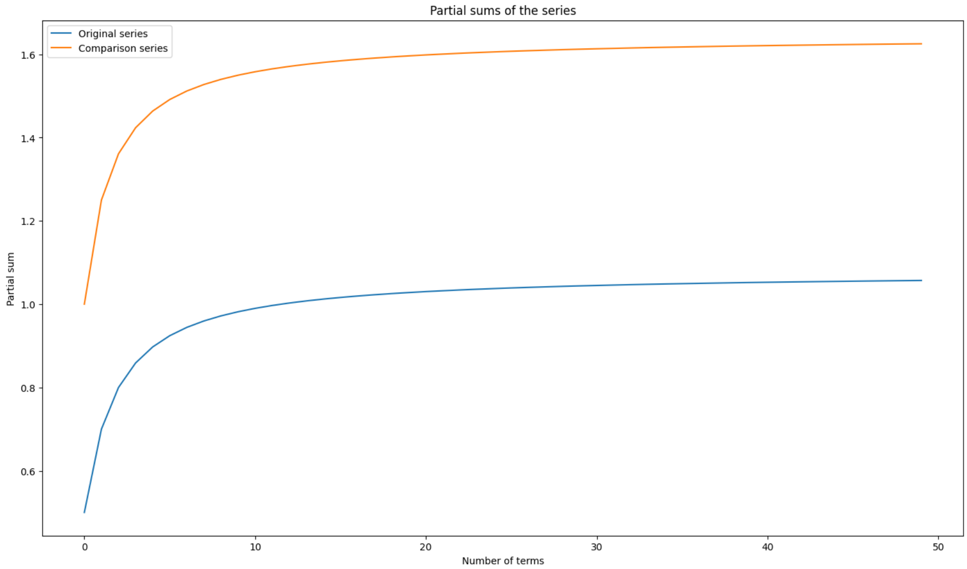
The **comparison test** is a method used to determine the convergence or divergence of a series by comparing it to another series whose convergence behavior is known. The comparison test states:

Suppose that $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ are series with positive terms, and suppose that $a_n \leq b_n$ for all n.

- If $\sum_{n=1}^{\infty} b_n$ converges, then $\sum_{n=1}^{\infty} a_n$ converges as well.
- If $\sum_{n=1}^{\infty} a_n$ diverges, then $\sum_{n=1}^{\infty} b_n$ diverges as well.

In other words, if the terms of the series $\sum_{n=1}^{\infty} a_n$ are always less than or equal to the terms of the series $\sum_{n=1}^{\infty} b_n$, and if $\sum_{n=1}^{\infty} b_n$ converges, then $\sum_{n=1}^{\infty} a_n$ must converge as well. Conversely, if $\sum_{n=1}^{\infty} a_n$ diverges, then $\sum_{n=1}^{\infty} b_n$ must also diverge.

The comparison test is often used to compare a given series to a p-series, since the convergence behavior of p-series is well-known. Specifically, if we have a series of the form $\sum_{n=1}^{\infty} \frac{1}{n^p}$ and another series $\sum_{n=1}^{\infty} a_n$ whose terms are always less than or equal to $\frac{1}{n^p}$, then we can use the comparison test to determine the convergence behavior of $\sum_{n=1}^{\infty} a_n$.



> The common infinite series that we use for reference are:
>
> 1. Geometric Series
> 2. Harmonic Serie
> 3. p-Series

2. Limit Comparison Test

The **limit comparison test** is another method used to determine the convergence or divergence of a series. Like the comparison test, it involves comparing the given series to another series whose convergence behavior is known. However, the limit comparison test allows for more flexibility in choosing the series to compare with.

Suppose that $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ are series with positive terms. Let $L = \lim_{n\to\infty} \frac{a_n}{b_n}$, where L is a positive finite number or ∞.

- If $0 < L < \infty$, then $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ either both converge or both diverge.
- If $L = 0$ and $\sum_{n=1}^{\infty} b_n$ converges, then $\sum_{n=1}^{\infty} a_n$ converges as well.
- If $L = \infty$ and $\sum_{n=1}^{\infty} b_n$ diverges, then $\sum_{n=1}^{\infty} a_n$ diverges as well.

The limit comparison test allows for more flexibility in choosing the series to compare with, since we only need the ratio of the terms to converge to a finite positive number. This means that we can often find a more manageable series to compare with than in the case of the comparison test.

3. Ratio Test

The **ratio test** is a test for the convergence or divergence of a series. The ratio test states:

Suppose that $\sum_{n=1}^{\infty} a_n$ is a series with positive terms, and let $L = \lim_{n\to\infty} \frac{a_{n+1}}{a_n}$ (the limit may or may not exist).

- If $L < 1$, then $\sum_{n=1}^{\infty} a_n$ converges absolutely.
- If $L > 1$ or $L = \infty$, then $\sum_{n=1}^{\infty} a_n$ diverges.
- If $L = 1$ or the limit does not exist, then the ratio test is inconclusive and we need to use another test to determine convergence or divergence.

Intuitively, the ratio test compares the terms of the series to the terms of a geometric series with common ratio L. If $L < 1$, then the terms of the series decay faster than the terms of a convergent geometric series, so the series converges. If $L > 1$, then the terms of the series grow faster than the terms of a divergent geometric series, so the series diverges. If $L = 1$, then the terms of the series decay at the same rate as the terms of a convergent geometric series, so the test is inconclusive.

4. Root Test

The **root test** is a test for the convergence or divergence of a series. The root test states:

Suppose that $\sum_{n=1}^{\infty} a_n$ is a series with positive terms, and let $L = \lim_{n\to\infty} \sqrt[n]{a_n}$ (the limit may or may not exist).

- If $L < 1$, then $\sum_{n=1}^{\infty} a_n$ converges absolutely.
- If $L > 1$ or $L = \infty$, then $\sum_{n=1}^{\infty} a_n$ diverges.
- If $L = 1$ or the limit does not exist, then the root test is inconclusive and we need to use another test to determine convergence or divergence.

Intuitively, the root test compares the terms of the series to the terms of a convergent geometric series with common ratio L. If $L < 1$, then the terms of the series decay faster than the terms of a convergent geometric series, so the series converges. If $L > 1$, then the terms of the series grow faster than the terms of a divergent geometric series, so the series diverges. If $L = 1$, then the terms of the series decay at the same rate as the terms of a convergent geometric series, so the test is inconclusive.

This one is very important, as many of the concepts and extensions we mention subsequently are based on the conclusions of this chapter, so make sure you read it carefully.