MODULE - II

CONTINOUS PROBABILITY DISTRIBUTION

Continous Random Variable (C.R.V)

A random variable is said to be continous if it assumes all possible values (infinite no) in the given (a, b)

Probability distribution function (P.D.F)

A function $f(x)$ of a cont random variable x is said to be a PDF if it satisfies the following conditions

1) $f(x) \geq 0$

2) $\int_{-\infty}^{\infty} f(x)\,dx = 1$

Cumulative Distribution Function or Distribution fn CDF

If x is CRV, then CDF is given by

$$\boxed{F(x) = P(X \leq x) = \int_{-\infty}^{x} f(x)\,dx}$$

$$f(x) = \frac{d}{dx}[F(x)]$$

If x is CRV with PDF $f(x)dx$ defined from $(-\infty, \infty)$ then

$$P(a < x < b) = P(a \leq x < b) = P(a < x \leq b) = P(a \leq x \leq b) = \int_{a}^{b} f(x)\,dx$$

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$$P(X<a) = P(X \le a) = \int_{-\infty}^{a} f(x)\,dx$$

and

$$P(X>a) = P(X \ge a) = \int_{a}^{\infty} f(x)\,dx$$

Mean and Variance of C.R.V

$$\text{Mean} = E(X) = \int_{-\infty}^{\infty} x f(x)\,dx$$

$$E(X^2) = \int_{-\infty}^{\infty} x^2 f(x)\,dx$$

$$\underset{(\sigma^2 x)}{\text{Variance}} = E(X^2) - [E(X)]^2$$

$E(X)$ or mean is also called the 1st order moment about the origin and $E(X^2)$ is called 2nd order moment about the origin.

A continuous random variable X has the P.D.F $f(x) = \dfrac{k}{1+x^2}$ $\quad -\infty < x < \infty$

i) Determine k

ii) $F(x)$

iii) $P(X \ge 0)$

i) $\displaystyle\int_{-\infty}^{\infty} \frac{k}{1+x^2}\,dx = 1$

$\displaystyle k\int_{-\infty}^{\infty} \frac{1}{1+x^2}\,dx = 1$

$\tan^{-1}\infty = \pi/2$

$\tan^{-1}(-\infty) = -\tan^{-1}\infty = \pi/2$

$k(\tan^{-1}x)_{-\infty}^{\infty} = 1 \Rightarrow k(\tan^{-1}\infty - \tan^{-1}-\infty) = 1$

$k(\pi/2 + \pi/2) = 1$

$k\pi = 1$

$k = 1/\pi$

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ii) $F(x) = \frac{1}{\pi}\int_{-\infty}^{x} \frac{1}{1+x^2}\,dx \quad = \frac{1}{\pi}\tan^{-1}x\Big]_{\pi/2}^{x}$

$= \frac{1}{\pi}\left[\tan^{-1}x + \frac{\pi}{2}\right]$

$= \frac{1}{\pi}(\tan^{-1}x) + \frac{1}{2}$

iii) $\frac{1}{\pi}\int_{0}^{\infty} \frac{1}{1+x^2}\,dx \quad = \frac{1}{\pi}\tan^{-1}x\Big]_{0}^{\infty}$

$= \frac{1}{\pi}\left[\tan^{-1}\infty - \tan^{-1}0\right]$

$= \frac{1}{\pi}(\tan^{-1}\infty)$

$= \frac{1}{\pi} \times \frac{\pi}{2}$

$= \frac{1}{2}$

Qn Find the value of k for PDF f(x) given below and hence find its mean and variance

$f(x) = kx^3 \quad 0 < x < 1$

$= 0$

ans Since it the PDF $\int_0^1 f(x)\,dx = 1$

$\Rightarrow \int_0^1 kx^3 = 1$

$k\left(\frac{x^4}{4}\right]_0^1) = 1$

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$k\frac{1}{4} = 1$

$\underline{\underline{k = 4}}$

$$\text{mean} = \int_0^1 x f(x)\,dx$$

$$= \cancel{4\int_0^1 \frac{x^4}{4}\,dx} = 0$$

$$= \cancel{4\int_0^1 x^4\,dx}$$

$$= \cancel{1\frac{x^5}{5}}$$

$$= \int_0^1 x\,4x^3\,dx$$

$$= 4\int_0^1 x^4\,dx$$

$$= 4\left[\frac{x^5}{5}\right]_0^1$$

$$= \underline{\underline{\frac{4}{5}}}$$

variance $\Rightarrow \int_0$

$$E(x^2) = \int_0^1 x^2 f(x)\,dx$$

$$= \int_0^1 x^2\,4x^3\,dx$$

$$= 4\int_0^1 x^5\,dx$$

$$= 4\left[\frac{x^6}{6}\right]_0^1$$

$$= \frac{4}{6}$$

variance $\sigma^2(x) = \frac{4}{6} - \left(\frac{4}{5}\right)^2 = \frac{8}{75} = \underline{\underline{0.261}}$

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An. Diameter of an electric cable X is assumed to be a continous random variable with PDF $f(x) = 6x(1-x) \quad 0 \le x \le 1$

$= 0$

1) Check whether $f(x)$ is PDF

$6x - 6x^2$

2) $P(0 < x < \frac{1}{2})$

$6(x - x^2)$

3) determine 'b' $P(X < b) = P(X \ge b)$

i) $\int_0^1$ it forms PDF, if $\int_0^1 6(x-x^2)\,dx = 1$

$$\Rightarrow 6\int_0^1 x - x^2\,dx = 1$$

$$\Rightarrow 6\left[\frac{x^2}{2}\right]_0^1 - \frac{x^3}{3}\Big]_0^1 = 6\left[\frac{1}{2} - \frac{1}{3}\right]$$

$$= 6 \times \frac{1}{6}$$

$$= \underline{\underline{1}}$$

ii) $6\int_0^{1/2}(x - x^2)\,dx = 6\left[\frac{x^2}{2}\right]_0^{1/2} - \frac{x^3}{3}\Big]_0^{1/2}\Big]$

$$= 6\left[\frac{1/4}{2} - \frac{1/8}{3}\right]$$

$$= 6 \times \frac{1}{12}$$

$$= \underline{\underline{\frac{1}{2}}}$$

iii) since $P(X < b) = P(X \ge b)$, value of each probability $= 1/2$

$\therefore P(X < b) = 1/2$

$6\int^b x - x^2\,dx = 1/2$

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$$6\left[\frac{b^2}{2}-\frac{b^3}{3}\right]=\frac{1}{2}$$

$$6\,\frac{3b^2-2b^3}{6}=\frac{1}{2}$$

$$3b^2-2b^3=1/2 \qquad \cancel{b^2/u}$$

$$\cancel{3u-2u^2=1/2}$$

$$\cancel{u=1.78}$$

$$\cancel{u=-0.28}$$

$$6b^2-4b^3=1$$

$$4b^3-6b^2=-1$$

$$4b^3-6b^2+1=0$$

$$\cancel{b=1.59,\ \frac{1+\sqrt{3}}{2},\ \frac{1-\sqrt{3}}{2},\ 1/2}$$

b

$$b=\frac{1+\sqrt{3}}{2},\ \frac{1-\sqrt{3}}{2},\ \frac{1}{2}.$$

$$b=\frac{1}{2}$$

Q. Find the mean and variance of $f(x)=1/2$ when $|x|\le 1$

$=0$ otherwise

$$\int_{-1}^{1} xf(x)\,dx=\int_{-1}^{1} x/2\,dx$$

$$=\left.\frac{x^2}{4}\right]_{-1}^{1}$$

$$=\frac{1}{4}-\frac{1}{4}$$

$$=\frac{1}{4}-\frac{1}{4}\ \underline{0}$$

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Sol $\Sigma f x^2$

$$\sigma^2(x) = E(x^2) - [E(x)]^2$$

$$E(x^2) = \int_{-1}^{1} x^2 \frac{1}{2} dx$$

$$= \frac{1}{2} \left[\frac{x^3}{3}\right]_{-1}^{1}$$

$$= \frac{1}{2}\left[\frac{1}{3} + \frac{1}{3}\right]$$

$$= \frac{2}{3} \times \frac{1}{2}$$

$$= \frac{1}{3}$$

$$\sigma^2(x) = \frac{1}{3} - 0$$

$$= \frac{1}{3}$$

Qn If the distrcbution fn of a random variable is given by $F(x) = 1 - \frac{1}{x^2}$

$= 0 \quad x \leq 1$

Find i) The PDF

ii) $P(X < 3)$

iii) $P(4 < x < 5)$

i) $f(x) = \frac{d}{dx}[F]$

$$= \frac{d}{dx}\left[1 - \frac{1}{x^2}\right]$$

$$= 2x^{-3} = \frac{2}{x^3}$$

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$f(x) = \frac{2}{x^3} \quad x > 1$

$= 0 \quad x \leq 1$

ii) $P(X < 3) = \int_1^3 \frac{2}{x^3}\, dx$

$= 2 \left[\frac{x^{-2}}{-2} \right]_1^3$

$= 2 \times \left[\frac{9}{-2} - \frac{1}{-2} \right] \quad = 2 \left[\frac{1}{-2x^2} \right]_1^3 \Big]$

$= 2 \times \frac{8}{-2} \quad = \left[\frac{1}{-9} + \frac{1}{1} \right]$

$= -$

$= 1 - \frac{1}{9}$

$= \frac{8}{9}$

iii) $P(4 < X < 5) = \int_4^5 \frac{2}{x^3}\, dx$

$= 2 \int_4^5 \frac{1}{x^3}\, dx$

$= 2 \left[\frac{1}{-2x^2} \right]_4^5 \Big]$

$= 2 \times \left[\frac{1}{-50} + \frac{1}{32} \right]$

$= \frac{1}{16} - \frac{1}{25}$

$= \underline{\underline{0.0225}}$

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Qn. Suppose that the error in the temp for a laboratory exp is a CRV with PDF $= f(x) = \frac{x^2}{3} \quad -1 < x < 2$

$= 0$

Obtain the Distribution fn. $F(x)$

Pf $P(X \le x) = \int_{-1}^{x} \frac{x^2}{3}\, dx$

$= \frac{1}{3}\left[\frac{x^3}{3}\right]_{-1}^{x}$

$= \frac{1}{3}\left[\frac{x^3}{3} + \frac{1}{3}\right]$

max value of :

$= \frac{x^3+1}{9} \quad -1 < x < 2$

$= 1 \quad x > 2$

$\int_{-1}^{2} f + \int_{2}^{\infty} f$

Q n. A continuous R.V has PDF $f(x) = kx^2 e^{-x} \quad x \ge 0$. Find

Find 1) The value of k

2) mean

3) variance

Note: gamma n is defined as $= \int_0^\infty e^{-x} x^{n-1}\, dx = \Gamma n$ where n is any positive no.

$\Gamma n = \int_0^\infty e^{-x} x^{n-1}\, dx$

$\Gamma n+1 = n!$ if n is an integer

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$f(x) = kx^2e^{-x}$

Since it is the PDF $f \Rightarrow \int_0^\infty kx^2e^{-x}dx = 1$

$$k\int_0^\infty x^2e^{-x}dx = 1$$

$$k\,\Gamma 3 = 1$$

$$k\,2 = 1$$

$$k = 1/2$$

i) mean $= \int_0^\infty x f(x)dx$

$$= \int_0^\infty \frac{1}{2}x^3e^{-x}dx$$

$$= \frac{1}{2}\int_0^\infty x^3e^{-x}dx$$

$$= \frac{1}{2}\Gamma 4$$

$$= \frac{3!}{2}$$

$$= \frac{6}{2} = 3$$

ii) variance $\sigma^2(x) = E(x^2) - [E(x)]^2$

$$E(x^2) = \int_0^\infty x^2 f(x)dx$$

$$= \int_0^\infty \frac{1}{2}x^4e^{-x}dx$$

$$= \frac{1}{2}\int_0^\infty x^4e^{-x}dx$$

$$= \frac{1}{2}\Gamma 5$$

$$= \frac{4!}{2} = 12$$

$$\sigma^2(x) = 12 - 9 = 3$$

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Q. Daily consumption of electric power in millions of kilowatt is a random variable having the PDF $= \frac{1}{9} x e^{-x/3}$ $x > 0$

$= 0$ otherwise

If the total production is million kilowatt, what is the probability this power supply will be inadequate on any given day.

ans. prob. that the power supply will inadequate $= P(X > 12)$

$= 1 - P(X \leq 12) \rightarrow ①$

$$P(X \leq 12) = \int_0^{12} \frac{1}{9} x e^{-x/3} \, dx$$

$$= \frac{1}{9} \int_0^{12} x e^{-x/3} \, dx$$

$$= \frac{1}{9} \left[\cancel{\int_0^{12} x e^{-x/3} - \int} \right.$$

$$= \frac{1}{9} \left[\frac{x e^{-x/3}}{-1/3} - \frac{e^{-x/3}}{1/9} \right]_0^{12} \Bigg]$$

$$= \frac{1}{9} \left[\frac{e^{-x/3} - 3x e^{-x/3}}{1/9} \right]_0^{12} \Bigg] - 1$$

$$= -e^{-x/3} - 3x e^{-x/3}$$

$$= 12 - e^{-4} - 36 e^{-4}$$

$$= e^{-4}[1 + 36] - e^{-1}[1 + 3 \cdot 0]$$

$$= -35 e^{-4} - 3 e^{-1} - 1$$

$$= 0.641$$

$$= \frac{1}{9} \left[\frac{12 e^{-4}}{-1/3} - \frac{e^{-4}}{1/9} + \frac{1}{1/9} \right]$$

$$= \frac{1}{9} \left[-36 e^{-4} - 9 e^{-4} + 9 \right]$$

$$= \frac{1}{9} \left[-45 e^{-4} + 9 \right]$$

$$= -5 e^{-4} + 1$$

from ①

$$1 - P(X \leq 12) = 1 + 5 e^{-4} - 1 = 5 e^{-4}$$

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Qn. Let the phase error of a tracking device having PDF;

$f(x) = \cos x \quad 0 < x < \pi/2$. Find $P(0 < x < \pi/4)$ and $P(x > \pi/3)$

$= 0$

1) $P(0 < x < \pi/4) = \int_0^{\pi/4} \cos x \, dx$

$= \sin x \Big]_0^{\pi/4}$

$= \frac{1}{\sqrt{2}} = 0.707$

1) $P(x > \pi/3) = 1 - P(x \leq \pi/3) \rightarrow (1)$

$P(X \leq \pi/3) = \int_0^{\pi/3} \cos x \, dx$

$= \sin x \Big]_0^{\pi/3}$

$= \frac{\sqrt{3}}{2}$

$P(x > \pi/3) = 1 - \frac{\sqrt{3}}{2}$

$= \frac{2 - \sqrt{3}}{2}$

$= 0.1339$

Qn. If X is a C.R.V, with PDF given by $f(x) = \frac{x}{8} \quad 0 < x < 2$

$= 1/4 \quad 2 < x < 4$

$= \frac{6-x}{8} \quad 4 < x < 6$

$= 0$

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Find the distribution fn.

when $0 < x < 2$; $F = \int_0^x f(x)\,dx$

$$= \int_0^x \frac{x}{8}\,dx$$

$$= \frac{1}{8}\left.\frac{x^2}{2}\right]_0^x$$

$$= \frac{x^2}{16}$$

when $2 < x < 4$; $F = \int_0^2 f(x)\,dx + \int_2^x f(x)\,dx$

$$= \int_0^2 \frac{x}{8}\,dx + \int_2^x \frac{1}{4}\,dx$$

$$= \left.\frac{x^2}{8}\right]_0^2 + \left.\frac{x}{4}\right]_2^x$$

$$= \frac{4}{8} + \frac{x}{4} - \frac{2}{4}$$

$$= \frac{x}{4}$$

when $4 < x < 6$; $F = \int_0^x f(x)\,dx$

$$= \int_0^2 \frac{x}{8}\,dx + \int_2^4 \frac{1}{4}\,dx + \int_4^x \frac{6-x}{8}\,dx$$

$$= \left.\frac{x^2}{8}\right]_0^2 + \left.\frac{x}{4}\right]_2^4 + \left.\frac{6x}{8} - \frac{x^2}{16}\right]_4^x$$

$$= \frac{4}{8} + \left(1 - \frac{1}{2}\right) +$$

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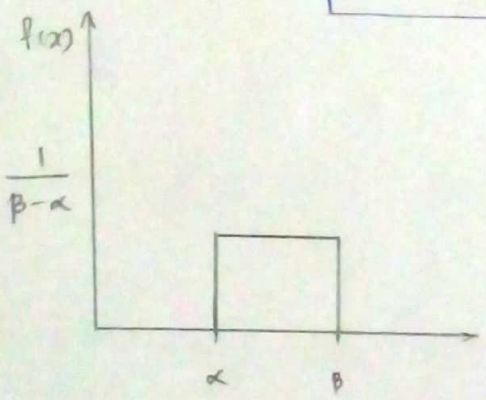
when $x > 6$ $F = 1$

Problem set 2.1 ; Qn 1, 5, 10, 11 Pg: 45

Qn UNIFORM DISTRIBUTION (RECTANGULAR DIST.)

A continuous R.V, X defined in the (α, β) are $(\alpha \leq x \leq \beta)$ is said to uniform dist. if its PDF $\boxed{f(x) = \frac{1}{\beta - \alpha} \; ; \; \alpha \leq x \leq \beta}$



If X follows uniform distribution with parameters α and β, then it is denoted by $\boxed{X \sim U(\alpha, \beta)}$

DISTRIBUTION FUNCTION F(x)

$$F(x) = P(X \leq x)$$

$$= \int_{\alpha}^{x} f(x)\,dx$$

$$= \int_{\alpha}^{x} \frac{1}{\beta - \alpha}\,dx = \left.\frac{x}{\beta - \alpha}\right]_{\alpha}^{x} = \frac{x}{\beta - \alpha} - \frac{\alpha}{\beta - \alpha} = \boxed{\frac{x - \alpha}{\beta - \alpha}}$$

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Q. Derive the mean and variance of uniform dist

OR

if $X \sim U(\alpha, \beta)$, then show that mean $= E(X) = \frac{\beta+\alpha}{2}$ and $\sigma^2(X) = \frac{(\beta-\alpha)^2}{12}$

ans: since $X \sim U(\alpha, \beta)$; $f(x) = \frac{1}{\beta-\alpha}$ $\quad \alpha \leq x \leq \beta$

$$\text{mean} = E(X) = \int_{\alpha}^{\beta} x f(x) dx$$

$$= \int_{\alpha}^{\beta} \frac{x}{\beta-\alpha} dx$$

$$= \frac{1}{\beta-\alpha} \left. \frac{x^2}{2} \right]_{\alpha}^{\beta}$$

$$= \frac{1}{\beta-\alpha} \left[\frac{\beta^2}{2} - \frac{\alpha^2}{2} \right.$$

$$= \frac{1}{\beta-\alpha} \frac{\beta^2-\alpha^2}{2}$$

$$= \frac{(\beta+\alpha)(\beta-\alpha)}{2(\beta-\alpha)}$$

$$= \frac{\beta+\alpha}{2}$$

Hence $\boxed{E(X) = \frac{\beta+\alpha}{2}}$

$$\sigma^2(X) = E(X^2) - [E(X)]^2 \longrightarrow (1)$$

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$$\int_{\alpha}^{\beta} x^2 f(x)\,dx$$

$$= \int_{\alpha}^{\beta} \frac{x^2}{\beta-\alpha}\,dx$$

$$= \frac{1}{\beta-\alpha} \left.\frac{x^3}{3}\right]_{\alpha}^{\beta}$$

$$= \frac{1}{\beta-\alpha}\left[\frac{\beta^3-\alpha^3}{3}\right]$$

$$\left[a^3-b^3 = (a-b)(a^2+ab+b^2)\right]$$

$$= \frac{1}{3(\beta-\alpha)}[\beta-\alpha][\beta^2+\alpha^2+\beta\alpha]$$

$$= \frac{[\beta+\alpha]^2}{3} \qquad \frac{\beta^2+\alpha^2+\beta\alpha}{3}$$

Substituting in ①

$$\sigma^2(x) = [\beta+\alpha]\frac{\beta^2+\alpha^2+\beta\alpha}{3} - \frac{[\beta^2+2\alpha\beta+\alpha^2]}{4}$$

$$= \frac{4\beta^2+4\alpha^2+4\beta\alpha-3\beta^2-6\alpha\beta-3\alpha^2}{12}$$

$$= \frac{\beta^2+\alpha^2-2\alpha\beta}{12}$$

$$= \frac{(\beta-\alpha)^2}{12}$$

Hence $\boxed{\sigma^2(x) = \frac{(\beta-\alpha)^2}{12}}$

$$\boxed{\sigma(x) = \frac{\beta-\alpha}{2\sqrt{3}}}$$

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Qn. If x is uniformly distributed R.V with mean 1 and variance $\frac{4}{3}$.

Find $P(|x-2|<2)$.

Ans. Let x is uniformly dist. in the interval (α, β)

Given, mean $= E(x) = \frac{\beta+\alpha}{2} = 1$

$$\beta+\alpha = 2 \longrightarrow (1)$$

Variance $= \sigma^2(x) = \frac{(\beta-\alpha)^2}{12} = \frac{4}{3}$

$$(\beta-\alpha)^2 = 16$$

$$\beta-\alpha = \pm 4$$

taking +ve value

$$\beta-\alpha = 4 \rightarrow (2)$$

(1) + (2)

$$2\beta = 6$$

$$\underline{\underline{\beta = 3}}$$

from (1)

$$\underline{\underline{\alpha = -1}}$$

$X \sim U(-1, 3)$

Hence $f(x) = \frac{1}{3-(-1)} = \frac{1}{4}$ $\quad -1 \le x \le 3$

$$P(|x-2|<2) = P(-2<x-2<2)$$

$$= P(0<x<4)$$

$$= \int_0^4 f(x)\,dx$$

$$= \int_0^3 \frac{1}{4}\,dx = \frac{x}{4}\Big]_0^4 = \frac{3}{4}$$

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Q. Find the distn. fn and the mean of the R.V having a unif. dist. $m(0,1)$.

~~f(x)=~~

$X \sim U(0,1)$

Ans.

So $f(x) = \frac{1}{1-0} = 1 \quad 0 \le x \le 1$

$\qquad\qquad\qquad 0 \quad$ otherwise

~~F(x)=~~

$$F(X \le x) = \int_0^x f(x)\,dx$$

$$= x\int_0^1 1\,dx$$

$$= \underline{\underline{x}}$$

$F(x) = 0$ when $x \le 0$

$\quad = x \qquad 0 \le x \le 1$

$\quad = 1 \qquad x > 1$

$$\text{mean} = E(x) = \int_0^1 x f(x)\,dx$$

$$= \int_0^1 x\,dx$$

$$= \frac{x^2}{2}\Big]_0^1$$

$$= \underline{\underline{\frac{1}{2}}}$$

Q. In certain exp. the error made in determining the ~~[illegible]~~ solubility of a substance is R.V having uniform density $\alpha = -0.005$, $\beta = 0.05$. What are the probabilities that such an error will be -

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i) b/w 0.01 and 0.015

ii) b/w −0.012 and 0.012

Ans: $X \sim U(-0.025, 0.025)$

Hence $f(x) = \dfrac{1}{0.05}$ $\quad -0.025 \le x \le 0.025$ $\qquad f(x) = \dfrac{1}{+0.025 + 0.025}$

i) $P(0.01 < x < 0.015) = \displaystyle\int_{0.01}^{0.015} \frac{1}{0.05}\,dx$

$$= \left.\frac{x}{0.05}\right]_{0.01}^{0.015}$$

$$= \frac{0.015 - 0.01}{0.05}$$

$$= \underline{\underline{0.1}}$$

~~ii) P(−0.025 < x <~~

~~ii) P(−0.012 < x~~

ii) $P(-0.012 < x < 0.012) = \displaystyle\int_{-0.012}^{0.012} \frac{1}{0.05}\,dx$

$$= \frac{0.012 + 0.012}{0.05}$$

$$= \frac{0.024}{0.05}$$

$$= \underline{\underline{0.48}}$$

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Sol. A bus arrives at every time minutes at bus stop. Assuming waiting time X for bus is uniformly distributed, find the probability that a person has to wait for the bus

i) for more than 7 minutes

ii) b/w 2 and 7 minutes.

ans. $X \sim U(0, 10)$

Hence $f(x) = \frac{1}{10} \quad (0 \le x \le 10)$

i) $P(X > 7) = \int_7^{10} \frac{1}{10} dx$

$= \frac{x}{10} \Big]_7^{10}$

$= \frac{10 - 7}{10}$

$= \frac{3}{10} = \underline{\underline{.3}}$

ii) $P(2 < X < 7) = \int_2^7 \frac{1}{10} dx$

$= \frac{x}{10} \Big]_2^7$

$= \frac{7 - 2}{10}$

$= \frac{5}{10}$

$= \underline{\underline{\frac{1}{2}}}$

Qn. If X is uniformly distributed in $(-\alpha, \alpha)$ with $\alpha > 0$, then α is.

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$P(X \le 1/2) = 0.7$

$X \sim U(-\alpha, \alpha)$

$$P(X \le 1/2) = \int_{-\alpha}^{1/2} f(x)\,dx = 0.7$$

$$\Rightarrow \int_{-\alpha}^{1/2} \frac{1}{2\alpha}\,dx = 0.7$$

$$\left.\frac{x}{2\alpha}\right]_{-\alpha}^{1/2} = 0.7$$

$$\frac{1/2}{2\alpha} + \frac{-\alpha}{2\alpha} = 0.7$$

$$\frac{0.5 + \alpha}{2\alpha} = 0.7$$

$$0.5 + \alpha = 1.4\alpha$$

$$0.5 = 1.3\alpha \cdot 4\alpha$$

$$\alpha = \frac{0.5}{1.3} \quad \frac{.5}{.4}$$

$$= \underline{1.25}$$

A string of length 10 m is divided into two parts of the part of random length X is uniformly dist. then find (i) $E(3X+2)$

(ii) $\text{var}(2X+3)$

(iii) $E(X(10-X))$

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$X \sim U(0, 10)$

$f(x) = \frac{1}{10} \quad (0 < x < 10)$

i) $E(X) = \frac{\beta + \alpha}{2}$

$E(X) = 5$

$\sigma^2(X) = \frac{(\beta - \alpha)^2}{12}$

$= \frac{100}{12}$

$= 8.8$

Using property expectation

i) $E(3X+2) = 3E(X) + 2$

$= 3 \times 5 + 2$

$= \underline{\underline{17}}$

ii) $var(2X+3) = 4\sigma^2(X)$

$= 4 \times 8.33$

$= 33.32$

iii) $E(X(10-X)) = 10E(X) - E(X)^2 \rightarrow (1)$

$E(X^2) = \int_0^{10} x^2 f(x)\, dx = \frac{1}{10}\int_0^{10} x^2\, dx$

$= \frac{x^3}{30}\Big]_0^{10} = \frac{1}{10}\frac{x^3}{3}\Big]_0^{10}$

$= \frac{1000}{30} = \frac{100}{3}$

$= 10$

substituting in (1) $= 10 \times 5 - 100/3 = 16.67$

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H.W

problem set 2.3 Qn: 8, 9, 10

Problem set 2.1

1. A R.V X defined in the interval (-3, 3) has PDF

$$f(x) = \begin{cases} \frac{1}{16}(3+x)^2, & -3 < x \leq -1 \\ \frac{1}{16}(6-2x^2), & -1 < x \leq 1 \\ \frac{1}{16}(3-x)^2, & 1 < x \leq 3 \\ 0, & \text{elsewhere} \end{cases}$$

Find the mean, S.D of X

III EXPONENTIAL DISTRIBUTION

A CRV X is said to follow exponential distribution if its PDF

$$f(x) = \frac{1}{\beta} e^{-x/\beta} \quad \text{where } x \geq 0.$$

$$= 0 \quad \text{otherwise.}$$

β is called the parameter of the distribution (survival parameter)

It is PDF because $f(x) \geq 0 \quad \forall x \geq 0$

and $\int_0^\infty f(x)\,dx = \int_0^\infty \frac{1}{\beta} \int_0^\infty \frac{1}{\beta} e^{-x/\beta}\,dx$

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$$= \frac{1}{\beta} \frac{e^{-x/\beta}}{-1/\beta} \Big]_0^\infty$$

$$= -e^{-x/\beta} \Big]_0^\infty \quad = -(e^{-\infty/\beta} - 1)$$

$$= 1$$

The CDF $F(x) = P(X \leq x)$

$$= \int_0^x \frac{1}{\beta} e^{-x/\beta} dx$$

$$= \frac{1}{\beta} \left[\frac{e^{-x/\beta}}{-1/\beta} \right]_0^x$$

$$= -\left[e^{-x/\beta} - 1 \right]$$

$$= 1 - e^{-x/\beta} \quad \text{when } x \geq 0$$

$$= 0 \quad \text{when } x < 0$$

* Derive the mean and variance of exponential distribution

or

Show that mean $= \beta$ and variance $= \beta^2$

<u>Proof</u>

Since X follows exponential distribution $f(x) = \frac{1}{\beta} e^{-x/\beta}$

$$\text{mean} = E(x) = \int_0^\infty x f(x) dx$$

$$= \int_0^\infty \frac{x e^{-x/\beta}}{\beta} dx$$

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$$= \frac{1}{\beta}\int_0^{\alpha} x e^{-x/\beta}\,dx$$

$$= \frac{1}{\beta}\left[\left(\frac{x e^{-x/\beta}}{-1/\beta} - \frac{e^{-x/\beta}}{1/\beta^2}\right)\right]_0^{\alpha}$$

$$= \frac{1}{\beta}\left[\frac{\alpha e^{-\alpha/\beta}}{-1/\beta} - \frac{e^{-\alpha/\beta}}{1/\beta^2} - \left[0 - \frac{1}{1/\beta^2}\right]\right]$$

$$= \frac{1}{\beta}\left[\cancel{1/}\ \frac{1}{\beta}\left[\frac{\beta^2}{\cancel{1/\beta^2}}\right] = \cancel{1/\beta}\ \beta\right.$$

Variance $= \sigma^2(X) = E(X)^2 - [E(X)]^2 \rightarrow (1)$

$$E(X)^2 = \int_0^{\alpha} x^2 f(x)\,dx$$

$$= \frac{1}{\beta}\int_0^{\alpha} x^2 e^{-x/\beta}\,dx$$

$$= \frac{1}{\beta}\left[\frac{x^2 e^{-x/\beta}}{-1/\beta} - \frac{2x e^{-x/\beta}}{1/\beta^2}\right]_0^{\alpha} + \frac{2e^{-x/\beta}}{(-1/\beta)^3}\Bigg]_0^{\alpha}$$

$$= \frac{1}{\beta}\left[\frac{\alpha^2 e^{-\alpha/\beta}}{-1/\beta} - \frac{2\alpha e^{-\alpha/\beta}}{1/\beta^2} + (0-0)\right] \frac{2e^{-\alpha/\beta}}{(-1/\beta)^3} - \left(\frac{2}{(-1/\beta)^3}\right)$$

$$= \frac{1}{\beta} \times 2\beta^3 = 2\beta^2$$

$$[E(X)]^2 = \beta^2$$

$$\sigma^2(X) = 2\beta^2 - \beta^2 = \beta^2$$

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Note :- We can also define the PDF exponential distribution using another parameter λ

Using the parameter λ, given by $f(x) = \lambda e^{-\lambda x}$ $\quad \lambda > 0 \quad x \geq 0$

$= 0$ $\quad [1/\beta = \lambda]$

λ is called the rate parameter

and mean $= 1/\lambda$

Variance $= 1/\lambda^2$

Eg. The time in hours required to repair a machine is exponentially distributed with mean 20. What is the probability that the required time;

1) exceeds 30 hrs

2) b/w 16 & 24 hrs.

Given

mean $= \beta$

$= 20$

$\therefore f(x) = \frac{1}{20} e^{-x/20} \quad x \geq 0$

① $P(X \geq 30) = \int_{30}^{\infty} f(x)\,dx$

$= \frac{1}{20} \int_{30}^{\infty} e^{-x/20}\,dx$

$= \frac{1}{20} \left[\frac{e^{-x/20}}{-1/20}\right]_{30}^{\infty} = \frac{1}{20}\left[-e^{-30/20}\right] = e^{-30/20}$

$= 0.223$

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(ii) $\int_{16}^{24} f(x)\,dx$

$= \frac{1}{20} \left. \frac{e^{-x/20}}{-1/20} \right]_{16}^{24}$

$= -\left[e^{-24/20} - e^{-16/20}\right]$

$= e^{-16/20} + e^{-24/20}$

$= \underline{\underline{0.148}}$

Qn The amount of time that a survillance camera run without having to reset is a R.V having exponential distribution with the parameter 50

Find the prob that such a camera will

1) have to reset in less than 20 days.

2) not have to be reset in atleast 60 days.

Given $\beta = 50$

$f(x) = \frac{1}{50} e^{-x/50}$

1) $P(X < 20) = \int_0^{20} f(x)\,dx$

$= \frac{1}{50} \int_0^{20} e^{-x/50}\,dx$

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$$= \frac{1}{50} \left. \frac{e^{-x/50}}{-1/50} \right]_0^{20}$$

$$= -\left[e^{-20/50} - 1\right]$$

$$= 1 - e^{-2/5}$$

$$= \underline{\underline{0.329}}$$

2) $P(X > 60) = \int_{60}^{\infty} f(x)\,dx$

$$= \frac{1}{50} \int_{60}^{\infty} e^{-x/50}\,dx$$

$$= \frac{1}{50} \left. \frac{e^{-x/50}}{-1/50} \right]_{60}^{\infty}$$

$$= -\left(0 - e^{-60/50}\right)$$

$$= e^{-6/5}$$

$$= \underline{\underline{0.301}}$$

Qn. The life in years of a certain type electrical switch has an E.D with mean 2. If 100 of these switches are installed in diff. s/ms, what is the prob. that atmost 30 fail during the 1st year.

Given, mean $= \beta$

$= 2$

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then $f(x)$ = probability of life span of electrical switch follows exponential distribution.

$$f(x) = \frac{1}{2} e^{-x/2} \quad x \geq 0$$

It follows binomial dis.

$$p = 1 - \frac{1}{2} e^{-x/2}$$

$$q = 1/2\, e^{-x/2}$$

$$n = 100$$

$$P(X \leq 30) = \sum_{x=0}^{30} 10C_x \left(1 - \frac{1}{2} e^{x/2}\right)^x \left(1/2\, e^{x/2}\right)^{10-x}$$

Qn. If X is an exponential distribution, with mean $= 1/\theta$ Find $P(X < 1 / X < 2)$.

$$\left[P(A/B) = \frac{P(A \cap B)}{P(B)} \right]$$

Given, $\beta = 1/\theta$

$\therefore f(x) = \theta e^{-\theta x}$

$$P(X<1 / X<2) = \frac{P(X<1 \cap X<2)}{P(X<2)}$$

$$= \frac{P(X<1)}{P(X<2)} \rightarrow ①$$

$$P(X<1) = \int_0^1 \theta e^{-\theta x} dx = \theta \int_0^1 e^{-\theta x} dx$$

$$= \theta \left[\frac{e^{-\theta x}}{-\theta} \right]_0^1 = \theta - (e^{-\theta} - 1)$$

$$= 1 - e^{-\theta}$$

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$$P(X < 2) = \int_0^2 \theta e^{-\theta x} dx$$

$$= \theta \int_0^2 e^{-\theta x} dx$$

$$= \left. \frac{\theta e^{-\theta x}}{-\theta} \right]_0^2$$

$$= -(e^{-2\theta} - 1)$$

$$= 1 - e^{-2\theta}$$

From ①

$$= \frac{1 - e^{-\theta}}{1 - e^{-2\theta}}$$

Pg 72 problem set 2.4 Qn, 1, 3, 4, 7

Problem 2.3

1. If X is uniformly dist. in $(-2, 2)$. Find.

i) $P(X < 1)$ ii) $P(|X - 1| \geq 1/2)$

$X \sim U(-2, 2)$

hence $f(x) = \frac{1}{2+2} = \frac{1}{4}$; $-2 < x < 2$

i) $P(X < 1) = \int_{-2}^{1} f(x) dx = \frac{1}{4} [x]_{-2}^{1} = \frac{1}{4} [1+2]$

$= 3/4$

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$P(-1/2 \le X - 1 \le 1/2) = P(1/2 \le X \le 3/2) = \int_{1/2}^{3/2} \frac{1}{2} dx = \frac{1}{2}[x]_{1/2}^{3/2}$

$= \frac{1}{4}\left[\frac{3-1}{2}\right] = \underline{\underline{1/4}}$

$P(|X-1| \ge 1/2) = 1 - P(|X-1| \le 1/2) = 1 - 1/4 = \underline{\underline{3/4}}$

9. The thickness of photo resist applied to wafers in semiconductor manufacturing at a particular location on the wafer is uniformly distributed b/w 0.205 and 0.215 micrometers. What thickness is exceeded by 10% of the wafers?

$X \sim U(0.205, 0.215)$ here

$f(x) = \frac{1}{0.215 - 0.205} = \frac{1}{0.10}, \quad 0.205 < x < 0.215$

$P(X > .10) = P(0.10 < X < 0.215)$

$= \int_{0.10}^{0.215} \frac{1}{0.01} dx = \frac{1}{0.01}[x]_{0.10}^{0.215} = \frac{1}{0.01}[0.215 - 0.10]$

$= \frac{0.115}{0.01} = \underline{\underline{11.5}}$

10 If $X \sim U(-3, 3)$ then find $P(X \le 2)$

$f(x) = \frac{1}{3+3} = \frac{1}{6}; \quad -3 \le x \le 3$

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$P(X \le 2) = P(-3 < X \le 2) = \int_{-3}^{2} \frac{1}{6}\, dx$

$= \frac{1}{6} x \Big]_{-3}^{2} = \frac{1}{6}[2+3] = \frac{5}{6}$

NORMAL DISTRIBUTION

A C.R.V X is said to follow normal distribution if its PDF $f(x) = 1$

PDF $f(x) = \frac{1}{\sqrt{2\pi}\,\sigma} e^{-(x-\mu)^2/2\sigma^2}$ $\quad -\infty < x < \infty$

Properties

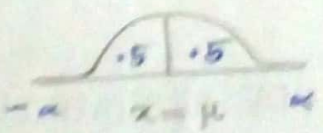
1. μ and σ are called the parameters of normal distr. denoted by

~~$X \sim N(\mu, \alpha$~~ $\quad X \sim N(\mu, \sigma)$

2. mean of the distribution is μ and variance is σ^2 and $SD = \sigma$

3. normal curve is bell shaped is symmetrical about the line

$x = \mu$



4. $\int_{-\infty}^{\infty} f(x)\,dx = 1$, the total area under the normal curve is unity

$\therefore \int_{-\infty}^{\mu} f(x)\,dx = \int_{\mu}^{\infty} f(x)\,dx = 1/2$

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5. Since $f(x) \geq 0$, the curve always lies above the x axis

6. max value of $f(x)$ attains at $x = \mu$ and $\therefore$ the max probability $= \frac{1}{\sqrt{2\pi}\,\sigma}$

7. As σ increases the curve flattens out and as σ decreases f increases and the curve has a sharp ~~has a~~ peak.

8. $P(a \leq x \leq b) = \int_a^b f(x)\,dx$.

9. $P(X \leq a) = \int_{-\infty}^{a} f(x)\,dx$

10. $P(X \geq a) = \int_a^{\infty} f(x)\,dx$.

11. $P(a \leq x \leq b) = \int_a^b f(x)\,dx =$ area under the normal curve b/w $x = a$ and $x = b$

Standard normal distribution

when $\mu = 0$ and $\sigma = 1$, then the PDF of normal dis. reduces to

$$f(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2} \quad -\infty < x < \infty$$

is called standard normal ~~no~~ distribution and is denoted by $X \sim N(0,1)$.

mean $= 0$ and S.D ~~or~~ variance $= 1$

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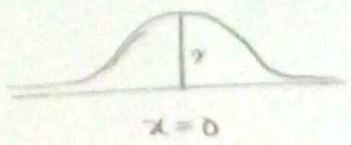
The standard normal curve is symmetrical about the line $x=0$



$$\therefore \int_{-\infty}^{0} f(x)\,dx = \int_{0}^{\infty} f(x)\,dx = 0.5$$

Transformation of normal dis. to standard normal dis.

Given

~~X follows X~~

$X \sim N(\mu, \sigma)$

Put $z = \dfrac{x-\mu}{\sigma}$

mean $= E\left[\dfrac{X-\mu}{\sigma}\right] = E(z)$

$= \dfrac{1}{\sigma}\left[E(X) - E(\mu)\right]$

$= \dfrac{1}{\sigma}\left[\mu - \mu\right]$

$= 0$

$Var(z) = var\left[\dfrac{x-\mu}{\sigma}\right]$

$= \dfrac{1}{\sigma^2} var[x]$

$= \dfrac{\sigma^2}{\sigma^2}$

$= 1$

$Z \sim N(0,1)$

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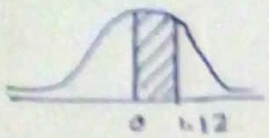
Valuation of probability using table

1) $P(0 < z < 1.12)$

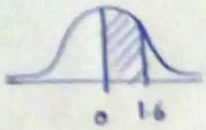


$= .3413$

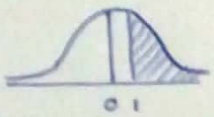
$= 0.3686$

2) $P(0 < z < .46)$

$= .4452$



3) $P(z > 1)$



$= 0.5 - .3413$

$= 0.1587$

4) $P(z < 1.75)$

$= .5.$

$= 0.5 + .4599$

$= 0.9599$

5) $P(-1.1 < z < 1.32)$

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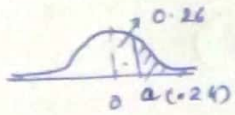
$= .3643 + .4066$

$= .7709$

* Find the value of ~~a~~ Prob a such that $P(Z > a) = 0.24$



Since the prob. $< .5$ a is positive

$a = 0.71$

1)Y

Sol. X is a normal variant with mean 20 and SD 5. Find the following

I) $P(16 \leq X \leq 22)$

II) $P(X < 23)$

iii) $P(X \geq 23)$

ans. Given $\mu = 20$

$\sigma = 5$

Put $z = \dfrac{X - \mu}{\sigma}$

$= \dfrac{X - 20}{5}$

I) $P(16 \leq X \leq 22) = P\left(\dfrac{16-20}{5} \leq X \leq \dfrac{22-20}{5}\right)$

$= P\left(\dfrac{-4}{5} \leq X \leq \dfrac{2}{5}\right)$

$= P(-.8 \leq X \leq .4)$

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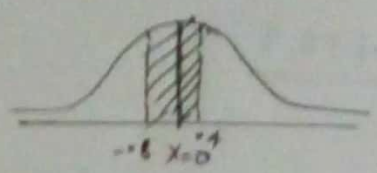
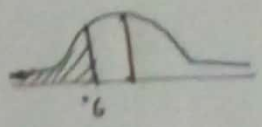
$= .2881 + .1554$

$= \underline{\underline{.4435}}$



ii) $P(X < 23) = \cancel{P(23 < 20}$

$= P\left(z < \frac{23-20}{5}\right)$



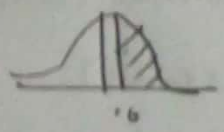
$= P(z < .6)$

$= 0.5 + \cancel{0.2148}\ 0.2257$

$= 0.7257$

iii) $P(X \geq 23) = \cancel{0.5 - 0.2257}\ \cancel{0.2743}$

$= \underline{0.2743}$



Qn. The time required to assemble a piece of machinery is a random variable approximately a normal distribution with mean 12.9 min. and SD 2 min what are the probabilities that the assembly of a piece machinery this kind will take

1) atleast 11.5 min

2) anywhere or b/w 11 to 14.8 min

$z = \frac{x - 12.9}{2}$

1) $P(X \geq \cancel{11.5})$

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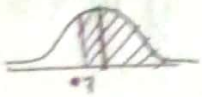
i) $P(X \geq 11.5) = \cancel{P\left(\frac{X-12.9}{2} \geq\right.}$ $P\left(Z \geq \frac{11.5-12.9}{2}\right)$

$= P(Z \geq .7)$

$= 0.5 + .2580$

$= \underline{\underline{.7580}}$



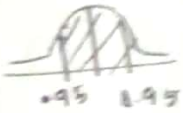
ii) $P\left(\frac{11-12.9}{2} < x < \frac{14.8-11}{2}\right) = P(-.95 < z < 0.95)$

$= (.3289 +$

$= .3289 + .3621$

$= .3289 + .3289$

$= \underline{.6578}$



Qn. The weekly wages 1000 work men are normally distributed around a mean of Rs 70 and with a SD of Rs 5.

Estimate the no. of workers whose weekly wages will be

i) b/w Rs 70 and 72

ii) more than Rs 75

iii) less than Rs 63

$Z = X - N$

$X \sim N(17, 5)$

$Z = \frac{X-70}{5}$

i) $P(70 < X < 72) = P\left(\frac{70-70}{5} < Z < \frac{72-70}{5}\right)$

$= P(0 < Z < .4)$

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$= 0 + .1554$

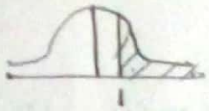
$= \underline{\underline{.1554}}$

$\Rightarrow$ for 1000 work men $= 1554$

ii) $P(X > 75) = P\left(Z > \frac{75-70}{5}\right)$

$= P(Z > 1)$

$= \underline{\underline{.3413}} = 0.5 - .3413$



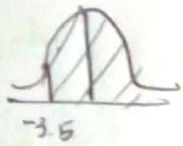
$= .1587$

$\Rightarrow .1587 \times 1000 = 159$

iii) $P(X < 63) = P\left(Z < \frac{63-70}{5}\right)$

$= P(Z < -1.4)$

$= .5 - .4192$



$= .5 - .4192$

$= 0.0808$

For 1000 men $= \underline{\underline{81}}$.

Qn. A random variable has a normal distribution with $\mu = 62.4$. Find the S.D. if the probability is 0.20 that it will take on a value greater than 79.2.

ans given $\mu = 62.4$

and let σ be the S.D

Then $z = \frac{X - 62.4}{\sigma}$

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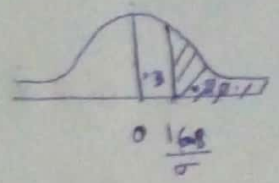
$P(x > 79.2) = 0.20$

$P(z > \frac{79.2 - 62.4}{\sigma}) = 0.20$

$P(z > \frac{16.8}{\sigma}) = 0.20$



From table value corresponding to $x = 0.3$

$\frac{16.8}{\sigma} = \cancel{8.45}\ .845$

$\frac{16.8}{.845} = \sigma$

$\sigma = \cancel{19.88}\ \cancel{21.35}\ 19.88$

6. A sample of 100 dry battery cells tested to find the length of life produced the following results

$\mu = 12$ hrs ; $\sigma = 3$ hrs. assuming the data to be normally distributed, what % of battery cells are expected to have life

1) more than 15 hrs

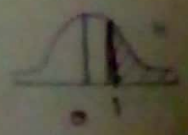
2) less than 6 hrs

3) b/w 10 and 14 hrs.

$z = \frac{x - 12}{3}$

1) $P(x > 5)$

$P(z > \frac{15-12}{3}) = P(z > 1)$



$= 0.5 - .3413$

$= .1587$

in % $= \frac{.1587}{1} \times 100 = \cancel{16.13189}\ 15.87$

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ii) $P(X < 6) = P\left(z < \frac{6-12}{3}\right)$

$= P(z < -2)$

$= 0.5 - .4981$

$= 0.5 - .4772$

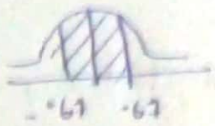
$= 0.0228$

$m\% = 0.0228 \times 100 = \underline{\underline{2.28}}$

iii) $P(10 < X < 14) = P\left(\frac{10-12}{3} < z < \frac{14-12}{3}\right)$

$= P(-.67 < z < .67)$

$= 2 \times .2486$



$= .4972$

$m\% = \underline{\underline{4972}}$.

Q. In a normal distribution 11% of the items are below 30 and 11% are above 60. Find the mean and S.D.

Let μ be mean and σ be the S.D.

then $z = \frac{x-\mu}{\sigma}$

Given $P(x < 30) = .11$

$P\left(z < \frac{30-\mu}{\sigma}\right) = .11$

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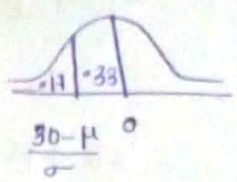
From table $-\left(\frac{30-\mu}{\sigma}\right) = .95$



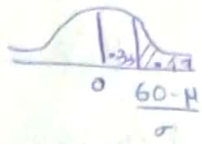
$-30+\mu = .95\sigma$

$\mu + .95\sigma = 30$

$\mu - .95\sigma = 30 \rightarrow ①$

Given $P(X > 60) = .17$

$P\left(Z > \frac{60-\mu}{\sigma}\right) = .17$



From table $\left(\frac{60-\mu}{\sigma}\right) = .95$

$60-\mu = .95\sigma$

$\cancel{\mu} - \mu - .95\sigma = -60$

$\mu + .95\sigma = 60 \rightarrow ②$

Solving ① & ②

$\mu = 45$

$\sigma = 15.78$

less than area

.5

value less than .5

-ve

greater than area

value more than

.5 -ve

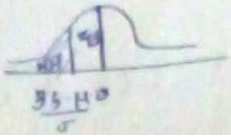
Q. For a normally distributed population 7% of the elem have their values less than 35 and 89% have their values less 63. Find the mean and SD

Let μ be mean and σ be SD

$Z = \frac{X-\mu}{\sigma}$

Given $P(X < 35) = 0.07$

$P\left(Z < \frac{35-\mu}{\sigma}\right) = 0.07$



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From table value corresponding to .43

$$-\left(\frac{35-\mu}{\sigma}\right) = 1.48$$

$$-35+\mu = 1.48\sigma$$

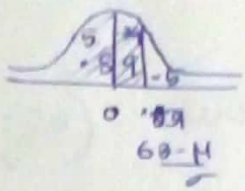
$$\mu - 1.48\sigma = 35 \rightarrow (1)$$

Given

$$P(X < 63) = .89$$

$$P\left(Z < \frac{63-\mu}{\sigma}\right) = .89$$

From table, value corresponding to .89



$$\frac{63-\mu}{\sigma} = 1.23$$

$$63-\mu = 1.23\sigma$$

$$-\mu - \mu - 1.23\sigma = -63$$

$$\mu + 1.23\sigma = 63 \rightarrow (2)$$

Solving (1) & (2)

$$\mu = 50.3 \quad ; \quad \sigma = 10.33$$

Qn. In a competitive examination 5000 students have appeared for a paper in engineering mathematics their average marks were 62 and SD 12. If there are only 100 vacancies, find the minimum marks that one should score inorder to get selected a

Given

$$\mu = 62$$
$$\sigma = 12$$

$$\therefore \quad z = \frac{X-62}{12}$$

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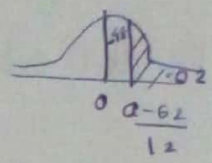
probability to get selected is $= \frac{1}{50} = 0.02$

ie $P(X > a) = 0.02$

$P\left(z > \frac{a-62}{12}\right) = 0.02$



From table value corresponding to .48 is 2.06

$\frac{a-62}{12} = 2.06$

$a - 62 = 12 \times 2.06$

$a = (12 \times 2.06) + 62$

$= \cancel{86.172}\ 86.2$

$= 86.72$

Qn. The ~~time~~ income dist. of workers in a certain factory was found to be normally distributed with mean Rs 1000 and SD 100. If 180 persons getting ~~Rs~~ above Rs 1200. How many persons were there in all.

$z = \frac{X - 1000}{100}$

$P(X > 1200) =$

$= P\left(z > \frac{1200-100}{100}\right)$

$= P(z > 2)$

$= 0.5 - .4772$

$= 0.0228$

10% 2.28 % of workers are getting more than 1200

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$\frac{180}{2.28} \times 100 = 7895$ = total persons

H.W

Problem set 2.2 Qn: 1, 5, 12

Problem set 2.4

1. The time (in hours) required to repair a machine is exponentially distributed with parameter $\beta = 2$ i) What is the probability that the repair time exceeds 2 hours?

ii) What is the conditional probability that a repair takes at least 10 hrs given that its duration exceeds 9 hrs?

Given $\beta = 2$, then $f(x) = \frac{1}{2} e^{-x/2}$; $x \geq 0$

i) $P(X > 2) = \int_2^{\infty} \frac{1}{2} e^{-x/2} dx$

$= \frac{1}{2} \left[\frac{e^{-x/2}}{-1/2} \right]_2^{\infty}$

$= -[0 - e^{-1}]$

$= e^{-1}$

$= \underline{\underline{0.3679}}$

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ii) $P(X \geq 10 \mid X > 9) = \dfrac{P(X \geq 10 \cap X > 9)}{P(X > 9)}$

$$= \frac{P(X \geq 10)}{P(X > 9)}$$

$$= \frac{\frac{1}{2}\int_{10}^{\infty} e^{-x/2}\,dx}{\frac{1}{2}\int_{9}^{\infty} e^{-x/2}\,dx}$$

$$= \frac{\left.\dfrac{e^{-x/2}}{-1/2}\right]_{10}^{\infty}}{\left.\dfrac{e^{-x/2}}{-1/2}\right]_{9}^{\infty}}$$

$$= \frac{-\frac{1}{2}\left[0 - e^{-5}\right]}{-\frac{1}{2}\left[0 - e^{-4.5}\right]}$$

$$= \frac{e^{-5}}{e^{-4.5}}$$

$$= \underline{\underline{.6065}}.$$

3. The time in hours required to repair a machine is exponentially distributed with parameter $\beta = 3$. What is the probability that the repair time exceeds 3 hours?

Given $\beta = 3$

then $f(x) = \frac{1}{3} e^{-x/3}$; $x > 0$

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$$P(X>3) = \int_3^{\infty} \frac{1}{3} e^{-x/3}\, dx$$

$$= \frac{1}{3}\int_3^{\infty} e^{-x/3}\, dx$$

$$= \frac{1}{3}\left[\frac{e^{-x/3}}{-1/3}\right]_3^{\infty}\Bigg]$$

$$= -\left[e^{-x/3}\right]_3^{\infty}\Bigg]$$

$$= -\left[0 - e^{-1}\right]$$

$$= \underline{\underline{e^{-1}}} = \underline{\underline{0.368}}$$

4. If a random variable X has the exponential distribution with mean $\mu = 1/2$. Calculate the probabilities that.

i) X will lie b/w 1 and 3

ii) X is greater than 0.5,

iii) X is almost 1.

Given $\mu = 1/2$

and $\mu = \beta$

$\therefore \beta = 1/2$ and $f(x) = \frac{1}{2} e^{-x/2}$; $x \geq 0$

i) $P(1 < X < 3) = \int_1^3 \frac{1}{2} e^{-x/2}\, dx$

$$= \frac{1}{2}\int_1^3 e^{-x/2}\, dx$$

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$$= \frac{1}{2} \left. \frac{e^{-x/2}}{-1/2} \right]_1^3$$

$$= -\left[\left. e^{-x/2} \right]_1^3 \right]$$

$$= -\left[e^{-3/2} - e^{-1/2} \right]$$

$$= -0.383$$

$\mu = 1/2 = B \qquad f(x) = 2e^{-2x}; \quad x \geq 0$

i) $P(1 < X < 3) = \int_1^3 2e^{-2x}\,dx$

$$= 2\int_1^3 e^{-2x}\,dx$$

$$= 2 \left. \frac{e^{-2x}}{-2} \right]_1^3$$

$$= -\left[\left. e^{-2x} \right]_1^3 \right]$$

$$= -\left[e^{-6} - e^{-2} \right]$$

$$= \underline{\underline{.133}}$$

ii) $P(X > 0.5) = \int_{0.5}^{\infty} 2e^{-2x}\,dx$

$$= 2\int_{0.5}^{\infty} e^{-2x}\,dx$$

$$= 2 \left. \frac{e^{-2x}}{-2} \right]_{0.5}^{\infty} = -\left(\left. e^{-2x} \right]_{0.5}^{\infty} \right) = -\left(e^{-2\times .5} \right)$$

$$= \underline{\underline{0.368}}$$

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$$P(X \leq 4) = \int_{-\infty}^{4} 2e^{-2x}\,dx$$

$$= 2\left[\frac{e^{-2x}}{-2}\right]_{-\infty}^{4}$$

$$= -\left(e^{-2x}\right)_{\infty}^{4}$$

$$= -\left(e^{-8} - 1\right)$$

$$= \underline{\underline{0.999}}$$

7. Let the mileage (in thousands of miles) of a particular type by a random variable x having PDF.

$$f(x) = \begin{cases} \frac{1}{20} e^{-x/20}, & x > 0 \\ 0, & x < 0 \end{cases}$$

Find the probability that one of these will last.

i) anywhere from 16,000 to 24000

ii) atleast 30,000 miles

Also find the iii) mean iv) variance

i) $P(16000 < x < 24000) = \frac{1}{20} \int_{16000}^{24000} e^{-x/20}\,dx$

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$$= \frac{1}{20}\left[\frac{e^{-x/20}}{1/20}\right]_{16000}^{24000} = \underline{\underline{0.148}}$$

ii) $P(X > 30) = \frac{1}{20}\int_{30}^{\alpha} e^{-x/20}\,dx = \left[-e^{-x/20}\right]_{30}^{\alpha}$

$$= e^{-3/2} = \underline{\underline{0.2231}}.$$

iii) mean $= \beta = \underline{\underline{20}}$

iv) variance $= \beta^2 = \underline{\underline{400}}$

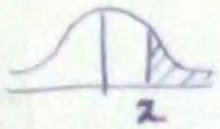
Problem set 2.2

1. The marks obtained by no. of students in a certain subject are approx. normally distributed with mean 65 and SD 5. Find probability that a student scores above 75.

Given

$\mu = 65$; $\sigma = 5$ $\quad \therefore \; z = \frac{x-65}{5}$

$P(X > 75) = P\left(z > \frac{75-65}{5}\right) = P(z > 2)$



$= 0.5 - 0.4772$

$= \underline{\underline{0.0228}}$.

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In an engineering examination, a student is considered to have failed, secured second class, first class and distinction, according as he scores less than 45%, b/w 45% and 60%, b/w 60% and 75% and above 75% respec. In a particular year, 10% of the students failed in exam. and 5% got distinction. Find the % of students who have got first class and second class.

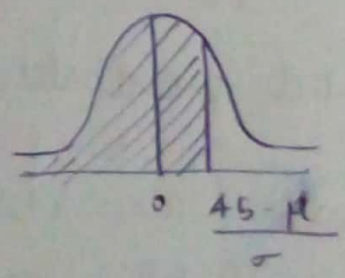
Given $P(X < 45) = 0.10$

$P(X > 75) = 0.05$

Let mean $= \mu$ and SD $= \sigma$ then $z = \frac{X - \mu}{\sigma}$

$P(X < 45) = 0.10$

$P\left(z < \frac{45-\mu}{\sigma}\right) = 0.10$



$P(X < 45) = a$

$0.5 - 0.10 = 0.4$

$-\left(\frac{45-\mu}{\sigma}\right) = 1.28$

$-(45-\mu) = 1.28\sigma$

$\mu - 1.28\sigma = 45 \rightarrow 0$

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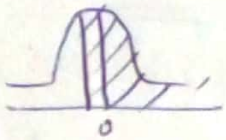
$P(X > 75) = 0.05$

$P\left(Z > \frac{75-\mu}{\sigma}\right) = 0.05$

$0.5 - 0.05 = 0.45$



$\frac{75-\mu}{\sigma} = 1.64$

$\mu + 1.64\sigma = 75 \rightarrow ②$

solving ① & ②

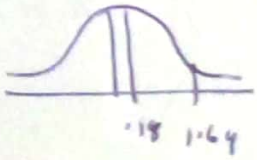
$\mu = 58.15, \sigma = 10.27$

$P(60 < X < 75) = P\left(\frac{60-58.15}{10.27} < Z < \frac{75-58.15}{10.27}\right)$

$= P(0.18 < Z < 1.64)$

$= 0.4495 - 0.0714$

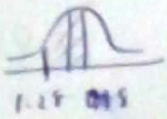
$= 0.3781$



$= 37.81\%$

$= 38\%$

$P(45 < X < 60) = P\left(\frac{45-58.15}{10.27} < Z < \frac{60-58.15}{10.27}\right)$



$= P(-1.28 < Z < 0.18)$

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$= 0.3997 + 0.0714$

$= .4711$

$= 47.11\%$

$= \underline{\underline{47\%}}$

12. The time required to assemble a piece of machinery is a random variable having approx. a normal distribution with $\mu = 12.9$ minutes and $\sigma = 2$ min. What are the probabilities that the assembly of a piece of machinery of this kind will take.

i) at least 11.5 minutes

ii) anywhere from 11.6 to 14.5 minutes

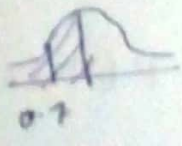
Given $\mu = 12.9$, $\sigma = 2$

$$z = \frac{X - 12.9}{2}$$

i) $P(X > 11.5) = P\left(z > \frac{11.5 - 12.9}{2}\right) = P(z > -0.7)$

$= 0.5 + 0.2580$

$= \underline{\underline{0.7580}}$



ii) $P(11.6 < X < 14.5) = P(-0.65 < X < 0.95)$

$= 0.1422 + 0.3289$

$= 1.5711$

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12/19

TUTORIAL

Q. If $f(x) = ke^{-x/\sigma}$ $\quad 0 \le x < \alpha$

$= 0$ $\quad$ elsewhere

is the PDF of a R.V. Show that the mean and S.D are equal to σ.

$$\int_0^\infty f(x)\,dx = 1$$

$$\int_0^\infty ke^{-x/\sigma} = 1$$

$$k\left.\frac{e^{-x/\sigma}}{-1/\sigma}\right]_0^\infty = 1$$

$$-\sigma k\,[0-1] = 1$$

$$\sigma k = 1$$

$$k = 1/\sigma$$

$$\text{mean} = \int_0^\infty x f(x)\,dx$$

$$= \int_0^\infty x k e^{-x/\sigma}\,dx$$

$$= k\int_0^\infty x e^{-x/\sigma}\,dx$$

$$= k\left[\left.x\frac{e^{-x/\sigma}}{-1/\sigma} - \frac{e^{-x}}{1/\sigma^2}\right]_0^\infty\right]$$

Substituting the value of k

$$= \frac{1}{\sigma}\left[\left.\frac{xe^{-x/\sigma}}{-1/\sigma} - \frac{e^{-x}}{1/\sigma^2}\right]_0^\infty\right]$$

$$= \frac{1}{\sigma}\left[0 - \frac{(-1)}{1/\sigma^2}\right]$$

$$= \frac{1}{\sigma} \times \frac{1}{1/\sigma^2}$$

$$= \underline{\underline{\sigma}}$$

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2. The CDF of C.R.V is given by

$\Rightarrow$

$F(x) = 0 \quad x \leq 2$

$= k(x-2) \quad 2 < x < 6$

$= 1 \quad x \geq 6$

6 i) Find the PDF $f(x)$

ii) Find k

iii) Find $P(3 \leq X \leq 5)$

$f(x) = \frac{d}{dx}(F)$

$= \frac{d}{dx}(0)$

$= 0$

* $f(x) = \frac{d}{dx}(k(x-2))$

$= k\left(\frac{d}{dx}(x-2)\right)$

$= k$

* $f(x) = \frac{d}{dx}(1)$

$= 0$

$\Rightarrow$

$f(x) = k \quad 2 < x < 6$

$= 0$ elsewhere

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i) Since it is PDF

$$\int_2^6 f(x)\,dx = 1$$

$$\int_2^6 k\,dx = 1$$

$$kx\Big]_2^6 = 1$$

$$6k - 2k = 1$$

$$4k = 1$$

$$k = 1/4$$

ii) $$\int_3^5 f(x)\,dx = \int_3^5 \frac{1}{4}\,dx$$

$$= \frac{x}{4}\Big]_3^5$$

$$= \frac{5}{4} - \frac{3}{4}$$

$$= \frac{2}{4}$$

$$= \frac{1}{2}$$

3. In a certain exam the % of passes and distinctions were 46 and 9 respectively. Estimate the avg. marks obtained by the candidates, the minimum pass and distinction marks being 40 &

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75 respectively. Assume normal distribution.

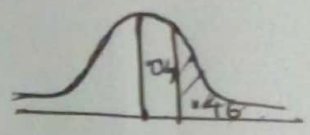
Let μ be the mean (avg marks)

Given

$$P(X \geq 40) = 0.46$$

$$z = \frac{X-\mu}{\sigma}$$

$$P\left(z \geq \frac{40-\mu}{\sigma}\right) = 0.46$$



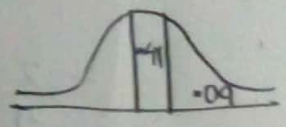
From table value corresponding to .04 is .1 $\quad z = \frac{40-\mu}{\sigma}$

$$\frac{40-\mu}{\sigma} = .1$$

$$\mu + .1\sigma = 40 \rightarrow (1)$$

$$P(X \geq 75) = .09$$

$$P\left(z \geq \frac{75-\mu}{\sigma}\right) = 0.09$$



$\frac{75-\mu}{\sigma}$

From table value corresponding .41

$$\frac{75-\mu}{\sigma} = 1.34$$

$$\mu + 1.34\sigma = 75 \rightarrow (2)$$

Solving (1) & (2)

$\mu = 87.17 \quad \sigma = 28.22$

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4. Let X be an exponential R.V with parameter λ, then show that $P(X > x+a / x > a) = P(X > x)$ [memoryless property of exponential distribution]

since X follows exponential distribution

$$f(x) = \lambda e^{-x\lambda} \quad ; x \geq 0$$

$$= 0 \quad \text{otherwise}$$

$$L.H.S = P(X > x+a | x > a)$$

$$= P(x+a \cap$$

$$= \frac{P(X > x+a \cap X > a)}{P(X > a)}$$

$$= \frac{\cancel{P(X > a)}}{\cancel{P(X > a)}} = \frac{P(X > x+a)}{P(X > a)}$$

$$= \frac{\int_{x+a}^{\infty} f(x)\,dx}{\int_{a}^{\infty} f(x)\,dx}$$

$$= \frac{\int_{x+a}^{\infty} \lambda e^{-x\lambda}\,dx}{\int_{a}^{\infty} \lambda e^{-x\lambda}\,dx} = \frac{\int_{x+a}^{\infty} e^{-x\lambda}\,dx}{\int_{a}^{\infty} e^{-x\lambda}\,dx} = \frac{\left.\frac{e^{-x\lambda}}{-\lambda}\right]_{x+a}^{\infty}}{\left.\frac{e^{-x\lambda}}{-\lambda}\right]_{a}^{\infty}}$$

$$= \frac{e^{-(x+a)\lambda}}{e^{-a\lambda}} = e^{-x\lambda} \rightarrow (1)$$

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$$R.H.S = P(X > x)$$

$$= \int_x^{\alpha} f(x)\,dx$$

$$= \int_x^{\alpha} \lambda e^{-\lambda x}\,dx$$

$$= \lambda \int_x^{\alpha} e^{-\lambda x}\,dx$$

$$= \lambda \left[\frac{e^{-\lambda x}}{-\lambda} \right]_x^{\alpha}$$

$$= -(0 - e^{-\lambda x})$$

$$= e^{-\lambda x} \rightarrow (2)$$

From (1) & (2) $\quad P(X > x+a \mid x > a) = P(X > x)$

5. A C.R.V X follows uniform distribution in the $(-k, k)$. Find k, if $P(X \geq 1) = 1/3$.

Since X follows uniform distribution $(-k, k)$

$$f(x) = \frac{1}{2k} \quad -k < x < k$$

$$= 0 \quad \text{otherwise}$$

$$P(X > 1) = 1/3$$

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$$\int_1^k f(x)\,dx = \frac{1}{3}$$

$$\int_1^k \frac{1}{2k}\,dx = \frac{1}{3}$$

$$\Rightarrow \frac{x}{2k}\Big]_1^k = \frac{k}{2k} - \frac{1}{2k}$$

$$\Rightarrow \quad = \frac{1}{2} - \frac{1}{2k}$$

$$= \frac{1}{3}$$

$$\frac{1}{2} - \frac{1}{2k} = \frac{1}{3}$$

$$\frac{1}{2} - \frac{1}{3} = \frac{1}{2k}$$

$$\frac{3-2}{6} = \frac{1}{2k}$$

$$\frac{1}{6} = \frac{1}{2k}$$

$$\underline{\underline{k = 3}}.$$

6 Diameter of a cylindrically shaped cable is a R.V X with P.D.F

$$f(x) = 6x(1-x) \quad 0 \le x \le 1$$

$$= 0 \quad \text{otherwise}$$

Find

1) the mean cable diameter.

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$$\text{mean} = E(x) = \int_0^1 x f(x)$$

$$= \int_0^1 x\, 6x(1-x)\, dx$$

$$= \int_0^1 6x^2 (1-x)\, dx$$

$$= \int_0^1 6x^3 - 6x^2\, dx$$

$$= -6 \int_0^1 x^3 - x^2\, dx$$

$$= -6 \left[\frac{x^4}{4} - \frac{x^3}{3} \right]_0^1 \Big]$$

$$= -6 \left[\frac{1}{4} - \frac{1}{3} \right]$$

$$= -6 \left[\frac{3-4}{12} \right]$$

$$= +\frac{1}{2}$$

7. Marks obtained in mathematics by 1000 students are normally distributed with mean 78% and S.D 11%.

Determine

i) How many students got marks above 90%

ii) What was the highest mark obtained by the lowest 10% of students.

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Given $\mu = .78$

$\sigma = .11$

i) $P(X > .90) = P\left(Z > \frac{.90 - .78}{.11}\right)$

$= P(Z > 1.$

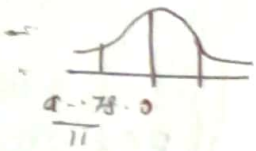
$= 0.5 - .3643$

$= \cancel{0.8643}$ $\underline{\underline{0.1357}}$

ii) Let a be the highest mark obtained by lowest 10% of the students

$P(X \leq a) = .1$

$P\left(Z \leq \frac{a - .78}{.11}\right) = .1$



$P\left(Z < \frac{a - .78}{.11}\right) = .1$

$a =$ ~~area~~ .11

value corresponding .4

$\frac{a - .78}{.11} = \cancel{0.1554}\ -1.28$

$a = -18.3$

8. Suppose a new machine is put into operation at time 0.

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If its life time is an exponential R.V with mean life 10 hrs

i) What is the probability that the machine will work continuously for one day.

ii) Suppose the machine has not failed by the end of 1st day what is the probability that it will work for the whole of next day.

mean $= \beta = 10$ hrs

$$f(x) = \frac{1}{10} e^{-x/10} \qquad x \geq 0$$

$$= 0 \qquad \text{otherwise}$$

i) $P(X > 24) = \frac{1}{10}\int_{24}^{\infty} e^{-x/10}\,dx$

$$= \frac{1}{10}\left[\frac{e^{-x/10}}{-1/10}\right]_{24}^{\infty}$$

$$= -\left[0 - e^{-24/10}\right]$$

$$= e^{-2.4}$$

$$= \underline{0.0907}$$

ii) Given that the machine has not failed for 1st day

$$P(X > 48 \mid X > 24) = \frac{P(X > 48 \cap X > 24)}{P(X > 24)}$$

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$$= \frac{P(X>48)}{P(X>24)}$$

$$= \frac{\frac{1}{10}\int_{48}^{\infty} e^{-x/10}\,dx}{e^{-2.4}}$$

$$= \frac{e^{-48/10}}{e^{-24}}$$

$$= \frac{e^{-4.8}}{e^{-2.4}}$$

$$= \underline{\underline{0.0907}}$$

9. X follows uniform dist with mean 1 and variance 4/3. If 3 independent observations are made. what is probability that all of them are negative.

Let $X \sim$ uniform dis. in the (α, β)

$$f(x) = \frac{1}{\beta - \alpha}$$

Given mean $= \frac{\alpha+\beta}{2} = 1$

$$\alpha + \beta = 2 \rightarrow (1)$$

$$\sigma = \frac{(\beta - \alpha)^2}{12} = 4/3$$

$$\alpha + \beta = 2$$

$$(\beta - \alpha)^2 = 16$$

$$\beta - \alpha = 4 \rightarrow (2)$$

Solving $\beta = 3$ and $\alpha = -1$

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$$P(x<0) = \int_{-1}^{0} f(x)\,dx$$

$$= \int_{-1}^{0} \frac{1}{4}\,dx$$

$$= \frac{x}{4}\Big]_{-1}^{0}$$

$$= \frac{+1}{4} \quad (-1, 3)$$

$n = 3$

$p =$ prob. that all are -ve $= \frac{1}{4} = .25$

$q = .75$

using B.D

$$f(x) = 3C_x (.25)^x (.75)^{3-x}$$

probability 3 of them are -ve

$$f(x) = 3C_3 (.25)^3$$

$$= \underline{0.015}$$

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