1/4/19

MODULE 3

Fourier Integrals and Transforms

Fourier Series

In $(-l, l)$ $f(x)$ can be expressed in the form,

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}$$

where $a_0 = \frac{1}{l}\int_{-l}^{l} f(t)\,dt$

$$a_n = \frac{1}{l}\int_{-l}^{l} f(t)\cos\frac{n\pi t}{l}\,dt$$

$$b_n = \frac{1}{l}\int_{-l}^{l} f(t)\sin\frac{n\pi t}{l}\,dt$$

$$\therefore f(x) = \frac{1}{2l}\int_{-l}^{l} f(t)\,dt + \sum_{n=1}^{\infty}\frac{1}{l}\int_{-l}^{l} f(t)\left[\cos\frac{n\pi t}{l}\cos\frac{n\pi x}{l} + \sin\frac{n\pi t}{l}\sin\frac{n\pi x}{l}\right]dt$$

Assume that $\int_{-l}^{l}|f(t)|\,dt$ is converging so that as $l \to \infty$,

$$\frac{1}{2l}\int_{-\infty}^{\infty} f(t)\,dt \to 0 \qquad \frac{\pi}{l} = \delta\omega$$

$$\lim_{\delta\omega \to 0} \sum_{n=1}^{\infty}\frac{1}{\pi}\left(\int_{-\infty}^{\infty} f(t)\cos n\,\delta\omega(t-x)\,dt\right)\delta\omega$$

$$f(x) = \frac{1}{\pi}\int_{0}^{\infty}\int_{-\infty}^{\infty} f(t)\cos\omega(t-x)\,dt\,d\omega \rightarrow \text{Fourier integral representation of } f(x).$$

$$f(x) = \frac{1}{\pi}\left[\int_{0}^{\infty}\int_{-\infty}^{\infty} f(t)\left(\cos\omega t\cos\omega x + \sin\omega t\sin\omega x\right)dt\,d\omega\right]$$

$$\boxed{f(x) = \frac{1}{\pi}\int_{0}^{\infty}\left[A(\omega)\cos\omega x + B(\omega)\sin\omega x\right]d\omega}$$

where $A(\omega) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(t)\cos\omega t\,dt \qquad B(\omega) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(t)\sin\omega t\,dt$

Case (1):- Suppose $f(x)$ is even

Scanned by CamScanner

Then $A(\omega) = \frac{2}{\pi}\int_0^\infty f(t)\cos\omega t\,dt$; $B(\omega) = 0$

$\therefore f(x) = \int_0^\infty A(\omega)\cos\omega x\,d\omega$ $\cancel{= \frac{2}{\pi}\int_0^\infty \int_0^\infty f(t)\cos}$ → Fourier cosine integral of $f(x)$

Case ②:- Suppose $f(x)$ is odd

Then $A(\omega) = 0$, $B(\omega) = \frac{2}{\pi}\int_0^\infty f(t)\sin\omega t\,dt$

$\therefore f(x) = \int_0^\infty B(\omega)\sin\omega x\,d\omega$ → Fourier sine integral of $f(x)$

① Represent $f(x)$ as a cosine integral

~~ans~~ $f(x) = \begin{cases} 1, & 0 < x, < 1 \\ 0, & x > 1 \end{cases}$

ans $f(x) = \int_0^\infty \left(A(\omega)\cos\omega x + B(\omega)\sin\omega x\right)d\omega$

where $A(\omega) = \frac{2}{\pi}\int_0^\infty f(t)\cos\omega t\,dt$ $\cancel{= \frac{1}{\pi}\int_0^\infty \frac{2}{\pi}\int_0^\infty}$

$= \frac{2}{\pi}\int_0^1 1\cdot\cos\omega t\,dt = \frac{2}{\pi\omega}[\sin\omega t]_0^1 = \frac{2\sin\omega}{\pi\omega}$

$B(\omega) = 0$

$\therefore f(x) = \int_0^\infty \frac{2\sin\omega\cos\omega x}{\pi\omega}\,d\omega$

② Find sine integral representatn of above function

ans $B(\omega) = \frac{2}{\pi}\int_0^\infty f(t)\sin\omega t\,dt = \frac{2}{\pi}\int_0^1 \sin\omega t\,dt = \frac{-2}{\pi\omega}[\cos\omega t]_0^1$

$= \frac{-2(\cos\omega - 1)}{\pi\omega} = \frac{2-2\cos\omega}{\pi\omega}$

$\therefore f(x) = \int_0^\infty \frac{(2-2\cos\omega)}{\pi\omega}\sin\omega x\,d\omega$

Scanned by CamScanner

2/4/19 ③ Find the Fourier sin integral of $f(x) = \begin{cases} x, & 0<x<a \\ 0, & x>a \end{cases}$

ans. ~~A(w)~~ $f(x) = \int_0^\infty [A(w)\cos wx + B(w)\sin wx]\,dw$

$A(w)=0$ $\quad B(w) = \frac{2}{\pi}\int_0^\infty f(t)\sin wt\,dt = \frac{2}{\pi}\int_0^a t\sin wt\,dt = \frac{2}{\pi}\left[-\frac{t}{w}\cos wt + \frac{\sin wt}{w^2}\right]_0^a$

$= \frac{2}{\pi}\left[-\frac{a\cos wa}{w} + \frac{\sin wa}{w^2}\right]$

$\therefore f(x) = \frac{2}{\pi}\int_0^\infty \left(\frac{\sin aw}{w^2} - \frac{a\cos aw}{w}\right)\sin wx\,dw$

④ S.T the integral represents the indicated function.

(i) $\int_0^\infty \frac{\cos xw + w\sin xw}{1+w^2}\,dw = \begin{cases} 0, & x<0 \\ \pi/2, & x=0 \\ \pi e^{-x}, & x>0 \end{cases}$

ans. $f(x) = \begin{cases} 0, & x<0 \\ \pi/2, & x=0 \\ \pi e^{-x}, & x>0 \end{cases} = RHS$

~~f(x)~~ In the LHS integral, there are both $\cos wx$ & $\sin wx$ terms. So it is in fourier integral form.

$f(x) = \int_0^\infty [A(w)\cos wx + B(w)\sin wx]\,dw$

$A(w) = \frac{1}{\pi}\int_{-\infty}^\infty f(t)\cos wt\,dt = \frac{1}{\pi}\int_{-\infty}^0 0 + \frac{1}{\pi}\int_0^\infty \pi e^{-t}\cos wt\,dt$

$= \frac{1}{\pi}\left[\frac{e^{-t}}{1+w^2}(-\cos wt + w\sin wt)\right]_0^\infty = \frac{1}{\pi}\left[-\frac{1}{1+w^2}(-1)\right] = \frac{1}{\pi(1+w^2)}$

$B(w) = \frac{1}{\pi}\int_{-\infty}^\infty f(t)\sin wt\,dt = \frac{1}{\pi}\int_0^\infty \pi e^{-t}\sin wt\,dt$

$= \frac{1}{\pi}\left[\frac{e^{-t}}{1+w^2}(-\sin wt - w\cos wt)\right]_0^\infty = \frac{w}{1+w^2}$

Scanned by CamScanner

$$\therefore f(x) = \int_0^\infty \frac{\cos\omega x + \omega \sin\omega x}{1+\omega^2}\, d\omega = \text{LHS}$$

(ii) $$\int_0^\infty \frac{1-\cos\pi\omega}{\omega} \sin x\omega\, d\omega = \begin{cases} \pi/2, & 0<x<\pi \\ 0, & x>\pi \end{cases}$$

ans. $$\text{RHS} = f(x) = \begin{cases} \pi/2, & 0<x<\pi \\ 0, & x>\pi \end{cases}$$

LHS is in sin integral form

$$f(x) = \int_0^\infty [A(\omega)\cos\omega x + B(\omega)\sin\omega x]\, d\omega$$

$A(\omega) = 0$

$$B(\omega) = \frac{2}{\pi}\int_0^\infty f(t)\sin\omega t\, dt = \frac{2}{\pi}\int_0^\pi \frac{\pi}{2}\sin\omega t\, dt = -\left[\frac{\cos\omega t}{\omega}\right]_0^\pi = \frac{1-\cos\pi\omega}{\omega}$$

$$\therefore f(x) = \int_0^\infty \frac{1-\cos\pi\omega}{\omega}\sin\omega x\, d\omega = \text{LHS}$$

(iii) $$\int_0^\infty \frac{\omega^3 \sin x\omega}{\omega^4+4}\, d\omega = \frac{1}{2}\pi e^{-x}\cos x, \quad x>0$$

ans. $$\text{RHS} = f(x) = \frac{\pi}{2} e^{-x}\cos x, \quad x>0$$

$$f(x) = \int_0^\infty [A(\omega)\cos\omega x + B(\omega)\sin\omega x]\, d\omega$$

LHS is in sin integral form

$A(\omega) = 0$

$$B(\omega) = \frac{2}{\pi}\int_0^\infty \frac{\pi}{2} e^{-t}\cos t \sin\omega t\, dt = \int_0^\infty e^{-t}\cos t \sin\omega t\, dt = \int_0^\infty [e^{-t}\sin(1+\omega)t - e^{-t}\sin(1-\omega)t]\, dt$$

$$= \int_0^\infty e^{-t}\sin(1+\omega)t\, dt - \int_0^\infty e^{-t}\sin(1-\omega)t\, dt = \frac{1}{2}[\mathrm{I} - \mathrm{II}]$$

$$\mathrm{I} = \left[\frac{e^{-t}}{1+(1+\omega)^2}\left(-\sin(1+\omega)t + (1+\omega)\cos(1+\omega)t\right)\right]_0^\infty$$

Scanned by CamScanner

$$= \frac{-(1+\omega)}{1+(1+\omega)^2}$$

$$II = \left[\frac{e^{-t}}{1+(1-\omega)^2} \left(-\sin(1-\omega)t + (1-\omega)\cos(1-\omega)t\right) \right]_0^\infty$$

$$= \frac{-\omega+1}{1+(1-\omega)^2}$$

$$I - II = \frac{-(1+\omega)}{1+(1+\omega)^2} - \frac{(1-\omega)}{1+(1-\omega)^2} = \frac{-(1+\omega)}{(\omega^2+2)+2\omega} - \frac{(1-\omega)}{(\omega^2+2)-2\omega}$$

$$= \frac{\cancel{\omega^2+2-2\omega} - (1-\omega)(\omega^2+2\omega+2) - (1+\omega)(\omega^2+2-2\omega)}{(\omega^2+2)^2 - 4\omega^2}$$

$$= \frac{\cancel{\omega^2+2-2\omega - (\omega^2+2\omega+2-\omega^3-2\omega^2-2\omega)}}{\omega^4+4\omega^2+4-4\omega^2}$$

$$= \frac{\cancel{\omega^2-2\omega-2+\omega^3+2\omega^2+2\omega-\omega^2-2+2\omega+\omega^3-2\omega+2\omega^2}}{\omega^4+4}$$

$$= \frac{2\omega^3}{\omega^4+4}$$

$$\therefore f(x) = \int_0^\infty \frac{\omega^3 \sin x\omega}{\omega^4+4}\, d\omega = LHS$$

(5) Find F.C.I representatn of $f(x) = \left\{ \frac{1}{1+x^2}, x>0 \right.$

ans $A(\omega) = \frac{2}{\pi}\int_0^\infty \frac{\cos\omega t\, dt}{1+t^2} = \cancel{\frac{2}{\pi}\left[\frac{\sin\omega t}{\omega(1+t^2)}\right]} = \cancel{\frac{2}{\pi}\left[\frac{\sin\omega t \tan^{-1}t}{1+t^2} + \int_0^\infty \sin\omega t \tan^{-1}t\, dt\right]}$

$$\cancel{= \frac{2}{\pi}\left[\frac{\sin\omega t \tan^{-1}t}{1+t^2} + \omega\left\{\frac{\tan^{-1}t \cos\omega t}{\omega} + \frac{1}{\omega}\int \frac{\cos\omega t\, dt}{1+t^2}\right\}\right]_0^\infty}$$

$$\cancel{A(\omega) = \frac{2}{\pi}\left[\cos\omega t \tan^{-1}t - \cos\omega t \tan^{-1}t\right] + \frac{\pi}{2\omega}A(\omega)}$$

$$\cancel{A(\omega) - \frac{\pi}{2\omega}A(\omega) = 0 = \frac{2}{\pi}}$$

Scanned by CamScanner

$$\cancel{= \frac{2}{\pi}\left[\frac{\sin\omega t}{1+t^2} - \tan^{-1}t\cos\omega t\right]_0^\infty + \frac{\pi}{2}A(\omega)}$$

$$\cancel{\therefore A(\omega)\left(1-\frac{\pi}{2}\right) = \frac{2}{\pi}\left[\right.}$$

(6) Express $f(x) = \begin{cases} 1, & |x| \le 1 \\ 0, & |x| > 1 \end{cases}$ as a F.I. Hence evaluate

$$\int_0^\infty \frac{\sin\omega}{\omega}\cos\omega x\, d\omega$$

ans. $$f(x) = \int_0^\infty [A(\omega)\cos\omega x + B(\omega)\sin\omega x]\, d\omega$$

$$A(\omega) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(t)\cos\omega t\, dt = \frac{1}{\pi}\int_{-1}^{1} 1\cos\omega t\, dt = \frac{1}{\pi\omega}[\sin\omega t]_{-1}^{1} = \frac{2\sin\omega}{\pi\omega}$$

$$B(\omega) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(t)\sin\omega t\, dt = \frac{1}{\pi}\int_{-1}^{1}\sin\omega t\, dt = \frac{-1}{\pi\omega}[\cos\omega t]_{-1}^{1} = 0$$

$$\therefore f(x) = \int_0^\infty \frac{2\sin\omega}{\pi\omega}\cos\omega x\, d\omega$$

$$\therefore \int_0^\infty \frac{\sin\omega}{\omega}\cos\omega x\, d\omega = \frac{\pi}{2}f(x)$$

(7) Find F.S.I of $f(\theta) = \begin{cases} \sin\theta, & 0 \le \theta \le \pi \\ 0, & \theta > \pi \end{cases}$

ans. $A(\omega) = 0$

$$B(\omega) = \frac{2}{\pi}\int_0^\infty f(t)\sin\omega t\, dt = \frac{2}{\pi}\int_0^\pi \sin t\sin\omega t\, dt = \frac{1}{\pi}\int_0^\pi [\cos(1-\omega)t - \cos(1+\omega)t]\, dt$$

$$= \frac{1}{\pi}\int_0^\pi \cos(1-\omega)t\, dt - \frac{1}{\pi}\int_0^\pi \cos(1+\omega)t\, dt = \frac{1}{\pi}\left[\frac{\sin(1-\omega)t}{1-\omega}\right]_0^\pi - \frac{1}{\pi}\left[\frac{\sin(1+\omega)t}{1+\omega}\right]_0^\pi$$

$$\cancel{= \frac{1}{\pi}[0] - \frac{1}{\pi}[0]} = \frac{\sin(1-\omega)\pi}{\pi(1-\omega)} - \frac{\sin(1+\omega)\pi}{\pi(1+\omega)} = \frac{(1+\omega)\sin(1-\omega)\pi - (1-\omega)\sin(1+\omega)\pi}{\pi(1-\omega^2)}$$

$$\cancel{\therefore f(\theta) = \int_0^\infty B(\omega)\sin\omega x\, d\omega} = \frac{(1+\omega)\sin(\pi-\pi\omega) - (1-\omega)\sin(\pi+\pi\omega)}{\pi(1-\omega^2)}$$

Scanned by CamScanner

$$= \frac{(1+\omega)\sin\pi\omega + (1-\omega)\sin\pi\omega}{\pi(1-\omega^2)} = \frac{2\sin\pi\omega}{\pi(1-\omega^2)}$$

$$\therefore f(\theta) = \int_0^\infty \frac{2\sin\pi\omega}{\pi(1-\omega^2)} \sin\omega\theta \, d\omega$$

(8) Solve $\int_0^\infty f(x)\cos\alpha x\,dx = e^{-\alpha}$

ans. $\int_0^\infty f(x)\cos\omega x\,dx = \frac{\pi}{2}A(\omega) = e^{-\omega}$

$$\Rightarrow A(\omega) = \frac{2}{\pi}\int_0^\infty f(x)\cos\omega x\,dx = \frac{2}{\pi}e^{-\omega}$$

$\cancel{f(x)} \Rightarrow B(\omega) = 0$

$$\therefore f(x) = \int_0^\infty \frac{2}{\pi}e^{-\omega}\cos\omega x\,d\omega = \text{F.C.I of } f(x)$$

$$= \frac{2}{\pi}\int_0^\infty e^{-\omega}\cos\omega x\,d\omega = \cancel{\frac{2}{\pi}\left[\cos\omega x\, e^{-\omega} + \right.}$$

$$= \frac{2}{\pi}\left[\frac{e^{-\omega}}{1+x^2}(-\cos\omega x + x\sin\omega x)\right]_{\omega=0}^{\infty}$$

$$= \frac{2}{\pi}\left[\frac{1}{x^2+1}\right] = \frac{2}{\pi(x^2+1)}$$

(9) Solve $\int_0^\infty f(x)\sin tx\,dx = \begin{cases} 1, & 0 \le t \le 1 \\ 2, & 1 \le t < 2 \\ 0, & t \ge 2 \end{cases}$

ans. $\int_0^\infty f(x)\sin tx = \begin{cases} 1, & 0 \le t < 1 \\ 2, & 1 \le t < 2 \\ 0, & t \ge 2 \end{cases} = B(t)\frac{\pi}{2}$

$$\therefore f(x) = \frac{2}{\pi}\int_0^\infty B(t)\sin tx\,dt = \frac{2}{\pi}\int_0^1 \sin tx\,dt + \frac{2}{\pi}\int_1^2 2\sin tx\,dt$$

$$= \frac{-2}{\pi x}\left[\cos tx\right]_0^1 - \frac{4}{\pi x}\left[\cos tx\right]_1^2 = \frac{-2}{\pi x}\left[\cos x - 1\right] - \frac{4}{\pi x}\left[\cos 2x - \cos x\right]$$

$$= \frac{2}{\pi x} - \frac{2}{\pi x}\cos x + \frac{4}{\pi x}\cos x - \frac{4}{\pi x}\cos 2x = \frac{2}{\pi}\left[\frac{1+\cos x - 2\cos 2x}{x}\right]$$

Scanned by CamScanner

4/4/19 ans (5)

$$f(x) = \frac{1}{1+x^2}, \quad x>0$$

$$f(x) = \int_0^\infty A(\omega)\cos\omega x\, d\omega$$

$$A(\omega) = \frac{2}{\pi}\int_0^\infty \frac{1}{1+t^2}\cos\omega t\, dt$$

$$\frac{d}{d\omega}A(\omega) = \frac{-2}{\pi}\int_0^\infty \frac{t\sin\omega t}{1+t^2}dt = \frac{-2}{\pi}\int_0^\infty \frac{t^2}{t(1+t^2)}\sin\omega t\, dt$$

$$= \frac{-2}{\pi}\int_0^\infty \left[\frac{1}{t} - \frac{1}{t(1+t^2)}\right]\sin\omega t\, dt$$

$$\frac{d^2}{d\omega^2}A(\omega) = \frac{-2}{\pi}\int_0^\infty \frac{t\cos\omega t}{t}dt + \frac{2}{\pi}\int_0^\infty \frac{t\cos\omega t\, dt}{t(1+t^2)} = \frac{-2}{\pi}\int_0^\infty \frac{t\cos\omega t}{t}dt + A(\omega)$$

$$A''(\omega) - A(\omega) = \frac{-2}{\pi}\int_0^\infty \cos\omega t\, dt = \frac{-2}{\pi\omega}\left[\sin\omega t\right]_0^\infty =$$

Fourier Transforms

Fourier integral representatn of $f(x)$

$$f(x) = \frac{1}{\pi}\int_{\omega=0}^{\infty}\int_{t=\infty}^{\infty} f(t)\cos\omega(x-t)\,dt\,d\omega$$

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(t)\cos\omega(x-t)\,dt\,d\omega \rightarrow (1)$$

Consider

$$0 = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(t)\sin\omega(x-t)\,dt\,d\omega \rightarrow (2)$$

$$(1) + i(2) \Rightarrow f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(t)\, e^{i\omega(x-t)}\,dt\,d\omega$$

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(t)\, e^{-i\omega t} e^{i\omega x}\,dt\,d\omega = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} F(\omega)\, e^{i\omega x}\,d\omega$$

where $\boxed{F(\omega) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(t)\, e^{-i\omega t}\,dt}$ $\rightarrow$ Fourier transform of $f(x)$

Scanned by CamScanner

Inverse fourier transform

$$\mathcal{F}^{-1}\{F(\omega)\} = f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(\omega) e^{i\omega x} d\omega$$

Fourier transform of $f(x)$

$$\mathcal{F}\{f(x)\} = F(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-i\omega x} dx$$

① Find the fourier transform of

$$f(x) = \begin{cases} e^{2ix}, & -1 < x < 1 \\ 0, & \text{otherwise} \end{cases}$$

ans. $$\mathcal{F}\{f(x)\} = F(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-i\omega x} dx = \frac{1}{\sqrt{2\pi}} \int_{-1}^{1} e^{2ix} \cdot e^{-i\omega x} dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-1}^{1} e^{ix(2-\omega)} dx = \frac{1}{\sqrt{2\pi}\,(i(2-\omega))} \left[e^{ix(2-\omega)}\right]_{-1}^{1}$$

$$= \frac{1}{i\sqrt{2\pi}\,(2-\omega)} \left(e^{i(2-\omega)} - e^{-i(2-\omega)}\right)$$

$$= \frac{1}{i\sqrt{2\pi}\,(2-\omega)} \left[\cancel{\cos(2-\omega)} + i\sin(2-\omega) - \cancel{\cos(2-\omega)} + i\sin(2-\omega)\right]$$

$$= \frac{2i\sin(2-\omega)}{i\sqrt{2\pi}\,(2-\omega)} = \frac{\sqrt{2}\sin(2-\omega)}{\sqrt{\pi}\,(2-\omega)}$$

② Find the fourier transform of the following:

(i) $$f(x) = \begin{cases} |x|, & -1 < x < 1 \\ 0, & \text{otherwise} \end{cases}$$

(ii) $$f(x) = \begin{cases} e^{x}, & -a < x < a \\ 0, & \text{otherwise} \end{cases}$$

(iii) $f(x) = e^{-|x|}$ $(-\infty < x < \infty)$

(iv) $$f(x) = \begin{cases} xe^{-x}, & -1 < x < 0 \\ 0, & \text{otherwise} \end{cases}$$

Scanned by CamScanner

ans. (i) $f(x) = \begin{cases} -x, & -1<x<0 \\ x, & 0<x<1 \\ 0, & \text{otherwise} \end{cases}$

$$F\{f(x)\} = F(\omega) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{-i\omega x}dx = \frac{1}{\sqrt{2\pi}}\left[\int_{-1}^{0} -xe^{-i\omega x}dx + \int_{0}^{1} xe^{-i\omega x}dx\right]$$

$$= \frac{1}{\sqrt{2\pi}}\left\{\left[\frac{xe^{-i\omega x}}{i\omega} - \frac{e^{-i\omega x}}{\omega^2}\right]_{-1}^{0} + \left[\frac{xe^{-i\omega x}}{-i\omega} + \frac{e^{-i\omega x}}{\omega^2}\right]_{0}^{1}\right\}$$

$$= \frac{1}{\sqrt{2\pi}}\left\{0 - \frac{1}{\omega^2} + \frac{e^{i\omega}}{i\omega} + \frac{e^{i\omega}}{\omega^2} + \frac{e^{-i\omega}}{-i\omega} + \frac{e^{-i\omega}}{\omega^2} - 0 - \frac{1}{\omega^2}\right\}$$

$$= \frac{1}{\sqrt{2\pi}}\left\{\frac{e^{i\omega} - e^{-i\omega}}{i\omega} + \frac{e^{i\omega} + e^{-i\omega}}{\omega^2} - \frac{2}{\omega^2}\right\}$$

$$= \frac{1}{\sqrt{2\pi}}\left[\frac{e^{i\omega} - e^{-i\omega}}{i\omega} + \frac{e^{i\omega} + e^{-i\omega} - 2}{\omega^2}\right]$$

$$= \frac{1}{\sqrt{2\pi}}\left[\frac{2i\sin\omega}{i\omega} + \frac{2\cos\omega - 2}{\omega^2}\right]$$

$$= \sqrt{\frac{2}{\pi}}\left[\frac{\omega\sin\omega + \cos\omega - 1}{\omega^2}\right]$$

(ii)

$$F\{f(x)\} = F(\omega) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{-i\omega x}dx = \frac{1}{\sqrt{2\pi}}\int_{-a}^{a} e^{x}\cdot e^{-i\omega x}dx$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-a}^{a} e^{x(1-i\omega)}dx = \frac{1}{\sqrt{2\pi}(1-i\omega)}\left[e^{x(1-i\omega)}\right]_{-a}^{a}$$

$$= \frac{e^{a(1-i\omega)} - e^{-a(1-i\omega)}}{\sqrt{2\pi}(1-i\omega)}$$

(iii) $f(x) = \begin{cases} e^{x}, & -\infty<x<0 \\ e^{-x}, & 0<x<\infty \end{cases}$

$$F\{f(x)\} = F(\omega) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{-i\omega x}dx = \frac{1}{\sqrt{2\pi}}\left[\int_{-\infty}^{0} e^{x}\cdot e^{-i\omega x}dx + \int_{0}^{\infty} e^{-x}\cdot e^{-i\omega x}dx\right]$$

$$= \frac{1}{\sqrt{2\pi}}\left[\int_{-\infty}^{0} e^{x(1-i\omega)}dx + \int_{0}^{\infty} e^{-x(1+i\omega)}dx\right]$$

Scanned by CamScanner

$$= \frac{1}{\sqrt{2\pi}} \left\{ \left[\frac{e^{x(1-i\omega)}}{(1-i\omega)} \right]_{-\infty}^{0} + \left[\frac{e^{-x(1+i\omega)}}{-(1+i\omega)} \right]_{0}^{\infty} \right\}$$

$$= \frac{1}{\sqrt{2\pi}} \left\{ \frac{1}{(1-i\omega)} + \frac{1}{(1+i\omega)} \right\} = \frac{1}{\sqrt{2\pi}} \left\{ \frac{1+i\omega+1-i\omega}{(1+\omega^2)} \right\} = \sqrt{\frac{2}{\pi}} \frac{1}{(1+\omega^2)}$$

(iv) $$F\{f(x)\} = F(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-i\omega x} dx = \frac{1}{\sqrt{2\pi}} \int_{-1}^{0} x e^{-x} \cdot e^{-i\omega x} dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-1}^{0} x e^{-x(1+i\omega)} dx = \frac{1}{\sqrt{2\pi}} \left[\frac{x e^{-x(1+i\omega)}}{-(1+i\omega)} - \frac{e^{-x(1+i\omega)}}{(1+i\omega)^2} \right]_{-1}^{0}$$

$$= \frac{1}{\sqrt{2\pi}} \left[0 - \frac{1}{(1+i\omega)^2} - \frac{e^{(1+i\omega)}}{(1+i\omega)} + \frac{e^{(1+i\omega)}}{(1+i\omega)^2} \right]$$

$$= \frac{1}{\sqrt{2\pi}} \left[\cancel{\cos(1+i} \; \frac{e^{(1+i\omega)} - 1}{(1+i\omega)^2} - \frac{e^{(1+i\omega)}}{(1+i\omega)} \right]$$

$$= \frac{1}{\sqrt{2\pi}} \left[\frac{e^{(1+i\omega)} - 1 - (1+i\omega) e^{(1+i\omega)}}{(1+i\omega)^2} \right]$$

5/4/19

Fourier Sine Transform & Cosine Transform

The fourier sine transform of $f(x)$

$$F_s\{f(x)\} = F_s(\omega) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} f(x) \sin \omega x \, dx$$

$$F_s^{-1}\{F_s(\omega)\} = f(x) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} F_s(\omega) \sin \omega x \, d\omega$$

Similarly, fourier cosine transform of $f(x)$

$$F_c\{f(x)\} = F_c(\omega) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} f(x) \cos \omega x \, dx$$

$$F_c^{-1}\{F_c(\omega)\} = f(x) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} F_c(\omega) \cos \omega x \, d\omega$$

① Find the fourier cosine transform of $f(x) = \begin{cases} x & \text{if } 0 < x < 2 \\ 0, & x > 2 \end{cases}$

Ans.

$$F_c\{f(x)\} = F_c(\omega) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} f(x) \cos \omega x \, dx$$

Scanned by CamScanner

$$= \sqrt{\frac{2}{\pi}} \int_0^2 x\cos\omega x\,dx = \sqrt{\frac{2}{\pi}} \left[\frac{x\sin\omega x}{\omega} + \frac{\cos\omega x}{\omega^2}\right]_0^2$$

$$= \sqrt{\frac{2}{\pi}} \left[\frac{2\sin 2\omega}{\omega} + \frac{\cos 2\omega}{\omega^2} - \frac{1}{\omega^2}\right]$$

$$= \sqrt{\frac{2}{\pi}} \left[\frac{2\omega\sin 2\omega + \cos 2\omega - 1}{\omega^2}\right]$$

② Find fourier sine transform of $e^{-ax}, a>0$

ans. $F_s\{f(x)\} = F_s(\omega) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x)\sin\omega x\,dx = \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-ax}\sin\omega x\,dx$

$$= \sqrt{\frac{2}{\pi}} \left[\cancel{\sin\omega x\left(\frac{e^{-ax}}{-a}\right) - \omega} \frac{e^{-ax}}{a^2+\omega^2}(-a\sin\omega x - \omega\cos\omega x)\right]_0^\infty$$

$$= \sqrt{\frac{2}{\pi}} \left[\frac{\omega}{a^2+\omega^2}\right]$$

③ Find F.S.T of $f(x) = 1/x, x>0$

ans. $F_s\{f(x)\} = F_s(\omega) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x)\sin\omega x\,dx = \sqrt{\frac{2}{\pi}} \int_0^\infty \frac{\sin\omega x}{x}\,dx$

Put $\omega x = t \Rightarrow x = t/\omega$

$\omega\,dx = dt \Rightarrow dx = dt/\omega$

When $x=0, t=0$

When $x=\infty, t=\infty$

$$\boxed{\int_0^\infty \frac{\sin t}{t}\,dt = \frac{\pi}{2}}$$

$$F_s(\omega) = \sqrt{\frac{2}{\pi}} \int_{t=0}^\infty \frac{\cancel{\omega}}{t}\sin t \frac{dt}{\cancel{\omega}} = \sqrt{\frac{2}{\pi}} \int_{t=0}^\infty \frac{\sin t}{t}\,dt = \sqrt{\frac{2}{\pi}} \times \frac{\pi}{2} = \sqrt{\frac{\pi}{2}}$$

④ Find $f(x)$ if $\int_0^\infty f(x)\sin\omega x\,dx = \begin{cases} 1, & 0\le\omega\le 1 \\ 2, & 1\le\omega\le 2 \\ 0, & \omega\ge 2 \end{cases}$

ans $F_s\{f(x)\} = F_s(\omega) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x)\sin\omega x\,dx$

$$\therefore \int_0^\infty f(x)\sin\omega x\,dx = \sqrt{\frac{\pi}{2}} F_s(\omega) = \begin{cases} 1, & 0\le\omega\le 1 \\ 2, & 1\le\omega\le 2 \\ 0, & \omega\ge 2 \end{cases} \Rightarrow F_s(\omega) = \begin{cases} \sqrt{2/\pi}, & 0\le\omega\le 1 \\ 2\sqrt{2/\pi}, & 1\le\omega\le 2 \\ 0, & \omega\ge 2 \end{cases}$$

Scanned by CamScanner

Then $f(x) = F\{F_s(\omega)\} = \sqrt{\frac{2}{\pi}}\int_0^\infty \sqrt{\frac{2}{\pi}} F_s(\omega)\sin\omega x\,d\omega$

$$= \sqrt{\frac{2}{\pi}}\left[\int_0^1 \sqrt{\frac{2}{\pi}}\sin\omega x\,d\omega + \int_1^2 2\sqrt{\frac{2}{\pi}}\sin\omega x\,d\omega\right]$$

$$= \frac{2}{\pi}\left[\int_0^1 \sin\omega x\,d\omega + 2\int_1^2 \sin\omega x\,d\omega\right] = \frac{2}{\pi}\left\{\left[\frac{-\cos\omega x}{x}\right]_0^1 - 2\left[\frac{\cos\omega x}{x}\right]_1^2\right\}$$

$$= \frac{2}{\pi}\left\{-\frac{\cos x}{x} + \frac{1}{x} - \frac{2\cos 2x}{x} + \frac{2\cos x}{x}\right\} = \frac{2}{\pi}\left\{\frac{1-\cos 2x}{x}\right\}$$

$$= \frac{2}{\pi x} \times 2\sin^2 x = \frac{4\sin^2 x}{\pi x} = \frac{2}{\pi}\left\{\frac{\cos x}{x} + \frac{1-2\cos 2x}{x}\right\}$$

$$= \frac{2}{\pi}\left\{\frac{1-2\cos 2x+\cos x}{x}\right\}$$

⑤ Find the F.C.T of

(i) $2e^{-3x} + 3e^{-2x}$

(ii) $f(x) = \begin{cases} \cos x, & 0<x<1 \\ 0, & x \geq 1 \end{cases}$

ans. (i) $F_c\{f(x)\} = F_c(\omega) = \sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\cos\omega x\,dx = \sqrt{\frac{2}{\pi}}\int_0^\infty (2e^{-3x}+3e^{-2x})\cos\omega x\,dx$

$$= 2\sqrt{\frac{2}{\pi}}\int_0^\infty e^{-3x}\cos\omega x\,dx + 3\sqrt{\frac{2}{\pi}}\int_0^\infty e^{-2x}\cos\omega x\,dx$$

$$= 2\sqrt{\frac{2}{\pi}}\left[\frac{e^{-3x}}{9+\omega^2}(-3\cos\omega x + \omega\sin\omega x)\right]_0^\infty + 3\sqrt{\frac{2}{\pi}}\left[\frac{e^{-2x}}{4+\omega^2}(-2\cos\omega x + \omega\sin\omega x)\right]_0^\infty$$

$$= 2\sqrt{\frac{2}{\pi}}\left[\frac{3}{9+\omega^2}\right] + 3\sqrt{\frac{2}{\pi}}\left[\frac{2}{4+\omega^2}\right] = 6\sqrt{\frac{2}{\pi}}\left[\frac{3}{9+\omega^2} + \frac{2}{4+\omega^2}\right]$$

$$= 6\sqrt{\frac{2}{\pi}}\left[\frac{5\omega^2+30}{\omega^4+13\omega^2+36}\right]$$

(ii) $F_c\{f(x)\} = F_c(\omega) = \sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\cos\omega x\,dx = \sqrt{\frac{2}{\pi}}\int_0^1 \cos x\cos\omega x\,dx$

$$= \sqrt{\frac{2}{\pi}} \times \frac{1}{2}\int_0^1 [\cos(1+\omega)x + \cos(1-\omega)x]\,dx$$

Scanned by CamScanner

$$= \frac{1}{\sqrt{2\pi}}\left[\int_0^1 \cos(1+\omega)x\,dx + \int_0^1 \cos(1-\omega)x\,dx\right]$$

$$= \frac{1}{\sqrt{2\pi}}\left\{\left[\frac{\sin(1+\omega)x}{1+\omega}\right]_0^1 + \left[\frac{\sin(1-\omega)x}{1-\omega}\right]_0^1\right\}$$

$$= \frac{1}{\sqrt{2\pi}}\left\{\frac{\sin(1+\omega)}{1+\omega} + \frac{\sin(1-\omega)}{1-\omega}\right\}$$

6/4/19

⑥ Find the F.S.T of $\frac{e^{-ax}}{x}$

ans: $F_s\{f(x)\} = F_s(\omega) = \sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\sin\omega x\,dx = \sqrt{\frac{2}{\pi}}\int_0^\infty \frac{e^{-ax}\sin\omega x}{x}\,dx \rightarrow ①$

$$\frac{d}{d\omega}F_s(\omega) = \sqrt{\frac{2}{\pi}}\int_0^\infty \frac{d}{d\omega}\left(\frac{e^{-ax}}{x}\sin\omega x\right)dx = \sqrt{\frac{2}{\pi}}\int_0^\infty \frac{e^{-ax}}{x}x\cos\omega x\,dx$$

$$= \sqrt{\frac{2}{\pi}}\left[\frac{e^{-ax}}{a^2+\omega^2}(-a\cos\omega x + \omega\sin\omega x)\right]_0^\infty$$

$$= \sqrt{\frac{2}{\pi}}\left[\frac{a}{a^2+\omega^2}\right]$$

$$\therefore F_s(\omega) = \sqrt{\frac{2}{\pi}}\int \frac{a}{a^2+\omega^2}\,d\omega = \sqrt{\frac{2}{\pi}}\left(\tan^{-1}\left(\frac{\omega}{a}\right) + c\right)$$

Put $\omega = 0 \Rightarrow F_s(0) = \sqrt{\frac{2}{\pi}}(0+c)$

But from ①, $F_s(0) = 0$

$\therefore \underline{c = 0}$

$$\therefore F_s(\omega) = \sqrt{\frac{2}{\pi}}\tan^{-1}\left(\frac{\omega}{a}\right)$$

⑦ Find the F.C.T of $e^{-x^2}, x>0$

ans. $F_c\{f(x)\} = F_c(\omega) = \sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\cos\omega x\,dx = \sqrt{\frac{2}{\pi}}\int_0^\infty e^{-x^2}\cos\omega x\,dx$

$$\frac{d}{d\omega}F_c(\omega) = \sqrt{\frac{2}{\pi}}\int_0^\infty \underbrace{-xe^{-x^2}}_{2nd}\underbrace{\sin\omega x}_{1st}\,dx$$

Put $-x^2 = t$

$-2x\,dx = dt$

$-x\,dx = \frac{dt}{2}$

$$\boxed{\int_0^\infty e^{-x^2}dx = \frac{\sqrt{\pi}}{2}}$$

$$\int -xe^{-x^2}dx = \frac{1}{2}\int e^t dt = \frac{1}{2}e^t = \frac{1}{2}e^{-x^2}$$

Scanned by CamScanner

$$= \sqrt{\frac{2}{\pi}}\left[\sin\omega x\,\frac{e^{-x^2}}{2} - \frac{\omega}{2}\int e^{-x^2}\cos\omega x\,dx\right]_0^\infty = -\sqrt{\frac{2}{\pi}}\int_0^\infty \frac{\omega}{2}e^{-x^2}\cos\omega x\,dx$$

$$= -\sqrt{\frac{2}{\pi}} \times \frac{\omega}{2}\int_0^\infty e^{-x^2}\cos\omega x\,dx = -\frac{\omega}{2}F_c(\omega)$$

$\frac{d}{d\omega}F_c(\omega) + \frac{\omega}{2}F_c(\omega) = 0$ is a 1st order v.s diff. eq in $F_c(\omega)$

$$\frac{dF_c(\omega)}{F_c(\omega)} = -\frac{\omega}{2}d\omega$$

Integrating both sides,

$$\log F_c(\omega) = -\frac{\omega^2}{4} + \log c$$

$$\frac{F_c(\omega)}{c} = e^{-\frac{\omega^2}{4}} \Rightarrow F_c(\omega) = ce^{-\omega^2/4}$$

Put $\omega = 0 \Rightarrow F_c(0) = c$

But $F_c(0) = \sqrt{\frac{2}{\pi}}\int_0^\infty e^{-x^2}dx = \sqrt{\frac{2}{\pi}} \times \frac{\sqrt{\pi}}{2} = \frac{1}{\sqrt{2}}$

$$\therefore F_c(\omega) = \frac{1}{\sqrt{2}}e^{-\omega^2/4}$$

Properties of Fourier Transform

$$\mathcal{F}\{f(x)\} = F(\omega) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{-i\omega x}dx$$

$$\mathcal{F}^{-1}\{F(\omega)\} = f(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} F(\omega)e^{-i\omega x}d\omega$$

(i) Linearity Property

$$\mathcal{F}\{af(x) + bg(x)\} = a\mathcal{F}\{f(x)\} + b\mathcal{F}\{g(x)\}$$

Proof:- $\mathcal{F}\{af(x)+bg(x)\} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}(af(x)+bg(x))e^{-i\omega x}dx$

$$= \frac{a}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{-i\omega x}dx + \frac{b}{\sqrt{2\pi}}\int_{-\infty}^{\infty} g(x)e^{-i\omega x}dx = a\mathcal{F}\{f(x)\} + b\mathcal{F}\{g(x)\} = \text{RHS}$$

(ii) Change of Scale Property

$\mathcal{F}\{f(ax)\} = \frac{1}{a}F\left(\frac{\omega}{a}\right)$ where $F(\omega) = \mathcal{F}\{f(x)\}$

Scanned by CamScanner

Proof:- $F\{f(ax)\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(ax) e^{-i\omega x} dx$ Put $ax = t \Rightarrow x = t/a$

$a\,dx = dt \Rightarrow dx = \frac{dt}{a}$

$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) e^{-\frac{i\omega t}{a}} \frac{dt}{a} = \frac{1}{a} \times \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) e^{-\frac{i\omega t}{a}} dt = \frac{1}{a} F\left(\frac{\omega}{a}\right) = RHS$

(iii) Shifting Property

$F\{f(x-a)\} = F(\omega) e^{-i\omega a}$ where $F(\omega) = F\{f(x)\}$

Proof:- $F\{f(x-a)\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x-a) e^{-i\omega x} dx$ Put $x-a = t \Rightarrow x = t+a$

$dx = dt$

$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) e^{-i\omega(t+a)} dt = e^{-i\omega a} \times \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) e^{-i\omega t} dt = e^{-i\omega a} F(\omega) = RHS$

$F^{-1}\{e^{-i\omega a} F(\omega)\} = f(x-a)$ where $f(x) = F^{-1}\{F(\omega)\}$

(iv) Frequency Shifting Property

$F\{e^{iax} f(x)\} = F(\omega - a)$ where $F(\omega) = F\{f(x)\}$

Proof:- $F\{e^{iax} f(x)\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{iax} f(x) e^{-i\omega x} dx = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) \cdot e^{-i(\omega-a)x} dx$

$= F(\omega - a) = RHS$

$\therefore F^{-1}\{F(\omega - a)\} = e^{iax} f(x)$ where $f(x) = F^{-1}\{F(\omega)\}$

Scanned by CamScanner

8/4/19

Tutorial - Module 3 & 4

① Find the cos transform of $\frac{e^{-ax}}{x}$

ans - $F_c\{f(x)\} = F_c(\omega) = \sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\cos\omega x\,dx = \sqrt{\frac{2}{\pi}}\int_0^\infty \frac{e^{-ax}}{x}\cos\omega x\,dx$

$\frac{d}{d\omega}F_c(\omega) = -\sqrt{\frac{2}{\pi}}\int_0^\infty e^{-ax}\sin\omega x\,dx = -\sqrt{\frac{2}{\pi}}\left[\frac{e^{-ax}}{a^2+\omega^2}(-a\sin\omega x - \omega\cos\omega x)\right]_0^\infty$

$= -\sqrt{\frac{2}{\pi}}\left[\frac{\omega}{a^2+\omega^2}\right]$

$\therefore F_c(\omega) = -\sqrt{2/\pi}\int\frac{\omega}{a^2+\omega^2}d\omega = -\sqrt{\frac{2}{\pi}}\times\frac{1}{2}\int\frac{dt}{t} = -\frac{1}{\sqrt{2\pi}}\log t = \frac{-1}{\sqrt{2\pi}}\log(a^2+\omega^2) + \log c$

~~Put $\omega=0 \Rightarrow F_c(0) =$~~ ~~Put $\omega = \pi/2$~~

② Find the F.T of $f(x) = \begin{cases} 1-x^2, & |x|<1 \\ 0, & |x|>1 \end{cases}$. Hence S.T

$\int_0^\infty \cos(\omega/2)\frac{\sin\omega - \omega\cos\omega}{\omega^3}d\omega = 3\pi/16$

ans. $F\{f(x)\} = F(\omega) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{-i\omega x}dx = \frac{1}{\sqrt{2\pi}}\int_{-1}^{1}(1-x^2)e^{-i\omega x}dx$

$= \frac{1}{\sqrt{2\pi}}\int_{-1}^{1} e^{-i\omega x}dx - x^2e^{-i\omega x}dx$

Since it is an even funct^n, F.T becomes FCT.

$F_c\{f(x)\} = F_c(\omega) = \sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\cos\omega x\,dx = \sqrt{\frac{2}{\pi}}\int_0^1(1-x^2)\cos\omega x\,dx$

$= \sqrt{\frac{2}{\pi}}\int_0^1[\cos\omega x - x^2\cos\omega x]\,dx = \sqrt{\frac{2}{\pi}}\left[\frac{\sin\omega x}{\omega} - \left(\frac{x^2\sin\omega x}{\omega} + \frac{2x\cos\omega x}{\omega^2} - \frac{2\sin\omega x}{\omega^3}\right)\right]_0^1$

$= \sqrt{\frac{2}{\pi}}\left[\frac{\sin\omega}{\omega} - \left(\frac{\sin\omega}{\omega} + \frac{2\cos\omega}{\omega^2} - \frac{2\sin\omega}{\omega^3}\right) - 0 + (0+0+0)\right]$

$= \sqrt{\frac{2}{\pi}}\left[\frac{-2\cos\omega}{\omega^2} + \frac{2\sin\omega}{\omega^3}\right] = +2\sqrt{\frac{2}{\pi}}\left[\frac{\sin\omega - \omega\cos\omega}{\omega^3}\right]$

Scanned by CamScanner

$$\int_0^\infty \cos(\omega/2)\,\frac{\sin\omega-\omega\cos\omega}{\omega^3}\,d\omega = \cancel{\int_0^\infty \sqrt{\frac{\pi}{8}}\cos\left(\frac{\omega}{2}\right)F_c(\omega)\,d\omega}$$

$$f(x) = \mathcal{F}_c^{-1}\{F_c(\omega)\} = \sqrt{\frac{2}{\pi}}\int_0^\infty F_c(\omega)\cos\omega x\,d\omega = \sqrt{\frac{2}{\pi}}\int_0^\infty 2\sqrt{\frac{2}{\pi}}\left(\frac{\sin\omega-\omega\cos\omega}{\omega^3}\right)\cos\omega x\,d\omega$$

$$= \frac{4}{\pi}\int_0^\infty\left(\frac{\sin\omega-\omega\cos\omega}{\omega^3}\right)\cos\omega x\,d\omega$$

$$\therefore \int_0^\infty\left(\frac{\sin\omega-\omega\cos\omega}{\omega^3}\right)\cos\omega x\,d\omega = \frac{\pi}{4}f(x)\begin{cases}\frac{\pi}{4}(1-x^2), & |x|<1\\ 0, & |x|>1\end{cases}$$

Put $x = 1/2$ on both sides,

$$\therefore \int_0^\infty \cos\left(\frac{\omega}{2}\right)\left(\frac{\sin\omega-\omega\cos\omega}{\omega^3}\right)d\omega = \frac{\pi}{4}\times\frac{3}{4} = \underline{\underline{3\pi/16}}$$

③ P.T $e^{-x}\cos x = \frac{2}{\pi}\int_0^\infty \frac{(\omega^2+2)}{\omega^4+4}\cos\omega x\,d\omega$

④ (i) $L\{\cos\sqrt{t}\}$ (ii) $L\{1/\sqrt{t}\}$ [Use $\Gamma(1/2)=\sqrt{\pi}$]

⑤ $L^{-1}\left\{\frac{(3s+1)e^{-2s}}{s^2(s^2+4)}\right\}$

Answers

③ $\cancel{\mathcal{F}_c\{f(x)\} = F_c(\omega) = \sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\cos\omega x\,d}$ $\quad f(x) = e^{-x}\cos x$

$$\mathcal{F}_c\{f(x)\} = F_c(\omega) = \sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\cos\omega x\,dx = \sqrt{\frac{2}{\pi}}\int_0^\infty e^{-x}\cos x\cos\omega x\,dx$$

$$= \sqrt{\frac{2}{\pi}}\int_0^\infty \frac{e^{-x}}{2}[\cos(1+\omega)x + \cos(1-\omega)x]\,dx$$

$$= \cancel{\sqrt{\frac{2}{\pi}}\int_0^\infty \frac{e^{-x}}{2}} = \frac{1}{\sqrt{2\pi}}\left[\int_0^\infty e^{-x}\cos(1+\omega)x\,dx + \int_0^\infty e^{-x}\cos(1-\omega)x\,dx\right]$$

$$= \frac{1}{\sqrt{2\pi}}\left\{\left[\frac{e^{-x}}{1+(1+\omega)^2}\left(-\cos(1+\omega)x + (1+\omega)\sin(1+\omega)x\right)\right]_0^\infty + \left[\frac{e^{-x}}{1+(1-\omega)^2}\left(-\cos(1-\omega)x + (1-\omega)\sin(1-\omega)x\right)\right]_0^\infty\right\}$$

Scanned by CamScanner

$$= \frac{1}{\sqrt{2\pi}}\left\{\frac{1}{\omega^2+2\omega+2}+\frac{1}{\omega^2-2\omega+2}\right\} = \frac{1}{\sqrt{2\pi}}\left\{\frac{2\omega^2+4}{\omega^4+4}\right\} = \sqrt{\frac{2}{\pi}}\left[\frac{\omega^2+2}{\omega^4+4}\right]$$

$$f(x) = F_c^{-1}\{F_c(\omega)\} = \sqrt{\frac{2}{\pi}}\int_0^\infty F_c(\omega)\cos\omega x\,d\omega = \frac{2}{\pi}\int_0^\infty \frac{(\omega^2+2)}{\omega^4+4}\cos\omega x\,d\omega$$

$$\therefore e^{-x}\cos x = \frac{2}{\pi}\int_0^\infty \frac{(\omega^2+2)}{\omega^4+4}\cos\omega x\,d\omega$$

④

(i) $$L\{\cos\sqrt{t}\} = L\left\{1-\frac{t}{2!}+\frac{t^2}{4!}-\frac{t^3}{6!}+\frac{t^4}{8!}\cdots\right\} = L\{1\}-\frac{1}{2}L\{t\}+\frac{1}{24}L\{t^2\}-\frac{1}{72}L\{t^3\}+\cdots$$

$$= \frac{1}{s}-\frac{1}{2s^2}+\frac{1}{12s^3}-\frac{1}{12s^4}+\cdots$$

(ii) $$L\{1/\sqrt{t}\} = L\{t^{-1/2}\} = \frac{\Gamma(-1/2+1)}{s^{-1/2+1}} = \frac{\Gamma(1/2)}{\sqrt{s}} = \sqrt{\pi/s}$$

⑤ $$L^{-1}\left\{\frac{(3s+1)e^{-3s}}{s^2(s^2+4)}\right\} = f(t-3)\,u(t-3) \text{ where } f(t) = L^{-1}\left\{\frac{3s+1}{s^2(s^2+4)}\right\}$$

$$L^{-1}\left\{\frac{3s+1}{s^2(s^2+4)}\right\} = \cancel{\int_{v=0}^{t} g(v)h(t-v)\,dv \text{ where } g(t) = L^{-1}\{G(s)\}}$$

$$G(s) = \frac{3s+1}{s^2+4} \qquad H(s) = \frac{1}{s^2}$$

$$g(t) = L^{-1}\{G(s)\} = 3L^{-1}\left\{\frac{s}{s^2+4}\right\}+\frac{1}{2}L^{-1}\left\{\frac{2}{s^2+4}\right\} = 3\cos 2t+\frac{1}{2}\sin 2t$$

$$h(t) = L^{-1}\{H(s)\} = L^{-1}\left\{\frac{1}{s^2}\right\} = \frac{t}{1!} = t$$

$$L^{-1}\left\{\frac{3s+1}{s^2(s^2+4)}\right\} = \int_{v=0}^{t} g(v)\,h(t-v)\,dv = \int_{v=0}^{t}\left(3\cos 2v+\frac{1}{2}\sin 2v\right)(t-v)\,dv$$

$$= 3t\int_{v=0}^{t}\cos 2v\,dv+\frac{t}{2}\int_{v=0}^{t}\sin 2v\,dv - 3\int_{v=0}^{t} v\cos 2v\,dv - \frac{1}{2}\int_{v=0}^{t} v\sin 2v\,dv$$

$$= \frac{3t}{2}\left[\sin 2v\right]_0^t - \frac{t}{4}\left[\cos 2v\right]_0^t - \cancel{3\left[\frac{v\sin 2v}{2}+\frac{\cos 2v}{4}\right]_0^t} - \cancel{\frac{1}{2}\left[\frac{-v\cos 2v}{2}\right.}$$

Scanned by CamScanner

$$-3\left[\frac{v\sin 2v}{2}+\frac{\cos 2v}{4}\right]_0^t-\frac{1}{2}\left[\frac{-v\cos 2v}{2}+\frac{\sin 2v}{4}\right]_0^t$$

$$=\cancel{\frac{3t}{2}\sin 2t}-\cancel{\frac{t}{4}\cos 2t}+\frac{t}{4}\cancel{-\frac{3t}{2}\sin 2t}-\frac{3}{4}\cos 2t+\frac{3}{4}+\cancel{\frac{t}{4}\cos 2t}-\frac{1}{8}\sin 2t$$

$$=\frac{t+3-3\cos 2t-2\sin 2t}{4}$$

Scanned by CamScanner