



# Optimal Matrix Multiplication

# Optimal Matrix Multiplication

- Given a sequence of matrices, find the most efficient way to multiply these matrices together.
- matrix multiplication is associative.
- no matter how we parenthesize the product, the result will be the same.
- the order in which we parenthesize the product affects the number of simple arithmetic operations needed to compute the product, or the efficiency

# Optimal Matrix Multiplication

- Given a sequence of matrices, find the most efficient way to multiply these matrices together.
  - A, B, C are 3 matrices
  - A is a  $10 \times 30$  matrix, B is a  $30 \times 5$  matrix, and C is a  $5 \times 60$  matrix
  - ABC
- matrix multiplication is associative.
  - $(AB)C$  or
  - $A(BC)$
- no matter how we parenthesize the product, the result will be the same.

# Optimal Matrix Multiplication

$$A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{pmatrix}$$

$$B = \begin{pmatrix} 7 & 8 & 1 \\ 5 & 6 & 2 \end{pmatrix}$$

$$AB = \begin{pmatrix} 1*7+2*5 & 1*8+2*6 & 1*1+2*2 \\ 3*7+4*5 & 3*8+4*6 & 3*1+4*2 \\ 5*7+6*5 & 5*8+6*6 & 5*1+6*2 \end{pmatrix}$$

- 18 Multiplications
- $3*2*3$  Multiplications

- $A (3 \times 2)$        $B (2 \times 3)$
- $3*2*3$  Multiplications

$$AB = (3 \times 3)$$

# Optimal Matrix Multiplication

A is a  $10 \times 30$  matrix, B is a  $30 \times 5$  matrix, and C is a  $5 \times 60$  matrix

AB =  $10 \times 30 \times 5$  multiplications & order will be  $10 \times 5$

- $(AB)C = (10 \times 30 \times 5) + (10 \times 5 \times 60) = 1500 + 3000 = 4500$  multiplications

BC =  $30 \times 5 \times 60$  multiplications & order will be  $30 \times 60$

- $A(BC) = (30 \times 5 \times 60) + (10 \times 30 \times 60) = 9000 + 18000 = 27000$  multiplications

# Optimal Matrix Multiplication

Find the optimal matrix multiplication of

$A_1(10 \times 100)$   $A_2(100 \times 5)$   $A_3(5 \times 50)$   $A_4(50 \times 1)$

# Optimal Matrix Multiplication

$A_1(10 \times 100)$   $A_2(100 \times 5)$   $A_3(5 \times 50)$   $A_4(50 \times 1)$

- $A_1(A_2(A_3A_4))$  - 1750 multiplications
- $(A_1A_2)(A_3A_4)$  - 5300 multiplications
- $((A_1A_2)A_3)A_4$  - 8000 multiplications
- $(A_1(A_2A_3))A_4$  - 75,500 multiplications
- $A_1((A_2A_3)A_4)$  - 31,000 multiplications

# Optimal Matrix Multiplication

$A_1(10 \times 100)$   $A_2(100 \times 5)$   $A_3(5 \times 50)$   $A_4(50 \times 1)$

- Using Dynamic programming



# Optimal Matrix Multiplication

$A_1(10 \times 100)$   $A_2(100 \times 5)$   $A_3(5 \times 50)$   $A_4(50 \times 1)$

Multiplications  $\longrightarrow$

0			
	0		
		0	
			0

Ordering/Pairing  $\longrightarrow$

0			
	0		
		0	
			0

# Optimal Matrix Multiplication

$A_1(10 \times 100)$     $A_2(100 \times 5)$     $A_3(5 \times 50)$     $A_4(50 \times 1)$

- $A_{12} = A_1 A_2$   
 $= 10 \times 100 \times 5 = 5000$
- $A_{23} = A_2 A_3$   
 $= 100 \times 5 \times 50 = 25000$
- $A_{34} = A_3 A_4$   
 $= 5 \times 50 \times 1 = 250$

0			
	0		
		0	
			0

0			
	0		
		0	
			0

# Optimal Matrix Multiplication

$A_1(10 \times 100)$     $A_2(100 \times 5)$     $A_3(5 \times 50)$     $A_4(50 \times 1)$

- $A_{12} = A_1 A_2$   
 $= 10 \times 100 \times 5 = 5000$
- $A_{23} = A_2 A_3$   
 $= 100 \times 5 \times 50 = 25000$
- $A_{34} = A_3 A_4$   
 $= 5 \times 50 \times 1 = 250$

0	5000		
	0	25000	
		0	250
			0

0	$A_1 A_2$ 10 x 5		
	0	$A_2 A_3$ 100 x 50	
		0	$A_3 A_4$ 5 x 1
			0

# Optimal Matrix Multiplication

$$A_1(10 \times 100) \quad A_2(100 \times 5) \quad A_3(5 \times 50) \quad A_4(50 \times 1)$$

$$A_{13} = A_1 A_2 A_3$$

- $A_{13} = (A_1 A_2) A_3$   
=

- $A_{13} = A_1 (A_2 A_3)$   
=

0	5000		
	0	25000	
		0	250
			0

0	$A_1 A_2$ 10 x 5		
	0	$A_2 A_3$ 100 x 50	
		0	$A_3 A_4$ 5 x 1
			0

# Optimal Matrix Multiplication

$$A_1(10 \times 100) \quad A_2(100 \times 5) \quad A_3(5 \times 50) \quad A_4(50 \times 1)$$

$$A_{13} = A_1 A_2 A_3$$

- $$A_{13} = (A_1 A_2) A_3$$

$$= 5000 + 10 \times 5 \times 50 = 7500$$
- $$A_{13} = A_1 (A_2 A_3)$$

$$= 10 \times 100 \times 50 + 25000 = 75000$$

0	5000		
	0	25000	
		0	250
			0

0	$A_1 A_2$ 10 x 5		
	0	$A_2 A_3$ 100 x 50	
		0	$A_3 A_4$ 5 x 1
			0

# Optimal Matrix Multiplication

$$A_1(10 \times 100) \quad A_2(100 \times 5) \quad A_3(5 \times 50) \quad A_4(50 \times 1)$$

$$A_{13} = A_1 A_2 A_3$$

- $$A_{13} = (A_1 A_2) A_3$$

$$= 5000 + 10 \times 5 \times 50 = 7500$$
- $$A_{13} = A_1 (A_2 A_3)$$

$$= 10 \times 100 \times 50 + 25000 = 75000$$

0	5000	7500	
	0	25000	
		0	250
			0

0	$A_1 A_2$ 10 x 5	$(A_1 A_2) A_3$ 10 x 50	
	0	$A_2 A_3$ 100 x 50	
		0	$A_3 A_4$ 5 x 1
			0

# Optimal Matrix Multiplication

$$A_1(10 \times 100) \quad A_2(100 \times 5) \quad A_3(5 \times 50) \quad A_4(50 \times 1)$$

$$m[i,j]=m[i,k] + m[k+1,j] + p_{i-1}p_kp_j$$

- $A_{13} = (A_1 A_2) A_3$

$$\begin{aligned} m[1,3] &= m[1,2] + m[3,3] + p_0 p_2 p_3 \\ &= 5000 + 0 + 10 * 5 * 50 = 7500 \end{aligned}$$

0	5000	7500	
	0	25000	
		0	250
			0

0	$A_1 A_2$ 10 x 5	$(A_1 A_2) A_3$ 10 x 50	
	0	$A_2 A_3$ 100 x 50	
		0	$A_3 A_4$ 5 x 1
			0

# Optimal Matrix Multiplication

$$A_1(10 \times 100) \quad A_2(100 \times 5) \quad A_3(5 \times 50) \quad A_4(50 \times 1)$$

$$A_{24} = A_2 A_3 A_4$$

- $$A_{24} = (A_2 A_3) A_4$$

$$= 25000 + 100 \times 50 \times 1 = 30000$$

- $$A_{24} = A_2 (A_3 A_4)$$

$$= 100 \times 5 \times 1 + 250 = 750$$

0	5000	7500	
	0	25000	750
		0	250
			0

0	$A_1 A_2$ 10 x 5	$(A_1 A_2) A_3$ 10 x 50	
	0	$A_2 A_3$ 100 x 50	$A_2 (A_3 A_4)$ 100 x 1
		0	$A_3 A_4$ 5 x 1
			0



# Optimal Matrix Multiplication

$$A_1(10 \times 100) \quad A_2(100 \times 5) \quad A_3(5 \times 50) \quad A_4(50 \times 1)$$

$$A_{14} = A_1 A_2 A_3 A_4$$

- $A_{14} = A_1(A_2 A_3 A_4)$
- $A_{14} = (A_1 A_2 A_3)A_4$
- $A_{14} = (A_1 A_2)(A_3 A_4)$

0	5000	7500	
	0	25000	750
		0	250
			0

0	$A_1 A_2$ 10 x 5	$(A_1 A_2) A_3$ 10 x 50	
	0	$A_2 A_3$ 100 x 50	$A_2 (A_3 A_4)$ 100 x 1
		0	$A_3 A_4$ 5 x 1
			0

# Optimal Matrix Multiplication

$$A_1(10 \times 100) \quad A_2(100 \times 5) \quad A_3(5 \times 50) \quad A_4(50 \times 1)$$

$$A_{14} = A_1 A_2 A_3 A_4$$

- $A_{14} = A_1(A_2 A_3 A_4)$   
 $= 10 \times 100 \times 1 + 750 = 1750$
- $A_{14} = (A_1 A_2 A_3) A_4$   
 $= 7500 + 10 \times 50 \times 1 = 8000$
- $A_{14} = (A_1 A_2)(A_3 A_4)$   
 $= 5000 + 250 + 10 \times 5 \times 1 = 5300$

0	5000	7500	
	0	25000	750
		0	250
			0

0	$A_1 A_2$ 10 x 5	$(A_1 A_2) A_3$ 10 x 50	
	0	$A_2 A_3$ 100 x 50	$A_2 (A_3 A_4)$ 100 x 1
		0	$A_3 A_4$ 5 x 1
			0

# Optimal Matrix Multiplication

$$A_1(10 \times 100) \quad A_2(100 \times 5) \quad A_3(5 \times 50) \quad A_4(50 \times 1)$$

$$A_{14} = A_1 A_2 A_3 A_4$$

- $$A_{14} = A_1(A_2 A_3 A_4)$$

$$= 10 \times 100 \times 1 + 750 = 1750$$
- $$A_{14} = (A_1 A_2 A_3) A_4$$

$$= 7500 + 10 \times 50 \times 1 = 8000$$
- $$A_{14} = (A_1 A_2)(A_3 A_4)$$

$$= 5000 + 250 + 10 \times 5 \times 1 = 5300$$

0	5000	7500	1750
	0	25000	750
		0	250
			0

0	$A_1 A_2$ 10 x 5	$(A_1 A_2) A_3$ 10 x 50	$A_1(A_2(A_3 A_4))$ 10 x 1
	0	$A_2 A_3$ 100 x 50	$A_2(A_3 A_4)$ 100 x 1
		0	$A_3 A_4$ 5 x 1
			0

# Optimal Matrix Multiplication

$$A_1(10 \times 100) \quad A_2(100 \times 5) \quad A_3(5 \times 50) \quad A_4(50 \times 1)$$

$$A_{14} = A_1 A_2 A_3 A_4$$

$$\text{Answer} = A_1(A_2(A_3A_4))$$

- 1750 multiplications

0	5000	7500	1750
	0	25000	750
		0	250
			0

0	$A_1A_2$ 10 x 5	$(A_1A_2)A_3$ 10 x 50	$A_1(A_2(A_3A_4))$ 10 x 1
	0	$A_2A_3$ 100 x 50	$A_2(A_3A_4)$ 100 x 1
		0	$A_3A_4$ 5 x 1
			0



Thank you