Master of Science in quantitative and financial modeling

'Travaux Pratiques'

EXAMINER: Prof. L. Laayouni **Due date: Monday March 18 2024**

FAMILY NAME:

GIVEN NAMES:

Instructions

1. Fill in the above clearly.
2. Do not tear pages from this book; all your writing — even rough work — must be handed in. You may do rough work for this paper anywhere in the booklet.
3. This is a closed book examination.
4. This examination booklet consists of this cover, Pages 1 through 5 containing questions; and Pages 6 and 7, which are blank.
5. You are expected to simplify your answers wherever possible. You are advised to spend the first few minutes scanning the problems.
6. A TOTAL OF 100 MARKS ARE AVAILABLE ON THIS EXAMINATION.

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1 /20	2 /20	3 /20	4 /20	5 /20				
								TOTAL /100

1. **[20 MARKS]** Find the LU-factorization of the following matrix $\mathbf{A}$ in $\mathbb{R}^{n\times n}$:

$$\mathbf{A} = \begin{bmatrix} 2 & -1 & 0 & \cdots & 0 \\ -1 & 2 & -1 & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & -1 & 2 & -1 \\ 0 & \cdots & 0 & -1 & 2 \end{bmatrix} \tag{1}$$

Is the matrix $\mathbf{A}$ positive definite, or is it positive semi-definite?

Write a Python script to solve a linear system associated with the above matrix.

2. [**20 MARKS**] Consider the following boundary value problem modeling the heat flow in a long pipe:

$$\begin{cases} y''(x) - p(x)y'(x) - q(x)y(x) = r(x), \quad x \in [a, b] \\ y(a) = \alpha, \ y(b) = \beta \end{cases} \tag{2}$$

   (a) Use a uniform discretization of the interval $[a, b]$ to derive the linear system corresponding to the model problem.
   (b) Solve the linear system using the Gaussian elimination method.
   (c) Solve the linear system using QR-factorization.
   (d) Solve the linear system using SVD-decomposition.
   (e) Compare the three methods.

   Remark: You are free to decide about the size of the matrix.

3. **[20 MARKS]** Solve the problem of fitting a polynomial $p(x) = \sum_{i=0}^{d} c_i x^{i-1}$ of degree d to data points $(x_i, yi),\ i = 1, \ldots, m$, in the plane by the method of normal equations and QR decomposition. Choose the degree of the polynomial to be $d = 5$ and then $d = 15$, choose the interval $x \in [-1, 1]$, discretize it using $N = 10$ or $N = 20$ points.

4. **[20 MARKS]**

   (a) Explain how Singular Value Decomposition (SVD) can be applied to compress color images. Discuss the process in the context of an RGB image.

   (b) Given a color image, implement an SVD-based compression algorithm in Python. Your implementation should:

   - Load a color image and separate it into R, G, and B channels.
   - Apply SVD to each channel and reconstruct the image using only the first k singular values and vectors.
   - Display the original and compressed images side by side for comparison, and compute the compression ratio.

   (c) Analyze the effect of varying the number of singular values (k) on the compression ratio and image quality. Use a specific color image for this analysis and provide visual and numerical results for at least three different values of k. Discuss the trade-off between compression ratio and image quality.

5. **[20 MARKS] Modeling option:** Consider the biharmonic equation

$$\Delta^2 u(s,t) := \Delta(\Delta u(s,t)) = f(s,t) \quad (s,t) \in \Omega, \tag{3}$$

with

$$u(s,t) = 0, \quad \Delta u(s,t) = 0 \quad (s,t) \in \partial\Omega.$$

Here Ω is the open unit square. The condition $\Delta u = 0$ is called the Navier boundary condition. Moreover,

$$\Delta^2 u = u_{xxxx} + 2u_{xxyy} + u_{yyyy}.$$

a) Let $v = -\Delta u$. Show that 3 can be written as a system

$$\begin{aligned} -\Delta v(s,t) &= f(s,t) \quad (s,t) \in \Omega \\ -\Delta u(s,t) &= v(s,t) \quad (s,t) \in \Omega \\ u(s,t) &= v(s,t) = 0 \quad (s,t) \in \partial\Omega. \end{aligned}$$

b) Discretizing with $T = \text{tridiag}(-1, 2, -1) \in \mathbb{R}^{m \times m}$, $h = \frac{1}{m+1}$, and $F = (f(jh, kh))_{j,k=1}^{m}$ we get two matrix equations

$$TV + VT = h^2 F, \quad TU + UT = h^2 V.$$

Data science or Financial engineering option: Consider the heat equation,

$$\frac{\partial u}{\partial t} = \alpha \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right).$$

Here, $u(x, y, t)$ represents the state variable (image intensity or option price), at spatial location (x, y) and time t, with α as the diffusion coefficient. This equation can be used both in data science (image denoising) and financial engineering (option pricing).

(a) Discretize the heat equation using the Finite Difference Method (FDM) for a 2D image grid.

(b) Construct matrices for the discrete Laplacian operator. Discuss the use of the Kronecker product for this purpose.

(c) Implement the code in Python, iterating over time steps using an explicit scheme.

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