Volume integration using contour boundaries

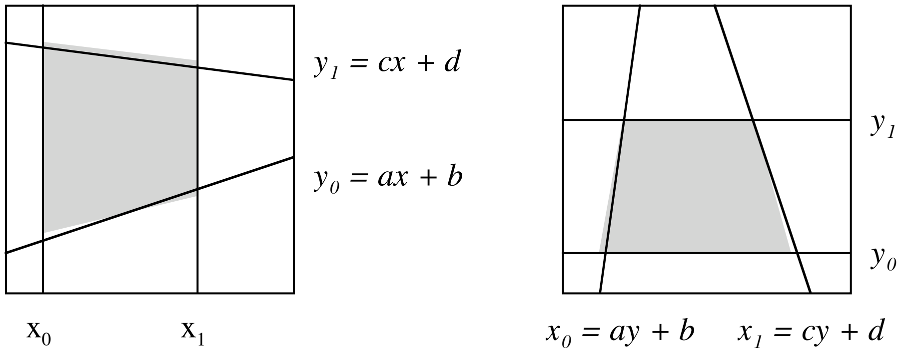

Figure 1. Integration of volume under bilinear surface bounded by x or y-trapezoid.

The surface will be approximated using a bilinear expression for each grid box:

$$z(x,y) = z_0 + z_x^{'}x + z_y^{'}y + z_{xy}^{''}xy \tag{0.1}$$

where z_0 is the grid value at the lower left corner where $x = y = 0$ (i.e., x,y are local grid box coordinates and go from 0, 0 to Δx, Δy), and the primed z's indicate the derivatives in x, y, and x and y, respectively. The general expression for the volume defined by the trapezoidal domain in Figure 1 for $0 < z < z(x,y)$ is

$$\Delta V = \int_{x=x_0}^{x=x_1} \int_{y=ax+b}^{y=cx+d} z(x,y)\,dx\,dy \tag{0.2}$$

We find

$$\Delta V = \frac{\Delta x \Delta y}{24}\left\{\begin{array}{l} 3z_{xy}^{''}\left(c^2-a^2\right)\left(x_1^4-x_0^4\right)+4\left[2z_x^{'}(c-a)+z_y^{'}\left(c^2-a^2\right)+2z_{xy}^{''}(cd-ab)\right]\left(x_1^3-x_0^3\right) \\ +6\left[2z_0(c-a)+2z_x^{'}(d-b)+2z_y^{'}(cd-ab)+z_{xy}^{''}\left(d^2-b^2\right)\right]\left(x_1^2-x_0^2\right) \\ +12\left[2z_0(d-b)+z_y^{'}\left(d^2-b^2\right)\right]\left(x_1-x_0\right) \end{array}\right\} \tag{0.3}$$

When the trapezoid is bounded by two horizontal lines we instead must integrate

$$\Delta V = \int_{y=y_0}^{y=y_1} \int_{x=ay+b}^{x=cy+d} z(x,y)\,dx\,dy \tag{0.4}$$

and we obtain

$$\Delta V = \frac{\Delta x \Delta y}{24}\left\{\begin{array}{l} 3z_{xy}^{''}\left(c^2-a^2\right)\left(y_1^4-y_0^4\right)+4\left[2z_y^{'}(c-a)+z_x^{'}\left(c^2-a^2\right)+2z_{xy}^{''}(cd-ab)\right]\left(y_1^3-y_0^3\right) \\ +6\left[2z_0(c-a)+2z_y^{'}(d-b)+2z_x^{'}(cd-ab)+z_{xy}^{''}\left(d^2-b^2\right)\right]\left(y_1^2-y_0^2\right) \\ +12\left[2z_0(d-b)+z_x^{'}\left(d^2-b^2\right)\right]\left(y_1-y_0\right) \end{array}\right\} \tag{0.5}$$

These two expressions (3 and 5) are general enough to handle all the possible scenarios of straight-line contours intersecting a grid box.