Spații vectoriale. Transformări ortogonale

④/Lista 10

$(\mathbb{C},+,\cdot)_{\mathbb{R}}$; $g:\mathbb{C}\times\mathbb{C}\to\mathbb{R}$ formă biliniară

$G=\begin{pmatrix}1 & 2\\ 2 & 5\end{pmatrix}$ matrice asociată lui g în rap cu $R_0=\{1,i\}$

a) $(\mathbb{C},g)$ s.v.e.r? (g produs scalar)

b) $u=2-i$ versor în rap cu g

c) $u^{\perp}$

d) să se ortonormeze R_0 în raport cu g

e) să se afle intersecția dintre cercul unitate în $(\mathbb{C},g_0)$ și în $(\mathbb{C},g)$

$z=x_1+ix_2$; $z'=y_1+iy_2$

$g:\mathbb{R}^2\times\mathbb{R}^2\to\mathbb{R}$

$g(z,z')=g((x_1,x_2),(y_1,y_2))=1\cdot x_1y_1+2\cdot x_1y_2+$
$+2x_2y_1+5x_2y_2$

Din ip g biliniară
$G=G^t$ $\Rightarrow$ g simetrică ①

$Q(z)=Q(x_1,x_2)=x_1^2+4x_1x_2+5x_2^2=$

$=\frac{1}{\Delta_1}x_1'^2+\frac{\Delta_1}{\Delta_2}x_2'^2=\frac{1}{1}x_1'^2+\frac{1}{1}x_2'^2=x_1'^2+x_2'^2>0$

$\Delta_1=1>0$

$\Delta_2=1>0$ $\qquad Q(x_1,x_2)=(x_1+2x_2)^2+x_2^2$

$(2,0)$ signatura $\Rightarrow$ poz def (2)

$g_0 : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$

$g_0((x_1,x_2),(y_1,y_2)) = x_1y_1 + x_2y_2$

prod scalar canonic

$\|(x_1,x_2)\| = \sqrt{g((x_1,x_2),(x_1,x_2))} = \sqrt{Q(x_1,x_2)}$

$Q_0((x_1,x_2)) = x_1^2 + x_2^2$

$g((x_1,x_2),(y_1,y_2)) = x_1y_1 + x_1y_2 + x_2y_1 + x_2y_2$

$Q(x_1,x_2) = x_1^2 + 4x_1x_2 + x_2^2 = (x_1+2x_2)^2 + x_2^2$

b) $Q_0((2,-1)) = 2^2+1 = 5$

$\| $

$Q_0(u)$

$\| $

$\|u\|^2_{g_0}$

$\|u\|^2_g = Q(u) = Q(2,-1) = 0+1 = 1 \Rightarrow u$ este versor

în rap cu g

c) $u^\perp = \{\underset{(y_1,y_2)}{y} \in \mathbb{R}^2 \mid g(\underset{(2,-1)}{u}, y) = 0\} = \{(y_1,0) \mid y_1 \in \mathbb{R}\} \equiv \mathbb{R}$

$\cancel{2y_1} + 2\cdot 2y_2 + \cancel{2(-1)y_1} + 5(-1)\cdot y_2 = -y_2 =$

$\mathbb{R}^2 = \langle\{u\}\rangle \oplus \mathbb{R}$

$\| $

$\langle\{u\}\rangle^\perp$

$$(0,1)$$

d) $R_0 = \{ \underset{\underset{(1,0)}{\|}}{\lambda}, \overset{(0,1)}{u} \}$

$f_1 = (1,0)$; $f_2 = (0,1)$

$e_1 = f_1$

$e_2 = f_2 - \dfrac{g(f_2, f_1)}{g(e_1, e_1)} e_1 = (0,1) - \dfrac{2}{1}(1,0) = (-2,1)$

$g(f_2, f_1) = g((0,1),(1,0)) = 2$

$g(e_1, e_1) = g((1,0),(1,0)) = 1$

~~$e_1 = (1,0)$~~ $\|e_1\| = \sqrt{g(e_1, e_1)} = 1$

~~e_2~~ $\|e_2\| = \sqrt{g(e_1, e_2)} = 1$

$\{e_1, e_2\}$ reper ortonormat în rap cu g
$\underset{\;}{\overset{\|}{u}}$

$S_{g_0} = \{ x \in \sigma \mid |x| = 1 \} = \{ (x_1, x_2) \subset \mathbb{R}^2 \} \mid g_0((x_1,x_2)(x_1$

$\| $

$Q_0(x_1, x_2))$

$\|$

$x_1^2 + x_2^2$

$S_g = \{ (x_1, x_2) \in \mathbb{R}^2 \mid Q(x_1, x_2) = 1 \}$

$\|$

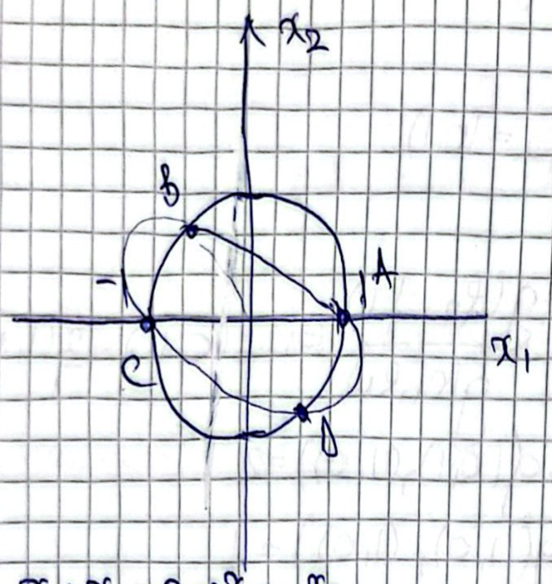
$x_1^2 + 4x_1x_2 + 5x_2^2 = 1$

$$\begin{cases} x_1^2 + x_2^2 = 1 \\ x_1^2 + 4x_1x_2 + 5x_2^2 = 1 \end{cases}$$

$4x_1x_2 + 4x_2^2 = 0$

$4x_2(x_1 + x_2) = 0$

I $x_2 = 0 \Rightarrow x_1^2 = 1 \Rightarrow x_1 = \pm 1$



II $x_1 + x_2 = 0 \Rightarrow x_1 = -x_2$

$x_2^2 + x_2^2 = 1 \Rightarrow x_2 = \pm \frac{1}{\sqrt{2}}$

$\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right), \left(\frac{-1}{\sqrt{2}}, \frac{-1}{2}\right)$

$S^1_{g_0} \cap S^1_g = \{ \underset{t=0}{A(1,0)}, \underset{t=\frac{3\pi}{4}}{B\left(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)}, \underset{t=\pi}{C(-1,0)},$
$\underset{t=\frac{7\pi}{4}}{D\left(\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right)} \}$

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$(M_2(\mathbb{R}), g)$, $g(A,B) = Tr(A^t \cdot B)$, $(\forall) A, B \in M_2(\mathbb{R})$

a) g produs scalar

b) $\mathcal{R} = \left\{ \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}$

să se ortonormeze ~~și apoi~~

$\Rightarrow A = \begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \end{pmatrix} \to (x_1, x_2, x_3, x_4)$

$$B = \begin{pmatrix} y_1 & y_2 \\ y_3 & y_4 \end{pmatrix}$$

$$A^t \cdot B = \begin{pmatrix} x_1 & x_3 \\ x_2 & x_4 \end{pmatrix} \begin{pmatrix} y_1 & y_2 \\ y_3 & y_4 \end{pmatrix} = \begin{pmatrix} x_1y_1 + x_3y_3 & x_1y_2 + x_3y_4 \\ x_2y_1 + x_4x_3 & x_2y_2 + x_4y_4 \end{pmatrix}$$

$$\to Tr(A^t \cdot B) = x_1y_1 + x_2y_2 + x_3y_3 + x_4y_4 \to$$

$$\to g(A,B) = g_0((x_1, x_2, x_3, x_4), (x_1, x_2, x_3, x_4))$$

$$f_1(1,0,2,1);\ f_2 = (0,-1,1,0),\ f_3 = (1,2,1,0);\ f_4 = (0,0,0,1)$$

$$e_1 = f_1$$

$$e_2 = f_2 - \frac{\langle f_1, e_1 \rangle}{\langle e_1, e_1 \rangle} e_1 = (0,-1,1,0) - \frac{2}{6}(1,0,2,1) =$$

$$= \left(-\frac{1}{3}, -1, \frac{1}{3}, -\frac{1}{3}\right) =$$

$$= \frac{1}{3}(-1,-3,1,-1)$$

$$e_3 = f_2 - \frac{\langle f_2, e_1 \rangle}{\langle e_1, e_1 \rangle} e_1 - \frac{\langle f_3, e_2 \rangle}{\langle e_2, e_2 \rangle} e_2 =$$

$$= (1,2,1,0) - \frac{3}{6}(-1,0,2,1) - \frac{\frac{1}{3}(-1-6+1)}{\frac{1}{9} \cdot 12} \cdot \frac{1}{3}(-1,3,1,-1)$$

$$= (1,2,1,0) - \frac{1}{2}(-1,0,2,1) + \frac{1}{2}(-1,-3,1,-1) =$$

$$= \left(0, \frac{1}{2}, \frac{1}{2}, -1\right) = \frac{1}{2}(0,1,1,-2)$$

$$e_4 = f_4 - \frac{\langle f_4, e_1\rangle}{\langle e_1, e_1\rangle} e_1 \ \frac{\langle f_4, e_2\rangle}{\langle e_2, e_2\rangle} e_2 \ \frac{\langle f_4, f e_3\rangle}{\langle e_3, e_3\rangle} e_3$$

$$e_4 = (0,0,0,1) - \frac{1}{6}(1,0,2,1) - \frac{-\frac{1}{\cancel{3}}}{\frac{4}{\cancel{3}}}\frac{1}{3}(-1,-3,1,-1) -$$

$$- \frac{-1}{\frac{1}{\cancel{2}} \cdot \frac{4}{3}} \frac{1}{\cancel{2}}(0,1,1,-2) =$$

$$= (0,0,0,1) - \left(\frac{1}{6}, 0, \frac{1}{3}, \frac{1}{6}\right) + \left(\frac{-1}{12}, \frac{-3}{12}, \frac{1}{12}, \frac{-1}{12}\right) +$$

$$+ \left(0, \frac{1}{3}, \frac{1}{3}, \frac{-2}{3}\right) = \left(-\frac{1}{4}, \frac{1}{12}, \frac{1}{12}, \frac{1}{12}\right) =$$

$$= \frac{1}{12}(-3, 1, 1, 1)$$

...

$$e_2' = \frac{1}{\sqrt{12}}(-1,-3,1,-1)$$

$$e_3' = \frac{1}{\sqrt{6}}(0,1,1,-2)$$

$$e_4' = \frac{1}{\sqrt{12}}(-3,1,1,1)$$

Lista 11

① $(\mathbb{R}^3, g_0)$ $f \in End(\mathbb{R}^3)$

$$A = [f]_{R_0, R_0} = \frac{1}{9}\begin{pmatrix} 8 & 1 & -4 \\ 1 & 8 & 4 \\ -4 & 4 & -7 \end{pmatrix}$$

a) $f \in O(\mathbb{R}^3)$

de speta 2 i.e $f = s \circ R_\varphi$

b) $\measuredangle$ de rotatie si axa de simetrie

c) un reper $R = \{e_1', e_2', e_3'\}$ ortonormat a.i.

$$[f]_{R,R} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & \cos\varphi & -\sin\varphi \\ 0 & \sin\varphi & \cos\varphi \end{pmatrix}$$

a) Avem că $A\cdot A^t = I_3$ și $\det A = -1$

$$\frac{1}{9^2}\begin{pmatrix}8&1&-4\\1&8&4\\-4&4&-7\end{pmatrix}\begin{pmatrix}8&1&-4\\1&8&4\\-4&4&-7\end{pmatrix} = \frac{1}{9^2}\begin{pmatrix}81&0&0\\0&81&0\\0&0&81\end{pmatrix} =$$

$$\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix} = I_3 \Rightarrow A\in O(3)$$

$$\det A = \frac{1}{9^3}\begin{vmatrix}8&1&-4\\1&8&4\\-4&4&-7\end{vmatrix} = \frac{1}{9^3}\begin{vmatrix}0&1&0\\-63&8&36\\-36&4&9\end{vmatrix} =$$

$$= \frac{1}{9}\begin{vmatrix}0&\frac{1}{9}&0\\-7&\frac{8}{9}&4\\-4&\frac{4}{9}&1\end{vmatrix} = -\frac{1}{9}\begin{vmatrix}-7&4\\-4&1\end{vmatrix} = -\frac{1}{9}\cdot 9 = -1$$

b) $\operatorname{Tr} A = \operatorname{Tr} A'$

$\frac{1}{9}(16-7) = -1 + 2\cos\varphi$

$\cos\varphi = 1 \Rightarrow \varphi = 0$

axă

$f(x) = -x$

$AX = -X$

$(A+I_3)X = \begin{pmatrix}0\\0\\0\end{pmatrix}$

$$\begin{cases}8x_1+x_2-4x_3 = -9x_1\\ x_1+8x_2+4x_3 = -9x_2\\ -4x_1+4x_2-7x_3 = -9x_3\end{cases}$$

$$\begin{cases} 17x_1+x_2-7x_3=0 \\ x_1+17x_2+4x_3=0 \\ -4x_1+4x_2+2x_3=0 \end{cases}$$

$$\det\begin{pmatrix} 17 & 1 & -1 \\ 1 & 17 & 4 \\ -4 & 4 & 2 \end{pmatrix} = 2\begin{vmatrix} 17 & 1 & -4 \\ 1 & 17 & 2 \\ -4 & 4 & 1 \end{vmatrix} = 0$$

$$\begin{cases} x_1+17x_2=-4x_3 \quad |\cdot 4 \\ -4x_1+4x_2=-2x_3 \end{cases}$$

$$/ \quad 4\cdot\cancel{18}x_2=-\cancel{18}x_3 \rightarrow x_2=-\frac{1}{4}x_3$$

$$x_1=-4x_3-17\left(-\frac{1}{4}\right)x_3=\frac{1}{4}x_3$$

$$\left(\frac{1}{4}x_3, -\frac{1}{4}x_3, x_3\right)=\frac{x_3}{4}(1,-1,4)$$

$$\boxed{e_1'=\frac{1}{3\sqrt{2}}(1,-1,4)}$$ versorul axei

c) $e_1^{\perp}=\{x\in\mathbb{R}^3 \mid x_1-x_2+4x_3=0\}=$

$$=\{(x_1, x_1+4x_3, 4x_3) \mid x_1, x_3\in\mathbb{R}\}=$$

$$=\langle \underbrace{(1,1,0)}_{f_2}, \underbrace{(0,4,1)}_{f_3} \rangle$$ $\{f_2, f_3\}$ reper în $e_1^{\perp}$

$e_2=f_2$

$$e_3=f_3-\frac{\langle f_3,e_2\rangle}{\langle e_2,e_2\rangle}e_2=(0,4,1)-\frac{\cancel{4}^{\,2}}{\cancel{2}}(1,1,0)=$$

$$=(-2,2,1)$$

$e_2' = \frac{1}{\sqrt{2}}(1,1,0)$

$e_3' = \frac{1}{3}(2,2,1)$

$R = \{e_1', e_2', e_3'\}$

$$A' = [f]_{R,R} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

12.11.2024.

Cursul 11

Endomorfisme simetrice

Def: $(E, \langle \cdot,\cdot \rangle)$ s.v.e.r, $f \in End(E)$

f s.n. endomorfism simetric $\Leftrightarrow \langle x, f(y) \rangle = \langle$

$(\forall)\ x, y$

Not $Sim(E)$ = mulțimea endomorfismelor simetrice