365Days

English Study Plan

Seminar – Săptămâna 1

Funcții:

$f : A \to B$

domeniu $\quad$ codomeniu

$x \to f(x)$

$Imf = \{ f(x) \mid x \in A \}$

$A_1 \subseteq A \to f(A_1) = \{ f(x) \mid x \in A_1 \}$

$B_1 \subseteq B \to f^{-1}(B_1) = \{ x \in A \mid f(x) \in B_1 \}$

(imaginea inversă / preimaginea lui B_1)

① F. injectivă dacă:

- $\forall x_1, x_2 \in A$ aî $f(x_1) = f(x_2) \Rightarrow x_1 = x_2$
- $\forall x_1, x_2 \in A$ aî $x_1 \neq x_2 \Rightarrow f(x_1) \neq f(x_2)$

Obs. ! orice funcție strict monotonă este injectivă

- orice paralelă la OX intersectează Gf în cel mult un punct

$\downarrow$

$f \neq$ inj. $\Leftrightarrow \exists$ o // la OX care $\cap$ Gf în mai multe puncte

② F. surjectivă dacă:

- $\forall y \in B, \exists x \in A$ aî $f(x) = y$
- $Imf = B$
- orice paralelă la OX dusă printr-un punct din B intersectează Gf în cel puțin un punct

$\downarrow$

$f \neq$ surj. dacă: $\exists$ o // la OX dusă printr-un punct din B care nu $\cap$ Gf în niciun punct

③ F. bijectivă dacă:

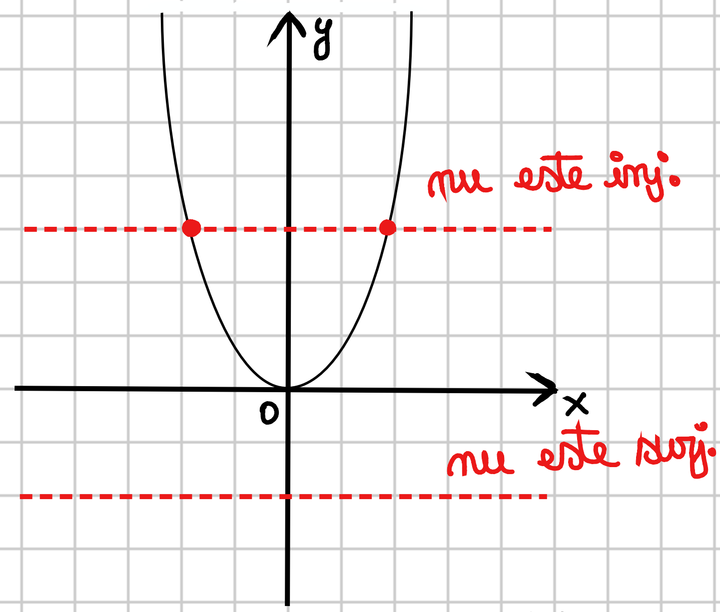
- inj. + bij.
- $\forall y \in B, \exists! x \in A$ aî $f(x)=y$ unic
- $\forall \parallel$ la Ox printr-un punct din B $\cap$ G_f într-un singur punct

f. bij. $\Rightarrow \exists f^{-1}: B \to A$ aî $f^{-1} \circ f = 1_A$

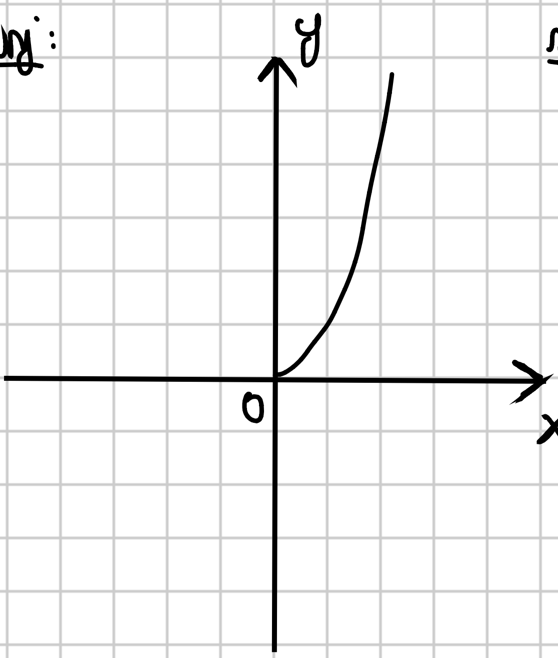
$f \circ f^{-1} = 1_B$

Aplicații:

① $f: \mathbb{R} \to \mathbb{R}, f(x) = x^2$

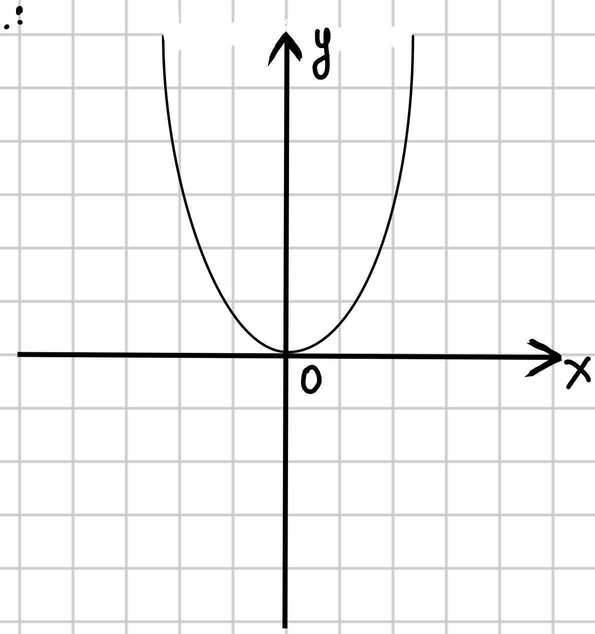


inj.:



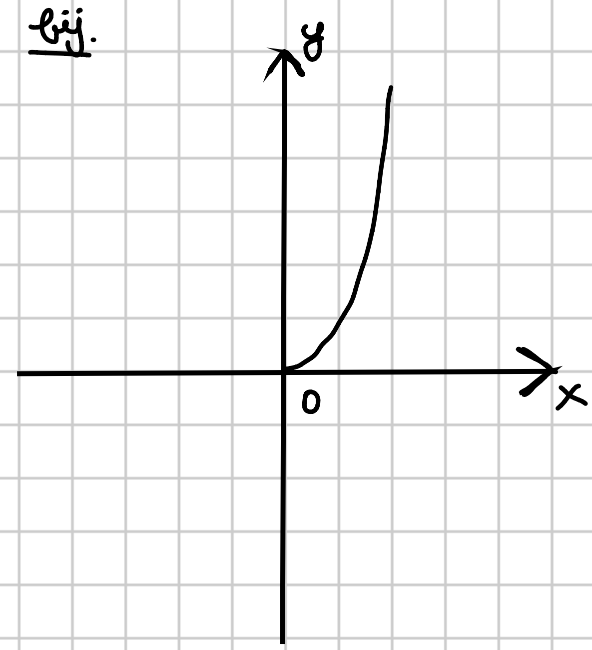
$f: [0, \infty) \to \mathbb{R}$

surj.:



$f: \mathbb{R} \to [0, \infty)$

bij.



$f: [0, \infty) \to [0, \infty)$

② $f: \mathbb{R} \to \mathbb{R}, f(x) = 4x - 3$

a) f = bij.? (liceu)

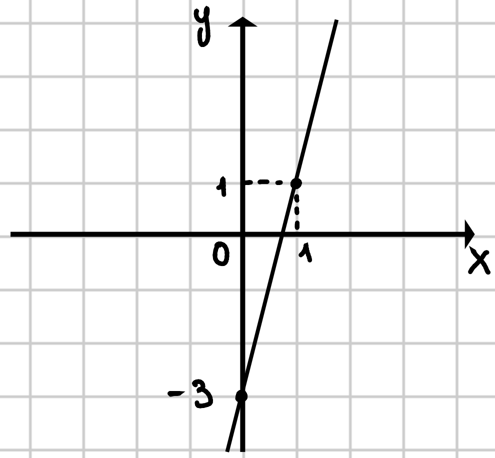
b) f = bij.? (G_f)

a) inj. $x_1, x_2 \in \mathbb{R}$ aî $f(x_1) = f(x_2) \Rightarrow 4x_1 - 3 = 4x_2 - 3 \Rightarrow x_1 = x_2$ $\Big\} \Rightarrow f$ bij.

surj. fie $y \in \mathbb{R}$ aî $f(x) = y \Leftrightarrow 4x - 3 = y \Rightarrow x = \frac{y+3}{4} \in \mathbb{R}$

$f^{-1}: \mathbb{R} \to \mathbb{R}, f^{-1}(x) = \frac{x+3}{4}$

b)



$x=0 \Rightarrow f(0)=-3$

$x=1 \Rightarrow f(1)=1$

c) $f:\mathbb{Z}\to\mathbb{Z},\ f(x)=4x-3$

$y=2 \Rightarrow x=\frac{5}{4}\notin\mathbb{Z}$

③ $f:\mathbb{R}\to\mathbb{R},\ f(x)=e^x$

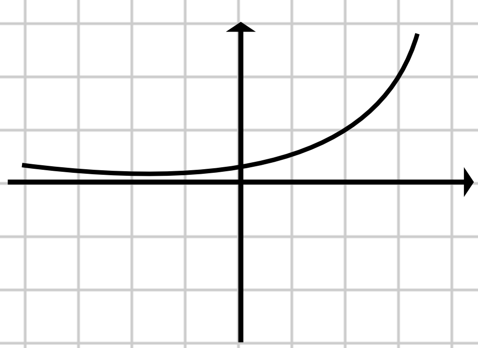
$f=$ bij.?

I) inj: $x_1,x_2\in\mathbb{R}$ a.î. $f(x_1)=f(x_2) \Rightarrow e^{x_1}=e^{x_2} \Leftrightarrow x_1=x_2$

surj: $y\in\mathbb{R}$ a.î. $f(x)=y \Leftrightarrow y=e^x$

pt $y=-1 \Rightarrow e^x=-1 \Rightarrow$ nu e surj. $\Rightarrow$ nu e bij.

II) $e^x>0 \Rightarrow$ nu e surj.



④ A, B - mulțimi finite

$|A|=a,\ |B|=b$

a) câte fct $f:A\to B$ $\exists$?

$A=\{i_1,i_2,\dots,i_a\}$

$$\left.\begin{array}{l} f(i_1)\to b \text{ variante} \\ f(i_2)\to \text{-\,\textquotedbl-} \\ \vdots \\ f(i_a)\to \text{-\,\textquotedbl-} \end{array}\right\} \Rightarrow \underbrace{b\cdot b\cdot\ldots\cdot b}_{\text{de } a \text{ ori}} = b^a \text{ fct}$$

b) câte fct inj. $f:A\to B$? $(a\le b)$

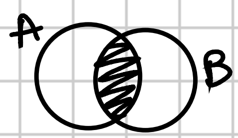
$$\left.\begin{array}{l} f(i_1)\to b \text{ var} \\ f(i_2)\to b-1 \text{ var} \\ \vdots \\ f(i_a)\to b-a+1 \text{ var}\end{array}\right\} \Rightarrow A_b^a = \frac{b!}{(b-a)!}$$

c) câte fct bij. $f:A\to B$? $(a=b)$

$A_b^a = A_b^b = \frac{b!}{0!} = b! = P_b$

$0! = 1$

$|A\cup B| = |A| + |B| - |A\cap B|$



$|A\cup B\cup C| = |A| + |B| + |C| - |A\cap B| - |A\cap C| - |B\cap C| + |A\cap B\cap C|$

$|A| = a + x + t + y$

$|B| = b + x + t + z$

$|C| = c + y + t - z$

Principiul includerii și al excluderii:

$$|A_1\cup A_2\cup\ldots\cup A_n| = \sum_{i=1}^{n}|A_i| - \sum_{1\le i<j\le n}|A_i\cap A_j| + \sum_{1\le i<j<k\le n}|A_i\cap A_j\cap A_k| - \ldots + (-1)^{n+1}|A_1\cap\ldots\cap A_n|$$

⑤ Câte fct surj. $\exists$?

$\hookrightarrow$ Câte fct nesurj. $\exists$?

f nu e surj. dacă $\exists y\in B$ a.î. $f(x)\ne y$, $\forall x\in A$

$B = \{1, 2, \ldots, b\}$

$B_1 = \{f:A\to B \,/\, f(x)\ne 1, \forall x\in A\}$

$\vdots$

$B_b = \{f:A\to B \,/\, f(x)\ne b, \forall x\in A\}$

$B_1\cup\ldots\cup B_b$ = mulțimea fct surjective $f:A\to B$

$|B_1\cup\ldots\cup B_b| \overset{not}{=} N$

$|B_i| = (b-1)^a$ variante

$|B_i \cap B_j| = (b-2)^a \ldots |B_1 \cap \ldots \cap B_b| = 0$

$|B_{i_1} \cap \ldots \cap B_{i_k}| = (b-k)^a$

$N = b\cdot(b-1)^a - C_b^2\cdot(b-2)^a + C_b^3\cdot(b-3)^a + \ldots + (-1)^{b+1}\cdot 0$

$$\boxed{N = \sum_{i=1}^{b} (-1)^{i+1} \cdot C_b^i \cdot (b-i)^a}$$

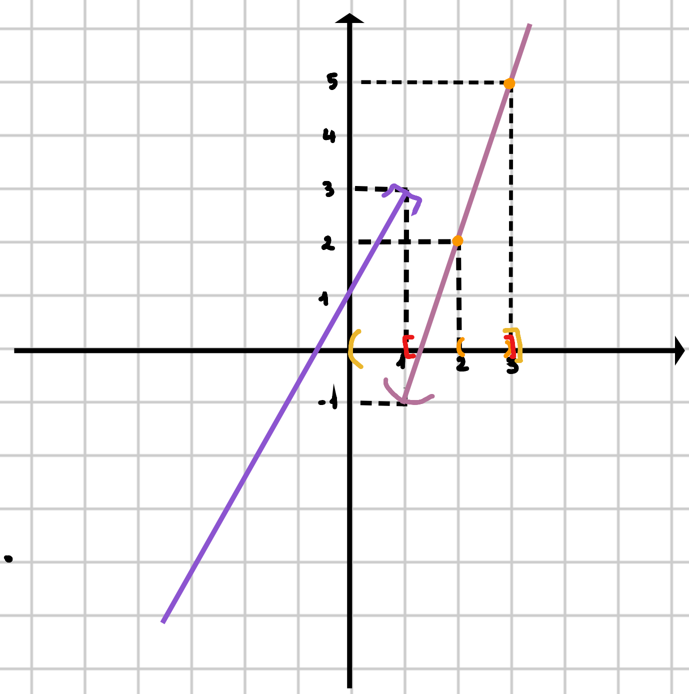
nr de fct surjective : $\boxed{b^a - N}$

⑥ $f:\mathbb{R}\to\mathbb{R},\ f(x) = \begin{cases} 2x+1, & x \le 1 \\ 3x-4, & x > 1 \end{cases}$

a) f - bij. ?

$\begin{cases} x = 1 \Rightarrow f(1) = 3 \\ x = 0 \Rightarrow f(0) = 1 \end{cases}$

$\begin{cases} x = 2 \Rightarrow f(2) = 2 \\ x = 3 \Rightarrow f(3) = 5 \\ \lim\limits_{\substack{x\to 1 \\ x>1}} f(x) = \lim\limits_{\substack{x\to 1 \\ x>1}} (3x-4) = -1 \end{cases}$



f. nu este inj. => f. nu este surj.

b) $f((2,3)) = (2,5)$

$f([1,3]) = \{3\} \cup (-1,5)$

$f((0,3]) = [1,3$

Seminar - Săptămâna 2

Recapitulare:

$f: A \to B$

$\operatorname{Im} f = \{ f(x) / x \in A \} = f(A)$

$A_1 \subseteq A \to f(A_1) = \{ f(x) / x \in A_1 \}$

$B_1 \subseteq B \to f^{-1}(B_1) = \{ x \in A \mid f(x) \in B_1 \}$

preimaginea lui B prin f

$\exists !$ chiar dacă $f \neq$ bij.

1. $f: \mathbb{R} \to \mathbb{R}, f(x) = \begin{cases} x+2, & x < 2 \\ x^2 - 5x + 6, & x \geq 2 \end{cases}$

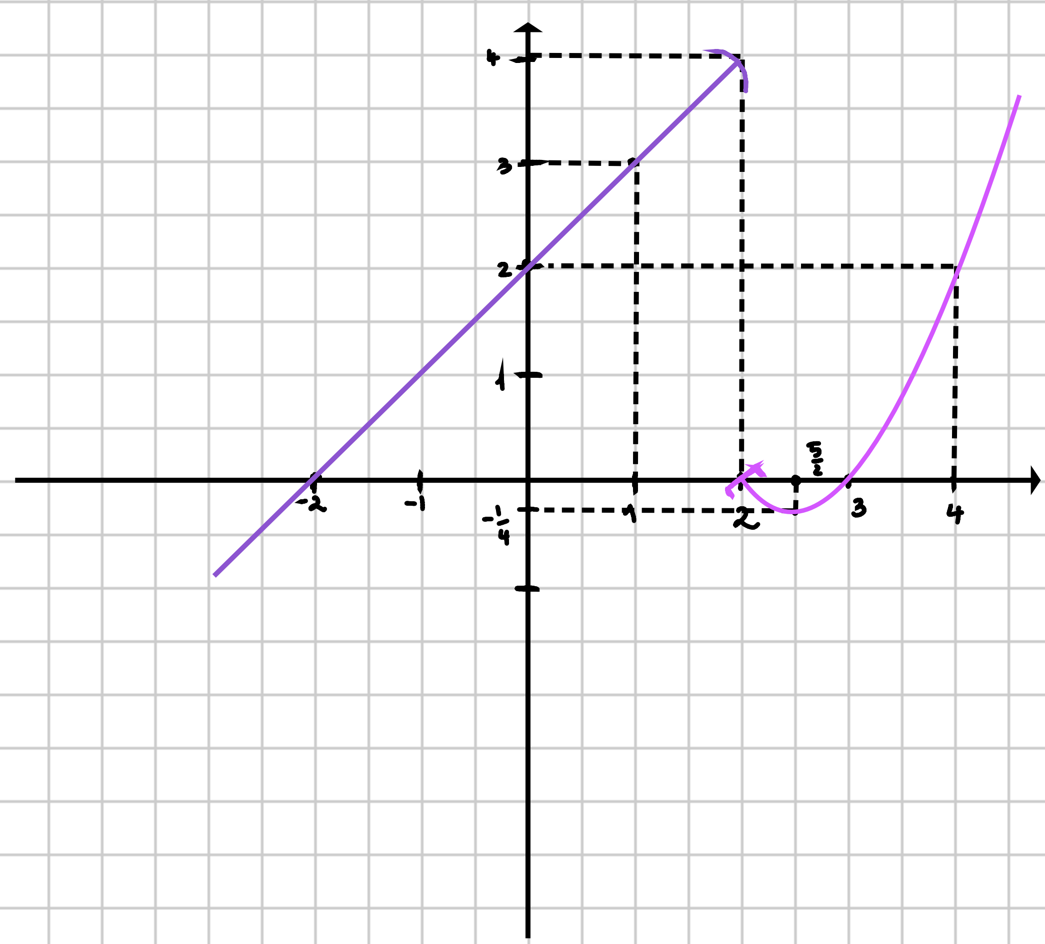
a) f - bij. ? $\not$

b) $f([1,2))$, $f([2,4])$, $f((1,3))$, $f^{-1}\left(\left(-\frac{1}{4}, 0\right)\right)$, $f^{-1}([-2,0])$, $f^{-1}((0,5))$

#

a) $x^2 - 5x + 6 = x^2 - 2x - 3x + 6 = x(x-2) - 3(x-2) = (x-2)(x-3)$

$\begin{cases} x = 0 \Rightarrow f(0) = 2 \\ x = 1 \Rightarrow f(1) = 3 \\ x = -2 \Rightarrow f(-2) = 0 \\ \lim\limits_{\substack{x \to 2 \\ x < 2}} x + 2 = 4 \end{cases}$



$V f\left(-\frac{b}{2a}, -\frac{\Delta}{4a}\right)$

$\frac{-b}{2a} = \frac{5}{2}$

$\Delta = 25 - 24 = 1 \Rightarrow \frac{-\Delta}{4a} = \frac{-1}{4}$

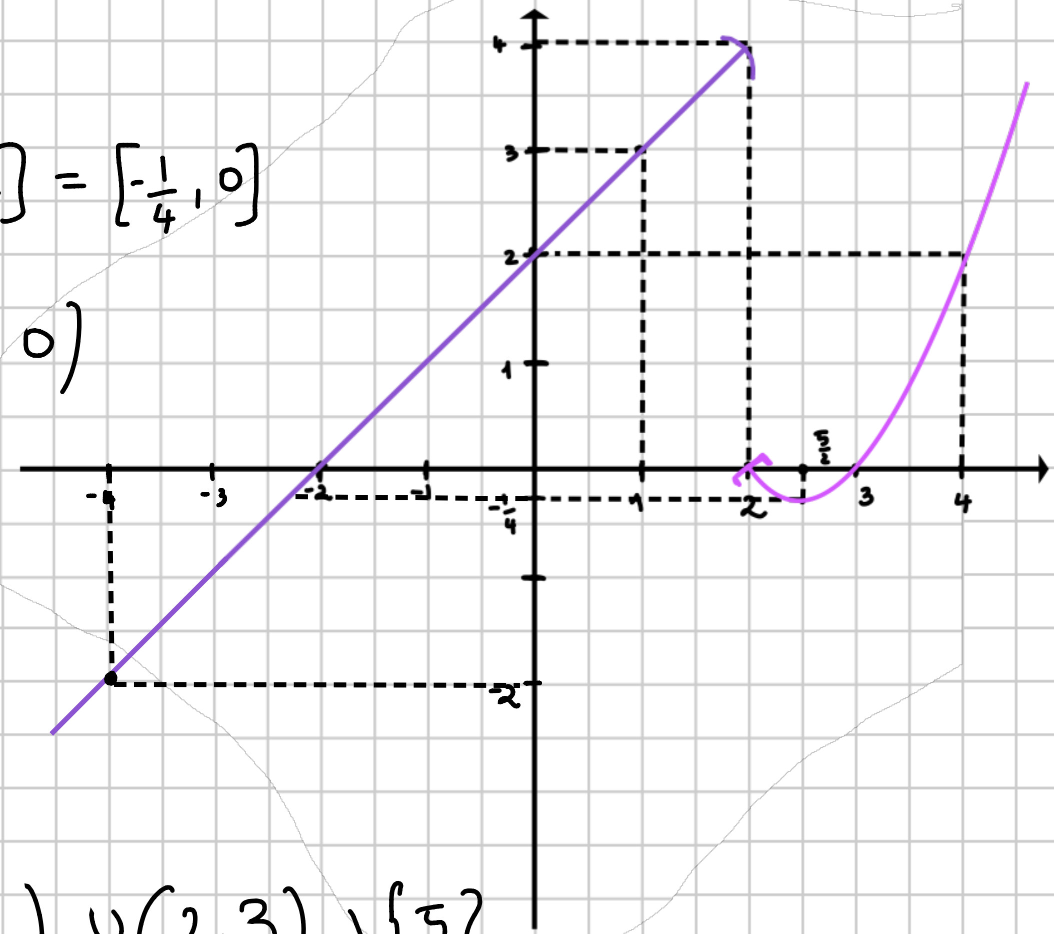
$x = 4 \Rightarrow f(4) = 16 - 20 + 6 = 2$

f surj dar nu inj $\Rightarrow$ f nu este bij.

b) $f([1,2)) = [3,4)$

$f([2,4]) = [-\frac{1}{4}, 0] \cup [0,2] = [-\frac{1}{4}, 0]$

$f((1,3)) = (3,4) \cup [-\frac{1}{4}, 0)$



$f^{-1}((-\frac{1}{4}, 0))$

$x+2 = -\frac{1}{4} \Rightarrow x = -\frac{1}{4} - 2 = -\frac{9}{4}$

$x+2 = 0 \Rightarrow x = -2$

$f^{-1}((-\frac{1}{4}, 0)) = (-\frac{9}{4}, -2) \cup (2,3) \setminus \{\frac{5}{2}\}$

$f^{-1}([-2,0)) = (-4,-2) \cup (2,3)$

$x+2 = -2 \Rightarrow x = -4$

$x+2 = 0 \Rightarrow x = -2$

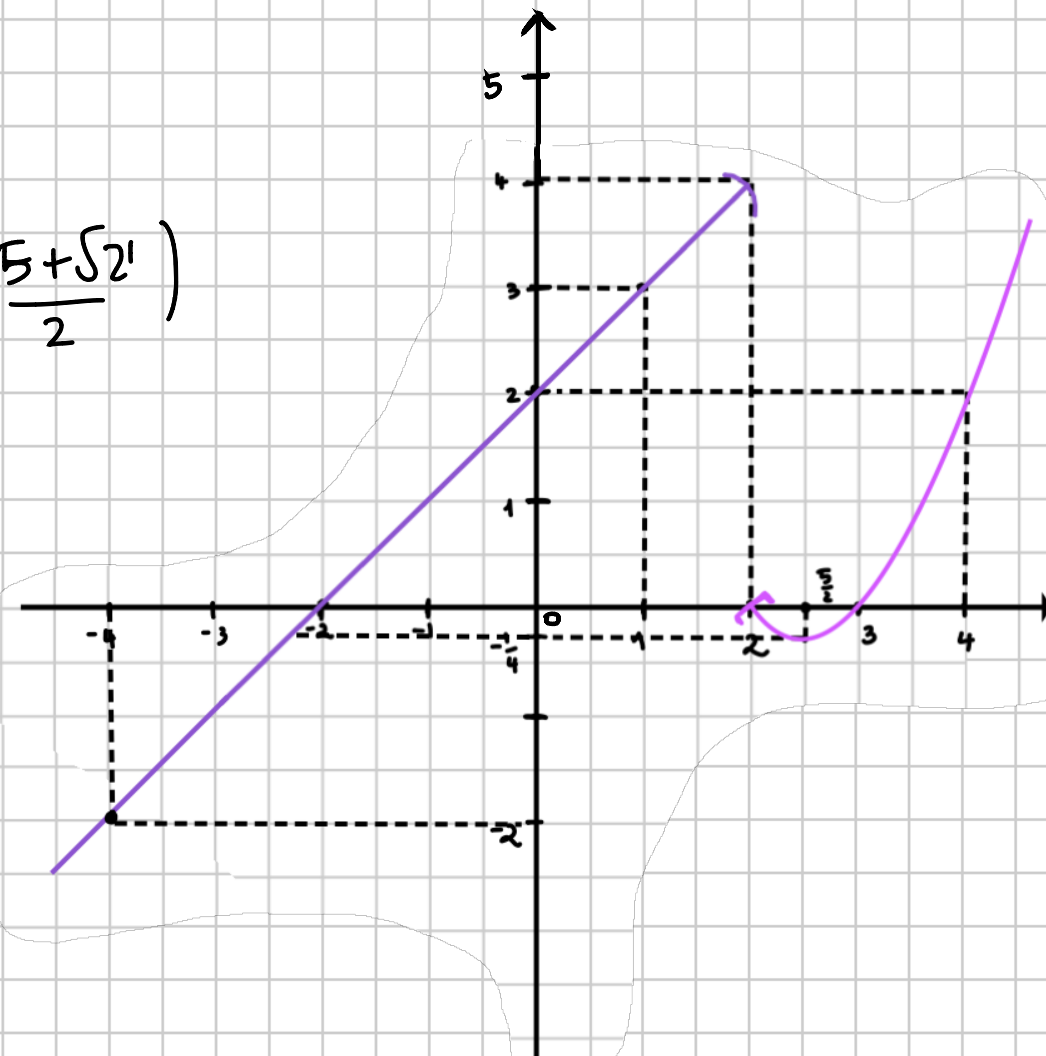
$f^{-1}((0,5)) = (-2,2) \cup (3, \frac{5+\sqrt{21}}{2})$

$x^2 - 5x + 6 = 5 \Rightarrow$

$\Rightarrow x^2 - 5x + 1 = 0$

$\Delta = 25 - 4 = 21$

$x_{1,2} = \frac{5 \pm \sqrt{21}}{2}$



(2) $f: A \to B$

dacă f-inj. $\Rightarrow \exists\, r: B \to A$ a.î. $r \circ f = 1_A$

(retracta lui f)

↳ unică dacă f-bij.

Dem. Construim funcția r

Dacă $b \in \operatorname{Im} f \Rightarrow \exists!\, a_b \in A$ cu $f(a_b) = b$

Def. $r(b) = a_b$

Dacă $b \notin \operatorname{Im} f \Rightarrow r(b) = a \in A$

$(r \circ f)(a_b) = r(f(a_b)) = r(b) = a_b$

P.I.E. (Principiul includerii și al excluderii)

$|A_1 \cup \ldots \cup A_n| = \sum |A| - \sum |A_i \cap A_j| + \sum |A_i \cap A_j \cap A_k| + \ldots + (-1)^{n+1} |A_1 \cap \ldots \cap A_n|$

1) $\varphi(n)$ = funcția Euler

$n \in \mathbb{N},\ n \geq 2$ ↳ nr de nr naturale $\leq n$ și prime cu n

Obs. $a, b \in \mathbb{N}$ sunt prime între ele dacă $(a, b) = 1$

ex: $\varphi(12) = 12\left(1 - \frac{1}{2}\right)\left(1 - \frac{1}{3}\right) = 12 \cdot \frac{1}{2} \cdot \frac{2}{3} = 4$

$$\varphi(n) = n\left(1 - \frac{1}{p_1}\right)\left(1 - \frac{1}{p_2}\right) \cdot \ldots \cdot \left(1 - \frac{1}{p_s}\right)$$

unde $n = p_1^{e_1} \ldots p_s^{e_s}$

det. nr de nr $\leq n$ și neprime cu n

$A_{p_1} = \{x = p_1 \cdot q \leq n\}$

$\vdots$

$A_{p_s} = \{x = p_s \cdot q \leq n\}$

$N = |A_{p_1} \cup A_{p_2} \cup A_{p_3} \cup \ldots \cup A_{p_s}|$

ex: $n = 12$: $p_1 = \{2, 4, 6, 8, 10, 12\} \to \frac{12}{2}$

$p_2 = \{3, 6, 9, 12\} \to \frac{12}{3}$

$p_3 = \{4, 8, 12\} \to \frac{12}{4}$

$\vdots$

$|A_{p_i}| = \frac{n}{p_i}$; $|A_{p_i} \cap A_{p_j}| = \frac{n}{p_i \cdot p_j}$; ... ; $|A_{p_{i_1}} \cap \dots \cap A_{p_{i_g}}| = \frac{n}{p_{i_1} \dots p_{i_g}}$

$$N = \sum_{i=1}^{s} \frac{n}{p_i} - \sum_{i,j} \frac{n}{p_i p_j} + \dots + (-1)^{s+1} \cdot \frac{n}{p_{i_1} \cdot \ldots \cdot p_{i_s}}$$

$$N = n\left(\frac{1}{p_1} + \frac{1}{p_2} + \dots + \frac{1}{p_s} - \frac{1}{p_1 p_2} - \frac{1}{p_1 p_3} - \dots - \frac{1}{p_{s-1} p_s} + \dots + (-1)^{s+1} \cdot \frac{1}{p_1 \cdot \ldots \cdot p_s}\right)$$

$$\varphi(n) = n - N = n\left(1 - \frac{1}{p_1} \dots\right)$$

MIM: V: $P(2)$: $n = p_1^{e_1} \cdot p_2^{e_2}$

$$\varphi(n) = n\left(1 - \frac{1}{p_1} - \frac{1}{p_2} + \dots + \frac{1}{p_1 p_2}\right) = n\left[1 - \frac{1}{p_1} - \frac{1}{p_2}(1 - p_1)\right] = n\left(1 - \frac{1}{p_1}\right)\left(1 - \frac{1}{p_2}\right)$$

D: Pp. $P(k)$ adev $\to$ dem $P(k+1)$

$P(k)$: $n\left(1 - \frac{1}{p_1}\right)\left(1 - \frac{1}{p_2}\right) \cdot \ldots \cdot \left(1 - \frac{1}{p_k}\right) = n\left(1 - \frac{1}{p_1} - \frac{1}{p_2} - \dots - \frac{1}{p_k} + \frac{1}{p_1 p_2} + \dots +\right.$

$$\left. + \underbrace{\frac{1}{p_{k-1} \cdot p_k} + \dots + (-1)^k \cdot \frac{1}{p_1 \cdot \ldots \cdot p_k}}_{\text{not } P}\right) \quad \Big| \; : n \;\; \cdot \left(1 - \frac{1}{p_{k+1}}\right)$$

$$\left(1 - \frac{1}{p_1}\right)\left(1 - \frac{1}{p_2}\right) \cdots \left(1 - \frac{1}{p_k}\right)\left(1 - \frac{1}{p_{k+1}}\right) = P - \frac{1}{p_{k+1}} + \frac{1}{p_1 \cdot p_{k+1}} + \frac{1}{p_2 \cdot p_{k+1}} +$$

$$+ \frac{1}{p_k \cdot p_{k+1}} - \frac{1}{p_1 p_2 p_{k+1}} - \dots - \frac{1}{p_{k-1} \cdot p_k \cdot p_{k+1}} + \dots + (-1)^{k+1} \cdot \frac{1}{p_1 \cdot \ldots \cdot p_{k+1}} \quad (A)$$

$$\Rightarrow \varphi(n) = n\left(1 - \frac{1}{p_1}\right)\left(1 - \frac{1}{p_2}\right) \cdots \left(1 - \frac{1}{p_s}\right)$$

PIE: $|A_1 \cup \ldots \cup A_m| = \sum |A_i| - \sum |A_i \cap A_j| + \sum |A_i \cap A_j \cap A_k| +$
$+ \ldots + (-1)^{m+1} |A_1 \cap \ldots \cap A_m|$

(2) A = mulțime cu n elemente

Câte partiții cu k elemente are mulțimea A?

Partiția unei mulțimi $M = M_1, M_2, \ldots, M_p \subseteq M$

1) $M_1 \cup \ldots \cup M_p = M$

2) $M_i \cap M_j = \emptyset, \forall i, j$

ex: $A = \{1, 2, 3, 4\}$, $k = 3$

$\{1\}, \{2,3\}, \{4\}$

$\{2\}, \{1,3\}, \{4\}$

$\{1, 2, 3, 4\} \rightarrow \{\_ \ \_ \ \_\}$

1.2.3 4
1.2 3.4
2.1 3.4
2.1 4.3
3.12.4
3.1.24

fie B = mulț. cu k el.

$f: A \to B$ surj. $\Rightarrow S = k^n - \sum_{i=1}^{k} (-1)^{i+1} \cdot (k-i)^n \cdot C_k^i$

$S(n,k) = \frac{S}{k!}$ (partiții cu k elem.)

(1) $f: \mathbb{R} \to \mathbb{R}$

$$f(x) = \begin{cases} x^2 - 2x - 7, & x \leq -1 \\ x + 3, & x > -1 \end{cases}$$

a) f. inj, surj, bij?

b) $f((-3,5)) = ?$, $f^{-1}((-2,4]) = ?$

a) $f(x) = x^2 - 2x - 7, x \leq -1$

$\Delta = 4 + 28 = 32$

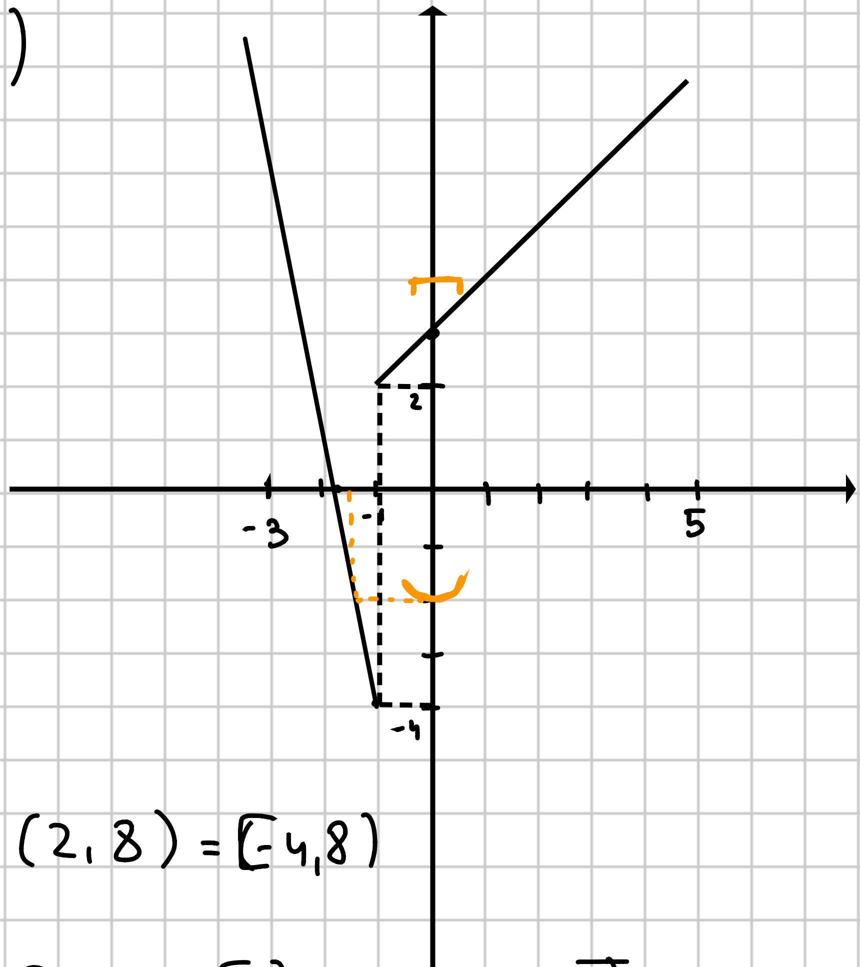
$x_{1,2} = \frac{2 \pm 4\sqrt{2}}{2} = 1 \pm 2\sqrt{2}$

$f(-1) = -4$

$V_f\left(\frac{-b}{2a}, \frac{-\Delta}{4a}\right) = (1, -8)$

$\frac{-b}{2a} = \frac{2}{2} = 1$

$\frac{-\Delta}{4a} = \frac{-32}{4} = -8$



f nu e inj $\left.\begin{matrix} \\ \\ \end{matrix}\right\} \Rightarrow$

f nu e surj

$\Rightarrow f$ nu e surj.

$f((-3,5)) = [-4,0) \cup (2,8) = [-4,8)$

$f^{-1}([-2,4]) = (1-2\sqrt{3}, 1-\sqrt{6}) \cup (-1, 1]$

$x^2 - 2x - 7 = -2 \Rightarrow x^2 - 2x - 5 = 0$

$\Delta = 4 + 20 = 24$

$x_{1,2} = \frac{2 \pm 2\sqrt{6}}{2} = 1 \pm \sqrt{6}$

$x^2 - 2x - 7 = 4 \Rightarrow x^2 - 2x - 11 = 0$

$\Delta = 4 + 44 = 48$

$x_{1,2} = \frac{2 \pm 4\sqrt{3}}{2} = 1 \pm 2\sqrt{3}$

Seminar - Săptămâna 3

① $f: \mathbb{R} \to \mathbb{R}$

$$f(x) = \begin{cases} -5x^2 - 50x - 121, & x \le -4 \\ x, & -4 < x \le 3 \\ -x + 4, & x > 3 \end{cases}$$

a) $\operatorname{Im} f = ?$

b) $f((-6,0))$, $f^{-1}((-2,2))$

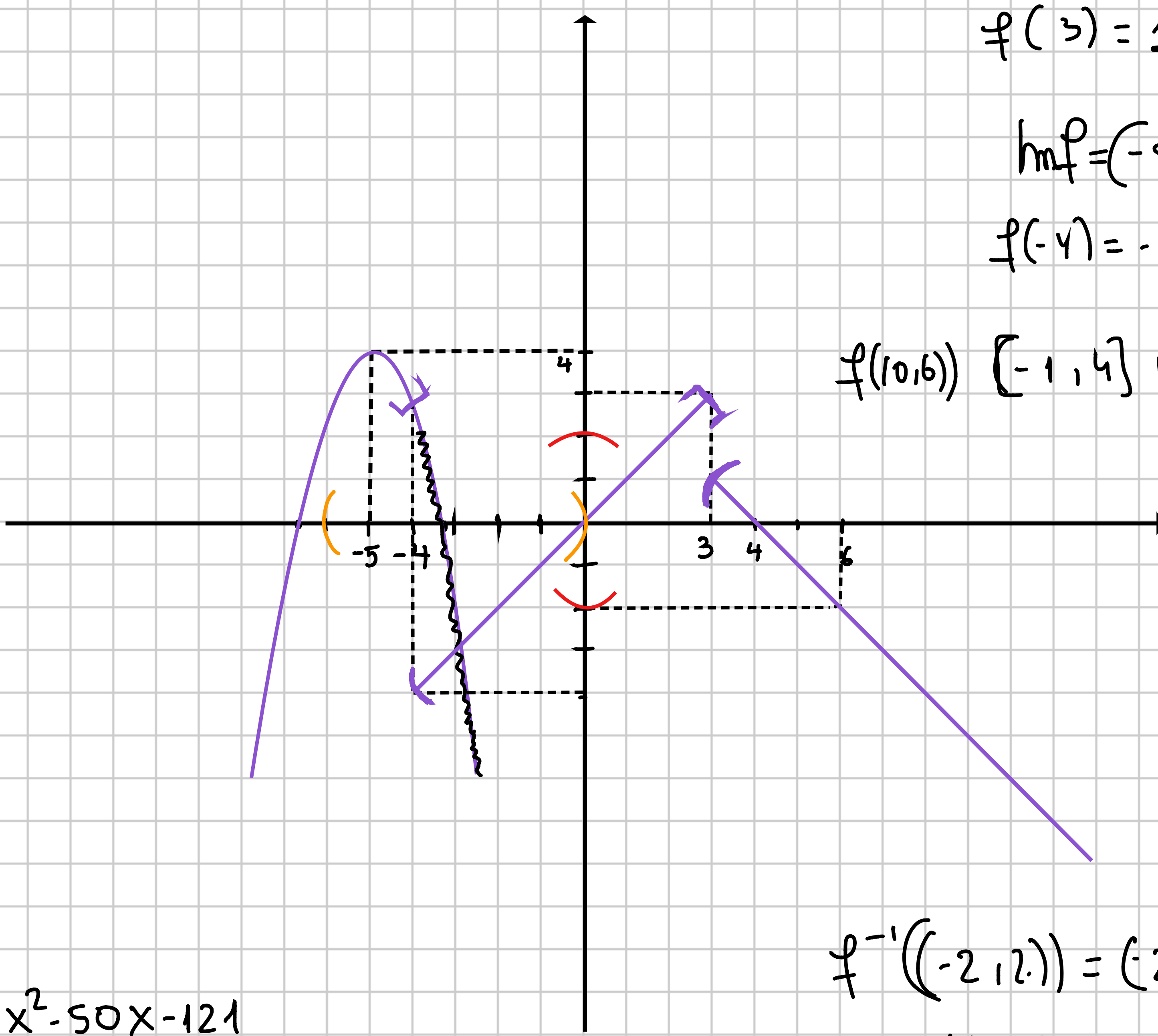
$f(3) = 3$, $f(-4) = -4$

$f(3) = 1$, $f(4) = 0$, $f(6) = -2$

$\operatorname{Im} f = (-\infty, 4]$

$f(-4) = -(80 - 200 + 121) =$
$= -1$

$f((-6,0))$ $[-1, 4] \cup (-4, 0) = (-4, 4]$



$f^{-1}((-2,2)) = (-2, 2) \cup (2, 6)$

$= -x + 4 = -2 \Rightarrow -x = -6 \Rightarrow x = 6$
$-x + 4 = 2 \Rightarrow x = 2$

$f(x) = -5x^2 - 50x - 121$

$5x^2 + 50x + 121 = 0$

$\Delta = 2500 - 2420 = 80$

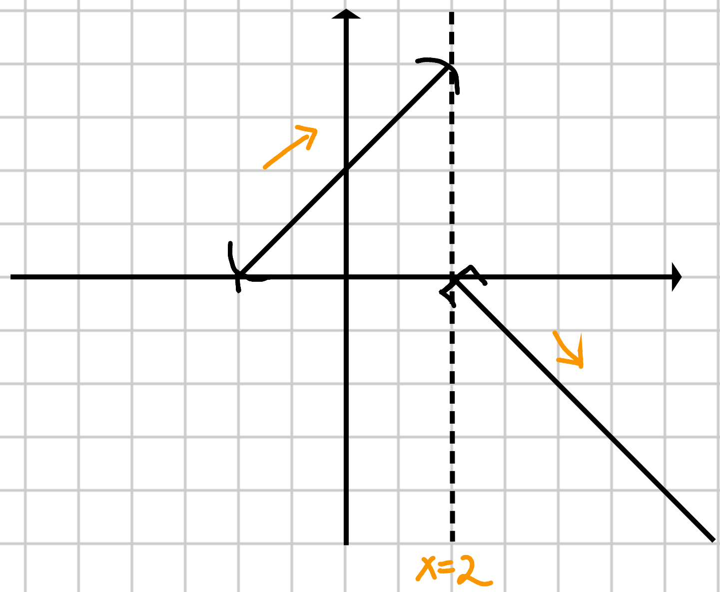
$x_{1,2} = \dfrac{-50 \pm 4\sqrt{5}}{10} = -5 \pm \dfrac{2\sqrt{5}}{5}$

$\dfrac{8{,}94}{5}$ ↗ $1{,}7$ < $-5 + 1{,}7 = -3{,}3$; $-5 - 1{,}7 = -6{,}7$

$V_f\left(-\dfrac{b}{2a}, -\dfrac{\Delta}{4a}\right)$; $-\dfrac{b}{2a} = \dfrac{50}{-10} = -5$; $-\dfrac{\Delta}{4a} = \dfrac{-80}{-20} = 4$

② Orice funcție monotonă este injectivă (A)

Orice funcție injectivă este monotonă (F)



$f(x)=\begin{cases} x+2, & -2<x<2 \\ -x+2, & x\geq 2 \end{cases}$

f. injectivă dar $f \neq$ monotonă

③ Ex de fct nesurj. :

a) $\mathbb{Z} \to \mathbb{Z}$

$f(x)=2x$

$f(x)=x^2$

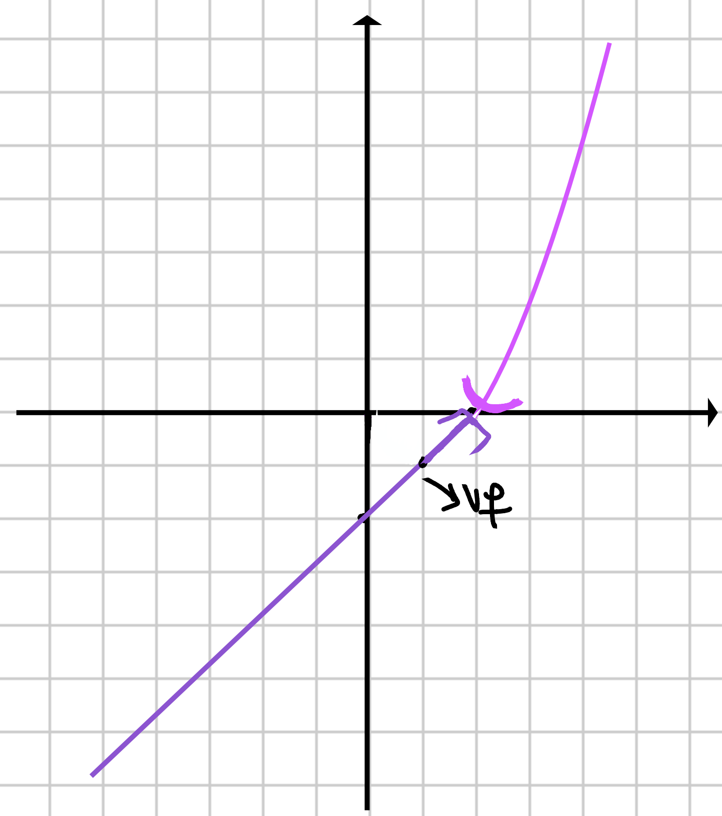
b) $\mathbb{R} \to \mathbb{R}$

$f(x)=x^2$

④ $f:\mathbb{R}\to\mathbb{R}$, $f(x)=\begin{cases} x-2, & x\leq 2 \\ x^2-2x, & x>2 \end{cases}$

a) f. inj., surj., bij. ?

b) $f^{-1}=?$



a) $x(x-2)=0 \quad \begin{array}{l} x=0 \\ x=2 \end{array}$

$V_f\left(\frac{-b}{2a}, \frac{-\Delta}{4a}\right)$; $\frac{-b}{2a}=\frac{2}{2}=1$; $\frac{-\Delta}{4a}=-\frac{4}{4}=-1$

este inj, surj, bij.

b) I $x-2=y \Rightarrow x=y+2$

$x\leq 2 \Rightarrow y\leq 0$

II $x^2 - 2x = y$

$x^2 - 2x - y = 0$

$\Delta = 4 + 4y$

$x_{1,2} = \dfrac{2 \pm \sqrt{4+4y}}{2} = 1 \pm \sqrt{1+y} = 1 + \sqrt{1+y}\ (x > 2)$

$1 + \sqrt{1+y} > 2 \Rightarrow \sqrt{1+y} > 1 \mid^2 \Rightarrow 1+y > 1 \Rightarrow y > 0$

$f^{-1}(x) = \begin{cases} x+2, & x \leq 0 \\ 1 + \sqrt{1+x}, & x > 0 \end{cases}$

Relații:

A, B = mulțimi nevide

$A \times B = \{(a, b) / a \in A, b \in B\}$

$\sim$ relație pe $A \times B$ dacă $\sim \subseteq A \times B$

nr. total de rel. pe A dacă $|A| = n$ este 2^{n^2}

$\underset{A}{a} \sim \underset{B}{b}$ dacă $(a, b) \in \sim$

Proprietăți:

$A \neq \emptyset$, $\sim$ = rel. pe A

1) reflexivitate: $\sim$ = reflexivă dacă $\forall x \in A$, $x \sim x$

2) simetrie: $\sim$ = simetrică dacă $\forall x, y \in A$ cu $x \sim y \Rightarrow y \sim x$

3) antisimetrie: $\sim$ = antisemetrică dacă $\forall x, y \in A$ cu $x \sim y$ și $y \sim x \Rightarrow$
$\Rightarrow x = y$

4) tranzitivitate: $\sim$ = tranzitivă dacă $\forall x, y, z \in A$ cu $x \sim y$ și $y \sim z \Rightarrow$
$\Rightarrow x \sim z$

⑤ Pe $\mathbb{R}$:

a) $x \sim y \iff x-y \in \mathbb{Z}$

(1): $\forall x \in \mathbb{R}$, $x-x=0 \in \mathbb{Z} \Rightarrow x \sim x$ ✓

(2): $\forall x,y \in \mathbb{R}$ cu $x \sim y \overset{?}{\Rightarrow} y \sim x$

$x \sim y \Rightarrow x-y \in \mathbb{Z} \Rightarrow -(x-y) \in \mathbb{Z} \Rightarrow y-x \in \mathbb{Z} \Rightarrow y \sim x$ ✓

(3): $\forall x,y \in \mathbb{R}$ cu $x \sim y$ și $y \sim x \overset{?}{\Rightarrow} x=y$

$x \sim y \Rightarrow x-y \in \mathbb{Z}$

$y \sim x \Rightarrow y-x \in \mathbb{Z}$

$$\left.\begin{array}{l}\text{Pt } x=1, y=2 \Rightarrow 1-2=-1 \in \mathbb{Z} \\ \qquad 2-1=1 \in \mathbb{Z} \\ \qquad \text{dar } -1 \neq 1\end{array}\right\} \Rightarrow \sim \text{ nu este antisimetrică}$$

(4): $\forall x,y,z \in \mathbb{R}$ cu $x \sim y$ și $y \sim z \overset{?}{\Rightarrow} x \sim z$

$$\left.\begin{array}{l}x \sim y \Rightarrow x-y \in \mathbb{Z} \\ y \sim z \Rightarrow y-z \in \mathbb{Z}\end{array}\right\} \oplus \Rightarrow x-y+y-z \in \mathbb{Z} \Rightarrow x-z \in \mathbb{Z} \Rightarrow$$

$\Rightarrow x \sim z$ ✓

b) $x \sim y \iff x \leq y$

(1): $\forall x \in \mathbb{R}$, $x \leq x$ ✓

(2): $\forall x,y \in \mathbb{R}$, $x \sim y \overset{?}{\Rightarrow} y \sim x$

$x \sim y \Rightarrow x \leq y$

$y \sim x \Rightarrow y \leq x$

Pt $x=1$, $y=2$, $1 \leq 2$ dar $2 \not\leq 1 \Rightarrow \sim$ nu este simetrică

(3): $\forall x,y \in \mathbb{R}$ cu $x \sim y$ și $y \sim x \overset{?}{\Rightarrow} x=y$

$$\left.\begin{array}{l}x \sim y \Rightarrow x \leq y \\ y \sim x \Rightarrow y \leq x\end{array}\right\} \Rightarrow x=y \checkmark$$

(4): $\forall x,y,z \in \mathbb{R}$, $x \sim y$ și $y \sim z \overset{?}{\Rightarrow} x \sim z$

$$\left.\begin{array}{l}x \sim y \Rightarrow x \leq y \\ y \sim z \Rightarrow y \leq z\end{array}\right\} \Rightarrow x \leq y \leq z \Rightarrow x \leq z \Rightarrow x \sim z \checkmark$$

c) $x \sim y \Leftrightarrow x < y$

(2): $\forall x, y \in \mathbb{R}$ cu $x \sim y \overset{?}{\Rightarrow} y \sim x$

$$\left.\begin{array}{l} x \sim y \Rightarrow x < y \\ y \sim x \Rightarrow y < x \end{array}\right\} \Rightarrow \text{contradicție} \Rightarrow \sim \text{- nu este simetrică}$$

(3): $\forall x, y \in \mathbb{R}$ cu $x \sim y$ și $y \sim x \overset{?}{\Rightarrow} x = y$

$$\left.\begin{array}{l} x \sim y \Rightarrow x < y \\ y \sim x \Rightarrow y < x \end{array}\right\} \Rightarrow \text{contradicție} \Rightarrow \text{nu e antisim.}$$

(6) $A = \{1, 2, 3\}$

$\sim = \{(1,2), (2,3)\}$

$\sim^* = \{(1,2), (2,3), (1,3)\}$ (închidere tranzitivă)

$1 \sim 2,\ 2 \sim 3 \Rightarrow 1 \sim 3$

$\chi_{A \Delta B} = (\chi_A - \chi_B)^2$

$A \Delta B = (A \setminus B) \cup (B \setminus A)$

1) $\chi_A = \chi_B \Rightarrow A = B$

2) $(\chi_A)^2 = \chi_A$

3) $\chi_\emptyset = 0$

4) $\chi_{C_E A} = 1 - \chi_A$

5) $\chi_{A \cap B} = \chi_A \cdot \chi_B$

6) $\chi_{A \cup B} = \chi_A + \chi_B - \chi_A \cdot \chi_B$

7) $\chi_{A \setminus B} = \chi_A (1 - \chi_B)$

$$\chi_{A \Delta B} = \chi_{(A \setminus B) \cup (B \setminus A)} = \chi_{A \setminus B} + \chi_{B \setminus A} - \chi_{A \setminus B} \cdot \chi_{B \setminus A} =$$

$$= \chi_A - \chi_A \cdot \chi_B + \chi_B - \chi_B \cdot \chi_A - \chi_A (1 - \chi_B) \cdot \chi_B (1 - \chi_A)$$

$$= \chi_A - \chi_A \cdot \chi_B + \chi_B - \chi_B \cdot \chi_A - \chi_A \cdot \chi_B (1 - \chi_A - \chi_B + \chi_A \cdot \chi_B)$$

$$= \chi_A + \chi_B - 2 \cdot \chi_A \cdot \chi_B - \chi_A \cdot \chi_B + \chi_A^2 \cdot \chi_B + \chi_A \cdot \chi_B^2 - \chi_A^2 \cdot \chi_B^2$$

$$= x_A + x_B - 3\, x_A \cdot x_B + 2 \cdot x_A \cdot x_B - x_A \cdot x_B$$

$$= x_A + x_B - 2 \cdot x_A \cdot x_B =$$

$$= x_A^2 - 2 \cdot x_A \cdot x_B + x_B^2$$

$$= (x_A - x_B)$$

Seminar - Săptămâna 4

def.
$\sim$ relație pe $A \neq \emptyset$

1) $\sim$ = reflexivă dacă $\forall x \in A \Rightarrow x \sim x$
2) $\sim$ = simetrică dacă $\forall x, y \in A$ cu $x \sim y \Rightarrow y \sim x$
3) $\sim$ = antisimetrică dacă $\forall x, y \in A$ cu $x \sim y$ și $y \sim x \Rightarrow x = y$
4) $\sim$ = tranzitivă dacă $\forall x, y, z \in A$ cu $x \sim y$ și $y \sim z \Rightarrow x \sim z$

def.
$\sim$ = relație de echivalență pe A dacă $\sim$ = reflexivă, simetrică și tranzitivă

pt $x \in A \Rightarrow [x] = \hat{x} = \{y \in A \mid x \sim y\}$ = "clasa de echivalență a lui x"

$A/_{\sim} = \{[x] \mid x \in A\}$ = mulțimea factor a lui $\sim$ = mulțimea tuturor claselor de echivalență

SCR = sistem complet de reprezentanți = mulțime care conține câte un reprezentant din fiecare clasă de echivalență = o submulțime a lui A

Aplicații

1) Pe $\mathbb{R}$: rel de echiv ?

a) $x \sim y \Leftrightarrow |x-y| < 2$

1) $\forall x \in \mathbb{R}$: $|x-x| < 2 \Rightarrow 0 < 2$ (A) $\Rightarrow x \sim x$

2) $\forall x, y \in \mathbb{R}$: $|x-y| < 2 \Rightarrow |-(-x+y)| < 2 \Rightarrow |y-x| < 2 \Rightarrow$
$\Rightarrow x \sim y$

~~3) $\forall x, y \in \mathbb{R}$: $|x-y| < 2$~~ ?
~~$|y-x| < 2$~~

4) $\forall x, y, z \in \mathbb{R}$: $|x-y| < 2$
$|y-z| < 2$ $\Rightarrow x \sim z$?

Pp. $x=1$, $y=2$, $z=3$

$|1-2|<2$
$|2-3|<2$ Ad dar $|1-3|<2$ Fals $\Rightarrow$

$\sim$ nu este tranzitivă $\Rightarrow$

$\Rightarrow$ $\sim$ nu este relație de echivalență

b) $x \sim y \Leftrightarrow x-y \in \mathbb{Z}$

1), 2), 4) Ad (din Seminar 3) $\Rightarrow$ $\sim$ este relație de echivalență

c) $x \sim y \Leftrightarrow x+y \in \mathbb{Z}$

1) $\forall x \in \mathbb{R}$: $x+x=2x$

dacă $x=\frac{1}{5} \Rightarrow 2x=\frac{2}{5} \notin \mathbb{Z} \Rightarrow$ $\sim$ nu este reflexivă $\Rightarrow$

$\Rightarrow$ $\sim$ nu este relație de echiv.

2

Pe $\mathbb{C}$: $z \sim w \Leftrightarrow |z|=|w|$

a) relație de echivalență ?

b) un SCR ?

a)

1) $\forall z \in \mathbb{C} : |z|=|z|$ Ad $\Rightarrow z \sim z$ Ad

2) $\forall z, w \in \mathbb{C} : |z|=|w| \Rightarrow |w|=|z|$ Ad $\Rightarrow w \sim z$

4) $\forall z, w, q \in \mathbb{C} \Rightarrow \begin{matrix} |z|=|w| \\ |w|=|q| \end{matrix} \Rightarrow |z|=|w|=|q| \Rightarrow z \sim q$ Ad

$\Rightarrow$ $\sim$ relație de echivalență

b) $[1]=\{z\in\mathbb{C} \mid |z|=1\}=\{a+bi\in\mathbb{C} \mid \sqrt{a^2+b^2}=1\}=$

$=\{\pm i, \pm 1\}$

$[a]$ pt fiecare $a\in[0,+\infty)$

Un SCR $=[0,\infty)$

③ Pe $\mathbb{C}$: $z \sim w \iff z - w \in \mathbb{R}$

a) $\sim$ rel. de echivalență

b) un SCR = ?

a)

1) $\forall z \in \mathbb{C}$: $z - z \in \mathbb{R} \Rightarrow 0 \in \mathbb{R}$

2) $\forall z, w \in \mathbb{C}$: $z - w \in \mathbb{R} \Rightarrow -(w - z) \in \mathbb{R} \Rightarrow w - z \in \mathbb{R}$

4) $\forall z, w, g \in \mathbb{C}$: $z - w \in \mathbb{R}$

$$\frac{\begin{matrix} z - w \in \mathbb{R} \\ w - g \in \mathbb{R} \end{matrix}}{z - g \in \mathbb{R}} \; (+)$$

$z - g \in \mathbb{R} \Rightarrow \sim$ rel. de echiv.

b) $z - w \in \mathbb{R} \iff \mathrm{Im}(z) = \mathrm{Im}(w)$

$[bi] = \{a + bi \mid a \in \mathbb{R}\}$

$SCR = \{bi \mid b \in \mathbb{R}\}$

sau

$SCR = \{2 + bi \mid b \in \mathbb{R}\}$

④ Pe $\mathbb{Z}$: $x \sim y \iff 3 / (x - y) \iff$ au același rest la împărțirea la 3 $\iff \hat{x} = \hat{y}$

a) rel. de echiv ?

b) un SCR ?

a)

1) $\forall x \in \mathbb{Z}$: $x - x = 0 \vdots 3 \Rightarrow x \sim x$ (Ad)

2) $\forall x, y \in \mathbb{Z}$: $3 / x - y \Rightarrow 3 / (-1)(y - x) \Rightarrow$

$3 / y - x \Rightarrow y \sim x$

4) $\forall x, y, z \in \mathbb{Z}: \begin{matrix} 3/x-y \\ 3/y-z \end{matrix} \Rightarrow 3/x-z$ (A) $\Rightarrow x \sim z$

$\oplus$

$\Rightarrow \sim$ - relație de echivalență

b) $[0] = \{\ldots, -3, 0, 3, \ldots\}$ (M_3)

$[1] = \{\ldots, -2, 1, 4, \ldots\}$ (M_{3+1})

$[2] = \{\ldots, -1, 2, 5, \ldots\}$ (M_{3+2})

Un SCR $= \{-3, 4, 5\}$

Alt SCR $= \{0, 1, 2\}$

c) $\mathbb{Z}/_{\sim} = \{[0], [1], [2]\} = \mathbb{Z}_3$

⑤ Pe $\mathbb{Z}$: $x \sim y \Leftrightarrow x \equiv y \pmod{n} \Leftrightarrow n / x-y$

a) $\sim$ rel de echiv?

b) Un SCR?

a)

1) $\forall x \in \mathbb{Z}: x - x \,\vdots\, n \Rightarrow x \sim x$

2) $\forall x, y \in \mathbb{Z}: x - y \,\vdots\, n \Rightarrow (-1)(y-x) \,\vdots\, n \Rightarrow y - x \,\vdots\, n \Rightarrow y \sim x$

4) $\forall x, y, z \in \mathbb{Z}: \left.\begin{matrix} x - y \,\vdots\, n \\ y - z \,\vdots\, n \end{matrix}\right\} \Rightarrow x - z \,\vdots\, n \Rightarrow x \sim z$

$\oplus$

$\Rightarrow \sim$ relație de echivalență

b) $[0] = \{n \cdot k, k \in \mathbb{Z}\}$

$[1] = \{1 + n \cdot k \,/\, k \in \mathbb{Z}\}$

$\vdots$

$[n-1] = \{(n-1) + n \cdot k \,/\, k \in \mathbb{Z}\}$

sau $\{n, n+1, n+2, 2n-1\}$

⑥ Pe $\mathbb{N} \times \mathbb{N}$: $(m, n) \sim (p, q) \iff m + q = n + p$

rel de echiv?

$\mathbb{N} \times \mathbb{N} /_{\sim} = ?$

$m + q = n + p \iff m - n = p - q$

$[(n, 0)] = \{(a, b) / a - b = n\}$

$n \in \mathbb{N}^*$

$[(0, n)] = \{(a, b) / a - b = -n\}$

$[(0, 0)] = \{(a, a) / a \in \mathbb{N}\}$

Un SCR $= \{(n, 0) / n \in \mathbb{N}^*\} \cup$

$\cup \{(0, n) / n \in \mathbb{N}^*\} \cup \{(0, 0)\}$

$\mathbb{N} \times \mathbb{N} /_{\sim} = \mathbb{Z}$

Seminar 5

def.

$\sim$ = relație de ordine pe $A \neq \emptyset$ dacă

$\sim$ = refl, antisim, tranz.

Obs. „$\leq$" relație de ordine pe A

1) A - mulțime parțial ordonată

2) Dacă, în plus, $\forall x, y \in A \Rightarrow x \leq y$ sau $y \leq x$

$\Rightarrow A$ = mulțime total ordonată (toate el sunt "comparabile")

1 $N \to x / y$
sem 4

$\hookrightarrow$ rel de ordine pe $\mathbb{N}$

a) $(\mathbb{N}, |)$ = Mulțime total ordonată (MTO) ?

Nu, $2 \nmid 3$ și $3 \nmid 2 \Rightarrow 2$ nu sunt comp

$\Rightarrow$ MPO

Def.

1) Element minimal = un el care nu este mai mare decât niciun alt el.

pot fi mai multe

2) Element minim = un el care este mai mic decât toate celelalte elem

unic, dacă $\exists$

3) Element maximal —//— mic

4) Element maxim —//— mare

b) elem. minimale, maximale, minimul, maximul (dacă $\exists$)?

minim: $A / x, \forall x \in \mathbb{N} \Rightarrow 1 / x, \forall x \in \mathbb{N} \Rightarrow$ 1-minim

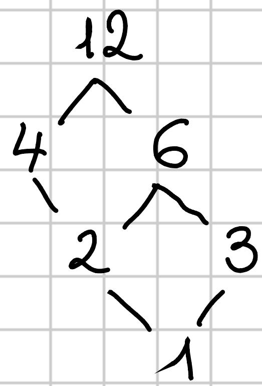
maxim: $x / B, \forall x \in \mathbb{N} \Rightarrow x / 0, \forall x \in \mathbb{N} \Rightarrow$ 0=maxim

② det. elem minimale, maximale, minim, maxim

a) $(D(12), /)$

$D(12) = \{1, 2, 3, 4, 6, 12\}$

Diagrama Hasse

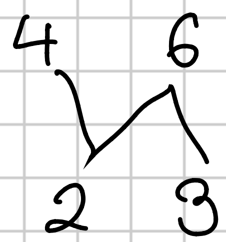


minim → 1

maxim → 12

b) $(DP(12), /)$

$DP = \{2, 3, 4, 6\}$



minimali → $\{2, 3\}$

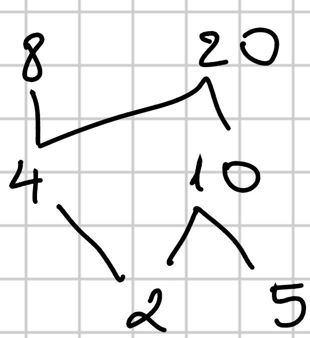
maximali → $\{4, 6\}$

c) $(DP(40), /)$

$DP = \{2, 4, 5, 8, 10, 20\}$

$40 = 2^3 \cdot 5^1 = 4 \cdot 2 = 8$

$DP = 8 - 2 = 6$ divizori proprii



minimali $= \{2, 5\}$

maximali $= \{8, 20\}$

d) $DP(60, |)$

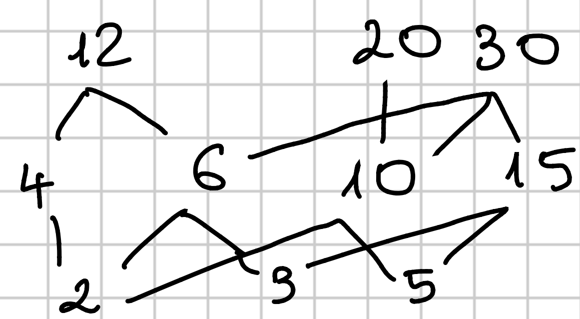
$60 = 2^2 \cdot 3^1 \cdot 5$

$nr\ div = (2+1)(1+1)(1+1) = 12$

$DP(60) = \{2, 3, 4, 5, 6, 10, 12, 15, 20, 30\}$

minimale $= \{2, 3, 5\}$

massimale $= \{12, 20, 30\}$

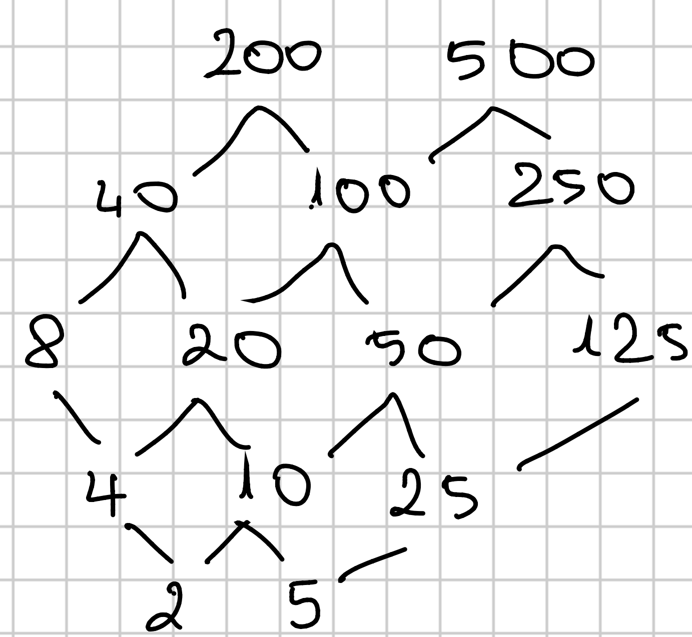


e) $DP(1000, |)$

$1000 = 2^3 \cdot 5^3$

$nr\ div = (3+1)(3+1) = 16 \to 14$ div propri

$DP(1000) = \{2, 4, 5, 8, 10, 20, 25, 40, 50, 125, 200, 250, 500\}$



minimale $= \{2, 5\}$

massimale $= \{200, 500\}$

Obs. $(DP(n), |)$

elem minimale : div primi

elem massimale $\frac{n}{p}$, p = div primi

③ MPO sau MTO

a) $(D(12), /)$ MPO $(2 \nmid 3,\ 3 \nmid 2 \Rightarrow$ nu sunt comparabile

b) $(DP(40), /)$ MPO $(2, 5)$

c) $(D(8), /) =$ MTO (Diagrama Hasse = lanț)

d) $(DP(256), /) =$ MTO (— // —)

Obs. $(D(n), /) =$ MTO dacă

$n = p^t$, $p =$ nr prim

$1 - p - p^2 - \dots - p^t = n$

(Diag. Hasse = lanț)

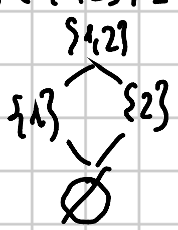
④ det e.m., e.max, e min, e max (dc $\exists$)

a) $(\mathbb{N}, \leq)$ $\to$ 0 = minim (unicul minimal)
$\to$ $\nexists$ maximale

b) $(\mathbb{Z}, \leq)$ $\nexists$ nimic

c) $(P(\{1,2\}), \subseteq)$

$P(\{1,2\}) = \{\emptyset, \{1\}, \{2\}, \{1,2\}\}$



minim: $\emptyset$

maxim: $\{1,2\}$

d) $(P(\{1,2\}) \setminus \emptyset, \subseteq)$

minimale: $\{1\}, \{2\}$

maxim: $\{1,2\}$

Aritmetica în $\mathbb{Z}$

• $/$ pe $\mathbb{Z}$

Refl. x/x ✓

AntiSim x/y și $y/x \Rightarrow x=y$?

AS dacă x/y și $y/x \Rightarrow x=y$? ✗

$-1/1$ & $1/-1 \Rightarrow 1 \neq -1$

Trans ✓

Teorema împărțirii cu rest (TÎR)

$a, b, c \in \mathbb{Z}$, $0 \leq r \leq |b|$

$a : b = q$, rest r

$a = q \cdot b + r$

ex: $a = -18$, $b = -5$

$-18 = -5 \cdot 4 + 2$

$-18 = -5 \cdot 3 - 3$ < 0

Def.

$gcd(a,b) = d$

1) d/a & d/b

2) $\forall d$ cu d_1/a & $d_1/b \Rightarrow d_1/d$

Def.

$lcm(a,b) = l$

1) a/l & b/l

2) $\forall l$ cu a/l_1 & $b/l_1 \Rightarrow l/l_1$

Obs. $a \cdot b = gcd(a,b) \cdot lcm(a,b)$

Algoritmul lui Euclid

input: $a, b \in \mathbb{Z}$

$\exists u, v \in \mathbb{Z}$ a.î. $gcd(a,b) = u \cdot a + v \cdot b$

output: $gcd(a,b) = d$

while $b \neq 0$

(a : b = q rest r)

$a \leftarrow b$

$b \leftarrow r$

$d \leftarrow a$

⑤ det $\gcd(a,b) = ?$

\+ det $u, v \in \mathbb{Z}$ aî $u \cdot a + v \cdot b = \gcd(a,b)$

a) $a = 348$

$b = 24$

$348 = \underbrace{24}_{a} \cdot 14 + \underbrace{12}_{b}$

$24 = 12 \cdot 2 + 0 \Rightarrow \gcd(348, 24) = 12$

u, v aî $348 \cdot u + 24 \cdot v = 12$?

$12 = 348 - 24 \cdot 14$

$12 = 348 \cdot \underset{u}{1} + 24 \cdot \underset{v}{(-14)}$

$\Rightarrow u_1 = 1$ & $v_1 = -14$

$\left.\begin{array}{l} \frac{348}{12} = 29 \Rightarrow v = -14 - 29 \cdot k \\ \frac{24}{12} = 2 \Rightarrow u = 1 + 2 \cdot k \end{array}\right\} \forall k \in \mathbb{Z}$

---

b) $a = 3737$, $b = 1517$

$3737 = \underbrace{1517}_{a} \cdot 2 + \underbrace{703}_{b}$

$1517 = \underbrace{703}_{a} \cdot 2 + \underbrace{111}_{b}$

$703 = \underbrace{111}_{a} \cdot 6 + \underbrace{37}_{b}$

$111 = 37 \cdot 3 + 0 \Rightarrow \gcd(3737, 1517) = 37$

$u, v \in \mathbb{Z}$ aî $3737 \cdot u + 1517 \cdot v = 37$

$3737 = \underset{a}{1517} \cdot 2 + \underset{b}{703}$

$1517 = \underset{a}{703} \cdot 2 + \underset{b}{111}$

$703 = \underset{a}{111} \cdot 6 + \underset{b}{37}$

$111 = 37 \cdot 3 + \underline{0} \Rightarrow$

$37 = 703 - \underline{111} \cdot 6$

$37 = \underline{703} - (1517 - 703 \cdot 2) \cdot 6 =$

$= 703 \cdot 13 - 1517 \cdot 6$

$37 = (3737 - 1517 \cdot 2) \cdot 13 - 1517 \cdot 6$

$= \underset{u}{3737 \cdot 13} - \underset{v}{1517 \cdot 32}$

$\Rightarrow u_1 = 13 \wedge v_1 = -32$

$\left.\begin{array}{l} \dfrac{3737}{37} = 101 \Rightarrow v = -32 - 101 \cdot k \\ \dfrac{1517}{37} = 41 \Rightarrow u = 13 + 41 \cdot k \end{array}\right\} \forall k \in \mathbb{Z}$

---

c) $a = 120$, $b = 23$ d) $a = 461$, $b = 153$

$120 = 23 \cdot 5 + 5$

$23 = 5 \cdot 4 + 3$

$5 = 3 \cdot 1 + 2$

$3 = 2 \cdot 1 + 1$

$2 = 2 \cdot 1 + 0 \Rightarrow \gcd(120, 23) = 1 \Rightarrow a, b$ - prime între ele

---

$u, v \in \mathbb{Z}$ a.î. $120 \cdot u + 23 \cdot v = 1$?

$1 = 3 - 2 \cdot 1 = 3 - (5 - 3 \cdot 1) \cdot 1 = -5 + 3 \cdot 2$

$1 = -5 + 3 \cdot 2 = -5 + (23 - 5 \cdot 4) \cdot 2 = 23 \cdot 2 - 5 \cdot 9$

$1 = 23 \cdot 2 - 5 \cdot 9 = 23 \cdot 2 - (120 - 23 \cdot 5) \cdot 9 =$

$= -120 \cdot 9 + 23 \cdot 47$

$1 = 120 \cdot (-9) + 23 \cdot 47$

$\Rightarrow u_1 = -9 \wedge v_1 = 47$

$\left.\begin{array}{l} \dfrac{120}{1} = 120 \Rightarrow v = 47 + 120k \\ \dfrac{23}{1} = 23 \Rightarrow u = -9 - 23k \end{array}\right\} \forall k \in \mathbb{Z}$

d) $a = 461$, $b = 153$

$461 = 153 \cdot 3 + 2$

$153 = 2 \cdot 76 + 1$

$2 = 2 \cdot 1 + 0 \Rightarrow \gcd(461, 153) = 1 \Rightarrow a$ și b sunt prime între ele

---

$u, v \in \mathbb{Z}$ aî $461 \cdot u + 153 \cdot v = 1$?

$1 = 153 - 2 \cdot 76 = 153 - (461 - 153 \cdot 3) 76$

$= 229 \cdot 153 - 461 \cdot 76$

$1 = 461 \cdot (-76) + 153 \cdot 229$

$\Rightarrow u_1 = -76$, $v_1 = 229$

$$\left.\begin{array}{l} \frac{461}{1} = 461 \Rightarrow v = 229 + 461 \cdot k \\ \frac{153}{1} = 153 \Rightarrow u = -76 - 153 \cdot k \end{array}\right\} \forall k \in \mathbb{Z}$$

Seminar - Săptămâna 6

1) din d (temă) $\Rightarrow \begin{cases} v = 229 + 461 \cdot k \\ u = -76 - 153 \cdot k \end{cases} \quad k \in \mathbb{Z}$

$(u, v) \in \{(-76 - 153k,\ 229 + 461k) \mid k \in \mathbb{Z}\}$

$\left[\begin{array}{l} 1 = (-76 - 153 \cdot k) \cdot 461 + (229 + 461 \cdot k) \cdot 153 \\ 1 = -76 \cdot 461 + 229 \cdot 153 \end{array}\right.$

din e (temă) $\Rightarrow \gcd(120, 23) = 1$

$u_1 = -9, \quad v_1 = 47$

$\left.\begin{array}{l} v = -9 - 120k \\ u = 47 + 23k \end{array}\right\} k \in \mathbb{Z}$

$\Rightarrow (u, v) = \{(47 + 23k,\ -9 - 120k) \mid k \in \mathbb{Z}\}$

e) $a = 1950, \quad b = 45$

$1950 = 45 \cdot 43 + 15$

$45 = 15 \cdot 3 + 0 \Rightarrow \gcd(1950, 45) = 15$

$15 = 1 \cdot 1950 - 45 \cdot 43 \Rightarrow u_1 = 1, \ v_1 = -43$

$\left.\begin{array}{l} \frac{1950}{15} = 130 \Rightarrow v = -43 - 130k \\ \frac{45}{15} = 3 \Rightarrow u = 1 + 3k \end{array}\right\} k \in \mathbb{Z}$

$\Rightarrow (u, v) = \{(1 + 3k,\ -43 - 130k) \mid k \in \mathbb{Z}\}$

2) Arătați că descompunerea în factori primi în $\mathbb{Z}$ este unică până la o rearanjare a termenilor și a semnelor

$n = \pm p_1^{\alpha_1} \cdot p_2^{\alpha_2} \cdot \ldots \cdot p_k^{\alpha_k}; \quad p_1, \ldots, p_k = \text{prim} (\in \mathbb{N})$

$\alpha_1, \ldots, \alpha_k \in \mathbb{N}^*$

$n = \pm q_1^{\beta_1} \cdot q_2^{\beta_2} \cdot \ldots \cdot q_t^{\beta_t}; \quad q_1, \ldots, q_t = \text{prim} (\in \mathbb{N})$

$\beta_1, \ldots, \beta_t \in \mathbb{N}^*$

$\pm$ depinde de $n \Rightarrow$ unic

$\alpha = \alpha_1 + \alpha_2 + \ldots + \alpha_k$

Inducție după $\alpha \geq 1$

$P(1)$: $\alpha = 1 \Rightarrow n = \pm p$
$p = \text{prim}$ $\Big\} \Rightarrow$ unică $\checkmark$

Pp. $P(\alpha - 1)$ (A) și dem. $P(\alpha)$ (A)

$n = \pm p_1^{\alpha_1} \cdot \ldots \cdot p_k^{k} \Rightarrow \underset{\text{prim}}{p_k} / n = \pm q_1^{\beta_1} \cdot \ldots \cdot q_t^{\beta_t} \Rightarrow \underset{j \in \{1, \ldots, t\}}{p_k / q_j} \rightarrow$ aleg. p_k / q_t

p_k, q_t - prim

$\Rightarrow p_k = q_t$

$n = \pm p_1^{\alpha_1} \cdot \ldots \cdot p_k^{\alpha_k}$ $\quad : p_k$

$n = \pm q_1^{\beta_1} \cdot \ldots \cdot q_t^{\beta_t}$

$: p_k = q_t$

$\underset{(:p_k)}{\Rightarrow} p_1^{\alpha_1} \cdot \ldots \cdot p_k^{\alpha_{k-1}} = q_1^{\beta_1} \cdot \ldots \cdot q_t^{\beta_{t-1}}$

$\alpha_1 + \alpha_2 + \ldots + \alpha_k - 1 = \alpha - 1$

$P(\alpha - 1)$ (A) $\Rightarrow$ descompunere unică $\Rightarrow k = t$, $p_i = q_i$, $\alpha_i = \beta_i$

$x^n - y^n = (x - y)(x^{n-1} + x^{n-2} \cdot y + \ldots + x \cdot y^{n-2} + y^{n-1})$

$x^n + y^n = (x + y)(x^{n-1} - x^{n-2} \cdot y + \ldots + x \cdot y^{n-2} + y^{n-1})$
(n impar)

(3) Arătați că $\gcd(2^a - 1, 2^b - 1) = 2^{\gcd(a,b)} - 1$

$x^n - 1 = (x-1)(x^{n-1} + x^{n-2} + \ldots + x + 1) \Rightarrow$

$\Rightarrow (x-1) \mid (x^n - 1)$, $\forall n \in \mathbb{N}$

pt $x = 2^b \Rightarrow (2^b - 1) \mid [(2^b)^n - 1]$, $\forall n \in \mathbb{N}$

$\Rightarrow (2^b - 1) \mid (2^{bn} - 1) \Rightarrow \boxed{(2^b - 1)(2^{b2} 1)}$

$\quad \| $
$(2^b - 1) \cdot Q$

$a \geq b$

$a = b \cdot q_1 + r_1$

$2^a - 1 = 2^{bq_1 + r_1} - 1$

$[2^a - 1 = (2^b - 1)$

$f:\mathbb{R}\to\mathbb{R},\ f(x)=\begin{cases} x^2-2x-7, & x<-1 \\ x+3, & x\ge -1 \end{cases}$

a) inj., surj., bij?

$x^2-2x-7=0$

$\Delta = 4+28=32$

$x_{1,2}=\dfrac{2\pm 4\sqrt{2}}{2}=1\pm 2\sqrt{2}$

$V_f\left(-\dfrac{b}{2a}, -\dfrac{\Delta}{4a}\right)$

$-\dfrac{b}{2a}=\dfrac{2}{2}=1$

$-\dfrac{\Delta}{4a}=-\dfrac{32}{4}=-8$

$x=0 \Rightarrow f(0)=3$

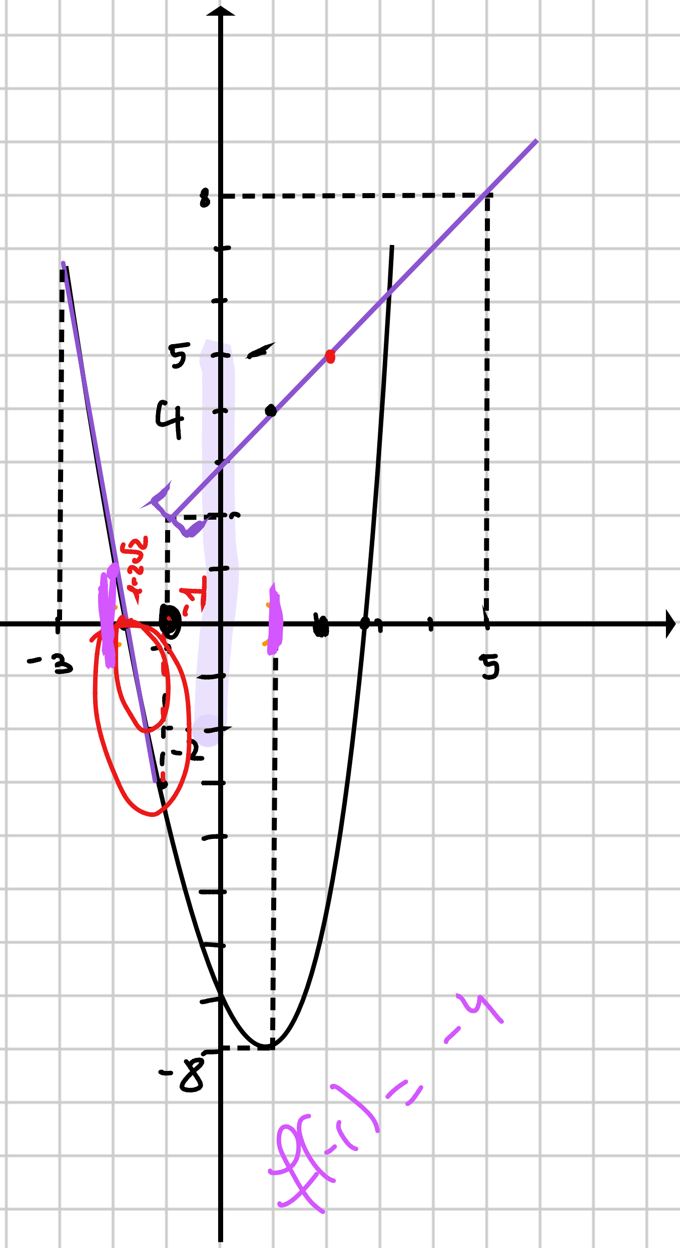
$x=-1 \Rightarrow f(-1)=2$

nu e inj, surj, bij.

$x^2-2x-7=-8$

$x^2-2x+1=0$

$x\cancel{=}1$



$f(-3)=9+6-7=8$

b) $f((-3,5))=(-4,8)\cup[2,8)=(-4,8)$

$f([-2,1))=(-4,+1]\cup(2,4)$

$f^{-1}((-8,0))=(1-2\sqrt{2},-1)$

$f^{-1}((-2,5])=(1-\sqrt{13},1-\sqrt{6})\cup[-1,2]$

$x^2-2x-7=-2 \Rightarrow x^2-2x-5=0$

$\Delta=4+20=24$

$x_{1,2}=\dfrac{2\pm 2\sqrt{6}}{2}=1\pm\sqrt{6}$

52 2
26 2
13

$x^2-2x-7=5 \Rightarrow x^2-2x-12=0$

$\Delta=4+48=52$, $x_{1,2}=\dfrac{2\pm\sqrt{52}}{2}=1\pm\sqrt{13}$

Test 132

1. $f:\mathbb{R}\to\mathbb{R}$, $f(x)=\begin{cases} x-3, & x\le 3\\ x^2-6, & x>3\end{cases}$

a) f inj

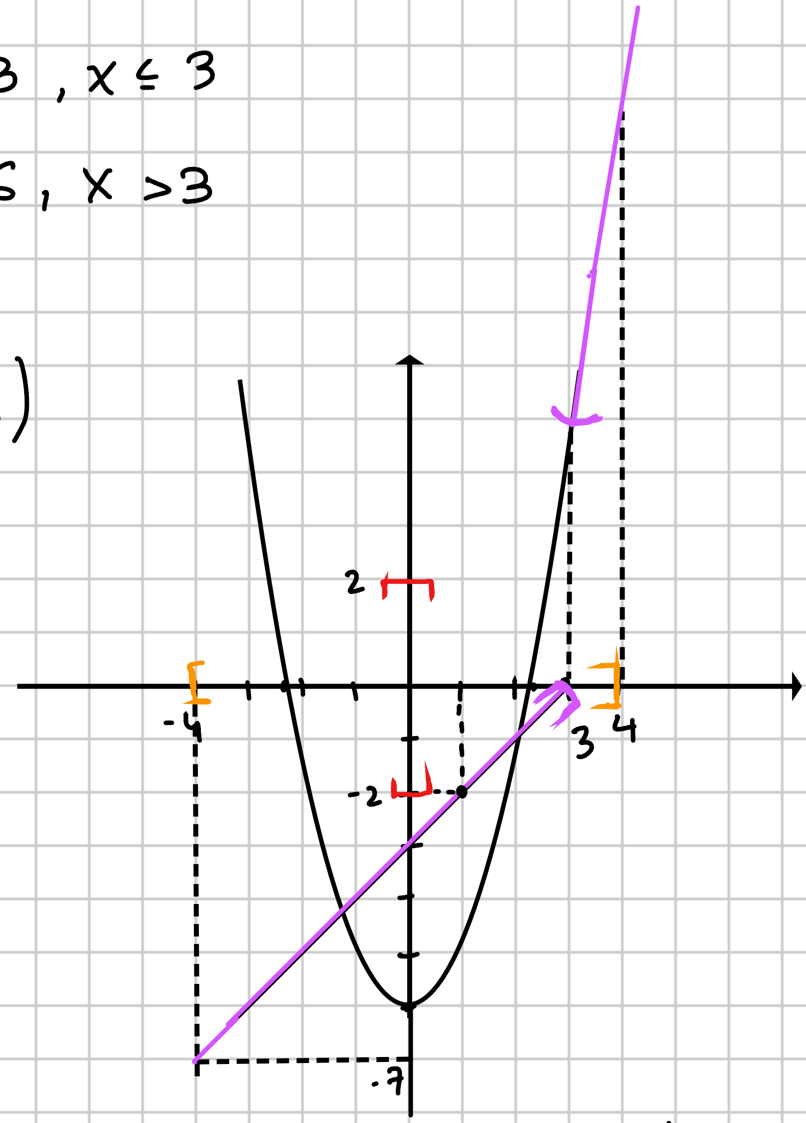
b) $f([-4,4])$, $f^{-1}([-2,2])$

a) $\begin{cases} f(3)=0\\ f(0)=-3\end{cases}$

$x^2=6 \Rightarrow x=\pm\sqrt{6}$

$\frac{-b}{2a}=0$

$\frac{-\Delta}{4a}=-\frac{24}{4}=-6$



nu e inj

b) $\begin{cases} f(-4)=-7\\ f(3)=0\end{cases}$

$\begin{cases} f(3)=3\\ f(4)=10\end{cases}$

$f([-4,4])=\boxed{[-7,0]\cup(3,10]}$

$x-3=-2 \Rightarrow x=1$

$x-3=0 \Rightarrow x=3$

$f^{-1}([-2,2])=[1,3]$

② $\mathbb{Z}\times\mathbb{Z}$, $(x_1, y_1) \sim (x_2, y_2) \iff 3x_1 + 2y_1 + 1 = 3x_2 + 2y_2 + 1$

a) rel. de echiv.

b) $\widehat{(1,1)}$

c) SCR

a)

(1): $\forall (x,y) \in \mathbb{Z}\times\mathbb{Z}$ $\Rightarrow 3x + 2y + 1 = 3x + 2y + 1$

$\Rightarrow (x,y) \sim (x,y) \Rightarrow \sim$ reflexivă

(2) $\forall (x_1, y_1), (x_2, y_2) \in \mathbb{Z}\times\mathbb{Z}$, $(x_1, y_1) \sim (x_2, y_2)$

$\Rightarrow 3x_1 + 2y_1 + 1 = 3x_2 + 2y_2 + 1 \Rightarrow 3x_2 + 2y_2 + 1 = 3x_1 + 2y_1 + 1$

$\Rightarrow (x_2, y_2) \sim (x_1, y_1) \Rightarrow$

$\Rightarrow \sim$ simetrică

(3): $\forall (x_1, y_1), (x_2, y_2), (x_3, y_3) \in \mathbb{Z}\times\mathbb{Z}$, $(x_1, y_1) \sim (x_2, y_2)$ și $(x_2, y_2) \sim (x_3, y_3)$

$$\left.\begin{array}{l} (x_1, y_1) \sim (x_2, y_2) \Rightarrow 3x_1 + 2y_1 + 1 = \underline{3x_2 + 2y_2 + 1} \\ (x_2, y_2) \sim (x_3, y_3) \Rightarrow \underline{3x_2 + 2y_2 + 1} = 3x_3 + 2y_3 + 1 \end{array}\right\} \Rightarrow (x_1, y_1) \sim (x_3, y_3)$$

$\Rightarrow \sim$ tranzitivă

$\Rightarrow \sim$ rel. de echivalență

b) $(x,y) \sim (1,1) \Rightarrow 3x + 2y + 1 = 3 + 2 + 1 \Rightarrow$

$\Rightarrow 3x + 2y = 5 \Rightarrow x = \frac{5-2y}{3}$

$$\widehat{(1,1)} = \left\{ \left(\frac{5-2y}{3}, y \right) \in \mathbb{Z}\times\mathbb{Z} \right\}$$

$\widehat{(1,1)} = \{ (x,y) \in \mathbb{Z}\times\mathbb{Z} \mid (1,1) \sim (x,y) \}$

$3+2+1 = 3x+2y+1$

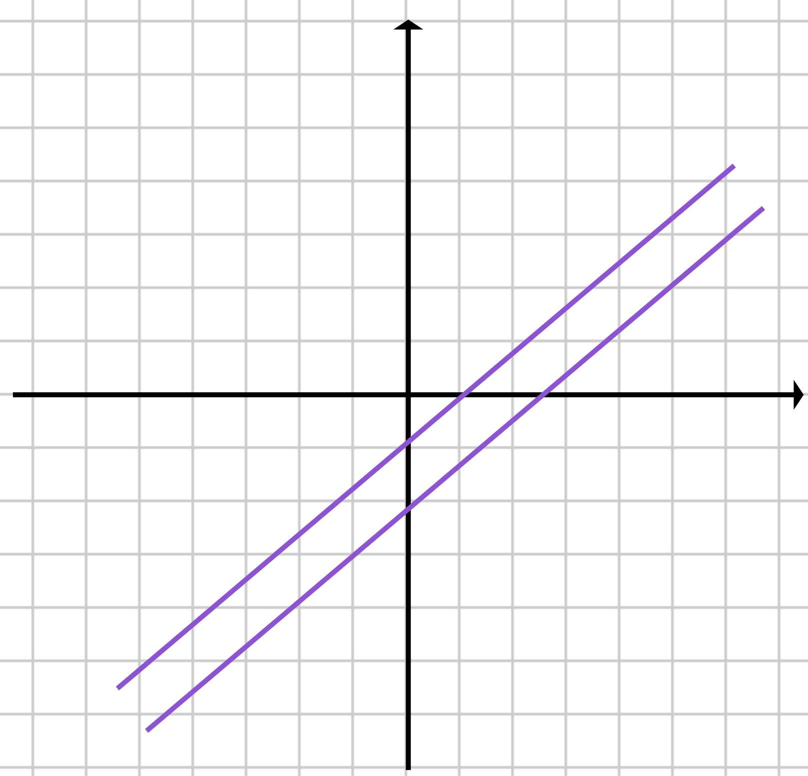
$3x+2y = 5$

$y = \dfrac{5-3x}{2}$

$\widehat{(1,1)} = \{ (x, \frac{5-3x}{2}) \in \mathbb{Z}\times\mathbb{Z} \mid x \text{ impar} \}$

c) SCR ?

câte un singur element din fiecare clasă de echivalență



fie $(x,y) \in \mathbb{Z}\times\mathbb{Z}$

Notăm $k = 3x+2y$

$(a,b) \sim (x,y) \iff 3a+2b = k$ $\qquad 3x_0+2y_0 = (3,2) = 1 \quad / \cdot k$

Lema lui Bezout

fie $a,b \in \mathbb{Z}$ fixate cu $(a,b)=1$

Ecuația $ax+by=1$ are soluție unică în $\mathbb{Z}\times\mathbb{Z}$

$\underline{I}$ k - par $\Rightarrow$ $(a, b) = \left(0, \frac{k}{2}\right)$ soluție

$\underline{II}$ k - impar $\Rightarrow$ $(a, b) = \left(1, \frac{k-3}{2}\right)$ soluție

SCR : $\left\{ \left(0, \frac{k}{2}\right) / k \text{ par} \right\} \cup \left\{ 1, \frac{k-3}{2} / k \text{ impar} \right\}$

$(x, y) \sim (a, b) \Leftrightarrow x^2 - y^2 = a^2 - b^2$ pe $\mathbb{R} \times \mathbb{R}$

fie $(a, b) \in \mathbb{R} \times \mathbb{R}$ fixat

$k = a^2 - b^2$

$(x, y) \in \mathbb{R}^2$ cu $x^2 - y^2 = a^2 - b^2 = k$

Diagrama Hasse pt $n=6$

$D(6) = \{1, 2, 3, 6\}$

$x \leq y \Rightarrow x$ se află mai jos

$x \leq y \leq z$

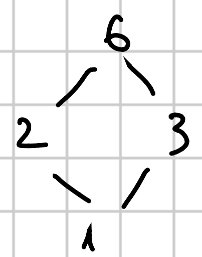
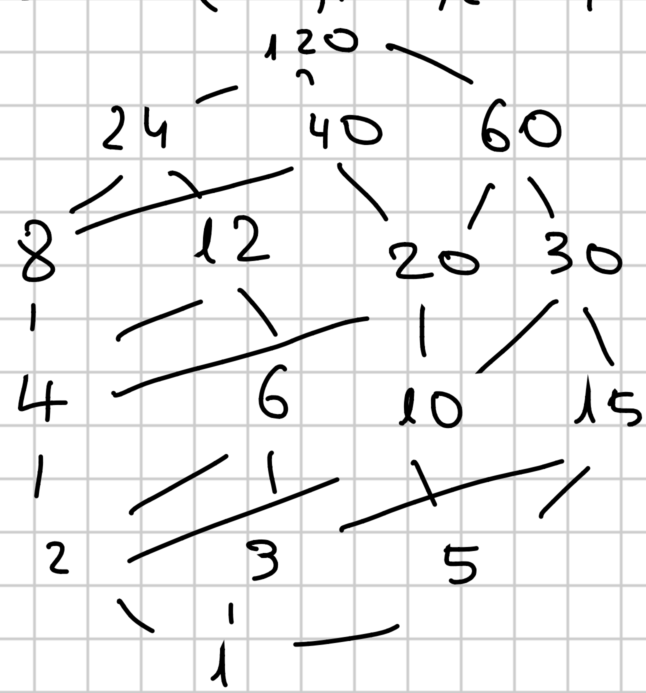


Diagrama Hasse pt 120

$D(120) = \{1, 2, 3, 4, 5, 6, 8, 10, 12, 15, 20, 24, 30, 40, 60, 120\}$

$120 = 20 \cdot 6 = 2^2 \cdot 5 \cdot 2 \cdot 3 = 2^3 \cdot 3 \cdot 5$

$\Rightarrow$ nr de divizori $= (3+1)(1+1)(1+1) = 16$



c) un SCR ?

3. $136u + 250v = \underset{\sim}{\gcd(136, 250)}$

$250 = 136 \cdot 1 + 114$

$136 = 114 \cdot 1 + 22$

$114 = 22 \cdot 5 + 4$

$22 = 4 \cdot 5 + 2$

$4 = 2 \cdot 2 + 0 \Rightarrow \gcd(136, 250) = 2$

$\frac{136}{2} = 78$

$\frac{250}{2} = 125$

$2 = 22 - 4 \cdot 5 = 22 - (114 - 22 \cdot 5) \cdot 5 = 22 \cdot 26 - 114 \cdot 5$

$2 = 22 \cdot 26 - 114 \cdot 5 = (136 - 114 \cdot 1) \cdot 26 - 114 \cdot 5 =$

$= 136 \cdot 26 - 114 \cdot 31$

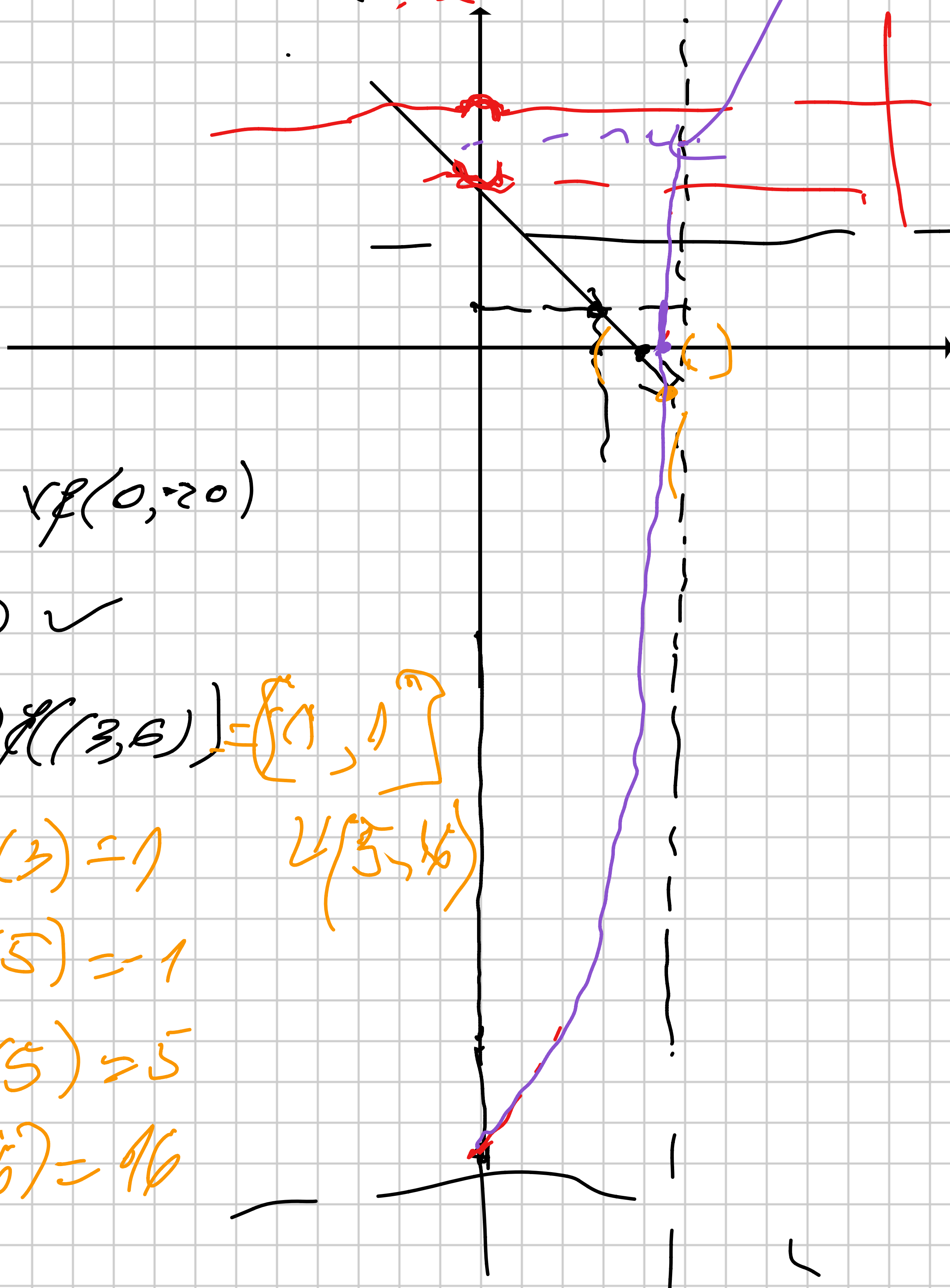
$2 = 136 \cdot 26 - 114 \cdot 31 = 136 \cdot 26 - (250 - 136 \cdot 1) \cdot 31 =$

$$\boxed{2 = 136 \cdot 57 - 250 \cdot 31}$$

$u_1 = 57, \ v_1 = -31$

$$\begin{cases} u = -31 - 78k \\ v = 57 + 125k \end{cases}, \ k \in \mathbb{Z}$$

$f: \mathbb{R} \to \mathbb{R},\ f(x) = \begin{cases} -x+4, & x \le 5 \\ x^2-20, & x > 5 \end{cases}$



$x^2 - 20 = 5$

$x > 5$

$x - 20 = 6$

$\cancel{x^2} = \pm\sqrt{26}$

$(5, \sqrt{26})$

$x^2 - 20 = 4$

$x^2 = 24$

$x = \pm\sqrt{24}$

$x = \pm 4, \ldots$

$V_f(0, -20)$

a) ✓

b) $f((3,6)) = [1, 1]$

$f(3) = 1$

$f(5) = -1$

$f(5) = 5$

$f(6) = 16$

$\cup (5, 16)$

$f^{-1}((4,6))$

$-x+4 = 4 \Rightarrow -x = 0$

$x = 0$

$-x+4 = 6 \Rightarrow x = -2$

$(-2, 0) \cup (5, \sqrt{26})$

② $\mathbb{R} \times \mathbb{R}$

$(x_1, y_1) \sim (x_2, y_2) \iff x_1^2 - y_1^2 = x_2^2 - y_2^2$

a) rel de echiv

(1) $\forall (x,y) \in \mathbb{R} \times \mathbb{R} \overset{?}{\Rightarrow} (x,y) \sim (x,y)$

$(x,y) \overset{\sim (x,y)}{\Rightarrow} x^2 - y^2 = x^2 - y^2 \Rightarrow (x,y) \sim (x,y)$

(2) $\forall (x_1, y_1), (x_2, y_2) \in \mathbb{R} \times \mathbb{R}$, $(x_1, y_1) \sim (x_2, y_2) \overset{?}{\Rightarrow} (x_2, y_2) \sim (x_1, y_1)$

$(x_1, y_1) \sim (x_2, y_2) \Rightarrow x_1^2 - y_1^2 = x_2^2 - y_2^2 \Rightarrow (x_2, y_2) \sim (x_1, y_1)$

(4) $\forall (x_1, y_1), (x_2, y_2), (x_3, y_3) \in \mathbb{R} \times \mathbb{R}$, $(x_1, y_1) \sim (x_2, y_2)$ și

$(x_2, y_2) \sim (x_3, y_3) \overset{?}{\Rightarrow} (x_1, y_1) \sim (x_3, y_3)$

$$\left.\begin{array}{l} (x_1, y_1) \sim (x_2, y_2) \Rightarrow x_1^2 - y_1^2 = x_2^2 - y_2^2 \\ (x_2, y_2) \sim (x_3, y_3) \Rightarrow x_2^2 - y_2^2 = x_3^2 - y_3^2 \end{array}\right\} \Rightarrow x_1^2 - y_1^2 = x_2^2 - y_2^2 = x_3^2 - y_3^2 \Rightarrow$$

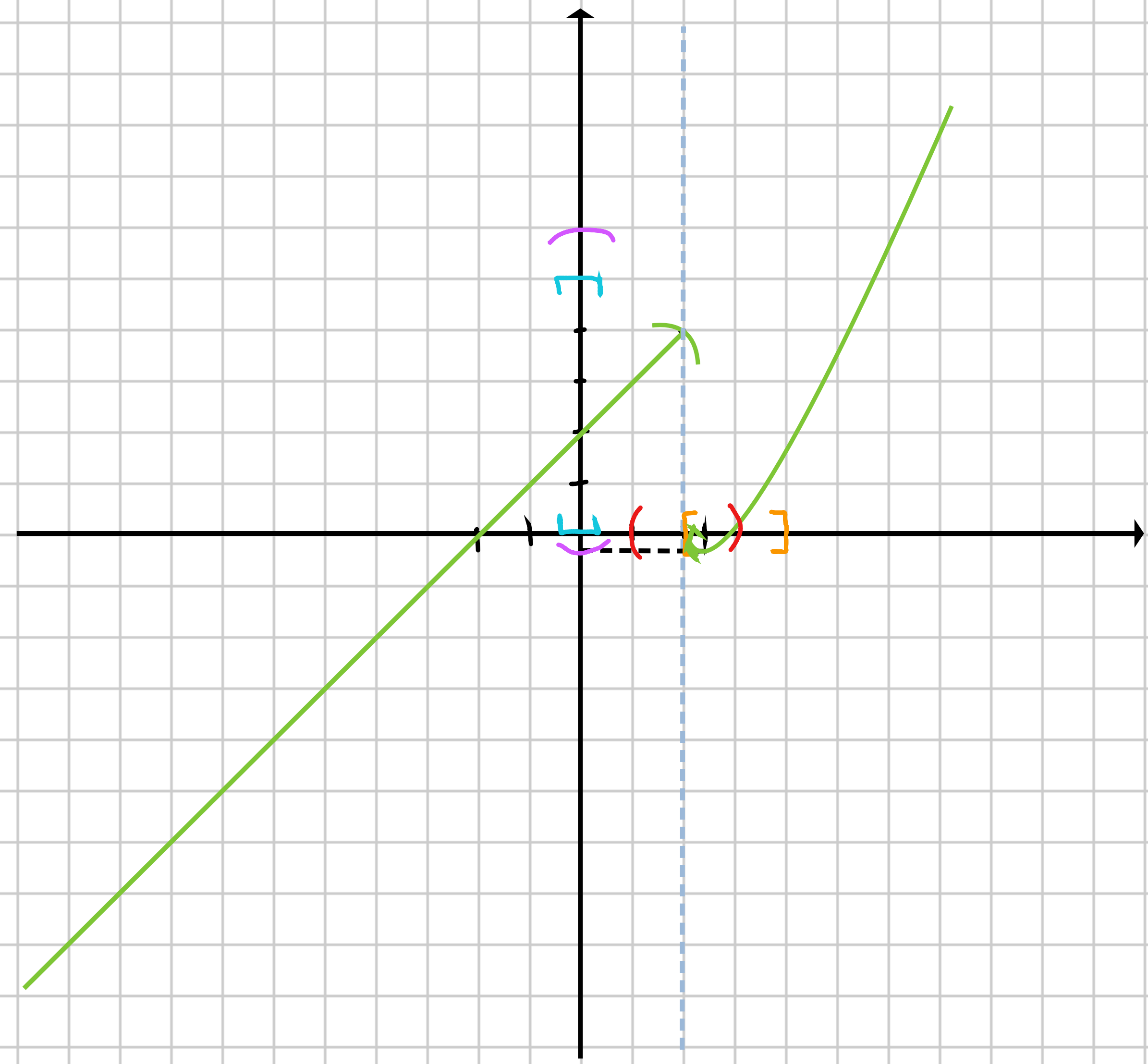
$\Rightarrow (x_1, y_1) \sim (x_3, y_3)$

b) $\widehat{(5,3)}$

$(5,3) \sim (x,y) \Rightarrow 25 - 9 = x^2 - y^2$

$f:\mathbb{R}\to\mathbb{R},\ f(x)=\begin{cases} x+2, & x<2 \\ x^2-5x+6, & x\ge 2 \end{cases}$

$f([2,4]),\ f((1,3)),\ f^{-1}\left(\left(-\frac{1}{4},6\right)\right),\ f^{-1}([0,5))$



$\begin{cases} f(2)=4 \\ f(0)=2 \\ f(-2)=0 \end{cases}$

$x^2-5x+6=0$

$\Delta=25-24=1$

$x_{1,2}=\frac{5\pm 1}{2}$ ⟨ 3, 2

$V_f\left(-\frac{b}{2a}, -\frac{\Delta}{4a}\right)$

$\frac{-b}{2a}=\frac{5}{2}$; $\frac{-\Delta}{4a}=-\frac{1}{4}$

f surj. dar nu inj.

$f([2,4])=\left[-\frac{1}{4},2\right]$

$f(4)=16-20+6=2$

$f((1,3))=(3,4)\cup\left[-\frac{1}{4},0\right)$

$f^{-1}\left(\left(-\frac{1}{4},6\right)\right)=\left(-\frac{9}{4},2\right)\cup\left(\frac{5}{2},5\right)$

$x+2=-\frac{1}{4} \Rightarrow x=-\frac{1}{4}-2=-\frac{9}{4}$

$x^2-5x+6=6 \Rightarrow x(x-5)=0 \begin{cases} x=0 \\ x=5 \end{cases}$

$f^{-1}([0,5])=[-2,2)\cup\left[3,\frac{5+\sqrt{21}}{2}\right]$

$x^2-5x+6=5 \Rightarrow x^2-5x+1=0$, $\Delta=25-4=21$, $x_{1,2}=\frac{5\pm\sqrt{21}}{2}$

$f:\mathbb{R}\to\mathbb{R}$, $f(x)=\begin{cases} x+2, & x<2 \\ x^2-5x+6, & x\ge 2\end{cases}$ $\quad f^{-1}([-2,0])$

$f([1,2))$

$f([2,4])$, $f((1,3))$, $f^{-1}((-\frac{1}{4},6))$, $f^{-1}([0,5))$

$$\begin{cases} f(2)=4 \\ f(0)=2 \\ f(-2)=0 \end{cases}$$

$x^2-5x+6=0$

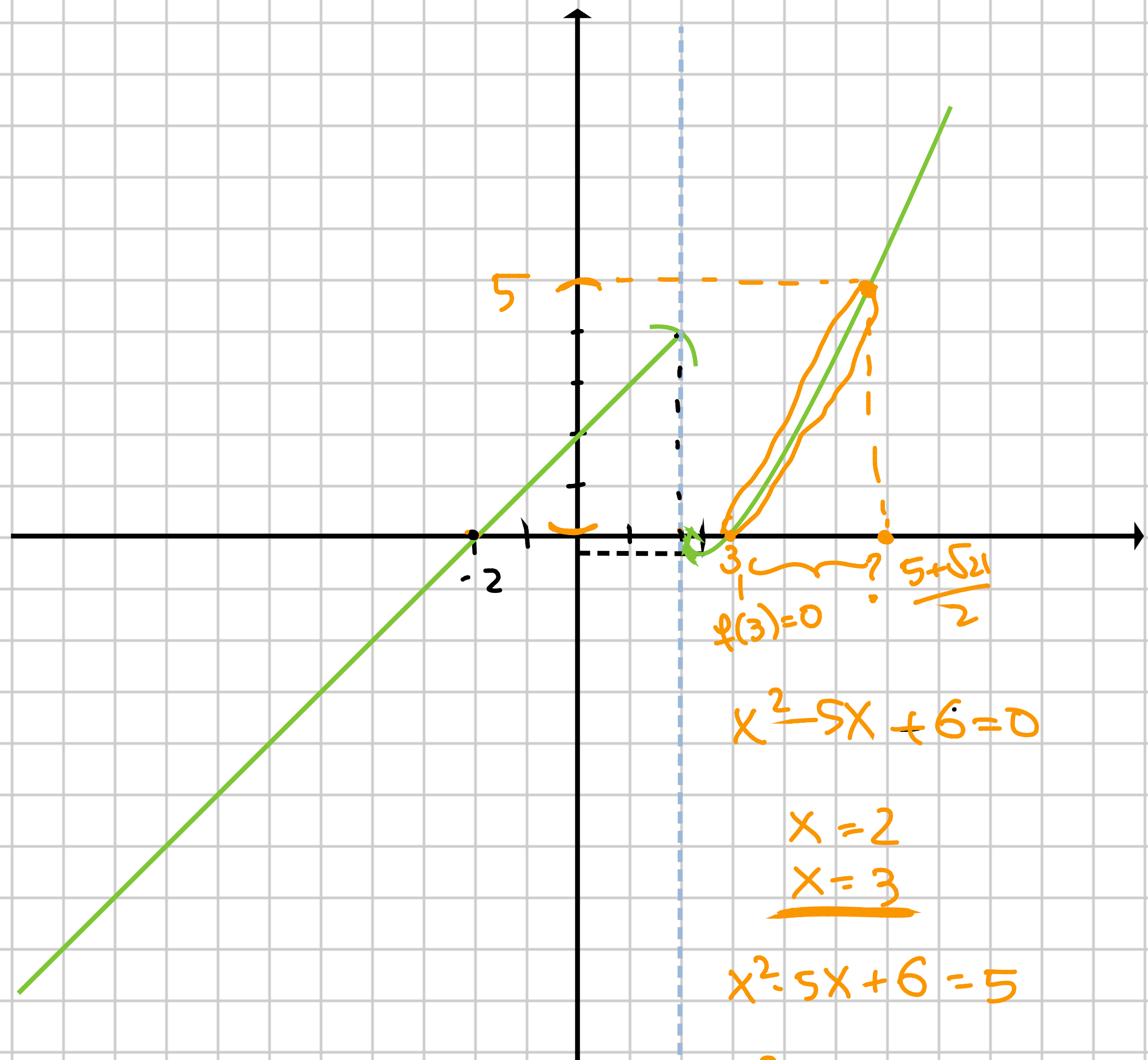
$\Delta=25-24=1$

$x_{1,2}=\frac{5\pm1}{2}$ $\begin{matrix} 3 \\ 2 \end{matrix}$

$V\left(-\frac{b}{2a}, -\frac{\Delta}{4a}\right)$

$-\frac{b}{2a}=\frac{5}{2}$; $-\frac{\Delta}{4a}=-\frac{1}{4}$

f surj. dar nu inj.



$x^2-5x+6=0$

$x=2$

$x=3$

$x^2-5x+6=5$

$x^2-5x+1=0$

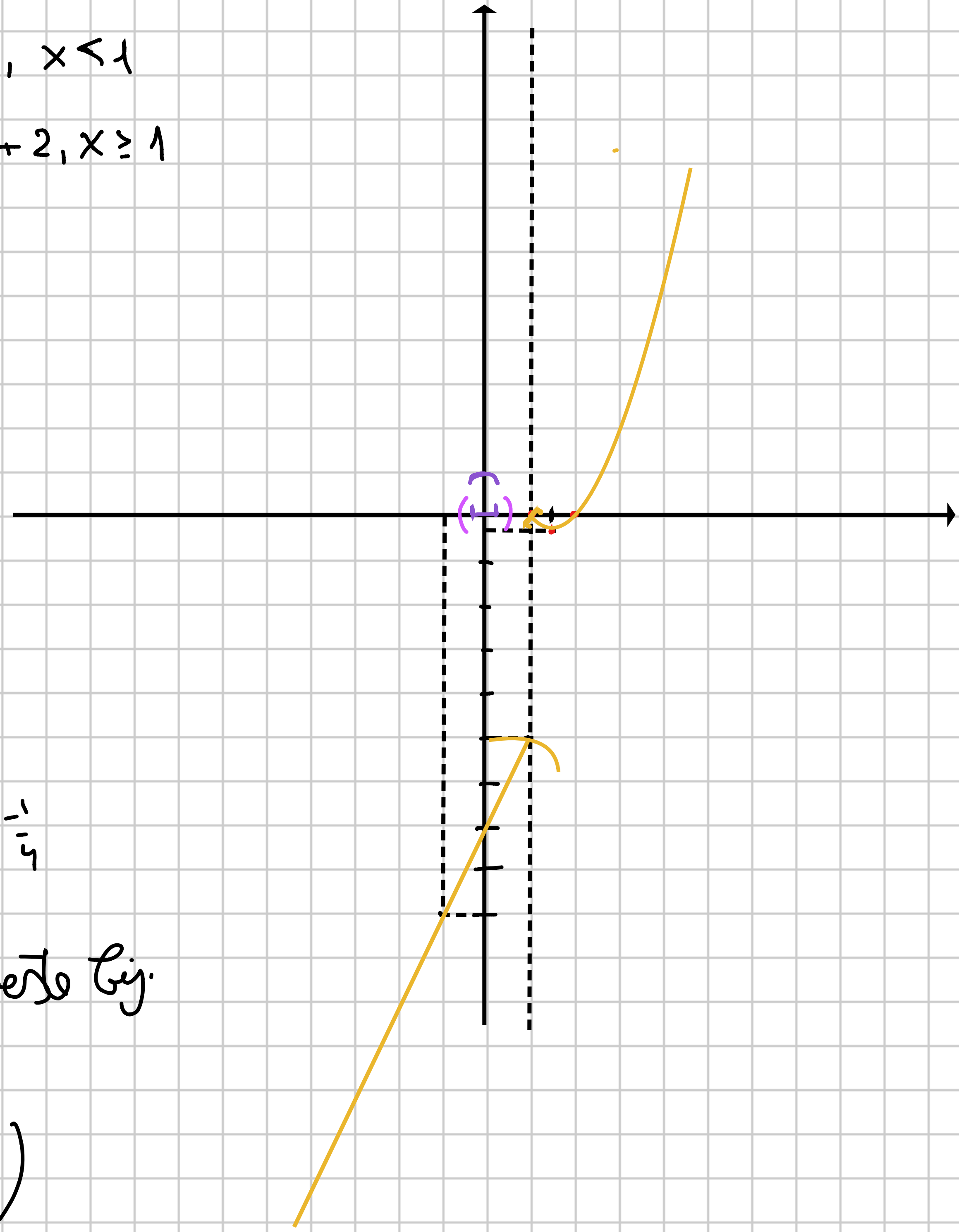
$\Delta=25-4=21$

$x_{1,2}=\frac{5\pm\sqrt{21}}{2}$

(ramura 2): $\left(3, \frac{5+\sqrt{21}}{2}\right)$

$f^{-1}((0,5)) = (-2,2)\cup\left(3,\frac{5+\sqrt{21}}{2}\right)$

① $f:\mathbb{R}\to\mathbb{R}$, $f(x)=\begin{cases}2x-7, & x<1\\ x^2-3x+2, & x\geq 1\end{cases}$



a) f. bij.?

$$\begin{cases}f(1)=-5\\ f(0)=-7\\ f(-1)=-9\end{cases}$$

$x^2-3x+2=0$

$\Delta=9-8=1$, $x_{1,2}=\frac{3\pm1}{2}$ ↗ 2, ↘ 1

$V\left(-\frac{b}{2a}, -\frac{\Delta}{4a}\right)$; $-\frac{b}{2a}=\frac{3}{2}$, $-\frac{\Delta}{4a}=-\frac{1}{4}$

f. nu este inj. }
f. nu este surj } => f nu este bij.

b) $f\left(\left(-\frac{1}{2},\frac{1}{2}\right)\right)$, $f^{-1}([0,1))$

$f\left(-\frac{1}{2}\right)=2\cdot\left(-\frac{1}{2}\right)-7=-8$

$f\left(\frac{1}{2}\right)=2\cdot\frac{1}{2}-7=-6$

} $\Rightarrow f\left(\left(-\frac{1}{2},\frac{1}{2}\right)\right)=(-8,-6)$

$x^2-3x+2=1 \Rightarrow x^2-3x+1=0$

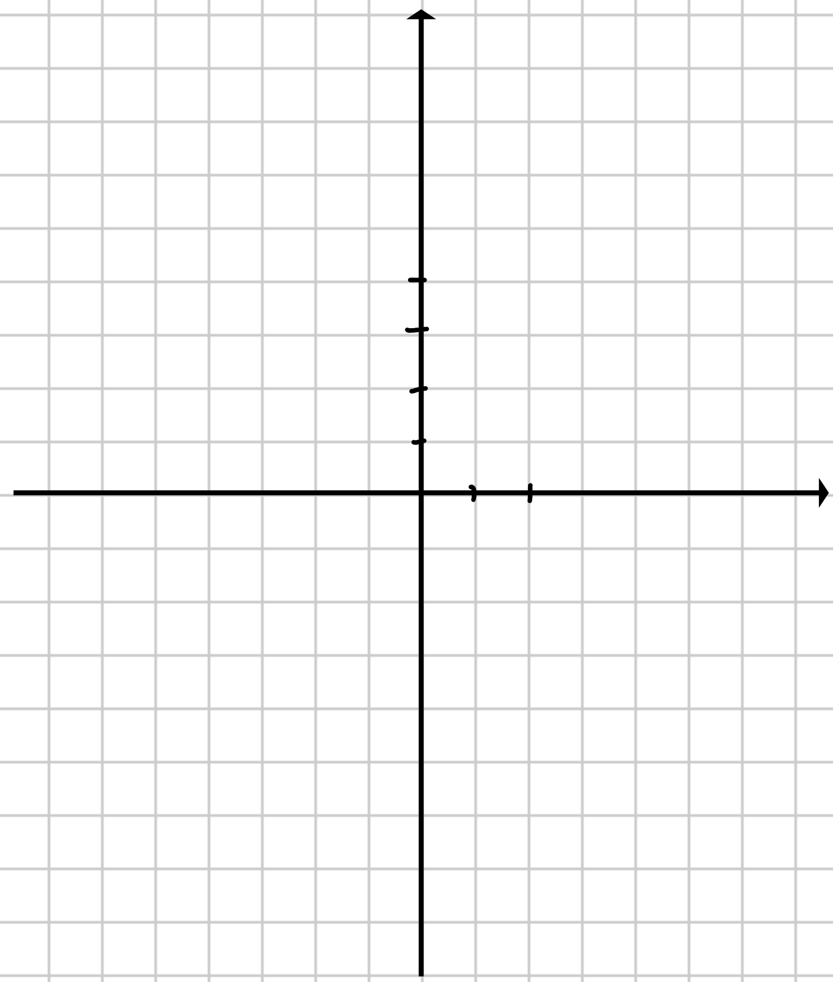
$\Delta=9-4=5$

$x_{1,2}=\frac{3\pm\sqrt{5}}{2}$

$f^{-1}([0,1))=\left[2,\frac{3+\sqrt{5}}{2}\right)$

$$f:\mathbb{R}\to\mathbb{R},\ f(x)=\begin{cases} x+2, & x<2 \\ x^2-5x+6, & x\geq 2 \end{cases}$$

$$\begin{cases} x=2 \Rightarrow f(2)=4 \\ x=0 \Rightarrow f(0)=2 \\ x=-2 \Rightarrow f(-2)=0 \end{cases}$$



Seminar 7

① Det. ultimele 2 cifre ale lui 32^{30}

$32^{30} \equiv ? \pmod{100}$

$(2^5)^{30} = 2^{150} = (2^{10})^{15} = 1024^{15} \Rightarrow \underline{32^{30} \equiv 1024^{15} \equiv 24^{15}}$

$24^{15} = (24^3)^{15} = 13824^5 \Rightarrow \underline{32^{30} \equiv 13824^5 \equiv 24^5}$

$24^5 = 24^3 \cdot 24^2 = 138\underline{24} \cdot 5\underline{76}$

$32^{30} \equiv 24 \cdot 76 \equiv 24$

② Det. restul împ. lui 38^{131} la → a) 3

b) 9

36 : 3 = 12, rest 0

a) $38^{131} \equiv 2^{131} \equiv 2^{130} \cdot 2 \equiv (2^2)^{65} \cdot 2 \equiv 4^{65} \cdot 2 \equiv 1^{65} \cdot 2 \equiv 2$

$\boxed{38^{131} \equiv ? \pmod 3}$

<u>Obs</u>. $38^{131} \equiv (-1)^{131} \equiv -1 \equiv 2$

b) $38^{131} \equiv 2^{131} \pmod 9$

$\boxed{38^{131} \equiv ? \pmod 9}$

$2^{131} \equiv (2^3)^{43} \equiv 8^{43} \cdot 4 \equiv (-1)^{43} \cdot 4 \equiv -1 \cdot 4 \equiv -4 \to 5$

③ Cu câte zero-uri se termină 25! ?

$25! = 1 \cdot 2 \cdot 3 \cdot \ldots \cdot 25$

$M_5 \to 5, 10, 15, 20, 25 \Rightarrow 5$

$\downarrow 5^2$

$M_{25} \to 25 \Rightarrow 1$

→ 6 zero-uri

<u>Temă</u>: det ultimele 3 cifre nenule ale lui 25!

④ Det. câtul și restul împărțirii lui f la g

$f = X^4 + 5X^2 + X - 1$

$g = 3X + 2$

$$
\begin{array}{l|l}
X^4 + 5X^2 + X - 1 & 3X + 2 \\
\hline
-X^4 - \frac{2}{3}X^3 & \frac{1}{3} \cdot X^3 - \frac{2}{9}X^2 + \frac{49}{27}X - \frac{71}{81} \\
\hline
/ \ -\frac{2}{3}X^3 + 5X^2 + X - 1 & \\
+\frac{2}{3}X^3 + \frac{4}{9}X^2 & \\
\hline
/ \ \frac{49}{9}X^2 + X - 1 & \\
-\frac{49}{9}X^2 - \frac{98}{27}X & \\
\hline
/ \ -\frac{71}{27}X - 1 & \\
+\frac{71}{27}X + \frac{142}{81} & \\
\hline
/ \ \frac{61}{81} & \\
= \text{rest} &
\end{array}
$$

b) $f = \hat{4}x^3 + \hat{3}x^2 - x + \hat{2}$

$g = \hat{2}x^2 + \hat{1}$

$\in \mathbb{Z}_7[x]$

$$\begin{array}{l|l}
\hat{4}x^3 + \hat{3}x^2 - x + \hat{2} & \hat{2}x^2 + \hat{1} \\
\hline
-\hat{4}x^3 - \hat{2}x & \underbrace{\hat{2}x + \hat{5}}_{\text{cât}} \\
\hline
/ \quad \hat{3}x^2 - \hat{3}x + \hat{2} & \\
-\hat{3}x^2 - \hat{5} & \\
\hline
\end{array}$$

$$-3\hat{x} - \hat{3} = \underbrace{\hat{4}x + \hat{4}}_{\text{rest}}$$

Seminar 8

marți, 17.12.2024 test la 10:00

① Verificați dacă $\hat{153}$ este inversabil în $\mathbb{Z}_{461}$ și det. inv, dc $\exists$

$\hat{x}$ inversabil în $\mathbb{Z}_n \Leftrightarrow gcd(x, n) = 1$

$u \cdot x + v \cdot n = 1 \ / \bmod n \Rightarrow \hat{u} \cdot \hat{x} = \hat{1}$

$461 = 153 \cdot 3 + 2 \quad \Rightarrow gcd(461, 153) = 1 \Rightarrow \exists \ \hat{153}$ în $\mathbb{Z}_{461}$

$153 = 2 \cdot 76 + 1$

$2 = 1 \cdot 2 + 0$

$1 = 153 - 2 \cdot 76 =$

$= 153 - 76(461 - 153 \cdot 3) =$

$= 153 - 76 \cdot 461 + 153 \cdot 228 =$

$= 461 \cdot (-76) + 153 \cdot 229$

$1 = 461 \cdot (-76) + 153 \cdot 229 \ / \bmod 461$

$\hat{1} = \hat{153} \cdot \hat{229} \Rightarrow$ inversul lui $\hat{153} = \hat{229}$

Temă: $\hat{23}$ în $\mathbb{Z}_{120}$

② Det. $gcd(f, g)$ și toți u, v aî $u \cdot f + v \cdot g = gcd(f, g)$

$f = X^4 + 2X^2 + X + 1$

$g = X^2 + 3$

$$
\begin{array}{l|l}
X^4 + 2X^2 + X + 1 & X^2 + 3 \\
\hline
-X^4 - 3X^2 & X^2 - 1 \\
\hline
/ \quad -X^2 + X + 1 & \\
\quad X^2 \qquad\quad + 3 & \\
\hline
\quad / \qquad X + 4 &
\end{array}
$$

$\Rightarrow X^4 + 2X^2 + X + 1 = (X^2+3)(X^2-1) + (X+4)$

$$\begin{array}{l|l} x^2+3 & x+4 \\ \cline{2-2} -x^2-4x & x-4 \\ \cline{1-1} \quad / \; -4x+3 & \\ \quad\;\; +4x+16 & \\ \cline{1-1} \qquad\quad 19 & \end{array}$$

$x^2+3=(x+4)(x-4)+19$

$x+4=19\left(\frac{1}{19}\cdot x+\frac{4}{19}\right)+0$

$\gcd(f,g)=19$ (constantă $\in \mathbb{R}$)

<u>Obs.</u> f,g - prime între ele

$19=\overset{g}{x^2+3}-(x+4)(x-4)$

$19=g-(f-g(x^2-1))(x-4)$

$19=g-f\cdot(x-4)+g(x^2-1)(x-4)$

$19=g-f\cdot(x-4)+g(x^3-4x^2-x+4)$

$19=-f\cdot(\underbrace{x-4}_{u_1})+g\cdot(\underbrace{x^3-4x^2-x+5}_{v_1})$

$\frac{x^4+2x^2+x+1}{19}=\frac{1}{19}\cdot x^4+\frac{2}{19}\cdot x^2+\frac{1}{19}\cdot x+\frac{1}{19} \Rightarrow v_1=x^3-4x^2-x+5+p\cdot\left(\frac{1}{19}\cdot x^4+\frac{2}{19}\cdot x^2+\frac{1}{19}\cdot x+\frac{1}{19}\right)$

$p\in\mathbb{R}[x]$

$\frac{x^2+3}{19}=\frac{1}{19}\cdot x^2+\frac{3}{19} \Rightarrow u=-x+4-p\cdot\left(\frac{1}{19}x^2+\frac{3}{19}\right)$

$p\in\mathbb{R}[x]$

③ Det. toate polinoamele de grad cel mult 3 din $\mathbb{R}[x]$ care dau:

a) restul 12 la împ. la $X-1$

b) restul $(x-1)$ la împ. la $(X-2)^2$

$f = (x-1)\cdot g + 12$ ← grad ≤ 2

grad ≤ 3 grad $=1$

$g = ax^2 + bx + c$

$a, b, c \in \mathbb{R}$

$$f = (x-1)(ax^2+bx+c)+12 =$$

$$= ax^3 + bx^2 + cx - ax^2 - bx - c + 12 =$$

$$= ax^3 + (b-a)x^2 + (c-b)x + 12 - c$$

$\cancel{f(1) = a + b - a + c - b + 12 - c = 12}$

b) $f = (x-2)^2 \cdot g$

grad $f \leq 3$ $+ (x-1)$

grad $g \leq 1$

$g = ax + b$, $a, b \in \mathbb{R}$

$$f = (x-2)^2(ax+b)+(x-1)$$

$$f = (x^2-4x+4)(ax+b) + (x-1)$$

$$f = ax^3 + bx^2 - 4ax^2 - 4bx + 4ax + 4b + x - 1$$

$$f = ax^3 + (b-4a)x^2 + (-4b+4a+1)x + 4b - 1, \ \forall a, b \in \mathbb{R}$$

④ Teoremă: $f \in \mathbb{Z}[x]$, $f = a_m \cdot x^m + a_{m-1} \cdot x^{m-1} + \ldots + a_1 \cdot x + a_0$

$\frac{u}{v}$ = rad. a lui f cu $(u, v) = 1$

$$\Rightarrow \begin{cases} u / a_0 \\ v / a_m \end{cases}$$

Dem.

$\frac{u}{v}$ răd $\Rightarrow f\left(\frac{u}{v}\right)=0$

$$a_n \cdot \frac{u^n}{v^n} + a_{n-1} \cdot \frac{u^{n-1}}{v^{n-1}} + \dots + a_1 \cdot \frac{u}{v} + a_0 = 0 \quad \Big| \cdot v^n$$

$$a_n \cdot u^n + a_{n-1} \cdot u^{n-1} \cdot v + \dots + a_1 \cdot u \cdot v^{n-1} + a_0 \cdot v^n = 0$$

$$u\,(\underbrace{a_n \cdot u^{n-1} + a_{n-1} \cdot u^{n-2} \cdot v + \dots + a_1 \cdot v^{n-1}}_{p}) = -a_0 \cdot v^n$$

$$\left.\begin{array}{l} v / u \cdot p \\ (u,v)=1 \end{array}\right\} \Rightarrow v/p \Rightarrow \left.\begin{array}{l} v / a_n \cdot u^{n-1} \\ (u,v)=1 \end{array}\right\} \Rightarrow \boxed{v / a_n}$$

$$\left.\begin{array}{l} u / -a_0 \cdot v^n \\ (u,v)=1 \end{array}\right\} \Rightarrow \boxed{u / a_0}$$

Consecință

- dacă $a_n = \pm 1$ și $\alpha \in \mathbb{Q}$ rădăcină $\Rightarrow \alpha \in \mathbb{Z}$
- dacă $\exists\ \alpha \in \mathbb{Z}$ răd divide termenul liber (a_0)

5. Găsiți o factorizare pt:

$$f = x^4 - 7x^3 + 15x^2 - x - 24 \in \mathbb{Z}[x]$$

$$D_{24} = \{\pm 1, \pm 2, \pm 3, \pm 4, \pm 6, \pm 8, \pm 12, \pm 24\}$$

$x = 1 \longrightarrow f(1) = 1 - 7 + 15 - 1 - 24 \neq 0$

$x = -1 \longrightarrow f(-1) = 0$

$\Downarrow$

$x+1 / f$

$\Downarrow$

împ de polinoame

$$f = (x+1)(x^3 - 8x^2 + 23x - 24)$$

$x=3 \Rightarrow f(3)=0$

	x^3	x^2	x^1	x^0
	1	-8	23	-24
3	1	-5	8	0

x^2-5x+8

$f=(x+1)(x-3)(x^2-5x+8)$

$\Delta=25-32<0$

⑥ Descompuneți x^n-1 în $\mathbb{Q}, \mathbb{R}, \mathbb{C}[x]$

pt $n=2, 3, 4, 5, 6$

$n=2 \Rightarrow x^2-1=0 \Rightarrow (x-1)(x+1)$ în $\mathbb{Q}, \mathbb{R}, \mathbb{C}$

$n=3 \Rightarrow x^3-1=0 \Rightarrow (x-1)(x^2+x+1)$ în $\mathbb{Q}, \mathbb{R}$

$\Delta=-3 \Rightarrow \varepsilon_{1,2}=\frac{-1 \pm i\sqrt{3}}{2}$

$(x-1)(x-\varepsilon_1)(x-\varepsilon_2)$ în $\mathbb{C}$

$n=4 \Rightarrow x^4-1=0 \Rightarrow (x-1)(x+1)(x-i)(x+i)$ în $\mathbb{C}$

$(x-1)(x+1)(x^2+1)$ în $\mathbb{C}$

$n=5 \Rightarrow x^5-1=0 \Rightarrow (x-1)(x^4+x^3+x^2+x+1)$ în $\mathbb{Q}, \mathbb{R}$

$x^4+x^3+x^2+x+1=0 \quad |:x^2$

$x^2+x+1+\frac{1}{x}+\frac{1}{x^2}=0$

$\left(x+\frac{1}{x}\right)+\left(x^2+\frac{1}{x^2}\right)+1=0$

$y^2+y+1-2=0$

$y=\frac{1}{x}+x$

$y^2+y-1=0$

$\Delta=5$

$y_{1,2}=\frac{-1\pm\sqrt{5}}{2} \Rightarrow x+\frac{1}{x}=$

Seminar Săptămâna 9

① Găsiți gcd (x^4-1, x^6-1)

V1) Alg. lui Euclid:

$$\begin{array}{r|l} x^6-1 & x^4-1 \\ \cline{2-2} -x^6+x^2 & x^2 \\ \cline{1-1} / \; x^2-1 & \end{array}$$

$x^6-1=(x^4-1)x^2+(x^2-1)$

$x^4-1=(x^2-1)(x^2+1)+0 \quad \Rightarrow gcd = x^2-1$

V2) $x^4-1=(x^2-1)(x^2+1)=(x-1)(x+1)(x^2+1)$

$x^6-1=(x^3-1)(x^3+1)=(x-1)(x^2+x+1)(x+1)(x^2-x+1)$

$\Rightarrow gcd=(x-1)(x+1)=x^2-1$

② Găsiți $gcd(x^5-1, x^6-1)$

V1) Alg. lui Euclid

$x^6-1=x(x^5-1)+(x-1)$ gcd

$x^5-1=(x-1)(x^4+x^3+x^2+x+1)+0$

V2) $x^5-1=\boxed{(x-1)}(x^4+x^3+x^2+x+1)$

$x^4+x^3+x^2+x+1=0 \quad | : x^2 \neq 0$

$x^2+x+1+\frac{1}{x}+\frac{1}{x^2}=0$

$\left(x+\frac{1}{x}\right)^2=x^2+2\cdot x\cdot\frac{1}{x}+\frac{1}{x^2}$

$\left(x+\frac{1}{x}\right)^2-2+\left(x+\frac{1}{x}\right)+1=0$

$x+\frac{1}{x}=t \Rightarrow t^2+t-1=0$

$\Delta=1+4=5 \Rightarrow t_{1,2}=\frac{-1\pm\sqrt{5}}{2}$

(1) $x+\frac{1}{x}=\frac{-1+\sqrt{5}}{2}$

$2x^2+2=-x+\sqrt{5}x \Rightarrow 2x^2+x(1-\sqrt{5})+2=0$

$\Delta=1-2\sqrt{5}+5-16=-2\sqrt{5}-10<0 \Rightarrow$ nu are răd.

$x + \frac{1}{x} = \frac{-1-\sqrt{5}}{2}$

$2x^2 + 2 = -x - \sqrt{5}x$

$2x^2 + x(1+\sqrt{5}) + 2 = 0$

$\Delta = 1 + 2\sqrt{5} + 5 - 16 = -16 + 2\sqrt{5} < 0$

$2\sqrt{5} = \sqrt{20}$

$\sqrt{16} < \sqrt{20} < \sqrt{25}$

$\Rightarrow$ nu are răd.

$\Rightarrow gcd = x - 1$

③ Det. polinoamele de grad cel mult 4 ireductibile în $\mathbb{Z}_2[x]$

grad 1 : $x + a$ $\to x$, $x + \hat{1}$

grad 2 : $x^2 + ax + b$

ired $\Leftrightarrow$ nu are răd.

$x = 0 \Rightarrow 0 + 0 + b = 1 \Rightarrow b = \hat{1}$

$x = 1 \Rightarrow \hat{1} + a + \hat{1} = \hat{1} \Rightarrow a = \hat{1}$

$x^2 + x + \hat{1}$

obs. $x^2 + \hat{1} = x^2 - \hat{1} = (x-1)(x+1) = (x+1)(x+1) = (x+1)^2$ în $\mathbb{Z}_2$

grad 3

ired $\Leftrightarrow$ nu are răd.

$x^3 + ax^2 + bx + c$

$x = \hat{1} \Rightarrow \hat{1} + a + b + c = \hat{1} \Rightarrow a + b + c = 0 \Rightarrow a + b = \hat{1}$

$x = \hat{0} \Rightarrow 0 + 0 + 0 + \hat{c} = \hat{1} \Rightarrow c = \hat{1}$

I $a = \hat{1}, b = \hat{0}, c = \hat{1} \Rightarrow x^3 + x^2 + \hat{1}$

II $a = \hat{0}; b = \hat{1}, c = \hat{1} \Rightarrow x^3 + x + \hat{1}$

grad 4

ired $\Leftrightarrow$ $\begin{cases} \text{nu are răd.} \\ \text{să nu fie prod de 2 polinoame ired de grad } 2 \neq (x^2+x+1)^2 \end{cases}$

$x^4+ax^3+bx^2+cx+d$

$x=\hat{1} \Rightarrow \hat{1}+a+b+c+\hat{1}=\hat{1} \Rightarrow a+b+c=\hat{1}$

$x=\hat{0} \Rightarrow \hat{0}+\hat{0}+\hat{0}+\hat{0}+d=\hat{1} \Rightarrow d=\hat{1}$

111
sau
001 | 010 | 100

I $a=\hat{1}, b=\hat{1}, c=\hat{1}, d=\hat{1} \Rightarrow x^4+x^3+x^2+x+\hat{1}$

I $a=\hat{1}, b=\hat{0}, c=\hat{0}, d=\hat{1} \Rightarrow x^4+x^3+\hat{1}$

$\vdots$

$\cancel{x^4+x^2+\hat{1}}$

$x^4+x+\hat{1}$

$(a+b+c)^2=a^2+b^2+c^2+\underbrace{2(ab+ac+bc)}_{\text{în } \mathbb{Z}_2 \,=0}$

$(x^2+x+\hat{1})^2=x^4+x^2+\hat{1}$

<u>Temă</u> grad 4 sau în $\mathbb{Z}_3$ același ex

(4) Stabiliți dacă $f=x^4+2x^3+3x^2+x+1$ este ireductibil în $\mathbb{Q}[x]$

$\begin{cases} (1) \text{ grad } 1 \cdot \text{grad } 3 \\ (2) \text{ grad } 2 \cdot \text{grad } 2 \end{cases}$

(1): $\frac{u}{v} \in \mathbb{Q}$ cu $u/1, v/1 \Rightarrow \alpha=\pm 1$

$\left.\begin{array}{l} f(1) \neq 0 \\ f(-1)=1-2+3-1+1 \neq 0 \end{array}\right\} \Rightarrow$ nu are răd.

(2): $f=g\cdot h$ (2 pol. de gr 2)

$\underbrace{(a_0+a_1x+a_2x^2)}_{g}\cdot\underbrace{(b_0+b_1x+b_2x^2)}_{h}$ -

$a_0 \cdot b_0 + x(a_0 b_1 + a_1 b_0) + x^2(a_0 b_2 + a_2 b_0 + a_1 b_1) + x^3(a_1 b_2 + a_2 b_1) +$
$x^4 \cdot a_2 b_2$

$\underbrace{a_0 b_0}_{1}$, $\underbrace{a_0 b_1 + a_1 b_0}_{1}$, $\underbrace{a_0 b_2 + a_2 b_0 + a_1 b_1}_{3}$, $\underbrace{a_1 b_2 + a_2 b_1}_{2}$, $\underbrace{a_2 b_2}_{1}$

$$\begin{cases} a_0 \cdot b_0 = 1 \Rightarrow a_0 = b_0 = \pm 1 & a_i, b_i \in \mathbb{Z} \\ a_0 b_1 + a_1 b_0 = 1 \quad a_0(b_1 + a_1) = 1 \Rightarrow b_1 + a_1 = \pm 1 \\ a_0 b_2 + a_2 b_0 + a_1 b_1 = 3 \\ a_1 b_2 + a_2 b_1 = 2 \quad a_2(b_1 + a_1) = 2 \Rightarrow a_1 + b_1 = \pm 2 \\ a_2 b_2 = 1 \Rightarrow a_2 = b_2 = \pm 1 \end{cases}$$

! Imposibil

$\Rightarrow f$ este ireductibil

(5) Stabiliți dacă $f = X^4 + 2X^3 + 4X^2 + X + 1$ este ired în $\mathbb{Q}[X]$

$f \equiv \underbrace{X^4 + X + \hat{1}}_{\text{ired în } \mathbb{Z}_2} \pmod 2$

$\Rightarrow f$ ired. în $\mathbb{Q}[X]$

(6) —//— $f = X^4 + 4X^3 + 10X^2 + 2X + 6$ ired în $\mathbb{Q}[X]$

$2 \mid 4, 10, 2, 6$
$2^2 = 4 \nmid 6$
$\Big\}$ Criteriul Eisenstein $\Rightarrow f$ ired.

(7) Găsiți inversul, dc $\exists$, pt $X + 2$ în $\mathbb{Q}[X]/(X^2-1)$

factorizat

judecăm în modulo X^2-1

Caut $g \in \mathbb{Q}[X]$ a.î. $\underbrace{(X+2)}_{f} \cdot g \equiv \hat{1} \ (\text{mod}\,(X^2-1))$

grad 2 $\Rightarrow g$ = grad 1

$g = aX + b$

$(X+2)(aX+b) = aX^2 + bX + 2aX + 2b$ (ne „ocupăm" de termenii de grad ≥ 2)

$= a(X^2-1) + a + bX + 2aX + 2b$

$f \cdot g = 1 \Rightarrow \overbrace{\underbrace{a(x^2-1)}_{=0} + a + bx + 2ax + 2b} = \hat{1}$

$\Rightarrow \overbrace{(b+2a)x + a + 2b} = \hat{1}$

identificăm coef.

$$\begin{cases} b+2a=0 \Rightarrow b=-2a \\ a+2b=1 \Rightarrow a-4a=1 \Rightarrow -3a=1 \Rightarrow a=-\frac{1}{3} \Rightarrow b=\frac{2}{3} \end{cases}$$

$\Rightarrow$ inversul lui $x+2$ este $-\frac{1}{3}x+\frac{2}{3}$

Verificare

$(x+2)\left(-\frac{1}{3}x+\frac{2}{3}\right) = \frac{1}{3}(2+x)(2-x) = \frac{1}{3}(4-x^2) = \frac{4}{3} - \frac{x^2}{3} = \underbrace{(x^2-1)\left(-\frac{1}{3}\right)}_{=0} \underbrace{-\frac{1}{3}+\frac{4}{3}}_{=1}$

(8) Găsiți inversul, dc ∃, pt x^2-1 în $\mathbb{Q}[x]/(x^3+1)$

$g = ax^2+bx+c$

$f \cdot g = (x^2-1)(ax^2+bx+c)$

$= ax^4+bx^3+cx^2-ax^2-bx-c$

$= \underbrace{ax(x^3+1)}_{0} - ax + \underbrace{b(x^3+1)}_{0} - b + cx^2 - ax^2 - bx\ c$

$f \cdot g = \hat{1} \Rightarrow \overbrace{x^2(c-a) + x(-a-b) + (-b-c)} = \hat{1}$

$$\begin{cases} c-a=0 \Rightarrow c=a \\ -a-b=0 \Rightarrow a=-b \\ -b-c=\hat{1} \Rightarrow -a+a=1 \Rightarrow 0=1 \text{ (Fals)} \end{cases}$$

$\Rightarrow x^2-1$ Nu este inv

(9) $f = x^2+1$

$\mathbb{Q}[x]/x^3+1$

$g = ax^2 + bx + c$

$f \cdot g = (x^2+1)(ax^2+bx+c) =$

$= ax^4 + bx^3 + cx^2 + ax^2 + bx + c =$

$= ax\underbrace{(x^4+1)}_{=0} - ax + b\underbrace{(x^3+1)}_{=0} - b + cx^2 + ax^2 + bx + c =$

$= x^2(a+c) + x(-a+b) + c - b = \hat{1}$

$$\begin{cases} a+c=0 \Rightarrow a=-c \\ -a+b=0 \Rightarrow b=a \\ c-b=\hat{1} \Rightarrow -a-a=1 \Rightarrow -2a=1 \Rightarrow a=-\frac{1}{2} \end{cases}$$

$b = -\frac{1}{2}$

$c = \frac{1}{2}$

inv. este $-\frac{1}{2}x^2 - \frac{1}{2}x + \frac{1}{2}$

Seminar - Săptămâna 10

$(A, +, \cdot)$ = inel comutativ $(\mathbb{Z}, \mathbb{Z}_n, K[x], \ldots)$

Def.

$\emptyset \neq J \subseteq A$ s.n. ideal dc.

1) $x + y \in J, \forall x, y \in A$

2) $\forall x \in J, \forall a \in A \Rightarrow x \cdot a \in J$

3)

Pt $S \subseteq A$ notăm $J = (S) \to$ idealul generat de el. din S

J = finit generat dc. $J = (\underbrace{S_1, S_2, \ldots, S_n}_{\text{nr finit}})$

J = principal dc. $J = (S_1)$

$\to$ J ideal în A (inel. comutativ)

$J = (S_1, S_2, \ldots, S_n)$

$J \ni \underbrace{i = S_1 \cdot a_1 + S_2 \cdot a_2 + \ldots + S_n a_n}_{\text{așa arată un elem. din } J}$ cu $a_1, \ldots, a_n \in A$

$(f_1, \ldots, f_n)$

Teoremă

Pt $A = \mathbb{Z}$ sau $K[x]$ (K = corp)

$(f_1, \ldots, f_n) = (\gcd(f_1, \ldots, f_n))$

$(f_1) \cap \ldots \cap (f_n) = (\operatorname{lcm}(f_1, \ldots, f_n))$

- $A/J \to$ „A factorizat la (idealul) J"
  $\hookrightarrow \hat{x} = x + J$

- Morfism de inel
  $\varphi: A_1 \to A_2$
  $\begin{cases} \varphi(a_1 + a_2) = \varphi(a_1) + \varphi(a_2) \\ \varphi(a_1 \cdot a_2) = \varphi(a_1) \cdot \varphi(a_2) \\ \varphi(1_{A_1}) = 1_{A_2} \end{cases}$

Teorema fundamentală de izomorfism (pt inele)

$\varphi: A_1 \to A_2$ morfism de inele $\Rightarrow A_1 / _{\ker \varphi} \simeq \operatorname{Im} \varphi$

$\ker \varphi = \{a_1 \in A_1 \mid \varphi(a_1) = 0\}$

↳ kernel (nucleul lui φ)

↳ Ker = ideal în A_1

---

(1) A = un inel comutativ, $a \in A$

A_2

Arătați că $A[x] / _{(x-a)} \simeq A$

A_1 Ker ↳ izomorf

<u>Caut</u>: $\varphi: A[x] \to A$ (surjectiv)

$f \to f(a)$ (îl evaluăm în a ?!)

$\varphi(f(x)) = f(a)$

$\ker \varphi = \{f(x) \in A[x] \mid \varphi(f(x)) = 0\} = \{f(x) \in A[x] \mid f(a) = 0\} =$

$= \{f(x) \in A[x] \mid a \text{ răd pt } f\} = (x-a)$

idealul lui $(x-a)$

a răd pt $f \Rightarrow f = (x-a) \cdot g$, $g \in A[x]$

$$\boxed{\begin{array}{l} \text{morf surj} \Rightarrow \operatorname{Im}\varphi = A_2 \\ A_1 / _{\ker \varphi} \simeq A_2 \end{array}}$$

Fie $b \in A$

$\left.\begin{array}{l} \varphi(f(x)) = b \\ \varphi(b) = b \end{array}\right\} \Rightarrow f(x)$ polinomul lui „b"

↳ surj

$(TFI) \Rightarrow A[x] / _{(x-a)} \simeq A$

② Ar. că $\mathbb{Z}[x]/(x^2+1) \simeq \mathbb{Z}[i]$

$\mathbb{Z}[i] = \{a+bi \mid a, b \in \mathbb{Z}\}$

Caut $\varphi: \mathbb{Z}[x] \to \mathbb{Z}[i]$ surjectiv

$\varphi(f(x)) = f(i)$

surj ?

$a+bi \in \mathbb{Z}[i]$

Caut $f(x)$ a.î. $f(i) = a+bi$

$f(x) = a+bx$ $\Rightarrow$ surj.

$\operatorname{Ker}\varphi = \{f(x) \in \mathbb{Z}[x] \mid \varphi(f(x)) = 0\} = \{f(x) \in \mathbb{Z}[x] \mid f(i) = 0\} =$
$= (x^2+1)$

$f(i) = 0 \Leftrightarrow (x^2+1)\cdot g$, $g \in \mathbb{Z}[x]$ $\Rightarrow$ (TFI) $\Rightarrow \mathbb{Z}[x]/(x^2+1) \simeq \mathbb{Z}[i]$

A = inel

$J = (p, q) = \{p\cdot x + q\cdot y \mid x, y \in A\}$

$J = (p) = \{p\cdot x \mid x \in A\}$

$J = (p_1, \dots, p_m) = \{p_1\cdot x_1 + p_2\cdot x_2 + \dots + p_m\cdot x_m \mid x_1, \dots, x_m \in A\}$

$\begin{cases} (1) = \text{ideal în } \mathbb{Z} \\ \mathbb{Z} = (1) \end{cases}$

__Obs.__ $(A, +, \cdot)$ = inel, $x \in A$

1) x = idempotent dacă $x^2 = x \Rightarrow x^n = x$, $n \geq 2$

2) x = nilpotent dacă $\exists k \in \mathbb{N}$ a.î. $x^k = 0$

3) x = divizor al lui 0 dacă $\exists y \in A$ a.î. $x\cdot y = 0$

$\mathbb{Z}_8 \to \hat{2}, \hat{4}$

③ Găsiți elementele idempotente în :

a) $\mathbb{Z}[X]/(X^2-1) \rightarrow \hat{f} = f + (X^2-1)$

Caut $\hat{f}$ cu $\hat{f}^2 = \hat{f}$

$\hat{f} = \widehat{aX+b}$

$\Rightarrow \hat{f}^2 = \hat{f} \Rightarrow (\widehat{aX+b})^2 = (\widehat{aX+b}) \Rightarrow \widehat{a^2X^2+2abX+b^2} = \widehat{aX+b}$

$\Rightarrow \widehat{a^2(X^2-1)+a^2+2abX+b^2} = \widehat{aX+b}$

$$\begin{cases} a^2+b^2 = b \\ 2ab = a \end{cases}$$

dacă $a=0 \Rightarrow b(b-1)=0 < \begin{matrix} b=0 \\ b=1 \end{matrix}$

dacă $a \neq 0 \Rightarrow 2b = 1 \Rightarrow b = \frac{1}{2} \notin \mathbb{Z}$

$\Rightarrow$ elem. idempotente sunt 0 și 1

b) $\mathbb{Z} \times \mathbb{Z}$

Caut (a,b) cu $(a,b)^2 = (a,b) \Rightarrow (a^2,b^2) = (a,b) \Rightarrow$

$\Rightarrow \begin{cases} a(a-1)=0 \Rightarrow a=0 \text{ sau } a=1 \\ b(b-1)=0 \Rightarrow b=0 \text{ sau } b=1 \end{cases}$ $\Rightarrow$ elem. idemp sunt $(0,0), (0,1), (1,0), (1,1)$

---

④ Verif. dacă

a) $(2X, X^3+3X) = \mathbb{R}[X]$

b) $(X^2-1, X+2) = \mathbb{Z}[X]$

c) $(X^3-1, 3X^2+2X-5)$, (X^2-2X+1, X^2-1) în $\mathbb{Q}[X]$

a) $J = (2X, X^3+3X)$

$J = \mathbb{R}[X]$ dacă $1 \in J$

$X^3+3X = 2X\left(\frac{1}{2}\cdot X^2 + \frac{3}{2}\right)$

$f \in J \Rightarrow f = 2x \cdot f' + (x^3+3x) \cdot g'$ cu $f', g' \in \mathbb{R}[x]$

$f = 2x \cdot f' + 2x\left(\frac{1}{2}x^2 + \frac{3}{2}\right) g'$

$f = 2x \cdot \left(f' + \left(\frac{1}{2} \cdot x^2 + \frac{3}{2}\right) \cdot g'\right) = (2x)$

$(2x) \to 2x \cdot$ ceva $\ulcorner$ ceva $0 \Rightarrow 0 \neq 1$

ceva $\neq 0 \Rightarrow 2x \cdot$ ceva are grad $\geq 1 \Rightarrow \neq$ grado

c) $(x^3-1) = \underline{(x-1)}(x^2+x+1)$

$3x^2+2x-5 = 3x^2-3x+5x-5 = 3x(x-1)+5(x-1) = \underline{(x-1)}(3x+5)$

$\gcd = x-1 \xRightarrow{Th.} J_1 = (x-1)$

$$\left.\begin{array}{l} x^2-2x+1 = (x-1)^2 \\ x^2-1 = (x-1)(x+1) \end{array}\right\} \Rightarrow \gcd = (x-1) \xRightarrow{Th} J_2 = (x-1)$$

$\Rightarrow J_1 = J_2$ ✓

⑤ Det. el. nilpotente în $\mathbb{Z}_n$

$n = p_1^{\alpha_1} \cdot p_2^{\alpha_2} \cdot \ldots \cdot p_k^{\alpha_k}$

$\hat{x}^k = \hat{0} = p_1^{\alpha_1} \cdot p_2^{\alpha_2} \cdot \ldots \cdot p_k^{\alpha_k}$

$p_1, p_2, \ldots, p_k \,/\, x \Rightarrow x = p_1 \cdot p_2 \cdot \ldots \cdot p_k \cdot p$

__ex:__ în $\mathbb{Z}_{48}$

$48 = 2^4 \cdot 3$

multiplu

x este nilpotent dacă $x = M_{2 \cdot 3} = M_6$

$\Rightarrow \hat{0}, \hat{6}, \hat{12}, \hat{18}, \hat{30}, \hat{36}, \hat{42}$

Seminar 11

① Ar. că a) $\mathbb{Q}[X] / (X^2-1) \simeq \mathbb{Q} \times \mathbb{Q}$

Th. $\varphi : A_1 \to A_2$ (surj.)

$A_1 / \text{Ker}\,\varphi \simeq A_2$

Caut $\varphi : \mathbb{Q}[X] \to \mathbb{Q} \times \mathbb{Q}$ surj.

$\varphi(f) = (f(1), f(-1))$

$X^2 - 1 = 0 \Rightarrow X_{1,2} = \pm 1$

$\text{Ker}\,\varphi = \{ f \in A_1 / \varphi(f) = 0 \}$

① surj.

$(a, b) \in \mathbb{Q} \times \mathbb{Q}$

Caut f a.î. $\begin{cases} f(1) = a \\ f(-1) = b \end{cases}$

$f = mX + n$ $\to$ $\left.\begin{matrix} m + n = a \\ -m + n = b \end{matrix}\right\} \overset{\oplus}{\Rightarrow} 2n = a+b \Rightarrow n = \frac{a+b}{2} \in \mathbb{Q}$

$m, n \in \mathbb{Q}$

$f = \frac{a-b}{2} \cdot X + \frac{a+b}{2} \Rightarrow \varphi =$ surj.

② $\text{Ker}\,\varphi = \{ f \in \mathbb{Q}[X] / \varphi(f(X)) = (0,0) \} = \{ f \in \mathbb{Q}[X] / f(1) = 0$ și $f(-1) = 0 \} =$
$= \{ f \in \mathbb{Q}[X] / 1$ și -1 răd pt $f \} = \{ f \in \mathbb{Q}[X] / f = (X^2-1) \cdot g, g \in \mathbb{Q}[X] \}$
$= (X^2-1)$

$\xRightarrow{\text{Teorema fundamentală de izomorfism}} \mathbb{Q}[X] / (X^2-1) \simeq \mathbb{Q} \times \mathbb{Q}$

b) $\mathbb{R}[x]/(x^2+1) \simeq \mathbb{C}$

$x^2+1=0 \Rightarrow x=\pm i$

Caut $\varphi: \mathbb{R}[x] \to \mathbb{C}$ surj.

$\varphi(f) = f(i)$

① surj. $a \in \mathbb{C}$

Caut $f \in \mathbb{R}[x]$ ai $f(i)=a$

$\downarrow$ polinom

$f = x + (-i+a)$

sau

$f = a \Rightarrow f$ surj.

② $\text{Ker}\,\varphi = \{ f \in \mathbb{R}[x] \mid \varphi(f) = 0\} = \{ f \in \mathbb{R}[x] \mid f(i) = 0\} =$

$= \{ f \in \mathbb{R}[x] \mid i$ răd pt $f \} = \{ f \in \mathbb{R}[x] \mid f = (x^2+1)\cdot g \} =$

$= (x^2+1) \xrightarrow{\text{T.F. de izomorfism}} \mathbb{R}[x]/(x^2+1) \simeq \mathbb{C}$

$\mathbb{R}[x]/(x^2+1)$

$\downarrow$

$\mathbb{R}[x]$ factorizat la x^2+1

$\hat{f} = f + (x^2+1)\cdot g$

$\uparrow$

$\mathbb{R}[x]$

c) $\underbrace{\mathbb{Z}[x]}/(x^2-1) \;\cancel{\simeq}\; \mathbb{Z}\times\mathbb{Z}$

2 elem. idempotente

$\downarrow$

4 elem. idempotente

d) $\mathbb{C}[x]/_{(x^2+1)} \simeq \mathbb{C}\times\mathbb{C}$ $\qquad x^2+1=0 \Rightarrow x=\pm i$

Caut $\varphi: \mathbb{C} \to \mathbb{C}\times\mathbb{C}$ surj.

$\varphi(f) = (f(i), f(-i))$

① surj. $(a,b) \in \mathbb{C}\times\mathbb{C}$

Caut $f \in \mathbb{C}[x]$ aî $(f(i), f(-i)) = (a,b) \Rightarrow \begin{cases} f(i)=a \\ f(-i)=b \end{cases} \Rightarrow$

$f = mx+n$

$\begin{cases} mi+n=a \\ -mi+n=b \end{cases}$ (+)

$2n = a+b \Rightarrow n = \frac{a+b}{2} \in \mathbb{C}$

$mi = a - \frac{a+b}{2} \Rightarrow mi = \frac{a-b}{2} \;|\cdot i \Rightarrow m = \frac{b-a}{2} \in \mathbb{C}$

$f = \frac{b-a}{2}\cdot i\cdot x + \frac{a+b}{2} \Rightarrow \varphi$ surj.

② $\operatorname{Ker}\varphi = \{f\in\mathbb{C}[x] \mid \varphi(f) = (0,0)\} = \{f\in\mathbb{C}[x] \mid i, -i \text{ rad pt } f\} =$
$= \{f\in\mathbb{C}[x] \mid f = (x^2+1)\cdot g,\ g\in\mathbb{C}[x]\} = (x^2+1) \xRightarrow{TFI} \mathbb{C}[x]/_{(x^2+1)} \simeq \mathbb{C}\times\mathbb{C}$

Obs. $J \subset K[x]$ ideal

J = principal dacă $J = (f)$ ← un singur generator

$(f_1, f_2, \dots, f_n) = (\gcd(f_1,\dots,f_n))$ DOAR DACĂ K = corp $(\mathbb{Q}, \mathbb{R}, \dots)$

Test: din sem. 7: restul împ. unei puteri la un nr

din sem 8: gcd pol., inversul, polinom de gr 4 ired.

din sem 9: tot fără gcd de polinoame

din sem 10: tot (elem. idemp, divizori 0, TFI, 2 ideale sunt egale ...)

② Arătați că $(2,x) \in \mathbb{Z}[x]$ nu este principal

Pp. că $\exists f \in \mathbb{Z}[x]$ a.î. $(f) = (2,x)$

$$\underbrace{2\cdot F + x\cdot G}_{\text{polinoame cu termenul liber par}} , F, G \in \mathbb{Z}[x]$$

1) grad $f = 0 \Rightarrow f = $ nr par

$\Rightarrow x \in$ (nr par) $\boxed{\text{FALS}}$

polinoame care au toți coef. pari

2) grad $f = 1 \Rightarrow f = x$

$\Rightarrow 2 \in (x)$ $\boxed{\text{FALS}}$

x polinom – nu are termen liber $\Rightarrow 2 \notin (x)$

Aplicații – LCR

$$\begin{cases} x \equiv a_1 \pmod{n_1} \\ x \equiv a_2 \pmod{n_2} \\ \vdots \\ x \equiv a_t \pmod{n_t} \end{cases} , (n_i, n_j) = 1, \forall i \neq j$$

sistemul are sol. unică mod $N \to x_0$

$N = n_1 \cdot n_2 \cdot \ldots \cdot n_t$

în $\mathbb{Z}$ $S = \{x_0 + N\cdot p / p \in \mathbb{Z}\}$

Algoritm:

Pas 1: $(n_i, n_j) = 1$

Pas 2: $N = n_1 \cdot \ldots \cdot n_t$

Pas 3: $n_i' = \frac{N}{n_i}$

Pas 4: k_i, inversul lui $n_i' \pmod{n_i}$

$k_i \cdot n_i' \equiv 1 \pmod{n_i}$ (Alg. lui Euclid pt nr mari)

Pas 5: $x_0 : m_1' \cdot a_1 \cdot k_1 + m_2' \cdot a_2 \cdot k_2 \cdot \ldots \cdot m_t' \cdot a_t \cdot k_t \pmod N$

③ Rezolvați în $\mathbb{Z}$

a) $\begin{cases} x \equiv 1 \pmod{15} \\ x \equiv 5 \pmod 8 \end{cases}$

- $((15, 8)) = 1$
- $N = 15 \cdot 8 = 120$

Încercări: $\{0, 1, 2, \ldots, 11.$

$\cancel{1}, \cancel{16}, \cancel{31}, \cancel{46}, 61$ ✓ $\Rightarrow x_0 = 61 \Rightarrow S = \{61 + 120p / p \in \mathbb{Z}\}$

$15 \cdot i + 1$

① ✓

② $N = 120$

③ $m_1' = \frac{120}{15} = 8 \qquad m_2' = \frac{120}{8} = 15$

④ $k_1 \cdot 8 \equiv 1 \pmod{15} \rightarrow k_1 = 2$

$k_2 \cdot 15 \equiv 1 \pmod 8 \rightarrow k_2 \cdot 7 \equiv 1 \pmod 8 \rightarrow k_2 = 7$

⑤ $x_0 = 8 \cdot 2 \cdot 1 + 15 \cdot 7 \cdot 5 = 16 + 525 = 541$

$S = \{61 + 120p / p \in \mathbb{Z}\}$

b) $\begin{cases} 2x \equiv 7 \pmod{15} \\ x \equiv 2 \pmod 8 \end{cases}$

$2x \equiv 7 \pmod{15} \;|\cdot 8 \Rightarrow \begin{cases} x \equiv \underset{a_1}{11} \pmod{\underset{m_1}{15}} \\ x \equiv \underset{a_2}{2} \pmod{\underset{m_2}{8}} \end{cases}$

$N = 15 \cdot 8 = 120$

Încercări: $\{0, \ldots, 119\}$

$\cancel{11}, 26$ ✓ $\qquad (26 = \underset{a_1}{11} + 1 \cdot \underset{m_1}{15} \xrightarrow[\text{mod } m_2]{\text{mod } 8} \underset{a_2}{5})$

$x_0 = 26 \Rightarrow S = \{26 + 120p, p \in \mathbb{Z}\}$

! c) $\begin{cases} 2x \equiv 4 \pmod{10} \\ x \equiv 1 \pmod{21} \end{cases}$ $\Leftrightarrow$ $2x : 10 = c$ rest 4 $\Leftrightarrow$ $2x = 10c + 4$

$\underline{x = 5c + 2}$

$\begin{cases} x \equiv 2 \pmod{5} \\ x \equiv 1 \pmod{21} \end{cases}$

$(5, 21) = 1$

$N = 5 \cdot 21 = 105$

Încercări : $\{0, \ldots, 104\}$

$\cancel{1}$, 22 ✓

$x_0 = 22 \Rightarrow S = \{22 + 105p \mid p \in \mathbb{Z}\}$

d) $\begin{cases} 2x \equiv 7 \pmod{10} \\ x \equiv 1 \pmod{21} \mid \cdot 2 \end{cases} \Rightarrow 2x \equiv 2 \pmod{21}$

not. $2x = t \Rightarrow \begin{cases} t \equiv 7 \pmod{10} \\ t \equiv 2 \pmod{21} \end{cases}$

$(10, 21) = 1$

$N = 10 \cdot 21 = 210$

Încercări : $\{0, \ldots, 209\}$

$7, 17, 27, 37, 47, 57, 67, 77, 87, 97, 107$ ✓

$t_0 = 105 \Rightarrow S = \{107 + 210p \mid p \in \mathbb{Z}\}$ NU E PAR $\Rightarrow$

$\Rightarrow$ nu putem împ. la 2 pt pt a afla sol pt $x \in \mathbb{Z}$

Seminar - Săptămâna 12

Curs : G_1, G_2 = grupuri

$(g_1, g_2) \in G_1 \times G_2$

$ord((g_1, g_2)) = lcm(ord(g_1), ord(g_2))$

• G = grup finit

1) $ord(G)$ = nr de elemente

2) $ord(g)$ = cel mai mic nr nat ≥ 1

$\mathbb{Z}_n$:

$$ord(\hat{x}) = \frac{n}{gcd(n, x)}$$

---

1) Det. în $G = (\mathbb{Z}_{24} \times \mathbb{Z}_{22}, +)$

$ord(\bar{4}, \hat{6})$

$$\left.\begin{array}{l} ord(\bar{4}) = \frac{24}{gcd(24,4)} = \frac{24}{4} = 6 \\ ord(\hat{6}) = \frac{22}{gcd(22,6)} = \frac{22}{2} = 11 \end{array}\right\} \Rightarrow ord(\bar{4}, \hat{6}) = lcm(6, 11) = 66$$

8) $ord(\bar{9}, \hat{3})$

$$\left.\begin{array}{l} ord(\bar{9}) = \frac{24}{gcd(24,9)} = \frac{24}{3} = 8 \\ ord(\hat{3}) = \frac{22}{gcd(22,3)} = \frac{22}{1} = 22 \end{array}\right\} \Rightarrow ord(\bar{9}, \hat{3}) = lcm(8, 22) = 88$$

2) Det. în $G = \mathbb{Z}_2 \times \mathbb{Z}_5 \times \mathbb{Z}_8$

$ord(\bar{1}, \hat{2}, \tilde{4})$

$$\left.\begin{array}{l} ord(\bar{1}) = \frac{2}{(2,1)} = \frac{2}{1} = 2 \\ ord(\hat{2}) = \frac{5}{(5,2)} = \frac{5}{1} = 5 \\ ord(\tilde{4}) = \frac{8}{(8,4)} = \frac{8}{4} = 2 \end{array}\right\} \Rightarrow ord(\bar{1}, \hat{2}, \tilde{4}) = lcm(2, 5, 2) = 10$$

③ Det. subgr. lui $(\mathbb{Z}_{16}, +)$

$\mathbb{Z}_{16} = \{\hat{0}, \hat{1}, \hat{2}, \dots, \hat{15}\}$

$<\hat{p}> = \{\hat{p} \cdot x \mid x \in \mathbb{Z}_{16}\} = <gcd(p, 16)>$

$<\hat{p_1}, \hat{p_2}> = \{\hat{p_1} x + \hat{p_2} \cdot y \mid x, y \in \mathbb{Z}_{16}\}$

nr impare - prime cu 16

$<\hat{1}> = \mathbb{Z}_{16} = <\hat{3}> = <\hat{5}> = <\hat{7}> = <\hat{9}> = <\hat{11}> = <\hat{13}> = <\hat{15}>$

$<\hat{2}> = <\hat{6}> = <\hat{10}> = <\hat{14}>$ ← tb să avem gcd 2, nr care nu se împ la 4

$<\hat{4}> = <\hat{12}>$

$<\hat{8}>$

$<\hat{0}> = \{0\}$

D(16)

$$\begin{array}{c} 16 \\ | \\ 8 \\ | \\ 4 \\ | \\ 2 \\ | \\ 1 \end{array} \qquad \begin{array}{c} \mathbb{Z}_{16} \\ | \\ <\hat{8}> \\ | \\ <\hat{4}> \\ | \\ <\hat{2}> \\ | \\ \{0\} \end{array}$$

temă $\mathbb{Z}_{24}$

Seminar - Săptămâna 13

Recapitulare

① $\hat{x} \in \mathbb{Z}_n$, $ord(\hat{x}) = \frac{n}{gcd(n,x)}$

$(g_1, g_2) \in G_1 \times G_2$, $ord(g_1, g_2) = lcm(ord(g_1), ord(g_2))$

Aplicații:

1) det. ord $(\hat{5}, \hat{3})$ în:

a) $(\mathbb{Z}_6 \times \mathbb{Z}_8, +)$

$\left.\begin{aligned} ord(\hat{5}) &= \frac{6}{1} = 6 \\ ord(\hat{3}) &= \frac{8}{1} = 8 \end{aligned}\right\} \Rightarrow ord(\hat{5}, \hat{3}) = lcm(6, 8) = 24$

b) $(\mathbb{Z}_{11} \times \mathbb{Z}_9, +)$

$\left.\begin{aligned} ord(\hat{5}) &= \frac{11}{gcd(5,11)} = \frac{11}{1} = 11 \\ ord(\hat{3}) &= \frac{9}{gcd(9,3)} = \frac{9}{3} = 3 \end{aligned}\right\} \Rightarrow ord(\hat{5}, \hat{3}) = lcm(11, 3) = 33$

2) det. elem. de ordin 4 din $(\mathbb{Z}_{12} \times \mathbb{Z}_{14}, +)$

(x, y) cu $ord(x, y) = 4$

$ord(x, y) = lcm(ord(x), ord(y))$

$x \in \mathbb{Z}_{12} \Rightarrow ord(x) \in D_{12} = \{1, 2, 3, 4, 6, 12\}$

$y \in \mathbb{Z}_{14} \Rightarrow ord(y) \in D_{14} = \{1, 2, 7, 14\}$

Rtc $lcm = 4 \Rightarrow ord(x), ord(y) \leq 4$

$\Rightarrow ord(x) \in \{1, 2, 3, 4\}$

$ord(y) \in \{1, 2\}$

Verificăm perechile:

$lcm(1,1) = 1$ $\quad$ $lcm(3,1) = 3$

$lcm(1,2) = 2$ $\quad$ $lcm(3,2) = 6$

$lcm(2,1) = 2$ $\quad$ $lcm(4,1) = 4$ ✓

$lcm(2,2) = 2$ $\quad$ $lcm(4,2) = 4$ ✓

Aflăm elementele posibile:

I: $ord(x) = 4$, $ord(y) = 1$

$ord(x) = 4 \Rightarrow \frac{12}{gcd(12,x)} = 4 \Rightarrow gcd(12,x) = 3 \Rightarrow x = \{\hat{3}, \hat{9}\}$

$ord(y) = 1 \Rightarrow \frac{14}{gcd(14,y)} = 1 \Rightarrow gcd(14,y) = 14 \Rightarrow y = \bar{0}$

Concluzie: $(\hat{3}, \bar{0}), (\hat{9}, \bar{0})$

II: $ord(y) = 2 \Rightarrow \frac{14}{gcd(14,y)} = 2 \Rightarrow gcd(14,y) = 7$

Concluzie: $(\hat{3}, \hat{7}), (\hat{9}, \bar{7})$

Teorie:

G_1, G_2 = grupuri ; $f : G_1 \to G_2$ = morfism de grupuri dacă

1) $f(1_{G_1}) = 1_{G_2}$

2) $f(a+b) = f(a) + f(b)$

$Ker\, f = \{g_1 \in G_1 / f(g_1) = 0\}$

$Im\, f = \{g_2 \in G_2 / f(g_2) = g_2$ pt $g_1 \in G_1\}$

3) Construiți morfisme de la:

a) $\mathbb{Z}$ la $\mathbb{Z}_3$

1. $f(0)=\hat{0}$

2. dc $f(1)=\hat{0} \Rightarrow f(2)=f(1)+f(1)=\hat{0} \Rightarrow f(k)=\hat{0}$

dc $f(1)=\hat{1} \Rightarrow f(2)=f(1)+f(1)=\hat{1}+\hat{1}=\hat{2} \Rightarrow f(k)=\hat{k}$

dc $f(1)=\hat{2} \Rightarrow f(2)=f(1)+f(1)=\hat{2}+\hat{2}=\hat{1} \Rightarrow f(k)=\widehat{2k}$

Concluzie: avem 3 morfisme de la $\mathbb{Z}$ la $\mathbb{Z}_3$:

$f(x)=\hat{0}$,

$f(x)=\hat{x}$, $\quad x \in \mathbb{Z}$

$f(x)=\widehat{2x}$,

b) $\mathbb{Z}_3$ la $\mathbb{Z}$

1. $f(\hat{0})=0$

2. $f(\hat{1})=a \in \mathbb{Z} \Rightarrow f(\hat{1}+\hat{1}+\hat{1})=3a=\hat{0} \Rightarrow a=\hat{0}$

Concluzie: avem 1 morfism de la $\mathbb{Z}_3$ la $\mathbb{Z}$

$f(x)=0$, $\hat{x} \in \mathbb{Z}_3$

c) $\mathbb{Z}_3$ la $\mathbb{Z}_6$

1. $f(\hat{0})=\bar{0}$

2. $f(\hat{1})=a \in \mathbb{Z}_6$

$$\left.\begin{array}{l} f(\hat{1}+\hat{1}+\hat{1}) = 3f(\hat{1}) = 3a \\ \qquad f(\hat{0})=\bar{0} \end{array}\right\} \Rightarrow 3a=\bar{0} \Leftrightarrow a \in \{\bar{0},\bar{2},\bar{4}\}$$

Concluzie: avem 3 morfisme de la $\mathbb{Z}_3$ la $\mathbb{Z}_6$:

$f(\hat{x})=\bar{0}$,

$f(\hat{x})=\overline{2x}$, $\quad x \in \mathbb{Z}_3$

$f(x)=\overline{4x}$,

4) Det. $\mathrm{Ker}\,f$ și $\mathrm{Im}\,f$ pt morfismele de la ex 3

$f: \mathbb{Z} \to \mathbb{Z}_3$

I $f(x) = \hat{0} \Rightarrow \mathrm{Im}\,f = \{\hat{0}\}$

$\mathrm{Ker}\,f = \mathbb{Z}$

II $f(x) = \hat{x} \Rightarrow \mathrm{Im}\,f = \mathbb{Z}_3 = \{\hat{0}, \hat{1}, \hat{2}\}$

$\mathrm{Ker}\,f = \{3x \mid x \in \mathbb{Z}\}$

III $f(x) = 2\hat{x} \Rightarrow \mathrm{Im}\,f = \mathbb{Z}_3 = \{\hat{0}, \hat{1}, \hat{2}\}$

$\mathrm{Ker}\,f = \{3x \mid x \in \mathbb{Z}\}$

$f: \mathbb{Z}_3 \to \mathbb{Z}$

$f(x) = \hat{0} \Rightarrow \mathrm{Im}\,f = \{\hat{0}\}$

$\mathrm{Ker}\,f = \mathbb{Z}_3 = \{\hat{0}, \hat{1}, \hat{2}\}$

$f: \mathbb{Z}_3 \to \mathbb{Z}_6$

I $f(x) = \bar{0} \Rightarrow \mathrm{Im}\,f = \{\hat{0}\}$

$\mathrm{Ker}\,f = \mathbb{Z}_3 = \{\hat{0}, \hat{1}, \hat{2}\}$

II $f(x) = \bar{2}\hat{x} \Rightarrow \mathrm{Im}\,f = \{\bar{0}, \bar{2}, \bar{4}\}$

$\mathrm{Ker}\,f = \{\hat{0}\}$

III $f(x) = 4\hat{x} \Rightarrow \mathrm{Im}\,f = \{\bar{0}, \bar{2}, \bar{4}\}$

$\mathrm{Ker}\,f = \{\hat{0}\}$

Recapitulare !

1. $(\forall)$ p prim, $(\forall)$ $e \in \mathbb{N}^*$, $\exists!$ (până la un izo.) corp cu p^e elemente

2. $K[X]/(f)$ = corp dacă f ired în K

K corp

3. $\mathbb{Z}_p$ = corp dacă p = prim

Se aplică pt $e > 1$?

5) Construiți un corp cu:

a) 4 elemente

$4 = 2^2 \Rightarrow p = 2$

Caut un polinom în $\mathbb{Z}_2[x]$ de gradul 2 ireductibil

$x^2 + x + \hat{1}$ ireductibil

$\Rightarrow \mathbb{Z}_2[x] / (x^2 + x + \hat{1})$ corp

Elementele din $\mathbb{Z}_2[x] / (x^2 + x + 1) = \{\hat{0}, \hat{1}, x, x + \hat{1}\}$

$\Rightarrow \mathbb{Z}_2[x] / (x^2 + x + \hat{1})$ corp cu 4 elemente

b) cu 9 elemente

$9 = 3^2 \Rightarrow p = 3$

Caut un polinom ireductibil în $\mathbb{Z}_3[x]$ de grd 2

$x^2 + 1$

$\mathbb{Z}_3[x] / x^2 + 1$ corp cu elem: $\{\hat{0}, \hat{1}, \hat{2}, x + \hat{1}, x + \hat{2}, x, \hat{2}x,$
$\hat{2}x + \hat{1}, \hat{2}x + \hat{2}\}$

Seminar 14

① Stabiliți dacă $\mathbb{Z}_2 \times \mathbb{Z}_6$ este ciclic

$\mathbb{Z}_2 \times \mathbb{Z}_6 \simeq \mathbb{Z}_{12}$

$\left.\begin{array}{l} \mathbb{Z}_2 : 1, 2 \text{ (ordine)} \\ \mathbb{Z}_6 : 1, 2, 3, 6 \end{array}\right\} \Rightarrow \mathbb{Z}_2 : 1, 2, 3, 6 \Rightarrow$ nu $\exists$ elem de ordin 12

$\Rightarrow \mathbb{Z}_2 \times \mathbb{Z}_6 \simeq \mathbb{Z}_{12} \Rightarrow$ nu este ciclic

Q: $\mathbb{Z}_n \times \mathbb{Z}_m = \mathbb{Z}_{n \cdot m}$ dacă $(n, m) = 1$

<u>Teoremă</u>

$\forall \sigma \in S_n$, σ se poate scrie ca produs de cicluri disjuncte

EX

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 4 & 6 & 2 & 1 & 9 & 3 & 5 & 7 & 8 \end{pmatrix} \in S_9$$

$\sigma = (1\ \ 4)(2\ \ 6\ \ 3)(5\ \ 9\ \ 8\ \ 7)$

! $c = (a_1, a_2, \ldots, a_k) =$ ciclu $\rightarrow$ k-lungimea ciclului $= ord(c)$

$\sigma = c_1 \cdot c_2 \cdot \ldots \cdot c_p$

• $\varepsilon(c) = (-1)^{k-1}$ (signatura)

-1 - impară

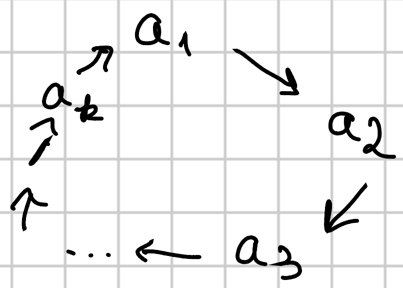
1 - pară

Pt σ:

• $ord(\sigma) =$ lcm (lung. ciclurilor din descompunere)

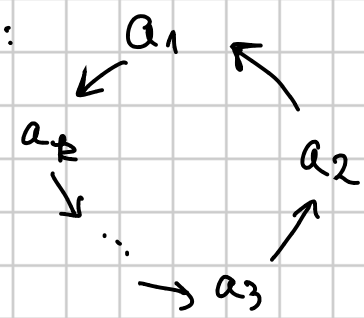
• $\varepsilon(\sigma) = \varepsilon(c_1) \cdot \varepsilon(c_2) \cdot \ldots \cdot \varepsilon(c_p)$

$c:$



$c = (a_1, a_2, \dots, a_k)$

$c^{-1}:$



$c = (a_k, a_{k-1}, \dots, a_1)$

$\sigma = (1\ \ 4)(2\ \ 6\ \ 3)(5\ \ 9\ \ 8\ \ 7)$

$ord(\sigma) = lcm(2, 3, 4) = 12$

$\varepsilon(\sigma) = (-1)^1 (-1)^2 (-1)^3$

Transpoziție = ciclu de lungime 2

$\forall \sigma \in S_n$, σ se poate scrie ca produs de transpoziții

$\sigma = c_1 \cdot c_2 \cdot \ldots \cdot c_p$ (cicluri disjuncte)

$c_i = (a_1\ a_2\ \dots\ a_k)$

1) $(a_1 a_2)(a_2 a_3)(a_3 a_4)\dots(a_{k-1} a_k)$

2) $(a_1 a_k)(a_1 a_{k-1})(a_1 a_{k-2}) \dots (a_1 a_3)(a_1 a_2)$

<u>dem</u> $c = (a_1\ a_2\ \ \ a_{k-1}\ a_k)$

$= (a_1\ a_2)(a_2\ a_3)(a_3\ a_4)\dots(a_{k-1}\ a_k)$

$a_1 \to a_1 \to \dots \to a_1 \to a_2$

$a_2 \to a_2 \to \dots a_2 \to a_3 \to a_3$

$\vdots$

$a_{k-1} \to a_k \to \dots \to a_k$

$a_k \to a_{k-1} \to a_{k-2} \to \dots \to a_3 \to a_2 \to a_1$

<u>ex</u> $\sigma = (1\ 4)(2\ 6\ 3)(5\ 9\ 8\ 7)$

$(1 4)(2 6)(6 3)(5 9)(9 8)(8 7)$

sau

$(1\ 4)(2\ 3)(2\ 6)(5\ 7)(5\ 8)(5\ 9)$

$$\begin{tikzcd} 1 \arrow[r] & 2 \arrow[d] \\ 4 \arrow[u] & 3 \arrow[l] \end{tikzcd}$$

$(1\ 2\ 3\ 4)^2$

$(1\ 3)(2\ 4)$

② $\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 \\ 2 & 5 & 9 & 4 & 7 & 10 & 1 & 11 & 3 & 8 & 6 \end{pmatrix}$

a) cicluri disjuncte + transpoziții

b) $ord(\sigma)$, $\varepsilon(\sigma)$

c) $\sigma^{2019} = ?$

d) $x \in S_{11}$, $x^4 = \sigma$

a) $\sigma = (1\ 2\ 5\ 7)(3\ 9)(4)(6\ 10\ 8\ 11)$

transpoziții: $(1\ 2)(2\ 5)(5\ 7)(3\ 9)(6\ 10)(10\ 8)(8\ 11)$

sau $(1\ 7)(1\ 5)(1\ 2)(3\ 9)(6\ 11)(6\ 8)(6\ 10)$

b) $ord(\sigma) = lcm(4, 2, 1, 4) = 4$

$\varepsilon(\sigma) = (-1)^3 \cdot (-1)^1 \cdot (-1)^0 \cdot (-1)^3 = -1$

c) $\sigma^{2019} = \sigma^{2016} \cdot \sigma^3 = \sigma^3 = \varepsilon^{-1} = C_1^{-1} \cdot C_2^{-1} \cdot C_3^{-1} = (7\ 5\ 2\ 1)(3\ 9) \cdot$

$\cdot (11\ 8\ 10\ 6)$

d) $x^4 = \sigma$

$\left.\begin{array}{l} \varepsilon(\sigma) = -1 \\ \varepsilon(x^4) = \varepsilon(x^4) = 1 \end{array}\right\} \Rightarrow$ nu $\exists$ sol

③ $\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 3 & 5 & 6 & 7 & 2 & 1 & 4 & 8 \end{pmatrix} \in S_8$

a) cicluri disjuncte și transpoziții

$\sigma = (1\ 3\ 6)(2\ 5)(4\ 7) = (1\ 3)(3\ 6)(2\ 5)(4\ 7)$

b) $ord(\sigma) = lcm(3, 2, 2) = 6$

$\varepsilon(\sigma) = (-1)^2 \cdot (-1)^1 \cdot (-1)^1 = 1$

c) $\sigma^{1004} = \sigma^{1002} \cdot \sigma^2 = \sigma^2 = c_1^2 \cdot c_2^2 \cdot c_3^2 =$

$= (1\ 6\ 3)$

$$\begin{tikzcd} & 1 \arrow[dr] & \\ 6 \arrow[ur] & & 3 \arrow[ll] \end{tikzcd} \qquad \begin{tikzcd} 2 \arrow[r, shift left] & 5 \arrow[l, shift left] \end{tikzcd}$$

$(2\ 5)^2 =$ permutarea identică

$(4, 7)^2$ — || —

d) $X_2 = \sigma$

$X = k_1 \cdot k_2 \cdot \ldots \cdot k_p$

$X^2 = k_1^2 \cdot k_2^2 \cdot \ldots \cdot k_p^2$

I dacă lungimea lui k_i este impară atunci lungimea lui k_i^2 va fi cât lungimea lui k_i

II dacă lungimea lui k_i este pară atunci k_i^2 este produs de 2 cicluri lungime k_i 2

$(1\ 3\ 6)$ nu provine dintr-un ciclu de lungime 6 pentru că e singurul ciclu de lungime 3 => $(1\ 3\ 6)$ provine dintr-un ciclu de lg 3 => $(1\ 6\ 3)$

$(2\ 5)(4\ 7)$ provin dintr-un ciclu de lungime pară (4) =

$= (2\ 4\ 5\ 7)(2\ 7\ 5\ 4)$

$$\begin{tikzcd} 2 \arrow[r] & 4 \arrow[d] \\ 7 \arrow[u] & 5 \arrow[l] \end{tikzcd}$$

$S = \{(1\ 6\ 3)(2\ 4\ 5\ 7), (1\ 6\ 3)(2\ 7\ 5\ 4)\}$

④ $\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 5 & 7 & 1 & 8 & 3 & 4 & 2 & 6 \end{pmatrix}$ în S_8

Se poate descompune σ ca produs de cicluri de lungime 3?

$\sigma = (1\ 5\ 3)(2\ 7)(4\ 8\ 6)$

$\varepsilon(\sigma) = (-1)^2 \cdot (-1) \cdot (-1)^2 = -1$

$\sigma = c_1 \cdot c_2 \cdot \ldots \cdot c_p$ $\Rightarrow$ imposibil

$\varepsilon(\sigma) = (-1)^2 \cdot \ldots \cdot (-1)^2 = 1$