$$\begin{pmatrix} 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix}$$

$\det A = 1 \neq 0 \Rightarrow f$ lin + f bij $\Rightarrow f \in Aut(\mathbb{R}_2[X])$

Seminar săpt 7

ex

$(V,+,\cdot)_{/\mathbb{K}}$ $\quad R = \{e_1, e_2\} \xrightarrow{C} R' = \{e_1 - e_2 = e_1',\ e_1 + 2e_2 = e_2'\}$

$(V^* = \{f : V \to K \mid f \text{ lin}\}, +, \cdot)_{/\mathbb{K}}$

$R^* = \{e_1^*, e_2^*\} \xrightarrow{D} R'^* = \{e_1'^*, e_2'^*\}$

Ce legătură există între C și D?

$e_i^*(e_j) = \delta_{ij}$ $\delta_{ij} = \begin{cases} 1, & i=j \\ 0, & i \neq j \end{cases}$

$e_i'^*(e_j') = \delta_{ij}$

$e_1'^* = a e_1^* + b e_2^*$

$e_1'^*(e_1') = 1 \Rightarrow (1)$

$e_1'^*(e_2') = 0 \quad (2)$

(1) $(a e_1^* + b e_2^*)(e_1 - e_2) = 1$

$a(e_1^*(e_1) - e_1^*(e_2)) + b(e_2^*(e_1) -$

$e_2^*(e_2)) = 1$

$\Leftrightarrow \quad a(1-0) + b(0-1) = 1$

$\Leftrightarrow \quad a - b = 1$

(2) $(a e_1^* + b e_2^*)(e_1 + 2e_2)$

$= a(e_1^*(e_1) + 2e_1^*(e_2))$

$+ b(e_2^*(e_1) + 2e_2^*(e_2))$

$= a(1 + 2\cdot 0) + b(0 + 2\cdot 1)$

$= a + 2b$

$= 0$

$\begin{cases} a - b = 1 \\ a + 2b = 0 \end{cases}$

$\Rightarrow 3b = -1 \Rightarrow b = -\frac{1}{3} \Rightarrow a = \frac{2}{3}$

$$e_2'^* = ce_1^* + de_2^* \Rightarrow \begin{cases} c-d=0 \\ c+2d=1 \end{cases}$$

$$3d = 1 \Rightarrow d = c = \frac{1}{3}$$

$$D = \begin{pmatrix} \frac{2}{3} & \frac{1}{3} \\ -\frac{1}{3} & \frac{1}{3} \end{pmatrix}$$

$$C = \begin{pmatrix} 1 & 1 \\ -1 & 2 \end{pmatrix} \quad (R')$$

$$C^T = \begin{pmatrix} 1 & -1 \\ 1 & 2 \end{pmatrix}$$

$$C^* = \begin{pmatrix} 2 & -1 \\ 1 & +1 \end{pmatrix}$$

$$C^{-1} = \frac{1}{3}\begin{pmatrix} 2 & -1 \\ 1 & 1 \end{pmatrix} \Rightarrow D = (C^{-1})^T$$

Proiecții și simetrii

p.o. $V = V_1 \oplus V_2 \longrightarrow V_1 \oplus V_2$

$p(x) = p(x_1 + x_2) = x_1$ proiecția pe V_1

$s(x) = s(x_1 + x_2) = 2x_1 - (x_1 + x_2)$

$= x_1 - x_2$ simet față de V_1

p proiecție $\Rightarrow s = 2p - id_V$ simetrie

$s \circ s = id$

$p \circ p = p$

$V = Im\,p \oplus Ker\,p$ $\quad R = R_1 \cup R_2$

$\ \ \shortparallel \qquad\quad \shortparallel$

$\ V_1 \qquad\ V_2$

$R_1 = \{e_1 \dots e_k\}$ $\quad Im\,p = V_1$

reper în

$R_2 = \{e_{k+1} \dots e_n\}$ $\quad Ker\,p = V_2$

$$\begin{cases} p(e_i) = e_i, & i = \overline{1,k} \\ p(e_j) = 0, & j = \overline{k+1,n} \end{cases}$$

$$A_p = [p]_{R,R} = \left(\begin{array}{c|c} I_k & 0 \\ \hline 0 & 0 \end{array}\right) \notin O_3$$

$$\begin{cases} s(e_i) = e_i, & i = \overline{1,k} \\ s(e_j) = -e_j & j = \overline{k+1,n} \end{cases}$$

$$A_s = [S]_{R,R} = \left(\begin{array}{c|c} I_k & 0 \\ \hline 0 & -I_{n-k} \end{array}\right) \in O_3$$

<u>La lucrare</u>

<u>ex</u> $(\mathbb{R}^3, +, \cdot)_{/\mathbb{R}}$ $\quad V_1 = \{x \in \mathbb{R}^3 \mid x_1 - x_2 + 2x_3 = 0\}$

p = proiecția pe V_1

s = simetria față de V_1

$p(1,2,5)$ $\quad$ $s(1,2,5)$

$\mathbb{R}^3 = V_1 \oplus V_2$

$x_1 = x_2 - 2x_3$

$V_1 = \{ (x_2 - 2x_3, x_2, x_3) \mid x_2, x_3 \in \mathbb{R} \}$

$= \langle \{ (1,1,0), (-2,0,1) \} \rangle$

$= \langle R_1 \rangle$

Așezăm pe col + ad un vect aî det $\neq 0$

$$\det \begin{vmatrix} 1 & -2 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{vmatrix} \neq 0$$

$V_2 = \langle \{ (1,0,0) \} \rangle = \langle R_2 \rangle$

$R = R_1 \cup R_2$ reper în $\mathbb{R}^3$

$$(1,2,5) = \overbrace{a(1,1,0) + b(-2,0,1)}^{x_1} + \overbrace{c(1,0,0)}^{x_2}$$

$$= (a - 2b + c, a, b)$$

$a = 2$

$b = 5$

$a - 2b + c = 1 \Rightarrow 2 - 10 + c = 1$

$-8 + c = 1 \Rightarrow c = 9$

$(1,2,5) = x_1 + x_2 \qquad \begin{cases} x_1 = (-8,2,5) \\ x_2 = (9,0,0) \end{cases}$

$$\begin{cases} p(1,2,5) = x_1 = (-8,2,5) \\ s(1,2,5) = x_1 - x_2 = (-17,2,5) \end{cases}$$

Endomorfisme. Diagonalizare

pr.1 $f \in End(\mathbb{R}^3)$

$R_0 = \{e_1, e_2, e_3\}$ reperul canonic în $\mathbb{R}^3$

a) $f(e_1) = e_2$

$f(e_2) = e_1 + e_2 + e_3$

$f(e_3) = e_2$

i) Există R a.î. $A = [f]_{R_0, R_0}$ se poate diagonaliza?

ii) $A^n = ?$

Soluție

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 0 \end{pmatrix} = [f]_{R_0, R_0}$$

$$P_A(\lambda) = \det(A - \lambda I_3)$$

$$= \begin{vmatrix} -\lambda & 1 & 0 \\ 1 & 1-\lambda & 1 \\ 0 & 1 & -\lambda \end{vmatrix} \quad e_1' = e_1 - e_3$$

$$= \begin{vmatrix} -\lambda & 1 & 0 \\ 0 & 1-\lambda & 1 \\ \lambda & 1 & -\lambda \end{vmatrix} \quad L_3' = L_3 + L_1$$

$$= \lambda \begin{vmatrix} -1 & 1 & 0 \\ 0 & 1-\lambda & 1 \\ 0 & 2 & -\lambda \end{vmatrix}$$

$$= \lambda \cdot (-1) \cdot (-1)^{1+1} \cdot \begin{vmatrix} 1-\lambda & 1 \\ 2 & -\lambda \end{vmatrix}$$

$$= -\lambda \left((1-\lambda)(-\lambda) - 2 \right)$$

$$= -\lambda \left(\lambda^2 - \lambda - 2 \right)$$

$$= -\lambda (\lambda+1)(\lambda-2)$$

m - multipl. rad.

$$\begin{cases} \lambda_1 = 0 & m_1 = 1 \\ \lambda_2 = -1 & m_2 = 1 \\ \lambda_3 = 2 & m_3 = 1 \end{cases}$$

$$V_{\lambda_1} = \{ x \in \mathbb{R}^3 \mid f(x) = 0 \} = \operatorname{Ker} f$$

$$\boxed{AX = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}}$$

$$\dim V_{\lambda_1} = 3 - rg\, A = 3 - 2 = 1 = m_{\lambda_1}$$

$$V_{\lambda_2} = \{ x \in \mathbb{R}^3 \mid f(x) = -x \}$$

$$AX = -X \Leftrightarrow (A + I_3) X = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Leftrightarrow$$

$$\boxed{\begin{pmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}}$$

$$\dim V_{\lambda_2} = 3 - rg(A + I_3) = 3 - 2 = 1 = m_2$$

$$\begin{vmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 1 \end{vmatrix} = 0 \Rightarrow rg\,(A + I_3) = 2$$

Scanned with CamScanner

$V_{\lambda_3} = \{ X \in \mathbb{R}^3 \mid f(x) = 2x \}$

$AX = 2X \Leftrightarrow (A - 2I_3) X = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

$$\boxed{\Leftrightarrow \begin{pmatrix} -2 & 1 & 0 \\ 1 & -1 & 1 \\ 0 & 1 & -2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}}$$

$$\begin{vmatrix} -2 & 1 & 0 \\ 1 & -1 & 1 \\ 0 & 1 & -2 \end{vmatrix} =$$

$\dim V_{\lambda_3} = 3 - 2 = 1 = m_3$

$R = R_1 \cup R_2 \cup R_3$ reper în $\mathbb{R}^3$,

R_K e reper în $V_{\lambda K}$ $K = \overline{1,3}$

• $$\begin{pmatrix} 0 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{cases} x_2 = 0 \\ x_1 + x_2 = -x_3 \end{cases} \Rightarrow x_1 = -x_3$$

$V_{\lambda_1} = \{ (-x_3, 0, x_3) \mid x_3 \in \mathbb{R} \}$

$= \langle \{ (-1, 0, 1) \} \rangle$

• $$\begin{pmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{cases} x_1 + x_2 = 0 \\ x_1 + 2x_2 = -x_3 \end{cases} \Rightarrow x_1 = x_3$$

$$x_2 = -x_3$$

$$V_{\lambda_2} = \{ (x_3, -x_3, x_3) \mid x_3 \in \mathbb{R} \}$$
$$= \langle \{ (1, -1, 1) \} \rangle$$

- $$\begin{pmatrix} -2 & 1 & 0 \\ 1 & -1 & 1 \\ 0 & 1 & -2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{cases} -2x_1 + x_2 = 0 \\ x_1 - x_2 = -x_3 \end{cases} \Rightarrow x_2 = 2x_3$$

$$-x_1 = -x_3 \Rightarrow x_1 = x_3$$

$$V_{\lambda_3} = \{ (x_3, 2x_3, x_3) \mid x_3 \in \mathbb{R} \} = \langle \{ (1, 2, 1) \} \rangle$$

$$R = \{ (-1, 0, 1), (1, -1, 1), (1, 2, 1) \}$$

$$A' = [g]_{R,R} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

ii) $R_0 \xrightarrow{C} R$

$$[g]_{R_0,R_0} = A$$

$$[g]_{R,R} = A'$$

$$A' = C^{-1} A C \Rightarrow A = C A' C^{-1}$$

$$A^n = (CA'C^{-1})(CA'C^{-1}) \cdots (CA'C^{-1})$$
$$= CA'^{n}C^{-1}$$

$$A'^{n} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & (-1)^n & 0 \\ 0 & 0 & 2^n \end{pmatrix}$$ $\square$

ex 5 (Endom + diagon)

$f \in End(\mathbb{R}^3)$

$\lambda_1 = 3 \quad \lambda_2 = -2 \quad \lambda_3 = 1$ sunt val. proprii

$v_1 = (-3, 2, 1) \quad v_2 = (-2, 1, 0)$

$v_3 = (-6, 3, 1)$ sunt vect proprii coresp.

$A = [f]_{R_0, R_0} = ?$

$R = \{v_1, v_2, v_3\}$ SLI (prop. curs)
$|R| = 3 = \dim R^3$ $\Rightarrow$

R reper în R^3

$f(v_1) = \lambda_1 v_1$
$f(v_2) = \lambda_2 v_2$
$f(v_3) = \lambda_3 v_3$

$$A' = [f]_{R,R} = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix} = \begin{pmatrix} 3 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$v_1 = -3e_1 + 2e_2 + e_3$

$v_2 = -2e_1 + e_2$

$v_3 = -6e_1 + 3e_2 + e_3$

$R_0 \xrightarrow{C} R \qquad C = \begin{pmatrix} -3 & -2 & -6 \\ 2 & 1 & 3 \\ 1 & 0 & 1 \end{pmatrix}$

$A' = C^{-1} A C \Rightarrow A = C A' C^{-1}$

Pt test: avem voie cu foaie A4 cu ce info. vrem noi