SEMINAR $\overline{X}$ - 14.12.2023

Spații vectoriale euclidiene.

Repere ortonormate

1) $g: \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$, $g(x,y) = ax_1y_1 + bx_1y_2 + bx_2y_1 + cx_2y_2$

a) $g \in L^s(\mathbb{R}^2, \mathbb{R}^2; \mathbb{R})$

b) g produs scalar $\Leftrightarrow \begin{cases} a > 0 \\ ac - b^2 > 0 \end{cases}$

a) $G = \begin{pmatrix} a & b \\ b & c \end{pmatrix} = G^T$

$g(x,y) = \sum_{i,j=1}^{2} g_{ij} x_i y_j \Leftrightarrow g$ formă biliniară simetrică $\Rightarrow$

$\Rightarrow g \in L^s(\mathbb{R}^2, \mathbb{R}^2; \mathbb{R})$

b) g pozitiv definită $\Leftrightarrow \begin{cases} \Delta_1 = a > 0 \\ \Delta_2 = ac - b^2 > 0 \end{cases}$

3) $(\mathbb{R}^3, g_0)$, $g: \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}$

$g_0(x,y) = x_1y_1 + x_2y_2 + x_3y_3$

$U = \{x \in \mathbb{R}^3 \mid x_1 + x_2 - x_3 = 0\}$

a) $U^\perp$

b) Să se determine un reper ortonormat $R = R_1 \cup R_2$ în $\mathbb{R}^3$, unde R_1 = reper ortonormat în U și R_2 = reper ortonormat în $U^\perp$

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a) $U^\perp = \{x \in \mathbb{R}^3 \mid g(x,y) = 0\ \forall y \in U\} = <\{(1,1,-1)\}>$

$(x_1,x_2,x_3)\,(1,1,-1)$ $\qquad \mathbb{R}^3 = U \oplus U^\perp$

b) $x_3 = x_1 + x_2$

$U = \{(x_1, x_2, x_1+x_2) \mid x_1; x_2 \in \mathbb{R}\} = <\{f_1, f_2\}>$ $\Rightarrow \{f_1, f_2\}$ este reper în U

$x_1\underbrace{(1,0,1)}_{f_1} + x_2\underbrace{(0,1,1)}_{f_2}$

$\dim U = 3-1 = 2$

Aplicăm procedeul Gram-Schmidt

$e_1 = f_1$

$e_2 = f_2 - \frac{<f_2, e_1>}{<e_1,e_1>} \cdot e_1 = (0,1,1) - \frac{1}{2}(1,0,1) = \left(-\frac{1}{2}, 1, \frac{1}{2}\right) =$

$= \frac{1}{2}(-1,2,1)$

$\{e_1, e_2\}$ reper ortogonal

$e_1' = \frac{e_1}{\|e_1\|} = \frac{1}{\sqrt{2}}(1,0,1)$ $\qquad R_1 = \{e_1', e_2'\}$ reper ortogonal în U

$R_2 = \{e_3'\}$ reper în $U^\perp$

$e_2' = \frac{1}{\sqrt{6}}(-1,2,1)$

$R = R_1 \cup R_2$ reper ortonormat în $\mathbb{R}^3$

4) $g : \mathbb{C} \times \mathbb{C} \to \mathbb{R}$ formă biliniară $G = \begin{pmatrix} 1 & 2 \\ 2 & 5 \end{pmatrix}$

a) $(\mathbb{C}, g)$ spațiu vectorial euclidian real

b) $u = (2 - 1)$ versor în raport cu g

c) $<\{u\}>^\perp$ $\qquad$ d) Să se ortonormeze R_0 în raport cu g

e) să se afle $\cap$ dintre cercul unitate în $(\mathbb{C}, g_0)$ și în $(\mathbb{C}, g)$

a) g formă biliniară

$G = G^t$

G pozitiv definită $\Leftrightarrow \Delta_1 = 1 > 0$, $\Delta_2 = \det G = 1 > 0$

$\Rightarrow g$ produs scalar $\Rightarrow (\mathbb{C}, g)$ sp. vect. euclidian real

b) $\|u\|_g = \sqrt{g(u,u)} = 1 \iff u$ versor în raport cu g

$g(z,w) = g((x_1,x_2),(y_1,y_2)) = x_1y_1 + 2x_1y_2 + 2x_2y_1 + 5x_2y_2$

$z = x_1 + ix_2 = (x_1, x_2)$

$w = y_1 + iy_2 = (y_1, y_2)$

$Q : \underset{\underset{\mathbb{R}^2}{\parallel\!|}}{\mathbb{C}} \to \mathbb{R},\ Q(z) = Q(x_1,x_2) = x_1^2 + 4x_1x_2 + 5x_2^2$

$\|u\|_g = \sqrt{Q(2,-1)} = \sqrt{4-8+5} = 1$

$<\{u\}>^{\perp} = \{ z \in \mathbb{C} \mid \underset{(x_1,x_2)(2,-1)}{g(z,u)=0} \} = \mathbb{R}$

$= 2x_1 - 2x_1 - 4x_2 - 5x_2 = 0$

$z = x_1 + ix_2 = x_1 \in \mathbb{R}$

d) $R_0 = \{ \underset{(1,0)}{1}, \underset{(0,1)}{i} \}$

$g : \mathbb{C} \times \mathbb{C} \to \mathbb{R}$

$g(\underset{(x_1,x_2)}{z}, \underset{(y_1,y_2)}{w}) = x_1y_1 + 5x_2y_2 + 2x_1y_2 + 2x_2y_1$

$g((1,0),(1,0)) = Q(1,0) = 1$

$Q(z) = x_1^2 + 4x_1x_2 + 5x_2^2$

$e_1 = f_1 = 1$

$e_2 = f_2 - \frac{g(f_2,e_1)}{g(e_1,e_1)} \cdot e_1 = i - \frac{2}{1} \cdot 1 = -2 + i = -u$

$\{e_1, e_2\}$ reper ortogonal $\Rightarrow$ ortonormat $Q(e_1) = Q(1,0) = 1$

$Q(e_2) = Q(-u) = Q(-2,1) = 4-8+5 = 1$

2) $S^1_{g_0} = \{ z \in \mathbb{C} \mid |z| = 1 \}$

$\|z\|_g = 1 \to \sqrt{x_1^2 + x_2^2}$

$z = x_1 + i x_2$, $g_0(z, w)$

$(x_1, x_2)\ (y_1, y_2)$

$S^1_g = \{ z \in \mathbb{C} \mid \|z\|_g = 1 \} = \{ z = x_1 + i x_2 \in \mathbb{C} \mid x_1^2 + 4x_1x_2 + 5x_2^2 = 1 \}$

$\sqrt{g(z,z)} = \sqrt{Q(z)}$

$S^1_{g_0} \cap S_g : \begin{cases} x_1^2 + x_2^2 = 1 \\ x_1^2 + 4x_1 x_2 + 5x_2^2 = 1 \end{cases} (-)$

$4x_1 x_2 + 4x_2^2 = 0 \Rightarrow 4x_2(x_1 + x_2) = 0$

$\Rightarrow x_2 = 0 \Rightarrow x_1 = \pm 1$

$x_2 = -x_1 \Rightarrow 2x_1^2 = 1 \Rightarrow x_1 = \pm \frac{1}{\sqrt{2}} = \pm \frac{\sqrt{2}}{2}$

Metoda 2 $\quad z = \cos t + i \sin t$

$\cos^2 t + 4 \sin t \cos t + 5 \sin^2 t = 1$

$4 \sin t (\sin t + \cos t) = 0$

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$(\mathbb{R}^3, g_0)$

$R = \{f_1 = (1,2,3),\ f_2 = (0,1,1),\ f_3 = (1,2,5)\}$

a) R repor în $\mathbb{R}^3$ să se ortonormeze

b) $f_1 \times f_2$

c) $f_1 \wedge f_2 \wedge f_3$

$$\det A = \begin{vmatrix} 1 & 0 & 1 \\ 2 & 1 & 2 \\ 3 & 1 & 5 \end{vmatrix} = 5 + \cancel{2} - 3 - \cancel{2} = 2 \neq 0.$$

Aplicăm Gram-Schmidt

$e_1 = f_1$

$$e_2 = f_2 - \frac{\langle f_2, e_1 \rangle}{\langle e_1, e_1 \rangle} \cdot e_1 = (0,1,1) - \frac{5}{14}(1,2,3)$$

$$= \left(-\frac{5}{14}, \frac{2}{7}, -\frac{1}{14}\right) = \frac{1}{14}(-5,4,-1)$$

$$e_3 = f_3 - \frac{\langle f_3, e_1 \rangle}{\langle e_1, e_1 \rangle} \cdot e_1 - \frac{\langle f_3, e_2 \rangle}{\langle e_1, e_2 \rangle} e_2$$

$$e_3 = (1,2,5) - \frac{1\cdot 1 + 2\cdot 2 + 3\cdot 5}{14}(1,2,3) - \frac{\cancel{\frac{1}{14}}(-5+8-5)}{\left(\frac{1}{14}\right)^{\cancel{2}}(25+16+1)} \cdot \frac{1}{\cancel{14}}(-5,4,-1)$$

$$e_3 = (1,2,5) - \frac{20}{14}(1,2,3) + \frac{2}{42}(-5,4,-1)$$

$$e_3 = \left(1 - \frac{10}{7} - \frac{5}{21},\ 2 - \frac{20}{7} + \frac{4}{21},\ 5 - \frac{30}{7} - \frac{1}{21}\right)$$

$$= \frac{1}{21}(21-30-5,\ 42-60+4,\ 105-90-1)$$

$$= \frac{1}{21}(-14,-14,14) = \frac{1}{3}(-2,-2,1) = \frac{2}{3}(-1,-1,1)$$

$e_1' = \frac{1}{\sqrt{14}}(1,2,3)$; $e_2' = \frac{1}{\sqrt{42}}(-5,4,1)$; $e_3' = \frac{1}{\sqrt{3}}(-1,-1,1)$

b) $f_1 \times f_2 = \begin{vmatrix} e_1 & e_2 & e_3 \\ 1 & 2 & 3 \\ 0 & 1 & 1 \end{vmatrix} = e_1 \begin{vmatrix} 2 & 3 \\ 1 & 1 \end{vmatrix} - e_2 \begin{vmatrix} 1 & 3 \\ 0 & 1 \end{vmatrix} + e_3 \begin{vmatrix} 1 & 2 \\ 0 & 1 \end{vmatrix}$

$= (-1,-1,1)$

c) $f_1 \wedge f_2 \wedge f_3 = ?$

$f_1 \wedge f_2 \wedge f_3 = \langle f_3, f_1 \times f_2 \rangle = 1 \cdot (-1) + 2(-1) + 5 \cdot 1$

$= -1 + 5 - 2 = 2$

sau

$f_1 \wedge f_2 \wedge f_3 = \begin{vmatrix} 1 & 2 & 3 \\ 0 & 1 & 1 \\ 1 & 2 & 5 \end{vmatrix} = 2$