$\boxed{C_1 - AG}$ 16 CTI

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Matrice. Determinanți. Rang. Forma eșalon

Fie $(\mathbb{K},+,\cdot)$ corp comutativ (Ex: $\mathbb{R}, \mathbb{C}, \mathbb{Z}_p, p = \text{prim}$)

$\det : \mathcal{M}_n(\mathbb{K}) \longrightarrow \mathbb{K}$

$$\det(A) = \sum_{\sigma \in S_n} \varepsilon_\sigma \, a_{1\sigma(1)} \cdot a_{2\sigma(2)} \cdot \ldots \cdot a_{n\sigma(n)}$$

$(S_n, \cdot)$ grupul permutărilor, $\sigma = \begin{pmatrix} 1 & 2 & \ldots & n \\ \sigma(1) & \sigma(2) & \ldots & \sigma(n) \end{pmatrix}$

$\varepsilon(\sigma) = (-1)^{m(\sigma)}$ $\qquad \sigma : \{1,..,n\} \to \{1,..,n\}$ bij.

$m(\sigma)$ = nr inversiunilor lui σ

(i,j) inversiune a lui $\sigma \Leftrightarrow \begin{cases} i < j \\ \sigma(i) > \sigma(j) \end{cases}$

$A_n = \{\sigma \in S_n \mid \sigma = \text{pară i.e. } \varepsilon(\sigma) = 1\}$, $|A_n| = \dfrac{n!}{2}$.

OBS

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21}$$
$$= \varepsilon(\sigma) a_{1\sigma(1)} a_{2\sigma(2)} + \varepsilon(\tau) a_{1\tau(1)} a_{2\tau(2)}$$

$\sigma, \tau \in S_2 \qquad \sigma = \begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix} = e, \quad \tau = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$

$m(\sigma) = 0, \ \varepsilon(\sigma) = 1 \qquad m(\tau) = 1, \ \varepsilon(\tau) = -1$

Def. $A \in \mathcal{M}_n(\mathbb{K})$ s.n. <u>nesingulară</u> $\Leftrightarrow \det A \neq 0$

s.n. <u>inversabilă</u> $\Leftrightarrow \exists A^{-1} \in \mathcal{M}_n(\mathbb{K})$

a.î. $A \cdot A^{-1} = A^{-1}A = I_n$.

Prop A nesingulară $\Leftrightarrow$ A inversabilă

Prop $A \in \mathcal{M}_n(\mathbb{K})$, $n \geq 2$, $\det A \neq 0$.

1) $\det(A^{-1}) = \frac{1}{\det A}$; 2) $\det(A^*) = (\det A)^{n-1}$

Dem

$A A^{-1} = I_n \mid \det \Rightarrow \det(A)\det(A^{-1}) = 1 \Rightarrow \det(A^{-1}) = \frac{1}{\det A}$

$A^{-1} = \frac{1}{\det A} \cdot A^* \mid \det \Rightarrow \underset{\frac{1}{\det A}}{\underset{\parallel}{\det(A^{-1})}} = \left(\frac{1}{\det A}\right)^n \det(A^*) \Rightarrow$

$\det(\alpha A) = \alpha^n \det A$

$\det(A^*) = (\det A)^{n-1}$.

Notații

1) $(GL(n, \mathbb{K}) = \{A \in \mathcal{M}_n(\mathbb{K}) \mid \det A \neq 0\}, \cdot)$ grupul general liniar.

2) $(O(n) = \{A \in \mathcal{M}_n(\mathbb{K}) \mid \underline{A \cdot A^T = I_n}\}, \cdot)$ grupul matricelor ortogonale.

$A \in O(n) \Rightarrow \det A = \pm 1$

3) $(SO(n) = \{A \in O(n) \mid \det A = 1\}, \cdot)$ grupul special ortogonal.

4) $(SL(n) = \{A \in GL(n, \mathbb{K}) \mid \det A = 1\}, \cdot)$ grupul special liniar.

$SO(n) = O(n) \cap SL(n)$.

Ex

a) $A = \begin{pmatrix} 1 & 2 & -1 \\ 1 & 3 & 2 \\ 0 & 1 & 2 \end{pmatrix}$

$\begin{vmatrix} 1 & 2 & -1 \\ 1 & 3 & 2 \\ 0 & 1 & 2 \end{vmatrix} \overset{L_2' = L_2 - L_1}{=} \begin{vmatrix} 1 & 2 & -1 \\ 0 & 1 & 3 \\ 0 & 1 & 2 \end{vmatrix} = 2 - 3 = -1 \neq 0$

$\Rightarrow A \in GL(3, \mathbb{R})$

b) $A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ 0 & \frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\frac{\pi}{3} & -\sin\frac{\pi}{3} \\ 0 & \sin\frac{\pi}{3} & \cos\frac{\pi}{3} \end{pmatrix}$

$\underset{A \in O(3)}{\underset{\Downarrow}{A \cdot A^T = I_3}}$, $\underset{A \in SO(3)}{\underset{\Downarrow}{\det A = 1}} \Rightarrow$

Def $A \in \mathcal{M}_{n,m}(\mathbb{K})$, $A \neq O_{n,m}$

$rg A = k$ $(k \leq \min\{n,m\}) \Leftrightarrow$

$\exists$ un minor de ord k nenul și toți minorii de ordin mai mare (dacă există) sunt nuli.

Convenție $rg(O_{n,m}) = 0$.

OBS $\exists\, C_n^{k+1} \cdot C_m^{k+1}$ minori de ordin $k+1$

Teoremă (de optimizare a determinării rangului)

$A \in \mathcal{M}_{n,m}(\mathbb{K})$.

$rg A = k \Leftrightarrow$ $\exists$ un minor de ord k nenul $(\Delta_k \neq 0)$ și toți minorii de ordin $k+1$ (dc $\exists$), care conțin Δ_k sunt nuli.

OBS $\exists\, (n-k)(m-k)$ minori de ordin $k+1$, care îl conțin pe Δ_k

Exemple

1) $A = \begin{pmatrix} a & 1 & 1 \\ 1 & a & 1 \\ 1 & 1 & a \end{pmatrix} \in \mathcal{M}_3(\mathbb{R})$ A pătratică: MARE $\Rightarrow$ MIC

$rg A = ?$

Sol.

$$\Delta = \begin{vmatrix} a & 1 & 1 \\ 1 & a & 1 \\ 1 & 1 & a \end{vmatrix} \overset{L_1' = L_1 + L_2 + L_3}{=} \begin{vmatrix} a+2 & a+2 & a+2 \\ 1 & a & 1 \\ 1 & 1 & a \end{vmatrix}$$

$$= (a+2) \begin{vmatrix} 1 & 1 & 1 \\ 1 & a & 1 \\ 1 & 1 & a \end{vmatrix} \overset{\substack{C_2' = C_2 - C_1 \\ C_3' = C_3 - C_1}}{=} (a+2) \begin{vmatrix} 1 & 0 & 0 \\ 1 & a-1 & 0 \\ 1 & 0 & a-1 \end{vmatrix} = (a+2) \begin{vmatrix} a-1 & 0 \\ 0 & a-1 \end{vmatrix}$$

$$= (a+2)(a-1)^2$$

I) $\Delta \neq 0 \Leftrightarrow a \in \mathbb{R} \setminus \{1, -2\}$ $rg A = 3$

II) $\Delta = 0 \Leftrightarrow a \in \{1, -2\}$.

a) $a=1$ $\quad A=\begin{pmatrix}1&1&1\\1&1&1\\1&1&1\end{pmatrix}$ $\quad rg A = 1$ (toți minorii de ord 2 sunt nuli ; $c_1=c_2=c_3$)

b) $a=-2$ $\quad A=\begin{pmatrix}\boxed{\begin{matrix}-2&1\\1&-2\end{matrix}}&\begin{matrix}1\\1\end{matrix}\\\begin{matrix}1&1\end{matrix}&-2\end{pmatrix}$ $\quad \begin{vmatrix}-2&1\\1&-2\end{vmatrix}=4-1=3$

$rg A = 2$

(2) $A=\begin{pmatrix}\boxed{\begin{matrix}1&1\\0&1\end{matrix}}&\begin{matrix}a&1\\-1&2\end{matrix}\\\begin{matrix}6&4\end{matrix}&\begin{matrix}8&3\end{matrix}\end{pmatrix}\in\mathcal{M}_{3,4}(\mathbb{R})$ $\quad rg A = ?$

A nu e pătratică : MIC → MARE

$\Delta_1=\begin{vmatrix}1&1&1\\0&1&2\\6&4&3\end{vmatrix}=\begin{vmatrix}1&1&1\\0&1&2\\0&-2&-3\end{vmatrix}=\begin{vmatrix}1&2\\-2&-3\end{vmatrix}=-3+4=1\neq 0$

$L_3' = L_3-6L_1$

$rg A = 3, \forall a\in\mathbb{R}$

(3) $A\in\mathcal{M}_n(\mathbb{R})$ $\quad A^3-A-I_n=O_n$

a) $rg A = ?$; b) $rg(A+I_n)$

<u>SOL</u>

a) $A^3-A=I_n \Rightarrow A(A^2-I_n)=I_n \quad |\det$

$\det A\cdot\det(A^2-I_n)=1 \Rightarrow \det A\neq 0$

$\Rightarrow rg A = n$

b) $A^3=A+I_n \quad |\det \Rightarrow (\underset{\neq}{\det A})^3=\det(A+I_n)$

$\Rightarrow \det(A+I_n)\neq 0 \Rightarrow rg(A+I_n)=n.$

<u>Def</u> Fie $A\in\mathcal{M}_{n,m}(\mathbb{K})$

Transformări elementare asupra liniilor matricei A

(T_1) transformări prin care se „·” o linie cu o ct $\neq 0$

$A=\begin{pmatrix}L_1\\\vdots\\L_i\\\vdots\\L_n\end{pmatrix}\xrightarrow{T_1}\begin{pmatrix}L_1\\\vdots\\\alpha L_i\\\vdots\\L_n\end{pmatrix}, \alpha\in\mathbb{R}^*$

(T2) transformări prin care se schimbă 2 linii între ele.

$$A = \begin{pmatrix} L_1 \\ \vdots \\ L_i \\ \vdots \\ L_j \\ \vdots \\ L_n \end{pmatrix} \xrightarrow{T_2} \begin{pmatrix} L_1 \\ \vdots \\ L_j \\ \vdots \\ L_i \\ \vdots \\ L_n \end{pmatrix}$$

(T3) transformări prin care la elementele unei linii se adună elem. altei linii, eventual „·" cu o const $\neq 0$

$$A = \begin{pmatrix} L_1 \\ \vdots \\ L_i \\ \vdots \\ L_j \\ \vdots \\ L_n \end{pmatrix} \xrightarrow{T_3} \begin{pmatrix} L_1 \\ \vdots \\ L_i \\ \vdots \\ L_j + \alpha L_i \\ \vdots \\ L_n \end{pmatrix}$$

OBS Analog pt. coloane.

Def $A, B \in \mathcal{M}_{n,m}(\mathbb{K})$

A și B s.n. matrice echivalente $\Leftrightarrow$ B se obține din A
(not $A \sim B$)
printr-un nr. finit de transf. elementare pe linii

Prop $A \sim B \Rightarrow rg A = rg B$.

Def. O matrice s.n. matrice în formă eșalon pe linii

dacă
$$A = \begin{pmatrix} \boxed{\times} & \cdots & & \\ 0 & \boxed{\times} & & \\ 0 & 0 & \boxed{\times} & \\ \vdots & \vdots & & \\ 0 & 0 & 0 & \end{pmatrix}$$

1. Toate liniile nule se află sub toate liniile nenule.
2. Pe fiecare linie nenulă, primul element din stânga s.n. pivot.
   Pivotul liniei l_{i+1} se află la dreapta pivotului liniei l_i

Forma eșalon s.n. forma eșalon redusă dc, în plus:
3) toți pivoții sunt 1;

4) Deasupra pivoților toate elem. sunt nule.

OBS

a) MATLAB → forma eșalon pt A : ref(A)
→ forma eșalon redusă pt A : rref(A)

b) scară : échelle (franceză).

Prop $\forall A \in \mathcal{M}_{n,m}(\mathbb{K})$ se poate ~~aduce la o~~ transforma într-o matrice eșalon (resp. eșalon redusă) printr-un nr. finit de transformări elementare pe linii.

OBS

a) forma eșalon (pe linii) nu e unică

b) forma eșalon redusă este unică.

Ex
$$A = \begin{pmatrix} \boxed{2} & -2 & 4 & -2 \\ 2 & 1 & 10 & 7 \\ -4 & 4 & -8 & 4 \\ 4 & -1 & 14 & 6 \end{pmatrix} \in \mathcal{M}_4(\mathbb{R})$$

a) Să se aducă la forma eșalon, resp. eșalon redusă) pe linii

b) rg(A) = ?

$$A \sim \begin{array}{l} \\ L_2 - L_1 \\ L_3 + 2L_1 \\ L_4 - 2L_1 \end{array} \begin{pmatrix} 2 & -2 & 4 & -2 \\ 0 & 3 & 6 & 9 \\ 0 & 0 & 0 & 0 \\ 0 & 3 & 6 & 10 \end{pmatrix} \sim \begin{pmatrix} \boxed{2} & -2 & 4 & -2 \\ 0 & \boxed{3} & 6 & 9 \\ 0 & 3 & 6 & 10 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$L_3 - L_2$

$$\sim \begin{pmatrix} \boxed{2} & -2 & 4 & -2 \\ 0 & \boxed{3} & 6 & 9 \\ 0 & 0 & 0 & \boxed{1} \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

f. eșalon.

$\frac{1}{2}L_1$; $\frac{1}{3}L_2$

$$\sim \begin{pmatrix} \boxed{1} & -1 & 2 & -1 \\ 0 & \boxed{1} & 2 & 3 \\ 0 & 0 & 0 & \boxed{1} \\ 0 & 0 & 0 & 0 \end{pmatrix} \sim \begin{pmatrix} \boxed{1} & -1 & 2 & 0 \\ 0 & \boxed{1} & 2 & 0 \\ 0 & 0 & 0 & \boxed{1} \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$L_2 - 3L_3$

$L_1 + L_3$

$$L_1 + L_2 \sim \begin{pmatrix} \boxed{1} & 0 & 4 & 0 \\ 0 & \boxed{1} & 2 & 0 \\ 0 & 0 & 0 & \boxed{1} \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

f. eșalon redusă

b) $rg\, A = 3$ (nr. pivoților)

<u>Algoritmul Gauss-Jordan de inversare a unei matrice</u>

Fie matricea dublă $(A|I_n) \sim (C|B)$ forma eșalon redusă.

<u>Teoremă</u>

Dacă $A \in \mathcal{M}_n(\mathbb{K})$ este inversabilă, at $C = I_n$ și $B = A^{-1}$.

<u>Ex</u> $A = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 2 & 1 \\ -1 & -1 & 2 \end{pmatrix}$, A^{-1} cu alg. G-J.

<u>SOL</u> $\det A = \begin{vmatrix} 1 & 2 & 3 \\ 0 & 2 & 1 \\ 0 & 1 & 5 \end{vmatrix} = 10 - 1 = 9 \Rightarrow A$ inversabilă

$$(A \mid I_3) = \left(\begin{array}{ccc|ccc} \boxed{1} & 2 & 3 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 & 1 & 0 \\ -1 & -1 & 2 & 0 & 0 & 1 \end{array}\right) \underset{L_3+L_1}{\sim} \left(\begin{array}{ccc|ccc} \boxed{1} & 2 & 3 & 1 & 0 & 0 \\ 0 & \boxed{2} & 1 & 0 & 1 & 0 \\ 0 & 1 & 5 & 1 & 0 & 1 \end{array}\right)$$

$$\overset{\frac{1}{2}L_2}{\sim} \left(\begin{array}{ccc|ccc} \boxed{1} & 2 & 3 & 1 & 0 & 0 \\ 0 & \boxed{1} & \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & 1 & 5 & 1 & 0 & 1 \end{array}\right) \underset{L_3-L_2}{\sim} \left(\begin{array}{ccc|ccc} \boxed{1} & 2 & 3 & 1 & 0 & 0 \\ 0 & \boxed{1} & \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & 0 & \boxed{\frac{9}{2}} & 1 & -\frac{1}{2} & 1 \end{array}\right)$$

$$L_3 \cdot \frac{2}{9} \sim \left(\begin{array}{ccc|ccc} \boxed{1} & 2 & 3 & 1 & 0 & 0 \\ 0 & \boxed{1} & \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & 0 & \boxed{1} & \frac{2}{9} & -\frac{1}{9} & \frac{2}{9} \end{array}\right)$$

$1 - \frac{2}{3}$

$\frac{1}{2} + \frac{1}{2} \cdot \frac{1}{9} = \frac{10}{2 \cdot 9}$

- $L_2 - \frac{1}{2} L_3$
  $L_1 - 3L_3$

$$\sim \left(\begin{array}{ccc|ccc} \boxed{1} & 2 & 0 & \frac{1}{3} & \frac{1}{3} & -\frac{2}{3} \\ 0 & \boxed{1} & 0 & -\frac{1}{9} & \frac{5}{9} & -\frac{1}{9} \\ 0 & 0 & \boxed{1} & \frac{2}{9} & -\frac{1}{9} & \frac{2}{9} \end{array}\right)$$

$L_1 - 2L_2$

$$\sim \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{5}{9} & -\frac{7}{9} & -\frac{4}{9} \\ 0 & 1 & 0 & -\frac{1}{9} & \frac{5}{9} & -\frac{1}{9} \\ 0 & 0 & 1 & \frac{2}{9} & -\frac{1}{9} & \frac{2}{9} \end{array}\right)$$

$$\underbrace{\qquad\qquad}_{A^{-1}}$$

$$\left(\frac{1}{3} + \frac{2}{9} \;;\; \frac{1}{3} - \frac{10}{9} \;;\; -\frac{2}{3} + \frac{2}{9} \right)$$

Polinom caracteristic. Teorema Hamilton-Cayley.

$A \in \mathcal{M}_n(\mathbb{C})$

$$P_A(X) = \det(A - XI_n) = \begin{vmatrix} a_{11}-X & a_{12} \dots & a_{1n} \\ \vdots & & \\ a_{n1} & a_{n2} \dots & a_{nn}-X \end{vmatrix}$$

$$= (-1)^n \left[X^n - \sigma_1 X^{n-1} + \sigma_2 X^{n-2} + \dots + (-1)^n \sigma_n \right]$$

polinomul caracteristic asociat lui A.

σ_k = suma minorilor diagonali de ord k, $k = \overline{1,n}$

$\sigma_1 = a_{11} + a_{22} + \dots + a_{nn} = Tr(A)$

$\sigma_2 = \sum_{1 \le i < j \le n} \begin{vmatrix} a_{ii} & a_{ij} \\ a_{ji} & a_{jj} \end{vmatrix}$ ($\exists C_n^2$ minori de ord 2)

$\sigma_3 = \sum_{1 \le i < j < k \le n} \begin{vmatrix} a_{ii} & a_{ij} & a_{ik} \\ a_{ji} & a_{jj} & a_{jk} \\ a_{ki} & a_{kj} & a_{kk} \end{vmatrix}$ ($\exists C_n^3$ minori de ord 3).

Cazuri particulare

1) $m=2$ $\quad P_A(X) = \begin{vmatrix} a_{11}-X & a_{12} \\ a_{21} & a_{22}-X \end{vmatrix} = X^2 - \overbrace{(a_{11}+a_{22})}^{Tr(A)}X + \overbrace{a_{11}a_{22}-a_{12}a_{21}}^{det(A)}$

2) $m=3$ $\quad P_A(X) = \begin{vmatrix} a_{11}-X & a_{12} & a_{13} \\ a_{21} & a_{22}-X & a_{23} \\ a_{31} & a_{32} & a_{33}-X \end{vmatrix} =$

$= -(X^3 - \sigma_1 X^2 + \sigma_2 X - \sigma_3)$

$(*)\begin{cases} \sigma_1 = a_{11}+a_{22}+a_{33} = Tr(A) \\ \sigma_2 = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} + \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix} + \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} = Tr(A^*) \text{ pt } m=3 \\ \sigma_3 = det(A) \end{cases}$

Teorema Hamilton-Cayley

$P_A(A) = O_m$, $\forall A \in \mathcal{M}_n(\mathbb{C})$.

$(-1)^m [A^n - \sigma_1 A^{n-1} + \sigma_2 A^{n-2} - \dots + (-1)^m \sigma_m \cdot I_n] = O_m \quad (1)$

Cazuri part

a) $m=2$ $\quad A^2 - \underset{Tr(A)}{\sigma_1} A + \underset{detA}{\sigma_2} I_2 = O_2$

b) $m=3$ $\quad (-1)^3 [A^3 - \underbrace{\sigma_1 A^2 + \sigma_2 A - \sigma_3}_{(*)} I_3] = O_3$

$\boxed{OBS}$ $(1) \mid \cdot A^{-1}$ $(detA \neq 0)$

$A^{n-1} - \sigma_1 A^{n-2} + \dots + (-1)^{n-1} \sigma_{n-1} \cdot I_n + (-1)^m \sigma_m \cdot A^{-1} = O_n \Rightarrow A^{-1} = \dots$

Ex.

$$A = \begin{pmatrix} 1 & 0 & 1 \\ 2 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \qquad A^{-1} \text{ cu TH-C.}$$

$$\begin{vmatrix} 1 & 0 & 1 \\ 2 & 1 & 1 \\ 1 & 1 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 \\ 2 & 1 & -1 \\ 1 & 1 & 0 \end{vmatrix} = 1 \neq 0$$

$$\sigma_1 = \operatorname{Tr} A = 3$$

$$\sigma_2 = \begin{vmatrix} 1 & 0 \\ 2 & 1 \end{vmatrix} + \begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix} + \begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix} = 1$$

$$\sigma_3 = \det A = 1.$$

$$(-1)^3 \left[A^3 - 3A^2 + A - I_3\right] = O_3 \quad | A^{-1}$$

$$A^2 - 3A + I_3 - A^{-1} = O_3 \Rightarrow A^{-1} = A^2 - 3A + I_3$$

$$= \begin{pmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & -1 & 1 \end{pmatrix}$$