CURS#2

3. Sisteme de ecuaţii liniare inferior/superior triunghiulare: metoda substituţiei ascendente/descendente.
4. Metoda de eliminare Gauss (MEG):
   (i) MEG fără pivotare: descrierea metodei; algoritm; limitări ale metodei.
   (ii) MEG cu pivotare parţială: descrierea metodei; algoritm; limitări ale metodei;

PROBLEME

1) Arătaţi că dacă $\mathbf{A} = \left(a_{ij}\right)_{i,j=\overline{1,n}} \in \mathscr{M}_n(\mathbb{R})$ este o matrice inferior (superior) triunghiulară inversabilă, atunci inversa sa $\mathbf{A}^{-1}$ este o matrice inferior (superior) triunghiulară.
2) Arătaţi că dacă $\mathbf{A} = \left(a_{ij}\right)_{i,j=\overline{1,n}}, \mathbf{B} = \left(b_{ij}\right)_{i,j=\overline{1,n}} \in \mathscr{M}_n(\mathbb{R})$ sunt matrice inferior (superior) triunghiulare, atunci produsul lor $\mathbf{A}\,\mathbf{B}$ este o matrice inferior (superior) triunghiulară.
3) Fie $\mathbf{M}_k \in \mathscr{M}_n(\mathbb{R})$ matricea corespunzătoare pasului $k = \overline{1, n-1}$ al MEGFP, i.e. o matrice de forma

$$\mathbf{M}_k = \mathbf{I}_n - \mathbf{m}^{(k)} \left(\mathbf{e}^{(k)}\right)^{\mathsf{T}}, \tag{1}$$

unde $\mathbf{I}_n \in \mathscr{M}_n(\mathbb{R})$ este matricea identitate şi

$$\mathbf{e}^{(k)} = \left(\delta_{ik}\right)_{i=\overline{1,n}} \in \mathbb{R}^n, \qquad \delta_{ik} = \begin{cases} 0 & \text{dacă } i \neq k \\ 1 & \text{dacă } i = k \end{cases} \tag{2a}$$

$$\mathbf{m}^{(k)} = \left(m_i^{(k)}\right)_{i=\overline{1,n}} \in \mathbb{R}^n, \qquad m_i^{(k)} = \begin{cases} 0 & \text{dacă } i = \overline{1,k} \\ a_{ik}^{(k)} / a_{kk}^{(k)} & \text{dacă } i = \overline{k+1,n}\,. \end{cases} \tag{2b}$$

Determinaţi $\left(\mathbf{M}_k\right)^{-1}$.

SISTEME DE ECUAȚII LINIARE

Vrem să rezolvăm <u>numeric</u> (cât mai acurat și mai stabil posibil) <u>sistemul de ecuații liniare pătratice:</u>

$$\boxed{A\underline{x} = \underline{b}} \qquad (1)$$

$A = (a_{ij})_{i,j=\overline{1,n}} \in \mathcal{M}_n(\mathbb{R})$ inv.

$\underline{b} = (b_1, b_2, \dots, b_n)^T \in \mathbb{R}^n$

$\underline{x} = (x_1, x_2, \dots, x_n)^T \in \mathbb{R}^n$

<u>Date</u> : $A \in \mathcal{M}_n(\mathbb{R})$, $\underline{b} \in \mathbb{R}^n$

<u>Necunoscute</u> : $\underline{x} \in \mathbb{R}^n$

<u>OBS</u> : Vrem să calculăm numeric soluția sistemului (1) fără a calcula inversa matricei A!

1. SISTEME TRIUNGHIULARE

DEFINIȚIE:

Matricea $A = (a_{ij})_{i,j=\overline{1,n}} \in M_n(\mathbb{R})$ sn

(i) <u>inferior triunghiulară</u> dacă

$$a_{ij} = 0, \quad \forall\, 1 \le i < j \le n$$

(ii) <u>superior triunghiulară</u> dacă

$$a_{ij} = 0, \quad \forall\, 1 \le j < i \le n$$

OBSERVAȚII:

1) $A = (a_{ij})_{i,j=\overline{1,n}}$ inf / sup triunghiulară $\Longrightarrow$

$$\det A = a_{11} a_{22} \dots a_{nn}$$

2) $A = (a_{ij})_{i,j=\overline{1,n}}$ inf / sup triunghiulară

A inversabilă $\Longleftrightarrow a_{ii} \neq 0, i = \overline{1,n}$

3) $A \in M_n(\mathbb{R})$ inf / sup triunghiulară

$\Longrightarrow A^{-1}$ inf / sup triunghiulară dacă $\exists$.

• Considerăm sistemul de ecuații liniare <u>inferior triunghiular</u>:

$$\underline{A}\,\underline{x} = \underline{b} \qquad (2)$$

$$\begin{cases} a_{11}x_1 & = b_1 \\ a_{21}x_1 + a_{22}x_2 & = b_2 \\ \vdots & \\ a_{k1}x_1 + a_{k2}x_2 + \dots + a_{kk}x_k & = b_k \\ \vdots & \\ a_{n1}x_1 + a_{n2}x_2 + \dots + a_{nn}x_n & = b_n \end{cases}$$

(E_1): $x_1 = b_1 / a_{11}$

(E_2): $a_{22}x_2 = b_2 - a_{21}x_1 \Rightarrow$

$$x_2 = (b_2 - a_{21}x_1)/a_{22}$$

$\vdots$

(E_k): $a_{kk}x_k = b_k - a_{k1}x_1 - \dots - a_{kk-1}x_{k-1} \Rightarrow$

$$x_k = \left(b_k - \sum_{j=1}^{k-1} a_{kj}x_j\right)/a_{kk}$$

⋮

(E_n): $a_{nn}x_n = b_n - a_{n1}x_1 - \dots - a_{n,n-1}x_{n-1} \Rightarrow$

$$x_n = \left(b_n - \sum_{j=1}^{n-1} a_{nj}x_j\right) / a_{nn}$$

<u>ALGORITM:</u>

(<u>Metoda substituției ascendente / Forward Substitution Method</u>)

$k=1$: $\quad x_k = b_k / a_{kk}$

$k=\overline{2,n}$: $\quad x_k = \left(b_k - \sum_{j=1}^{k-1} a_{kj}x_j\right) / a_{kk}$

<u>OBSERVAȚIE</u>:

În cazul sistemelor de ecuații liniare <u>superior triunghiulare</u>, se obține un algoritm similar denumit <u>metoda substituției descendente / Backward Substitution Method</u>.

2. METODA DE ELIMINARE GAUSS (MEG)

$$A\underline{x} = \underline{b} \qquad (1)$$

Date: $A = (a_{ij})_{i,j=\overline{1,n}}$ inversabilă

$\underline{b} = (b_1, b_2, \ldots, b_n)^T \in \mathbb{R}^n$

Neacunoscute: $\underline{x} = (x_1, x_2, \ldots, x_n)^T \in \mathbb{R}^n$

TRANSFORMĂRI ELEMENTARE:

a) Permutarea a două linii/ecuații E_i și E_k:

$$E_i \longleftrightarrow E_k$$

b) Înmulțirea unei linii/ecuații cu un scalar $\alpha \in \mathbb{R}^*$:

$$\alpha E_i \longrightarrow E_i$$

c) Adunarea unei linii/ecuații,

E_i, cu o altă linie/ecuație, E_k, înmulțită cu un scalar, $\alpha \in \mathbb{R}^*$:

$$E_i + \alpha E_k \longrightarrow E_i$$

IDEEA (MEG):

Folosind transformări elementare, să se modifice sistemul (1) într-un sistem echivalent superior triunghiular:

$$U\underline{x} = \underline{\tilde{b}} \qquad (2)$$

$U = (u_{ij})_{i,j=\overline{1,n}} \in \mathcal{M}_n(\mathbb{R})$

$u_{ij} = 0, \quad \forall\, 1 \le j < i \le n$

$\tilde{b} = (\tilde{b}_1, \tilde{b}_2, \ldots, \tilde{b}_n)^T \in \mathbb{R}^n$

OBSERVAȚIE:

Sistemul (2) se rezolvă folosind metoda substituției descendente.

2.1. MEG FĂRĂ PIVOTARE (MEGFP)

<u>Date</u>: $A = (a_{ij})_{i,j=\overline{1,n}} \in \mathcal{M}_n(\mathbb{R})$

$\underline{b} = (b_i)_{i=\overline{1,n}} \in \mathbb{R}^n$

$\overline{A} := [A \mid \underline{b}] \in \mathcal{M}_{n,n+1}(\mathbb{R})$

<u>Pasul 1</u>:

$$\overline{A} \equiv \overline{A}^{(1)} = \left[\begin{array}{c|ccc|c} a_{11}^{(1)} & a_{12}^{(1)} & \cdots & a_{1n}^{(1)} & b_1^{(1)} \\ \hline a_{21}^{(1)} & a_{22}^{(1)} & \cdots & a_{2n}^{(1)} & b_2^{(1)} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{n1}^{(1)} & a_{n2}^{(1)} & \cdots & a_{nn}^{(1)} & b_n^{(1)} \end{array}\right] \sim$$

$$\sim \overline{A}^{(2)} = \left[\begin{array}{c|ccc|c} a_{11}^{(1)} & a_{12}^{(1)} & \cdots & a_{1n}^{(1)} & b_1^{(1)} \\ \hline 0 & a_{22}^{(2)} & \cdots & a_{2n}^{(2)} & b_2^{(2)} \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & a_{n2}^{(2)} & \cdots & a_{nn}^{(2)} & b_n^{(2)} \end{array}\right]$$

$$i=\overline{2,n}: \begin{cases} (E_i - m_{i1}^{(1)} E_1) \longrightarrow E_i \\ m_{i1}^{(1)} := a_{i1}^{(1)} / a_{11}^{(1)} \end{cases} \Longleftrightarrow$$

$$\boxed{\begin{array}{l} i=\overline{2,n}: \\ \quad m_{i1}^{(1)} := a_{i1}^{(1)} / a_{11}^{(1)} \\ \quad b_i^{(2)} := b_i^{(1)} - m_{i1}^{(1)} b_1^{(1)} \\ \quad j=\overline{2,n}: \\ \qquad a_{ij}^{(2)} := a_{ij}^{(1)} - m_{i1}^{(1)} a_{1j}^{(1)} \end{array}}$$

<u>OBS</u>: Definim

$$m_{i1}^{(1)} := \begin{cases} 0 & , i=1 \\ a_{i1}^{(1)} / a_{11}^{(1)} & , i=\overline{2,n} \end{cases}$$

$$\underline{m}^{(1)} := \left(m_{i1}^{(1)}\right)_{i=\overline{1,n}} \in \mathbb{R}^n$$

$$\boxed{\begin{array}{l} \overline{A}^{(2)} = M^{(1)} \overline{A}^{(1)} \\ \qquad M^{(1)} := I_n - \underline{m}^{(1)} \left(\underline{e}^{(1)}\right)^T \end{array}}$$

$$M^{(1)} := \left[\begin{array}{c|ccc} 1 & 0 & \dots & 0 \\ \hline -m_{21}^{(1)} & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ -m_{n1}^{(1)} & 0 & \dots & 1 \end{array}\right]$$

<u>OBS</u>:

- $M^{(1)} := I_n - \underline{m}^{(1)}(\underline{e}^{(1)})^T$ inferior triunghiulară și inversabilă
- $\boxed{(M^{(1)})^{-1} = I_n + \underline{m}^{(1)}(\underline{e}^{(1)})^T}$

<u>Pasul 2</u>: Analog cu Pasul 1, dar pentru matricea

$$\left[\begin{array}{c|ccc|c} a_{22}^{(2)} & a_{23}^{(2)} & \dots & a_{2n}^{(2)} & b_2^{(2)} \\ \hline a_{32}^{(2)} & a_{33}^{(2)} & \dots & a_{3n}^{(2)} & b_3^{(2)} \\ \vdots & \vdots & & \vdots & \vdots \\ a_{n2}^{(2)} & a_{n3}^{(2)} & \dots & a_{nn}^{(2)} & b_n^{(2)} \end{array}\right]$$

$\in \mathcal{M}_{n-1}(\mathbb{R})$ $\quad\quad$ $\in \mathcal{M}_{n-1,1}(\mathbb{R}) \equiv \mathbb{R}^{n-1}$

Analog cu Pasul 1, obținem

$$i=\overline{3,n}: \begin{cases} \left(E_i - m_{i2}^{(2)} E_2\right) \longrightarrow E_i \\ m_{i2}^{(2)} := a_{i2}^{(2)} / a_{22}^{(2)} \end{cases}$$

$$\boxed{\begin{array}{l} i=\overline{3,n}: \\ \quad m_{i2}^{(2)} := a_{i2}^{(2)} / a_{22}^{(2)} \\ \quad b_i^{(3)} := b_i^{(2)} - m_{i2}^{(2)} b_2^{(2)} \\ \quad j=\overline{3,n}: \\ \qquad a_{ij}^{(3)} := a_{ij}^{(2)} - m_{i2}^{(2)} a_{2j}^{(2)} \end{array}}$$

<u>OBS</u>: Definim

$$m_{i2}^{(2)} := \begin{cases} 0 & , i=1,2 \\ a_{i2}^{(2)} / a_{22}^{(2)} & , i=\overline{3,n} \end{cases}$$

$$\underline{m}^{(2)} := \left(m_{i2}^{(2)}\right)_{i=\overline{1,n}} \in \mathbb{R}^n$$

$$\boxed{\begin{aligned}\overline{A}^{(3)} &= M^{(2)}\overline{A}^{(2)} = M^{(2)}M^{(1)}\overline{A}^{(1)} \\ M^{(2)} &:= I_n - \underline{m}^{(2)}\left(\underline{e}^{(2)}\right)^T\end{aligned}}$$

$$M^{(2)} := \left[\begin{array}{cc|cccc} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ \hline 0 & -m_{32}^{(2)} & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & -m_{n2}^{(2)} & 0 & \cdots & 1 \end{array}\right]$$

<u>OBS</u>:

- $M^{(2)} = I_n - \underline{m}^{(2)}\left(\underline{e}^{(2)}\right)^T$ inferior triunghiulară și inversabilă
- $\boxed{\left(M^{(2)}\right)^{-1} = I_n + \underline{m}^{(2)}\left(\underline{e}^{(2)}\right)^T}$

Pasul n-1 :

În mod analog, obținem

$$\overline{A}^{(n)} := [\underbrace{\tilde{U}}_{\text{superior triungh.}} | \tilde{\underline{b}}]$$

$$= M^{(n-1)} \overline{A}^{(n-1)}$$

$$= M^{(n-1)} M^{(n-2)} \overline{A}^{(n-2)} = \dots =$$

$$= \left(M^{(n-1)} M^{(n-2)} \dots M^{(1)}\right) \overline{A}^{(1)}$$

$$= \left(\underbrace{M^{(n-1)} M^{(n-2)} \dots M^{(1)}}_{\text{inferior triungh.}}\right) [A | \underline{b}]$$

ALGORITM (MEGFP):

Date: $A = (a_{ij})_{i,j=\overline{1,n}}$, $\underline{b} = (b_i)_{i=\overline{1,n}}$

$k = \overline{1, n-1}$:

$i = \overline{k+1, n}$:

$m := a_{ik} / a_{kk}$

$b_i := b_i - m\, b_k$

$j = \overline{k+1, n}$:

$a_{ij} := a_{ij} - m\, a_{kj}$

$a_{ik} = 0$

OBS (limitarea MEGFP):

$$\begin{cases} \varepsilon x_1 + x_2 = 1 \\ x_1 + x_2 = 2 \end{cases}, \quad \varepsilon = 0\,(10^{-200}) \text{ ie } \quad 1 \pm \varepsilon \simeq 1$$

$$\left[\begin{array}{cc|c} \varepsilon & 1 & 1 \\ 1 & 1 & 2 \end{array}\right] \underset{\sim}{\left(E_2 - \frac{1}{\varepsilon}E_1\right) \to E_2}$$

$$\left[\begin{array}{cc|c} \varepsilon & 1 & 1 \\ 0 & 1-\frac{1}{\varepsilon} & 2-\frac{1}{\varepsilon} \end{array}\right] \Rightarrow$$

$$\begin{cases} \varepsilon x_1 + x_2 = 1 \\ \left(1-\frac{1}{\varepsilon}\right)x_2 = 2 - \frac{1}{\varepsilon} \end{cases} \Rightarrow$$

$$\frac{\varepsilon - 1}{\varepsilon} x_2 = \frac{2\varepsilon - 1}{\varepsilon} \Leftrightarrow$$

$$x_2 = \frac{2\varepsilon - 1}{\varepsilon} \cdot \frac{\varepsilon}{\varepsilon - 1} = \frac{2\varepsilon - 1}{\varepsilon - 1} \simeq \frac{-1}{-1} = 1$$

$$\Leftrightarrow \boxed{x_2 = 1}$$

$\varepsilon x_1 + x_2 = 1 \iff \varepsilon x_1 + 1 = 1 \iff$

$\varepsilon x_1 = 0 \iff \boxed{x_1 = 0}$

<u>Verificare</u>:

$$\begin{cases} \varepsilon \cdot 0 + 1 = 1 & \checkmark \\ 0 + 1 = 2 & ? \end{cases}$$

<u>Cauza</u>:

Pivotul, $a_{11}^{(1)} = \varepsilon = O(10^{-200}) \ll 1$, este <u>f mic</u>!

Nu se ține seama de valoarea pivotului în comparație cu elementele de pe <u>coloana sa</u>!

2.2. MEG CU PIVOTARE PARȚIALĂ (MEGPP)

IDEEA (MEGPP):

La fiecare pas al MEGFP, $k=\overline{1,n-1}$, alege ca pivot elementul de pe coloana k (i.e. coloana pivotului) care este cel mai mare în valoare absolută:

(i) Determină $l \in \overline{k,n}$ a.î.

$$|a_{lk}^{(k)}| = \max_{j=\overline{k,n}} |a_{jk}^{(k)}|$$

(ii) • Dacă $l = k \implies$ MEGFP la pasul k

• Dacă $l \neq k$ (i.e. $l > k$) $\implies$

$E_k \longleftrightarrow E_l$ + MEGFP la pasul k

OBS (remedierea limitării MEGFP):

$$\begin{cases} \varepsilon x_1 + x_2 = 1 \\ x_1 + x_2 = 2 \end{cases}, \quad \varepsilon = O(10^{-200}), \text{ ie } \varepsilon \ll 1$$

$$\left[\begin{array}{cc|c} \varepsilon & 1 & 1 \\ 1 & 1 & 2 \end{array}\right] \overset{E_1 \leftrightarrow E_2}{\sim} \left[\begin{array}{cc|c} 1 & 1 & 2 \\ \varepsilon & 1 & 1 \end{array}\right] \sim$$

$$\overset{(E_2 - \varepsilon E_1) \to E_2}{\sim} \left[\begin{array}{cc|c} 1 & 1 & 2 \\ 0 & 1-\varepsilon & 1-2\varepsilon \end{array}\right] \Rightarrow$$

- $\underline{k=1}$:

$$\max_{i=1,2} |a_{i1}^{(1)}| = \max_{i=1,2} \{|\varepsilon|, 1\} = 1$$

$$= a_{21}^{(1)} \Rightarrow l = \max_{j=1,2} |a_{j1}^{(1)}| = 2$$

- $l = 2 \neq 1 = k \Rightarrow E_1 \leftrightarrow E_2$ + MEGFP

- Sistemul de rezolvat devine:

$$\begin{cases} x_1 + x_2 = 2 \\ (1-\varepsilon) x_2 = 1 - 2\varepsilon \end{cases} \Rightarrow$$

$$x_2 = \frac{1-2\varepsilon}{1-\varepsilon} \simeq \frac{1}{1} = 1 \Rightarrow \boxed{x_2 = 1}$$

$$x_1 + x_2 = 2 \Rightarrow x_1 + 1 = 2 \Rightarrow \boxed{x_1 = 1}$$

<u>Verificare</u>:

$$\begin{cases} \varepsilon x_1 + x_2 = 1 \\ x_1 + x_2 = 2 \end{cases} \Leftrightarrow \begin{cases} \varepsilon + 1 = 1 \checkmark \\ 1 + 1 = 2 \checkmark \end{cases}$$

<u>ok</u>!

<u>Obs</u>: Înmulțirea la stânga a lui $\overline{A}^{(k)} := [A^{(k)} \mid \underline{b}^{(k)}]$ cu o matrice <u>permutare simplă</u>, P_{kl}, $l > k$, reprezintă, în fapt, interschimbul liniilor/ecuațiilor E_k și E_l:

$$P_{kl} = [\underline{e}^{(1)} \ldots \underline{e}^{(k-1)} \; \underline{e}^{(l)} \; \underline{e}^{(k+1)} \ldots \underline{e}^{(l-1)} \; \underline{e}^{(k)} \; \underline{e}^{(l+1)} \ldots \underline{e}^{(n)}]$$

unde $\underline{e}^{(j)} = (\delta_{ij})_{i=\overline{1,n}} \in \mathbb{R}^n$,

$$\delta_{ij} = \begin{cases} 1, & i = j \\ 0, & \text{altfel} \end{cases} \quad \text{kronecker}$$

Pt $k<\ell$, $P_{k\ell} :=$

$$
\begin{array}{cc}
\left[\begin{array}{ccc|c|ccc|c|ccc}
1 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0\\
\vdots & \ddots & \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots\\
0 & \cdots & 1 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0\\
\hline
0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 1 & 0 & \cdots & 0\\
\hline
0 & \cdots & 0 & 0 & 1 & \cdots & 0 & 0 & 0 & \cdots & 0\\
\vdots & & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & & \vdots\\
0 & \cdots & 0 & 0 & 0 & \cdots & 1 & 0 & 0 & \cdots & 0\\
\hline
0 & \cdots & 0 & 1 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0\\
\hline
0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 1 & \cdots & 0\\
\vdots & & \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & \ddots & \vdots\\
0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 1
\end{array}\right]
&
\begin{array}{l}
\\ \\ \\
(e^{(\ell)})^T\\
\\ \\ \\
(e^{(k)})^T\\
\\ \\ \\
\end{array}
\end{array}
$$

col k: $\underline{e}^{(\ell)}$ col ℓ: $\underline{e}^{(k)}$

$(P_{k\ell})^T = P_{k\ell}$; $(P_{k\ell})^2 = I_n$;

$(P_{k\ell})^{-1} = P_{k\ell}^T = P_{k\ell}$

<u>Obs</u>:

În cazul MEGPP, obținem:

$$\overline{A}^{(n)} := [U \mid \underline{\tilde{b}}]$$
$$= M^{(n-1)} P^{(n-1)} \overline{A}^{(n-1)}$$
$$= M^{(n-1)} P^{(n-1)} M^{(n-2)} P^{(n-2)} \overline{A}^{(n-2)}$$
$$= \ldots =$$
$$= M^{(n-1)} P^{(n-1)} M^{(n-2)} P^{(n-2)} \ldots M^{(1)} P^{(1)} \underline{A}^{(1)}$$
$$= M^{(n-1)} P^{(n-1)} M^{(n-2)} P^{(n-2)} \ldots M^{(1)} P^{(1)} [A \mid \underline{b}]$$

unde:

- $M^{(k)} := I_n - \underline{m}^{(k)} (\underline{e}^{(k)})^T, \ k = \overline{1, n-1};$
- $P^{(k)} := P_{k\ell}$ cu $k < \ell$,
  – permutarea simplă de la pasul $k = \overline{1, n-1}$

ALGORITM (MEGPP):

Date: $A = (a_{ij})_{i,j=\overline{1,n}}$; $\underline{b} = (b_i)_{i=\overline{1,n}}$

$k = \overline{1, n-1}$:

$\quad l : |a_{lk}| = \max\limits_{i=\overline{k,n}} |a_{ik}| \quad (k \le l)$

$\quad A := P_{lk} A \quad (\text{ie } (E_l) \leftrightarrow (E_k))$

$\quad i = \overline{k+1, n}$:

$\quad\quad m := a_{ik} / a_{kk}$

$\quad\quad b_i := b_i - m b_k$

$\quad\quad j = \overline{k+1, n}$:

$\quad\quad\quad a_{ij} := a_{ij} - m a_{kj}$

$\quad\quad$ end

$\quad\quad a_{ik} = 0$

$\quad$ end

end