Endomorfisme simetrice

Teoremă (C_{10} - AG)

$(E, \langle \cdot, \cdot \rangle)$ s.v.e.r, $f \in Sim(E)$

$\Rightarrow$ toate rădăcinile polinomului caracteristic sunt reale

Dem. $\mathcal{R} = \{e_1, \dots, e_n\}$ reper ortonormat în E.

$A = [f]_{\mathcal{R},\mathcal{R}}$ și $P(\lambda) = \det(A - \lambda I_n) = 0$. Fie λ rădăcină.

Fie $AX = \lambda X$, $X = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$

$(A - \lambda I_n) X = O_{n,1}$ este SLO

$$\begin{pmatrix} a_{11} - \lambda & \dots & a_{1n} \\ \vdots & & \\ a_{n1} & \dots & a_{nn} - \lambda \end{pmatrix} \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}$$

Înmulțim la stânga cu matricea:

$$\begin{pmatrix} \overline{x_1} & \dots & 0 \\ \vdots & & \\ 0 & \dots & \overline{x_n} \end{pmatrix}$$

Prin calcul obținem:

$$\begin{pmatrix} \overline{x_1} & \dots & 0 \\ 0 & \dots & \overline{x_n} \end{pmatrix} \begin{pmatrix} (a_{11} - \lambda) x_1 + \dots + a_{1n} x_n \\ \vdots \\ a_{n1} x_1 + \dots + (a_{nn} - \lambda) x_n \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}$$

Rezultă

$$\begin{cases} (a_{11} - \lambda) x_1 \overline{x_1} + a_{12} x_2 \overline{x_1} + \dots + a_{1n} x_n \overline{x_1} = 0 \\ \vdots \\ a_{n1} x_1 \overline{x_n} + a_{n2} x_2 \overline{x_n} + \dots + (a_{nn} - \lambda) x_n \overline{x_n} = 0 \end{cases} \quad \oplus$$

$$\sum_{k,j=1}^{n} a_{kj} x_k \overline{x_j} = \lambda \underbrace{\sum_{k=1}^{n} x_k \overline{x_k}}_{\in \mathbb{R}} \quad (\text{Prop: } z \cdot \overline{z} = |z|^2 \in \mathbb{R})$$

$$\|$$

$$\sum_{k<j} a_{kj} x_k \overline{x_j} + \sum_{k>j} a_{kj} x_k \overline{x_j} + \sum_{k=1}^{n} a_{kk} x_k \overline{x_k}$$

$(A = A^T \in \mathcal{M}_n(\mathbb{R}))$

$$\sum_{k<j} a_{kj} (\underbrace{x_k \overline{x_j} + x_j \overline{x_k}}_{\in \mathbb{R}}) + \sum_{k=1}^{n} a_{kk} \underbrace{x_k \overline{x_k}}_{\in \mathbb{R}} = \lambda \sum_{k=1}^{n} x_k \overline{x_k} \Rightarrow \lambda \in \mathbb{R}$$

Scanat cu CamScanner

Teorema de descompunere polară

$(E, \langle \cdot, \cdot \rangle)$ s.v.e.r

$$\forall f \in Aut(E) \Rightarrow \begin{array}{l} \exists h \in Sim(E) \\ \exists t \in O(E) \end{array} \text{ a.î. } f = h \circ t$$

OBS

$$\forall A \in GL(n, \mathbb{R}), \begin{array}{l} \exists B \in \mathcal{M}_n(\mathbb{R}),\ B = B^T \\ \exists C \in O(n) \end{array} \text{ a.î. } A = B \cdot C$$

Lemă

$f \in Sim(E)$, poz. def ($[f]_{R,R}$ poz. definită sau Q forma pătratică asociată poz. definită) $\Rightarrow$ $\exists h \in Sim(E)$ poz. def a.î. $f = h^2$

Dem (Lemă) $\mathcal{R} = \{e_1, \ldots, e_n\}$ reper orton. a.î. $A_f = [f]_{R,R}$
$= \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}$ (f este diagonalizabil)

$Q_f : E \to \mathbb{R}$ f. pătratică asociată

$Q_f(x) = \lambda_1 x_1^2 + \ldots + \lambda_n x_n^2$, $x = \sum_{i=1}^{n} x_i e_i$ (sign este $(n, 0)$)

Q_f este p. def $\Rightarrow \lambda_1 > 0, \ldots, \lambda_n > 0$

Fie $h \in End(E)$, $[h]_{R,R} = A_h = \begin{pmatrix} \sqrt{\lambda_1} & & 0 \\ & \ddots & \\ 0 & & \sqrt{\lambda_n} \end{pmatrix}$

$h \in Sim(E)$

$A_{h^2} = A_h \cdot A_h = A_f$, $Q_h(x) = \sqrt{\lambda_1} x_1^2 + \ldots + \sqrt{\lambda_n} x_n^2$ este poz. def $\Rightarrow$ h este poz. def. și

$$f = h^2$$

Scanat cu CamScanner

Dem (teoremă)

Fie $R = \{e_1, \dots, e_n\}$ reper orton, $A_f = [f]_{R,R} \in GL(n, R)$)

Fie $\tilde{f} \in End(E)$ aî $A_{\tilde{f}} = A_f \cdot A_f^T \overset{not}{=} B$ ⊛

$B = B^T \Rightarrow \tilde{f} \in Sim(E)$ ⊛⊛

Dem că $\tilde{f}$ este p.def. definită.

Fie $Q_{\tilde{f}} : E \to \mathbb{R}$ forma pătratică asociată

$Q_{\tilde{f}}(e_i) = \langle e_i, \tilde{f}(e_i) \rangle = \langle e_i, \sum_{j=1}^{n} b_{ji} e_j \rangle =$
$i = \overline{1,n}$

$= \sum_{j=1}^{n} b_{ji} \langle e_i, e_j \rangle = b_{ii} \overset{⊛}{=} \sum_{k=1}^{n} a_{ik} a_{ik} = \sum_{k=1}^{n} a_{ik}^2 > 0$

($\langle e_i, e_j \rangle = \delta_{ij}$)

(linia i a lui A_f nu poate fi nulă, $A_f \in GL(n, R)$)

Deci $Q_{\tilde{f}}(x) > 0, \forall x \neq 0_E \Rightarrow \tilde{f}$ p.def $\overset{Lema}{\underset{⊛⊛}{\Longrightarrow}}$

$\exists h \in Sim(E)$, p.def aî $\tilde{f} = h^2$

$B = A_f \cdot A_f^T = A_h \cdot A_h$

Fie $t = h^{-1} \circ f$. Dem că $t \in O(n)$

$A_t \cdot A_t^T = A_{h^{-1} \circ f} (A_{h^{-1} \circ f})^T = A_{h^{-1}} \cdot A_f (A_{h^{-1}} \cdot A_f)^T$

$= A_{h^{-1}} \cdot \underbrace{A_f \cdot A_f^T}_{A_{\tilde{f}} = A_{h^2}} \cdot A_{h^{-1}} = A_{h^{-1}} \cdot A_h \cdot A_h \cdot A_{h^{-1}} = I_n.$

(h sim)

$\Rightarrow t \in O(n)$

Deci $\underset{Aut(E)}{f} = \underset{Sim(E)}{h} \circ \underset{O(E)}{t}$.