5.12.2024

Seminarul 10

Forme pătratice. Formă canonică. Spații vectoriale cu produs scalar. Procedeul Gram-Schmidt

• $f : \mathbb{R}^3 \to \mathbb{R}$

$Q(x) = 2x_1^2 + 5x_2^2 + 2x_3^2 - 4x_1x_2 - 2x_1x_3 + 4x_2x_3$

a) $G = ?$

$$G = \begin{pmatrix} 2 & -2 & -1 \\ -2 & 5 & 2 \\ -1 & 2 & 2 \end{pmatrix}$$

b) $g : \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}$

formă polară asociată

$g(x, y) = 2^{-1}[Q(x+y) - Q(x) - Q(y)]$

$$\sum_{i,j=1}^{3} g_{ij} x_i y_j = 2x_1y_1 - 2x_1y_2 - x_1y_3 - 2x_2y_1 + 5x_2y_2 + 2x_2y_3 - x_3y_1 + 2x_3y_2 + 2x_3y_3$$

$$Q(x) = \sum_{i=1}^{3} g_{ii} x_i^2 + 2 \sum_{i<j} g_{ij} x_i x_j$$

~~Ker~~

c) Este g nedegenerată?

$g \in L^s(\mathbb{R}^3, \mathbb{R}^3; \mathbb{R})$

$\ker g = \{x \in \mathbb{R}^3 \mid g(x, y) = 0, \ x, y \in \mathbb{R}^3\}$

g nedegenerată $\Leftrightarrow \ker g = \{0_{\mathbb{R}^3}\}$

$$\begin{vmatrix} 2 & -2 & -1 \\ -2 & 5 & 2 \\ -1 & 2 & 2 \end{vmatrix} \underline{\underline{\begin{matrix} C_1+2C_3 \\ C_2-2C_3 \end{matrix}}} \begin{vmatrix} 0 & 0 & -1 \\ 2 & 1 & 2 \\ 3 & -2 & 2 \end{vmatrix} =$$

$$= - \begin{vmatrix} 2 & 1 \\ 3 & -2 \end{vmatrix} \neq 0 \Rightarrow G \text{ nedegenerată}$$
$$\Rightarrow g \text{ nedegenerată}$$

d) Q formă canonică

Jacobi

$\Delta_1 = 2 \neq 0$

$$\Delta_2 = \begin{vmatrix} 2 & -2 \\ -2 & 5 \end{vmatrix} = 10-4 = 6 \neq 0$$

$$\Delta_3 = \begin{vmatrix} 2 & -2 & -1 \\ -2 & 5 & 2 \\ -1 & 2 & 2 \end{vmatrix} = 7 \neq 0$$

$(\exists)$ R reper $\subset \mathbb{R}^3$ a.î. $Q(x) = \frac{1}{\Delta_1} x_1'^2 + \frac{\Delta_1}{\Delta_2} x_2'^2 + \frac{\Delta_2}{\Delta_3} x_3'^2$

$Q(x) = \frac{1}{2} x_1'^2 + \frac{1}{3} x_2'^2 + \frac{6}{7} x_3'^2$

$(3,0)$ signatura

Q poz. definită

Met II Gauss

$Q(x) = 2(x_1^2 - 2x_1x_2 - x_1x_3) + 5x_2^2 + 2x_3^2 + 4x_2x_3 =$

$= 2\left(x_1 - x_2 - \frac{1}{2}x_3\right)^2 - 2x_2^2 - \frac{1}{2}x_3^2 + 2x_2x_3 + 5x_2^2 + 2x_3^2 + 4x_2x_3$

$3x_2^2 + \frac{3}{2}x_3^2 + x_2x_3$

$= \ldots \quad 3\left(x_2^2 + \frac{2}{3}x_2x_3 + \frac{x_3^2}{9}\right) - \frac{x_3^2}{3} + \frac{3}{2}x_3^2$

$$Q(x) = 2\left(x_1 - x_2 - \frac{1}{2}x_3\right)^2 + 3\left(x_2 + \frac{1}{3}x_3\right)^2 + \frac{7}{3}x_3^2$$

$x_1' = x_1 - x_2 - \frac{1}{2}x_3$

$x_2' = x_2 + \frac{1}{3}x_3$

$x_3' = x_3$

$Q(x) = 2x_1'^2 + 3x_2'^2 + \frac{7}{3}x_3'^2$

(3,0) signature

(2)

$Q: \mathbb{R}^3 \to \mathbb{R}, \quad Q(x) = 2x_1x_2 - 6x_1x_3 - 6x_2x_3$

forma canonica

$$G = \begin{pmatrix} 0 & 1 & -3 \\ 1 & 0 & -3 \\ -3 & -3 & 0 \end{pmatrix}$$

Gauss $\begin{cases} x_1' = x_1 + x_2 \\ x_2' = x_1 - x_2 \\ x_3' = x_3 \end{cases} \Rightarrow \begin{cases} x_1 = \frac{1}{2}(x_1' + x_2') \\ x_2 = \frac{1}{2}(x_1' - x_2') \\ x_3 = x_3' \end{cases}$

$Q(x) = 2(x_1'^2 - x_2'^2) - 6x_1'x_3'$

$= \frac{1}{2}(x_1'^2 - 12x_1'x_3' + 36x_3'^2) - \frac{36}{2}x_3'^2 - \frac{1}{2}x_2'^2$

$Q(x) = \frac{1}{2}(x_1' - 6x_3')^2 - \frac{1}{2}x_2'^2 - 18x_3'^2$

$x_1'' = x_1' - 6x_3'$

$x_2'' = x_2'$

$x_3'' = x_3'$

$Q(x) = \frac{1}{2} x_1''^2 - \frac{1}{2} x_2''^2 - 18 x_3''^2$

$S(1,2) \Rightarrow Q$ nu e poz def

② ~~$Q: \mathbb{R}^3 \to \mathbb{R}$~~

$g: \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}$ f. biliniară

$G = \begin{pmatrix} 3 & 2 & 0 \\ 2 & 2 & 2 \\ 0 & 2 & 1 \end{pmatrix}$ matricea asociată

este perechea $(\mathbb{R}^3, g)$ sp. vect euclidian real?
(este g prod scalar)

$G = G^t \Rightarrow g$ simetrică
g biliniară

$\Bigg| \Rightarrow g \in L^s(\mathbb{R}^3, \mathbb{R}^3; \mathbb{R})$
f bil sim.

$Q(x) = 3x_1^2 + 2x_2^2 + x_3^2 + 4x_1x_2$

$\Delta_1 = 3$

$\Delta_2 = \begin{vmatrix} 3 & 2 \\ 2 & 2 \end{vmatrix} = 2 > 0$

$\Delta_3 = \begin{vmatrix} 3 & 2 & 0 \\ 2 & -2 & 2 \\ 0 & 0 & 1 \end{vmatrix} = -10 < 0$

sign $(2,1) \Rightarrow$ nu este poz def

③ $(\mathbb{R}^3, g_0)$ s.v.e.r

g_0 - produsul scalar canonic

$g_0 : \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}$ $g_0(x,y) = \sum_{i=1}^{3} x_i y_i$

$U = \{x \in \mathbb{R}^3 \mid x_1 + x_2 - x_3 = 0\}$

a) $U^\perp = ?$

b) R ortonormat $= R_1 \cup R_2$ în $\mathbb{R}^3$

unde R_1 rep ortonormat în U

R_2 rep ortonormat în $U^\perp$

$\mathbb{R}^3 = U \oplus U^\perp$

a) $U^\perp = \{x \in \mathbb{R}^3 \mid g_0(x,y) = 0, (\forall) y \in U\}$

$U^\perp = \{(1,1,-1)\}$

b) $U = \{(x_1, x_2, x_1 + x_2) \mid x_1, x_2 \in \mathbb{R}\} \Rightarrow$

$\Rightarrow U = \langle \{ \underbrace{(1,0,1)}_{f_1}, \underbrace{(0,1,1)}_{f_2} \} \rangle$

Aplicăm procedeul de ortogonalizare Gram-Schmidt

$e_1 = f_1 = (1,0,1)$ $1 \cdot 0 + 0 \cdot 1 + 1 \cdot 1$

$e_2 = f_2 - \frac{\langle f_2, f_1 \rangle}{\langle e_1, e_1 \rangle} e_1 = (0,1,1) - \frac{1}{2}(1,0,1) =$

produs scalar $= \left(-\frac{1}{2}, 1, \frac{1}{2}\right) = \frac{1}{2}(-1, 1, 1)$

$e_1' = \frac{e_1}{\|e_1\|} = \frac{1}{\sqrt{2}}(1,0,1)$ ①

Def: $\|x\| = \sqrt{g_0(x,x)} = \sqrt{x_1^2 + x_2^2 + x_3^2}$

$e_2' = \frac{e_2}{\|e_2\|} = \frac{1}{\sqrt{6}}(-1,2,1)$ ②

Obs: $V = \alpha V'$, $\alpha > 0$

$\frac{V}{\|V\|} = \frac{V'}{\|V'\|}$

Din ①, ② $\Rightarrow$ $R_1 = \{e_1', e_2'\}$ reper ortonormat în U

$R_2 = \{\frac{1}{\sqrt{3}}(1,1,-1)\}$ reper ortonormat în $U^\perp$

$R = R_1 \cup R_2$ reper ort. în $\mathbb{R}^3$

$R = \{e_1', e_2', \frac{1}{\sqrt{3}}(1,1,-1)\}$

Obs: $U^\perp = \{x \in \mathbb{R}^3 \mid \begin{cases} g_0(x, f_1) = 0 \\ g_0(x_2, f_2) = 0 \end{cases}\} =$

$= \{x \in \mathbb{R}^3 \mid \begin{cases} x_1 + x_3 = 0 \\ x_2 + x_3 = 0 \end{cases}\} =$

$= \{(-x_3, -x_3, x_3) \mid x_3 \in \mathbb{R}\} = \langle\{(-1,-1,1)\}\rangle$

Obs: $R_0 \xrightarrow{A} R$

$\det A > 0$

$$A = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ 0 & \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{3}} \end{pmatrix}$$

$$\det A = \frac{1}{6}\begin{vmatrix} 1 & -1 & -1 \\ 0 & 2 & 1 \\ 1 & 1 & -1 \end{vmatrix} \overset{L_3-L_1}{=} \frac{1}{6}\begin{vmatrix} 1 & -1 & -1 \\ 0 & 2 & 1 \\ 0 & 2 & -2 \end{vmatrix} =$$

$$= \frac{1}{6}(-4-2) = -1$$

$$R_0 \xrightarrow{A} R$$

R nu este pozitiv orientat ca R_0 ($\det A < 0$)

⑤ $(\mathbb{R}^3, g_0)$, s.e.r

$R = \{f_1 = (1,2,3), f_2 = (0,1,1), f_3 = (1,2,5)\}$

a) R reper cin $\mathbb{R}^3$

Să se ortonormeze

b) $f_1 \times f_2$, $f_1 \wedge f_2 \wedge f_3$

$$\begin{vmatrix} 1 & 0 & 1 \\ 2 & 1 & 2 \\ 3 & 1 & 5 \end{vmatrix} \overset{C_3-C_1}{=} \begin{vmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 1 & 2 \end{vmatrix} = \begin{vmatrix} 1 & 0 \\ 1 & 2 \end{vmatrix} = 2 \neq 0$$

Obs: $R_0 \xrightarrow{A} R$

$\det A > 0 \to R_0, R$ la fel orientate

$$e_1 = f_1 \to e_1' = \frac{1}{\sqrt{14}}(1,2,3)$$

$$e_2 = f_2 \to e_2' = f_2 - \frac{\langle f_2, e_1\rangle}{\langle e_1, e_1\rangle} e_1$$

$$e_2 = (0,1,1) - \frac{5}{14}(1,2,3) =$$

$$= \left(-\frac{5}{14}, \frac{2}{7}, -\frac{1}{14}\right) = \frac{1}{14}(-5,4,-1)$$

$$e_2' = \frac{e_2}{\|e_2\|} = \frac{1}{\sqrt{42}}(-5,4,-1)$$

$$e_2 = f_3 - \frac{\langle f_3, e_1\rangle}{\langle e_1, e_1\rangle} e_1 - \frac{\langle f_3, e_2\rangle}{\langle e_2, e_2\rangle} e_2$$

$$e_3' = \frac{e_3}{\|e_3\|}$$

$$f_1 \times f_2 = \begin{vmatrix} i & j & k \\ 1 & 2 & 3 \\ 0 & 1 & 1 \end{vmatrix} = \quad \begin{array}{|l} i = (1,0,0) \\ j = (0,1,0) \\ k = (0,0,1) \end{array}$$

$$= \left(\begin{vmatrix} 2 & 3 \\ 1 & 1 \end{vmatrix}, -\begin{vmatrix} 1 & 3 \\ 0 & 1 \end{vmatrix}, \begin{vmatrix} 1 & 2 \\ 0 & 1 \end{vmatrix} \right) =$$

$$= (-1,-1,1)$$

$$e_3' = \frac{1}{\sqrt{3}}(-1,-1,1)$$

$$f_1 \wedge f_2 \wedge f_3 = \begin{vmatrix} \overbrace{1 \quad 2 \quad 5}^{\text{coord lui } f_3} \\ 1 \quad 2 \quad 3 \\ 0 \quad 1 \quad 1 \end{vmatrix} =$$

sau

$$= g_0(f_3, f_1 \times f_2) = -1 - 2 + 5 = 2.$$