$$x_n = \frac{2n+1}{n-3}$$

$$\lim_{n\to\infty} x_n = 2$$

Fie $\varepsilon > 0$ fixat arbitrar

$$|x_n - 2| = \left|\frac{2n+1}{n-3} - \overset{n-3}{2}\right| = \left|\frac{2n+1-2n+6}{n-3}\right| = \left|\frac{7}{n-3}\right|$$

Vrem $\left|\frac{7}{n-3}\right| < \varepsilon \iff \frac{7}{n-3} < \varepsilon \iff 7 < \varepsilon\cdot(n-3) \iff \frac{7}{\varepsilon} < n-3 \iff n > \frac{7}{\varepsilon} + 3 \iff n > \frac{7+3\varepsilon}{\varepsilon}$

Fie $N_\varepsilon = \left[\frac{7+3\varepsilon}{\varepsilon}\right] + 1 \Rightarrow \forall n \geq n_\varepsilon$ avem $|x_n - 2| < \varepsilon \to \lim_{n\to\infty} x_n = 2$

$$\frac{1}{2^n} \to 0$$

$\lim_{n\to\infty} a_n = l \iff \forall \varepsilon > 0, \exists N_\varepsilon \in \mathbb{N}$ a.î. $\forall n \geq N_\varepsilon$, $|a_n - l| < \varepsilon \iff l - \varepsilon < a_n < l + \varepsilon$

$\lim_{n\to\infty} \underbrace{\frac{1}{2^n}}_{a_n} = 0 \iff \forall \varepsilon > 0, \exists N_\varepsilon \in \mathbb{N}$ a.î., $\forall n \geq N_\varepsilon$, $|a_n| < \varepsilon$

Fie $\varepsilon > 0$ arbitrar

Vrem $|a_n| < \varepsilon \iff \frac{1}{2^n} < \varepsilon \iff 2^n > \frac{1}{\varepsilon} \iff n > \log_2 \frac{1}{\varepsilon}$

Fie $N_\varepsilon = \left[\log_2 \frac{1}{\varepsilon}\right] + 1 \Rightarrow \forall n \geq n_\varepsilon$, avem $|a_n| < \varepsilon \Rightarrow a_n \xrightarrow{n\to\infty} 0$.

$x_n = \frac{n^2+\sin n}{3n^2+4}$

$\lim_{n\to\infty} x_n = \frac{1}{3}$

Fie $\varepsilon>0$. Fie $N_\varepsilon = \left[\sqrt{\frac{10}{9\varepsilon}}\right]+1 \Rightarrow$

$\lim_{n\to\infty} x_n = l \iff \forall \varepsilon>0, \exists N_\varepsilon \in \mathbb{N}$ a.î. $\forall n \geq N_\varepsilon$, avem $|x_n - l| < \varepsilon$

$\Rightarrow \forall n \geq N_\varepsilon$, avem $n > \sqrt{\frac{10}{9\varepsilon}} \Rightarrow \frac{10}{9n^2} < \varepsilon$

Vrem: $\left|x_n - \frac{1}{3}\right| < \varepsilon$

$\left|x_n - \frac{1}{3}\right| = \left|\frac{3\sin n - 4}{9n^2+12}\right| < \frac{10}{9n^2} < \varepsilon$

$3\sin n - 4 < 10$

$$\left|x_n - \frac{1}{3}\right| = \left|\frac{n^2+\sin n}{3n^2+4} - \frac{1}{3}\right| = \left|\frac{\cancel{3n^2}+3\sin n - \cancel{3n^2}-4}{9n^2+12}\right| = \left|\frac{3\sin n - 4}{9n^2+12}\right| \leq \frac{10}{9n^2+12} \leq \frac{10}{9n^2} < \frac{1}{n}$$

$\frac{10}{9n^2} < \varepsilon \Rightarrow 10 < 9\varepsilon n^2 \Rightarrow n^2 > \frac{10}{9\varepsilon} \Rightarrow n > \frac{\sqrt{10}}{3\sqrt{\varepsilon}}$ Alegem $N_\varepsilon = \left[\frac{\sqrt{10}}{3\sqrt{\varepsilon}}\right]+1$

pt $\forall n > n_\varepsilon$ $\left|x_n - \frac{1}{3}\right| < \varepsilon$

$x_n = \frac{n^2+4n+1}{n+2}$

$\lim_{n\to\infty} x_n = \infty$

Fie $\varepsilon > 0$,

Vrem N_ε a.î. $x_n > \varepsilon$

$\frac{n^2+4n+1}{n+1} > \frac{n^2+n}{n+1} > \frac{n(\cancel{n+1})}{\cancel{n+1}} > n > \varepsilon$ (*)

$n > \varepsilon$

Fie $N_\varepsilon = [\varepsilon] + 1$ atunci $\forall\, n \geq N_\varepsilon \Rightarrow n \geq N_\varepsilon = [\varepsilon] + 1 > \varepsilon$

Din (*) $\Rightarrow \lim_{n\to\infty} x_n = \infty$

$\lim_{n\to\infty} x_n = l \iff \forall \varepsilon > 0, \exists N_\varepsilon \in \mathbb{N}$ a.î. $\forall n \geq N_\varepsilon$, avem $|x_n - l| < \varepsilon$

$\lim_{n\to\infty} x_n = \infty \iff \forall \varepsilon > 0, \exists N_\varepsilon \in \mathbb{N}$ a.î. $\forall n \geq N_\varepsilon$, avem $x_n > \varepsilon$

$\lim_{n\to\infty} x_n = -\infty \iff \forall \varepsilon < 0, \exists N_\varepsilon \in \mathbb{N}$ a.î. $n \geq N_\varepsilon$, avem $x_n < \varepsilon$

$x_n > \varepsilon \Rightarrow n > \frac{\varepsilon}{4}$

$$\lim_{n\to\infty} \frac{3n^2+7n+9}{n^2+5n+1} = 3$$

Fie $\varepsilon > 0$, alegem $N_\varepsilon = \left\lfloor \frac{8}{\varepsilon} \right\rfloor + 1$.

fie $n \geq N_\varepsilon \Rightarrow \frac{1}{n} \leq \frac{1}{N_\varepsilon}$

$$\left| \frac{3n^2+7n+9}{n^2+5n+1} - 3 \right| = \left| \frac{\cancel{3n^2}+7n+9-\cancel{3n^2}-15n-3}{n^2+5n+1} \right| = \left| \frac{7n-15n+6}{n^2+5n+1} \right| = \left| \frac{-8n+6}{n^2+5n+1} \right| = \frac{8n-6}{n^2+5n+1} \leq \frac{8n}{n^2} = \frac{8}{n}$$

$$\leq \frac{8}{N_\varepsilon} < \varepsilon$$

$\lim_{n\to\infty} x_n = l \Leftrightarrow \forall \varepsilon > 0, \exists N_\varepsilon \in \mathbb{N}$ a.î. $\forall n \geq N_\varepsilon$, avem $|x_n - l| < \varepsilon$

$\lim_{n\to\infty} x_n = \infty \Leftrightarrow \forall \varepsilon > 0, \exists N_\varepsilon \in \mathbb{N}$ a.î. $\forall n \geq N_\varepsilon$, avem $x_n > \varepsilon$

$\lim_{n\to\infty} x_n = -\infty \Leftrightarrow \forall \varepsilon < 0, \exists N_\varepsilon \in \mathbb{N}$ a.î. $n \geq N_\varepsilon$, avem $x_n < \varepsilon$

Spunem că $(x_n)_n$ are limită dacă $\exists\, l \in \overline{\mathbb{R}}$ a.î.
$\lim\limits_{n\to\infty} x_n = l$

Spunem că $(x_n)_n$ conv dacă $l \in \mathbb{R}$

$(x_n)_n$ conv $\iff \exists\, l \in \mathbb{R}$ a.î. $\forall \varepsilon > 0,\ \exists N_\varepsilon \in \mathbb{N}$ a.î. $\forall n \geq N_\varepsilon$, avem $|x_n - l| < \varepsilon$

$(x_n)_n$ nu e conv $\iff \forall\, l \in \mathbb{R},\ \exists \varepsilon > 0$ a.î. $\forall N_\varepsilon \in \mathbb{N}$ a.î. $\exists n > N_\varepsilon$ cu $|x_n - l| \geq \varepsilon$

$\lim\limits_{n\to\infty} x_n = l \iff \forall (x_{k(n)})_n$ un subșir al lui $(x_n)_n$, $x_{k(n)} \xrightarrow{n\to\infty} l$

$\mathcal{L}((x_n)_n)$

$\lim\limits_{n\to\infty} x_n = l \iff \forall \varepsilon > 0,\ \exists N_\varepsilon \in \mathbb{N}$ a.î. $\forall n \geq N_\varepsilon$, avem $|x_n - l| < \varepsilon$

$\lim\limits_{n\to\infty} x_n = \infty \iff \forall \varepsilon > 0,\ \exists N_\varepsilon \in \mathbb{N}$ a.î. $\forall n \geq N_\varepsilon$, avem $x_n > \varepsilon$

$\lim\limits_{n\to\infty} x_n = -\infty \iff \forall \varepsilon < 0,\ \exists N_\varepsilon \in \mathbb{N}$ a.î. $n \geq N_\varepsilon$, avem $x_n < \varepsilon$

$a_n = (-1)^n$

$a_{2n} = 1 \xrightarrow{n\to\infty} 1$

$a_{2n+1} = -1 \xrightarrow{n\to\infty} -1$ $\Rightarrow$

$\Rightarrow (a_n)_n$ nu e conv

1) $a_1 \in (0,1)$, $a_{n+1} = a_n - a_n^3$ $n \geq 1$

$a_1 = \frac{1}{2} \in (0,1)$

$a_2 = a_1 - a_1^3 = \frac{1}{2} - \frac{1}{2^3} = \frac{4-1}{8} = \frac{3}{8} \in (0,1)$

$a_3 = a_2 - a_2^3 = \frac{3}{8} - \frac{27}{8^3} = \frac{3 \cdot 8^2 - 3^3}{8^3} = \frac{3(8^2 - 3^2)}{8^3} = \frac{165}{512} \in (0,1)$

$a_{n+1} - a_n = \cancel{a_n} - a_n^3 - \cancel{a_n} = -a_n^3 < 0$

Dem $a_n \in (0,1)$ $\forall n \in \mathbb{N}$

$n = 1 \Rightarrow a_1 \in (0,1)$ (A)

$n \to n+1$

$P(n)$: $a_n \in (0,1)$ pp adevărat

$P(n+1)$: $a_{n+1} \in (0,1)$ de dem

$a_{n+1} = a_n - a_n^3$

$a_{n+1} = a_n(1 - a_n^2)$

$a_n \in (0,1) \Rightarrow a_n^2 \in (0,1) \Rightarrow 1 - a_n^2 \in (0,1)$

$a_n \in (0,1)$ $\Big\} \Rightarrow (1 - a_n^2) a_n \in (0,1)$

dar $a_n(1 - a_n^2) = a_{n+1}$

$\Rightarrow P(n+1)$ adev $\forall n \in \mathbb{N}$

$\Rightarrow (a_n)_n \downarrow$

dar $(a_n)_n \in (0,1)$ $\xrightarrow{T.W.}$ $(a_n)_n$ convergent

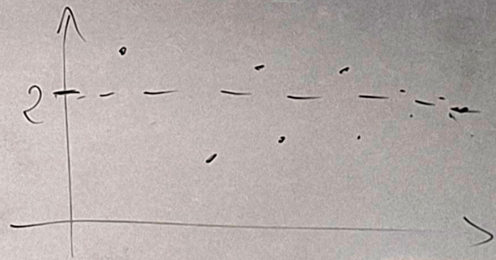
$\Rightarrow \exists\, l \in \mathbb{R}$ a.î. $\lim_{n \to \infty} a_n = l$

Dacă $\lim_{n \to \infty} a_n = l$

$\Rightarrow l = l - l^3 \mid -l$

$\Rightarrow -l^3 = 0 \Rightarrow l = 0$

$x_1=0,\ x_{n+1}=\sqrt{6-x_n}$



$x_1=0$

$x_2=\sqrt{6}$

$x_3=\sqrt{6-\sqrt{6}}$

$x_4=\sqrt{6-\sqrt{6-\sqrt{6}}}$

$x_5=\sqrt{6-\sqrt{6-\sqrt{6-\sqrt{6}}}}$

Dem că $(x_{2n+1})_n \nearrow$

Vrem $x_{2n+1}-x_{2n-1}=\sqrt{6-x_{2n}}-x_{2n-1}=$

$=\sqrt{6-\sqrt{6-x_{2n-1}}}-x_{2n-1}\geq 0 \iff \sqrt{6-\sqrt{6-x_{2n-1}}}\geq x_{2n-1} \iff 6-\sqrt{6-x_{2n-1}}\geq x_{2n-1}^2 \iff$

$\iff 6-x_{2n-1}^2\geq\sqrt{6-x_{2n-1}} \iff$

$\iff \left(6-x_{2n-1}^2\right)^2\geq 6-x_{2n-1}$

$x_{2n-1}^4-12x_{2n-1}^2+36\geq 6-x_{2n-1}$

$x_{2n-1}^4-12x_{2n-1}^2+x_{2n-1}+30\geq 0$

Dem. prin ind că $(x_{2n+1})_n\leq 2,\ \forall n\in\mathbb{N}$

$\underline{n=0}$: $x_{2\cdot 0+1}=x_1=0\leq 2$

$\underline{n\to n+1}$: Ştim că $x_{2n+1}\leq 2$ şi vrem $x_{2n+3}\leq 2$

$x_{2n+3}=\sqrt{6-\sqrt{6-x_{2n+1}}}$

$x_{2n+1}\leq 2 \Rightarrow 6-x_{2n+1}\geq 4 \Rightarrow \sqrt{6-x_{2n+1}}\geq 2 \Rightarrow 6-\sqrt{6-x_{2n+1}}\leq 6-2=4 \Rightarrow \underbrace{\sqrt{6-\sqrt{6-x_{2n+1}}}}_{= x_{2n+3}}\leq 2$

$f(x)=x^4-12x^2+x+30.$

$f(x)\geq 0\ \forall x\in[0,2].$