$v_1 = -3e_1 + 2e_2 + e_3$
$v_2 = -2e_1 + e_2$
$v_3 = -6e_1 + 3e_2 + e_3$

$$R_0 \xrightarrow{C} R \quad C = \begin{pmatrix} -3 & -2 & -6 \\ 2 & 1 & 3 \\ 1 & 0 & 1 \end{pmatrix}$$

$$A' = C^{-1} A C \Rightarrow A = C A' C^{-1}$$

Pt test: avem voie cu foaie A4 cu ce info. vrem noi

Seminar 8
Forme pătratice. Forme biliniare
Forma canonică
(Metoda Gauss / Jacobi)

ex1 $Q: \mathbb{R}^3 \to \mathbb{R}$, $Q(x) = x_1^2 + x_2^2 + x_3^2 + x_1x_2 + x_1x_3 + x_2x_3$

pg1 a) $G = ?$ matr. asoc în rap cu R_0

b) $g: \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}$ forma polară asoc

c) Q la o formă canonică (G/J)

Q poz def? Generalizare

sol $Q(x) = \sum_{i=1}^{3} g_{ii} x_i^2 + 2\sum_{i<j} g_{ij} x_i x_j$

a) $G = \begin{pmatrix} 1 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & 1 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & 1 \end{pmatrix}$

b) $g(x,y) = \frac{1}{2}\left(Q(x+y) - Q(x) - Q(y)\right)$

$G(x,y) = \sum_{i,j=1}^{3} g_{ij} x_i x_j$

$g(x,y) = x_1y_1 + \frac{1}{2}x_1y_2 + \frac{1}{2}x_1y_3$

$+ \frac{1}{2}x_2y_1 + x_2y_2 + \frac{1}{2}x_2y_3$

$+ \frac{1}{2}x_3y_1 + \frac{1}{2}x_3y_2 + x_3y_3$

c) Met I - Jacobi

$\Delta_1 = |1| = 1 \neq 0$

$\Delta_2 = \begin{vmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & 1 \end{vmatrix} = \frac{3}{4} \neq 0$

$\Delta_3 = \det G = 2\begin{vmatrix} 1 & \frac{1}{2} & 1 \\ \frac{1}{2} & 1 & 1 \\ \frac{1}{2} & \frac{1}{2} & 1 \end{vmatrix} = \frac{1}{2} \neq 0$

Merge Jacobi

$\exists$ reper în $\mathbb{R}^3$ a.î. $Q(x) = \frac{1}{\Delta_1} x_1'^2 + \frac{\Delta_1}{\Delta_2} x_2'^2 + \frac{\Delta_2}{\Delta_3} x_3'^2$

$= 1 x_1'^2 + \frac{4}{3} x_2'^2 + \frac{3}{2} x_3'^2$

$\Downarrow$

$(3,0)$ signatura $\Rightarrow$ ~~poz~~ definită

Met II : Gauss

Ne ducem la Q, subl. toți termenii lui x_1 și facem un pp

$$Q(x) = \underline{x_1^2} + x_2^2 + x_3^2 + \underline{x_1x_2} + \underline{x_1x_3} + x_2x_3$$

$$Q(x) = \left(x_1 + \frac{1}{2}x_2 + \frac{1}{2}x_3\right)^2 \underbrace{- \left(\frac{1}{4}x_2^2\right) - \left(\frac{1}{4}x_3^2\right) - \frac{1}{2}x_2x_3 + x_2^2 + x_3^2 + x_2x_3}_{\underline{\frac{3}{4}x_2^2} + \frac{3}{4}x_3^2 + \underline{\frac{1}{2}x_2x_3}}$$

$$= \left(x_1 + \frac{1}{2}x_2 + \frac{1}{2}x_3\right)^2 + \underbrace{\frac{3}{4}\left(x_2^2 + \frac{2}{3}x_2x_3\right) + \frac{3}{4}x^3}_{\frac{3}{4}\left(x_2 + \frac{1}{3}x_3\right)^2 - \frac{1}{12}x_3^2 + \frac{9}{12}x_3^2}$$

$$Q(x) = \left(x_1 + \frac{1}{2}x_2 + \frac{1}{2}x_3\right)^2 + \frac{3}{4}\left(x_2 + \frac{1}{3}x_3\right)^2 + \frac{2}{3}x_3^2$$

Fie schimbarea de reper:

$$\begin{cases} y_1 = x_1 + \frac{1}{2}x_2 + \frac{1}{2}x_3 \\ y_2 = x_2 + \frac{1}{3}x_3 \\ y_3 = x_3 \end{cases}$$

13iunie $Q(x) = y_1^2 + \frac{3}{4} y_2^2 + \frac{2}{3} y_3^2$

signatura $(3,0)$

ex2 $Q: \mathbb{R}^3 \to \mathbb{R}$ $\quad Q(x) = 2x_1x_2 - 6x_1x_3 - 6x_2x_3$

pg1 Să se aducă la o formă canonică (G | J)

Preciz. signatura.

! Obs

$$Q(x) = x_1^2 + x_2^2 + x_3^2 + x_1x_2 + x_1x_3 + x_2x_3$$

signatura: $(3,0)$

$Q: \mathbb{R}^n \to \mathbb{R} \Rightarrow$

$$Q(x) = \sum_{i=1}^{n} x_i^2 + \sum_{i<j} x_i x_j \Rightarrow$$

signatura $(n,0)$

$$G = \begin{pmatrix} 0 & 1 & -3 \\ 1 & 0 & -3 \\ -3 & -3 & 0 \end{pmatrix}$$

• $g_{12} \neq 0$

• $\begin{cases} y_1 = x_1 + x_2 \\ y_2 = x_1 - x_2 \\ y_3 = x_3 \end{cases} \Rightarrow \begin{cases} x_1 = \frac{1}{2}(y_1 + y_2) \\ x_2 = \frac{1}{2}(y_1 - y_2) \\ x_3 = y_3 \end{cases}$

$$Q(x) = \frac{1}{2}(y_1^2 - y_2^2) - 6 y_3 y_1$$

$$= \frac{1}{2}(y_1^2 - 12 y_3 y_1) - \frac{1}{2} y_2^2$$

$$= \frac{1}{2}(y_1 - 6y_3)^2 - 18 y_3^2 - \frac{1}{2} y_2^2$$

$z_1 = y_1 - 6y_3$

$z_2 = y_2$

$z_3 = y_3$

$$\Rightarrow Q(x) = \frac{1}{2} z_1^2 - \frac{1}{2} z_2^2 - 18 z_3^2$$

signatura $(1,2) \Rightarrow Q$ nu e poz definit.

Exercițiu (extindere)

$\ker g = ?$ g - forma polară asoc lui Q

$\ker g = \{ x \in \mathbb{R}^3 \mid g(x,y) = 0 \ \forall y \in \mathbb{R}^3 \}$

g sm nedegenerată $\Leftrightarrow \ker g = \{ 0_{\mathbb{R}^3} \} \Leftrightarrow$ $\det G \neq 0$

$$\det G = \begin{vmatrix} 0 & 1 & -3 \\ 1 & 0 & -3 \\ -3 & -3 & 0 \end{vmatrix} \overset{C_3 + = 3C_2}{=} \begin{vmatrix} 0 & 1 & 0 \\ 1 & 0 & -3 \\ -3 & -3 & -9 \end{vmatrix}$$

$$= 1 \cdot (-1)^{1+2} \begin{vmatrix} 1 & -3 \\ -3 & -9 \end{vmatrix} \neq 0$$

$\Rightarrow g$ e degenerată

$$\begin{matrix} g(x, e_1) = 0 \\ g(x, e_2) = 0 \\ g(x, e_3) = 0 \end{matrix} \Rightarrow \begin{cases} x_2 - 3x_3 = 0 \\ x_1 - 3x_3 = 0 \\ -3x_1 - 3x_2 = 0 \end{cases}$$

$\dim \ker g = 3 - \operatorname{rg} G = 3 - 2 = 1$

$$g(x,y) = x_1 y_2 - 3x_1 y_3 + x_2 y_1 - 3x_2 y_3$$
$$- 3x_3 y_1 - 3x_3 y_2$$