CURS#8

11. Metode de ortogonalizare:
    (v) metoda Gram-Schimdt modificată;
    (vi) metoda reflexiilor (Householder);

PROBLEME

1) Cazul particular al problemei celor mai mici pătrate (PCMMP) în care $n = 1$ şi $m > n$ se formulează astfel:

   Daţi fiind vectorii $\mathbf{a}, \mathbf{b} \in \mathbb{R}^m$ (în fapt, $\mathbf{a} \in \mathscr{M}_{m,1}(\mathbb{R}) \equiv \mathbb{R}^m$), determinaţi $\widehat{x} \in \mathbb{R}$ astfel încât

$$\widehat{x} = \arg\min_{x\in\mathbb{R}} \frac{1}{2} \|\mathbf{a}\, x - \mathbf{b}\|_2^2 \,. \tag{1}$$

   Presupunând că $\mathbf{a}, \mathbf{b} \in \mathbb{R}^m \setminus \{\mathbf{0}_m\}$, rezolvaţi PCMMP (1) folosind sistemul de ecuaţii normale asociat şi arătaţi că norma vectorului eroare reziduală este dată de

$$\|\mathbf{r}\|_2^2 \coloneqq \|\mathbf{b} - \mathbf{a}\,\widehat{x}\|_2^2 = \|\mathbf{b}\|_2^2 \sin^2\theta \,, \tag{2}$$

   unde $\theta \in [0, \pi]$ este unghiul făcut de vectorii $\mathbf{a}$ şi $\mathbf{b}$.

   Care este interpretarea geometrică a relaţiei (2)?

2) Fie $\mathbf{A} \in \mathscr{M}_{m,n}(\mathbb{R})$, $m \geq n$ şi $\operatorname{rang}(\mathbf{A}) = n$.
   (i) Arătaţi că matricea $\mathbf{A}^\mathsf{T}\mathbf{A}$ este pozitiv definită.
   (ii) Arătaţi că matricea sistemului augmentat nu poate fi pozitiv definită.

3) Considerăm următoarea partiţie a matricei ortogonale $\mathbf{Q} \in \mathscr{M}_{m+n}(\mathbb{R})$

$$\mathbf{Q} = \begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{0}_{n,m} & \mathbf{C} \end{bmatrix}, \tag{3}$$

   unde $\mathbf{A} \in \mathscr{M}_m(\mathbb{R})$, $\mathbf{B} \in \mathscr{M}_{m,n}(\mathbb{R})$, $\mathbf{0}_{n,m} \in \mathscr{M}_{n,m}(\mathbb{R})$ şi $\mathbf{C} \in \mathscr{M}_n(\mathbb{R})$.

   Arătaţi că matricele $\mathbf{A}$ şi $\mathbf{C}$ sunt ortogonale, iar $\mathbf{B} = \mathbf{0}_{m,n}$.

4) Fie vectorul $\mathbf{a} \in \mathbb{R}^m$, $m \geq 1$, considerat ca o matrice, i.e. $\mathbb{R}^m \equiv \mathscr{M}_{m,1}(\mathbb{R})$.
   (i) Scrieţi factorizarea QR a lui $\mathbf{a} \in \mathbb{R}^m \equiv \mathscr{M}_{m,1}(\mathbb{R})$ punând în evidenţă, în mod explicit, matricele $\mathbf{Q}$ şi $\mathbf{R}$.

(ii) Care este soluţia în sensul celor mai mici pătrate a problemei

$$\mathbf{a}\,\mathbf{x} = \mathbf{b}\,, \tag{4}$$

unde $\mathbf{a} \in \mathbb{R}^m \equiv \mathscr{M}_{m,1}(\mathbb{R})$ şi $\mathbf{b} \in \mathbb{R}^m$ sunt date.

5) Demonstraţi unicitatea factorizării QR a matricei $\mathbf{A} \in \mathscr{M}_{m,n}(\mathbb{R})$, unde $m \geq n$ şi $\operatorname{rang}(\mathbf{A}) = n$, pornind de la construcţia factorizării QR a acesteia prin metoda Gram-Schmidt modificată.

6) Fie $\mathbf{A} \coloneqq [\mathbf{a}_1\ \mathbf{a}_2\ \ldots\ \mathbf{a}_n] \in \mathscr{M}_{m,n}(\mathbb{R})$, $m \geq n$ şi $\operatorname{rang}(\mathbf{A}) = n$, unde $\mathbf{a}_k \coloneqq \big(a_{ik}\big)_{i=\overline{1,m}} \in \mathbb{R}^m$, $k = \overline{1,n}$. Considerăm *factorizarea QR* a matricei $\mathbf{A}$, i.e. $\mathbf{A} = \mathbf{Q}\,\mathbf{R}$, unde $\mathbf{Q} \coloneqq [\mathbf{q}_1\ \mathbf{q}_2\ \ldots\ \mathbf{q}_n] \in \mathscr{M}_{m,n}(\mathbb{R})$, $m \geq n$, $\mathbf{Q}^\mathsf{T}\mathbf{Q} = \mathbf{I}_n$ şi $\mathbf{q}_k \coloneqq \big(q_{ik}\big)_{i=\overline{1,m}} \in \mathbb{R}^m$, $k = \overline{1,n}$, iar $\mathbf{R} = \big(r_{ij}\big)_{i,j=\overline{1,n}} \in \mathscr{M}_n(\mathbb{R})$ este o matrice superior triunghiulară cu $r_{kk} > 0$, $k = \overline{1,n}$.

În cazul metodelor Gram-Schmidt clasică/standard, respectiv modificată, definim matricele proiecţie ortogonală $\mathbf{Q}_k \coloneqq \mathbf{q}_k\mathbf{q}_k^\mathsf{T} \in \mathscr{M}_m(\mathbb{R})$, $k = \overline{0, n-1}$, cu convenţia $\mathbf{Q}_0 \equiv \mathbf{0} \in \mathscr{M}_m(\mathbb{R})$.

(i) Arătaţi că metoda Gram-Schmidt clasică/standard este echivalentă cu

$$\mathbf{q}_k\, r_{kk} = \Big[\mathbf{I}_m - \big(\mathbf{Q}_1 + \mathbf{Q}_2 + \ldots + \mathbf{Q}_{k-1}\big)\Big]\, \mathbf{a}_k\,, \quad k = \overline{1,n}\,. \tag{5}$$

(ii) Arătaţi că metoda Gram-Schmidt modificată este echivalentă cu

$$\mathbf{q}_k\, r_{kk} = \big(\mathbf{I}_m - \mathbf{Q}_{k-1}\big)\big(\mathbf{I}_m - \mathbf{Q}_{k-2}\big)\ldots\big(\mathbf{I}_m - \mathbf{Q}_1\big)\mathbf{a}_k\,, \quad k = \overline{1,n}\,. \tag{6}$$

(iii) Metoda Gram-Schmidt clasică/standard se poate stabiliza prin aplicarea sa în mod repetat, i.e. prin aşa-numita *rafinare iterativă a metodei Gram-Schmidt clasice/standard*

$$\mathbf{q}_k\, r_{kk} = \Big[\mathbf{I}_m - \big(\mathbf{Q}_1 + \mathbf{Q}_2 + \ldots + \mathbf{Q}_{k-1}\big)\Big]^\ell\, \mathbf{a}_k\,, \quad k = \overline{1,n}\,, \tag{7}$$

unde $\ell \in \mathbb{N}$ şi $\ell \geq 2$, cu menţiunea că $\ell = 2$ este, de regulă, suficient.

Arătaţi că, din punct de vedere matematic, cele trei formulări ale metodei Gram-Schmidt (5)–(7) sunt echivalente.

3.2. FACTORIZAREA QR: METODA GRAM-SCHMIDT MODIFICATĂ

Fie $A \in \mathcal{M}_{m,n}(\mathbb{R})$, $m \geq n$, inversabilă la stânga ($\Leftrightarrow$ rang $A = n$).

Vrem să construim matricele Q și R date de factorizarea QR a lui A folosind <u>metoda Gram-Schmidt modificată</u>.

① $$A = [a_1 \,|\, \underbrace{a_2 \dots a_n}_{=:A_2}] = [a_1 \,|\, A_2] \in \mathcal{M}_{m,n}(\mathbb{R})$$

$$a_k := (a_{ik})_{i=\overline{1,m}} \in \mathbb{R}^m \equiv \mathcal{M}_{m,1}(\mathbb{R}), \ k=\overline{1,n}$$

$$A_2 := [a_2 \dots a_n] \in \mathcal{M}_{m,n-1}(\mathbb{R})$$

② $$Q = [q_1 \,|\, \underbrace{q_2 \dots q_n}_{=:Q_2}] = [q_1 \,|\, Q_2] \in \mathcal{M}_{m,n}(\mathbb{R})$$

$$q_k = (q_{ik})_{i=\overline{1,m}} \in \mathbb{R}^m \equiv \mathcal{M}_{m,1}(\mathbb{R}), \ k=\overline{1,n}$$

$$Q_2 := [q_2 \dots q_n] \in \mathcal{M}_{m,n-1}(\mathbb{R})$$

$$I_n = Q^T Q = \begin{bmatrix} \underline{q}_1^T \\ Q_2^T \end{bmatrix} \begin{bmatrix} \underline{q}_1 & Q_2 \end{bmatrix}$$

$$= \begin{bmatrix} \underline{q}_1^T \underline{q}_1 & \underline{q}_1^T Q_2 \\ Q_2^T \underline{q}_1 & Q_2^T Q_2 \end{bmatrix} \Longrightarrow$$

$$\boxed{\begin{aligned} &\underline{q}_1^T \underline{q}_1 = 1 \\ &\underline{q}_1^T Q_2 = [\underline{q}_1^T \underline{q}_2 \cdots \underline{q}_1^T \underline{q}_n] = \underline{0}_{n-1}^T \\ &Q_2^T Q_2 = I_{n-1} \end{aligned}}$$

③

$$R = \begin{bmatrix} r_{11} & \underline{R}_{12}^T \\ \underline{0}_{n-1} & R_{22} \end{bmatrix} \in \mathcal{M}_n(\mathbb{R}) \, ; \; r_{kk} > 0, \, k = \overline{1,n}$$

$r_{11} > 0$

$\underline{R}_{12}^T = [r_{12} \cdots r_{1n}] \in \mathcal{M}_{1,n-1}(\mathbb{R})$

$R_{22} = (r_{ij})_{i,j=\overline{2,n}} \in \mathcal{M}_{n-1}(\mathbb{R})$

④ Folosind partiționările de la ①-③, vrem să rezolvăm $A=QR$ pentru a obține matricele Q și R:

$$[\underline{a}_1 \; A_2] = [\underline{q}_1 \; Q_2] \begin{bmatrix} r_{11} & \underline{R}_{12}^T \\ \underline{0}_{n-1} & R_{22} \end{bmatrix}$$

$$= [\underline{q}_1 r_{11} \quad \underline{q}_1 \underline{R}_{12}^T + Q_2 R_{22}] \Rightarrow$$

$$\begin{cases} \underline{a}_1 = \underline{q}_1 r_{11} \\ A_2 = \underline{q}_1 \underline{R}_{12}^T + Q_2 R_{22} \end{cases}$$

Ecuația #1: $\boxed{\underline{a}_1 = \underline{q}_1 r_{11}}$

$$\underline{a}_1^T \underline{a}_1 = (\underline{q}_1 r_{11})^T (\underline{q}_1 r_{11}) = r_{11} (\underline{q}_1^T \underline{q}_1) r_{11}$$

$$= r_{11}^2 \Rightarrow r_{11} = \pm \|\underline{a}_1\|_2$$

Cum $r_{kk} > 0$, $k=\overline{1,n}$, rezultă:

$$\boxed{r_{11} = \|\underline{a}_1\|_2} \quad (1)$$

$$\boxed{\underline{q}_1 = \underline{a}_1 / \|\underline{a}_1\|_2} \quad (2)$$

$\underline{\text{Ecuația \#2}}$: $\boxed{A_2 = \underline{q}_1 \underline{R}^T_{12} + Q_2 R_{22}}$

$$\underline{q}_1^T A_2 = \underline{q}_1^T \left(\underline{q}_1 \underline{R}^T_{12} + Q_2 R_{22} \right)$$

$$= \underbrace{\left(\underline{q}_1^T \underline{q}_1\right)}_{=1} \underline{R}^T_{12} + \underbrace{\left(\underline{q}_1^T Q_2\right)}_{= \underline{O}^T_{n-1}} R_{22} \Rightarrow$$

$$\boxed{\underline{R}^T_{12} = \underline{q}_1^T A_2} \quad \text{i.e.} \quad \boxed{r_{1j} = \underline{q}_1^T \underline{a}_j, j = \overline{2,n}} \quad (3)$$

Prin urmare, Ecuația #2 se reduce la:

$$\boxed{A_2 - \underline{q}_1 \underline{R}^T_{12} = Q_2 R_{22}} \qquad (4)$$

unde:

- $Q_2 := \left[\underline{q}_2 \ \underline{q}_3 \cdots \underline{q}_n\right] \in \mathcal{M}_{m,n-1}(\mathbb{R})$

  $Q_2^T Q_2 = I_{n-1}$ (ortogonală);

- $R_{22} = (r_{ij})_{i,j=\overline{2,n}} \in \mathcal{M}_{n-1}(\mathbb{R})$

  superior triunghiulară cu

  $r_{kk} > 0, k = \overline{2,n}$

OBSERVAȚII:

1) În relația (2) trebuie ca $\|\underline{a}_1\|_2 \neq 0$ $\Leftrightarrow$ $\underline{a}_1 \neq \underline{0}_m$ ceea ce este o consecință a faptului că rang $A = n$. ✓

2) Relația (4) reprezintă factorizarea QR a matricei $A_2 - \underline{q}_1 \underline{R}_{12}^T \in \mathcal{M}_{m,n-1}(\mathbb{R})$ și trebuie demonstrat că <u>există</u>!

Este suficient să demonstrăm că $A_2 - \underline{q}_1 \underline{R}_{12}^T \in \mathcal{M}_{m,n-1}(\mathbb{R})$ este <u>inversabilă la stânga</u>.

—//—

Fie $\underline{x} \in \mathbb{R}^{n-1}$ a.î.

$$\left(A_2 - \underline{q}_1 \underline{R}_{12}^T\right) \underline{x} = \underline{0}_m$$

$$\left(A_2 - \frac{1}{r_{11}} \underline{a}_1 \underline{R}_{12}^T\right) \underline{x} = \underline{0}_m$$

$$\underline{a}_1 \left(-\frac{1}{r_{11}} \underline{R}_{12}^T \underline{x}\right) + A_2 \underline{x} = \underline{0}_m$$

$$\underbrace{\begin{bmatrix} \underline{a}_1 & A_2 \end{bmatrix}}_{= A \in \mathcal{M}_{m,n}(\mathbb{R})} \underbrace{\begin{bmatrix} -\frac{1}{r_{11}} \underline{R}_{12}^T \underline{x} \\ \underline{x} \end{bmatrix}}_{=: \underline{y} \in \mathbb{R}^n} = \underline{0}_m \implies$$

$$A \underline{y} = \underline{0}_m$$

Cum A este inversabilă la stânga, rezultă $\underline{y} = \underline{0}_n$, i.e. $\underline{x} = \underline{0}_{n-1}$.

Prin urmare, rezultă că matricea $(A_2 - \underline{a}_1 \underline{R}_{12}^T) \in \mathcal{M}_{m,n-1}(\mathbb{R})$ este inversabilă la stânga. $\square$

3) Factorizarea se poate face pentru $\underline{a} := A = \mathcal{M}_{m,1}(\mathbb{R}) \equiv \mathbb{R}^m$, i.e. pentru $n=1$.

$$\underline{a} = \underline{Q} R : \begin{cases} \underline{Q} \in \mathcal{M}_{m,1}(\mathbb{R}) \equiv \mathbb{R}^m, & \underline{Q}^T \underline{Q} = 1 \\ R \in \mathcal{M}_1(\mathbb{R}) \equiv \mathbb{R}, & R > 0 \end{cases}$$

$$\|\underline{a}\|_2^2 = \underline{a}^T \underline{a} = (\underline{Q} R)^T (\underline{Q} R) = R^2 \implies$$

Luăm $R = \|\underline{a}\|_2$ și $\underline{Q} = \underline{a} / \|\underline{a}\|_2$ $\square$

ALGORITM (Metoda Gram-Schmidt modificată):

Date: $A = [\underline{a}_1\ \underline{a}_2 \dots \underline{a}_n] \in \mathcal{M}_{m,n}(\mathbb{R})$

$m \geq n$, rang $A = n$

$\underline{a}_k = (a_{ik})_{i=\overline{1,m}} \in \mathbb{R}^m$, $k = \overline{1,n}$

① Definește $A = [\underline{a}_1\ A_2] \in \mathcal{M}_{m,n}(\mathbb{R})$

$$\begin{cases} \underline{a}_1 \in \mathbb{R}^m \\ A_2 = [\underline{a}_2\ \underline{a}_3 \dots \underline{a}_n] \in \mathcal{M}_{m,n-1}(\mathbb{R}) \end{cases}$$

② Definește $Q = [\underline{q}_1\ Q_2] \in \mathcal{M}_{m,n}(\mathbb{R})$

$$\begin{cases} \underline{q}_1 \in \mathbb{R}^m \\ Q_2 = [\underline{q}_2\ \underline{q}_3 \dots \underline{q}_n] \in \mathcal{M}_{m,n-1}(\mathbb{R}) \end{cases}$$

③ Definește

$$R = \begin{bmatrix} r_{11} & \underline{R}_{12}^T \\ \underline{O}_{n-1} & R_{22} \end{bmatrix} \in \mathcal{M}_n(\mathbb{R})$$

$$\begin{cases} r_{11} > 0 \\ \underline{R}_{12}^T = [r_{12}\ r_{13} \dots r_{1n}] \in \mathcal{M}_{1,n-1}(\mathbb{R}) \\ R_{22} = (r_{ij})_{i,j=\overline{2,n}} \in \mathcal{M}_{n-1}(\mathbb{R}) \end{cases}$$

④ Calculează

$$r_{11} = \|\underline{a}_1\|_2 = (\underline{a}_1^T \underline{a}_1)^{1/2}$$

$$\underline{q}_1 = \frac{\underline{a}_1}{r_{11}} = \frac{\underline{a}_1}{\|\underline{a}_1\|_2}$$

$$\underline{R}_{12}^T = \underline{q}_1^T A_2$$

⑤ Calculează

$$A := A_2 - \underline{q}_1 \underline{R}_{12}^T \in \mathcal{M}_{m,n-1}(\mathbb{R})$$

matrice inversabilă la stânga și repetă pașii ① - ④.

<u>OBSERVAȚII:</u>

1) În cazul <u>metodei Gram-Schmidt clasice / standard</u>, la pasul $k=\overline{1,n}$ al algoritmului se determină <u>coloana $k=\overline{1,n}$ a matricii R</u>.

2) În cazul <u>metodei Gram-Schmidt modificate</u>, la pasul $k=\overline{1,n}$ al algoritmului se det. <u>linia $k=\overline{1,n}$ a lui R</u>.

3.3. METODA REFLEXIILOR (HOUSEHOLDER)

DEFINIȚIE:

Fie matricea / transformarea Householder asociată vectorului $\underline{v} \in \mathbb{R}^m \setminus \{\underline{0}_m\}$

$$\boxed{H_{\underline{v}} := H(\underline{v}) = I_m - 2\,\frac{\underline{v}\,\underline{v}^T}{\underline{v}^T\underline{v}} \in \mathcal{M}_m(\mathbb{R})}$$

OBSERVAȚII:

1) În definiția matricei / transformării Householder, se poate considera un vector unitar, i.e.

$$\tilde{\underline{v}} := \frac{\underline{v}}{\|\underline{v}\|_2} \in \mathbb{R}^m, \quad \|\tilde{\underline{v}}\|_2 = 1$$

$$H_{\tilde{\underline{v}}} := H(\tilde{\underline{v}}) = I_m - 2\,\frac{\underline{v}}{\|\underline{v}\|_2}\left(\frac{\underline{v}}{\|\underline{v}\|}\right)^T$$
$$= I_m - 2\,\tilde{\underline{v}}\,\tilde{\underline{v}}^T \in \mathcal{M}_m(\mathbb{R})$$

2) $\forall \underline{x} \in \mathbb{R}^m$, $H_{\underline{v}}\underline{x}$, $\underline{x}$ și $\underline{v}$ coplanari.

$\forall \underline{x} \in \mathbb{R}^m$

$$H_{\underline{v}}\underline{x} = \left(I_m - 2\frac{\underline{v}\,\underline{v}^T}{\underline{v}^T\underline{v}}\right)\underline{x}$$

$$= \underline{x} - 2\underline{v}\left(\frac{\underline{v}^T\underline{x}}{\underline{v}^T\underline{v}}\right) \quad \square$$

3) Hiperplanul de normală $\underline{v} \in \mathbb{R}^m \setminus \{\underline{0}_m\}$

$$\underline{v}^\perp := \{\underline{x} \in \mathbb{R}^m \mid \underline{v}^T\underline{x} = 0\} \subset \mathbb{R}^m$$

este invariant în raport cu matricea / transformarea Householder asociată vectorului $\underline{v}$, ie

$$\boxed{H_{\underline{v}}\underline{x} \in \underline{v}^\perp, \ \forall \underline{x} \in \underline{v}^\perp}$$

Fie $\underline{x} \in \underline{v}^\perp \Rightarrow \underline{v}^T\underline{x} = 0 \Rightarrow$

$$H_{\underline{v}}\underline{x} = \left(I_m - 2\frac{\underline{v}\,\underline{v}^T}{\underline{v}^T\underline{v}}\right)\underline{x}$$

$$= \underline{x} - 2\underline{v}\,\frac{\underline{v}^T\underline{x}}{\underline{v}^T\underline{v}} \underset{\underline{v}^T\underline{x}=0}{=} \underline{x} \in \underline{v}^{\perp} \qquad \square$$

4) $H_{\underline{v}}\underline{x}$ este reflexia lui $\underline{x} \in \mathbb{R}^m$ față de hiperplanul de normală $\underline{v} \in \mathbb{R}^m \setminus \{\underline{0}_m\}$, $\underline{v}^{\perp}$, ie

$$\boxed{\underline{v}^T(H_{\underline{v}}\underline{x}) = -\underline{v}^T\underline{x}, \quad \forall \underline{x} \in \mathbb{R}^m}$$

$\forall\, \underline{x} \in \mathbb{R}^m$,

$$\underline{v}^T(H_{\underline{v}}\underline{x}) = \underline{v}^T\left[\left(I_m - 2\frac{\underline{v}\,\underline{v}^T}{\underline{v}^T\underline{v}}\right)\underline{x}\right]$$

$$= \underline{v}^T\left[\underline{x} - \frac{2}{\underline{v}^T\underline{v}}(\underline{v}\,\underline{v}^T)\underline{x}\right]$$

$$= \underline{v}^T\underline{x} - \frac{2}{\underline{v}^T\underline{v}}\,\underbrace{\underline{v}^T(\underline{v}\,\underline{v}^T)\underline{x}}_{=(\underline{v}^T\underline{v})(\underline{v}^T\underline{x})}$$

$$= \underline{v}^T\underline{x} - 2\,\underline{v}^T\underline{x} = -\underline{v}^T\underline{x} \qquad \square$$

5) Fie $\underline{v} \in \mathbb{R}^m \setminus \{\underline{0}_m\}$.

Definim <u>vectorul unitar</u> asociat:

$$\underline{w} := \frac{\underline{v}}{\|\underline{v}\|} \in \mathbb{R}^m \setminus \{\underline{0}_m\}$$

$$\|\underline{w}\| = 1$$

și matricea/transformarea Householder asociată lui $\underline{w}$:

$$\boxed{H_{\underline{w}} := I_m - 2\,\underline{w}\,\underline{w}^T \in \mathcal{M}_m(\mathbb{R})}$$

(i) Construim

$$(\underline{w}\,\underline{w}^T)\,\underline{x} = \underline{w}\,(\underline{w}^T\underline{x}) = \mathrm{pr}_{\underline{w}}\,\underline{x}$$

(ii) Construim

$$(I_m - \underline{w}\,\underline{w}^T)\,\underline{x} = \underline{x} - (\underline{w}\,\underline{w}^T)\,\underline{x}$$

$$= \underline{x} - \underline{w}\,(\underline{w}^T\underline{x}) \in \underline{w}^{\perp}$$

(iii) Construim

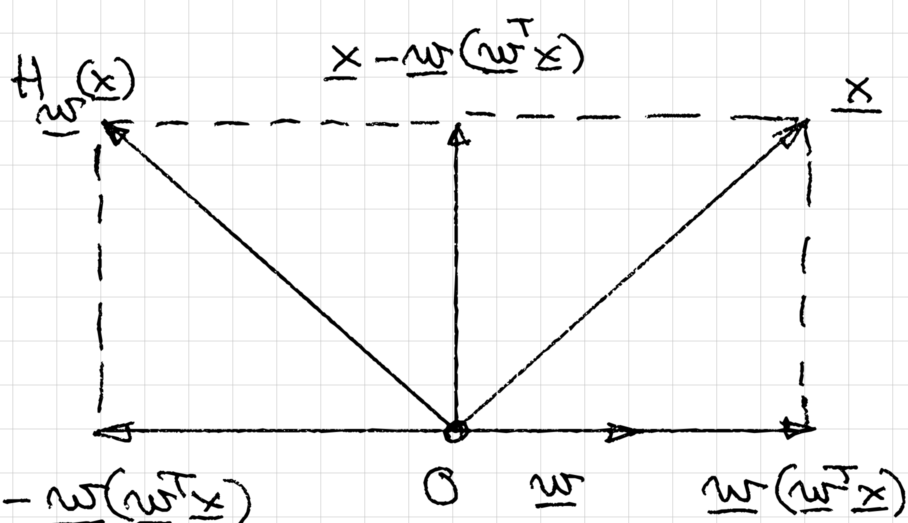
$$-(\underline{w}\,\underline{w}^T)\,\underline{x} = -\underline{w}\,(\underline{w}^T\underline{x}) = -pr_{\underline{w}}\,\underline{x}$$

(iv) Construim

$$(I_m - \underline{w}\,\underline{w}^T)\,\underline{x} + \left[-\underline{w}\,(\underline{w}^T\underline{x})\right] =$$

$$\left[\underline{x} - \underline{w}(\underline{w}^T\underline{x})\right] + \left[-\underline{w}\,(\underline{w}^T\underline{x})\right] = H_{\underline{w}}(\underline{x})$$

<u>Interpretare geometrică</u>:



LEMA #1:

Matricea/transformarea Householder asociată vectorului $\underline{v} \in \mathbb{R}^m \setminus \{\underline{0}_m\}$ este simetrică și ortogonală.

Dem:

$$H_{\underline{v}} := I_m - 2\frac{\underline{v}\,\underline{v}^T}{\underline{v}^T\underline{v}} \Rightarrow$$

- $$H_{\underline{v}}^T = \left(I_m - 2\frac{\underline{v}\,\underline{v}^T}{\underline{v}^T\underline{v}}\right)^T = I_m - 2\frac{(\underline{v}\,\underline{v}^T)^T}{\underline{v}^T\underline{v}}$$

$$= I_m - 2\frac{\underline{v}\,\underline{v}^T}{\underline{v}^T\underline{v}} = H_{\underline{v}} \Rightarrow \text{simetrie}$$

- $$H_{\underline{v}}^T H_{\underline{v}} = \left(I_m - 2\frac{\underline{v}\,\underline{v}^T}{\underline{v}^T\underline{v}}\right)\left(I_m - 2\frac{\underline{v}\,\underline{v}^T}{\underline{v}^T\underline{v}}\right)$$

$$= I_m - 2\frac{\underline{v}\,\underline{v}^T}{\underline{v}^T\underline{v}} - 2\frac{\underline{v}\,\underline{v}^T}{\underline{v}^T\underline{v}} + 4\frac{\underline{v}\,\underline{v}^T}{\underline{v}^T\underline{v}}\,\frac{\underline{v}\,\underline{v}^T}{\underline{v}^T\underline{v}}$$

$$= I_m - 4\frac{\underline{v}\,\underline{v}^T}{\underline{v}^T\underline{v}} + 4\frac{\underline{v}(\underline{v}^T\underline{v})\underline{v}^T}{(\underline{v}^T\underline{v})^2} = I_m \Rightarrow \text{ortog}$$

$\square$

LEMA#2 :

Fie $1 \leq k \leq m$ și $H_k \in \mathcal{M}_k(\mathbb{R})$ o matrice / transformare Householder.

Atunci

$$H := \begin{bmatrix} I_{m-k} & 0_{m-k,k} \\ 0_{k,m-k} & H_k \end{bmatrix} \in \mathcal{M}_m(\mathbb{R})$$

este matrice / transformare Householder.

Dem :

$H_k \in \mathcal{M}_k(\mathbb{R})$ matrice Householder $\Rightarrow$

$\exists\, \underline{v}_k \in \mathbb{R}^k \setminus \{\underline{0}_k\} : H_k = I_k - 2\dfrac{\underline{v}_k \underline{v}_k^T}{\underline{v}_k^T \underline{v}_k}$

Fie $\underline{v} = \begin{bmatrix} \underline{0}_{m-k} \\ \underline{v}_k \end{bmatrix} \in \mathbb{R}^m \Rightarrow \underline{v} \neq \underline{0}_m$.

$$\underline{v}^T \underline{v} = \begin{bmatrix} \underline{0}^T_{m-k} & \underline{v}_k^T \end{bmatrix} \begin{bmatrix} \underline{0}_{m-k} \\ \underline{v}_k \end{bmatrix}$$

$$= \underline{0}^T_{m-k} \underline{0}_{m-k} + \underline{v}_k^T \underline{v}_k = \underline{v}_k^T \underline{v}_k$$

$$\underline{v}\,\underline{v}^T = \begin{bmatrix} \underline{0}_{m-k} \\ \underline{v}_k \end{bmatrix} \begin{bmatrix} \underline{0}_{m-k}^T & \underline{v}_k^T \end{bmatrix}$$

$$= \begin{bmatrix} \underline{0}_{m-k}\,\underline{0}_{m-k}^T & \underline{0}_{m-k}\,\underline{v}_k^T \\ \underline{v}_k\,\underline{0}_{m-k}^T & \underline{v}_k\,\underline{v}_k^T \end{bmatrix}$$

$$= \begin{bmatrix} 0_{m-k} & 0_{m-k,k} \\ 0_{k,m-k} & \underline{v}_k\underline{v}_k^T \end{bmatrix}$$

$$H = I_m - 2\,\frac{\underline{v}\,\underline{v}^T}{\underline{v}^T\underline{v}} =$$

$$= \begin{bmatrix} I_{m-k} & 0_{m-k,k} \\ 0_{k,m-k} & I_k \end{bmatrix} - 2 \begin{bmatrix} 0_{m-k} & 0_{m-k,k} \\ 0_{k,m-k} & \dfrac{\underline{v}_k\underline{v}_k^T}{\underline{v}_k^T\underline{v}_k} \end{bmatrix}$$

$$= \begin{bmatrix} I_{m-k} & 0_{m-k,k} \\ 0_{k,m-k} & H_k \end{bmatrix}$$

$\square$

LEMA #3:

$\forall\, \underline{x} \in \mathbb{R}^m \setminus \{\underline{0}_m\}$, $\exists\, H \in \mathcal{M}_m(\mathbb{R})$ matrice Householder a.î.

$$H\underline{x} = \alpha\, \underline{e}_1$$

$$\alpha \in \mathbb{R}^* \quad \text{și} \quad \underline{e}_1 = (1\ 0 \dots 0)^T \in \mathbb{R}^m$$

Interpretare geometrică:

Pentru orice vector nenul $\underline{x} \in \mathbb{R}^m \setminus \{\underline{0}_m\}$, există un hiperplan de normală $\underline{v} \in \mathbb{R}^m \setminus \{\underline{0}_m\}$, $\mathcal{H} = \underline{v}^\perp$, $\dim \mathcal{H} = m-1$, care trece prin $\underline{0}_m \in \mathbb{R}^m$ și pentru care reflexia lui $\underline{x}$ față de hiperplanul $\mathcal{H} = \underline{v}^\perp$, $H_{\underline{v}}\underline{x}$, se găsește pe dreapta x_1, i.e. $\{\alpha\, \underline{e}_1 \mid \alpha \in \mathbb{R}\}$.

OBS: În fapt, trebuie demonstrat că $\exists\, \underline{v} \in \mathbb{R}^m \setminus \{\underline{0}_m\}$: $H_{\underline{v}} = I_m - 2\dfrac{\underline{v}\,\underline{v}^T}{\underline{v}^T\underline{v}}$.

Dem:

Pentru a determina $\mathcal{H} = \underline{v}^{\perp}$, trebuie determinat $\underline{v} \in \mathbb{R}^m \setminus \{\underline{0}_m\}$, a.î.

$$H_{\underline{v}} := I_m - 2\frac{\underline{v}\,\underline{v}^T}{\underline{v}^T\underline{v}} \in \mathcal{M}_m(\mathbb{R})$$

Cf observației 2), $H_{\underline{v}}\underline{x} = \alpha\underline{e}_1$, $\underline{x}$ și $\underline{v}$ sunt coplanari $\Rightarrow$

Deci $\underline{e}_1$, $\underline{x}$ și $\underline{v}$ sunt coplanari $\Rightarrow$

$\exists\, c \in \mathbb{R}^*$: $\underline{v} = \underline{x} + c\,\underline{e}_1$

- $\underline{v}^T\underline{x} = (\underline{x} + c\underline{e}_1)^T\underline{x} = \underline{x}^T\underline{x} + c\,\underline{e}_1^T\underline{x}$

$= \|\underline{x}\|_2^2 + c\,x_1$

unde $\underline{x} = (x_1 . x_2 \dots x_m)^T \in \mathbb{R}^m$

- $\underline{v}^T\underline{v} = (\underline{x} + c\underline{e}_1)^T(\underline{x} + c\underline{e}_1)$

$= \underline{x}^T\underline{x} + 2c(\underline{e}_1^T\underline{x}) + c^2(\underline{e}_1^T\underline{e}_1)$

$= \|\underline{x}\|^2 + 2c\,x_1 + c^2$

$$\bullet \ \alpha \underline{e}_1 = H_{\underline{v}} \underline{x} = \left(I_m - 2 \frac{\underline{v}\,\underline{v}^T}{\underline{v}^T \underline{v}} \right) \underline{x}$$

$$= \underline{x} - 2 \frac{\underline{v}\,\underline{v}^T}{\underline{v}^T \underline{v}} \underline{x} = \underline{x} - \frac{2}{\underline{v}^T \underline{v}} \underline{v} (\underline{v}^T \underline{x})$$

$$= \frac{\underline{x} (\underline{v}^T \underline{v}) - 2 \underline{v} (\underline{v}^T \underline{x})}{\underline{v}^T \underline{v}}$$

$$= \frac{(\|\underline{x}\|^2 + 2 c x_1 + c^2) \underline{x} - 2 (\|\underline{x}\|^2 + c x_1) \underline{v}}{\|\underline{x}\|^2 + 2 c x_1 + c^2}$$

$$= \frac{(\|\underline{x}\|^2 + 2 c x_1 + c^2) \underline{x} - 2 (\|\underline{x}\|^2 + c x_1) (\underline{x} + c \underline{e}_1)}{\|\underline{x}\|^2 + 2 c x_1 + c^2}$$

$$= \frac{(c^2 - \|\underline{x}\|^2) \underline{x} - 2c (\|\underline{x}\|^2 + c x_1) \underline{e}_1}{\|\underline{x}\|^2 + 2 c x_1 + c^2} \Rightarrow$$

$$\begin{cases} c^2 - \|\underline{x}\|^2 = 0 \\ \|\underline{x}\|^2 + 2 c x_1 + c^2 \neq 0 \end{cases} \Rightarrow$$

$$\begin{cases} c^2 = \|\underline{x}\|^2 \neq 0 \Rightarrow \boxed{c \neq 0} \\ \|\underline{x}\|^2 + 2 c x_1 + c^2 = 2 c x_1 + 2 c^2 = \\ \quad = 2c (x_1 + c) \neq 0 \end{cases} \Rightarrow$$

$$\begin{cases} \boxed{c = \pm \|\underline{x}\|} \\ c + x_1 \neq 0 \end{cases}$$

Dacă $\underline{x} = (x_1\ 0 \dots 0)^T \in \mathbb{R}^m \Rightarrow$

Trebuie ca $\boxed{\text{sign}(c) = \text{sign}(x_1)}$

Prin urmare, obținem:

$$\boxed{c = \text{sign}(x_1) \cdot \|\underline{x}\|}$$

Astfel, am obținut

$$\alpha \underline{e}_1 = H_{\underline{v}} \underline{x} = \frac{-2c(cx_1 + \|\underline{x}\|^2)\underline{e}_1}{\|\underline{x}\|^2 + 2cx_1 + c^2}$$

$$= \frac{-2\,\text{sign}(x_1)\|\underline{x}\|(\text{sign}(x_1)\|\underline{x}\|\, x_1 + \|\underline{x}\|^2)\underline{e}_1}{\|\underline{x}\|^2 + 2\,\text{sign}(x_1)\|\underline{x}\|\, x_1 + (\text{sign}(x_1)\|\underline{x}\|)^2}$$

$$= \frac{-2\,\text{sign}(x_1)\|\underline{x}\|^2(\text{sign}(x_1)\, x_1 + \|\underline{x}\|)\underline{e}_1}{2\|\underline{x}\|(\text{sign}(x_1)\, x_1 + \|\underline{x}\|)}$$

$$= -\operatorname{sign}(x_1)\, \|\underline{x}\|\, \underline{e}_1 \quad \Rightarrow$$

$$\boxed{\alpha = -\operatorname{sign}(x_1)\, \|\underline{x}\| = -c}$$

În concluzie, am obținut:

$$\boxed{\begin{array}{l}
\underline{\text{Date}}: \quad \underline{x} \in \mathbb{R}^m \setminus \{\underline{0}_m\} \\[2ex]
c := \operatorname{sign}(x_1)\, \|\underline{x}\|_2 \\
\underline{v} := \underline{x} + c\, \underline{e}_1 \\[2ex]
H_{\underline{v}} := I_m - 2\, \dfrac{\underline{v}\,\underline{v}^T}{\underline{v}^T \underline{v}} \\
\left(\text{unde } H_{\underline{v}}\underline{x} = -c\, \underline{e}_1 \right)
\end{array}}$$

<u>APLICAȚIE</u>: FACTORIZAREA QR – METODA REFLEXIILOR (HOUSEHOLDER)