Exerciții - limite

1) Det. $\underline{\lim}\, x_n$, $\overline{\lim}\, x_n$ și prec. dacă există $\lim\limits_{n\to\infty} x_n$:

a) $x_n = \dfrac{(1-(-1)^n)\cdot 2^n+1}{2^n+3}$, $\forall n \in \mathbb{N}$

Sol:

$$x_{2k} = \frac{(1-(-1)^{2k})\cdot 2^{2k}+1}{2^{2k}+3} = \frac{(1-1)\cdot 2^{2k}+1}{2^{2k}+3} = \frac{0\cdot 2^{2k}+1}{2^{2k}+3}$$

$$= \frac{1}{2^{2k}+3} \xrightarrow[k\to+\infty]{} \frac{1}{2^{\infty}+3} = \frac{1}{\infty} = \boxed{0}$$

$$x_{2k+1} = \frac{(1-(-1)^{2k+1})\cdot 2^{2k+1}+1}{2^{2k+1}+3} = \frac{(1+1)\cdot 2^{2k+1}+1}{2^{2k+1}+3}$$

$$= \frac{2^{2k+2}+1}{2^{2k+1}+3} = \frac{\cancel{2^{2k}}\left(2+\overset{\to 0}{\frac{1}{2^{2k+2}}}\right)}{\cancel{2^{2k}}\left(1+\underset{\to 0}{\frac{1}{2^{2k+1}}}\right)} \xrightarrow[k\to+\infty]{} \frac{2}{1} = \boxed{2}$$

$\Rightarrow$ ~~Lim~~ $\mathbb{N} = 2\mathbb{N} \cup (2\mathbb{N}+1)$

$\mathcal{L}((x_n)_n) = \{0, 2\} \Rightarrow \underline{\lim}\, x_n = 0$, $\overline{\lim}\, x_n = 2$ $\Big| \Rightarrow \underline{\lim}\, x_n \neq \overline{\lim}\, x_n$

$\Downarrow$

$(\nexists) \lim\limits_{n\to\infty} x_n$ $\square$

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b) $x_n = \dfrac{2+(-1)^n}{1+n^{(-1)^n}} + \sin\dfrac{n\pi}{2},\ \forall n \in \mathbb{N}$

Sol:

$x_{4k} = \dfrac{2+(-1)^{4k}}{1+(4k)^{(-1)^{4k}}} + \sin\left(\dfrac{4k\pi}{2}\right)$

$= \dfrac{2+1}{1+4k} + \sin(2k\pi) = \dfrac{3}{1+4k} + \sin(2k\pi)$, with $\dfrac{3}{1+4k} \to 0$ and $\sin(2k\pi) \to 0$

$\xrightarrow[k\to\infty]{} \boxed{0}$

$x_{4k+1} = \dfrac{2+(-1)^{4k+1}}{1+(4k+1)^{(-1)^{4k+1}}} + \sin\left(\dfrac{(4k+1)\pi}{2}\right)$

$= \dfrac{2-1}{1+\underbrace{\frac{1}{4k+1}}_{\to 0}} + \sin\left(\dfrac{4k\pi}{2}+\dfrac{\pi}{2}\right) \xrightarrow[k\to\infty]{} \dfrac{1}{1} + \sin\dfrac{\pi}{2} = 1+1 = \boxed{2}$

$x_{4k+2} = \dfrac{2+(-1)^{4k+2}}{1+(4k+2)^{(-1)^{4k+2}}} + \sin\left(\dfrac{(4k+2)\pi}{2}\right)$

$= \underbrace{\dfrac{2}{1+4k+2}}_{\to 0} + \sin\pi \xrightarrow[k\to\infty]{} \sin\pi = \boxed{0}$

$x_{4k+3} = \dfrac{2+(-1)^{4k+3}}{1+(4k+3)^{(-1)^{4k+3}}} + \sin\left(\dfrac{(4k+3)\pi}{2}\right)$

$= \dfrac{2-1}{1+\underbrace{\frac{1}{4k+3}}_{\to 0}} + \sin\left(\dfrac{3\pi}{2}\right) \xrightarrow[k\to\infty]{} 1+(-1) = \boxed{0}$

$\mathbb{N} = 4\mathbb{N} \cup (4\mathbb{N}+1) \cup (4\mathbb{N}+2) \cup (4\mathbb{N}+3)$

$\mathcal{L}((x_n)_n) = \{0, 2\}$

$\Rightarrow \left.\begin{array}{l} \overline{\lim}\, x_n = 2 \\ \underline{\lim}\, x_n = 0 \end{array}\right\} \Rightarrow \overline{\lim}\, x_n \neq \underline{\lim}\, x_n \Rightarrow (\nexists) \lim\limits_{n\to\infty} x_n$

□

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(...TĂ/MĂRIRE 2024)

c) $$x_n = \frac{n^2+1}{2n^2+3n+1}\sin\left(\frac{(-1)^n\pi}{2}\right) + \frac{n^3+2}{3n^3+3n+4}\cos\left(\frac{n\pi}{3}\right)$$

• $$\boxed{x_{6k}} = \underbrace{\frac{(6k)^2+1}{2(6k)^2+3(6k)+1}}_{\to \frac{1}{2}}\sin\left(\frac{(-1)^{6k}\pi}{2}\right) + \underbrace{\frac{(6k)^3+2}{3(6k)^3+3(6k)+4}}_{\to\frac{1}{3}}\underbrace{\cos\left(\frac{6k\pi}{3}\right)}_{\to 1}$$

$$\xrightarrow{k\to\infty} \frac{1}{2}\sin\left(\frac{\pi}{2}\right) + \frac{1}{3}\cancel{\cos(2\pi)}\cdot 1 = \frac{1}{2} + \frac{1}{3} = \boxed{\frac{5}{6}}$$

• $$\boxed{x_{6k+1}} = \underbrace{\frac{(6k+1)^2+1}{2(6k+1)^2+3(6k+1)+1}}_{\to\frac{1}{2}}\sin\left(\frac{(-1)^{6k+1}\pi}{2}\right) + \underbrace{\frac{(6k+1)^3+2}{3(6k+1)^3+3(6k+1)+4}}_{\to\frac{1}{3}}\cos\left(\frac{(6k+1)\pi}{3}\right)$$

$$\xrightarrow{k\to\infty} \frac{1}{2}\sin\left(-\frac{\pi}{2}\right) + \frac{1}{3}\cos\left(\frac{6k\pi}{3} + \frac{\pi}{3}\right) \quad \Big\} \cos(x + 2k\pi) = \cos(x)$$

$$= -\frac{1}{2} + \frac{1}{3}\cos\left(\frac{\pi}{3}\right) = -\frac{1}{2} + \frac{1}{3}\cdot\frac{1}{2} = -\frac{1}{2} + \frac{1}{6} = -\frac{2}{6} = \boxed{-\frac{1}{3}}$$

(mai direct de acum)

• $$\boxed{x_{6k+2}} = \frac{1}{2}\sin\left(\frac{\pi}{2}\right) + \frac{1}{3}\cos\left(\frac{2\pi}{3}\right) = \frac{1}{2} + \frac{1}{3}\cdot\left(-\frac{1}{2}\right) = \frac{1}{2} - \frac{1}{6} = \frac{2}{6} = \boxed{\frac{1}{3}}$$

• $$\boxed{x_{6k+3}} = \frac{1}{2}\sin\left(-\frac{\pi}{2}\right) + \frac{1}{3}\cos(\pi) = -\frac{1}{2} + \frac{1}{3}(-1) = -\frac{1}{2} - \frac{1}{3} = \boxed{-\frac{5}{6}}$$

• $$\boxed{x_{6k+4}} = \frac{1}{2}\sin\left(\frac{\pi}{2}\right) + \frac{1}{3}\cos\left(\frac{4\pi}{3}\right) = \frac{1}{2}\cdot 1 + \frac{1}{3}\cdot\left(-\frac{1}{2}\right) = \frac{1}{2} - \frac{1}{6} = \boxed{\frac{1}{3}}$$

• $$\boxed{x_{6k+5}} = \frac{1}{2}\sin\left(-\frac{\pi}{2}\right) + \frac{1}{3}\cos\left(\frac{5\pi}{3}\right) = \frac{1}{2}\cdot(-1) + \frac{1}{3}\cdot\frac{1}{2} = -\frac{1}{2} + \frac{1}{6} = \boxed{-\frac{1}{3}}$$

$$\mathbb{N} = 6\mathbb{N} \cup (6\mathbb{N}+1) \cup (6\mathbb{N}+2) \cup (6\mathbb{N}+3) \cup (6\mathbb{N}+4) \cup (6\mathbb{N}+5)$$

$$\mathcal{L}((x_n)_n) = \left\{-\frac{5}{6}, -\frac{1}{3}, \frac{1}{3}, \frac{5}{6}\right\}$$

$$\Rightarrow \left.\begin{array}{l} \overline{\lim}\, x_n = \frac{5}{6} \\ \underline{\lim}\, x_n = -\frac{5}{6} \end{array}\right\} \overline{\lim}\, x_n \neq \underline{\lim}\, x_n \Rightarrow (\nexists) \lim_{n\to\infty} x_n \quad \square$$

ETI

Exerciții → serii

2) Studiați convergența seriilor

(lucrarea finală)

IV/a) $\displaystyle\sum_{n=1}^{\infty} \frac{n!\,(n+3)!}{(2n+1)!\cdot x^n}$, $x \in (0, \infty)$.

Sol: Folosim criteriul raportului

Fie $x_n = \dfrac{n!\,(n+3)!}{(2n+1)!\cdot x^n}$, $\forall n \in \mathbb{N}^*$

$$\lim_{n\to\infty} \frac{x_{n+1}}{x_n} = \lim_{n\to\infty} \frac{\overset{n+1}{\cancel{(n+1)!}}\ \overset{n+1}{\cancel{(n+4)!}}}{\underset{(2n+2)(2n+3)}{\cancel{(2n+3)!}}\cdot \underset{x}{\cancel{x^{n+1}}}} \cdot \frac{\cancel{(2n+1)!}\cdot \cancel{x^n}}{\cancel{n!}\ \cancel{(n+3)!}}$$

$$= \lim_{n\to\infty} \underbrace{\frac{(n+1)(n+4)}{(2n+2)(2n+3)}}_{\frac{1}{4}} \cdot \frac{1}{x} = \frac{1}{4x}.$$

Conform Criteriului raportului avem:

1) Dacă $\dfrac{1}{4x} < 1$ (i.e. $x \in \left(\frac{1}{4}, \infty\right)$), seria este conv.

2) Dacă $\dfrac{1}{4x} > 1$ (i.e. $x \in \left(0, \frac{1}{4}\right)$), seria este div.

3) Dacă $\dfrac{1}{4x} = 1$ (i.e. $x = \frac{1}{4}$) crit. nu decide

Fie $x = \frac{1}{4} \Rightarrow x_n = \dfrac{n!\,(n+3)!}{(2n+1)!\left(\frac{1}{4}\right)^n} = \dfrac{n!\,(n+3)!\cdot 4^n}{(2n+1)!}$ $\forall n \in \mathbb{N}^*$

(fol. crit Raabe-D.)

$$\lim_{n\to\infty} n\left(\frac{x_n}{x_{n+1}} - 1\right) = \lim_{n\to\infty} n\left(\frac{n!\,(n+3)!\cdot 4^n}{(2n+1)!}\cdot\frac{(2n+3)!}{(n+1)!\,(n+4)!\cdot 4^{n+1}}\right)$$

$$= \lim_{n\to\infty} n\left(\frac{(2n+2)(2n+3)}{(n+1)(n+4)\cdot 4} - 1\right) = \lim_{n\to\infty} n\left(\frac{4n^2+10n+6}{4n^2+20n+16} - 1\right)$$

$$= \lim_{n\to\infty} n\left(\frac{4n^2+10n+6-4n^2-20n-16}{4n^2+20n+16}\right) = \lim_{n\to\infty} \frac{-10n^2-10n}{4n^2+20n+16}$$

$= -\dfrac{10}{4} = -\dfrac{5}{2} < 1 \Rightarrow$ Conform Crit. Raabe-Duhamel seria e divergentă.

Am obținut:

$$\sum_{n=1}^{\infty} \frac{n!(n+3)!}{(2n+1)!\, x^n} \begin{cases} \text{convergentă, dacă } x \in \left(\frac{1}{4}, \infty\right) \\ \text{divergentă, dacă } x \in \left(0, \frac{1}{4}\right] \end{cases}$$ $\square$

I b) $\sum_{n=1}^{\infty} e^{-n^2}$

Sol: Fol. crit de comp cu ineg (se poate și cu lim)

Fie $x_n = e^{-n^2}$ și $y_n = e^{-n}$

! $e^{n^2} \geq e^{n}$, deci $\frac{1}{e^{n^2}} \leq \frac{1}{e^n}$.

~~Conform Crit de Comp cu ineg., $\sum$~~

Deci $\sum_{n=1}^{\infty} y_n = \sum_{n=1}^{\infty} e^{-n} = \sum_{n=1}^{\infty} \frac{1}{e^n} = \underbrace{\sum_{n=1}^{\infty} \left(\frac{1}{e}\right)^n}_{\text{conv}}$ $\Big\}$ $\frac{1}{e} \in (-1,1)$ serie geometrică

$\Rightarrow$ Conform Crit de Comp cu ineg, $\sum_{n=1}^{\infty} x_n$ e convergentă $\square$

II c) $\sum_{n=1}^{\infty} a^n \left(1+\frac{1}{n}\right)^n$, $a>0$

Sol: Fol. crit ~~raportului raportului~~ radicalului

Fie $x_n = a^n\left(1+\frac{1}{n}\right)^n \Rightarrow \lim_{n\to\infty} \sqrt[n]{x_n} = \lim_{n\to\infty} \sqrt[n]{a^n\left(1+\frac{1}{n}\right)^n}$

$= \lim_{n\to\infty} \sqrt[n]{a^n} \cdot \sqrt[n]{\left(1+\frac{1}{n}\right)^n} = \lim_{n\to\infty} a\cdot\left(1+\frac{1}{n}\right) = a \cdot 1 = a$

Conform Criteriului ~~Raportului~~ Radicalului:

1) Dacă $l<1$ (i.e. $a<1$), at $\sum_n x_n$ este conv

2) Dacă $l>1$ (i.e. $a>1$), at $\sum_n x_n$ este div.

3) Dacă $l=1$ crit nu decide

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Fie $a=1 \Rightarrow \sum\limits_{n}^{\infty}\left(1+\frac{1}{n}\right)^{n} \xrightarrow[n\to\infty]{} e \Rightarrow$ divergentă

$\Rightarrow$ Am aflat

$$\sum_{n}^{\infty} a^{n}\left(1+\frac{1}{n}\right)^{n}, \quad \begin{cases} \text{conv, pt } a\in(0,1) \\ \text{div, pt } a\in[1,\infty) \end{cases}$$

$\square$

III d) $\sum\limits_{n=2}^{\infty} \frac{1}{n(\ln n)^{p}}, \; p>0$

Sol: Fol. crit. condensării

Fie $x_n = \frac{1}{n(\ln n)^{p}}$

$$\sum_{n}^{\infty} 2^{n}\cdot x_{2^{n}} = \sum_{n}^{\infty} \cancel{2^{n}}\cdot\frac{1}{\cancel{2^{n}}(\ln 2^{n})^{p}} = \sum_{n}^{\infty}\frac{1}{(n\ln 2)^{p}}$$

$$= \sum_{n}^{\infty} \underbrace{\frac{1}{(\ln 2)^{p}}}_{=C}\cdot\frac{1}{n^{p}} = C\cdot\sum_{n}^{\infty}\frac{1}{n^{p}} \quad \begin{cases} \text{conv pt } p>1 \\ \text{div pt } p\le 1 \end{cases}$$

const >0
nu se ia în calcul

Cum $\sum\limits_{n}^{\infty} x_n \sim \sum\limits_{n}^{\infty} 2^{n} x_{2^{n}} \Rightarrow \sum\limits_{n}^{\infty} x_n \begin{cases} \text{conv pt } p>1 \\ \text{div pt } p\le 1 \end{cases}$

$\square$