b) $\det A_n$, ptr. $\forall n \geq 2$

$$A_n = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & \cdots & n \\ -1 & 2 & 3 & 4 & 5 & \cdots & n \\ -1 & -2 & -3 & 4 & 5 & \cdots & n \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ -1 & -2 & -3 & -4 & -5 & \cdots & n \end{pmatrix}$$

$$V_n = \begin{pmatrix} 1 & 1 & 1 & \cdots & 1 \\ -1 & 1 & 1 & \cdots & 1 \\ -1 & -1 & 1 & \cdots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -1 & -1 & -1 & \cdots & 1 \end{pmatrix}$$

$$j \cdot v_j = j \cdot (1, 1, \ldots, 1, -1, \ldots, -1)^T$$

$A_n = V_n \cdot D_n$

$V_n = [v_1, v_2, \ldots, v_n]$

$D_n = \operatorname{diag}(1, 2, \ldots, n)$

$\det(A_n) = \det V_n \cdot \det D_n = \det V_n \cdot (1 \cdot 2 \cdot \ldots \cdot n) =$
$= n! \cdot \det V_n$

b) $\det A_n$, pt n. $\forall n \geq 2$

$$A_n = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & \cdots & n \\ -1 & 2 & 3 & 4 & 5 & \cdots & n \\ -1 & -2 & 3 & 4 & 5 & \cdots & n \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ -1 & -2 & -3 & -4 & -5 & \cdots & n \end{pmatrix}$$

$$\det A_n = n! \cdot \det V_n$$

$$\det V_n = \begin{vmatrix} 1 & 1 & 1 & \cdots & 1 \\ -1 & 1 & 1 & \cdots & 1 \\ -1 & -1 & 1 & \cdots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -1 & -1 & -1 & \cdots & 1 \end{vmatrix} \begin{matrix} C_2 \leftarrow C_2 - C_1 \\ C_3 \leftarrow C_3 - C_1 \\ C_n \leftarrow C_n - C_1 \end{matrix} \begin{vmatrix} 1 & 0 & 0 & \cdots & 0 \\ -1 & 1 & 0 & \cdots & 0 \\ -1 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -1 & 0 & 0 & \cdots & 1 \end{vmatrix}$$

pt n. $i = \overline{2, n}$

$L_i \leftarrow L_i + L_1$

$$\begin{matrix} L_2 \to L_2 + L_1 \\ 2^{n-1} \\ L_3 \to L_3 + L_1 \\ \cdots \\ L_n \to L_n + L_1 \end{matrix} \begin{vmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{vmatrix} = 2^{n-1} \cdot \det I_n$$

$\Rightarrow \det V_n = 2^{n-1} \cdot 1 = 2^{n-1}$

$\Rightarrow \det A_n = n! \, 2^{n-1}$, $\forall n \geq 2$

b) $\det A_n$, ptr. $\forall n \geq 2$

$$A_n = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & \cdots & n \\ -1 & 2 & 3 & 4 & 5 & \cdots & n \\ -1 & -2 & 3 & 4 & 5 & \cdots & n \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ -1 & -2 & -3 & -4 & -5 & \cdots & n \end{pmatrix}$$

ptr. liniile $1, \dots, j-1$: $1-1=0$

linia j : $1-(-1)=2$

liniile $j+1, \dots, n$: $-1-(-1)=0$

$$v_j - v_{j-1} = 2\cdot[0, 0, \dots, \underbrace{1}_{\text{poz. } j}]$$

ptr $1 \leq j \leq n$

c) $n \geq 2$

$B = (b_{ij})_{1 \leq i,j \leq n} \in \mathcal{M}_n(\mathbb{R})$

a.î. $b_{ij} \in \{-1, 1\}$; $\forall\, i, j$

Să ar. că dc. n = impar, at. $B \cdot A_n$ n-are 0 pe linia poz.

Dc. n = par, care este nr. max. di poz. nule ale lui $B \cdot A_n$?

$$(B \cdot A_n)_{ik} = \sum_{j=1}^{n} b_{ij} \cdot a_{ik} \qquad a_{jk} = \begin{cases} k ; & j \leq k \\ -k ; & j > k \end{cases}$$

$$(B \cdot A_n)_{ik} = b_{i1} \cdot a_{1k} + b_{i2} \cdot a_{2k} + \dots + b_{in} \cdot a_{nk}$$

Col k din A_n:

$$\begin{pmatrix} a_{1k} \\ \vdots \\ a_{nk} \end{pmatrix} \begin{pmatrix} k \\ k \\ k \\ \vdots \\ -k \\ -k \\ \vdots \\ -k \end{pmatrix}$$

Linia i din B:

$(b_{i1}, \dots, b_{in})$

c) $n \geq 2$

$B = (b_{ij})_{1 \leq i,j \leq n} \in \mathcal{M}_n(\mathbb{R})$

a.î. $b_{ij} \in \{-1, 1\}$; $\forall\, i, j$

$$(B \cdot A_n)_{ik} = \sum_{j=1}^{n} b_{ij}\, a_{jk}$$

Pt.n. fiecare linie i, k :

$$(B \cdot A_n)_{ik} = k \left(\sum_{j=1}^{k} b_{ij} - \sum_{j=k+1}^{n} b_{ij} \right) ; \quad k \neq 0$$

$$(B \cdot A_n)_{ik} = 0 \iff \sum_{j=1}^{k} b_{ij} = \sum_{j=k+1}^{n} b_{ij} \quad \text{i.e.} \quad \sum_{j=1}^{k} b_{ij} = \frac{1}{2} \sum_{j=1}^{n} b_{ij}$$

Pt.n. o linie i fixată :

$S_i(k) := \sum_{j=1}^{k} b_{ij}$ $(1 \leq k \leq n)$ → suma primelor k elem. de pe lin. i din B

$T_i := \sum_{j=1}^{n} b_{ij}$ → suma tuturor celor n elem. din lin. i

$$\boxed{S_i(k) = \frac{T_i}{2}}$$

c) $n \geq 2$

$B = (b_{ij})_{1 \leq i,j \leq n} \in \mathcal{M}_n(\mathbb{R})$

a.î. $b_{ij} \in \{-1, 1\}$; $\forall i, j$

$-1 \equiv 1 \pmod 2$

$-1 - 1 \equiv 0 \pmod 2$

$-2 \equiv 0 \pmod 2$

$$\boxed{S_i(k) = \frac{T_i}{2}}$$

→ Ptr. ∀ k

$$S_i(k) = \sum_{j=1}^{k} b_{ij} = \sum_{j=1}^{k} 1 = k \pmod 2$$

$\Rightarrow S_i(k)$ are aceeași paritate cu k (ptr. ∀ linie i)

Mai mult)

$T_i = S_i(n) = n \pmod 2$

$\Rightarrow T_i$ = par doar dc n = par

$\overset{\text{paritat}}{\Rightarrow}$ $S_i(k) = \frac{T_i}{2}$

$\Rightarrow$ Toate par. k tb. să aibă aceeași paritate (care pot anula (B · A))

→ ~~fie~~ pare, fie impare

c) $n \geq 2$

$B = (b_{ij})_{1 \leq i,j \leq n} \in \mathcal{M}_n(\mathbb{R})$

a.î. $b_{ij} \in \{-1, 1\}$; $\forall i, j$

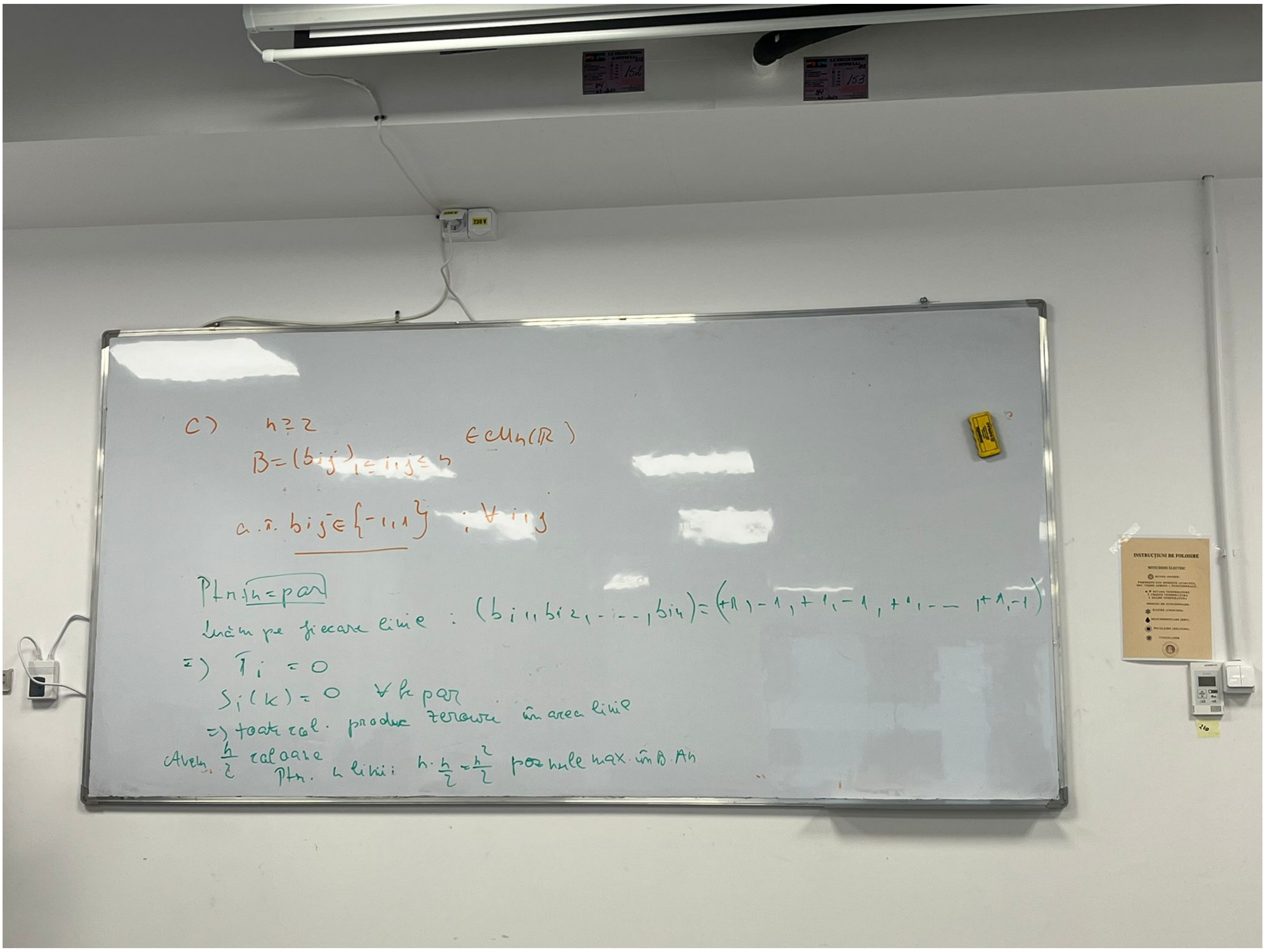
$\Rightarrow$ ptr. n impar $\Rightarrow$ nu avem poz. nule

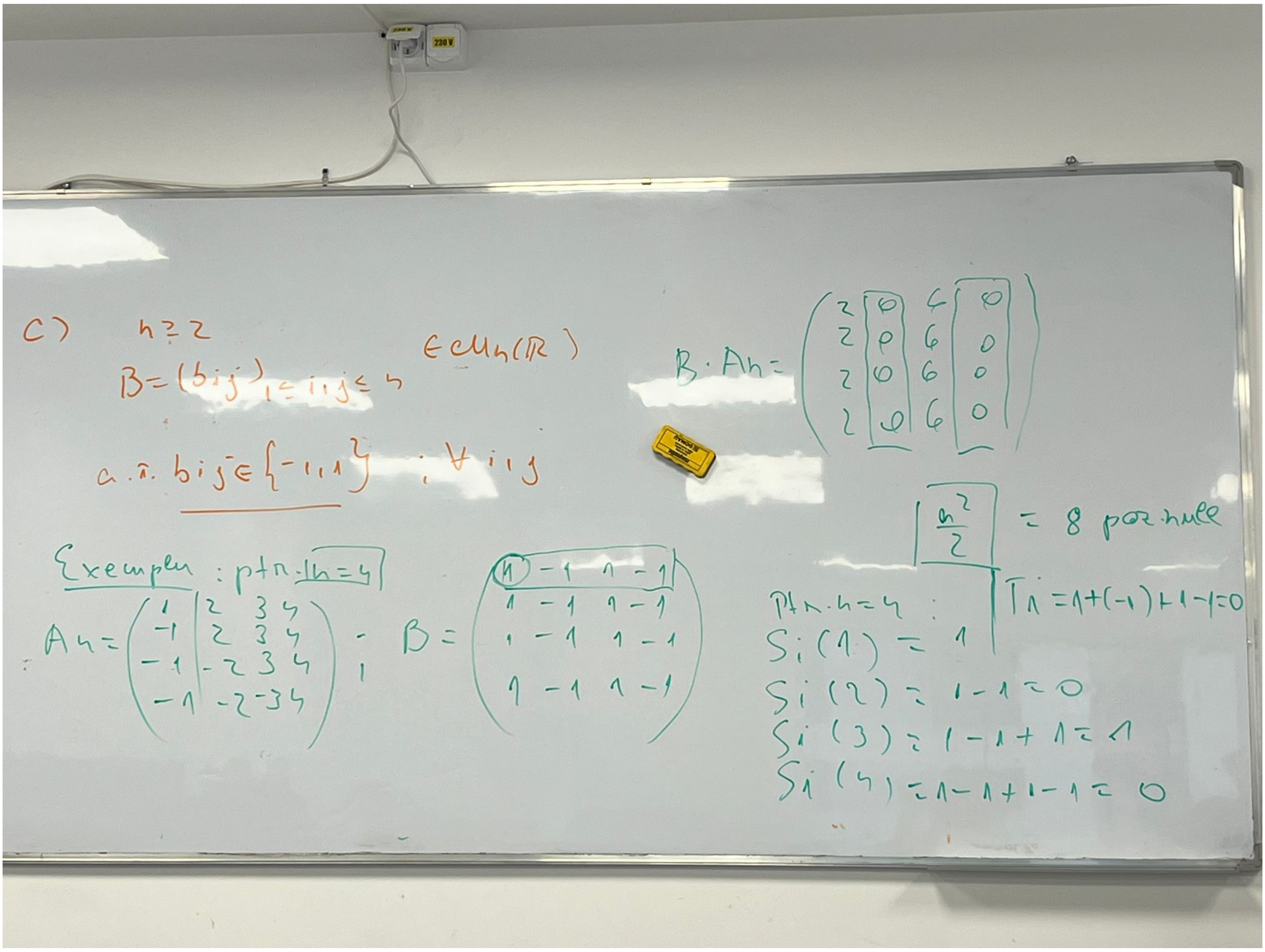
2)

$\Rightarrow$ Nr. poz. k = cel mult nr. de poz. pare sau impare din $\{1, \dots, n\}$ i.e. $\left[\frac{n}{2}\right] = \begin{cases} \frac{n}{2}, & \text{ptr. } n = \text{par} \\ \left[\frac{n}{2}\right] + 1, & \text{altfel} \end{cases}$

Obs. dc. n = impar $\Rightarrow T_i$ impar

$\Rightarrow \nexists\, k$ cu $S_i(k) = \frac{T_i}{2} \notin \mathbb{Z}$


c) $n \geq 2$
$B = (b_{ij})_{1 \leq i,j \leq n} \in \mathcal{M}_n(\mathbb{R})$
a.î. $b_{ij} \in \{-1, 1\}$; $\forall i, j$
Ptr. $\boxed{n = par}$
Luăm pe fiecare linie : $(b_{i1}, b_{i2}, \dots, b_{in}) = (+1, -1, +1, -1, +1, \dots, +1, -1)$
$\Rightarrow \bar{T}_i = 0$
$S_i(k) = 0 \quad \forall k$ par
$\Rightarrow$ toate col. produc zerouri în acea linie
Avem $\frac{n}{2}$ coloane
Ptr. n linii: $n \cdot \frac{n}{2} = \frac{n^2}{2}$ poziții max. în $B \cdot A_n$



c) n ≥ 2
B = (bij) 1 ≤ i,j ≤ n ∈ Mn(R)
a.î. bij ∈ {-1, 1} ; ∀ i, j
B · An =
2 0 6 0
2 0 6 0
2 0 6 0
2 0 6 0
n²/2 = 8 poz. nule
Exemplu : ptr. n = 4
A4 =
1 2 3 4
-1 2 3 4
-1 -2 3 4
-1 -2 -3 4
B =
1 -1 1 -1
1 -1 1 -1
1 -1 1 -1
1 -1 1 -1
ptr. n = 4 :
T1 = 1+(-1)+1-1=0
S1(1) = 1
S1(2) = 1 - 1 = 0
S1(3) = 1 - 1 + 1 = 1
S1(4) = 1 - 1 + 1 - 1 = 0

Examen 2023 $n \geq 3$, $A_n \in \mathcal{M}_n(\mathbb{R})$

- 1 pe poz (i,j) cu $i+j \leq n+1$
- 1 pe (i,i) $i=\overline{1,n}$
- 0, în rest

a) $\det A_3$, $\det A_4$

$$A_3 = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

$$A_3 = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$$

1, $i+j \leq 4$.
$(1,1), (1,2), (1,3), (2,1), (2,2)$
$(3,1), (3,3)$

$$\det A_3 = \begin{vmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{vmatrix} = 1+0+0-1-1 = -1 \neq 0.$$

$$A_4 = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix} \quad i+j \leq 5$$

$$\det A_4 \overset{L_3=L_2}{=} 0.$$

b) A_3 inv + A_3^{-1}.

$$(A_3 \mid I_3) \longrightarrow (I_3 \mid A_3^{-1})$$

Examen 2023 $n \geq 3$, $A_n \in \mathcal{M}_n(\mathbb{R})$

- 1 pe poz (i,j) cu $i+j \leq n+1$
- 1 pe (i,i) $i = \overline{1,n}$
- 0, în rest

c) $\forall n \geq 2$ A_{2n} are liniile n și $n+1$ egale

1, $i+j \leq 2n+1$

1, (i,i)

$$L_n = (a_{n1}, a_{n2}, \dots \overset{\text{Col } n}{\overset{\downarrow}{a_{nn}}}\ \overset{\text{Col } n+1}{\overset{\downarrow}{a_{n\,n+1}}} \dots a_{n\,2n})$$

$(n,1), (n,2) \dots (n,n), (n,n+1)$

$$L_n = (1, 1, \dots \overset{n}{\overset{\downarrow}{1}}, \overset{n+1}{\overset{\downarrow}{1}}, 0 \dots 0)$$

$$L_{n+1} = (a_{n+1\,1}\ a_{n+1\,2} \dots a_{n+1\,n}, a_{n+1\,n+1}, a_{n+1\,n+2} \dots a_{n+1\,2n})$$

1, $i+j \leq 2n+1$.

1, (i,i)

$1 \to (n+1,1), (n+1,2) \dots (n+1,n), (n+1,n+1)$

$$L_{n+1} = (1, 1 \dots \overset{n}{\overset{\downarrow}{1}}, \overset{n+1}{\overset{\downarrow}{1}}, 0 \dots 0)$$

$\Rightarrow L_n = L_{n+1}$

Examen 2023 $n \geq 3$, $A_n \in \mathcal{M}_n(\mathbb{R})$

- 1 pe poz (i,j) cu $i+j \leq n+1$
- 1 pe (i,i) $i=\overline{1,n}$
- 0, în rest

a) $\det(A_n) = ?$ $\forall n \geq 3$

n par. A_{2n}, $L_n = L_{n+1} \Rightarrow \det(A_{2n}) = 0$.

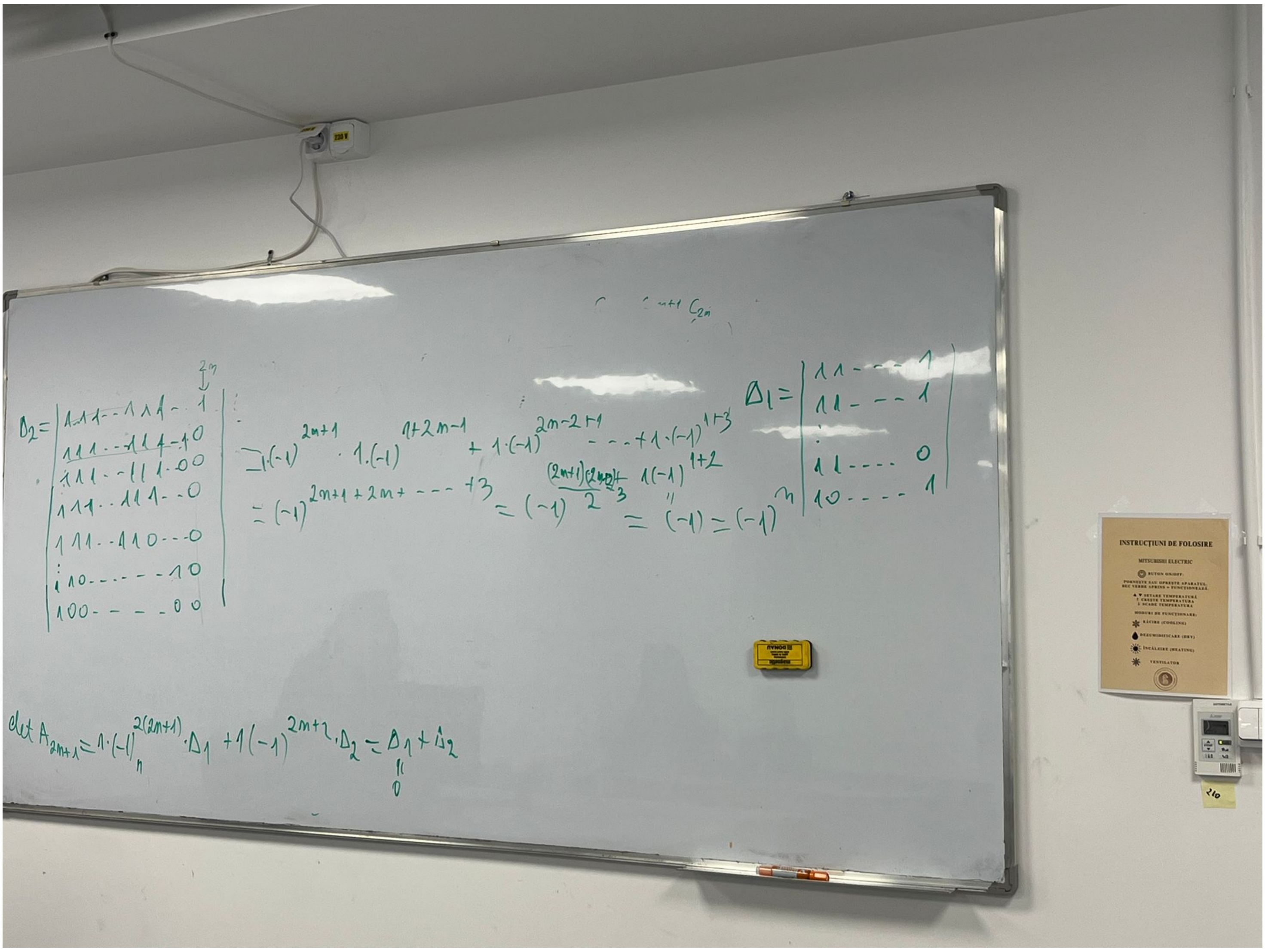
n impar A_{2n+1}.

1, $i+j \leq 2n+2$

1, (i,i)

$$A_{2n+1} = \begin{array}{r} \\ \\ \\ \\ \\ L_n \rightarrow \\ L_{n+1} \rightarrow \\ L_{2n} \rightarrow \\ \\ \end{array} \begin{array}{c} \begin{array}{ccccccccc} & & & & C_n & C_{n+1} & & C_{2n} & \end{array} \\ \begin{pmatrix} 1 & 1 & 1 & \cdots & 1 & 1 & \cdots & 1 & 1 \\ 1 & 1 & 1 & \cdots & 1 & 1 & \cdots & 1 & 0 \\ 1 & 1 & 1 & \cdots & 1 & 1 & \cdots & 0 & 0 \\ \vdots & & & & & & & & \\ 1 & 1 & 1 & \cdots & 1 & 1 & 1 \cdots & & 0 \\ 1 & 1 & 1 & \cdots & 1 & 1 & 0 \cdots & & 0 \\ 1 & 1 & 0 & \cdots & \cdots & \cdots & \cdots & 1 & 0 \\ 1 & 0 & 0 & \cdots & \cdots & \cdots & \cdots & 0 & 1 \end{pmatrix} \end{array}$$

$$\det A_{2n+1} = 1 \cdot (-1)_n^{2(2n+1)} \cdot \Delta_1 + 1(-1)^{2n+2} \cdot \Delta_2 = \Delta_1 + \Delta_2$$


$\Delta_2 = \begin{vmatrix} 1 1 1 \cdots 1 1 1 \cdots 1 \\ 1 1 1 \cdots 1 1 1 \cdots 1 0 \\ 1 1 1 \cdots 1 1 1 \cdots 0 0 \\ 1 1 1 \cdots 1 1 1 \cdots 0 \\ 1 1 1 \cdots 1 1 0 \cdots 0 \\ \vdots \\ 1 1 0 \cdots 1 0 \\ 1 0 0 \cdots 0 0 \end{vmatrix}$
$= 1 \cdot (-1)^{2m+1} \cdot 1 \cdot (-1)^{1+2m-1} + 1 \cdot (-1)^{2m-2+1} \cdots + 1 \cdot (-1)^{1+3} + 1(-1)^{1+2}$
$= (-1)^{2m+1+2m+\cdots+3} = (-1)^{\frac{(2m+1)(2m+2)}{2} - 3} = (-1) = (-1)^n$
$\Delta_1 = \begin{vmatrix} 1 1 \cdots 1 \\ 1 1 \cdots 1 \\ \vdots \\ 1 1 \cdots 0 \\ 1 0 \cdots 1 \end{vmatrix}$
$\det A_{2m+1} = 1 \cdot (-1)^{2(2m+1)} \cdot \Delta_1 + 1(-1)^{2m+2} \cdot \Delta_2 = \Delta_1 + \Delta_2$
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