→ Examen Algebră Liniară, 20 ian. 2025

② Ptr. fiecare $n \in \mathbb{N}, n \geq 2$, fie matricea

$A_n = (a_{ij})_{1 \le i,j \le n} \in \mathcal{M}_n(\mathbb{R})$

ptr. care $\begin{cases} a_{ij} = j, & \text{dc } 1 \le i \le j \le n \\ a_{ij} = -j, & \text{dc } 1 \le j < i \le n \end{cases}$

(a) Să se calculeze $\det(A_3)$, să se ar. că A_3 = invb. și să i se calculeze inversa.

(b) Să se calculeze $\det(A_n)$, ptr. orice $n \geq 2$

(a) $(a_{ij})_{1 \le i,j \le n}$

$$a_{ij} = \begin{cases} j, & \text{dc } i \le j \\ -j, & \text{dc } i > j \end{cases} \qquad A_3 = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

i.e. $\begin{cases} a_{1j} = a_{2j} = \dots = a_{jj} = j \\ a_{(j+1)j} = a_{(j+2)j} = \dots = a_{nj} = -j \end{cases}$

$$A_3 = \begin{pmatrix} 1 & 2 & 3 \\ -1 & 2 & 3 \\ -1 & -2 & 3 \end{pmatrix}$$

$$\det A_3 = \begin{vmatrix} 1 & 2 & 3 \\ -1 & 2 & 3 \\ -1 & -2 & 3 \end{vmatrix} = 6 + 6 - 6 - (-6 - 6 - 6) = 6 - (-18) = 24$$

$\Rightarrow \det A_3 \neq 0 \Rightarrow A_3$ este inversabilă

Ptr. calculul inversei, avem 2 metode:

→ cu Gauss-Jordan

→ cu adjugata, formula din liceu:

<u>Met. I</u> $A^{-1} = \dfrac{1}{\det A} \cdot A^*$ → ca în liceu

Fie $C = (c_{ij})$, unde $c_{ij} = (-1)^{i+j} M_{ij}$

M_{ij} = minorul obț. prin eliminarea rândului i și coloanei j

$$c_{11} = (-1)^{1+1} \cdot \begin{vmatrix} 2 & 3 \\ -2 & 3 \end{vmatrix} = 12$$

$$c_{12} = (-1)^{1+2} \cdot \begin{vmatrix} -1 & 3 \\ -1 & 3 \end{vmatrix} = -0 = 0$$

$$c_{13} = (-1)^{1+3} \cdot \begin{vmatrix} -1 & 2 \\ -1 & -2 \end{vmatrix} = 4$$

$$c_{21} = (-1)^{2+1} \cdot \begin{vmatrix} 2 & 3 \\ -2 & 3 \end{vmatrix} = -12$$

$$c_{22} = (-1)^{2+2} \cdot \begin{vmatrix} 1 & 3 \\ -1 & 3 \end{vmatrix} = +6$$

$$c_{23} = (-1)^{2+3} \cdot \begin{vmatrix} 1 & 2 \\ -1 & -2 \end{vmatrix} = 0$$

$$c_{31} = (-1)^{3+1} \cdot \begin{vmatrix} 2 & 3 \\ 2 & 3 \end{vmatrix} = 0$$

$$c_{32} = (-1)^{3+2} \cdot \begin{vmatrix} 1 & 3 \\ -1 & 3 \end{vmatrix} = -6$$

$$c_{33} = (-1)^{3+3} \cdot \begin{vmatrix} 1 & 2 \\ -1 & 2 \end{vmatrix} = 4$$

$$C = \begin{pmatrix} 12 & 0 & 4 \\ -12 & 6 & 0 \\ 0 & -6 & 4 \end{pmatrix} \Rightarrow \text{matricea cofactorilor}$$

$$A_3^* = C^T = \begin{pmatrix} 12 & -12 & 0 \\ 0 & 6 & -6 \\ 4 & 0 & 4 \end{pmatrix}$$

Știm de la calculul anterior al det. că $\det A_3 = 24 \Rightarrow A_3^{-1} = \frac{1}{24} \cdot \begin{pmatrix} 12 & -12 & 0 \\ 0 & 6 & -6 \\ 4 & 0 & 4 \end{pmatrix}$

$$\Rightarrow A_3^{-1} = \begin{pmatrix} 1/2 & -1/2 & 0 \\ 0 & 1/4 & -1/4 \\ 1/6 & 0 & 1/6 \end{pmatrix}$$

<u>Met. II</u> $\Rightarrow$ Gauss Vrem prin reduceri de linii să obț. $(A_3 \mid I_3) \rightsquigarrow (I_3 \mid A_3^{-1})$

$$(A_3 \mid I_3) = \left(\begin{array}{ccc|ccc} \boxed{1} & 2 & 3 & 1 & 0 & 0 \\ -1 & 2 & 3 & 0 & 1 & 0 \\ -1 & -2 & 3 & 0 & 0 & 1 \end{array}\right) \xrightarrow[L_3 \to L_3 + L_1]{L_2 \to L_2 + L_1}$$

$$\left(\begin{array}{ccc|ccc} \boxed{1} & 2 & 3 & 1 & 0 & 0 \\ 0 & 4 & 6 & 0 & 1 & 0 \\ 0 & 0 & 6 & 1 & 0 & 1 \end{array}\right) \xrightarrow[L_3 \leftarrow \frac{1}{6} L_3]{L_2 \leftarrow \frac{1}{4} L_2} \left(\begin{array}{ccc|ccc} \boxed{1} & 2 & 3 & 1 & 0 & 0 \\ 0 & \boxed{1} & 3/2 & 1/4 & 1/4 & 0 \\ 0 & 0 & 1 & 1/6 & 0 & 1/6 \end{array}\right)$$

$$\xrightarrow[L_2 \leftarrow L_2 - \frac{3}{2} L_3]{L_1 \leftarrow L_1 - 3L_2} \left(\begin{array}{ccc|ccc} \boxed{1} & 2 & 0 & 1/2 & 0 & -1/2 \\ 0 & \boxed{1} & 0 & 0 & 1/4 & -1/4 \\ 0 & 0 & \boxed{1} & 1/6 & 0 & 1/6 \end{array}\right) \xrightarrow{L_1 \leftarrow L_1 - 2L_2}$$

$$\left(\begin{array}{ccc|ccc} \boxed{1} & 0 & 0 & 1/2 & -1/2 & 0 \\ 0 & \boxed{1} & 0 & 0 & 1/4 & -1/4 \\ 0 & 0 & \boxed{1} & 1/6 & 0 & 1/6 \end{array}\right) \to A_3^{-1}$$

$$\Rightarrow A_3^{-1} = \begin{pmatrix} 1/2 & -1/2 & 0 \\ 0 & 1/4 & -1/4 \\ 1/6 & 0 & 1/6 \end{pmatrix}$$

V: $A_3 \cdot A_3^{-1} = I_3$

Calc. $v_j - v_{j-1}$ ptr. $1 \le j \le n$.

(b) $\det A_n$ ptr. $\forall n \ge 2$

$$A_n = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & \dots & n \\ -1 & 2 & 3 & 4 & 5 & \dots & n \\ -1 & -2 & 3 & 4 & 5 & \dots & n \\ -1 & -2 & -3 & 4 & 5 & \dots & n \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ -1 & -2 & -3 & -4 & -5 & \dots & n \end{pmatrix}$$

Obs. că coloana j din A_n este este:

$j \cdot v_j = j(\underbrace{1,1,\dots,1}_{j\text{ ori}}, \underbrace{-1,-1,\dots,-1}_{n-j\text{ ori}})^T$; $v_j = \begin{pmatrix} 1 \\ 1 \\ -1 \\ \vdots \\ -1 \end{pmatrix} \begin{matrix} \} \text{ 1 repetat de } j \text{ ori} \\ \} \text{ -1 de } n-j \text{ ori} \end{matrix}$

Putem scrie $A_n = V_n \cdot D_n$

unde $V_n = [v_1 v_2 \dots v_n]$

$D_n = diag(1,2,\dots,n)$

$\det(A_n) = \det(V_n) \cdot \det(D_n) = \det(V_n) \cdot (1 \cdot 2 \cdot \dots \cdot n) =$

$= n! \cdot \det(V_n)$

$$V_n = \begin{pmatrix} 1 & 1 & 1 & \dots & 1 \\ -1 & 1 & 1 & \dots & 1 \\ -1 & -1 & 1 & \dots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -1 & -1 & -1 & \dots & 1 \end{pmatrix}$$

Aplic ptr. $j = n, n-1, \dots, 2$ op. pe col. $C_j \to C_j - C_{j-1}$

$$\begin{vmatrix} 1 & 1 & 1 & \dots & 1 \\ -1 & 1 & 1 & \dots & 1 \\ -1 & -1 & 1 & \dots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -1 & -1 & -1 & \dots & 1 \end{vmatrix} \xrightarrow[\substack{C_3 \leftarrow C_3 - C_2 \\ \dots \\ C_n \leftarrow C_n - C_{n-1}}]{C_2 \leftarrow C_2 - C_1} 2^{n-1} \cdot \begin{vmatrix} 1 & 0 & 0 & \dots & 0 \\ -1 & 1 & 0 & \dots & 0 \\ -1 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -1 & 0 & 0 & \dots & 1 \end{vmatrix}$$

Calc. $v_j - v_{j-1}$ ptr. $1 \le j \le n$:

· ptr. liniile $1, \dots, j-1$: $1-1=0$

linia j: $1-(-1)=2$

liniile $j+1, \dots, n$: $-1-(-1)=0$

$\Rightarrow v_j - v_{j-1} = 2 \cdot e_j = 2 \cdot [0, 0, \dots, \underset{\text{poz. } j}{1}]$

$\Rightarrow$ După toate $c_j \leftarrow c_j - c_{j-1}$, ptr. $\forall j \ge 2$:

$c_1 = v_1$

$c_j = 2 \cdot e_j$; ptr. $j = \overline{2,n}$

$\tilde{V}_n = [v_1; 2e_2; 2e_3; \dots; 2e_n] = 2^{n-1} \cdot [v_1; e_2; \dots; e_n]$

$\det(\tilde{V}_n) = 2^{n-1} \det[v_1; e_2; \dots; e_n]$

$$\begin{vmatrix} 1 & 0 & 0 & \cdots & 0 \\ -1 & 1 & 0 & \cdots & 0 \\ -1 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -1 & 0 & 0 & \cdots & 1 \end{vmatrix} \xrightarrow[\substack{L_3 \to L_3 + L_1 \\ \cdots \\ L_n \to L_n + L_1}]{\substack{\text{i.e. ptr. } i=\overline{2,n} : L_i \leftarrow L_i + L_1 \\ L_2 \to L_2 + L_1}} \begin{vmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{vmatrix} = I_n$$

$\Rightarrow \det V_n = \det I_n = 1$

$\Rightarrow \det(V_n) = 2^{n-1} \cdot 1 = 2^{n-1}$

$\det(\tilde{V}_n)$

$\Rightarrow \det(A_n) = n! \cdot \det V_n = n! \cdot 2^{n-1}$; $\forall n \geq 2$

c) Fie $n \geq 2$, $B = (b_{ij})_{1 \leq i,j \leq n} \in \mathcal{M}_n(\mathbb{R})$

a.î. $b_{ij} \in \{1, -1\}$ ptr. orice i, j

Să se ar. dc. n = impar, at. $B \cdot A_n$ n-are 0 pe nicio poziție. Dc. n = par, care este nr. maxim posibil de poz. nule ale lui $B \cdot A_n$?

(R) $A_n = \begin{cases} j, & j \geq i \\ -j, & j < i \end{cases}$ (a_{ij})

col. j, cu lin. i

Linia i din B: $(b_{i1}, b_{i2}, \dots, b_{in})$

Col. k din A_n $\begin{pmatrix} a_{1k} \\ \vdots \\ a_{nk} \end{pmatrix}$

$B = (b_{ij})$, știm $b_{ij} \in \{1, -1\}$

i devine lin. j

$(B \cdot A_n)_{ik} = \sum_{j=1}^{n} b_{ij} a_{jk}$ $\quad a_{jk} = \begin{cases} k, & j \leq k \\ -k, & j > k \end{cases}$

$(B \cdot A_n)_{ik} = b_{i1} a_{1k} + b_{i2} a_{2k} + \dots + b_{in} a_{nk}$ col. k

Ptr. fiecare linie i:

$(B \cdot A_n)_{ik} = k \Big(\underbrace{\sum_{j=1}^{k} b_{ij} - \sum_{j=k+1}^{n} b_{ij}}_{} \Big) \Big| : k$ $\quad \begin{pmatrix} k \\ k \\ \vdots \\ k \\ -k \\ -k \\ \vdots \\ -k \end{pmatrix}$

elem. va fi 0 la egalitate

(Eliminăm $k \neq 0$)

$(BA_n)_{ik} = 0 \iff \sum_{j=1}^{k} b_{ij} = \frac{1}{2} \sum_{j=1}^{n} b_{ij}$

Suma primilor k = suma ultimilor $n-k$

Ptr. o linie i fixată, notăm:

$$\begin{cases} S_i(k) := \sum_{j=1}^{k} b_{ij} \ (1 \leq k \leq n) \to \text{suma primelor } k \text{ elem. din lin. } i \text{ a matricei } B \\ T_i := \sum_{j=1}^{n} b_{ij} \to \text{suma tuturor elem. } (n) \text{ din linia } i \end{cases}$$

Cond. ptr. anulare: (ca elem. (i,k) să fie 0)

$(BA_n)_{ik}=0 \iff S_i(k)=\frac{T_i}{2}$

Pasul 1

Ptr. fiecare $b_{ij}\in\{\pm1\}$ obs. că

$1\equiv -1 \pmod 2$

$\Rightarrow$ Ptr. $\forall k$: $S_i(k)=\sum_{j=1}^{k} b_{ij}=\sum_{j=1}^{k} 1=k \pmod 2$

$\Rightarrow S_i(k)$ are aceeași paritate cu k (ptr. $\forall$ o linie i)

Mai mult,

~~$T_i=S_i(n)\equiv n \pmod 2$~~

~~$\Rightarrow T_i=S_n(n)\equiv n \pmod 2$~~

$T_i=\sum_{j=1} b_j\equiv n \pmod 2$

$\Rightarrow T_i=S_i(n)\equiv n \pmod 2 \Rightarrow T_i=$ par doar dc

$n=$ par (de aia problema este pusă doar ptr. $n=$ par)

$\overset{\text{pas 1}}{\Longrightarrow}$ (paritate) k tb să aibă aceeași paritate ca $\frac{T_i}{2}$ $\left(S_i(k)=\frac{T_i}{2}\right)$

Toate poz. k care pot anula (BA) în acea linie tb. să fie de aceeași paritate

$\rightarrow$ fie pare, fie impare

$\Rightarrow$ Nr. poz. k = cel mult nr. elem. pare / impare din $\{1, \dots, n\}$

i.e. $[\frac{n}{2}] = \begin{cases} \frac{n}{2}, & \text{ptr. } n = \text{par} \\ [\frac{n}{2}], & \text{altfel} \end{cases}$

$\Rightarrow$ Ptr. $\forall$ linie i:

$\{k : (BA_n)_{ik} = 0\} \leq \frac{n}{2}$

<u>Obs</u>: dc. n = impar $\Rightarrow T_i$ = impar

și $\nexists\, k$ cu $S_i(k) = \frac{T_i}{2}$

$\notin \mathbb{Z}$

$\Rightarrow$ ptr. n impar nu avem poz. nule

Ptr. n = par, luăm pe fiecare linie același

șir cu semne alternante

$(b_{i1}, b_{i2}, \dots, b_{in}) = (+1, -1, +1, -1, \dots, +1, -1)$

$\Rightarrow T_i = 0$ și $S_i(k) = 0$, $\forall\, k$ par

$\Rightarrow$ toate col. produc zerouri în acea linie

Avem exact $\frac{n}{2}$ col. de acest fel pe fiecare

linie

Ptr. n linii: $n \cdot \frac{n}{2} = \frac{n^2}{2}$ poz. nule $\;=\;$ nr. max. de poz. nule în BA_n

Exemple: ptr. $n=4$

$$A_4 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ -1 & 2 & 3 & 4 \\ -1 & -2 & 3 & 4 \\ -1 & -2 & -3 & 4 \end{pmatrix} ; \quad B = \begin{pmatrix} 1 & -1 & 1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & 1 & -1 \end{pmatrix}$$

$$B \cdot A_4 = \begin{pmatrix} 2 & 0 & 6 & 0 \\ 2 & 0 & 6 & 0 \\ 2 & 0 & 6 & 0 \\ 2 & 0 & 6 & 0 \end{pmatrix}$$

Mat. B ; ptr. lin.:

$+1, -1, +1, -1$

$\frac{4^2}{2} = 8$ par. nule

Ptr. $n=4$: $Si(1) = 1$

$Si(2) = 1 + (-1) = 0$

$Si(3) = 1 + (-1) + 1 = 1$

$Si(4) = 1 + (-1) + 1 + (-1) = 0$

$Ti = 1 - 1 + 1 - 1 = 0$