Transformări ortogonale. Endomorfisme simetrice

1

$(\mathbb{R}^3, g_0)$ s.n v e r

$f \in End(\mathbb{R}^3)$

$$A = [f]_{R_0, R_0} = \frac{1}{9}\begin{pmatrix} 8 & 1 & -4 \\ 1 & 8 & 4 \\ -4 & 4 & -7 \end{pmatrix}$$

R_0 = reperul canonic

a) Arăt. că $f \in O(\mathbb{R}^3)$, de speța 2 i.e. $f = s \circ R_\varphi$

b) $\varphi = ?$ axa $= ?$

c) $R = ?$ $\quad R = \{e_1, e_2, e_3\}$ orton.

$$[f]_{R, R} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & \cos\varphi & -\sin\varphi \\ 0 & \sin\varphi & \cos\varphi \end{pmatrix}$$

Sol:

$$A^T \cdot A = \frac{1}{9^2}\begin{pmatrix} 8 & 1 & -4 \\ 1 & 8 & 4 \\ -4 & 4 & -7 \end{pmatrix}\begin{pmatrix} 8 & 1 & -4 \\ 1 & 8 & 4 \\ -4 & 4 & -7 \end{pmatrix}$$

$$= \frac{1}{9^2}\begin{pmatrix} 81 & 0 & 0 \\ 0 & 81 & 0 \\ 0 & 0 & 81 \end{pmatrix} = I_3 \Rightarrow A \in O(3)$$

$$\det(A) = \frac{1}{9^3}\begin{vmatrix} 0 & -63 & -36 \\ 1 & 8 & 4 \\ 0 & 36 & 9 \end{vmatrix} = \frac{9^2}{9^3}\begin{vmatrix} 0 & -7 & -4 \\ 1 & 8 & 4 \\ 0 & 4 & 1 \end{vmatrix}$$

$$= -\frac{1}{9}\begin{vmatrix} -7 & -4 \\ 4 & 1 \end{vmatrix} = -\frac{1}{9} \cdot 9 = (-1)$$

$\Rightarrow f$ de speța 2

b) $Tr(A) = \frac{1}{9} \cdot 9 = 1 = -1 + 2\cos(\varphi) \Rightarrow \cos(\varphi) = 1$

$\Rightarrow \varphi = 0$

$f(x) = (-x)$

$AX = -X$

$$\begin{pmatrix} 8 & 1 & -4 \\ 1 & 8 & 4 \\ -4 & 4 & -7 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -9x_1 \\ -9x_2 \\ -9x_3 \end{pmatrix}$$

$$\begin{cases} 8x_1 + x_2 - 4x_3 = -9x_1 \\ x_1 + 8x_2 + 4x_3 = -9x_2 \\ -4x_1 + 4x_2 - 7x_3 = -9x_3 \end{cases} \Leftrightarrow \begin{cases} 17x_1 + x_2 - 4x_3 = 0 \\ x_1 + 17x_2 + 4x_3 = 0 \\ -4x_1 + 4x_2 + 2x_3 = 0 \end{cases}$$

$$B = \begin{pmatrix} 17 & 1 & -4 \\ 1 & 17 & 4 \\ -4 & 4 & 2 \end{pmatrix}$$

$$\det(B) = 2 \cdot \begin{vmatrix} 17 & 1 & -2 \\ 1 & 17 & 2 \\ -4 & 4 & 1 \end{vmatrix} = 2 \cdot \begin{vmatrix} 18 & 18 & 0 \\ 9 & 9 & 0 \\ -4 & 4 & 1 \end{vmatrix} = 0$$

$$\Delta_{cor} = \begin{vmatrix} 17 & 1 \\ 1 & 17 \end{vmatrix} \neq 0 \Rightarrow rg(B) = 2$$

$$\begin{cases} 17x_1 + x_2 = 4x_3 \\ x_1 + 17x_2 = -4x_3 \end{cases} \Rightarrow 16x_1 = 4x_3 \Rightarrow x_1 = \frac{1}{4}x_3$$

$x_1 + x_2 = 0 \Rightarrow x_2 = (-x_1)$

$$V_{-1} = \left\{ \left(\frac{1}{4}x_3, -\frac{1}{4}x_3, x_3\right) \middle| x_3 \in \mathbb{R} \right\} = \left\{ \frac{x_3}{4}(1, -1, 4) \middle| x_3 \in \mathbb{R} \right\}$$

$$e_1 = \frac{1}{\sqrt{18}}(1, -1, 4) = \frac{1}{3\sqrt{2}}(1, -1, 4)$$

versorul axei

$$e_1^{\perp} = \{ x \in \mathbb{R}^3 \mid g_0(x, e_1) = 0 \}$$

$$= \{ x \in \mathbb{R}^3 \mid x_1 - x_2 + 4x_3 = 0 \}$$

$$x_2 = x_1 + 4x_3$$

$$= \{ (x_1, x_1 + 4x_3, 4x_3) \mid x_1, x_3 \in \mathbb{R} \}$$

$$x_1 \underbrace{(1, 1, 0)}_{f_2} + x_3 \underbrace{(0, 4, 1)}_{f_3}$$

$\mathcal{R} = \{ f_2, f_3 \}$ reper în $e_1^{\perp}$

Aplicăm Gram Schmidt

$$e_2' = f_2 = (1, 1, 0)$$

$$e_3' = f_3 - \frac{\langle f_3, e_2' \rangle}{\langle e_2', e_2' \rangle} e_2' =$$

$$= (0, 4, 1) - \frac{\cancel{4}^{2}}{\cancel{2}_{1}} (1, 1, 0) = (-2, 2, 1)$$

$$e_2 = \frac{1}{\sqrt{2}} (1, 1, 0)$$

$$e_3 = \frac{1}{3} (-2, 2, 1)$$

$$[f]_{\mathcal{R}, \mathcal{R}} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

f = simetria față de $e_1^{\perp}$

② $(\mathbb{R}^3, g_0)$ s.v.e.r. $u = (1, 1, 0)$

a) $\langle \{u\} \rangle^{\perp} = ?$

Precizați un reper ortonormat

b) Să se det $f \in O(\mathbb{R}^3)$ de speță 1, care are rotația de unghi $t = \frac{1}{2}$, și axa u

Sol:

$u^{\perp} = \{ x \in \mathbb{R}^3 \mid g(x, u) = 0 \} = \{ x \in \mathbb{R}^3 \mid x_1 + x_2 = 0 \}$

$= \{ (x_1, x_1, x_3 \mid x_1, x_3 \in \mathbb{R} \} =$

$= x_1 \underbrace{(1, -1, 0)}_{f_2} + x_3 \underbrace{(0, 0, 1)}_{f_3} \mid x_1, x_3 \in \mathbb{R} \}$

$\langle f_2, f_3 \rangle = 0$

$\{ \langle f_2, f_3 \rangle \}$ reper ortogonal în $u^{\perp}$

$e_2 = \frac{1}{\sqrt{2}} \cdot (1, -1, 0)$

$e_3 = (0, 0, 1)$

$\{ e_2, e_3 \}$ reper ortonormat în $u^{\perp}$

$e_1 = \frac{1}{\sqrt{2}} (1, 1, 0)$

$$[f]_{R, R} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos(\frac{\pi}{2}) & -\sin(\frac{\pi}{2}) \\ 0 & \sin(\frac{\pi}{2}) & \cos(\frac{\pi}{2}) \end{pmatrix} = \underset{\substack{\parallel \\ A'}}{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}}$$

$R = \{ e_1, e_2, e_3 \}$

$A = [f]_{R_0, R_0}$

$R_0 \xrightarrow{C} R \qquad A' = C^T A C$

$A = C A' C^T$

$$C = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

(punem pe coloane vectorii e_1, e_2, e_3)

$$= \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & \sqrt{2} \end{pmatrix}$$

$$A = \frac{1}{2} \begin{pmatrix} 1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & \sqrt{2} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & \sqrt{2} \end{pmatrix}$$

$$= \frac{1}{2} \begin{pmatrix} 1 & 0 & -1 \\ 1 & 0 & 1 \\ 0 & \sqrt{2} & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & \sqrt{2} \end{pmatrix} =$$

$$= \frac{1}{2} \begin{pmatrix} 1 & 1 & -\sqrt{2} \\ 1 & 1 & \sqrt{2} \\ \sqrt{2} & -\sqrt{2} & 0 \end{pmatrix}$$

$f : \mathbb{R}^3 \to \mathbb{R}^3$

$f(X) = \frac{1}{2} \left(x_1 + x_2 - \sqrt{2} + x_3 ,\ x_1 + x_2 + \sqrt{2} x_3 ,\ \sqrt{2} x_1 - \sqrt{2} x_2 \right)$

③

$(\mathbb{R}, g_0)$, $f \in \text{End}(\mathbb{R}^3)$

$A = [f]_{\mathcal{R}_0, \mathcal{R}_0} = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$

a) $f \in \text{Sim}(\mathbb{R}^3)$. , $f = ?$

b) $Q : \mathbb{R}^3 \to \mathbb{R}$, formă pătratică asociată lui

f $(\langle x, f(x) \rangle = Q(x),\ \forall x \in \mathbb{R}^3)$

c) Q la forma canonică

efectuând transf. ort. $h = ?$

(schimbare de reper orton.)

SOL:

a) $A = A^T \Rightarrow f \in Sim(\mathbb{R}^3)$

$f : \mathbb{R}^3 \to \mathbb{R}^3$, $f(x) = (x_1 + x_3, x_2, x_1 + x_3)$

(din $A \cdot \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$)

b) $Q : \mathbb{R}^3 \to \mathbb{R}$, $Q(x) = x_1^2 + x_2^2 + x_3^2 + 2x_1x_3$

(Metoda valorilor proprii)

$$P_A(\lambda) = \det(A) - \lambda I_3 = 0 = \begin{vmatrix} 1-\lambda & 0 & 1 \\ 0 & 1-\lambda & 0 \\ 1 & 0 & 1-\lambda \end{vmatrix} =$$

$$= (1-\lambda) \begin{vmatrix} 1-\lambda & 1 \\ 1 & 1-\lambda \end{vmatrix} = (1-\lambda)((1-\lambda^2) - 1) =$$

$$= (1-\lambda)(-\lambda(2-\lambda)) = -\lambda(1-\lambda)(2-\lambda)$$

$\lambda_1 = 0$, $m_1 = 1$

$\lambda_2 = 1$, $m_2 = 1$

$\lambda_3 = 2$, $m_3 = 1$

$V_{\lambda_1} = \{ x \in \mathbb{R}^3 \mid Ax = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \}$

$x_1 = -x_3$ $\qquad$ $e_1 = \frac{1}{\sqrt{2}}(-1, 0, 1)$

$x_2 = 0$

$V_{\lambda_2} = \{ x \in \mathbb{R}^3 \mid Ax = x \} = \{ (0, x_2, 0) \mid x_2 \in \mathbb{R} \}$

$e_2 = (0, 1, 0)$

$(A - I_3)x = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

$$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$V_{\lambda_3} = \{ X \in \mathbb{R}^3 \mid AX = 2X \} = \{ (x_1, 0, x_1) \mid x_1 \in \mathbb{R} \}$

$e_3 = \frac{1}{\sqrt{2}} (1, 0, 1)$

$\mathcal{R} = \{ e_1 = \frac{1}{\sqrt{2}} (-1, 0, 1), e_2 = (0, 1, 0), e_3 = \frac{1}{\sqrt{2}} (1, 0, 1) \}$

$C \uparrow$

$\mathcal{R}_0 = \{ e_1^0, e_2^0, e_3^0 \}$

$h(e_i^0) = e_i, \ \forall i = \overline{1,3}$

$C = [h]_{\mathcal{R}_0, \mathcal{R}_0}$

$$C = \frac{1}{\sqrt{2}} \begin{pmatrix} -1 & 0 & 1 \\ 0 & \sqrt{2} & 0 \\ 1 & 0 & 1 \end{pmatrix}$$

$h : \mathbb{R}^3 \to \mathbb{R}^3$

$h(X) = \frac{1}{\sqrt{2}} \left(-x_1 + x_3, \sqrt{2}\, x_2, x_1 + x_3 \right)$

$h \in O(\mathbb{R}^3)$

$$A' = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$ din $\begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix}$ cu multiplicități m_1, m_2, m_3

$Q(X) = x_2'^2 + 2 x_3'^2$

$(2,0)$

Q nu e poz def

dci g, forma polară asoc.

lui Q nu este prod scalar

(4) $(\mathbb{R}^3, g_0)$, $f \in End(\mathbb{R}^3)$

$f : \mathbb{R}^3 \to \mathbb{R}^3$

$$f(x) = \underset{\substack{\parallel \\ (x_1 - x_2 + 2x_3)\cdot v}}{g_0(x, u)\cdot v}, \quad v = (1, -1, 2)$$

$$f(x) = (x_1 - x_2 + 2x_3, -x_1 + x_2 - 2x_3, 2x_1 - 2x_2 + 4x_3)$$

a) Dem $f \in Sim(\mathbb{R}^3)$, $f = ?$

b) $Q : \mathbb{R}^3 \to \mathbb{R}$ formă păt. asoc. lui f.
Aduc Q la o formă canonică, efectuând o transf. ortogonală h

Sol:

Fie $A = [f]_{R_0, R_0}$

$$A = \begin{pmatrix} 1 & -1 & 2 \\ -1 & 1 & -2 \\ 2 & -2 & 4 \end{pmatrix} = A^T \Rightarrow f \in Sim(\mathbb{R}^3)$$

$$Q(x) = x_1^2 + x_2^2 + 4x_3^2 - 2x_1x_2 + 4x_1x_3 - 4x_2x_3$$

$$\det(A - \lambda I_3) = \begin{vmatrix} 1-\lambda & -1 & 2 \\ -1 & 1-\lambda & -2 \\ 2 & -2 & 4-\lambda \end{vmatrix} = 0$$

$$\begin{vmatrix} -\lambda & -1 & 2 \\ -\lambda & 1-\lambda & -2 \\ 0 & -2 & 4-\lambda \end{vmatrix} = 0$$

$$-\lambda \cdot \begin{vmatrix} 1 & -1 & 2 \\ 0 & 2-\lambda & -4 \\ 0 & -2 & 4-\lambda \end{vmatrix} = 0$$

$-\lambda\ ((2-\lambda)(4-\lambda)-8)=0$

$-\lambda\ (8-2\lambda-4\lambda+\lambda^2-8)=0$

$-\lambda(\lambda^2-6\lambda)=0$

$-\lambda^2(\lambda-6)=0$

$\lambda_1=0,\ m_1=2$

$\lambda_2=6,\ m=1$

$V_{\lambda_1}=\{x\in\mathbb{R}^3 \mid AX=0_{3,1}\}$

$x_1=x_2-2x_3$

$(x_2-2x_3,\ x_2,\ x_3)=x_2(1,\ 1,\ 0)$

$f_1=(1,1,0)$

$f_2=(-2,\ 0,\ 1)$

$e_1'=(1,1,\ 0)$

$e_2'=f_2-\dfrac{\langle e_2,e_1'\rangle}{\langle e_1',e_1'\rangle}\cdot e_1'$

$=(-2,\ 0,\ 1)+(1,\ 1,0)$

$=(-1,\ 1,\ 1)$

$e_1=\dfrac{1}{\sqrt{2}}(1,1,0)$

$e_2=\dfrac{1}{\sqrt{3}}(-1,\ 1,\ 1)$

$V_{\lambda_2}=\{x\in\mathbb{R}^3 \mid AX=6X\}$

$$\begin{pmatrix} \boxed{\begin{matrix} -5 & -1 \\ -1 & -5 \end{matrix}} & \begin{matrix} 2 \\ -2 \end{matrix} \\ \begin{matrix} 2 & -2 \end{matrix} & -2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{cases} -5x_1 - x_2 + 2x_3 = 0 \\ -x_1 - 5x_2 - 2x_3 = 0 \end{cases}$$

$$-6x_1 - 6x_2 = 0 \Rightarrow x_2 = (-x_1)$$

$$-4x_1 = -2x_3 \Rightarrow x_3 = 2x_1$$

$$x_1 = \frac{1}{2} x_3$$

$$A' = \begin{pmatrix} 0 & & 0 \\ & 0 & \\ 0 & & 6 \end{pmatrix}$$

$\{e_1^0, e_2^0, e_3^0\}$
$\parallel$
$\mathcal{R}_0 \xrightarrow{C} \mathcal{R} = \{e_1, e_2, e_3\}$

$h \in O(\mathbb{R}^3)$, $h(e_i^0) = e_i \quad \forall i = \overline{1,3}$

$C = [h]_{\mathcal{R}_0, \mathcal{R}_0}$