SEMINAR $\overline{\text{VII}}$ - 10.11.2023

Proiecții. Simetrii. Vectori proprii. Valori proprii.

Diagonalizare

2. $f: \mathbb{R}^3 \to \mathbb{R}^3$, $f(x) = (x_1 + x_2 - x_3, -x_1 - x_2 + x_3, x_1 + x_2 + x_3)$

a) $[f]_{R,R} = A'$

$R = \{e_1' = e_1 + e_2 + e_3,\ e_2' = e_1 + e_3,\ e_3' = e_1 + e_2\}$

b) $\mathbb{R}^3 = \operatorname{Im} f \oplus W$

$s: \mathbb{R}^3 \to \mathbb{R}^3$ simetrie față de W

$s(0, 1, 1) = ?$

c) $\mathbb{R}^3 = f(V') \oplus U$

$V' = \left\{ x \in \mathbb{R}^3 \;\middle|\; \begin{cases} x_1 + 2x_2 + x_3 = 0 \\ -x_1 + x_2 + 2x_3 = 0 \end{cases} \right\}$

$p: \mathbb{R}^3 \to \mathbb{R}^3$ proiecția pe $f(V')$

$p(2, -1, 3) = ?$

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a) Metoda 1

$R_0 = \{e_1, e_2, e_3\} \to R = \{e_1' = e_1 + e_2 + e_3,\ e_2' = e_1 + e_3,\ e_3' = e_1 + e_2\}$

$A = [f]_{R_0, R_0}$

$A' = [f]_{R, R}$

$$\boxed{A' = C^{-1} \cdot A \cdot C} \Rightarrow C = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$$

$$\underbrace{\begin{pmatrix} 1 & 1 & -1 \\ -1 & -1 & 1 \\ 1 & 1 & 1 \end{pmatrix}}_{A} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} x_1 + x_2 - x_3 \\ -x_1 - x_2 + x_3 \\ x_1 + x_2 + x_3 \end{pmatrix}$$

OBSERVAȚIE:

$A^n = ?$

$A = C A' C^{-1}$

$$A^n = \underbrace{(C A' C^{-1})(C A' C^{-1}) \dots (C A' C^{-1})}_{\text{de } n \text{ ori}} = C \cdot (A')^n C^{-1}$$

Dacă A' diag.

$$A' = \begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix} \Rightarrow A^n = \begin{pmatrix} a^n & 0 & 0 \\ 0 & b^n & 0 \\ 0 & 0 & c^n \end{pmatrix}$$

$f(e_1') = f(1,1,1) = (1, -1, 3) = a e_1' + b e_2' + c e_3' =$

$= a(1,1,1) + b(1,0,1) + c(1,1,0)$

$= (a + b + c,\ a + c,\ a + b) = (1, -1, 3)$

$$\begin{cases} a + b + c = 1 \\ a + c = -1 \\ a + b = 3 \end{cases} \Rightarrow b = 2 \Rightarrow a = 1 \Rightarrow c = -2$$

$$A' = \begin{pmatrix} 1 & \cdot & \cdot \\ 2 & \cdot & \cdot \\ -2 & \cdot & \cdot \end{pmatrix}$$

$f(e_2') = f(1,0,1) = \dots$

$f(e_3') = f(1,1,0) = \dots$

b) $\text{Im} f = \{ y \in \mathbb{R}^3 \mid \exists\, x \in \mathbb{R}^3,\ f(x) = y \}$

$$\begin{cases} x_1 + x_2 - x_3 = y_1 \\ -x_1 + x_2 + x_3 = y_2 \\ x_1 + x_2 + x_3 = y_3 \end{cases} \qquad A = \left(\begin{array}{ccc} 1 & 1 & -1 \\ -1 & -1 & 1 \\ 1 & 1 & 1 \end{array}\right) \left|\begin{array}{c} y_1 \\ y_2 \\ y_3 \end{array}\right.$$

$$\Delta p = \begin{vmatrix} -1 & 1 \\ 1 & 1 \end{vmatrix} = -1 - 1 = -2 \neq 0$$

$$\Delta c = 0 \Rightarrow \begin{vmatrix} 1 & -1 & y_1 \\ -1 & 1 & y_2 \\ 1 & 1 & y_3 \end{vmatrix} = 0 \Rightarrow \begin{vmatrix} 1 & -1 & y_1 \\ -1 & 1 & y_2 \\ 1 & 1 & y_3 \end{vmatrix} = \begin{vmatrix} 1 & -1 & y_1 \\ 0 & 0 & y_1 + y_2 \\ 1 & 1 & y_3 \end{vmatrix} =$$

$$= (y_1 + y_2)(-1) \begin{vmatrix} 1 & -1 \\ 1 & 1 \end{vmatrix} = -(y_1 + y_2)(-2) = 0$$

$\text{Im} f = \{ y \in \mathbb{R}^3 \mid y_1 + y_2 = 0 \} \Rightarrow y_2 = -1$

$= \{ (y_1, -y_1, y_3) \mid y_1, y_3 \in \mathbb{R} \} \Rightarrow y_1(1,-1,0) + y_3(0,0,1)$

$\Rightarrow SG \Rightarrow \dim \text{Im} f = 2 \Rightarrow R_1 = \{ (1,-1,0), (0,0,1) \}$ reper în $\text{Im} f$

Extindem la un reper în $\mathbb{R}^3$

$$\begin{vmatrix} 1 & 0 & 1 \\ -1 & 0 & 0 \\ 0 & 1 & 0 \end{vmatrix} \neq 0 \Rightarrow R_1 \cup \{ e_1 \} \text{ reper în } \mathbb{R}_3$$

$$\Rightarrow W = \langle \{ e_1 \} \rangle$$

$$(0,1,1) = \underbrace{a(1,-1,0) + b(0,0,1)}_{v_1} + \underbrace{c(1,0,0)}_{v_2} = (a+c, -a, b)$$

$\quad \| $

$v_1 + v_2$

$\cap \quad \cap$

Inf $\quad$ W

$$\begin{cases} a+c=0 & \Rightarrow c=1 \\ -a=1 & \Rightarrow a=-1 \\ b=1 \end{cases}$$

$v_1 = (-1,1,0) + (0,0,1) = (-1,1,1)$

$v_2 = (1,0,0)$

$s, p : V_1 \oplus V_2 \to V_1 \oplus V_2$, $p(v_1+v_2) = v_1$ proiecția pe V_1

$s(v_1+v_2) = v_1 - v_2$ simetria față de V_1

$s(v_1+v_2) = -v_1 + v_2 = (1,-1,-1) + (1,0,0) = (2,-1,-1)$

$s = 2p - id_{\mathbb{R}^3}$, p-proiecția pe W

$p(v_1+v_2) = v_2$

$$s(v_1+v_2) = 2v_2 - \underbrace{id(v_1+v_2)}_{-v_1-v_2} = 2v_2 - v_1 - v_2 = v_2 - v_1$$

c) $\mathbb{R}^3 = f(V') \oplus U$ $\qquad$ dim $V' = 3-2=1$

$$\begin{cases} x_1 + 2x_2 + x_3 = 0 \\ -x_1 + x_2 + 2x_3 = 0 \end{cases}$$

$$\begin{cases} x_1 + 2x_2 = -x_3 \\ -x_1 + x_2 = -2x_3 \end{cases} (+)$$

$3x_2 = -3x_3$

$x_2 = -x_3$

$x_1 = x_3$

$V' = \{(x_3, -x_3, x_3) \mid x_3 \in \mathbb{R}\}$

$(x_3, -x_3, x_3) = x_3(1,-1,1)$

$V' = \langle \{(1,-1,1)\} \rangle$

$f(1,-1,1) = (-1,1,-1)$

$f(V') = \langle \{(-1,1,1)\} \rangle$

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$$rg\begin{pmatrix} -1 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} = 3 \Rightarrow SLi \Rightarrow U = <\{e_2, e_1\}>$$

$$\{(-1,1,1), (0,1,0), (1,0,0)\}$$

$$(2,-1,3) = \underbrace{a(-1,1,1)}_{u_1} + \underbrace{b(0,1,0) + c(1,0,0)}_{u_2} =$$

$$\parallel$$

$$u_1 + u_2 = (-a+c, a+b, a)$$

$$\begin{matrix} u_1 & u_2 \\ \cap & \cap \\ f(U') & U \end{matrix}$$

$$\begin{cases} -a+c = 2 & \Rightarrow c = 5 \\ a+b = -1 & \Rightarrow b = -4 \\ a = 3 \end{cases}$$

$$u_1 = 3(-1,1,1) = (-3,3,3)$$

$$u_2 = -4(0,1,0) + 5(1,0,0) = (0,-4,0) + (5,0,0)$$
$$= (5,-4,0)$$

$$p(u_1 + u_2) = u_1 = (-3,3,3)$$

5. $f: \mathbb{R}^4 \to \mathbb{R}^4 \quad f(x) = (x_2 - x_3 + x_4,\ x_2 - x_3 + x_4,\ x_4,\ x_4)$

a) Să se afle valorile proprii

b) Precizați care sunt subspațiile proprii

c) $\exists$ un reper R în $\mathbb{R}^4$ a.î. $[f]_{R,e}$ este diagonală

$$\underbrace{\begin{pmatrix} 0 & 1 & -1 & 1 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}}_{A} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} x_2 - x_3 + x_4 \\ x_2 - x_3 + x_4 \\ x_4 \\ x_4 \end{pmatrix}$$

$$P(\lambda) = \det(A - \lambda I_4) = \begin{vmatrix} \lambda & 1 & -1 & 1 \\ 0 & \lambda-1 & -1 & 1 \\ 0 & 0 & -\lambda & 1 \\ 0 & 0 & 0 & \lambda-1 \end{vmatrix} = 0 \Rightarrow (-\lambda)^2(1-\lambda)^2 = 0$$

$\lambda_1 = 0,\ m_1 = 2$

$\lambda_2 = 1,\ m_2 = 2$

$V_{\lambda_1} = \{ x \in \mathbb{R}^4 \mid f(x) = 0 \cdot x = 0 \} = \operatorname{Ker} f$

$= \operatorname{Ker} f = \{(x_1, x_2, x_2, 0) \mid x_1, x_2 \in \mathbb{R}\}$

$AX = 0_{4,1} = \begin{cases} x_2 - x_3 + x_4 = 0 \Rightarrow x_2 = x_3 \\ x_4 = 0 \end{cases}$

$\dim V_{\lambda_1} = 4 - rg\, A = 4 - 2 = 2 = m_1$

$x_1(1,0,0,0) + x_2(0,1,1,0)$

$R_1 = \{(1,0,0,0), (0,0,1,0)\}$ reper în V_1

$V_{\lambda_2} = \{ x \in \mathbb{R}^4 \mid f(x) = 1 \cdot x \}$

$AX = X$

$(A - i_4)X = 0_{4,1}$

$$\begin{pmatrix} -1 & 1 & -1 & 1 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$\begin{cases} -x_1 + x_2 - x_3 + x_4 = 0 \\ -x_3 + x_4 = 0 \end{cases}$ $(-)$

$-x_1 + x_2 = 0$

$x_1 = x_2$

$x_3 = x_4$

$V_{\lambda_2} = \{(x_1, x_1, x_3, x_3) \mid x_1, x_3 \in \mathbb{R}\}$

$\left. \begin{array}{l} x_1(1,1,0,0) + x_3(0,0,1,1) \\ \dim V_{\lambda_2} = 4 - 2 = 2 \end{array} \right\} \Rightarrow$

$\Rightarrow R_2 = \{(1,1,0,0), (0,0,1,1)\}$

reper în V_{λ_2}

$R = R_1 \cup R_2 = \{(1,0,0,0), (0,1,1,0), (1,1,0,0), (0,0,1,1)\}$

$$[f]_{R,R} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

7. $f: \mathbb{R}^3 \to \mathbb{R}^3$, $f(x) = (x_2, x_3, 2x_1 - 5x_2 + 4x_3)$

Să se arate că f nu este un endomorfism diagonalizabil.

$$\underbrace{\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 2 & -5 & 4 \end{pmatrix}}_{A = [f]_{R_0, R_0}} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} x_2 \\ x_3 \\ 2x_1 - 5x_2 + 4x_3 \end{pmatrix}$$

$$P(\lambda) = \det(A - \lambda I_3) = \begin{vmatrix} -\lambda & 1 & 0 \\ 0 & -\lambda & 1 \\ 2 & -5 & 4-\lambda \end{vmatrix} = 0$$

$$(-1)^3 [\lambda^3 - \sigma_1 \lambda^2 + \sigma_2 \lambda - \sigma_3] = 0$$

$$\sigma_1 = 4$$

$$\sigma_2 = \begin{vmatrix} 0 & 1 \\ -5 & 4 \end{vmatrix} + \begin{vmatrix} 0 & 0 \\ 2 & 4 \end{vmatrix} + \begin{vmatrix} 0 & 1 \\ 0 & 0 \end{vmatrix} = 5$$

$$\sigma_3 = (-1)^{1+2} \begin{vmatrix} 0 & 1 \\ 2 & 4 \end{vmatrix} = 2$$

$$\lambda^3 - 4\lambda^2 + 5\lambda - 2 = 0$$

$$\lambda^3 - \lambda^2 - 3\lambda^2 + 3\lambda + 2\lambda - 2 = \lambda^2(\lambda - 1) - 3\lambda(\lambda-1) + 2(\lambda - 1)$$

$$= (\lambda - 1)(\lambda^2 - 3\lambda + 2) = 0 \Rightarrow (\lambda-1)^2(\lambda - 2) = 0$$

$\lambda_1 = 1 \quad m = 2$

$\lambda_2 = 2 \quad m = 1$

$V_{\lambda_1} = \{ x \in \mathbb{R}^3 \mid f(x) = 1 \cdot x \}$

$AX = X \Leftrightarrow (A - I_3)\, X = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

$$\begin{pmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 2 & -5 & 3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$\dim V_{\lambda_2} = 3 - 2 = 1 \neq m_1$

$V_{\lambda_1} = \{ x \in \mathbb{R}^3 \mid f(x) = 2x \}$

$AX = 2X \Rightarrow (A - 2I_3)\, X = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

$$\begin{pmatrix} -2 & 1 & 0 \\ 0 & -2 & 1 \\ 2 & -5 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$\dim V_{\lambda_2} = 3 - 2 = 1$