Tutoriat 7

1) a) Studiați continuitatea funcției f

b) Determinați $\frac{\partial f}{\partial x}$, $\frac{\partial f}{\partial y}$

c) Studiați diferențiabilitatea funcției f.

i) $f: \mathbb{R}^2 \to \mathbb{R}$

$$f(x,y) = \begin{cases} \frac{xy^3}{\sqrt{x^4+y^4}} & ; (x,y) \neq (0,0) \\ 0 & ; (x,y) = (0,0) \end{cases}$$

Sol.

a) f cont pe $\mathbb{R}^2 \setminus \{(0,0)\}$ (op. cu fct. elementare)

Studiem cont lui f în $(0,0)$

Fie $(x,y) \in \mathbb{R}^2 \setminus \{(0,0)\}$

$$|f(x,y) - f(0,0)| = \left| \frac{xy^3}{\sqrt{x^4+y^4}} - 0 \right| = \overbrace{\frac{y^2}{\sqrt{x^4+y^4}}}^{\leq 1} \cdot |xy|$$

$$\sqrt{x^4+y^4} \geq \sqrt{y^4} = y^2$$

$$\leq 1 \cdot |xy| \xrightarrow[(x,y) \to (0,0)]{} 0 \Rightarrow f \text{ cont în } (0,0).$$

b) Fie $(x,y) \in \mathbb{R}^2 \setminus \{(0,0)\}$

$$\frac{\partial f}{\partial x}(x,y) = \left(\frac{xy^3}{\sqrt{x^4+y^4}} \right)'_x = \frac{(xy^3)'_x \sqrt{x^4+y^4} - xy^3 \left(\sqrt{x^4+y^4}\right)'_x}{x^4+y^4}$$

$$= \frac{\sqrt{/}\, y^3\sqrt{x^4+y^4} - (xy^3)\frac{4x^3}{2\sqrt{x^4+y^4}}}{x^4+y^4} = \frac{y^3(x^4+y^4) - 2x^4y^3}{(x^4+y^4)\sqrt{x^4+y^4}}$$

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$$= \frac{x^4y^3 + y^7 - 2x^4y^3}{(x^4+y^4)\sqrt{x^4+y^4}} = \frac{y^7 - x^4y^3}{(x^4+y^4)\sqrt{x^4+y^4}}$$

$$\frac{\partial f}{\partial y}(x,y) = \left(\frac{xy^3}{\sqrt{x^4+y^4}}\right)'_y = \frac{(y^3x)'_y\sqrt{x^4+y^4} - (y^3x)\left(\sqrt{x^4+y^4}\right)'_y}{x^4+y^4} =$$

$$= \frac{3y^2x\sqrt{x^4+y^4} - (y^3x)\frac{4y^3}{2\sqrt{x^4+y^4}}}{x^4+y^4} = \frac{3y^2x(x^4+y^4) - 2y^6x}{(x^4+y^4)\sqrt{x^4+y^4}} =$$

$$= \frac{3y^2x^5 + 3y^6x - 2y^6x}{(x^4+y^4)\sqrt{x^4+y^4}} = \frac{3y^2x^5 + y^6x}{(x^4+y^4)\sqrt{x^4+y^4}}$$

$$\frac{\partial f}{\partial x}(0,0) = \lim_{t\to 0} \frac{f((0,0)+te_1) - f(0,0)}{t}$$

$$= \lim_{t\to 0} \frac{f(t,0) - f(0,0)}{t} = \lim_{t\to 0} \frac{\frac{t\cdot 0}{\sqrt{t^4+0}} - 0}{t} = \lim_{t\to 0} \frac{0-0}{t}$$

$$= 0$$

$$\frac{\partial f}{\partial y}(0,0) = \lim_{t\to 0} \frac{f((0,0)+te_2) - f(0,0)}{t} = \lim_{t\to 0} \frac{f(0,t) - f(0,0)}{t}$$

$$= \lim_{t\to 0} \frac{\frac{0\cdot t^3}{\sqrt{0+t^4}} - 0}{t} =$$

$$= \lim_{t\to 0} \frac{0-0}{t} = 0$$

c) $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}$ cont pe $\mathbb{R}\setminus\{(0,0)\}$ (ope cu fct elementare)
$\mathbb{R}^2\setminus\{(0,0)\}$ deschisă
$\Rightarrow$ f diferențiabilă pe $\mathbb{R}^2\setminus\{(0,0)\}$

Studiem diferențiabilitatea lui f în $(0,0)$.

Dacă f ar fi diferențiabilă în $(0,0)$, atunci

$$df(0,0): \mathbb{R}^2 \to \mathbb{R}, \quad df(0,0)(u,v) =$$

$$= {}^t\left[\begin{pmatrix} \frac{\partial f}{\partial x}(0,0) & \frac{\partial f}{\partial y}(0,0) \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}\right] = {}^t\left[\begin{pmatrix} 0 & 0 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}\right] = 0 \cdot u + 0 \cdot v = 0$$

$$\lim_{(x,y)\to(0,0)} \frac{f(x,y) - f(0,0) - df(0,0)((x,y)-(0,0))}{\|(x,y)-(0,0)\|} =$$

$$= \lim_{(x,y)\to(0,0)} \frac{\frac{xy^3}{\sqrt{x^4+y^4}} - 0 - 0}{\sqrt{x^2+y^2}} = \lim_{(x,y)\to(0,0)} \frac{xy^3}{\sqrt{x^4+y^4} \cdot \sqrt{x^2+y^2}} = E(x,y)$$

Studiem $|E(x,y)| = \dfrac{|xy^3|}{\sqrt{x^4+y^4} \cdot \sqrt{x^2+y^2}} = \underbrace{\dfrac{y^2}{\sqrt{x^4+y^4}}}_{\leq 1} \cdot \underbrace{\dfrac{|x|}{\sqrt{x^2+y^2}}}_{\leq 1} \cdot |y|$

$\sqrt{x^4+y^4} \geq \sqrt{y^4} = y^2$

$\sqrt{x^2+y^2} \geq \sqrt{x^2} = |x|$

$0 \leq |-|$

$\leq 1 \cdot 1 \cdot |y| \xrightarrow[(x,y)\to(0,0)]{} 0 \; (*)$ $\quad \Big\} \; 0 \leq |E(x,y)| \leq 0 \; (*)$

Deci $\displaystyle\lim_{(x,y)\to(0,0)} \frac{xy^3}{\sqrt{x^4+y^4} \cdot \sqrt{x^2+y^2}} = 0.$

Prin urmare, f este diferențiabilă în $(0,0)$ $\square$

T7

ii) $f(x,y) = \begin{cases} \dfrac{x^2y^2}{\sqrt{x^4+y^6}} & , (x,y) \neq (0,0) \\ 0 & , (x,y) = (0,0) \end{cases}$

Sol:

a) f cont pe $\mathbb{R}^2 \setminus \{(0,0)\}$ (op. cu fct elementare)

Studiem cont lui f în (0, 0).

Fie $(x,y) \in \mathbb{R}^2 \setminus \{(0,0)\}$

$$|f(x,y) - f(0,0)| = \left| \frac{x^2y^2}{\sqrt{x^4+y^6}} - 0 \right| = \underbrace{\frac{x^2}{\sqrt{x^4+y^6}}}_{\leq 1} \cdot y^2$$

$$\sqrt{x^4+y^6} \geq \sqrt{x^4} = |x^2| = x^2$$

$\leq 1 \cdot y^2 \xrightarrow[(x,y)\to(0,0)]{} 0 \Rightarrow$ f. cont în (0,0).

b) Fie $(x,y) \in \mathbb{R}^2 \setminus \{(0,0)\}$

$$\frac{\partial f}{\partial x}(x,y) = \left(\frac{x^2y^2}{\sqrt{x^4+y^6}} \right)'_x = \frac{(x^2y^2)'_x \sqrt{x^4+y^6} - (x^2y^2) \cdot (\sqrt{x^4+y^6})'_x}{x^4+y^6}$$

$$\overset{\sqrt{}/}{=} \frac{2xy^2\sqrt{x^4+y^6} - (x^2y^2)\dfrac{4x^3}{2\sqrt{x^4+y^6}}}{x^4+y^6} = \frac{2xy^2(x^4+y^6) - (x^2y^2) \cdot 2x^3}{(x^4+y^6)\sqrt{x^4+y^6}}$$

$$= \frac{2x^5y^2 + 2xy^8 - 2x^5y^2}{(x^4+y^6)\sqrt{x^4+y^6}} = \frac{2xy^8}{(x^4+y^6)\sqrt{x^4+y^6}}$$

$$\frac{\partial f}{\partial y}(x,y) = \left(\frac{x^2y^2}{\sqrt{x^4+y^6}} \right)'_y = \frac{(x^2y^2)'_y \sqrt{x^4+y^6} - (x^2y^2)(\sqrt{x^4+y^6})'_y}{x^4+y^6}$$

$$\overset{\sqrt{}/}{=} \frac{2yx^2\sqrt{x^4+y^6} - (x^2y^2)\dfrac{6y^5}{2\sqrt{x^4+y^6}}}{x^4+y^6} = \frac{2yx^2(x^4+y^6) - (x^2y^2) \cdot 3y^5}{(x^4+y^6)\sqrt{x^4+y^6}}$$

$$= \frac{2yx^6 + 2y^7x^2 - 3y^7x^2}{(x^4+y^6)\sqrt{x^4+y^6}} = \frac{2yx^6 - y^7x^2}{(x^4+y^6)\sqrt{x^4+y^6}}$$

$$\frac{\partial f}{\partial x}(0,0) = \lim_{t\to 0} \frac{f((0,0)+te_1) - f(0,0)}{t} = \lim_{t\to 0} \frac{f(t,0)-f(0,0)}{t}$$

$$= \lim_{t\to 0} \frac{\frac{t^2\cdot 0}{\sqrt{t^4+0}} - 0}{t} = \lim_{t\to 0} \frac{0-0}{t} = 0$$

$$\frac{\partial f}{\partial y}(0,0) = \lim_{t\to 0} \frac{f((0,0)+te_2) - f(0,0)}{t} = \lim_{t\to 0} \frac{f(0,t)-f(0,0)}{t}$$

$$= \lim_{t\to 0} \frac{\frac{0\cdot t^2}{\sqrt{0+t^6}} - 0}{t} = \lim_{t\to 0} \frac{0-0}{t}$$

c) $\left.\begin{array}{l}\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \text{ cont pe } \mathbb{R}^2 \setminus \{(0,0)\} \\ \mathbb{R}^2 \setminus \{(0,0)\} \text{ deschisă}\end{array}\right\} \Rightarrow f$ diferențiabilă pe $\mathbb{R}^2 \setminus \{(0,0)\}$

Studiem diferențiabilitatea lui f în $(0,0)$.

Dacă f ar fi diferențiabilă în $(0,0)$, at.

$$df_{(0,0)} : \mathbb{R}^2 \to \mathbb{R}, \quad df_{(0,0)}(u,v) =$$

$${}^t\left[\begin{pmatrix} \frac{\partial f}{\partial x}(0,0) & \frac{\partial f}{\partial y}(0,0)\end{pmatrix}\begin{pmatrix} u \\ v\end{pmatrix}\right] = 0\cdot u + 0\cdot v = 0$$

$$\lim_{(x,y)\to(0,0)} \frac{f(x,y) - f(0,0) - df_{(0,0)}((x,y)-(0,0))}{\|(x,y)-(0,0)\|} =$$

$$= \lim_{(x,y)\to(0,0)} \frac{\frac{x^2y^2}{\sqrt{x^4+y^6}} - 0 - 0}{\sqrt{x^2+y^2}} = \lim_{(x,y)\to(0,0)} \frac{x^2y^2}{\sqrt{x^4+y^6}\cdot\sqrt{x^2+y^2}} = E(x,y)$$

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Studiem $\left|E(x,y)\right| = \left|\dfrac{x^2y^2}{\sqrt{x^4+y^6}\cdot\sqrt{x^2+y^2}}\right| =$

$$= \underbrace{\frac{x^2}{\sqrt{x^4+y^6}}}_{\leq 1} \cdot \underbrace{\frac{|y|}{\sqrt{x^2+y^2}}}_{\leq 1} \cdot |y|$$

$$\sqrt{x^4+y^6} \geq \sqrt{x^4} = |x^2| = x^2$$

$$\sqrt{x^2+y^2} \geq \sqrt{y^2} = |y|$$

$$\leq 1 \cdot 1 \cdot |y| \xrightarrow[(x,y)\to(0,0)]{} 0$$

Deci $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^2y^2}{\sqrt{x^4+y^6}\cdot\sqrt{x^2+y^2}} = 0$

Prin urmare, f este diferențiabilă în $(0,0)$ $\square$

EXAMEN 2025

iii) $f(x,y) = \begin{cases} \dfrac{x^{11}y}{\sqrt{x^{20}+y^4}}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases}$

Sol:

a) $\left|f(x,y) - f(0,0)\right| = \left|\dfrac{x^{11}y}{\sqrt{x^{20}+y^4}} - 0\right| = \left|\dfrac{x^{11}y}{\sqrt{x^{20}+y^4}}\right| =$

$$= \underbrace{\frac{x^{10}}{\sqrt{x^{20}+y^4}}}_{\leq 1} \cdot |xy| \leq 1 \cdot |xy| \xrightarrow[(x,y)\to(0,0)]{} 0$$

$$\sqrt{x^{20}+y^4} \geq \sqrt{x^{20}} = |x^{10}| = x^{10}$$

$\Rightarrow f$ cont în $(0,0)$

b) $\frac{\partial f}{\partial x}(x,y) = \left(\frac{x^{11} y}{\sqrt{x^{20}+y^4}} \right)'_x = \frac{(x^{11}y)'_x \sqrt{x^{20}+y^4} - (x^{11}y)\left(\sqrt{x^{20}+y^4}\right)'_x}{x^{20}+y^4}$

$$= \frac{(11x^{10}y)\sqrt{x^{20}+y^4} - (x^{11}y)\cdot \frac{20x^{19}}{2\sqrt{x^{20}+y^4}}}{x^{10}+y^4}$$

$$= \frac{11x^{30}y + 11x^{10}y^5 - 10x^{30}y}{(x^{20}+y^4)\sqrt{x^{20}+y^4}} = \frac{11x^{10}y^5 + x^{30}y}{(x^{20}+y^4)\sqrt{x^{20}+y^4}}$$

$$\frac{\partial f}{\partial y}(x,y) = \left(\frac{x^{11} y}{\sqrt{x^{20}+y^4}} \right)'_y = \frac{(yx^{11})'_y \sqrt{x^{20}+y^4} - (yx^{11})\left(\sqrt{x^{20}+y^4}\right)'_y}{x^{20}+y^4} =$$

$$= \frac{x^{11}\sqrt{x^{20}+y^4} - (yx^{11})\frac{4y^3}{2\sqrt{x^{20}+y^4}}}{x^{20}+y^4} = \frac{x^{31} + y^4x^{11} - 2y^4x^{11}}{x^{20}+y^4}$$

$$= \frac{x^{31} - y^4x^{11}}{x^{20}+y^4}$$

$$\frac{\partial f}{\partial x}(0,0) = \lim_{t\to 0} \frac{f((0,0)+te_1) - f(0,0)}{t} = \lim_{t\to 0} \frac{f(t,0)-f(0,0)}{t}$$

$$= \lim_{t\to 0} \frac{\frac{t^{11}\cdot 0}{\sqrt{t^{20}+0}} - 0}{t} = \lim_{t\to 0} \frac{0-0}{t} = 0$$

$$\frac{\partial f}{\partial y}(0,0) = \lim_{t\to 0} \frac{f((0,0)+te_2) - f(0,0)}{t} = \lim_{t\to 0} \frac{f(0,t)-f(0,0)}{t}$$

$$= \lim_{t\to 0} \frac{\frac{0\cdot t}{\sqrt{0+t^4}} - 0}{t} = \lim_{t\to 0} \frac{0-0}{t} = 0$$

c) (...)

$df(0,0) : \mathbb{R}^2 \to \mathbb{R}$, $df(0,0)(u,v) =$

$$= \left[\begin{pmatrix} \frac{\partial f}{\partial x}(0,0) & \frac{\partial f}{\partial y}(0,0) \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} \right]^t = 0\cdot u + 0\cdot v = 0$$

$$\lim_{(x,y)\to(0,0)} \frac{f(x,y)-f(0,0)-df(0,0)\left((x,y)-(0,0)\right)}{\|(x,y)-(0,0)\|} =$$

$$= \lim_{(x,y)\to(0,0)} \frac{\frac{x^{11}y}{\sqrt{x^{20}+y^4}} - 0 - 0}{\sqrt{x^2+y^2}} = \lim_{(x,y)\to(0,0)} \frac{x^{11}y}{\sqrt{x^{20}+y^4}\cdot\sqrt{x^2+y^2}}$$

$$= \lim_{(x,y)\to(0,0)} \frac{x^{10}}{\sqrt{x^{20}+y^4}} \cdot \frac{y}{\sqrt{x^2+y^2}} \cdot x = E(x,y)$$

Studiem $|E(x,y)| = \left| \overset{\le 1}{\frac{x^{10}}{\sqrt{x^{20}+y^4}}} \cdot \overset{\le 1}{\frac{y}{\sqrt{x^2+y^2}}} \cdot x \right| \le$

$$\sqrt{x^{20}+y^4} \geq \sqrt{x^{20}} = |x^{10}| = x^{10}$$

$$\sqrt{x^2+y^2} \geq \sqrt{y^2} = |y|$$

$$\le 1 \cdot 1 \cdot x \xrightarrow[(x,y)\to(0,0)]{} 0$$

Deci $\lim\limits_{(x,y)\to(0,0)} \cancel{\frac{x^{11}y}{\sqrt{x^{20}+y^4}}} \frac{x^{11}y}{\sqrt{x^{20}+y^4}\cdot\sqrt{x^2+y^2}} = 0$

Prin urmare, f este diferențiabilă în $(0,0)$. $\square$