CURS#11

13. Descompunerea valorilor singulare (DVS): algoritm DVS (naiv); instabilitatea algoritmului; Lema Schur şi DVS; proprietăţi; DVS pentru matrice complexe; matrice de rang nemaxim şi DVS.
14. Analiza senzitivităţii/stabilităţii sistemelor de ecuaţii liniare pătratice: perturbări şi inverse ale matricelor.

PROBLEME

1) Fie $\mathbf{A} \in \mathscr{M}_{m,n}(\mathbb{R})$ şi descompunerea valorilor singulare asociată

$$\mathbf{A} = \mathbf{U}\boldsymbol{\Sigma}\mathbf{V}^{\mathsf{T}},$$

unde

$$\begin{aligned}
&\mathbf{U} \coloneqq [\mathbf{u}_1\ \mathbf{u}_2\ \ldots\ \mathbf{u}_m] \in \mathscr{M}_m(\mathbb{R}): \quad \mathbf{U}^{\mathsf{T}}\mathbf{U} = \mathbf{I}_m;\\
&\mathbf{V} \coloneqq [\mathbf{v}_1\ \mathbf{v}_2\ \ldots\ \mathbf{v}_n] \in \mathscr{M}_n(\mathbb{R}): \quad \mathbf{V}^{\mathsf{T}}\mathbf{V} = \mathbf{I}_n;\\
&\boldsymbol{\Sigma} \coloneqq \operatorname{diag}(\sigma_1, \sigma_2, \ldots, \sigma_{\min(m,n)}) \in \mathscr{M}_{m,n}(\mathbb{R}):\\
&\qquad \sigma_1 \geq \sigma_2 \geq \ldots \geq \sigma_r > \sigma_{r+1} = \ldots = \sigma_{\min(m,n)} = 0.
\end{aligned}$$

Arătaţi că au loc identităţile:

(a) $\operatorname{rang}(\mathbf{A}) = \operatorname{rang}(\boldsymbol{\Sigma}) = r$;

(b) $\operatorname{Ker}(\mathbf{A}) = \operatorname{span}\{\mathbf{v}_r, \mathbf{v}_{r+1}, \ldots, \mathbf{v}_{\min(m,n)}\}$;

(c) $\operatorname{Im}(\mathbf{A}) = \operatorname{span}\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_r\}$;

(d) $\mathbf{A} = \displaystyle\sum_{i=1}^{r} \sigma_i\big(\mathbf{u}_i\mathbf{v}_i^{\mathsf{T}}\big)$, unde $\mathbf{u}_i\mathbf{v}_i^{\mathsf{T}} \in \mathscr{M}_{m,n}(\mathbb{R})$ cu $\operatorname{rang}\big(\mathbf{u}_i\mathbf{v}_i^{\mathsf{T}}\big) = 1$, $i = \overline{1,r}$.

(e) $\|\mathbf{A}\|_{\mathrm{F}} = \sqrt{\sigma_1^2 + \sigma_2^2 + \ldots + \sigma_{\min(m,n)}^2}$;

(f) $\|\mathbf{A}\|_2 = \sigma_1 = \sigma_{\max}(\mathbf{A})$;

(g) $\displaystyle\min_{\substack{\mathbf{x}\in\mathbb{R}^n:\\ \mathbf{x}\neq\mathbf{0}_n}} \frac{\|\mathbf{A}\mathbf{x}\|_2}{\|\mathbf{x}\|_2} = \sigma_{\min(m,n)} = \sigma_{\min}(\mathbf{A})$.

__OBS:__ Fie DVS a matricei $A \in \mathcal{M}_{m,n}(\mathbb{R})$ dată de relația (1), unde

$$U := \text{col}\left[\underline{u}_1\ \underline{u}_2 \dots \underline{u}_m\right] \in \mathcal{M}_m(\mathbb{R}), \quad (2)$$
$$\underline{u}_i \in \mathbb{R}^m,\ i = \overline{1,m}$$

$$V := \text{col}\left[\underline{v}_1\ \underline{v}_2 \dots \underline{v}_n\right] \in \mathcal{M}_n(\mathbb{R}), \quad (3)$$
$$\underline{v}_i \in \mathbb{R}^n,\ i = \overline{1,n}$$

și $r := \min\{m, n\}$. Atunci

(i) $\sigma_i^2,\ i = \overline{1,r}$, și $\underline{v}_i \in \mathbb{R}^n,\ i = \overline{1,r}$, sunt <u>valorile proprii și vectorii proprii ai matricei $A^T A \in \mathcal{M}_n(\mathbb{R})$</u>.

(ii) $\sigma_i^2,\ i = \overline{1,r}$, și $\underline{u}_i \in \mathbb{R}^m,\ i = \overline{1,r}$, sunt <u>valorile proprii și vectorii proprii ai matricei $A A^T \in \mathcal{M}_m(\mathbb{R})$</u>.

__Dem:__ Din DVS (1), rezultă $AV = U\Sigma \Longrightarrow$

$$A\underline{v}_i = \sigma_i \underline{u}_i, \quad i = \overline{1,r} \quad (4)$$

$$\underline{u}_i = \frac{1}{\sigma_i} A \underline{v}_i, \quad i = \overline{1,r} \quad (5)$$

Prin transpunerea DVS (1), obținem

$$V^T A^T U = \Sigma^T \qquad (6)$$

și, prin urmare, $A^T U = V \Sigma^T \Rightarrow$

$$A^T \underline{u}_i = \sigma_i \underline{v}_i, \quad i = \overline{1,r} \qquad (7)$$

$$\underline{v}_i = \frac{1}{\sigma_i} A^T \underline{u}_i, \ i = \overline{1,r} \qquad (8)$$

(i) Inserând (5) în (7), obținem

$$\boxed{A^T A \ \underline{v}_i = \sigma_i^2 \underline{v}_i, \quad i = \overline{1,r}}$$

(ii) Analog, substituind (8) în (4), obținem

$$\boxed{A A^T \ \underline{u}_i = \sigma_i^2 \underline{u}_i, \quad i = \overline{1,r}}$$

□

INTERPRETARE GEOMETRICĂ:

Valorile singulare ale lui $A \in \mathcal{M}_{m,n}(\mathbb{R})$, notate cu $\sigma_i(A)$, $i = \overline{1,r}$, unde $r := \min\{m, n\}$, sunt semiaxele hiperelipsoidului

$$E := \{ A\underline{x} \in \mathbb{R}^m \mid \underline{x} \in \mathbb{R}^n, \|\underline{x}\|_2 = 1 \}.$$

<u>OBS</u>: DVS <u>nu este unică</u>!

$$\boxed{A = \sum_{i=1}^{r} \sigma_i \underline{u}_i \underline{v}_i^T = \sum_{i=1}^{r} \sigma_i (-\underline{u}_i)(-\underline{v}_i^T)}$$

<u>ALGORITM (DVS)</u>:

<u>Date</u>: $A \in \mathcal{M}_{m,n}(\mathbb{R})$, $r := \min\{m,n\}$

(I) <u>$r := n$ $(m \geq n)$</u>:

(1) Determină $\lambda_i \geq 0$, $i = \overline{1,r}$, și $\underline{v}_i \in \mathbb{R}^n$, $i = \overline{1,r}$, ortonormalizați prin procedeul Gram-Schmidt, a.î.

$$\boxed{\begin{aligned} &(A^T A)\, \underline{v}_i = \lambda_i \underline{v}_i, \quad i = \overline{1,n} \\ &\begin{cases} \lambda_i \geq 0, & i = \overline{1,n} \\ \underline{v}_i^T \underline{v}_j = \delta_{ij}, & i,j = \overline{1,n} \end{cases} \end{aligned}}$$

(2) Definește

$$\boxed{\begin{aligned} &V := \mathrm{col}\,[\underline{v}_1\ \underline{v}_2 \ldots \underline{v}_n] \in \mathcal{M}_n(\mathbb{R}) : V^T V = I_n \\ &\sigma_i := \sqrt{\lambda_i}, \quad i = \overline{1,n} \end{aligned}}$$

③ Determină

$$\boxed{\underline{u}_i := \frac{1}{\sigma_i} A\underline{v}_i \in \mathbb{R}^m, \quad i=\overline{1,n}}$$

și completează până la o bază ortonormată a lui $\mathbb{R}^m$ prin procedeul Gram-Schmidt

$$\boxed{\begin{aligned} &U := \operatorname{col}[\underline{u}_1\ \underline{u}_2 \ldots \underline{u}_n\ \underline{u}_{n+1} \ldots \underline{u}_m] \in \mathcal{M}_m(\mathbb{R}): \\ &\qquad U^T U = I_m \end{aligned}}$$

④ Definește

$$\boxed{\begin{aligned} &\Sigma := \begin{bmatrix} \Sigma_n \\ O_{m-n,n} \end{bmatrix} \in \mathcal{M}_{m,n}(\mathbb{R}) \\ &\qquad \Sigma_n := \operatorname{diag}(\sigma_1, \sigma_2, \ldots, \sigma_n) \in \mathcal{M}_n(\mathbb{R}) \end{aligned}}$$

Ⅱ $\underline{r := m \ (m \leq n)}$:

① Determină $\lambda_i \geq 0$, $i=\overline{1,m}$, și $\underline{u}_i \in \mathbb{R}^m$, $i=\overline{1,m}$, ortonormalizați prin procedeul Gram-Schmidt, a.î.

$$\boxed{\begin{array}{l}(AA^T)\,\underline{u}_i = \lambda_i \underline{u}_i, \quad i=\overline{1,m} \\ \begin{cases}\lambda_i \geq 0, & i=\overline{1,m} \\ \underline{u}_i^T \underline{u}_j = \delta_{ij}, & i,j = 1,m\end{cases}\end{array}}$$

② Determină

$$\boxed{\begin{array}{l}U := \mathrm{col}\,[\underline{u}_1\ \underline{u}_2 \ldots \underline{u}_m] \in \mathcal{M}_m(\mathbb{R}) : U^T U = I_m \\ \sigma_i := \sqrt{\lambda_i}, \quad i=\overline{1,m}\end{array}}$$

③ Determină

$$\boxed{\underline{v}_i := \frac{1}{\sigma_i} A^T \underline{u}_i, \quad i=\overline{1,m}}$$

și completează până la o bază ortonormată a lui $\mathbb{R}^n$ prin procedeul Gram-Schmidt

$$\boxed{\begin{array}{l}V := \mathrm{col}\,[\underline{v}_1\ \underline{v}_2 \ldots \underline{v}_m\ \underline{v}_{m+1} \ldots \underline{v}_n] \in \mathcal{M}_n(\mathbb{R}) : \\ V^T V = I_n\end{array}}$$

④ Definește

$$\boxed{\begin{array}{l}\Sigma := [\Sigma_m\ \ 0_{m,n-m}] \in \mathcal{M}_{m,n}(\mathbb{R}) : \\ \Sigma_m := \mathrm{diag}(\sigma_1, \sigma_2, \ldots, \sigma_m)\end{array}}$$

<u>OBS</u>: DVS determinată prin calculul valorilor și vectorilor proprii corespunzători lui $A^TA \in \mathcal{M}_n(\mathbb{R})$, $m \geq n$, respectiv lui $AA^T \in \mathcal{M}_m(\mathbb{R})$, $n \geq m$, este <u>instabilă</u>!

<u>EXEMPLU</u> ($n=2$):

$A = [\underline{a}_1 \ \underline{a}_2] \in \mathcal{M}_{m,2}(\mathbb{R})$,

$$\underline{a}_1, \underline{a}_2 \in \mathbb{R}^m : \begin{cases} \underline{a}_1^T \underline{a}_2 = \cos\theta \\ \|\underline{a}_1\|_2 = \|\underline{a}_2\| = 1 \end{cases}$$

$$A^TA = \begin{bmatrix} \underline{a}_1^T \\ \underline{a}_2^T \end{bmatrix} \begin{bmatrix} \underline{a}_1 & \underline{a}_2 \end{bmatrix} = \begin{bmatrix} \underline{a}_1^T\underline{a}_1 & \underline{a}_1^T\underline{a}_2 \\ \underline{a}_2^T\underline{a}_1 & \underline{a}_2^T\underline{a}_2 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & \cos\theta \\ \cos\theta & 1 \end{bmatrix} \Longrightarrow$$

$$0 = \det(A^TA - \lambda I_2) = \begin{vmatrix} 1-\lambda & \cos\theta \\ \cos\theta & 1-\lambda \end{vmatrix}$$

$$= (1-\lambda)^2 - \cos^2\theta = (1-\cos\theta-\lambda)(1+\cos\theta-\lambda)$$

$$\left.\begin{aligned} &= \left(2\sin^2\frac{\theta}{2} - \lambda\right)\left(2\cos^2\frac{\theta}{2} - \lambda\right) \\ &\theta \in [0,\pi] \Longrightarrow \theta/2 \in [0,\pi/2) \end{aligned}\right\} \Longrightarrow$$

$$\boxed{\begin{aligned} \sigma_1 &:= \sqrt{\lambda_1} = \sqrt{2}\,\sin\frac{\theta}{2} \\ \sigma_2 &:= \sqrt{\lambda_2} = \sqrt{2}\,\cos\frac{\theta}{2} \end{aligned}}$$

<u>valorile singulare ale lui A</u>

Obținem, prin calcule, <u>vectorii singulari la dreapta ai lui A</u>:

$$\boxed{\underline{v}_1 := \frac{1}{\sqrt{2}}\begin{bmatrix} -1 \\ 1 \end{bmatrix}; \quad \underline{v}_2 := \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix}}$$

și <u>vectorii singulari la stânga ai lui A</u>:

$$\boxed{\underline{u}_i := \frac{1}{\sigma_i} A\,\underline{v}_i, \quad i=1,2}$$

Dpdv numeric, dacă $\theta \in [0, \sqrt{\epsilon})$, unde $\epsilon > 0$ este precizia mașinii, atunci

$$\cos\theta \simeq \cos 0 + (\cos' 0)\,\theta + (\cos'' 0)\,\frac{\theta^2}{2}$$
$$= 1 + 0 - \frac{\theta^2}{2} = 1 - \frac{\epsilon}{2} \simeq 1 \Longrightarrow$$

$$\boxed{\cos\theta \simeq 1} \Longrightarrow \boxed{\begin{aligned} \sigma_1 &\simeq 0 \\ \sigma_2 &= \sqrt{2} \end{aligned}}$$

$\sigma_1 \simeq 0 \longrightarrow$ se pierde <u>$\sqrt{2}\sin\frac{\theta}{2} > 0$!</u>

OBS: Teorema #1 (DVS) se mai poate demonstra folosind

LEMA (Schur):

$\forall A \in \mathcal{M}_n(\mathbb{R})$: A are vectori proprii din $\mathbb{R}^n$,

$\exists U \in \mathcal{M}_n(\mathbb{R})$: $U^TU = UU^T = I_n$

$$\boxed{U^TAU = T \text{ sup. triunghiulară}}$$

Dem: Inducție după $n \geq 1$.

COROLAR:

$\forall A \in \mathcal{M}_n(\mathbb{R})$: $A^T = A$,

$\exists U \in \mathcal{M}_n(\mathbb{R})$: $U^TU = UU^T = I_n$

$$\boxed{U^TAU = D \text{ diagonală}}$$

Dem: Din Lema Schur, obținem

$U^TAU = T$ sup. triunghiulară

$T^T = (U^TAU)^T = U^TA^TU = U^TAU = T \Rightarrow$

$\Rightarrow T$ inf. triunghiulară

$\square$

PROPRIETĂȚI:

Fie DVS a lui $A \in \mathcal{M}_{m,n}(\mathbb{R})$

$$A = U \Sigma V^T$$

$$\begin{cases} U := [\underline{u}_1\ \underline{u}_2 \dots \underline{u}_m] \in \mathcal{M}_m(\mathbb{R}) : U^T U = I_m \\ V := [\underline{v}_1\ \underline{v}_2 \dots \underline{v}_n] \in \mathcal{M}_n(\mathbb{R}) : V^T V = I_n \\ \Sigma := \operatorname{diag}(\sigma_1, \sigma_2, \dots, \sigma_{\min\{m,n\}}) \in \mathcal{M}_{m,n}(\mathbb{R}) \\ \quad \sigma_1 \geq \sigma_2 \geq \dots \geq \sigma_r > \sigma_{r+1} = \dots = \sigma_{\min\{m,n\}} = 0 \end{cases}$$

Atunci:

(a) $\operatorname{rang} A = r = \operatorname{rang} \Sigma$

(b) $\operatorname{Ker} A = \operatorname{span}\{\underline{v}_{r+1}, \dots, \underline{v}_n\}$

(c) $\operatorname{Im} A = \operatorname{span}\{\underline{u}_1, \dots, \underline{u}_r\}$

(d) $A = \sum_{i=1}^{r} \sigma_i (\underline{u}_i \underline{v}_i^T)$

$$\begin{cases} \underline{u}_i \underline{v}_i^T \in \mathcal{M}_{m,n}(\mathbb{R}), \ i = \overline{1,r} \\ \operatorname{rang}(\underline{u}_i \underline{v}_i^T) = 1, \ i = \overline{1,r} \end{cases}$$

(e) $\|A\|_F = \sqrt{\sigma_1^2 + \sigma_2^2 + \ldots + \sigma_{\min\{m,n\}}^2}$

(f) $\|A\|_2 = \sigma_1 = \sigma_{max}(A)$

(g) $\displaystyle\min_{\substack{\underline{x} \in \mathbb{R}^n: \\ \underline{x} \neq \underline{0}_n}} \frac{\|A\underline{x}\|_2}{\|\underline{x}\|_2} = \sigma_{\min\{m,n\}} = \sigma_{\min}(A)$

<u>OBS</u> (<u>DVS redusă / thin SVD</u>):

Fie DVS a lui $A \in \mathcal{M}_{m,n}(\mathbb{R})$, $m \geq n$,

$A = U \Sigma V^T$:

$$\begin{cases} U := [\underline{u}_1\ \underline{u}_2 \ldots \underline{u}_m] \in \mathcal{M}_m(\mathbb{R}): \ U^T U = I_m \\ V = [\underline{v}_1\ \underline{v}_2 \ldots \underline{v}_n] \in \mathcal{M}_n(\mathbb{R}): \ V^T V = I_n \\ \Sigma = \operatorname{diag}(\sigma_1, \sigma_2, \ldots, \sigma_n) \in \mathcal{M}_{m,n}(\mathbb{R}): \\ \qquad \sigma_1 \geq \sigma_2 \geq \ldots \geq \sigma_n \geq 0 \end{cases}$$

Atunci are loc <u>DVS redusă</u>:

$$\boxed{\begin{aligned} & A = U_1 \Sigma_1 V: \\ & \begin{cases} U_1 := [\underline{u}_1\ \underline{u}_2 \ldots \underline{u}_n] \in \mathcal{M}_{m,n}(\mathbb{R}) \\ \Sigma_1 := \operatorname{diag}(\sigma_1, \sigma_2, \ldots, \sigma_n) \in \mathcal{M}_n(\mathbb{R}) \end{cases} \end{aligned}}$$

DVS PENTRU $\mathcal{M}_{m,n}(\mathbb{C})$

DEFINIȚII:

(i) Sn <u>matricea hermitiană</u> a matricei

$$Q = (q_{ij})_{\substack{i=\overline{1,n} \\ j=\overline{1,n}}} \in \mathcal{M}_n(\mathbb{C}) \text{ matricea}$$

$$Q^H := \overline{Q}^T = (\overline{q}_{ji})_{\substack{i=\overline{1,n} \\ j=\overline{1,n}}} \in \mathcal{M}_n(\mathbb{C})$$

(ii) $Q \in \mathcal{M}_n(\mathbb{C})$ sn <u>matrice unitară</u> dacă

$$Q^H Q = Q Q^H = I_n$$

TEOREMA #3 (<u>DVS pt $\mathcal{M}_{m,n}(\mathbb{C})$</u>):

$\forall A \in \mathcal{M}_{m,n}(\mathbb{C})$

$\exists U := [\underline{u}_1 \dots \underline{u}_m] \in \mathcal{M}_m(\mathbb{C}) : U^H U = U U^H = I_m$

$\exists V := [\underline{v}_1 \dots \underline{v}_n] \in \mathcal{M}_n(\mathbb{C}) : V^H V = V V^H = I_n$

$\exists \Sigma := \operatorname{diag}(\sigma_1, \dots, \sigma_{\min\{m,n\}}) \in \mathcal{M}_{m,n}(\mathbb{R})$:

$$\sigma_1 \geq \sigma_2 \geq \dots \geq \sigma_{\min\{m,n\}} \geq 0$$

$$\boxed{A = U \Sigma V^H}$$

MATRICE DE RANG NEMAXIM şi DVS

Erorile de rotunjire şi erorile (perturbările) datelor conduc la problema netrivială de determinare numerică a rangului unei matrice.

DEFINIȚIE:

Pentru $\varepsilon > 0$ suficient de mic, definim ε-rangul matricei $A \in \mathcal{M}_{m,n}(\mathbb{R})$, $m \geq n$, prin

$$\boxed{\operatorname{rang}(A, \varepsilon) := \min_{\|A - B\|_2 \leq \varepsilon} \operatorname{rang}(B)}$$

DEFINIȚIE:

Spunem că $A \in \mathcal{M}_{m,n}(\mathbb{R})$, $m \geq n$, calculată cu virgulă mobilă, este de rang numeric nemaxim dacă

$$\boxed{\begin{array}{l} \operatorname{rang}(A, \varepsilon) < \min\{m, n\} \\ \varepsilon := \epsilon \|A\|_2, \ \epsilon \text{ precizia maşinii} \end{array}}$$

OBS: Rangul numeric nemaxim și ε-rangul unei matrice pot fi caracterizate elegant prin intermediul DVS a matricei respective.

TEOREMA #2:

Fie DVS a matricei $A \in \mathcal{M}_{m,n}(\mathbb{R})$.

Dacă $k < r = \text{rang}(A)$ și

$$\boxed{A_k := \sum_{i=1}^{k} \sigma_i \underline{u}_i \underline{v}_i^T}$$

atunci

$$\boxed{\min_{\text{rang}(B)=k} \|A-B\|_2 = \|A-A_k\|_2 = \sigma_{k+1}}$$

Dem:

- $U^T A_k V = \text{diag}(\sigma_1, \dots, \sigma_k, 0, \dots, 0) \in \mathcal{M}_{m,n}(\mathbb{R})$

$\Rightarrow \boxed{\text{rang}\, A_k = k}$ (1)

DVS a lui $A \in \mathcal{M}_{m,n}(\mathbb{R}) \Rightarrow$

$U^T A V = \mathrm{diag}(\sigma_1, \dots, \sigma_k, \sigma_{k+1}, \dots, \sigma_{\min\{m,n\}}) \in \mathcal{M}_{m,n}(\mathbb{R})$

Prin urmare, obținem:

$$U^T A V - U^T A_k V = U^T (A - A_k) V =$$

$$= \mathrm{diag}(0, \dots, 0, \sigma_{k+1}, \dots, \sigma_{\min\{m,n\}}) \Longrightarrow$$

$$\boxed{\|A - A_k\|_2 = \sigma_{k+1}} \quad (2)$$

- Fie $B \in \mathcal{M}_{m,n}(\mathbb{R})$, $\mathrm{rang}(B) = k \Rightarrow$

$\exists\, \underline{x}_1, \underline{x}_2, \dots, \underline{x}_{n-k} \in \mathbb{R}^n$:

$$\begin{cases} \underline{x}_i^T \underline{x}_j = \delta_{ij}, & i,j = \overline{1, n-k} \\ \ker B = \mathrm{span}\{\underline{x}_1, \underline{x}_2, \dots, \underline{x}_{n-k}\} \subset \mathbb{R}^n \end{cases}$$

Cum $\dim \mathrm{span}\{\underline{x}_1, \underline{x}_2, \dots, \underline{x}_{n-k}\} = n-k$ și $\dim \mathrm{span}\{\underline{v}_1, \underline{v}_2, \dots, \underline{v}_{k+1}\} = k+1$, rezultă

$$\mathrm{span}\{\underline{x}_1, \underline{x}_2, \dots, \underline{x}_{n-k}\} \cap$$

$$\cap\, \mathrm{span}\{\underline{v}_1, \underline{v}_2, \dots, \underline{v}_{k+1}\} \neq \{\underline{0}_n\}.$$

Fie $\underline{z} \in \mathrm{span}\{\underline{x}_1, \underline{x}_2, \dots, \underline{x}_{n-k}\} \cap$

$\cap\, \mathrm{span}\{\underline{v}_1, \underline{v}_2, \dots, \underline{v}_{k+1}\}$ a.î. $\|\underline{z}\|_2 = 1$.

Atunci $B\underline{z} = \underline{0}_m$ și $A\underline{z} = \sum_{i=1}^{k+1} \sigma_i \underline{u}_i (\underline{v}_i^T \underline{z})$

$$\|A-B\|_2^2 \geq \|(A-B)\underline{z}\|_2^2 = \|A\underline{z} - \underbrace{B\underline{z}}_{=\underline{0}_m}\|_2^2$$

$$= \|A\underline{z}\|_2^2 = \sum_{i=1}^{k+1} \sigma_i^2 (\underline{v}_i^T \underline{z})^2$$

$$\geq \sigma_{k+1}^2 \sum_{i=1}^{k+1} (\underline{v}_i^T \underline{z})^2 = \sigma_{k+1}^2 \|\underline{z}\|_2^2 = \sigma_{k+1}^2$$

Dar $\underline{z} = \sum_{i=1}^{k+1} (\underline{v}_i^T \underline{z}) \underline{v}_i \Rightarrow \sum_{i=1}^{k+1} (\underline{v}_i^T \underline{z})^2 = \|\underline{z}\|_2^2$

Prin urmare, am obținut:

$$\boxed{\|A-B\|_2 \geq \sigma_{k+1}, \quad \forall B \in \mathcal{M}_{m,n}(\mathbb{R}) : \operatorname{rang} B = k} \quad (3)$$

Din relațiile (1)-(3) obținem rezultatul Teoremei #2.

$\square$

<u>OBS</u>:

1) $\sigma_{min}(A)$ = distanța, în $\|\cdot\|_2$, de la $A \in \mathcal{M}_{m,n}(\mathbb{R})$ la mulțimea matricelor de rang nemaxim ($< \min\{m,n\}$).

2) $\{A \in \mathcal{M}_{m,n}(\mathbb{R}) \mid \operatorname{rang} A = \min\{m,n\}\}$

este deschisă și densă în $\mathcal{M}_{m,n}(\mathbb{R})$.

3) Dacă $r_\varepsilon := \text{rang}(A, \varepsilon)$ este ε-rangul matricei $A \in \mathcal{M}_{m,n}(\mathbb{R})$, atunci

$$\boxed{\sigma_1 \geq \ldots \geq \sigma_{r_\varepsilon} > \varepsilon \geq \sigma_{r_\varepsilon+1} \geq \ldots \geq \sigma_{\min\{m,n\}}}$$

4) Dacă $\varepsilon \in (0, \epsilon]$, unde ϵ este <u>precizia mașinii</u>, și $r_\varepsilon := \text{rang}(A, \varepsilon)$ este ε-rangul matricei $A \in \mathcal{M}_{m,n}(\mathbb{R})$, atunci

$$\boxed{\sigma_k \simeq 0, \quad k \geq r_\varepsilon + 1}$$

4. PERTURBĂRI și INVERSE ALE MATRICELOR

LEMA #1 (Banach):

Dacă $F \in \mathcal{M}_n(\mathbb{R})$ cu $\|F\|_p < 1$, atunci

(i) $I_n - F \in \mathcal{M}_n(\mathbb{R})$ inversabilă.

(ii) $$\boxed{(I_n - F)^{-1} = \sum_{k=0}^{\infty} F^k}$$

(iii) Are loc relația

$$\boxed{\|(I_n - F)^{-1}\|_p \leq \frac{1}{1 - \|F\|_p}}$$

Dem:

(i) Pp $I_n - F \in \mathcal{M}_n(\mathbb{R})$ este singulară $\Rightarrow$

$\exists\, \underline{x} \in \mathbb{R}^n \setminus \{\underline{0}_n\} : (I_n - F)\,\underline{x} = \underline{0}_n$

$\exists\, \underline{x} \in \mathbb{R}^n \setminus \{\underline{0}_n\} : \underline{x} = F\underline{x}$

$$\left.\begin{array}{l} \|\underline{x}\|_p = \|F\underline{x}\|_p \leq \|F\|_p \|\underline{x}\|_p \\ \|\underline{x}\|_p > 0 \end{array}\right\} \Rightarrow$$

$\Rightarrow \|F\|_p \geq 1$ ⨳ $\Rightarrow I_n - F$ inversabilă

(ii) Fie $N \in \mathbb{N}$. $\Rightarrow$

$$\left(\sum_{k=0}^{N} F^k\right)(I_n - F) = \sum_{k=0}^{N} F^k - \sum_{k=1}^{N+1} F^k$$

$$= I_n - F^{N+1} \quad (*)$$

$$0 \leq \lim_{N\to\infty} \|F^N\|_p \leq \lim_{N\to\infty} (\|F\|_p)^N = 0$$

$$\Rightarrow \lim_{N\to\infty} F^N = O_n$$

Trecând (*) la limită pt $N \to \infty$, obținem

$$\left(\lim_{N\to\infty} \sum_{k=0}^{N} F^k\right)(I_n - F) = I_n,$$ i.e relația dorită.

(iii) $$\|(I_n - F)^{-1}\|_p = \left\|\sum_{k=0}^{\infty} F^k\right\|_p \leq \sum_{k=0}^{\infty} \|F^k\|_p$$

$$\leq \sum_{k=0}^{\infty} (\|F\|_p)^k = \lim_{N\to\infty} \frac{1 - (\|F\|_p)^{N+1}}{1 - \|F\|_p} = \frac{1}{1 - \|F\|_p}$$

$\square$

COROLAR #2:

$$\boxed{\begin{gathered}\forall F \in \mathcal{M}_n(\mathbb{R}) : \|F\|_p < 1, \\ \|(I_n - F)^{-1} - I_n\|_p \le \frac{\|F\|_p}{1 - \|F\|_p}\end{gathered}}$$

Dem:

$$\|(I_n - F)^{-1} - I_n\|_p = \|\sum_{k=0}^{\infty} F^k - I_n\|_p = \|\sum_{k=1}^{\infty} F^k\|_p$$

$$= \|F \sum_{k=0}^{\infty} F^k\|_p \le \|F\|_p \cdot \|\sum_{k=0}^{\infty} F^k\|_p =$$

$$= \|F\|_p \, \|(I_n - F)^{-1}\|_p \underset{\text{Lema Banach}}{\le} \frac{\|F\|_p}{1 - \|F\|_p} \qquad \square$$

OBS: O perturbare $O(\varepsilon)$ a lui I_n, F, induce o perturbare $O(\varepsilon)$ a inversei perturbării, $(I_n - F)^{-1}$, față de I_n.

TEOREMA #2:

Dacă $A \in \mathcal{M}_n(\mathbb{R})$ este inversabilă și $E \in \mathcal{M}_n(\mathbb{R})$ a.î. $r := \|A^{-1}E\|_p < 1$, atunci

(i) $A+E \in \mathcal{M}_n(\mathbb{R})$ inversabilă;

(ii) $$\boxed{\|(A+E)^{-1} - A^{-1}\|_p \leq \frac{\|E\|_p \, \|A^{-1}\|_p^2}{1-r}}$$

Dem:

(i) $$A+E = A\left[I_n - \underbrace{(-A^{-1}E)}_{=:F \in \mathcal{M}_n(\mathbb{R})\,:\, \|F\|_p < 1}\right] = A(I_n - F)$$

Din Lema Banach, rezultă că $I_n - F \in \mathcal{M}_n(\mathbb{R})$ este inversabilă.

Cum $A \in \mathcal{M}_n(\mathbb{R})$ inversabilă, obținem că $A+E = A(I_n - F) \in \mathcal{M}_n(\mathbb{R})$ inversabilă.

(ii) $$(A+E)^{-1} = [A(I_n - F)]^{-1} = (I_n - F)^{-1} A^{-1} \Rightarrow$$

$$(A+E)^{-1} - A^{-1} = (I_n - F)^{-1} A^{-1} - A^{-1}$$

$$= \left[(I_n - F)^{-1} - I_n\right] A^{-1} \Rightarrow$$

$$\|(A+E)^{-1} - A^{-1}\|_p = \|[(I_n - F)^{-1} - I_n] A^{-1}\|_p$$

$$\leq \|(I_n - F)^{-1} - I_n\|_p \; \|A^{-1}\|_p$$

$$\underset{\uparrow}{\leq} \frac{\|F\|_p}{1-\|F\|_p} \|A^{-1}\|_p = \frac{\|-A^{-1}E\|_p}{1-r} \|A^{-1}\|_p$$

Lema Banach

$$\leq \frac{\|A^{-1}\|_p \, \|E\|_p}{1-r} \|A^{-1}\|_p = \frac{\|E\|_p \, \|A^{-1}\|_p^2}{1-r}$$

$\square$