$$f:\mathbb{R}^2\to\mathbb{R},\ f(x,y)=\begin{cases}\frac{x^4y^3}{\sqrt{x^{12}+y^8}}, x^2+y^2\neq 0\\ 0, x=y=0\end{cases}$$

f cont. pe $\mathbb{R}^2\setminus\{(0,0)\}$

$$\lim_{(x,y)\to(0,0)} f(x,y)=\lim_{(x,y)\to(0,0)}\frac{x^4y^3}{\sqrt{x^{12}+y^8}}\overset{\frac{\infty}{\infty}}{=}?$$

Alegem:

$$y=x\Rightarrow\lim_{x\to 0} f(x,x)=\lim_{x\to 0}\frac{x^4x^3}{\sqrt{x^{12}+x^8}}=\lim_{x\to 0}\frac{x^7}{x^4\sqrt{x^4+1}}=\lim_{x\to 0}\frac{x^3}{\sqrt{x^4+1}}=0$$

$$y=-x\Rightarrow\lim_{x\to 0} f(x,-x)=\lim_{x\to 0}\frac{x^4(-x)^3}{\sqrt{x^{12}+(-x)^8}}=\lim_{x\to 0}-\frac{x^7}{x^4\sqrt{x^4+1}}=\lim_{x\to 0}-\frac{x^3}{\sqrt{x^4+1}}=0$$

$$y=\sqrt{x}\Rightarrow\lim_{x\to 0} f(x,\sqrt{x})=\lim_{x\to 0}\frac{x^4\sqrt{x^3}}{\sqrt{x^{12}+x^4}}=\lim_{x\to 0}\frac{x^{4+\frac{3}{2}}}{x^2\sqrt{x^8+1}}=\lim_{x\to 0}\frac{x^{2+\frac{3}{2}}}{\sqrt{x^8+1}}=0$$

Este posibil ca $\lim_{(x,y)\to(0,0)} f(x,y)=l(=0)\in\mathbb{R}$

Evaluam

$$0\ \leq\ |f(x,y)-l|\ \leq\ g(x,y)$$

$$(x,y)\to(0,0)$$

$$0$$

$$0\leq|f(x,y)-0|=\left|\frac{x^4y^3}{\sqrt{x^{12}+y^8}}\right|\leq\left|\frac{x^4y^3}{\sqrt{2|x^6y^4|}}\right|=\frac{x^4y^3}{x^3y^2\sqrt{2}}=\frac{xy}{\sqrt{2}}$$

$$0\ \leq\ |f(x,y)-l|\ \leq\frac{xy}{\sqrt{2}}$$

$$(x,y)\to(0,0)$$

$$0$$

$$\left.\begin{array}{r}\Rightarrow\lim_{(x,y)\to(0,0)}|f(x,y)-0|=0\Rightarrow\lim_{(x,y)\to(0,0)} f(x,y)=0\\ f(0,0)=0\end{array}\right|\Rightarrow f\text{ cont pe }(0,0)$$

$$\frac{\partial f}{\partial x}(x,y)=\frac{4x^3y^3\sqrt{x^{12}+y^8}-x^4y^3\frac{1}{2\sqrt{x^{12}+y^8}}12x^{11}}{x^{12}+y^8}=4x^3y^3\frac{2x^{12}+2y^8-3x^{12}}{(x^{12}+y^8)\sqrt{x^{12}+y^8}}=4x^3y^3\frac{2y^8-x^{12}}{(x^{12}+y^8)^{1+\frac{1}{2}}}$$

$$\frac{\partial f}{\partial y}(x,y)=\frac{3x^4y^2\sqrt{x^{12}+y^8}-x^4y^3\frac{1}{2\sqrt{x^{12}+y^8}}8y^7}{x^{12}+y^8}=x^4y^2\frac{3x^{12}+3y^8-4y^8}{(x^{12}+y^8)\sqrt{x^{12}+y^8}}=x^4y^2\frac{3x^{12}-y^8}{(x^{12}+y^8)^{1+\frac{1}{2}}}$$

$e_1=(1,0)$

$e_2=(0,1)$

$$\lim_{t\to 0}\frac{f((0,0)+te_1)-f((0,0))}{t}=\lim_{t\to 0}\frac{f((0,0)+t(1,0))-f((0,0))}{t}=\lim_{t\to 0}\frac{f((t,0))-f((0,0))}{t}=$$

$$=\lim_{t\to 0}\frac{\frac{t^4*0^3}{\sqrt{t^{12}+0^8}}-0}{t}=\lim_{t\to 0}\frac{0}{t}=0\Rightarrow\exists\frac{\partial f}{\partial x}(0,0)=0$$

$$\lim_{t\to 0}\frac{f((0,0)+te_2)-f((0,0))}{t}=\lim_{t\to 0}\frac{f((0,0)+t(0,1))-f((0,0))}{t}=\lim_{t\to 0}\frac{f((0,t))-f((0,0))}{t}=$$

$$=\lim_{t\to 0}\frac{\frac{0^4 * t^3}{\sqrt{0^{12}+t^8}}-0}{t}=\lim_{t\to 0}\frac{0}{t}=0 \Rightarrow \exists\frac{\partial f}{\partial y}(0,0)=0$$

$\exists\dfrac{\partial f}{\partial x}(x,y),\dfrac{\partial f}{\partial y}(x,y) pe \mathbb{R}^2\backslash\{(0,0)\}$

$\dfrac{\partial f}{\partial x},\dfrac{\partial f}{\partial y}$ functii continue pe $\mathbb{R}^2\backslash\{(0,0)\}$

$\mathbb{R}^2\backslash\{(0,0)\}$ multime deschisa

$\Rightarrow f$ diferentiabila pe $\mathbb{R}^2\backslash\{(0,0)\}$

Fie $T:\mathbb{R}^2\to\mathbb{R}, T((x,y))=\dfrac{\partial f}{\partial x}(0,0)x+\dfrac{\partial f}{\partial x}(0,0)y=0x+0y=0,\forall(x,y)\in\mathbb{R}^2$

$$\lim_{(x,y)\to(0,0)}\frac{|f(x,y)-f(0,0)-T((x,y)-(0,0))|}{||(x,y)-(0,0)||}=\lim_{(x,y)\to(0,0)}\frac{\left|\frac{x^4y^3}{\sqrt{x^{12}+y^8}}-0-T((x,y))\right|}{||(x,y)||}=$$

$$=\lim_{(x,y)\to(0,0)}\frac{\left|\frac{x^4y^3}{\sqrt{x^{12}+y^8}}-0\right|}{\sqrt{x^2+y^2}}=\lim_{(x,y)\to(0,0)}\underbrace{\frac{x^4y^3}{\sqrt{x^{12}+y^8}\sqrt{x^2+y^2}}}_{g(x,y)}$$

$$0\le g(x,y)=\frac{x^4y^3}{\sqrt{x^{12}+y^8}\sqrt{x^2+y^2}}\le\frac{x^4y^2|y|}{\sqrt{2|x^6y^4|}\sqrt{x^2+y^2}}=\frac{x^4y^2|y|}{x^3y^2\sqrt{2}\sqrt{x^2+y^2}}\le\frac{x|y|}{\sqrt{2x^2}}=\frac{y}{\sqrt{2}}$$

$$0\le g(x,y)\le\frac{y}{2}$$

$(x,y)\to(0,0)$

0

$$\Rightarrow\lim_{(x,y)\to(0,0)}\frac{|f(x,y)-f(0,0)-T((x,y)-(0,0))|}{||(x,y)-(0,0)||}=0\Rightarrow f \text{ e diferentiabila in } (0,0)$$

$f(x,y)=x^4+x^2+2xy+y^2,\forall x,y\in\mathbb{R}^2$

f cont. pe$\mathbb{R}^2$

Df = mult punct de discont.

$Df=\emptyset$

$\dfrac{\partial f}{\partial x}(x,y)=(x^4+x^2+2xy+y^2)'_x=4x^3+2x+2y$

$\dfrac{\partial f}{\partial y}(x,y)=(x^4+x^2+2xy+y^2)'_y=2x+2y$

$\dfrac{\partial f}{\partial x},\dfrac{\partial f}{\partial y}$ functii cont pe$\mathbb{R}^2$

$\mathbb{R}^2$mult desc

$\Rightarrow f$ dif pe$\mathbb{R}^2$

D_1 = mult pct in care functia este dif

$D_1=\emptyset$

$$\begin{cases} \frac{\partial f}{\partial x}(x,y)=0 \\ \frac{\partial f}{\partial x}(x,y)=0 \end{cases} \Leftrightarrow \begin{cases} 4x^3+2x+2y=0 \\ 2x+2y=0 \end{cases} \Leftrightarrow \begin{cases} 4x^3+2x+2x=0 \\ 2y=-2x \end{cases} \Leftrightarrow \begin{cases} 4x^3=0 \\ 2y=-2x \end{cases} \Leftrightarrow \begin{cases} x=0 \\ y=0 \end{cases}$$

$\Rightarrow (0,0) \in \mathbb{R}^2$

$C=\{(0,0)\}$

$$\frac{\partial^2 f}{\partial^2 x}(x,y)=\frac{\partial f}{\partial x}\left(\frac{\partial f}{\partial x}\right)(x,y)=(4x^3+2x+2y)'_x=12x^2+2$$

$$\frac{\partial^2 f}{\partial x \partial y}(x,y)=\frac{\partial f}{\partial x}\left(\frac{\partial f}{\partial y}\right)(x,y)=(2x+2y)'_x=2$$

$$\frac{\partial^2 f}{\partial^2 y}(x,y)=\frac{\partial f}{\partial y}\left(\frac{\partial f}{\partial y}\right)(x,y)=(2x+2y)'_y=2$$

$$\frac{\partial^2 f}{\partial y \partial x}(x,y)=\frac{\partial f}{\partial y}\left(\frac{\partial f}{\partial x}\right)(x,y)=(4x^3+2x+2y)'_y=2$$

$\mathbb{R}^2$ mult desc

$\frac{\partial^2 f}{\partial^2 x}, \frac{\partial^2 f}{\partial x \partial y}, \frac{\partial^2 f}{\partial^2 y}, \frac{\partial^2 f}{\partial y \partial x}$ functii cont pe $\mathbb{R}^2$

$$H_f(0,0)=\begin{pmatrix} \frac{\partial^2 f}{\partial^2 x}(0,0) & \frac{\partial^2 f}{\partial x \partial y}(0,0) \\ \frac{\partial^2 f}{\partial y \partial x}(0,0) & \frac{\partial^2 f}{\partial^2 y}(0,0) \end{pmatrix} = \begin{array}{c} \Delta_1 \\ \begin{pmatrix} \boxed{2} & 2 \\ 2 & 2 \end{pmatrix} \\ \Delta_2 \end{array}$$

$\Delta_1=2>0$

$\Delta_2=0$

$f(x,y)-f(0,0)\ ?\ 0$

$f(x,y)-f(0,0)=x^4+x^2+2xy+y^2=x^4+(x+y)^2 \geq 0 \Rightarrow (0,0)$ punct de minim local