$$\sum_{n=0}^{\infty} \frac{(1+a)(1+2a)\cdots(1+na)}{(1+b)(1+2b)\cdots(1+nb)} \quad a,b>0$$

$$\lim_{n\to\infty} x_n = ?$$

$$\lim_{n\to\infty} \frac{x_{n+1}}{x_n} = \lim_{n\to\infty} \frac{\dfrac{(1+b)(1+2b)\cdots(1+nb)(1+(n+1)a)}{(1+b)(1+2b)\cdots(1+nb)(1+(n+1)b)}}{\dfrac{(1+a)(1+2a)\cdots(1+na)}{(1+b)(1+2b)\cdots(1+nb)}} = \lim_{n\to\infty} \frac{na+a+1}{nb+b+1} \Rightarrow$$

$$= \frac{a}{b} \Rightarrow l = \frac{a}{b}$$

Discuție !

$$l > 1 \iff \frac{a}{b} > 1 \iff a > b \Rightarrow \sum_{n=0}^{\infty} x_n \text{ convergentă}$$

$$l < 1 \iff \frac{a}{b} < 1 \iff a < b \Rightarrow \sum_{n=0}^{\infty} x_n \text{ divergentă}$$

$$l = 1 \iff \frac{a}{b} = 1 \iff a = b \Rightarrow \sum_{n=0}^{\infty} \cancel{x_n} \text{ nu poate fi determinat}$$

$$x_n = \frac{(1+a)(1+2a)\cdots(1+na)}{(1+a)(1+2a)\cdots(1+na)} = 1 \xrightarrow{n\to\infty} 1 \Rightarrow \sum_{n=0}^{\infty} \text{ divergentă}$$

Deci :

dacă $a > b$, atunci $\sum_{n=0}^{\infty} x_n$ convergentă

dacă $a \le b$, atunci $\sum_{n=0}^{\infty} x_n$ divergentă

$$\lim \sum_{n=0}^{\infty} \frac{(n^2+n+1)^n}{(2\cdot n^2)^n}$$

$$\lim_{n\to\infty} x_n = ?$$

$$\lim_{n\to\infty} \sqrt[n]{x_n} = \lim_{n\to\infty} \sqrt[n]{\frac{(n^2+n+1)^n}{(2\cdot n^2)^n}} = \lim_{n\to\infty} \frac{n^2+n+1}{2\cdot n^2} = \frac{1}{2}$$

Discuție:

$$\frac{1}{2} > 1 (=) \; l > 1 =) \sum_{n=0}^{\infty} x_n \text{ convergentă}$$

$$\frac{1}{2} < 1 (=) \; l < 1 =) \sum_{n=0}^{\infty} x_n \text{ divergentă}$$

$$\frac{1}{2} = 1 (=) \; l = 1 =) \sum_{n=0}^{\infty} x_n \text{ nu putem determina}$$

$$x_n = \frac{(n^2+n+1)^n}{(n^2)^n} =) \lim_{x_n \to \infty} \frac{n^{2n}\left(1+\frac{1}{n}+\frac{1}{n^2}\right)^n}{n^{2n}} = 1 =) \sum_{n=0}^{\infty} x_n \text{ divergentă}$$

Deci:

$$\text{pt } 2 < 1, \sum_{n=0}^{\infty} x_n \text{ conv}$$

$$\text{pt } 2 \geqslant 1, \sum_{n=0}^{\infty} x_n \text{ div}$$

$$\sum_{n=0}^{\infty} tg\left(\sqrt{n^2+n+1} - \sqrt{n^2+n-1}\right)$$

$$\lim_{n\to\infty} s_n = ?$$

$$\sqrt{A} - \sqrt{B} = \frac{A-B}{\sqrt{A}+\sqrt{B}} \implies \sqrt{n^2+n+1} - \sqrt{n^2+n-1} = \frac{n^2+n+1-n^2-n+1}{\sqrt{n^2+n+1} + \sqrt{n^2+n-1}} =$$

$$= \frac{2}{\sqrt{n^2+n+1} + \sqrt{n^2+n+1}}$$

$$y_n = \frac{1}{n}$$

folosim crit. comp. cu limită $\implies$

$$\lim_{n\to\infty} \frac{x_n}{y_n}$$

$$\lim_{n\to\infty} \frac{tg\left(\frac{2}{\sqrt{n^2+n+1} + \sqrt{n^2+n-1}}\right)}{\frac{1}{n}} = \lim_{n\to\infty} \frac{tg\left(\frac{2}{\sqrt{n^2+n+1} + \sqrt{n^2+n-1}}\right)}{\frac{2}{\sqrt{n^2+n+1} + \sqrt{n^2+n-1}}} \cdot \cancel{\sqrt{n^2+1}}$$

$$\cdot \frac{\frac{2}{\sqrt{n^2+n+1} + \sqrt{n^2+n-1}}}{\frac{1}{n}} = \lim_{n\to\infty} \frac{2n}{\sqrt{n^2+n+1} + \sqrt{n^2+n-1}} = \lim_{n\to\infty} \frac{2n}{2\sqrt{n^2+n}\sqrt{}}$$

$$\lim_{t\to 0} \frac{tg(t)}{t} = 1$$

$$= \lim_{n\to\infty} \frac{2n}{n\left(\sqrt{1+\frac{1}{n}+\frac{1}{n^2}} + \sqrt{1+\frac{1}{n}-\frac{1}{n^2}}\right)} = \lim_{n\to\infty} \frac{2}{2} = 1$$

$$l = 1 \implies \sum_{n=0}^{\infty} x_n \sim \sum_{n=0}^{\infty} y_n$$

$$\lim_{n\to\infty} y_n = 0 \implies \sum_{n=0}^{\infty} y_n \text{ conv} \implies$$

$$\implies \sum_{n=0}^{\infty} x_n \text{ conv.}$$

$$\sum_{n=0}^{\infty} \left(\frac{an}{n+1} \right)^{n(n+1)}$$

$$\lim_{n \to \infty} s_x = ?$$

$$\lim_{n \to \infty} \sqrt[n]{f_n} = \lim_{n \to \infty} \left(\frac{an}{n+1} \right)^{n+1} = \lim_{n \to \infty} \left(1 - 1 + \frac{an}{n+1} \right)^{n+1} =$$

$$= \lim_{n \to \infty} \left(1 + \frac{an - n - 1}{n+1} \right)^{n+1} = \lim_{n \to \infty} \left(1 + \frac{n(a-1) - 1}{n+1} \right)^{n+1} =$$

$$= \lim_{n \to \infty} \left(1 + \frac{n(a-1)+1}{n+1} \right)^{\frac{n+1}{n(a-1)-1} \cdot (n(a-1)-1)} = e^{\lim_{n \to \infty} (n(a-1)-1)} =$$

$$= e^{\infty} = \infty \Rightarrow l > 1 \Rightarrow \sum_{n=0}^{\infty} f_n \text{ conv}$$

caz de exceptie $a = 1 \Rightarrow e^{\lim_{n \to \infty} (n(1-1)-1)} \Rightarrow e^{-1} = \frac{1}{e} < 1 \Rightarrow$

$$\Rightarrow \sum_{n=0}^{\infty} f_n \text{ div.}$$

$$\sum_{n=1}^{\infty} \frac{e^n \cdot n!}{n^n} \qquad (\text{Ce i-eu făcut lui Cernea / Ilieș?})$$

Stirling Lupulaș, o !

$$\lim_{n \to \infty} x_n = ?$$

$$\lim_{x \to \infty} \sqrt[n]{x_n} = \lim_{n \to \infty} \sqrt[n]{\frac{e^n \cdot n!}{n^n}} = \lim_{n \to \infty} \frac{e \sqrt[n]{n!}}{n} \approx \lim_{n \to \infty} \frac{e \frac{n}{e}}{n} = \frac{e}{e}$$

Discuție:

$$\frac{e}{e} < 1 \Rightarrow \ell < 1 \Rightarrow \sum_{n=0}^{\infty} x_n \text{ conv}$$

$$\frac{e}{e} > 1 \Rightarrow \ell > 1 \Rightarrow \sum_{n=0}^{\infty} x_n \text{ div}$$

$$\frac{e}{e} = 1 \Rightarrow \ell = 1 \Rightarrow \text{ nu știm}$$

$$\frac{e}{e} = 1 \Leftrightarrow e = e \Rightarrow x_n = \frac{e^n \cdot n!}{n^n}$$

$$\lim_{n \to \infty} x_n = \lim_{n \to \infty} \frac{e^n \cdot n!}{n^n} \underset{\text{Stirling}}{\approx} \lim_{n \to \infty} \sqrt{2\pi n} = \infty \Rightarrow \sum_{n=0}^{\infty} x_n \text{ div}$$