EXAMEN 2022

1) $T:\mathbb{R}^4 \to \mathbb{R}^4$ aplicație liniară

$$A = \begin{pmatrix} 0 & 3 & -2 & 1 \\ -1 & 3 & 0 & 0 \\ -1 & 1 & 2 & 0 \\ -2 & 3 & -2 & 3 \end{pmatrix}$$

a) P_A, Spec A
b) V_λ, $\dim V_\lambda$
c) Să se arate că T izomorfism liniar. Este T diagonalizabilă?
d) forma canonică Jordan a lui T.

a) $$P_A(\lambda) = \det(A - \lambda I_4) = \begin{vmatrix} -\lambda & 3 & -2 & 1 \\ -1 & 3-\lambda & 0 & 0 \\ -1 & 1 & 2-\lambda & 0 \\ -2 & 3 & -2 & 3-\lambda \end{vmatrix} = \underbrace{1\cdot(-1)^{1+4}}_{-1} \cdot \underbrace{\begin{vmatrix} -1 & 3-\lambda & 0 \\ -1 & 1 & 2-\lambda \\ -2 & 3 & -2 \end{vmatrix}}_{\Delta_1} + \underbrace{(3-\lambda)\cdot(-1)^{4+4}}_{3-\lambda} \cdot \underbrace{\begin{vmatrix} -\lambda & 3 & -2 \\ -1 & 3-\lambda & 0 \\ -1 & 1 & 2-\lambda \end{vmatrix}}_{\Delta_2}$$

$$\Delta_1 = \begin{vmatrix} -1 & 3-\lambda & 0 \\ -1 & 1 & 2-\lambda \\ -2 & 3 & -2 \end{vmatrix} \xlongequal[L_3 \to L_3 - 2L_1]{L_2 \to L_2 - L_1} \begin{vmatrix} -1 & 3-\lambda & 0 \\ 0 & -2+\lambda & 2-\lambda \\ 0 & -3+2\lambda & -2 \end{vmatrix} = (-1)\cdot(-1)^{1+1} \cdot \begin{vmatrix} -2+\lambda & 2-\lambda \\ -3+2\lambda & -2 \end{vmatrix} =$$

$$\Delta_1 = -\left[-2(\lambda-2) - (2-\lambda)(-3+2\lambda)\right]$$
$$\Delta_1 = -\left[-2(\lambda-2) - (\lambda-2)(3-2\lambda)\right]$$
$$\Delta_1 = -\left[(\lambda-2)(-2-3+2\lambda)\right]$$
$$\Delta_1 = -(\lambda-2)(-5+2\lambda)$$

$$\Delta_2 = \begin{vmatrix} -\lambda & 3 & -2 \\ -1 & 3-\lambda & 0 \\ -1 & 1 & 2-\lambda \end{vmatrix} \xlongequal{L_3 \to L_3 - L_2} \begin{vmatrix} -\lambda & 3 & -2 \\ -1 & 3-\lambda & 0 \\ 0 & -2+\lambda & 2-\lambda \end{vmatrix} = (\lambda-2)\cdot \begin{vmatrix} -\lambda & 3 & -2 \\ -1 & 3-\lambda & 0 \\ 0 & 1 & -1 \end{vmatrix}$$

$$\Delta_2 = (\lambda-2)\left(\lambda(3-\lambda) + 2 + 0 - 0 - 0 - 3\right)$$
$$\Delta_2 = (\lambda-2)\left(\lambda(3-\lambda) - 1\right)$$

$$P_A(\lambda) = (\lambda-2)(2\lambda-5) + (3-\lambda)(\lambda-2)(3\lambda-\lambda^2-1)$$
$$P_A(\lambda) = (\lambda-2)\left[2\lambda-5 + (3-\lambda)(3\lambda-\lambda^2-1)\right]$$
$$P_A(\lambda) = (\lambda-2)\left(2\lambda-5+9\lambda-3\lambda^2-3-3\lambda^2+\lambda^3+\lambda\right)$$
$$P_A(\lambda) = (\lambda-2)\left(\lambda^3-6\lambda^2+12\lambda-8\right)$$
$$P_A(\lambda) = (\lambda-2)(\lambda-2)^3 = (\lambda-2)^4$$

Deci $\boxed{P_A(\lambda) = (\lambda-2)^4}$ $\Rightarrow m_a(2) = 4$

$P_A(\lambda) = 0 \Rightarrow \lambda_1 = \lambda_2 = \lambda_3 = \lambda_4 = 2$

b) $V_{\lambda_1} = \{ v \in \mathbb{R}^4 \mid Av = 2v \}$

$(A - 2I_4)v = 0$

Fie $v = \begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix} \Rightarrow \begin{pmatrix} -2 & 3 & -2 & 1 \\ -1 & 1 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ -2 & 3 & -2 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix} = 0$

$$\Rightarrow \begin{cases} -2x+3y-2z+t=0 \\ -x+\ \ y=0 \\ -x+y=0 \Rightarrow x=y \\ -2x+3y-2z+t=0 \end{cases} \Rightarrow \quad x-2z+t=0 \Rightarrow x=2z-t$$

$$\Rightarrow V_{\lambda_1}=\{(2z-t,\ 2z-t,\ z,\ t)\mid z,t\in\mathbb{R}\}$$

$$\Rightarrow V_{\lambda_1}=\langle(2,2,1,0),(-1,-1,0,1)\rangle$$

$$\Rightarrow \dim V_{\lambda_1}=2 \Rightarrow m_g(\lambda_1)=m_g(2)=2$$

Teorie! $\quad T$ diagonalizabilă ?

$f:V\to V$ endomorfism diagonalizabil:

$$\begin{cases} (1)\ \ m_a(\lambda_1)+m_a(\lambda_2)+\dots+m_a(\lambda_p)=n=\dim_K V \\ (2)\ \ m_a(\lambda_i)=m_g(\lambda_i) \end{cases}$$

$T:\mathbb{R}^4\to\mathbb{R}^4$

Cum $\left.\begin{matrix} m_a(2)=4 \\ m_g(2)=2 \end{matrix}\right\} \Rightarrow 4=2$ fals $\Rightarrow$ T nu este diagonalizabilă

c) T izomorfism liniar $\Leftrightarrow$ T aplicație liniară și bijectivă

T aplicație liniară

$$f(X)=AX,\quad X=\begin{pmatrix}x\\y\\z\\t\end{pmatrix},\quad A=\begin{pmatrix}0&3&-2&1\\-1&3&0&0\\-1&1&2&0\\-2&3&-2&3\end{pmatrix}$$

Fie $X_1, X_2, X_3, X_4 \in \mathbb{R}^4$, $\alpha_1,\alpha_2,\alpha_3,\alpha_4\in\mathbb{R}$

$$X_1=\begin{pmatrix}x_1\\y_1\\z_1\\t_1\end{pmatrix},\ X_2=\begin{pmatrix}x_2\\y_2\\z_2\\t_2\end{pmatrix},\ X_3=\begin{pmatrix}x_3\\y_3\\z_3\\t_3\end{pmatrix},\ X_4=\begin{pmatrix}x_4\\y_4\\z_4\\t_4\end{pmatrix}$$

$$f(\alpha_1X_1+\alpha_2X_2+\alpha_3X_3+\alpha_4X_4)=A(\alpha_1X_1+\alpha_2X_2+\alpha_3X_3+\alpha_4X_4)=A(\alpha_1X_1)+A(\alpha_2X_2)+A(\alpha_3X_3)+A(\alpha_4X_4)=(A\alpha_1)X_1+(A\alpha_2)X_2+(A\alpha_3)X_3+(A\alpha_4)X_4$$

$$=(\alpha_1A)X_1+(\alpha_2A)X_2+(\alpha_3A)X_3+(\alpha_4A)X_4=\alpha_1(AX_1)+\alpha_2(AX_2)+\alpha_3(AX_3)+\alpha_4(AX_4)=\alpha_1f(X_1)+\alpha_2f(X_2)+\alpha_3f(X_3)+\alpha_4f(X_4)\Rightarrow f \text{ aplicație liniară}$$

(= morfism de spații vectoriale)

$T:\mathbb{R}^4\to\mathbb{R}^4$

$$f\begin{pmatrix}x\\y\\z\\t\end{pmatrix}=\underbrace{\begin{pmatrix}0&3&-2&1\\-1&3&0&0\\-1&1&2&0\\-2&3&-2&3\end{pmatrix}}_{A}\begin{pmatrix}x\\y\\z\\t\end{pmatrix}$$

$$A=\begin{pmatrix}0&3&-2&1\\-1&3&0&0\\-1&1&2&0\\-2&3&-2&3\end{pmatrix},\quad \det A=\begin{vmatrix}0&3&-2&1\\-1&3&0&0\\-1&1&2&0\\-2&3&-2&3\end{vmatrix}\xlongequal{L_3\to L_3-3L_1}\begin{vmatrix}0&3&-2&1\\-1&3&0&0\\-1&1&2&0\\-2&-6&4&0\end{vmatrix}=1\cdot(-1)^{1+4}\cdot\begin{vmatrix}-1&3&0\\-1&1&2\\-2&-6&4\end{vmatrix}=(-1)\cdot(-2)\cdot\underbrace{\begin{vmatrix}-1&3&0\\-1&1&2\\1&3&-2\end{vmatrix}}_{8}=16,\ \det A\neq0\Rightarrow T \text{ e surjectivă (1)}$$

$T:\mathbb{R}^4\to\mathbb{R}^4 \Rightarrow T$ e injectivă (2)

$\xRightarrow{(1),(2)}$ T izomorfism liniar

d) Forma canonică Jordan a lui T

Avem $\lambda_1=1$, $m_a(\lambda_1)=4$, $m_g(\lambda_1)=\dim V_{\lambda_1}=2$

Determin m-minim a.î.:

$\dim \operatorname{Ker} N^m=m_a(\lambda_1)=4$, unde $N=A-2I_4$

$\dim \operatorname{Ker} N^m=4-\operatorname{rang} N^m \Rightarrow \operatorname{rang} N^m=0 \Rightarrow N^m=O_4$

$$\Rightarrow N=\begin{pmatrix}-2&3&-2&1\\-1&1&0&0\\-1&1&0&0\\-2&3&-2&1\end{pmatrix}\Rightarrow N^2=N\cdot N=\begin{pmatrix}-2&3&-2&1\\-1&1&0&0\\-1&1&0&0\\-2&3&-2&1\end{pmatrix}\cdot\begin{pmatrix}-2&3&-2&1\\-1&1&0&0\\-1&1&0&0\\-2&3&-2&1\end{pmatrix}=\begin{pmatrix}1&-2&2&-1\\1&-2&2&1\\1&-2&2&1\\1&-2&2&-1\end{pmatrix}$$

$$N^3=N^2\cdot N=\begin{pmatrix}1&-2&2&-1\\1&-2&2&1\\1&-2&2&1\\1&-2&2&-1\end{pmatrix}\begin{pmatrix}-2&3&-2&1\\-1&1&0&0\\-1&1&0&0\\-2&3&-2&1\end{pmatrix}=\begin{pmatrix}0&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix}=O_4$$

$$\Rightarrow \boxed{m=3}$$

$$\text{Avem: }\begin{cases}\gamma_1+\gamma_2+\gamma_3=m_g(\lambda_1)=2=r\\ \gamma_1+2\gamma_2+3\gamma_3=m_a(\lambda_1)=4\end{cases}$$

$\gamma_3\neq0$

Met I

$$2r-\gamma_1=4-\operatorname{rang}\underbrace{(A-2I_4)^2}_{\operatorname{rg}N^2=1}$$

$$4-\gamma_1=3\Rightarrow\gamma_1=1$$

$$3r-2\gamma_1-\gamma_2=4-\operatorname{rang}\underbrace{(A-2I_4)^3}_{0}$$

$$6-2-\gamma_2=4$$

$$\Rightarrow\gamma_2=0\Rightarrow\gamma_3=1$$

Met II

$$\left.\begin{matrix}\gamma_3\neq0\\3\gamma_3\le4\end{matrix}\right\}\Rightarrow\gamma_3=1\Rightarrow\begin{cases}\gamma_1+\gamma_2=1\\\gamma_1+2\gamma_2=1\end{cases}\Rightarrow-\gamma_2=0\Rightarrow\gamma_2=0$$

$$\Rightarrow\gamma_1=1$$

Deci forma canonică Jordan a lui T este:

$$\begin{pmatrix}\boxed{2}&0&0&0\\0&2&1&0\\0&0&2&1\\0&0&0&2\end{pmatrix}$$