$$f:\mathbb{R}^2\to\mathbb{R}, f(x,y) \quad = \quad \begin{cases} \frac{x^3y^7}{\sqrt{x^{16}+y^{16}}}, x^2+y^2\neq 0 \\ 0, x=y=0 \end{cases}$$

f cont pe $\mathbb{R}^2\backslash\{(0,0)\}$

$$\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{(x,y)\to(0,0)} \frac{x^3y^7}{\sqrt{x^{16}+y^{16}}} \overset{\frac{\infty}{\infty}}{=} ?$$

Alegem:

$$y=x \Rightarrow \lim_{x\to 0} f(x,x) = \lim_{x\to 0} \frac{x^3x^7}{\sqrt{x^{16}+x^{16}}} = \lim_{x\to 0} \frac{x^10}{x^8\sqrt{2}} = \lim_{x\to 0} \frac{x^2}{\sqrt{2}} = 0$$

$$y=-x \Rightarrow \lim_{x\to 0} f(x,-x) = \lim_{x\to 0} \frac{x^3(-x)^7}{\sqrt{x^{16}+(-x)^{16}}} = \lim_{x\to 0} -\frac{x^{10}}{x^8\sqrt{2}} = \lim_{x\to 0} -\frac{x^2}{\sqrt{2}} = 0$$

$$y=\sqrt{x} \Rightarrow \lim_{x\to 0} f(x,\sqrt{x}) = \lim_{x\to 0} \frac{x^3x^{7/2}}{\sqrt{x^{16}+x^{16/2}}} = \lim_{x\to 0} \frac{x^{13/2}}{x^4\sqrt{x^8+1}} = \lim_{x\to 0} \frac{x^{5/2}}{\sqrt{x^8+1}} = 0$$

Este posibil ca

$$\lim_{(x,y)\to(0,0)} f(x,y) = l(=0)\in\mathbb{R}$$

Evaluăm:

$$\begin{tikzcd} 0 \arrow[dr] & \leq |f(x,y)-l| \leq & g(x,y) \arrow[dl, "(x,y)\to(0,0)"] \\ & 0 & \end{tikzcd}$$

$$x^{16}\leq x^{16}+y^{16} \Leftrightarrow \sqrt[16]{x^{16}} \leq \sqrt[16]{x^{16}+y^{16}} \Leftrightarrow |x| \leq \sqrt[16]{x^{16}+y^{16}} \Leftrightarrow |x| \leq (x^{16}+y^{16})^{\frac{3}{16}}$$

Procedăm la fel și pentru

$$|y|^7 \leq (x^{16}+y^{16})^{\frac{7}{16}}$$

$$\Rightarrow \left|x^3y^7\right| \leq (x^{16}+y^{16})^{\frac{3}{16}+\frac{7}{16}} \Leftrightarrow \left|x^3y^7\right| \leq (x^{16}+y^{16})^{\frac{5}{8}}$$

$$0\leq |f(x,y)-0| = \left|\frac{x^3y^7}{\sqrt{x^{16}+y^{16}}}\right| \leq \left|\frac{(x^{16}+y^{16})^{\frac{5}{8}}}{(x^{16}+y^{16})^{\frac{1}{2}}}\right| = (x^{16}+y^{16})^{\frac{5}{8}-\frac{1}{2}} = (x^{16}+y^{16})^{\frac{1}{8}} = \sqrt[8]{x^{16}+y^{16}}$$

$$\begin{tikzcd} 0 \arrow[dr] & \leq |f(x,y)-l| \leq \arrow[d] & \sqrt[8]{x^{16}+y^{16}} \arrow[dl, "(x,y)\to(0,0)"] \\ & 0 & \end{tikzcd}$$

$$\Rightarrow \lim_{(x,y)\to(0,0)} |f(x,y)-0| = 0 \Rightarrow \left.\begin{array}{r} \lim\limits_{(x,y)\to(0,0)} f(x,y) = 0 \\ f(0,0)=0 \end{array}\right| \Rightarrow \text{f cont pe } (0,0)$$

$$f : \mathbb{R}^2 \to \mathbb{R}, f(x,y) = \begin{cases} \frac{x^2y^2}{\sqrt{x^4+y^{10}}}, x^2+y^2 \neq 0 \\ 0, x=y=0 \end{cases}$$

f cont. pe $\mathbb{R}^2 \backslash \{(0,0)\}$

$$\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{(x,y)\to(0,0)} \frac{x^2y^2}{\sqrt{x^2+y^{10}}} \overset{\frac{\infty}{\infty}}{=} ?$$

Alegem:

$$y = x \Rightarrow \lim_{x\to 0} f(x,x) = \lim_{x\to 0} \frac{x^2x^2}{\sqrt{x^4+x^{10}}} = \lim_{x\to 0} \frac{x^4}{x^2\sqrt{1+x^6}} = \lim_{x\to 0} \frac{x^2}{\sqrt{1+x^6}} = 0$$

$$y = -x \Rightarrow \lim_{x\to 0} f(x,-x) = \lim_{x\to 0} \frac{x^2(-x)^2}{\sqrt{x^4+(-x)^{10}}} = \lim_{x\to 0} \frac{x^4}{x^2\sqrt{1+x^6}} = \lim_{x\to 0} \frac{x^2}{\sqrt{1+x^6}} = 0$$

$$y = \sqrt{x} \Rightarrow \lim_{x\to 0} f(x,\sqrt{x}) = \lim_{x\to 0} \frac{x^2x}{\sqrt{x^4+x^5}} = \lim_{x\to 0} \frac{x^3}{x^2\sqrt{1+x}} = \lim_{x\to 0} \frac{x}{\sqrt{1+x}} = 0$$

Este posibil ca

$$\lim_{(x,y)\to(0,0)} f(x,y) = l(=0) \in \mathbb{R}$$

Evaluam

$$\begin{tikzcd} 0 \arrow[dr] & \leq |f(x,y)-l| \leq & g(x,y) \arrow[dl, "(x,y)\to(0,0)"] \\ & 0 & \end{tikzcd}$$

$$0 \leq |f(x,y)-0| = \left| \frac{x^2y^2}{\sqrt{x^4+y(10)}} \right| \leq \left| \frac{x^2y^2}{\sqrt{x^4}} \right| = \frac{x^2y^2}{x^2} = y^2$$

$$\begin{tikzcd} 0 \arrow[dr] & \leq |f(x,y)-l| \leq \arrow[d] & y^2 \arrow[dl, "(x,y)\to(0,0)"] \\ & 0 & \end{tikzcd}$$

$$\Rightarrow \lim_{(x,y)\to(0,0)} |f(x,y)-0| = 0 \Rightarrow \left. \begin{array}{r} \lim_{(x,y)\to(0,0)} f(x,y) = 0 \\ f(0,0) = 0 \end{array} \right| \Rightarrow \text{f cont pe } (0,0)$$

$$f:\mathbb{R}^2 \to \mathbb{R}, f(x,y) \quad = \quad \begin{cases} \frac{x^9y^6}{x^{14}+y^{14}}, x^2+y^2 \neq 0 \\ 0, x=y=0 \end{cases}$$

f cont pe $\mathbb{R}^2 \setminus \{(0,0)\}$

$$\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{(x,y)\to(0,0)} \frac{x^9y^6}{x^{14}+y^{14}} \overset{\frac{\infty}{\infty}}{=} ?$$

Alegem:

$$y=x \Rightarrow \lim_{x\to 0} f(x,x) = \lim_{x\to 0} \frac{x^9x^6}{x^{14}+x^{14}} = \lim_{x\to 0} \frac{x^{1}5}{2x^{1}4} = \lim_{x\to 0} \frac{x}{2} = 0$$

$$y=-x \Rightarrow \lim_{x\to 0} f(x,-x) = \lim_{x\to 0} \frac{x^9(-x)^6}{x^{14}+(-x)^{14}} = \lim_{x\to 0} \frac{x^{15}}{2x^{14}} = \lim_{x\to 0} \frac{x}{2} = 0$$

$$y=\sqrt{x} \Rightarrow \lim_{x\to 0} f(x,\sqrt{x}) = \lim_{x\to 0} \frac{x^9x^3}{x^{14}+x^7} = \lim_{x\to 0} \frac{x^{12}}{x^7(x^7+1)} = \lim_{x\to 0} \frac{x^5}{x^7+1} = 0$$

Este posibil ca

$$\lim_{(x,y)\to(0,0)} f(x,y) = l(=0) \in \mathbb{R}$$

Evaluăm:

$$\begin{tikzcd} 0 \arrow[dr] & \le|f(x,y)-l|\le & g(x,y) \arrow[dl, "{(x,y)\to(0,0)}"] \\ & 0 & \end{tikzcd}$$

$$x^{14} \le x^{14}+y^{14} \Leftrightarrow \sqrt[14]{x^{14}} \le \sqrt[14]{x^{14}+y^{14}} \Leftrightarrow |x| \le \sqrt[14]{x^{14}+y^{14}} \Leftrightarrow \left|x^9\right| \le (x^{14}+y^{14})^{\frac{9}{14}}$$

Procedăm la fel și pentru

$$\left|y^6\right| \le (x^{14}+y^{14})^{\frac{6}{14}}$$

$$\Rightarrow \left|x^9y^6\right| \le (x^{14}+y^{14})^{\frac{9}{14}+\frac{6}{14}} \Leftrightarrow \left|x^9y^6\right| \le (x^{14}+y^{14})^{\frac{15}{14}}$$

$$0 \le |f(x,y)-0| = \left|\frac{x^9y^6}{\sqrt{x^{14}+y^{14}}}\right| \le \left|\frac{(x^{14}+y^{14})^{\frac{15}{14}}}{x^{14}+y^{14}}\right| = (x^{14}+y^{14})^{\frac{15}{14}-1} = (x^{14}+y^{14})^{\frac{1}{14}} = \sqrt[14]{x^{14}+y^{14}}$$

$$\begin{tikzcd} 0 \arrow[dr] & \le|f(x,y)-l|\le \arrow[d] & \sqrt[14]{x^{14}+y^{14}} \arrow[dl, "{(x,y)\to(0,0)}"] \\ & 0 & \end{tikzcd}$$

$$\Rightarrow \lim_{(x,y)\to(0,0)} |f(x,y)-0| = 0 \Rightarrow \left.\begin{array}{r} \lim\limits_{(x,y)\to(0,0)} f(x,y) = 0 \\ f(0,0) = 0 \end{array}\right| \Rightarrow \text{f cont pe } (0,0)$$