Gal - Sem 2 | Teorema H-C. T. Laplace. Sisteme | 7.03.2024

(1) Fie a) $A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}$

$A^{-1} = ?$

$P_A(A) = (-1)^3 \left(A^3 - \sigma_1 A^2 + \sigma_2 A - \sigma_3 I_3 \right) = O_3$

$A^2 - Tr(A)A + \sigma_2 \cdot I_3 = \sigma_3 \cdot A^{-1}$

$\sigma_1 = Tr(A) = 2$

$\sigma_2 = \begin{vmatrix} 1 & 1 \\ 0 & 0 \end{vmatrix} + \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} + \begin{vmatrix} 0 & 1 \\ 1 & 1 \end{vmatrix} = 0 + 1 + -1 = 0$

$\sigma_3 = det(A) = 1 \cdot (-1)^{2+3} \cdot \begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix} = -1 \cdot (1-0) = -1$

$A^2 - 2A = -A^{-1} \quad | \cdot (-1) \Rightarrow$

$\Rightarrow -A^2 + 2A = A^{-1}$

$A^{-1} = -\begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}^2 + 2 \cdot \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix} = -\begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix} + 2 \cdot \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix} =$

$= -\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 2 \end{pmatrix} + \begin{pmatrix} 2 & 2 & 0 \\ 0 & 0 & 2 \\ 0 & 2 & 2 \end{pmatrix} = \boxed{\begin{pmatrix} 1 & 1 & -1 \\ 0 & -1 & 1 \\ 0 & 1 & 0 \end{pmatrix}}$

③ $A=\begin{pmatrix} 0 & 1 & 1 & 0 \\ -1 & 2 & 0 & 1 \\ -1 & 0 & -2 & 1 \\ 0 & -1 & -1 & 0 \end{pmatrix}$

a) Polinomul caracteristic

$$P_A(x) = \det(A-x) = \begin{vmatrix} -x & 1 & 1 & 0 \\ -1 & 2-x & 0 & 1 \\ -1 & 0 & -2-x & 1 \\ 0 & -1 & -1 & -x \end{vmatrix} = x^4 - \sigma_1 x^3 + \sigma_2 x^2 - \sigma_3 x - \sigma_4$$

$\sigma_1 = Tr(A) = 0$

$$\sigma_2 = \begin{vmatrix} 0 & 1 \\ -1 & 2 \end{vmatrix} + \begin{vmatrix} 0 & 1 \\ -1 & -2 \end{vmatrix} + \begin{vmatrix} 0 & 0 \\ 0 & 0 \end{vmatrix} + \begin{vmatrix} 2 & 0 \\ 0 & -2 \end{vmatrix} + \begin{vmatrix} 2 & 1 \\ -1 & 0 \end{vmatrix} + \begin{vmatrix} -2 & 1 \\ -1 & 0 \end{vmatrix} =$$

$$= 1 + 1 + 0 + (-4) + 1 + 1 = 0$$

$$\sigma_3 = \begin{vmatrix} 0 & 1 & 1 \\ -1 & 2 & 0 \\ -1 & 0 & -2 \end{vmatrix} + \begin{vmatrix} 0 & 1 & 0 \\ -1 & 2 & 1 \\ 0 & -1 & 0 \end{vmatrix} + \begin{vmatrix} 0 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & -1 & 0 \end{vmatrix} + \begin{vmatrix} 2 & 0 & 1 \\ 0 & -2 & 1 \\ -1 & -1 & 0 \end{vmatrix} =$$

$$= \begin{vmatrix} 0 & 0 & 1 \\ -1 & 2 & 0 \\ -1 & 2 & -2 \end{vmatrix} + 1\cdot(-1)^{1+2}\cdot\begin{vmatrix} -1 & 1 \\ 0 & 0 \end{vmatrix} + 0 + \begin{vmatrix} 2 & 0 & 1 \\ -2 & -2 & 0 \\ -1 & -1 & 0 \end{vmatrix} =$$

$$= 1\cdot(-1)^{1+3}\cdot\begin{vmatrix} -1 & 2 \\ -1 & 2 \end{vmatrix} + 0 + 0 + 1\cdot(-1)^{1+3}\cdot\begin{vmatrix} -2 & -2 \\ -1 & -1 \end{vmatrix} = 0$$

$$\sigma_4 = \det(A) = \begin{vmatrix} 0 & 1 & 1 & 0 \\ -1 & 2 & 0 & 1 \\ -1 & 0 & -2 & 1 \\ 0 & -1 & -1 & 0 \end{vmatrix} \underset{L_3-L_2}{=\!=} \begin{vmatrix} 0 & 1 & 1 & 0 \\ -1 & 2 & 0 & 1 \\ 0 & -2 & -2 & 0 \\ 0 & -1 & -1 & 0 \end{vmatrix} = 0 \quad (L_3 \text{ prop. cu } L_4)$$

Polinomul caracteristic $= x^4$

b) $A^{100} = ?$ Th H-C $A^4 = O_4 \Rightarrow A^{100} = O_4$

$$\boxed{P_A(A) = O_n}$$

④ $A=\begin{pmatrix} 1 & -2 \\ 2 & 0 \end{pmatrix}$; $B = A^4 - 3A^3 + 3A^2 - 2A + 8J_2$

Det. $a, b \in \mathbb{R}$ a.î. $B = aA + bJ_2$ $\boxed{\text{Metoda 1}}$

$P_A(x) = x^2 - \sigma_1 x + \sigma_2 = x^2 - x + 4$

$q(x) = x^4 - 3x^3 + 3x^2 - 2x + 8$

$q = P_A \cdot c + \underset{ax+b}{R}$

$$\begin{array}{l|l} x^4 - 3x^3 + 3x^2 - 2x + 8 & x^2 - x + 4 \\ \hline -x^4 + x^3 - 4x^2 & x^2 - 2x - 3 \\ \hline / \;\; -2x^3 - x^2 - 2x + 8 & \\ \;\;\; +2x^3 - 2x^2 + 8x & \\ \hline / \;\; -3x^2 + 6x + 8 & \\ \;\;\; +3x^2 - 3x + 12 & \\ \hline / \;\; 3x + 20 & \end{array}$$

$\Rightarrow a = 3$

$b = 20$

$B = 3A + 20 I_2$

$\boxed{\text{Metoda } 2}$

$A^2 - A + 4I_2 = O_2$

$A^2 = A - 4I_2$

$A^3 = A^2 - 4A = A - 4I_2 - 4A = -3A - 4I_2$

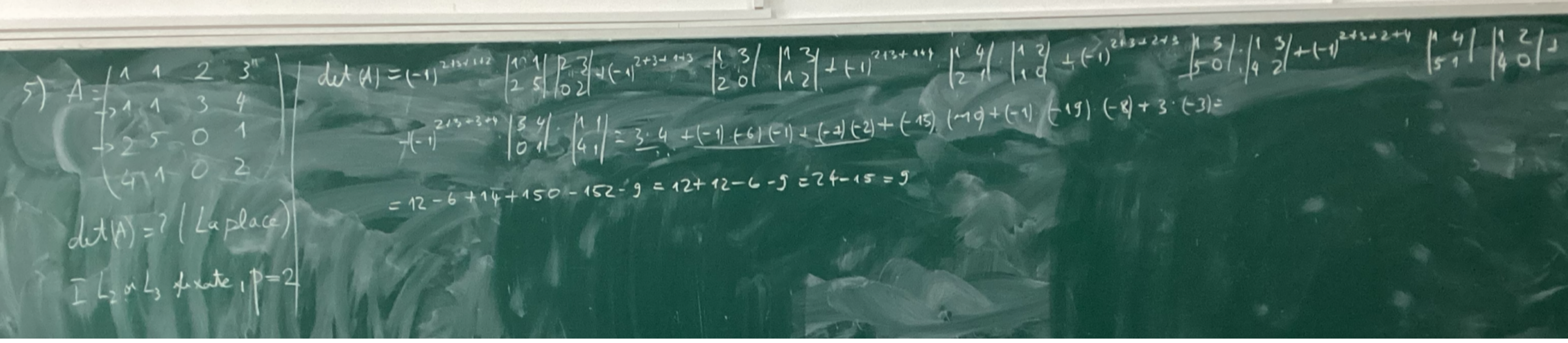
$A^4 = -3A^2 - 4A = -3(A - 4I_2) - 4A = -3A + 12I_2 - 4A = -7A + 12I_2$

$B = A^4 - 3A^3 + 3A^2 - 2A + 8I_2$

$B = (-7A + 12I_2) - 3(-3A - 4I_2) + 3(A - 4I_2) - 2A + 8I_2 =$

$= \underline{-7}A + 12I_2 + \underline{9}A + \cancel{12I_2} + \underline{3}A \cancel{-12I_2} \underline{-2}A + 8I_2 =$

$= 3A + 20I_2$



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$A = \begin{pmatrix} 1 & 5 \\ 0 & 4 \end{pmatrix}$. Calculați A^n, utilizând Th H-C

$Tr(A) = 5$

$\det(A) = 4$

$A^n = x_n A + y_n \cdot I_2 \quad / \cdot A$

$A^{n+1} = x_n A^2 + y_n A \Leftrightarrow A^n = x_n (5A - 4I_2) + y_n A =$

$= A(5x_n + y_n) - 4x_n I_2$

$A^{n+1} = x_{n+1} + y_{n+1} \cdot I_2$

$\begin{cases} x_{n+1} = 5x_n + y_n \\ y_{n+1} = -4x_n \end{cases} \Leftrightarrow \begin{cases} x_{n+1} = 5x_n - 4x_{n-1} \\ y_{n+1} = -4x_n \end{cases} \Leftrightarrow \boxed{x_{n+1} - 5x_n + 4x_{n-1} = 0}$

$\forall n \geq 2$

$A = 1 \cdot A + 0 \cdot I_2 \Rightarrow x_1 = 1$
$y_1 = 0$

$A^2 = 5A - 4I_2 \Rightarrow x_2 = 5$
$y_2 = -4$

$t^2 - 5t + 4 = 0 \Rightarrow t_1 = 1$
$t_2 = 4$

$x_n = C_1 t_1^n + C_2 \cdot t_2^n = C_1 + C_2 \cdot 4^n \quad \forall n \in \mathbb{N}^*$

$n = 1 \Rightarrow 1 = C_1 + 4C_2$
$n = 2 \Rightarrow 5 = C_1 + 16C_2$

$\Rightarrow 4 = 12C_2 \Rightarrow C_2 = \frac{1}{3} \Rightarrow C_1 = -\frac{1}{3}$

$x_n = \frac{1}{3}\left(-1+4^n\right)$

$y_n = -\frac{4}{3}\left(4^{n-1}-1\right)$

Deci, $A^n = \frac{4^n-1}{3} A - \frac{4^n-4}{3} I_2$

(11) $X^{2024} = A = \begin{pmatrix} 2 & -4 \\ -1 & 2 \end{pmatrix}$; $X \in \mathcal{M}_2(\mathbb{R})$

a) $X \in \mathcal{M}_2(\mathbb{R})$, $X = ?$

$\det\left(X^{2024}\right) = \det(A) = 0 \Rightarrow \det(X)^{2024} = 0 \Rightarrow \det(X) = 0$

$X^2 = tr(X) \cdot X$

$X^n = tr^{n-1}(X) \cdot X$

$X^{2024} = tr^{2023}(X) \cdot X$

$A = tr^{2023}(X) \cdot X \;/^{tr} \Rightarrow tr(A) = tr^{2024}(X) \Rightarrow$

$\Rightarrow tr^{2024}(X) = 4$

$tr^{2024}(X) - 4 = 0 \Rightarrow$ 2 soluții : $tr\, X = \pm 4^{\frac{1}{2024}}$

$$X_{1,2} = \frac{A}{\pm 4^{\frac{2023}{2024}}}$$

(*) are 2024 rădăcini complexe $\Rightarrow \exists$ 2024 matrice X

Fie $\triangle ABC$ cu lungimile lat. a, b, c

Fie sist $\begin{cases} ay + bx = c \\ cx + az = b \\ bz + cy = a \end{cases}$

Dem. $\forall \triangle ABC$, sist. are soluție unică (x_0, y_0, z_0) a.î. $x_0, y_0, z_0 \in (-1, 1)$

Fie $A = \underbrace{\left(\begin{array}{ccc|c} b & a & 0 & c \\ c & 0 & a & b \\ 0 & c & b & a \end{array}\right)}_{\bar{A}}$

$\det(A) = 0 + 0 + 0 - 0 - abc - abc = -2abc \neq 0$

$\text{rang } A = \text{rang } \bar{A} = 3 \Rightarrow$ SCD

$$\Delta_x = \begin{vmatrix} c & a & 0 \\ b & 0 & a \\ a & c & b \end{vmatrix} = a^3 - ac^2 - ab^2$$

$$\Delta_y = \begin{vmatrix} b & c & 0 \\ c & b & a \\ 0 & a & b \end{vmatrix} = b^3 - a^2 b - bc^2$$

$$\Delta_z = \begin{vmatrix} b & a & c \\ c & 0 & b \\ 0 & c & a \end{vmatrix} = c^3 - b^2 c - a^2 c$$

$$x = \frac{a^3 - ac^2 - ab^2}{-2abc} = \frac{a^2 - c^2 - b^2}{-2bc} = \frac{b^2 + c^2 - a^2}{2bc} =$$

$$y = \frac{a^2 + c^2 - b^2}{2ac}$$

$$z = \frac{a^2 + b^2 - c^2}{2ab}$$

$(x_0, y_0, z_0) = (\cos A, \cos B, \cos C) \in (-1, 1)$

$$\begin{cases} x + 2y = m + 1 \\ 2x - 3y = m - 1 \\ mx + y = 3 \end{cases}$$

$m = ?$ a.î. Si

Fie $A = \left(\begin{array}{cc|c} 1 & 2 & m+1 \\ 2 & -3 & m-1 \\ m & 1 & 3 \end{array}\right)$

$\operatorname{rg} A = 2$, pt. că $\Delta_p = \begin{vmatrix} 1 & 2 \\ 2 & -3 \end{vmatrix} = -7 \neq 0$

$$\Delta_c = \begin{vmatrix} 1 & 2 & m+1 \\ 2 & -3 & m-1 \\ m & 1 & 3 \end{vmatrix} = -9 + \cancel{2m} + 2 + 2m^2 - \cancel{2m} + 3m^2 + 3m - m + 1 - 12$$

$= 5m^2 + 2m - 18 \neq 0$

$5m^2 + 2m - 18 = 0$

$\Delta = 4 - 4 \cdot 5 \cdot (-18) = 4 + 20 \cdot 18 = 4 + 360 = 364$

$\sqrt{\Delta} = \sqrt{364} = 2\sqrt{91}$

$m_{1,2} = \dfrac{-2 \pm 2\sqrt{91}}{10} = \dfrac{-1 \pm \sqrt{91}}{5}$

$m \in \mathbb{R} \setminus \left\{ \dfrac{-1 \pm \sqrt{91}}{5} \right\}$

$$\begin{array}{r|l} 364 & 2 \\ 182 & 2 \\ 91 & 91 \end{array}$$