21. 11. 2024.

Seminarul 8

Vectori proprii. Valori proprii. Diagonalizare

(8) $f \in End(\mathbb{R}^3)$

$R_0 = \{e_1, e_2, e_3\}$

$f(e_1) = e_3$ — coef repr prima coloană a lui $[f]_{R_0, R_0}$

$f(e_2) = e_2$

$f(e_3) = e_1$

Precizați dacă $(\exists)$ R reper din $\mathbb{R}^3$ a.î. $\underbrace{[f]_{R,R}}_{\Lambda}$ este diagonală

$[f]_{R_0,R_0} = A$

$$A = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \qquad f(e_1) = e_1 \cdot 0 + 0 \cdot e_2 + 1 \cdot e_3$$

Obs: $f(x) = y \Rightarrow Y = AX$

$f : \mathbb{R}^3 \to \mathbb{R}^3$, $f(x_1, x_2, x_3) = (x_3, x_2, x_1)$

$$A \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = f(x_1, x_2, x_3)$$

I. $P_f(\lambda) = \det(A - \lambda I_3) = \begin{vmatrix} -\lambda & 0 & 1 \\ 0 & 1-\lambda & 0 \\ 1 & 0 & -\lambda \end{vmatrix} = (-\lambda+1) \begin{vmatrix} -\lambda & 1 \\ 1 & -\lambda \end{vmatrix} =$

$= (-\lambda+1)(\lambda^2 - 1) = (1-\lambda)(\lambda-1)(\lambda+1)$

$= -(\lambda-1)^2(\lambda+1)$

$\lambda_1 = 1$; $m_1 = 2$

$\lambda_2 = -1$; $m_2 = 1$

II Subspațiile proprii

$V_{\lambda_1} = \{X \in \mathbb{R}^3 \mid AX = \lambda_1 X\}$ $\quad X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$

$V_{\lambda} = \{x \in \mathbb{R}^3 \mid f(x) = \lambda x\}$

$AX = \lambda_1 X \Rightarrow \underbrace{(A - \lambda_1 I_3)}X = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

$A - I_3 = \begin{pmatrix} -1 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & -1 \end{pmatrix}$

$\dim V_{\lambda_1} = 3 - \underbrace{rg(A - I_3)}_{1} = 3 - 1 = 2 = m_1 \Rightarrow$ se diagonalizează

$\boxed{\mathbb{R}^3 = V_{\lambda_1} \oplus V_{\lambda_2}}$ Care e reperul în care se realizează diagonalizarea? (pt lucrare)

$V_{\lambda_1} = \{x \in \mathbb{R}^3 \mid -x_1 + x_3 = 0\}$

$(A - I_3)X = \begin{pmatrix} -1 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} x_3 - x_1 \\ 0 \\ x_3 - x_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

$\{(x_1, x_2, x_1)\}$, $x_1, x_2 \in \mathbb{R}$

$x_1(1, 0, 1) + x_2(0, 1, 0)$ SG

$\mathcal{R}_1 = \{(1, 0, 1); (0, 1, 0)\}$

reper în V_{λ_1}

$V_{\lambda_2} = \{ x \in \mathbb{R}^3 \mid AX = -X \}$

$(A + I_3)X = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

$A + I_3 = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 1 \end{pmatrix}$

$\dim V_{\lambda_2} = 3 - rg(A + I_3) = 3 - 2 = 1 = m_2$

$(A + I_3)X = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow \begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

$\begin{pmatrix} x_1 + x_3 \\ 2x_2 \\ x_1 + x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

$V_{\lambda_2} = \{ x \in \mathbb{R}^3 \mid x_1 + x_3 = 0 \; x_2 = 0 \} \Rightarrow x_1 = -x_3$

$\Rightarrow V_{\lambda_2} = \langle \{ (1, 0, -1) \} \rangle$ S.G

$\Rightarrow R_2 = \{ (1, 0, -1) \}$ reper in V_{λ_2}

Recap: I $P_f(x) = \det(A - \lambda I_3)$ $\lambda_1 = 1 ; m_1 = 2$

$\lambda_2 = 1 ; m_2 = 1 ; \lambda_1, \lambda_2 \in \mathbb{R}$

II $V_{\lambda_1} = \{ x \in \mathbb{R}^3 \mid AX = 1 \cdot X \} ; \dim V_{\lambda_1} = 2 = m_1$

$V_{\lambda_2} = \{ x \in \mathbb{R}^3 \mid AX = -X \} \; \dim V_{\lambda_2} = 1 = m_2$ $\xRightarrow{Th}$ A este diagonalizabil

A că $\mathbb{R}^3 = V_{\lambda_1} \oplus V_{\lambda_2}$

$\mathcal{R} = \mathcal{R}_1 \cup \mathcal{R}_2 = \{(1,0,1),(0,1,0),(1,0,-1)\}$
reper în $\mathbb{R}^3$

$$[f]_{\mathcal{R},\mathcal{R}} = A' = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_1 & 0 \\ 0 & 0 & \lambda_2 \end{pmatrix} = \begin{array}{c} m_1 \\ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \\ m_2 \end{array}$$

Obs: $\mathcal{R}_0 = \{e_1, e_2, e_3\} \xrightarrow{C} \mathcal{R} = \{e_1', e_2', e_3'\}$

$A' = C^{-1} A C$

la ex anterior

$\mathcal{R} = \{\underset{e_1'}{(1,0,1)}, \underset{e_2'}{(0,1,0)}, \underset{e_3'}{(1,0,-1)}\}$

$$\begin{cases} e_1' = e_1 + e_3 \\ e_2' = e_2 \\ e_3' = e_1 - e_3 \end{cases} \qquad C = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & -1 \end{pmatrix}$$

(12) $f \in End[\mathbb{R}^3]$

$\lambda_1 = 3$, $\lambda_2 = -2$, $\lambda_3 = 1$ val proprii

$v_1 = (-3,2,1)$, $v_2 = (-2,1,0)$, $v_3 = (-6,3,1)$ vectorii proprii coresp.

Care e matricea $A = [f]_{\mathcal{R}_0,\mathcal{R}_0}$?

$f(v_1) = \lambda_1 v_1$

$f(v_2) = \lambda_2 v_2$

$f(v_3) = \lambda_3 v_3$

$\mathcal{R} = \{v_1, v_2, v_3\}$ SLI (vectori proprii coresp. la val. proprii distincte - vezi curs)

$\dim \mathbb{R}^3 = |\mathcal{R}| = 3$ $\Rightarrow$ $\mathcal{R}$ reper în $\mathbb{R}^3$

$$[f]_{\mathcal{R},\mathcal{R}} = A' = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix} = \begin{pmatrix} 3 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\mathcal{R}_0 \xrightarrow{C} \mathcal{R} = \{v_1 = -3e_1 + 2e_2 + e_3,\ v_2 = -2e_1 + e_2,\ v_3 = -6e_1 + 3e_2 + e_3\}$$

coeficienții

$$C = \begin{pmatrix} -3 & -2 & -6 \\ 2 & 1 & 3 \\ 1 & 0 & 1 \end{pmatrix}$$

$A' = C^{-1}AC$

$\Rightarrow A = CA'C^{-1} \Rightarrow A = \dots$ (se face calcul)

Dacă se cere A^n $\Rightarrow$ $A^n = (CA'C^{-1})(CA'C^{-1}) \cdot \ldots \cdot (CA'C^{-1})$

$A^n = CA'^n C^{-1}$

$$A'^n = \begin{pmatrix} 3^n & 0 & 0 \\ 0 & (-2)^n & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

(3 curs) $f: \mathbb{R}^3 \to \mathbb{R}^3$, $f(x) = (x_1 - x_2 + x_3, x_1 - x_2 + x_3, x_3)$

a) $[f]_{R_0, R_0} = A = ?$

b) $\ker f = ?$

$\operatorname{Im} f = ?$

Reper

c) $\mathbb{R}^3 = \ker f \oplus W$

$W = ?$

$p: \ker f \oplus W \to \ker f$ proiecția pe $\ker f$

$s: \mathbb{R}^3 \to \mathbb{R}^3$ simetria față de $\ker f$

Calc $p(1,0,3)$; $s(1,0,3)$

a) $f(x) = y \Rightarrow Y = AX \Rightarrow \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \underbrace{\begin{pmatrix} 1 & -1 & 1 \\ 1 & -1 & 1 \\ 0 & 0 & 1 \end{pmatrix}}_{[f]_{R_0, R_0}} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$

b) $\ker f = \left\{ x \in \mathbb{R}^3 \mid AX = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \right\} \Rightarrow \begin{cases} x_1 - x_2 + x_3 = 0 \Rightarrow x_1 = x_2 \\ x_3 = 0 \end{cases}$

$\dim \ker f = 3 - \operatorname{rg} A = 3 - 2 = 1$

$\ker f = \{(x_1, x_1, 0) \mid x_1 \in \mathbb{R}\} = \langle \{(1,1,0)\} \rangle$

$x_1(1,1,0)$

H

Extindem $\mathcal{R}_1$ la un reper în $\mathbb{R}^3$

$$rg\begin{vmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = 3 \text{ (max)}$$

$\Rightarrow \mathcal{R}_1 \cup \{e_2, e_3\} \to$ reper în $\mathbb{R}^3$

$\{f(e_2), f(e_3)\}$ reper în $Im f$

$f(0,1,0) = (-1,-1,0)$ $\Rightarrow \mathcal{R}_2 = \{(-1,-1,0),(1,1,1)\}$
$f(e_3) = (1,1,1)$ reper în $Im f$.

M II $Im f = \{y \in \mathbb{R}^3 \mid (\exists)\, x \in \mathbb{R}^3 \text{ a.î. } f(x) = y\}$

$$\begin{cases} x_1 - x_2 + x_3 = y_1 \\ x_1 - x_2 + x_3 = y_2 \\ x_3 = y_3 \end{cases}$$

$$\Delta_C = \begin{vmatrix} -1 & -1 & y_1 \\ -1 & 1 & y_2 \\ 0 & 1 & y_3 \end{vmatrix} \overset{L_1 - L_2}{=\!=} \begin{vmatrix} 0 & 0 & y_1 - y_2 \\ -1 & 1 & y_2 \\ 0 & 1 & y_3 \end{vmatrix} = 0$$

$\Rightarrow \boxed{y_1 - y_2 = 0}$

$\Rightarrow Im f = \{y \in \mathbb{R}^3 \mid y_1 - y_2 = 0\} = \{(y_1, y_1, y_3) \mid y_1, y_3 \in \mathbb{R}\}$

$= y_1(1,1,0) + y_3(0,0,1) \Rightarrow Im f = \langle \underbrace{\{(1,1,0),(0,0,1)\}}_{SG} \rangle$

$\dim Im f = |\mathcal{R}_2| = 2$ $\mathcal{R}_2$

$\Downarrow$

$\mathcal{R}_2$ reper.

c) $W \langle \{e_2, e_3\} \rangle$ (vezi b)

$R = \{(1,1,0), \underbrace{(0,1,0), (0,0,1)}_{e\ u}\}$ reper

$(1,0,3) = a(1,1,0) + \underbrace{b(0,1,0) + c(0,0,1)}_{v} =$
$= (a, a+b, c)$

$\Rightarrow a = 1, c = 3, b = -1$

$(1,0,3) = \underset{\ker f}{u} + \underset{W}{v} = (1,1,0) + (0,-1,3)$

$\begin{cases} p(u+v) = u = (1,1,0) \\ s(u+v) = u - v = (1,1,0) - (0,-1,3) = (1,2,-3) \end{cases}$

Obs: $s(x) = 2p(x) - id_{R^3}$
$s(x) = 2p(x) - id_R(x) = 2x_1 - (x_1 + x_2) = x_1 - x_2$

d) $p' : \ker f \oplus W \to W$ proiectia pe W de-a lungul lui $\ker f$

$s' : \underbrace{\ker f \oplus W}_{R^3} \to R^3$ simetria fata de W

$p'(\underset{\ker f}{u} + \underset{W}{v}) = v = (0,-1,3)$

$s'(\underset{\ker f}{u} + \underset{W}{v}) = 2p'(u+v) - (u+v) = v - u = (0,-1,3) - (1,1,0) =$
$= (-1,-2,3)$