separat)

Seminar 3

1. Stud. convergența seriilor

a) $\sum_{n=1}^{\infty} \tau(n) x^n$, $x>0$, $\tau(n)$ = reprez ~~suma~~ nr. divizorilor lui n

b) $\sum_{n=1}^{\infty} \frac{1}{n\sqrt{n+1}} x^n$, $x>0$

Sol:

$$1 \leq \tau(n) \leq \cancel{n} \, n \,/\cdot x^n$$

$$x^n \leq x^n \cdot \tau(n) \leq n x^n \quad |\sqrt[n]{\ }$$

$$x \leq x\sqrt[n]{\tau(n)} \leq x\sqrt[n]{n}$$

$x_n = \sigma(n)\, x^n$

Aplicam crit radicalului

$\lim_{n\to\infty} \sqrt[n]{x_n} = ?$

$1 \le \sigma(n) \le n \quad | \cdot x^n$

$x^n \le x^n \sigma(n) \le n x^n \quad | \sqrt[n]{}$

$\sqrt[n]{x^n} \le \sqrt[n]{x_n} \le x \sqrt[n]{n}, \ \forall\, n \in \mathbb{N}^*$

$\sqrt[n]{x^n} \xrightarrow{n\to\infty} x$, $\sqrt[n]{x_n} \to x$, $x\sqrt[n]{n} \xrightarrow{n\to\infty} x$

I. Dacă $x<1$ (i.e. $x \in (0;1)$), atunci $\sum_{n\ge 1} x_n$ e conv.

II. Dacă $x>1$ (i.e. $x\in(1;\infty)$) at, $\sum_n x_n$ e diverg.

III. Dacă $x=1$ atunci criteriul nu decide

Fie $x=1 \Rightarrow \sum_{n=1}^{\infty} \sigma(n)$

$x_n = \sigma(n)$

$\sigma(n) \ge 1 \quad \forall n \in \mathbb{N}^*$, atunci $\lim_{n\to\infty} x_n \ge 1 \ne 0$

$\overset{\text{cf}}{\Rightarrow} \sum_{n=1}^{\infty} \sigma(n)$ e divergent

crit suf de diverg

$\square$

b) $\sum_{n=1}^{\infty} \frac{1}{n\sqrt{n+1}}\, x^n, \ x>0$

$x_n = \frac{x^{n}}{(n)\sqrt{n+1}}$

Aplicăm crit. raportului

$$\lim_{n\to\infty} \frac{x_{n+1}}{x_n} = \lim_{n\to\infty} \frac{x^{n+1}}{\cancel{(n+1)}\sqrt{n+2}} \cdot \frac{\cancel{n}\sqrt{n+1}}{\cancel{x^n}} = x$$

$\sqrt{n+1}$

~~Dacă~~ I. Dacă $x<1$ $(x\in(0;1))$ $\Rightarrow$ $\sum_{n=1}^{\infty} x_n$ este converg

II. Dacă $x>1$ $(x\in(1;+\infty))$ $\Rightarrow$ $\sum_{n=1}^{\infty} x_n$ este diverg

III. Dacă $x=1$ ~~este~~, criteriul nu decide

$$x_n = \frac{1}{n\sqrt{n+1}} \quad \forall n\in\mathbb{N}^*$$

$$y_n = \frac{1}{n\sqrt{n}} \text{ (e converg seria)} \to \frac{1}{n^{3/2}}$$

$$\lim_{n\to\infty} \frac{x_n}{y_n} = \lim_{n\to\infty} \frac{1}{\cancel{n}\sqrt{n+1}} \cdot \cancel{n}\sqrt{n} = 1 \in (0;1)$$

cf criteriului de comp cu limită $\Rightarrow$ $\sum_{n}^{\infty} x_n \sim \sum_{n}^{\infty} y_n$

$\sum_{n}^{\infty} y_n$ e converg (serie armonică cu $\alpha>1$) $\Rightarrow$

cf $\sum_{n}^{\infty} x_n$ e converg $\square$

c) $\displaystyle\sum_{n=1}^{\infty} \frac{a^n+n}{3^n+n^3}$; $a>0$

Sol $\left(\displaystyle\sum_{n=1}^{\infty} \frac{a^n+n}{3^n+n^3} = \sum_{n=1}^{\infty} \frac{a^n}{3^n+n^3} + \frac{n}{3^n+n^3} \right)$

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Fie $x_n = \frac{a^n + n}{3^n + n^3}$

$a_n = \frac{a^n}{3^n + n^3} + \frac{n}{3^n + n^3}$

$x_n = \frac{a^n}{3^n + n^3}$ $y_n = \frac{n}{3^n + n^3}$

$a_n = x_n + y_n \quad \forall\, n \in \mathbb{N}^*$

Studiem conv $\sum_{n=1}^{\infty} x_n$, $\sum_n y_n$

Considerăm $z_n = \frac{n}{n^3} \quad \forall\, n \in \mathbb{N}^*$

$y_n < z_n \quad \forall\, n \in \mathbb{N}^*$

$\sum_n z_n = \sum_n \frac{\cancel{n}}{\cancel{n}n^2} = \sum_n \frac{1}{n^2}$ conv (serie armonică generalizată cu $\alpha = 2$)

Deci $\sum_n a_n \sim \sum_n x_n$

Fie $t_n = \frac{a^n}{3^n} \quad \forall\, n \in \mathbb{N}^*$

$\lim_{n\to\infty} \frac{x_n}{t_n} = \lim_{n\to\infty} \frac{\cancel{a^n}}{3^n + n^3} \cdot \frac{3^n}{\cancel{a^n}}$

$\lim_{n\to\infty} \frac{\cancel{3^n}}{\cancel{3^n}\left(1 + \underbrace{\frac{n^3}{3^n}}_{\to 0}\right)} = 1 \in (0;\infty) \Rightarrow$

cf crit de comp cu limită $\Rightarrow \sum_n x_n \sim \sum_n t_n$

$q = \frac{a}{3}$

$t_n = \frac{a^n}{3^n} = \left(\frac{a}{3}\right)^n \to$ serie ~~armonică~~ geometrică ~~generalizată~~

1) Pt $a \in (0;3) \Rightarrow \frac{a}{3} \in (0;1) \Rightarrow \sum_n t_n$ e convergent, deci $\sum_n x_n$ e converg, deci $\sum_n x_n$ e converg.

2) Pt $a \in (3;\infty) \Rightarrow \sum_n t_n$ e divergent, deci $\sum_n x_n$ e divergent, deci $\sum_n x_n$ e diverg

d) $\sum_{n=1}^{\infty} \frac{\sin \frac{1}{n(n+1)}}{\left(\cos \frac{1}{n}\right)\left(\cos \frac{1}{n+1}\right)}$

$$x_n = \frac{\sin \frac{1}{n(n+1)}}{\left(\cos \frac{1}{n}\right)\cos \frac{1}{n+1}} = \frac{\sin\left(\frac{1}{n} - \frac{1}{n+1}\right)}{\cos \frac{1}{n} \cos \frac{1}{n+1}}$$

$$= \frac{\sin \frac{1}{n} \cos \frac{1}{n+1} - \sin \frac{1}{n+1} \cos \frac{1}{n}}{\cos \frac{1}{n} \cos \frac{1}{n+1}}$$

$$= \frac{\sin \frac{1}{n}}{\cos \frac{1}{n}} - \frac{\sin \frac{1}{n+1}}{\cos \frac{1}{n+1}}$$

$$= \operatorname{tg} \frac{1}{n} - \operatorname{tg} \frac{1}{n+1} \quad \forall n \in \mathbb{N}^*$$

$$s_n = x_1 + x_2 + \dots + x_n \quad \forall n \in \mathbb{N}$$

$$s_n = \left(\operatorname{tg} 1 - \cancel{\operatorname{tg} \frac{1}{2}}\right) + \left(\cancel{\operatorname{tg} \frac{1}{2}} - \cancel{\operatorname{tg} \frac{1}{3}}\right) + \dots + \left(\cancel{\operatorname{tg} \frac{1}{n}} - \operatorname{tg} \frac{1}{n+1}\right)$$

$$= \operatorname{tg} 1 - \operatorname{tg} \frac{1}{n+1}$$

$$\lim_{n \to \infty} s_n = \lim_{n \to \infty} \operatorname{tg} 1 - \underbrace{\operatorname{tg} \frac{1}{n+1}}_{0} = \operatorname{tg} 1 \in \mathbb{R}$$

Deci, $\sum_n x_n$ e converg

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e) $\sum_{n=1}^{\infty} \frac{\sin \frac{1}{\sqrt{n}}}{\left(\cos \frac{1}{n}\right)\left(\cos \frac{1}{n+1}\right)}$ > 0, $\forall n \in \mathbb{N}^*$ avem $\frac{1}{\sqrt{n}}, \frac{1}{n}, \frac{1}{n+1} \in \left(0; \frac{\pi}{2}\right)$

Fie $x_n = \frac{\sin \frac{1}{\sqrt{n}}}{\left(\cos \frac{1}{n}\right)\left(\cos \frac{1}{n+1}\right)}$ $\forall n \in \mathbb{N}^*$

Considerăm $y_n = \frac{1}{\sqrt{n}}$ $\forall n \in \mathbb{N}^*$

$$\lim_{n \to \infty} \frac{x_n}{y_n} = \lim_{n \to \infty} \frac{\sin \frac{1}{\sqrt{n}}}{\cos \frac{1}{n} \cdot \cos \frac{1}{n+1}} \cdot \frac{\sqrt{n}}{1}$$

$\frac{\sin n x}{2n}$

$$= \lim_{n \to \infty} \underbrace{\frac{\sin \frac{1}{\sqrt{n}}}{\frac{1}{\sqrt{n}}}}_{\to 1} \cdot \underbrace{\frac{1}{\cos \frac{1}{n} \cdot \cos \frac{1}{n+1}}}_{\to 1}$$

$= 1 \in (0; \infty)$

(Am folosit lim remarc. $\lim_{x \to 0} \frac{\sin x}{x} = 1$)

Cf criteriului de comp cu limită avem

$$\sum_{n=1}^{\infty} x_n \sim \sum_{n=1}^{\infty} y_n$$

Dar $\sum_{n=1}^{\infty} y_n = \sum_{n=1}^{\infty} \frac{1}{n^{1/2}}$ diverg (serie armon cu $\alpha = 1/2$)

Deci $\sum_{n=1}^{\infty} x_n$ diverg $\square$

f) $\sum_{n=1}^{\infty} \frac{\cos nx}{n^{\lambda}}$, $x \in$

Vom aplica criteriul Abel-Dirichlet (I)

Fie $x_n = \frac{1}{n^{\lambda}}$ $\forall n \in \mathbb{N}^*$

și $y_n = \cos nx \quad \forall\, n \in \mathbb{N}^*$

(x_n) descrescător

$\lim_{n\to\infty} x_n = 0 \qquad (1)$

? $\exists\, M > 0$ a.î. $\forall\, n \in \mathbb{N}^*$, avem $|y_1 + \dots + y_n| \le M$

(nu tb să depindă M de n)

M nu poate depinde de n, dar poate depinde de x!

$y_1 + y_2 + \dots + y_n = ?$

$\cos x + \cos nx = \cos\frac{(n+1)x}{2} - \sin\frac{(n-1)x}{2}$

$\cos x + \cos 2x + \dots + \cos nx$

$\cos a - \cos b = -2 \sin\frac{a+b}{2} \sin\frac{a-b}{2}$

~~Consi~~ Fie $z = \cos x + i \sin x$

$z^2 = \cos 2x + i \sin 2x$ (Moivre)

$\vdots$

$z^n = \cos nx + i \sin 2x$

$y_1 + \dots + y_n = \mathrm{Re}(z + z^2 + \dots + z^n)$

Pp $z \neq 1$ i.e $x \in \mathbb{R} \setminus \{2k\pi \mid k \in \mathbb{Z}\}$

$z + z^2 + \dots + z^n = z \cdot \frac{z^{n} - 1}{z - 1}$

$= \frac{z^{n+1} - z}{z - 1}$

$$= \frac{\cos(n+1)x + i\sin(n+1)x - \cos x - i\sin x}{\cos x + i\sin x - 1}$$

$$\overset{*}{=} \frac{-\cancel{2}\sin\frac{(n+2)x}{2}\sin\frac{n}{2}x + i\cancel{2}\cos\frac{n+2}{2}\cdot\sin\frac{n}{2}\cdot x}{-\cancel{2}\sin^2\frac{x}{2} + i\cancel{2}\sin\frac{x}{2}\cos\frac{x}{2}}$$

$\cos x - 1 = -2\sin^2\frac{x}{2}$

$$= \frac{\sin\frac{n}{2}\cdot x\left(-\sin\frac{(n+2)}{2}x + i\cos\frac{n+2}{2}x\right)}{\sin\frac{x}{2}\left(-\sin\frac{x}{2} + i\cos\frac{x}{2}\right)}$$

$$= \frac{\sin\frac{n}{2}x}{\sin\frac{x}{2}}\cdot\frac{\cos\frac{n+2}{2}x + i\sin\frac{n+2}{2}x}{\cos\frac{x}{2} + i\sin\frac{x}{2}}$$

$$= \frac{\sin\frac{n}{2}x}{\sin\frac{x}{2}}\cdot\frac{(\cos x + i\sin x)^n}{\cos\frac{x}{2} + i\sin\frac{x}{2}}$$

$$= \frac{\sin\frac{n}{2}x}{\sin\frac{x}{2}}\cdot\left(\cos\frac{n+1}{2}x + i\sin\frac{n+1}{2}x\right)$$

$$y_1 + y_2 + \dots + y_n = Re\,(z_1 + z^2 + \dots + z^n)$$

$$= \frac{\sin\frac{n}{2}x}{\sin\frac{x}{2}}\cdot\cos\frac{n+1}{2}x$$

$$|y_1 + y_2 + \dots + y_n| = \frac{\left|\sin\frac{n}{2}x\right|}{\left|\sin\frac{x}{2}\right|}\cdot\left|\cos\frac{n+1}{2}x\right|$$

$$\leq \frac{1}{\left|\sin\frac{x}{2}\right|}$$

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Alegem $M = \frac{1}{|\sin \frac{x}{2}|} > 0$

Deci, $\forall\ n \in \mathbb{N}^*$ avem $|y_1 + y_2 + \dots + y_n| \leq M$

Am tratat doar cazul $x \in \mathbb{R} \setminus \{2k\pi \mid k \in \mathbb{Z}\}$

Fie $x \in \{2k\pi \mid k \in \mathbb{Z}$

$$\sum_{n=1}^{\infty} \frac{\cos nx}{n^\lambda} = \sum_{n=1}^{\infty} \frac{1}{n^\lambda} \text{ (armonică generalizată)}$$

$\to$ converg pt $\lambda \in (1; \infty)$

$\to$ diverg pt $x \in (0; 1]$

$\square$

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