Partea I

FORMA CANONICĂ JORDAN (METODA NUCLEULUI STABIL)

francez

Se aplică când o matrice nu se diagonalizează.

Exemplu: $A = \begin{pmatrix} 1 & -1 & -1 \\ -3 & -4 & -3 \\ 4 & 7 & 6 \end{pmatrix}$

I Calculăm Spec A / $\sigma(A) = \{-1, 2\}$

$m_a(\lambda_1 = -1) = 1$
$m_a(\lambda_2 = 2) = 1$

Th: $m_g \leq m_a$ $\Rightarrow m_g(\lambda_1) = 1$

Pt $\lambda_2 = 2$ caut $V(\lambda_2)$

Noi făceam $(A v^t = \lambda_2 v^t)$

Definim $M = A - \lambda_2 I_3$

$\operatorname{Ker} N = \{ v \in \mathbb{R}^3 \mid N \cdot v^t = 0 \}$

$N \in \mathcal{M}_3(\mathbb{R})$

Ne amintim ($f: V \to W$ aplicație liniară
$\operatorname{Ker} f = \{ v \in V \mid f(v) = 0_W \}$
Dacă $A = M_f^{bc} \Rightarrow f(v) = A v^t$)

$\operatorname{Ker} M = \{ v \in \mathbb{R}^3 \mid M v^t = 0 \}$

$(A - \lambda_2 I_3) v^t = 0 \Leftrightarrow A v^t - \lambda_2 I_3 \cdot v^t = 0$

$A v^t = \lambda_2 v^t$

$\Rightarrow \operatorname{Ker} M = \underbrace{V(\lambda_2)}_{\text{subspațiul invariant}}$

$\operatorname{Ker} M = \{\underbrace{(1, 0, -1)}_{\text{spațiul generat}}\} = V(\lambda_2) \Rightarrow m_g(\lambda_2) = 1 \neq m_a(\lambda_2) \Rightarrow A$ nu se diagonalizează, deci A se jordanizează

Th: $\exists$ șirul:
$V(\lambda_2) = \operatorname{Ker} M \subset \operatorname{Ker} M^2 \subset \ldots \subset \operatorname{Ker} M^s = \operatorname{Ker} M^{s+1} = \operatorname{Ker} M^{s+2} = \ldots$ a.î. dim $\operatorname{Ker} M^s = m_a(\lambda_2)$

↑ șirul se stabilizează

La noi: $\underbrace{V(\lambda_2) = \operatorname{Ker} M}_{\substack{\dim 1 \\ m_g(\lambda_2) = 1}} \subset \underbrace{\operatorname{Ker} M^2}_{\substack{\dim 2 \\ m_a(\lambda_2) = 2}} = \operatorname{Ker} M$

Not $\operatorname{Ker} M^s = V^{\lambda_2}$ (cel care stabilizează șirul)

Bază în $V(\lambda_2) = \{(1, 0, -1)\}$
Pot completa baza lui Ker M la o bază a lui $\operatorname{Ker} M^2$

$M^2 v^t = M(M v^t)$

Aleg $u = (-2, 1, 0) \Rightarrow \{(1, 0, -1),\ u = (-2, 1, 0)\}$ bază a lui $\operatorname{Ker} M^2$

$B = \{(1, 0, -1),\ u = (-2, 1, 0),\ M u^t\}$ bază a lui $\mathbb{R}^3$

Acum facem ca la diagonalizare:

Fie $T = M_{B_C \to B}$ $\Rightarrow$ $\boxed{J = T^{-1}AT} = \begin{pmatrix} \boxed{-1} & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 1 & 0 \end{pmatrix}$ bloc Jordan de mărime 1; bloc Jordan de mărime 2

se obține în baza B

Obs că nu este o matrice diagonală!

$A \in \mathcal{M}_3(\mathbb{R})$

Cazuri posibile pentru forma Jordan:

1) $\sigma(A) = \{\lambda\}$, $m_a(\lambda) = 3$

$m_g(\lambda) = 2$ sau 1

$$J = \begin{pmatrix} \lambda & 0 & 0 \\ 1 & \lambda & 0 \\ 0 & 1 & \lambda \end{pmatrix} \to 1 \text{ bloc Jordan de mărime } 3$$

2) $\sigma(A) = \{\lambda_1, \lambda_2\}$ $\quad m_a(\lambda_1) = 1 = m_g(\lambda_1)$

$m_a(\lambda_2) = 2$

$m_g(\lambda_2) = 1$

$$\Rightarrow J_2 = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 1 & \lambda_2 \end{pmatrix}$$

(blocuri de mărime 1 și 2)

3) $\sigma(A) = \{\lambda_1, \lambda_2, \lambda_3\}$

$m_a(\lambda_i) = 1 \Rightarrow m_g = 1 \Rightarrow J = \text{Diag}$

$$\Rightarrow J = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix}$$

Partea a II-a

ALGORITMUL DE DETERMINARE A FORMEI JORDAN A UNEI MATRICI

(! Lucrăm mereu în cadrul: $P_A(X)=(X-a_1)\cdots(X-a_n)$, $a_1,\dots,a_n\in K$
$A\in M_n(K)$

Fie λ val proprie a lui A

$V^\lambda=\{v\in V \mid \exists\, m\in\mathbb{N}^* \text{ a.î. } (A-\lambda I)^m v=0\}$

$\cup|$

$V_\lambda=\{v\in V \mid Av=\lambda v \iff (A-\lambda I)v=0\}$

Puteți nota
$a(\lambda)$ = multiplicitatea algebrică
$g(\lambda)$ = multiplicitatea geometrică,
respectiv $ma(\lambda)$, $mg(\lambda)$

Doar fiți consecvenți în notație!

Fapte:

1. $\dim_K V^\lambda=a(\lambda)$
2. $V^\lambda=\bigcup_{m\geq 1}\ker(A-\lambda I)^m$
3. $\exists\, m$ minim a.î. $V^\lambda=\ker(A-\lambda I)^m$
4. $V=V^{\lambda_1}\oplus\dots\oplus V^{\lambda_k}$, unde $\lambda_1,\dots,\lambda_k$ sunt valorile proprii ale lui A

Alg:

* se det. polinomul caracteristic
* se det. valorile proprii

Pt fiecare valoare proprie λ vom face:

1) Determinăm m minim a.î. $V^\lambda=\ker(A-\lambda I)^m$, i.e determinăm m minim a.î. $\dim(\ker(A-\lambda I)^m)=a(\lambda)=m_a(\lambda)$

2) $r=\dim(\underbrace{\ker(A-\lambda I)}_{V_\lambda})=g(\lambda)=mg(\lambda)$

$\tilde{N}=A-\lambda I$

$N=\tilde{N}|_{V^\lambda}$ $\qquad \ker\tilde{N}=\ker N$

$r=\dim(\ker N)$

3) Determinăm $\gamma_1,\dots,\gamma_m$ din următoarele relații:

$2r-\gamma_1=\dim(\ker N^2)$

$3r-2\gamma_1-\gamma_2=\dim(\ker N^3)$

$\vdots$

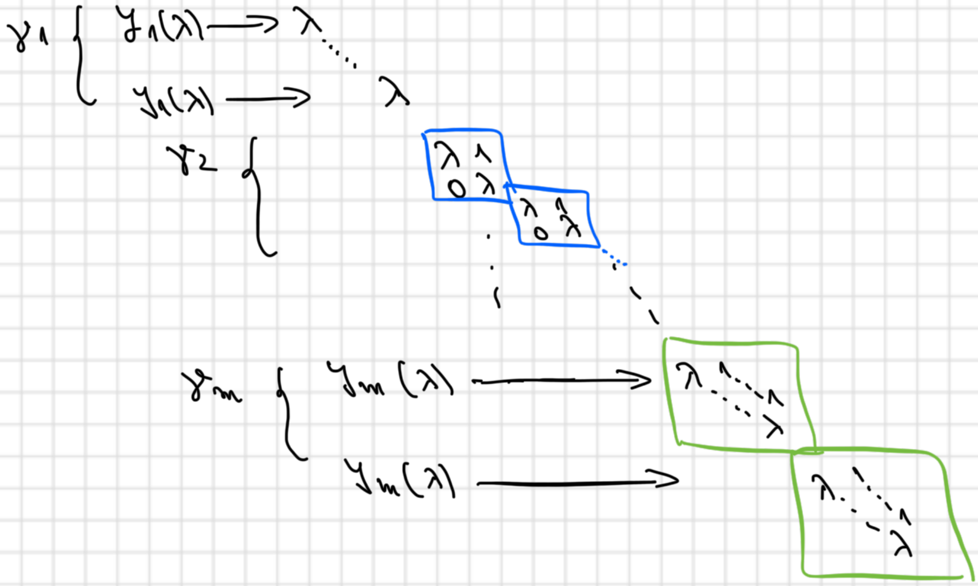
$mr-(m-1)\gamma_1-\dots-\gamma_{m-1}=\dim(\ker N^m)$

și $\gamma_m=r-\gamma_1-\dots-\gamma_{m-1}$

4) Asociate lui λ vor fi

γ_1 celule Jordan $J_1(\lambda)$
γ_2 —||— $J_2(\lambda)$
$\vdots$
γ_m —||— $J_m(\lambda)$

! $\begin{cases}\gamma_1+\dots+\gamma_m=g(\lambda)=mg(\lambda)=r\\ \gamma_1+2\gamma_2+\dots+m\gamma_m=\dim(\ker N^m)=\dim V^\lambda=a(\lambda)=ma(\lambda)\end{cases}$



OBS! 1) Avem $a(\lambda)$ apariții ale lui λ pe diagonala principală

$$a(\lambda) = \gamma_1 + 2\gamma_2 + \dots + m\gamma_m$$

2) Avem $g(\lambda)$ celule Jordan

$$g(\lambda) = r$$

Ex ①
$$A = \begin{pmatrix} 1 & 1 & 1 & -2 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$P_A = \det(A - \lambda I_4)$$

$$P_A = \begin{vmatrix} 1-\lambda & 1 & 1 & -2 \\ 0 & 1-\lambda & 0 & -1 \\ 0 & 0 & 1-\lambda & 1 \\ 0 & 0 & 0 & 1-\lambda \end{vmatrix} = (1-\lambda)^4$$

(matrice superior triunghiulară)

$\det(A - \lambda I_4) = 0 \Rightarrow \lambda = 1$ valoare proprie
$\Rightarrow ma(\lambda) = 4$

$$m(g) = \dim V_\lambda = \dim \operatorname{Ker}(A - \lambda I_4) = \dim \operatorname{Ker}(A - I_4)$$

$$N = A - I = \begin{pmatrix} 0 & 1 & 1 & -2 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\dim \operatorname{Ker}(A - I) + \underbrace{\dim \operatorname{Im}(A - I)}_{\underbrace{\operatorname{rang}(A-I)}_{=2}} = \underbrace{\dim V}_{=4}$$ (T rang-defect)

$$\Rightarrow \dim \operatorname{Ker}(A - I) = 2 = r$$

Pasul 1: Determinăm m minim a.î.

$\dim \operatorname{Ker} N^m = ma(\lambda) = 4$

$N = A - I$

$\underbrace{\dim \operatorname{Ker} N^m}_{4} + \underbrace{\dim \operatorname{Im} N^m}_{= rg N^m} = 4 \Rightarrow rg N^m = 4-4=0$

$\Leftrightarrow N^m = O_4$

$N^2 = O_4 \Rightarrow m=2$

Pasul 2: Avem de determinat γ_1, γ_2

$2r - \gamma_1 = \dim \operatorname{ker} N^2$

$4 - \gamma_1 = 4 \Rightarrow \gamma_1 = 0$

$\boxed{\gamma_m \neq 0!}$

$$\begin{cases} \gamma_1 + \gamma_2 = r = 2 \\ \gamma_1 + 2\gamma_2 = ma(\lambda) = 4 \end{cases}$$

$\Rightarrow \gamma_2 = 2$

Pasul 3:

deci avem: 0 celule Jordan $J_1(\lambda)$

2 celule Jordan $J_2(\lambda)$
$\underset{\gamma_2}{\shortparallel}$

$\Rightarrow$ forma Jordan a lui A este:

$$\begin{pmatrix} \boxed{\begin{matrix}\lambda & 1 \\ 0 & \lambda\end{matrix}} & \\ & \boxed{\begin{matrix}\lambda & 1 \\ 0 & \lambda\end{matrix}} \end{pmatrix}$$

② $A = \begin{pmatrix} 5 & -1 & 0 & 0 \\ 9 & -1 & 0 & 0 \\ 0 & 0 & 7 & -2 \\ 0 & 0 & 12 & -3 \end{pmatrix}$

$$P_A(\lambda) = \det(A - \lambda I_4) = \begin{vmatrix} 5-\lambda & -1 & 0 & 0 \\ 9 & -1-\lambda & 0 & 0 \\ 0 & 0 & 7-\lambda & -2 \\ 0 & 0 & 12 & -3-\lambda \end{vmatrix}$$

bloc de matrici pătrate

P: $M = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$, $A, B \in M_n(K)$ $\Rightarrow \det M = \det A \cdot \det B$

$P_A(\lambda) = \begin{vmatrix} 5-\lambda & -1 \\ 9 & -1-\lambda \end{vmatrix} \cdot \begin{vmatrix} 7-\lambda & -2 \\ 12 & -3-\lambda \end{vmatrix}$

$P_A(\lambda) = (-(5-\lambda)(1+\lambda) + 9)(-(7-\lambda)(3+\lambda) + 24)$

$P_A(\lambda) = (-(5+5\lambda-\lambda-\lambda^2)+9)(-(21+7\lambda-3\lambda-\lambda^2)+24)$

$P_A(\lambda) = (-5+\lambda^2-4\lambda+9)\cdot(-21-4\lambda+\lambda^2+24)$

$P_A(\lambda) = (\lambda^2-4\lambda+4)(\lambda^2-4\lambda+3)$
$P_A(\lambda) = (\lambda-2)^2(\lambda-1)(\lambda-3)$

$\lambda_1=1$, $\lambda_2=2$, $\lambda_3=3$

$m_a(\lambda_1)=1$
$\left(m_g \leq \underset{1}{\underset{\parallel}{m_a}} \Rightarrow m_g(\lambda_1)=1 \right)$

$m_a(\lambda_2)=2$, $\underset{m_g(\lambda_3)=1}{\underset{\parallel}{m_a(\lambda_3)=1}}$

Pt $\lambda_1=1$:

* determinăm m minim a.î. $\dim \operatorname{Ker} N^m = m_a(\lambda_1)=1$

unde $N=A-1\cdot I_4 = A-I_4$

$m=1$ deoarece $\dim \operatorname{Ker} N = m_g(\lambda_1)=1$

* avem de aflat $\gamma_1=1 \Rightarrow$ o singură celulă Jordan $J_1(1)$

Pt $\lambda_3=1$, identic cu cazul de mai sus ... 1 celulă Jordan $J_1(3)$

Pt $\lambda_2=2$

Det m minim a.î. $\dim \operatorname{Ker} N^m = m_a(\lambda_2)=2$

$N=A-2I_4$

$$N=\begin{pmatrix} 3 & -1 & 0 & 0 \\ 9 & -3 & 0 & 0 \\ 0 & 0 & 5 & -2 \\ 0 & 0 & 12 & -5 \end{pmatrix}$$

$\dim \operatorname{Ker} N = 4 - rg\, N = 1 = r$

$\dim \operatorname{Ker} N^2 = 4 - \underset{2}{\underset{\parallel}{rg\, N^2}} = 2 = m_a(\lambda_2)$

$$N^2=\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix}$$

* avem de det γ_1, γ_2

$2r - \gamma_1 = \dim(\operatorname{Ker} N^2)$

$\begin{cases} \gamma_1 + \overset{\neq 0}{\gamma_2} = r = 1 \\ \gamma_1 + 2\cdot\gamma_2 = 2 \end{cases} \Rightarrow \begin{matrix} \gamma_1 = 0 \\ \gamma_2 = 1 \end{matrix} \Rightarrow$ avem o singură celulă $J_2(\underset{\lambda}{\underset{\parallel}{2}}) \rightsquigarrow \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix}$

* o formă Jordan a lui A este:

$$\begin{pmatrix} \boxed{\begin{matrix} 2 & 1 \\ 0 & 2 \end{matrix}} & & \\ & 1 & \\ & & 3 \end{pmatrix}$$

$$\begin{pmatrix} 1 & & \\ & \boxed{\begin{matrix} 2 & 1 \\ 0 & 2 \end{matrix}} & \\ & & 3 \end{pmatrix}$$

3) $A = \begin{pmatrix} 1 & -1 & -2 & 3 \\ 0 & 0 & -2 & 3 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & -1 & 2 \end{pmatrix}$

$P_A = \det(A - \lambda I_4) = \underset{(\text{calculați})}{\dots} = (\lambda-1)^4 \Rightarrow \lambda = 1$ v.p, $m_a(\lambda) = 4$

$N = A - I = \begin{pmatrix} 0 & -1 & -2 & 3 \\ 0 & -1 & -2 & 3 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & -1 & 1 \end{pmatrix}$

Pt $\lambda = 1$:
determinăm m minim a.î. $\dim \operatorname{Ker} N^m = m_a(1) = 4$

Deci vrem m minim a.î. $4 - \operatorname{rg} N^m = 4$, i.e. m minim a.î. $N^m = O_4$

$N^2 = \begin{pmatrix} 0 & -1 & -1 & 2 \\ 0 & -1 & -1 & 2 \\ 0 & -1 & -1 & 2 \\ 0 & -1 & -1 & 2 \end{pmatrix}$

$N^3 = O_4$

deci $\boxed{m = 3}$

Avem de determinat $\gamma_1, \gamma_2, \gamma_3$

$r = m_g(1) = \dim \operatorname{Ker} N = 4 - \underset{\overset{\|}{2}}{\operatorname{rg} N} = 2$

$\begin{cases} \gamma_1 + \gamma_2 + \overset{\neq 0}{\gamma_3} = 2 \\ \gamma_1 + 2\gamma_2 + 3\gamma_3 = 4 \end{cases}$

$\gamma_1 = 2 \cdot 2 - \dim \operatorname{Ker} N^2$

$\gamma_1 = 4 - 3 = 1$

$3r - 2\gamma_1 - \gamma_2 = \dim \operatorname{Ker} N^3 = 4$

$6 - 2 - \gamma_2 = 4 \Rightarrow \gamma_2 = 0$

$\gamma_3 = 1$

Deci

$\gamma_1 = 1, \ \gamma_2 = 0, \ \gamma_3 = 1$

$\Rightarrow$ 1 celulă Jordan $J_1(1)$
1 celulă Jordan $J_3(1)$

! Forma Jordan a unei matrici e unică până la permutarea blocurilor Jordan de pe diagonală.

O formă Jordan este

$J_1(1)$

$$\begin{pmatrix} \boxed{1} & & & \\ & 1 & 1 & 0 \\ & 0 & 1 & 1 \\ & 0 & 0 & 1 \end{pmatrix}$$

$J_3(1)$

Să se determine polinomul caracteristic, valorile proprii, vectorii proprii corespunzători, forma canonică Jordan.

- $A = \begin{pmatrix} -1 & 1 & 0 & 0 \\ -1 & 2 & 1 & 0 \\ 5 & -2 & 2 & 0 \\ 0 & 1 & 2 & 1 \end{pmatrix}$ $V = \mathbb{R}^4$

$$P_T = \det(A - \lambda I_4) = \begin{vmatrix} -1-\lambda & 1 & 0 & 0 \\ -1 & 2-\lambda & 1 & 0 \\ 5 & -2 & 2-\lambda & 0 \\ 0 & 1 & 2 & 1-\lambda \end{vmatrix} \overset{\text{Dez după } C_4}{=\!=} (1-\lambda)\cdot(-1)^{4+4} \cdot \begin{vmatrix} -1-\lambda & 1 & 0 \\ -1 & 2-\lambda & 1 \\ 5 & -2 & 2-\lambda \end{vmatrix}$$

$= (1-\lambda)\left[(-1-\lambda)(2-\lambda)^2 + 0 + 5 - 0 + 2(-1-\lambda) + 2-\lambda\right]$

$= (1-\lambda)\left[(-1-\lambda)(4+\lambda^2-4\lambda) + 5 - 2\lambda - \lambda\right]$

$= (1-\lambda)\left[(-1-\lambda)(4+\lambda^2-4\lambda) + 5 - 3\lambda\right]$

$= (1-\lambda)\left[-4 - \cancel{\lambda^2} + \cancel{4\lambda} - \cancel{4}\lambda - \cancel{\lambda^3} + \cancel{4\lambda^2} + 5 - 3\lambda\right]$

$= (1-\lambda)(-\lambda^3 + 3\lambda^2 - 3\lambda + 1)$

$= (\lambda-1)(\lambda^3 - 3\lambda^2 + 3\lambda - 1)$

$= (\lambda-1)^4$

$P_T = 0 \Rightarrow \lambda = 1$

$m_a(\lambda) = m_a(1) = 4$

$\dim \operatorname{Ker} N^m = m_a(\lambda_1) = 4$

$\boxed{N = A - I_4}$

$N = \begin{pmatrix} -2 & 1 & 0 & 0 \\ -1 & 1 & 1 & 0 \\ 5 & -2 & 1 & 0 \\ 0 & 1 & 2 & 0 \end{pmatrix}$, $\det N = 0$, cât este rang $N = ?$

$\Delta_3 = \begin{vmatrix} -2 & 1 & 0 \\ -1 & 1 & 1 \\ 5 & -2 & 1 \end{vmatrix} = 0$

$\Delta_2 = \begin{vmatrix} -2 & 1 \\ -1 & 1 \end{vmatrix} = -1 \neq 0 \Rightarrow rg\, N = 2$

$\underbrace{\dim \operatorname{Ker} N^m}_{4} + \underbrace{\dim \operatorname{Im} N^m}_{rg\, N^m} = \underbrace{\dim V}_{4} \Rightarrow rg\, N^m = 0$

$N^2 = N \cdot N = \begin{pmatrix} -2 & 1 & 0 & 0 \\ -1 & 1 & 1 & 0 \\ 5 & -2 & 1 & 0 \\ 0 & 1 & 2 & 0 \end{pmatrix}\begin{pmatrix} -2 & 1 & 0 & 0 \\ -1 & 1 & 1 & 0 \\ 5 & -2 & 1 & 0 \\ 0 & 1 & 2 & 0 \end{pmatrix} = \begin{pmatrix} 3 & -1 & 1 & 0 \\ 6 & -2 & 2 & 0 \\ -3 & 1 & -1 & 0 \\ 9 & -3 & 3 & 0 \end{pmatrix}$

$N^3 = N^2 N = \begin{pmatrix} 3 & -1 & 1 & 0 \\ 6 & -2 & 2 & 0 \\ -3 & 1 & -1 & 0 \\ 9 & -3 & 3 & 0 \end{pmatrix}\begin{pmatrix} -2 & 1 & 0 & 0 \\ -1 & 1 & 1 & 0 \\ 5 & -2 & 1 & 0 \\ 0 & 1 & 2 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = O_4$

$\Rightarrow N^3 = O_4 \Rightarrow m = 3$

Determinăm $\gamma_1, \gamma_2, \gamma_3$

$$\begin{cases} \gamma_1 + \gamma_2 + \gamma_3 = r = 2 \\ \gamma_1 + 2\gamma_2 + 3\gamma_3 = m_a(\lambda_1) = 4 \end{cases}$$

$\underbrace{\dim \operatorname{Ker} N}_{r} + \underbrace{\dim \operatorname{Im} N}_{rg\, N = 2} = \underbrace{\dim V}_{4} \Rightarrow \dim \operatorname{Ker} N = 2 \Rightarrow r = 2$

$$2n-\gamma_1 = 4-\underbrace{rang(A-I_4)^2}_{rang\, N^2=1}$$

$$4-\gamma_1=3 \Rightarrow \boxed{\gamma_1=1}$$

$$3n-2\gamma_1-\gamma_2 = 4-\underbrace{rang(A-I_4)^3}_{=0}$$

$$6-2-\gamma_2=4$$

$$\Rightarrow \boxed{\gamma_2=0} \xrightarrow{\gamma_1+\gamma_2+\gamma_3=2} \boxed{\gamma_3=1}$$

Deci $\gamma_1=1$, $\gamma_2=0$, $\gamma_3=1$

$\Rightarrow$ 1 celulă Jordan $J_1(1)$

1 celulă Jordan $J_3(1)$

$$J_T = \begin{pmatrix} \boxed{1} & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$J_1(1)$, $J_3(1)$

Fișă de lucru:

Încercați și pentru:

a) $A=\begin{pmatrix} 4 & 5 & -2 \\ -2 & -2 & 1 \\ -1 & -1 & 1 \end{pmatrix}$ c) $C=\begin{pmatrix} 3 & 0 & 8 \\ 3 & -1 & 6 \\ -2 & 0 & -5 \end{pmatrix}$ e) $E=\begin{pmatrix} 1 & 2 & 0 \\ 0 & 2 & 0 \\ -2 & -2 & -1 \end{pmatrix}$ g) $G=\begin{pmatrix} 0 & 1 & 0 & 0 \\ 11 & 6 & -4 & -4 \\ 22 & 15 & -8 & -9 \\ -3 & -2 & 1 & 2 \end{pmatrix}$

b) $B=\begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 \end{pmatrix}$ d) $D=\begin{pmatrix} -4 & 2 & 10 \\ -4 & 3 & 7 \\ -3 & 1 & 7 \end{pmatrix}$ f) $F=\begin{pmatrix} 4 & 6 & 0 \\ -3 & -5 & 0 \\ -3 & -6 & 1 \end{pmatrix}$