$$f:\mathbb{R}^2\to\mathbb{R},\, f(x,y)=\begin{cases}\frac{x^2y^2}{\sqrt{x^4+y^{10}}}, x^2+y^2\neq 0\\ 0, x=y=0\end{cases}$$

f functia este continua pe $\mathbb{R}^2$ (vezi tut 4)

$$\frac{\partial f}{\partial x}(x,y)=\frac{2xy^2\sqrt{x^4+y^{10}}-x^2y^2\frac{1}{2\sqrt{x^4+y^{10}}}4x^3}{x^4+y^{10}}=2x\frac{y^2(x^4+y^{10})-x^4y^2}{(x^4+y^{10})\sqrt{x^4+y^{10}}}=2x\frac{y^{12}}{(x^4+y^{10})\sqrt{x^4+y^{10}}}$$

$$\frac{\partial f}{\partial y}(x,y)=\frac{2x^2y\sqrt{x^4+y^{10}}-x^2y^2\frac{1}{2\sqrt{x^4+y^{10}}}10y^9}{x^4+y^{10}}=y\frac{2x^2(x^4+y^{10})-5x^2y^{10}}{(x^4+y^{10})\sqrt{x^4+y^{10}}}=y\frac{2x^6-3x^2y^{10}}{(x^4+y^{10})\sqrt{x^4+y^{10}}}$$

$e_1=(1,0)$

$e_2=(0,1)$

$$\lim_{t\to 0}\frac{f((0,0)+te_1)-f((0,0))}{t}=\lim_{t\to 0}\frac{f((0,0)+t(1,0))-f((0,0))}{t}=\lim_{t\to 0}\frac{f((t,0))-f((0,0))}{t}=$$

$$=\lim_{t\to 0}\frac{\frac{t^2*0^2}{\sqrt{t^4+0^{10}}}-0}{t}=\lim_{t\to 0}\frac{0}{t}=0\Rightarrow\exists\frac{\partial f}{\partial x}(0,0)=0$$

$$\lim_{t\to 0}\frac{f((0,0)+te_2)-f((0,0))}{t}=\lim_{t\to 0}\frac{f((0,0)+t(0,1))-f((0,0))}{t}=\lim_{t\to 0}\frac{f((0,t))-f((0,0))}{t}=$$

$$=\lim_{t\to 0}\frac{\frac{0^2*t^2}{\sqrt{0^4+t^{10}}}-0}{t}=\lim_{t\to 0}\frac{0}{t}=0\Rightarrow\exists\frac{\partial f}{\partial y}(0,0)=0$$

$\exists\frac{\partial f}{\partial x}(x,y),\frac{\partial f}{\partial y}(x,y) pe \mathbb{R}^2\setminus\{(0,0)\}$

$\frac{\partial f}{\partial x},\frac{\partial f}{\partial y}$ functii continue pe $\mathbb{R}^2\setminus\{(0,0)\}$

$\mathbb{R}^2\setminus\{(0,0)\}$ multime deschisa

$\Rightarrow f$ diferentiabila pe $\mathbb{R}^2\setminus\{(0,0)\}$

Fie $T:\mathbb{R}^2\to\mathbb{R},\, T((x,y))=\frac{\partial f}{\partial x}(0,0)x+\frac{\partial f}{\partial x}(0,0)y=0x+0y=0,\forall(x,y)\in\mathbb{R}^2$

$$\lim_{(x,y)\to(0,0)}\frac{|f(x,y)-f(0,0)-T((x,y)-(0,0))|}{||(x,y)-(0,0)||}=\lim_{(x,y)\to(0,0)}\frac{\left|\frac{x^2y^2}{\sqrt{x^4+y^{10}}}-0-T((x,y))\right|}{||(x,y)||}=$$

$$=\lim_{(x,y)\to(0,0)}\frac{\left|\frac{x^2y^2}{\sqrt{x^4+y^{10}}}-0\right|}{\sqrt{x^2+y^2}}=\lim_{(x,y)\to(0,0)}\underbrace{\frac{x^2y^2}{\sqrt{x^4+y^{10}}\sqrt{x^2+y^2}}}_{g(x,y)}$$

$$0\le g(x,y)=\frac{x^2y^2}{\sqrt{x^4+y^{10}}\sqrt{x^2+y^2}}\le\frac{x^2y^2}{\sqrt{x^4}\sqrt{x^2+y^2}}=\frac{y^2}{\sqrt{x^2+y^2}}\le\frac{y^2}{\sqrt{y^2}}=y$$

$$\begin{tikzcd}
0 \arrow[r, phantom, "\le"] \arrow[dr] & g(x,y) \arrow[r, phantom, "\le"] \arrow[d, dashed] & y \arrow[dl, "{(x,y)\to(0,0)}"] \\
& 0 &
\end{tikzcd}$$

$$\Rightarrow\lim_{(x,y)\to(0,0)}\frac{|f(x,y)-f(0,0)-T((x,y)-(0,0))|}{||(x,y)-(0,0)||}=0\Rightarrow f \text{ e diferentiabila in } (0,0)$$

$f:\mathbb{R}^2 \to \mathbb{R}, f(x,y) = \begin{cases} \frac{x^5y^2}{x^8+y^4}, x^2+y^2 \neq 0 \\ 0, x=y=0 \end{cases}$

$$\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{(x,y)\to(0,0)} \frac{x^5y^2}{x^8+y^4} \overset{\infty}{\underset{\infty}{=}} ?$$

Alegem:

$$y = x \Rightarrow \lim_{x\to 0} f(x,x) = \frac{x^5x^2}{x^8+x^4} = \lim_{x\to 0}\frac{x^3}{x^4+1} = 0$$

$$y = -x \Rightarrow \lim_{x\to 0} f(x,-x) = \frac{x^5x^2}{x^8+x^4} = \lim_{x\to 0}\frac{x^3}{x^4+1} = 0$$

$$y = \sqrt{x} \Rightarrow \lim_{x\to 0} f(x,\sqrt{x}) = \frac{x^5x}{x^8+x^2} = \lim_{x\to 0}\frac{x^4}{x^6+1} = 0$$

Este posibil ca:

$$\lim_{(x,y)\to(0,0)} f(x,y) = l(=0) \in \mathbb{R}$$

Evaluam:

$$0 \leq |f(x,y) - l| \leq g(x,y)$$

$$(x,y) \to (0,0)$$

$$0$$

$$0 \leq |f(x,y) - l| = \left|\frac{x^5y^2}{x^8+y^4}\right| = |x|\frac{x^4y^2}{x^8+y^4} \leq |x|\frac{1}{2}$$

$$0 \leq |f(x,y) - l| \leq \frac{|x|}{2}$$

$$(x,y) \to (0,0)$$

$$0$$

$$\frac{\partial f}{\partial x}(x,y) = \frac{5x^4y^2(x^8+y^4) - 8x^{12}y^2}{(x^8+y^4)^2} = x^4y^2\frac{5y^4 - 3x^8}{(x^8+y^4)^2}$$

$$\frac{\partial f}{\partial y}(x,y) = \frac{2x^5y(x^8+y^4) - 4x^5y^5}{(x^8+y^4)^2} = 2x^5y\frac{x^8 - y^4}{(x^8+y^4)^2}$$

$e_1 = (1,0)$

$e_2 = (0,1)$

$$\lim_{t\to 0}\frac{f((0,0)+te_1) - f((0,0))}{t} = \lim_{t\to 0}\frac{f((0,0)+t(1,0)) - f((0,0))}{t} = \lim_{t\to 0}\frac{f((t,0)) - f((0,0))}{t} =$$

$$= \lim_{t\to 0}\frac{\frac{t^5*0^2}{\sqrt{t^8+0^4}} - 0}{t} = \lim_{t\to 0}\frac{0}{t} = 0 \Rightarrow \exists \frac{\partial f}{\partial x}(0,0) = 0$$

$$\lim_{t\to 0}\frac{f((0,0)+te_2) - f((0,0))}{t} = \lim_{t\to 0}\frac{f((0,0)+t(0,1)) - f((0,0))}{t} = \lim_{t\to 0}\frac{f((0,t)) - f((0,0))}{t} =$$

$$= \lim_{t\to 0}\frac{\frac{0^5*t^2}{\sqrt{0^8+t^4}} - 0}{t} = \lim_{t\to 0}\frac{0}{t} = 0 \Rightarrow \exists \frac{\partial f}{\partial y}(0,0) = 0$$

$\exists \frac{\partial f}{\partial x}(x,y), \frac{\partial f}{\partial y}(x,y) pe \mathbb{R}^2 \backslash \{(0,0)\}$

$\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}$ functii continue pe $\mathbb{R}^2 \backslash \{(0,0)\}$

$\mathbb{R}^2 \backslash \{(0,0)\}$ multime deschisa

$\Rightarrow f$ diferentiabila pe $\mathbb{R}^2 \backslash \{(0,0)\}$

Fie $T:\mathbb{R}^2\to\mathbb{R}, T((x,y))=\frac{\partial f}{\partial x}(0,0)x+\frac{\partial f}{\partial x}(0,0)y=0x+0y=0, \forall(x,y)\in\mathbb{R}^2$

$$\lim_{(x,y)\to(0,0)}\frac{|f(x,y)-f(0,0)-T((x,y)-(0,0))|}{||(x,y)-(0,0)||}=\lim_{(x,y)\to(0,0)}\frac{\left|\frac{x^5y^2}{x^8+y^4}-0-T((x,y))\right|}{||(x,y)||}=$$

$$=\lim_{(x,y)\to(0,0)}\frac{\left|\frac{x^5y^2}{x^8+y^4}\right|-0}{\sqrt{x^2+y^2}}=\lim_{(x,y)\to(0,0)}\underbrace{\frac{x^5y^2}{(x^8+y^4)\sqrt{x^2+y^2}}}_{g(x,y)}$$

$y=x^2\Rightarrow\lim_{x\to0}\frac{x^5x^4}{(x^8+x^8)\sqrt{x^2+x^4}}=\lim_{x\to0}\frac{x^9}{2x^9\sqrt{1+x^2}}=\frac{1}{2}\neq0$

$\Rightarrow\lim_{(x,y)\to(0,0)}\frac{|f(x,y)-f(0,0)-T((x,y)-(0,0))|}{||(x,y)-(0,0)||}\neq0\Rightarrow f$ nu e diferentiabila in $(0,0)$

$f:\mathbb{R}^2\to\mathbb{R}, f(x,y)=\begin{cases}\frac{x^3y^7}{\sqrt{x^{16}+y^{16}}}, x^2+y^2\neq0\\ 0, x=y=0\end{cases}$

f functia este continua pe $\mathbb{R}^2$(vezi tut 4)

$$\frac{\partial f}{\partial x}(x,y)=\frac{3x^2y^7\sqrt{x^{16}+y^{16}}-x^3y^7\frac{1}{2\sqrt{x^{16}+y^{16}}}16x^{15}}{x^{16}+y^{16}}=x^2y^7\frac{-5x^{16}+3y^{16}}{(x^{16}+y^{16})\sqrt{x^{16}+y^{16}}}$$

$$\frac{\partial f}{\partial y}(x,y)=\frac{7x^3y^6\sqrt{x^{16}+y^{16}}-x^3y^7\frac{1}{2\sqrt{x^{16}+y^{16}}}16y^{15}}{x^{16}+y^{16}}=x^3y^6\frac{-x^{16}+7y^{16}}{(x^{16}+y^{16})\sqrt{x^{16}+y^{16}}}$$

$e_1=(1,0)$

$e_2=(0,1)$

$$\lim_{t\to0}\frac{f((0,0)+te_1)-f((0,0))}{t}=\lim_{t\to0}\frac{f((0,0)+t(1,0))-f((0,0))}{t}=\lim_{t\to0}\frac{f((t,0))-f((0,0))}{t}=$$

$$=\lim_{t\to0}\frac{\frac{t^3*0^7}{\sqrt{t^{16}+0^{16}}}-0}{t}=\lim_{t\to0}\frac{0}{t}=0\Rightarrow\exists\frac{\partial f}{\partial x}(0,0)=0$$

$$\lim_{t\to0}\frac{f((0,0)+te_2)-f((0,0))}{t}=\lim_{t\to0}\frac{f((0,0)+t(0,1))-f((0,0))}{t}=\lim_{t\to0}\frac{f((0,t))-f((0,0))}{t}=$$

$$=\lim_{t\to0}\frac{\frac{0^3*t^7}{\sqrt{0^{16}+t^{16}}}-0}{t}=\lim_{t\to0}\frac{0}{t}=0\Rightarrow\exists\frac{\partial f}{\partial y}(0,0)=0$$

$\exists\frac{\partial f}{\partial x}(x,y),\frac{\partial f}{\partial y}(x,y) pe \mathbb{R}^2\backslash\{(0,0)\}$

$\frac{\partial f}{\partial x},\frac{\partial f}{\partial y}$ functii continue pe $\mathbb{R}^2\backslash\{(0,0)\}$

$\mathbb{R}^2\backslash\{(0,0)\}$ multime deschisa

$\Rightarrow f$ diferentiabila pe $\mathbb{R}^2\backslash\{(0,0)\}$

Fie $T:\mathbb{R}^2\to\mathbb{R}, T((x,y))=\frac{\partial f}{\partial x}(0,0)x+\frac{\partial f}{\partial x}(0,0)y=0x+0y=0, \forall(x,y)\in\mathbb{R}^2$

$$\lim_{(x,y)\to(0,0)}\frac{|f(x,y)-f(0,0)-T((x,y)-(0,0))|}{||(x,y)-(0,0)||}=\lim_{(x,y)\to(0,0)}\frac{\left|\frac{x^3y^7}{\sqrt{x^{16}+y^{16}}}-0-T((x,y))\right|}{||(x,y)||}=$$

$$= \lim_{(x,y)\to(0,0)} \frac{\left|\frac{x^3y^7}{\sqrt{x^{16}+y^{16}}} - 0\right|}{\sqrt{x^2+y^2}} = \lim_{(x,y)\to(0,0)} \underbrace{\frac{|x^3y^7|}{\sqrt{x^{16}+y^{16}}\sqrt{x^2+y^2}}}_{g(x,y)}$$

$$x^{16} \le x^{16}+y^{16} \Leftrightarrow \sqrt[16]{x^{16}} \le \sqrt[16]{x^{16}+y^{16}} \Leftrightarrow |x| \le \sqrt[16]{x^{16}+y^{16}} \Leftrightarrow |x|^3 \le (x^{16}+y^{16})^{\frac{3}{16}}$$

Procedăm la fel și pentru $|y|^7 \le (x^{16}+y^{16})^{\frac{7}{16}}$

$$\Rightarrow \left|x^3y^7\right| \le (x^{16}+y^{16})^{\frac{3}{16}+\frac{7}{16}} = (x^{16}+y^{16})^{\frac{5}{8}}$$

$$0 \le g(x,y) = \frac{x^3y^7}{\sqrt{x^{16}+y^{16}}\sqrt{x^2+y^2}} \le \frac{(x^{16}+y^{16})^{\frac{5}{8}}}{\sqrt{x^{16}+y^{16}}\sqrt{x^2+y^2}} = \frac{(x^{16}+y^{16})^{\frac{5}{8}-\frac{1}{2}}}{\sqrt{x^2+y^2}} = \frac{\sqrt[8]{x^{16}+y^{16}}}{\sqrt{x^2+y^2}} \le$$

$$\le \frac{\sqrt[8]{(x^2+y^2)^8}}{\sqrt{x^2+y^2}} = \frac{x^2+y^2}{\sqrt{x^2+y^2}} = \sqrt{x^2+y^2}$$

$$0 \le g(x,y) \le \sqrt{x^2+y^2}$$

$$(x,y) \to (0,0)$$

$$0$$

$$\Rightarrow \lim_{(x,y)\to(0,0)} \frac{|f(x,y) - f(0,0) - T((x,y)-(0,0))|}{||(x,y)-(0,0)||} = 0 \Rightarrow f \text{ e diferentiabila in } (0,0)$$