Seminar 9

1. Să se dezvolte în serie de puteri ale lui x funcț. următoare:

a) $f:(-2;2) \to \mathbb{R}$ $f(x) = \frac{1}{2-x}$

$$f(x) = \frac{1}{2-x} = \frac{1}{2}\left(\frac{1}{1-\frac{x}{2}}\right)$$

notăm $\frac{x}{2} = a$

$x \in (-2;2) \Rightarrow \frac{x}{2} \in (-1;1)$

$$f(x) = \frac{1}{2}\left(\frac{1}{1-a}\right) = \frac{1}{2}\sum_{n=0}^{\infty} a^n = \sum_{n=0}^{\infty}\frac{1}{2}\left(\frac{x}{2}\right)^n$$

$$= \sum_{n=0}^{\infty} \frac{1}{2^{n+1}} x^n$$

b) $f: [-1;1) \to \mathbb{R}$ $\quad f(x) = \ln(1-x)$

$$f'(x) = (\ln(1-x))' = -\frac{1}{1-x} = -\sum_{n=0}^{\infty} x^n$$

$$= \sum_{n=0}^{\infty} (-1)\, x^n \quad \forall\, x \in (-1;1)$$

Integrăm „termen cu termen" și obținem că $\exists\, C \in \mathbb{R}$ a.î. $f(x) = \sum_{n=0}^{\infty} (-1) \frac{x^{n+1}}{n+1} + C \ \forall$ $x \in (-1;1)$

$$\left.\begin{aligned} f(0) &= \ln 1 = 0 \\ f(0) &= \sum_{n=0}^{\infty} (-1) \frac{0^{n+1}}{n+1} + C = C \end{aligned}\right| \Rightarrow C = 0$$

Deci, $f(x) = \sum_{n=0}^{\infty} (-1) \frac{x^{n+1}}{n+1} \ \forall\, x \in (-1;1)$

• pt $x = -1$ seria devine $\sum_{n=0}^{\infty} (-1) \frac{(-1)^{n+1}}{n+1}$

~~Cf. Th. II a lui Abel~~ $= \sum_{n=0}^{\infty} (-1)^n \cdot \frac{(-1)^2}{n+1}$

~~$\lim_{x \to -1,\ x > -1}$~~

$= \sum_{n=0}^{\infty} (-1)^n \cdot \frac{1}{n+1}$ cf. crit. Leibniz converg

Cf. Th. II a lui Abel:

$$\lim_{\substack{x \to -1 \\ x > -1}} f(x) = \sum_{n=0}^{\infty} (-1) \frac{(-1)^{n+1}}{n+1}.$$

Dar $\lim_{\substack{x \to -1 \\ x > -1}} f(x) = \lim_{\substack{x \to -1 \\ x > -1}} \ln(1-x) = \ln 2 = f(-1)$

Deci, $f(x) = \sum_{n=0}^{\infty} (-1) \frac{x^{n+1}}{n+1}$ pt $\forall$ $x \in [-1;1)$

2. a) Studiati cont lui f

b) Det $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}$

c) Stud. dif. lui f

unde:

i) $f: \mathbb{R}^2 \to \mathbb{R}$ $f(x,y) = \begin{cases} \frac{xy}{x^2+y^2}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases}$

<u>Sol</u> a) Vezi seminar 6

b) $\frac{\partial f}{\partial x}(x,y) = \left(\frac{xy}{x^2+y^2}\right)'_x = \frac{(xy)'_x \cdot (x^2+y^2) - xy \cdot (x^2+y^2)'_x}{(x^2+y^2)^2}$

$= \frac{y \cdot (x^2+y^2) - xy \cdot 2x}{(x^2+y^2)^2}$

$\frac{\partial f}{\partial x}(0,0) = \lim_{t \to 0} \frac{f((0,0) + t \cdot e_1) - f(0,0)}{t}$

$= \lim_{t \to 0} \frac{f((0,0) + (t,0)) - f(0,0)}{t}$

$= \lim_{t \to 0} \frac{f(t,0) - f(0,0)}{t} = \lim_{t \to 0} \frac{f(t,0) - 0}{t}$

$= \lim_{t \to 0} \frac{\frac{0 \cdot t}{t^2+0^2}}{t} = 0$

$\frac{\partial f}{\partial y}(x,y) = \left(\frac{xy}{x^2+y^2}\right)'_y = \frac{x(x^2+y^2) - xy \cdot 2y}{(x^2+y^2)^2}$

$\frac{\partial f}{\partial y}(0,0) = \lim_{t \to 0} 0$ $\quad \frac{\frac{1}{n}\cdot\frac{1}{n}}{\left(2\frac{1}{n^2}\right)^2} = \frac{\frac{1}{n^2}}{4\cdot\frac{1}{n^4}} = \frac{1}{n^2}\cdot\frac{n^4}{4} = \frac{n^2}{4}$

pt $(x,y) \neq (0,0)$

$\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}$ cont $\mathbb{R}^2 \setminus \{(0,0)\}$ deschisă $\xrightarrow[\text{difer.}]{\text{crit}}$ f diferențiabilă pe $\mathbb{R}^2 \setminus \{(0,0)\}$

Pp că f este dif în $(0,0)$

Atunci $\exists\ df(0,0) : \mathbb{R}^2 \to \mathbb{R}$

$df(0,0)(u,v) = \left[\left(\underset{\substack{\| \\ 0}}{\frac{\partial f}{\partial x}(0,0)} \quad \underset{\substack{\| \\ 0}}{\frac{\partial f}{\partial y}(0,0)}\right)\begin{pmatrix} u \\ v \end{pmatrix}\right] = 0$

Arătăm că $\lim_{(x,y)\to(0,0)} \frac{f(x,y) - f(0,0) - df(0,0)\big((x,y)-(0,0)\big)}{\|(x,y)-(0,0)\|} = 0$

Fie

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sau: Studiem dif lui f în $(0,0)$

f nu e cont în $(0,0)$ $\Rightarrow$ f nu e dif în origine

ii) $f:\mathbb{R}^2\to\mathbb{R}$ $\quad f(x,y)=\begin{cases}\dfrac{x^5\cdot y^2}{x^8+y^4} & ; (x,y)\neq(0,0)\\ 0 & (x,y)=(0,0)\end{cases}$

grad sus $\leq$ grad jos $\Rightarrow$ nu e cont

Sol a) f cont pe $\mathbb{R}^2\setminus\{(0,0)\}$ (op. cu f-ții elem)

Studiem cont lui f în $(0,0)$

$f(x,y) = \dfrac{x\cdot x^4\cdot y^2}{x^8+y^4}$ $\to$ nu rod jos la examen

V_1 Fie $(x,y)\in\mathbb{R}^2\setminus\{(0;0)\}$

$|f(x,y)-f(0,0)| = \left|\dfrac{x^5\cdot y^2}{x^8+y^4}-0\right| = \left|\dfrac{x^5y^2}{x^8+y^4}\right|$

$= |x|\cdot \dfrac{x^4y^2}{x^8+y^4} \leq |x|\cdot\dfrac{1}{2}$ $(x\to 0)$

$\leq\frac{1}{2}$ (expl: $\dfrac{x^8+y^4}{2}\geq\sqrt{x^8\cdot y^4}$

$\Rightarrow \dfrac{x^8+y^4}{2}\geq x^4y^2$

deci $\dfrac{x^4y^2}{x^8+y^4}\leq\dfrac{1}{2}$)

$\Rightarrow f$ cont în $(0,0)$

V_2 Fie $(x,y)\in\mathbb{R}^2\setminus\{(0;0)\}$ $\to$ merge pt puteri pare

$|f(x,y)-f(0,0)| = \left|\dfrac{x^5y^2}{x^8+y^4}-0\right|$

$=\dfrac{|x^5y^2|}{x^8+y^4} = \left(\dfrac{|x|^8}{x^8+y^4}\right)^{\frac{5}{8}}\cdot\left(\dfrac{|y|^4}{x^8+y^4}\right)^{\frac{2}{4}}\cdot(x^8+y^4)^{\frac{5}{8}+\frac{2}{4}-1}$ >0 $\downarrow$ $\frac{1}{8}$

$=\underbrace{\left(\dfrac{x^8}{x^8+y^4}\right)^{\frac{5}{8}}\cdot\left(\dfrac{y^4}{x^8+y^4}\right)^{\frac{2}{4}}}_{\leq 1}\cdot(x^8+y^4)^{\frac{9}{8}-1}\leq(x^8+y^4)^{\frac{1}{8}} \xrightarrow[(x,y)\to(0,0)]{} 0$

≤ 1 (Expl: $x^8 \le x^8 + y^4 \Rightarrow \frac{x^8}{x^8+y^4} \le 1 \Rightarrow \left(\frac{x^8}{x^8+y^4}\right)^{\frac{5}{8}} \le 1$)

$\Rightarrow f$ cont în origine

b) Se calc derivatele parțiale

c) $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}$ cont pe $\mathbb{R}^2 \setminus \{(0,0)\}$

$\mathbb{R}^2 \setminus \{(0,0)\}$ deschisă

$\Rightarrow f$ dif pe $\mathbb{R}^2 \setminus \{(0,0)\}$

Stud. dif lui f în $(0,0)$

Dacă f ar fi dif în $(0,0)$, atunci $df(0,0): \mathbb{R}^2 \to \mathbb{R}$, $df(0,0)(u,v)$

$$= \left[\left(\underbrace{\frac{\partial f}{\partial x}(0,0)}_{=0} \quad \underbrace{\frac{\partial f}{\partial y}(0,0)}_{=0}\right)\begin{pmatrix} u \\ v \end{pmatrix}\right]$$

$= 0 \cdot u + 0 \cdot v = 0$

$$\lim_{(x,y)\to(0,0)} \frac{f(x,y) - f(0,0) - df(0,0)((x,y)-(0,0))}{\|(x,y)-(0,0)\|}$$

$$\lim_{(x,y)\to(0,0)} \frac{\frac{x^5y^2}{x^8+y^4} - 0 - 0}{\sqrt{x^2+y^2}} = \lim_{(x,y)\to(0,0)}$$

$$\frac{x^5y^2}{(x^8+y^4)\sqrt{x^2+y^2}}$$

Alegem $(x_n, y_n) = \left(\frac{1}{n}, \frac{1}{n^2}\right) \; \forall n \in \mathbb{N}^*$

Avem $\lim_{n\to\infty} (x_n, y_n) = (0,0)$ și

$$\left(\left|\frac{x^5y^2}{(x^8+y^4)\sqrt{x^2+y^2}}\right| \leq \left|\frac{x^5y^2}{x^8+y^4}\right| \cdot \left|\frac{1}{\sqrt{x^2+y^2}}\right| \right.$$

$$\leq \frac{1}{2}|x| \left|\frac{1}{\sqrt{x^2+y^2}}\right| = \frac{1}{2}\left|\frac{x}{\sqrt{x^2+y^2}}\right|$$

$$0 \leq x^2 \leq x^2+y^2 \quad /\sqrt{\ }$$

$$\left. 0 \leq x \leq \sqrt{x^2+y^2} \quad / \right)$$

$$\lim_{n\to\infty} \frac{x_n^5 y_n^2}{(x_n^8+y_n^4)\sqrt{x_n^2+y_n^2}} = \lim_{n\to\infty} \frac{\frac{1}{n^5}\cdot\frac{1}{n^4}}{\left(\frac{1}{n^8}+\frac{1}{n^8}\right)\sqrt{\frac{1}{n^2}+\frac{1}{n^4}}}$$

$$= \lim_{n\to\infty} \frac{\frac{1}{n^9}}{\frac{2}{n^8}\sqrt{\frac{n^2+1}{n^4}}} = \lim_{n\to\infty} \frac{\frac{1}{n^9}}{\frac{2}{n^8}\frac{\sqrt{n^2+1}}{n^2}}$$

$$= \lim_{n\to\infty} \frac{1}{n^9} \frac{n^{10}\, n^{8}}{2\sqrt{n^2+1}} = \frac{1}{2} \neq 0$$

Deci, $\lim\limits_{(x,y)\to(0,0)} \frac{x^5y^2}{(x^8+y^4)\sqrt{x^2+y^2}} \neq 0 \Rightarrow f$

nu e dif în $(0,0)$

iii) $f: \mathbb{R}^2 \to \mathbb{R}$ $f(x,y) = \begin{cases} \frac{y^3}{x^4+y^2} & : (x,y) \neq (0,0) \\ 0 & : (x,y) = (0,0) \end{cases}$

a) f cont pe $\mathbb{R}^2 \setminus \{(0,0)\}$ (op cu fct elem)

Studiem cont lui f în $(0,0)$

Fie $(x,y) \in \mathbb{R}^2 \setminus \{(0,0)\}$

$$|f(x,y) - f(0,0)| = \left|\frac{y^3}{x^4+y^2} - 0\right|$$

$$= \left|\frac{y^3}{x^4+y^2}\right| = |y| \cdot \underbrace{\left|\frac{y^2}{x^4+y^2}\right|}_{\leq 1} \leq |y| \xrightarrow{(x,y)\to(0,0)} 0$$

(expl: $y^2 \leq x^4+y^2 \Rightarrow \frac{y^2}{x^4+y^2} \leq 1$)

$\Rightarrow f$ cont în $(0,0)$

b) + c) Rezolvăm noi! $\square$

b)

• pt $(x,y) \in \mathbb{R}^2 \setminus \{(0,0)\}$

$$\frac{\partial f}{\partial x} = \left(\frac{y^3}{x^4+y^2}\right)'_x = \frac{(y^3)'_x \cdot (x^4+y^2)_x - y^3 \cdot (x^4+y^2)'_x}{(x^4+y^2)^2}$$

$$= \frac{-y^3 \cdot 4x^3}{(x^4+y^2)^2} = \frac{-4x^3y^3}{(x^4+y^2)^2}$$

$$\frac{\partial f}{\partial y} = \left(\frac{y^3}{x^4+y^2}\right)'_y = \frac{(y^3)'_y(x^4+y^2) - y^3(x^4+y^2)'_y}{(x^4+y^2)^2}$$

$$= \frac{3y^2(x^4+y^2) - y^3 \cdot 2y}{(x^4+y^2)^2} = \frac{3y^2(x^4+y^2) - 2y^4}{(x^4+y^2)^2}$$

$$= \frac{3x^4y^2 + y^4}{(x^4+y^2)^2}$$

• în pct $(0,0)$

$$\frac{\partial f}{\partial x}(0,0) = \lim_{t\to 0} \frac{f((0,0)+t\cdot e_1) - f(0,0)}{t}$$

$$= \lim_{t\to 0} \frac{f(t,0) - \overbrace{f(0,0)}^{0}}{t}$$

$$= \lim_{t\to 0} \frac{\frac{0^3}{t^4+0^2}}{t} = 0$$

$$\frac{\partial f}{\partial y}(0,0) = \lim_{t\to 0} \frac{f((0,0)+t\cdot e_2) - f(0,0)}{t}$$

$$= \lim_{t\to 0} \frac{f(0,t) - f(0,0)}{t} = \lim_{t\to 0} \frac{\frac{t^3}{0^4+t^2}}{t} = 1$$

c)

$\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}$ cont pe $\mathbb{R}^2 \setminus \{0,0\}$ | crit de $\Rightarrow$ ~~[illegible]~~ diferent. | f este dif pe $\mathbb{R}^2 \setminus \{0,0\}$

$\mathbb{R}^2 \setminus \{0,0\}$ mult. deschisă

Studiem dif în $(0,0)$.

Dacă f este dif în $(0,0)$, atunci $df(0,0): \mathbb{R}^2 \to \mathbb{R}$

$$df(0,0)(u,v) = \left[\left(\underbrace{\frac{\partial f}{\partial x}(0,0)}_{0} \quad \underbrace{\frac{\partial f}{\partial y}(0,0)}_{1}\right)\begin{pmatrix} u \\ v \end{pmatrix}\right]$$

$$= u\cdot 0 + v\cdot 1 = v$$

$$\lim_{(x,y)\to(0,0)} \frac{f(x,y) - f(0,0) - df(0,0)\left((x,y)-(0,0)\right)}{\|(x,y)-(0,0)\|}$$

$$\lim_{(x,y)\to(0,0)} \frac{\frac{y^3}{x^4+y^2} - 0 - y}{\sqrt{x^2+y^2}} = \lim_{(x,y)\to(0,0)} \frac{y^3 - yx^4 - y^3}{(x^4+y^2)\sqrt{x^2+y^2}}$$

$$= \lim_{(x,y)\to(0,0)} \frac{-yx^4}{(x^4+y^2)\sqrt{x^2+y^2}}$$

Alegem $(x_n, y_n) =$