$f: A \to \mathbb{R},\ f(x,y) = x$ f cont

$$\iint_A f(x,y)\,dx\,dy = \int_1^{\sqrt[3]{2}} \left(\int_0^y x\,dx \right) dy = \int_1^{\sqrt[3]{2}} \frac{x^2}{2} \Big|_0^y dy$$

$$= \int_1^{\sqrt[3]{2}} \frac{y^2}{2} dy = \frac{y^3}{6} \Big|_1^{\sqrt[3]{2}} = \frac{1}{6}$$

16.01.2024

Curs 14

Teorema de permutare a limitei cu integrala (Cazul multidimensional)

Fie $p \in \mathbb{N}^*$, $\emptyset \neq A \in \mathcal{J}(\mathbb{R}^p)$; $f_n, f: A \to \mathbb{R}$, $\forall n \in \mathbb{N}$ a.î.

1) f_n integr. Riemann și mărg (pe A)

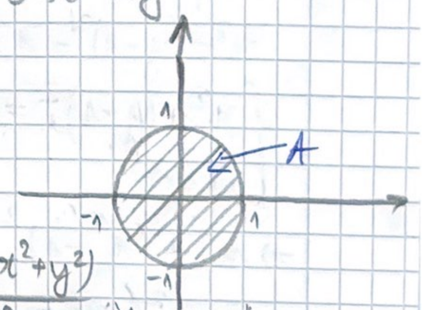
2) $f_n \xrightarrow[n\to\infty]{u} f$

Atunci f este integr. Riemann și mărg. și

$$\lim_{n\to\infty} \int_A f_n(x)\,dx = \int_A f(x)\,dx.$$

Ex: Fie $A = B[(0,0), 1] = \overline{B}((0,0), 1) = \{(x,y) \in \mathbb{R}^2 \mid x^2 + y^2 \leq 1\}$

Det $\lim\limits_{n\to\infty} \iint_A \frac{\cos(n(x+y)) + 2(x^2+y^2)}{n^2 + nx^2 + y^2} dx\,dy$



Sol:

A convexă și mărg $\Rightarrow A \in \mathcal{J}(\mathbb{R}^2)$

A compactă

Fie $f_n: A \to \mathbb{R}$, $f_n(x,y) = \frac{\cos(n(x+y)) + 2(x^2+y^2)}{n^2 + nx^2 + y^2}$ $\forall n \in \mathbb{N}^*$

f_n cont $(\forall) n \in \mathbb{N}^*$
$A \in \mathcal{J}(\mathbb{R}^2)$ și A compactă $\Bigg| \Rightarrow f_n$ integrabilă Riemann pe A și f_n mărginită $\forall n \in \mathbb{N}^*$

CS: Fie $(x,y) \in A$.

$$0 \leq |f_n(x,y)| = \frac{|\cos(n(x+y)) + 2(x^2+y^2)|}{n^2 + nx^2 + y^2} \leq \frac{|\cos(n(x+y))| + 2|x^2+y^2|}{n^2 + nx^2 + y^2}$$

$$\leq \frac{1+2}{n^2} = \frac{3}{n^2} \quad \forall n \in \mathbb{N}^*$$

Avem $0 \leq |f_n(x,y)| \leq \frac{3}{n^2}$ $\forall n \in \mathbb{N}^*$

Deci $\lim\limits_{n\to\infty} |f_n(x,y)| = 0$ Deci $\lim\limits_{n\to\infty} f_n(x,y) = 0$

CV: $\sup_{(x,y)\in A} |f_n(x,y) - f(x,y)| = \sup_{(x,y)\in A} \left| \frac{\cos(n(x+y)) + 2(x^2+y^2)}{n^2+nx^2+y^2} \right|$

$= \sup_{(x,y)\in A} \frac{|\cos(n(x+y)) + 2(x^2+y^2)|}{n^2+nx^2+y^2} \le \frac{3}{n^2} \xrightarrow[n\to\infty]{} 0$

Deci $f_n \xrightarrow[n\to\infty]{u} f$

Cf Teoremei de perm. a limitei cu integrala - Cazul multidimensional, avem că f integr. R. și mărginită și $\lim_{n\to\infty} \iint_A f_n(x,y)\,dx\,dy = \iint_A f(x,y)\,dx\,dy = \iint_A 0\,dx\,dy = 0$.

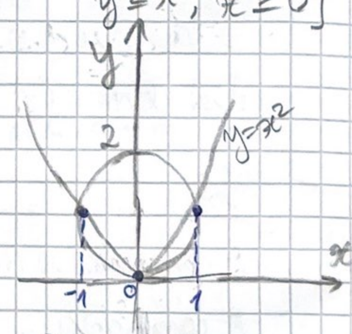
Seminar

1) Det.

a) $\iint_A (1-y)\,dx\,dy$, unde $A = \{(x,y)\in\mathbb{R}^2 \mid x^2+(y-1)^2 \le 1, y \le x^2, x \ge 0\}$

Sol:

Det punctele de intersecție dintre $x^2+(y-1)^2 = 1$ și $y = x^2$.



$\begin{cases} x^2+(y-1)^2 = 1 \\ y = x^2 \end{cases} \Rightarrow$

$\Rightarrow x^2 + (x^2-1)^2 - 1 = 0 \quad \Rightarrow x^2 + x^4 - 2x^2 + 1 - 1 = 0$

$\Rightarrow x^2(x^2-1) = 0 \quad \Rightarrow x \in \{-1, 0, 1\}$.

$x_1 = -1 \quad \Rightarrow y_1 = 1$

$x_2 = 0 \quad \Rightarrow y_2 = 0$

$x_3 = 1 \quad \Rightarrow y_3 = 1$

$x^2+(y-1)^2 \le 1 \quad \Rightarrow (y-1)^2 \le 1-x^2 \quad \Rightarrow -\sqrt{1-x^2} \le y-1 \le \sqrt{1-x^2}$

$1+\sqrt{1-x^2} \le y \le 1+\sqrt{1-x^2}$

$A = \{(x,y)\in\mathbb{R}^2 \mid x\in[0,1],\ 1+\sqrt{1-x^2} \le y \le 1+\sqrt{1-x^2}\}$

Fie $\alpha, \beta : [0,1] \to \mathbb{R}$, $\alpha(x) = 1 - \sqrt{1-x^2}$,

$\beta(x) = x^2$

α, β cont

$A \in \mathcal{J}(\mathbb{R}^2)$ și A compactă

Fie $f : A \to \mathbb{R}$, $f(x,y) = 1-y$

f cont

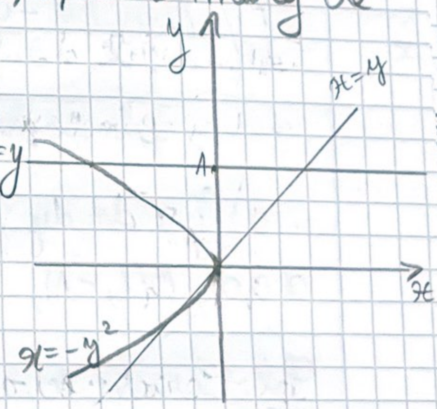
$$\iint_A f(x,y)\,dx\,dy = \int_0^1 \left(\int_{1-\sqrt{1-x^2}}^{x^2} (1-y)\,dy \right) dx =$$

$$= \int_0^1 \left((-1) \cdot \frac{(1-y)^2}{2} \Bigg|_{y=1-\sqrt{1-x^2}}^{y=x^2} \right) dx =$$

$$= -\frac{1}{2} \int_0^1 \left[(1-x^2)^2 - (1-1+\sqrt{1+x^2})^2 \right] dx = -\frac{1}{2}\int_0^1 (1+x^4-2x^2-1+x^2)\,dx$$

$$= -\frac{1}{2}\int_0^1 (x^4-x^2)\,dx = -\frac{1}{2}\left(\frac{1}{5}-\frac{1}{3}\right) = \frac{1}{15}$$

b) $\iint_A y\,dx\,dy$, unde A e mult. plană mărg de $x=-y^2$, $x=y$ și $y=1$.



Fie $\varphi, \psi : [0,1] \to \mathbb{R}$, $\varphi(y) = -y^2$, $\psi(y) = y$

φ, ψ cont

$A \in \mathcal{J}(\mathbb{R}^2)$ și A compactă

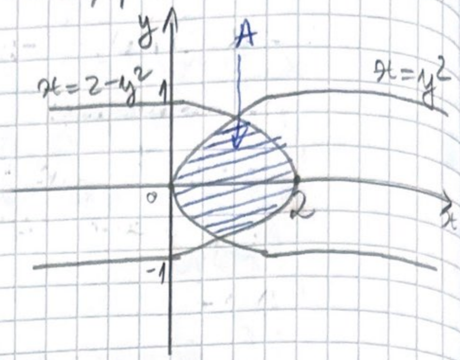
Fie $f : A \to \mathbb{R}$, $f(x,y) = y$

f cont.

$$\iint_A f(x,y)\,dx\,dy = \int_0^1 \left(\int_{-y^2}^{y} y\,dx \right) dy = \int_0^1 \left(yx \Big|_{x=-y^2}^{x=y} \right) dy =$$

$$= \int_0^1 y(y+y^2)\,dy = \int_0^1 y^2+y^3\,dy = \frac{1}{3} + \frac{1}{4} = \frac{7}{12} \quad \square$$

c) $\iint_A xy\, dx\, dy$; unde A e mulț. plană limitată de $x=y^2$ și $x=2-y^2$.



Det punctele de intersecție dintre $x=2-y^2$ și $x=y^2$.

$x=y^2=2-y^2 \Rightarrow y=\pm 1,\ x=1$.

$A=\{(x,y) \mid y\in[-1,1],\ y^2\le x\le 2-y^2\}$

Fie $\varphi, \psi: [-1,1]\to\mathbb{R}$, $\varphi(y)=y^2$, $\psi(y)=2-y^2$.

φ, ψ continue

$A\in J(\mathbb{R}^2)$, A compactă

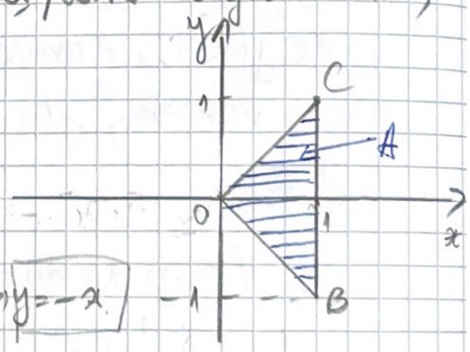
Fie $f: A\to\mathbb{R}$, $f(x,y)=xy$.

f continuă.

$$\iint_A f(x,y)\,dx\,dy = \int_{-1}^{1}\left(\int_{y^2}^{2-y^2} xy\,dx\right)dy = \int_{-1}^{1}\left(\frac{x^2 y}{2}\Bigg|_{x=y^2}^{x=2-y^2}\right)dy$$

$$=\underbrace{\int_{-1}^{1}\left(\frac{(2-y^2)^2 y - y\cdot y^4}{2}\right)dy}_{\text{funcție impară}} = 0$$

d) $\iint_A x\, dx\, dy$; unde A e mulț. plană mărg. de $\triangle OBC$, $O(0,0)$, $B(1,-1)$, $C(1,1)$



Sol: Scriem ecuațiile dreptelor OB, OC, (și BC).

OB: $\dfrac{y-y_0}{y_B-y_0}=\dfrac{x-x_0}{x_B-x_0} \Rightarrow \dfrac{y}{-1}=\dfrac{x}{1} \Rightarrow \boxed{y=-x}$

OC: $\dfrac{y-y_0}{y_C-y_0}=\dfrac{x-x_0}{x_C-x_0} \Rightarrow \dfrac{y}{1}=\dfrac{x}{1} \Rightarrow \boxed{y=x}$

BC: $\dfrac{y-1}{-2}=\dfrac{x-1}{1-1} \Rightarrow \boxed{x=1}$

$A=\{(x,y)\in\mathbb{R}^2 \mid x\in[0,1],\ -x\le y\le x\}$

Fie $\alpha,\beta:[0,1]\to\mathbb{R}$, $\alpha(x)=-x$, $\beta(x)=x$

α,β continue

A măs. Jordan și compactă

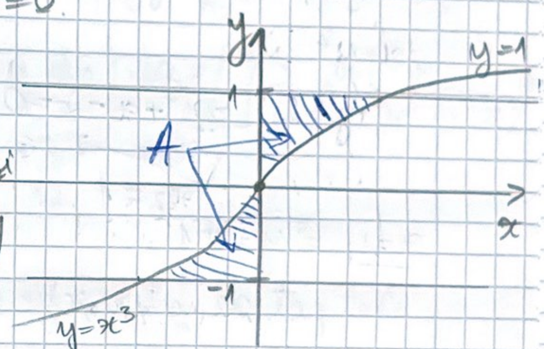
Fie $f:A\to\mathbb{R}$, $f(x,y)=x$

f cont

$$\iint_A f(x,y)\,dx\,dy=\int_0^1\left(\int_{-x}^{x} x\,dy\right)dx=\int_0^1 xy\Big|_{y=-x}^{y=x}dx=$$

$$=\int_0^1 2x^2\,dx=\frac{2x^3}{3}\Big|_0^1=\frac{2}{3}.$$

e) $\iint_A e^{y^4}\,dx\,dy$, unde A e mulț. plană mărg. de $x=y^3$, $y=1$, $y=-1$ și $x=0$.



Sol: $A=A_1\cup A_2$, unde

$A_1=\{(x,y)\in\mathbb{R}^2 \mid y\in[-1,0],\ y^3\le x\le 0\}$ și

$A_2=\{(x,y)\in\mathbb{R}^2 \mid y\in[0,1],\ 0\le x\le y^3\}$

Fie $\varphi_1,\psi_1:[-1,0]\to\mathbb{R}$,

$\varphi_1(y)=y^3$, $\psi_1(y)=0$.

φ_1,ψ_1 cont.

$A_1\in\mathcal{J}(\mathbb{R}^2)$ și A_1 compactă

Fie $\varphi_2,\psi_2:[0,1]\to\mathbb{R}$, $\varphi_2(y)=0$, $\psi_2(y)=y^3$

φ_2,ψ_2 cont

$A_2\in\mathcal{J}(\mathbb{R}^2)$ și A_2 compactă

Deci $A=A_1\cup A_2\in\mathcal{J}(\mathbb{R}^2)$ și A compactă

$A_1\cap A_2=\{(0,0)\}=\{0\}\times\{0\}\ \Rightarrow \mu(A_1\times A_2)=0\cdot 0=0$

Fie $f: A \to \mathbb{R}$, $f(x,y) = e^{y^4}$, f cont

$$\iint_A f(x,y)\,dx\,dy = \iint_{A_1} f(x,y)\,dx\,dy + \iint_{A_2} f(x,y)\,dx\,dy$$

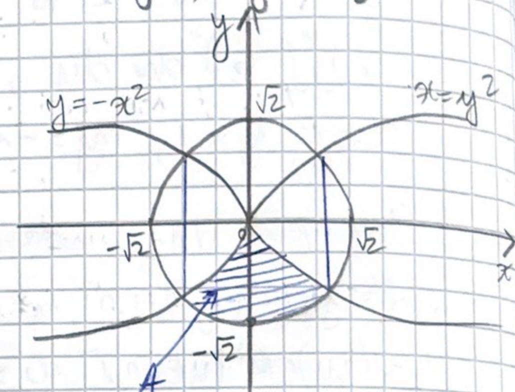
$$\iint_{A_1} f(x,y)\,dx\,dy = \int_{-1}^{0}\left(\int_{-y^3}^{0} e^{y^4}\,dx\right)dy = \int_{-1}^{0} e^{y^4}\, x\Big|_{x=-y^3}^{x=0}\Big)dy = \int_{-1}^{0} e^{y^4}(0-y^3)\,dy$$

$$= -\frac{1}{4}\int_{-1}^{0} 4y^3 e^{y^4}\,dy = -\frac{1}{4} e^{y^4}\Big|_{y=-1}^{y=0} = -\frac{1}{4}(1-e) = \frac{e-1}{4}.$$

$$\iint_{A_2} f(x,y)\,dx\,dy = \int_0^1\left(\int_0^{y^3} e^{y^4}\,dx\right)dy = \dots = \frac{e-1}{4}$$

$$\iint_A f(x,y)\,dx\,dy = \frac{e-1}{4} + \frac{e-1}{4} = \frac{e-1}{2}.$$

f) $\iint_A y\,dx\,dy$, unde $A = \{(x,y) \in \mathbb{R}^2 \mid x^2+y^2 \le 2,\ x \le y^2,\ x \ge -y^2,\ y \le 0\}$



Det pct de intersecție dintre $x = y^2$ și $x^2+y^2 = 2$.

$$\begin{cases} x = y^2 \\ x^2+y^2 = 2 \end{cases} \Rightarrow x^2 + x - 2 = 0$$

$\cancel{x_1 = -2}$

$x_2 = 1$

$\underline{x=1} \Rightarrow y = \pm 1.$

Det pct de intersecție dintre $x = -y^2$ și $x^2+y^2 = 2$

$$\begin{cases} x = -y^2 \\ x^2+y^2 = 2 \end{cases} \Rightarrow x^2 - x - 2 = 0 \qquad x_{1,2} = \begin{cases} \cancel{2} \\ -1 \end{cases}$$

$x = -1 \Rightarrow y = \pm 1.$

$x^2+y^2 \le 2 \Rightarrow y^2 \le 2 - x^2 \Rightarrow -\sqrt{2-x^2} \le y \le \sqrt{2-x^2}$

$x \le y^2 \Rightarrow y^2 \ge x \Rightarrow y \le -\sqrt{x}$ sau $y \ge \sqrt{x}$.

$x \ge -y^2 \Rightarrow -y^2 \le x \Rightarrow y^2 \ge -x \Rightarrow y \le -\sqrt{-x}$ sau $y \ge -\sqrt{x}$

$A = A_1 \cup A_2$, unde $A_1 = \{(x,y) \in \mathbb{R}^2 \mid x \in [-1, 0],\ -\sqrt{2-x^2} \le y \le -\sqrt{-x}$

Fie $\alpha_1, \beta_1 : [-1, 0] \to \mathbb{R}$, $\alpha_1(x) = -\sqrt{2-x^2}$

$\beta_1(x) = -\sqrt{-x}$.

α_1, β_1 cont $\Rightarrow A_1$ cont și măs. Jordan.

Fie $\alpha_2, \beta_2 : [0, 1] \to \mathbb{R}$ $\Rightarrow \alpha_2, \beta_2$ cont $\Rightarrow A_2$ cont și măs. J.

$A = A_1 \cup A_2 \in J(\mathbb{R}^2)$ și comp.

$A_1 \cap A_2 = \{(0, y) \in \mathbb{R}^2 \mid y \in [-\sqrt{2}, 0]\} = \{0\} \times [-\sqrt{2}, 0]$

$\mu(A_1 \cap A_2) = 0 \cdot \sqrt{2} = 0$

Fie $f : A \to \mathbb{R}$, $f(x, y) = y$, f cont.

$\iint_A f(x,y)\,dx\,dy = \iint_{A_1} f(x,y)\,dx\,dy + \iint_{A_2} f(x,y)\,dx\,dy$.

$\iint_{A_1} f(x,y)\,dx\,dy = \int_{-1}^{0} \left(\int_{-\sqrt{2-x^2}}^{-\sqrt{-x}} y\,dy \right) dx = \int_{-1}^{0} \left(\frac{1}{2} y^2 \Big|_{y=-\sqrt{2-x^2}}^{y=-\sqrt{-x}} \right) dx$

$= \int_{-1}^{0} \frac{1}{2}(-x - 2 + x^2)\,dx = \ldots = -\frac{7}{12}$.

$\iint_{A_2} f(x,y)\,dx\,dy = \int_0^1 \left(\int_{-\sqrt{2-x^2}}^{-\sqrt{x}} y\,dy \right) dx = \int_0^1 \frac{1}{2}(x - 2 + x^2)\,dx = -\frac{7}{12}$

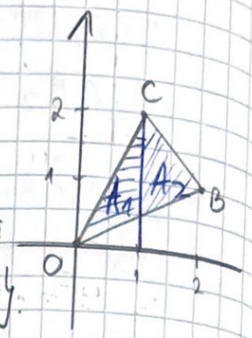
Deci $\iint_A f(x,y)\,dx\,dy = -\frac{7}{12} - \frac{7}{12} = -\frac{7}{6}$.

g) $\iint_A x\,dx\,dy$, unde A e mulț. plană limitată ΔOBC
$O(0,0)$, $B(2,1)$, $C(1,2)$.

Scriem ec dreptelor OB, OC, BC.

OB: $\dfrac{y-y_0}{y_B-y_0} = \dfrac{x-x_0}{x_B-x_0} \Leftrightarrow \dfrac{y-0}{1-0} = \dfrac{x-0}{2-0} \Leftrightarrow y = \dfrac{x}{2}$

OC: $\dfrac{y-y_0}{y_C-y_0} = \dfrac{x-x_0}{x_C-x_0} \Leftrightarrow y = 2x$



BC: $y = 3-x$.

$A = A_1 \cup A_2$; $A_1 = \{(x,y)\in\mathbb{R}^2 \mid x\in[0,1], \frac{x}{2} \le y \le 2x\}$ și
$A_2 = \{(x,y)\in\mathbb{R}^2 \mid x\in[1,2], \frac{x}{2} \le y \le 3-x\}$.

Fie $\alpha_1, \beta_1 : [0,1] \to \mathbb{R}$, $\alpha_1(x) = \frac{x}{2}$, $\beta_1(x) = 2x$,
$A_1 \in J(\mathbb{R}^2)$ și A_1 compactă

Fie $\alpha_2, \beta_2 : [1,2] \to \mathbb{R}$, $\alpha_2(x) = \frac{x}{2}$, $\beta_2(x) = 3-x$,
$A_2 \in J(\mathbb{R}^2)$ și A_2 compactă

$A = A_1 \cup A_2 \in J(\mathbb{R}^2)$ și A compactă

Fie $\Delta(x_\Delta, y_\Delta) \in OB$ a.î. $x_\Delta = 1$

Avem $y_\Delta = \frac{x_\Delta}{2} = \frac{1}{2}$

$A_1 \cap A_2 = [C\Delta] = \{1\} \times [\frac{1}{2}, 2] \Rightarrow \mu(A_1 \cap A_2) = 0(2-\frac{1}{2}) = 0$

Fie $f : A \to \mathbb{R}$, $f(x,y) = x$, f cont

$\iint_A f(x,y)\,dx\,dy = \iint_{A_1} f(x,y)\,dx\,dy + \iint_{A_2} f(x,y)\,dx\,dy$

$\iint_{A_1} f(x,y)\,dx\,dy = \int_0^1 \left(\int_{\frac{x}{2}}^{2x} x\,dy \right) dx = \int_0^1 \left(xy \Big|_{y=\frac{x}{2}}^{y=2x} \right) dx =$

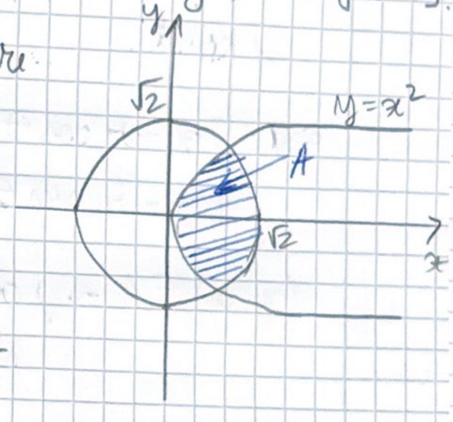
$= \int_0^1 x\left(2x - \frac{x}{2}\right) dx = \int_0^1 \left(2x^2 - \frac{x^2}{2}\right) dx = 2\cdot\frac{x^3}{3}\Big|_0^1 - \frac{1}{2}\cdot\frac{x^3}{3}\Big|_0^1 = \frac{1}{2}$

$$\iint_{A_2} f(x,y)\,dx\,dy = \int_1^2 \left(\int_{\frac{x}{2}}^{3-x} x\,dy \right) dx = \int_1^2 \left(xy \Big|_{y=\frac{x}{2}}^{y=3-x} \right) dx =$$

$$= \int_1^2 x\left(3-x-\frac{x}{2}\right) dx = \int_1^2 \left(3x - x^2 - \frac{x^2}{2}\right) dx = \left(3\frac{x^2}{2} - \frac{x^3}{3} - \frac{1}{2}\cdot\frac{x^3}{3}\right)\Big|_1^2$$

$= 1.$

h) $\iint_A y\,dx\,dy$, unde $A = \{(x,y) \in \mathbb{R}^2 \mid x \geq y^2,\ x^2+y^2 \leq 2\}$.



Sol: Det punctele de intersecție dintre $x \geq y^2$ și $x^2+y^2 \leq 2$.

$$\begin{cases} x = y^2 \\ x^2+y^2 = 2 \end{cases} \Rightarrow x^2 + x - 2 = 0 \quad \begin{matrix} \to x_1 = 1 \\ \to x_2 = -2 \end{matrix}$$

$y^2 = 1 \Rightarrow y = \pm 1$.

$A = \{(x,y) \in \mathbb{R}^2 \mid y \in [-1,1]\}$.

$x^2+y^2 \leq 2 \Rightarrow x^2 \leq 2-y^2 \Rightarrow -\sqrt{2-y^2} \leq x \leq \sqrt{2-y^2}$

$A = \{(x,y) \in \mathbb{R}^2 \mid y \in [-1,1],\ y^2 \leq x \leq \sqrt{2-y^2}\}$

Fie $\varphi, \psi : A \to \mathbb{R}$, $\varphi(x) = y^2$, $\psi(x) = \sqrt{1-x^2}$

φ, ψ cont.

$A \in \mathcal{J}(\mathbb{R}^2)$ și compactă

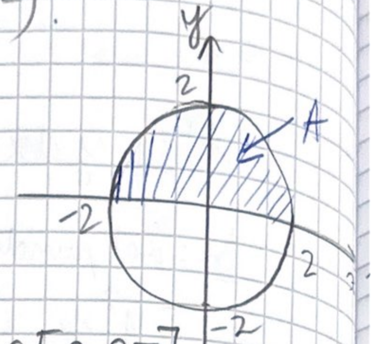
Fie $f : A \to \mathbb{R}$, $f(x) = y$ cont.

$$\iint_A f(x,y)\,dx\,dy = \int_{-1}^1 \left(\int_{y^2}^{\sqrt{2-y^2}} y\,dx \right) dy =$$

$$= \int_{-1}^1 \left(yx \Big|_{x=y^2}^{x=\sqrt{2-y^2}} \right) dy = \underbrace{\int_{-1}^1 y\left(\sqrt{1-y^2} - y^2\right) dy}_{\text{fct impară}} = 0$$

② Det: a) $\iint_A e^{-x^2-y^2}\,dx\,dy$,
unde $A=\{(x,y)\in\mathbb{R}^2 \mid x^2+y^2\le 4,\ y\ge 0\}$.

Sol: A convexă și mărg. $\Rightarrow A\in \mathcal{J}(\mathbb{R}^2)$.
A compactă
Fie $f: A\to\mathbb{R}$, $f(x,y)=e^{-x^2-y^2}$
f cont.



SV: $\begin{cases} x=r\cos\theta \\ y=r\sin\theta \end{cases}$, $r\in[0,\infty)$, $\forall\, \theta\in[0,2\pi]$.

$(x,y)\in A \Rightarrow \begin{cases} x^2+y^2\le 4 \\ y\ge 0 \end{cases} \Rightarrow \begin{cases} r^2\cos^2\theta+r^2\sin^2\theta\le 4 \\ r\sin\theta\ge 0 \end{cases}$

$\Rightarrow \begin{cases} r^2\le 4 \\ r\sin\theta\ge 0 \end{cases} \Rightarrow \begin{cases} r\in[-2,2] \\ \theta\in[0,\pi] \end{cases}$

Fie $B=[0,2]\times[0,\pi]$

$\iint_A f(x,y)\,dx\,dy=\iint_B r f(r\cos\theta, r\sin\theta)\,dr\,d\theta=$
$=\int_0^2\left(\int_0^\pi r\cdot e^{-r^2}\,d\theta\right)dr=\int_0^2 \pi r e^{-r^2}\,dr=-\frac{\pi}{2}\int_0^2(-2r)e^{-r^2}\,dr=$
$=-\frac{\pi}{2}e^{-r^2}\Big|_{r=0}^{r=2}=-\frac{\pi}{2}\left(e^{-4}-1\right)=\frac{\pi}{2}\left(1-e^{-4}\right)$

b) $\iint_A \sqrt{1-\frac{x^2}{9}-\frac{y^2}{4}}\, dx\,dy$,

unde $A=\{(x,y)\in\mathbb{R}^2 \mid \frac{x^2}{9}+\frac{y^2}{4}\le 1,\ x\ge 0,\ y\ge 0\}$.

Sol: A convexă și mărg

$\Rightarrow A\in \mathcal{J}(\mathbb{R}^2)$, A compactă

Fie $f: A\to\mathbb{R}$, $f(x,y)=\sqrt{1-\frac{x^2}{9}-\frac{y^2}{4}}$

SV: $\begin{cases} x=3r\cos\theta \\ y=2r\sin\theta \end{cases}$, $r\in[0,\infty)$, $\theta\in[0,2\pi]$

$(x,y)\in A \Rightarrow \begin{cases} \frac{x^2}{9}+\frac{y^2}{4}\le 1 \\ x\ge 0 \\ y\ge 0 \end{cases} \Rightarrow \begin{cases} \frac{\cancel{9}r^2\cos^2\theta}{\cancel{9}}+\frac{\cancel{4}r^2\sin^2\theta}{\cancel{4}}\le 1 \\ 3r\cos\theta\ge 0 \\ 2r\sin\theta\ge 0 \end{cases} \Rightarrow$

$\Rightarrow \begin{cases} r^2\le 1 \\ 3r\cos\theta\ge 0 \\ 2r\sin\theta\ge 0. \end{cases} \Rightarrow \begin{cases} r\in[0,1] \\ \theta\in[0,\frac{\pi}{2}] \end{cases}$

Fie $B=[0,1]\times[0,\frac{\pi}{2}]$

$\iint_A f(x,y)\,dx\,dy=\iint_B 3\cdot 2r\, f(3r\cos\theta, 2r\sin\theta)\cdot dr\,d\theta=$

$=\int_0^1\left(\int_0^{\frac{\pi}{2}} 6r\sqrt{1-r^2}\,d\theta\right)dr=\int_0^1 \frac{\pi}{\cancel{2}}\cdot \overset{3}{\cancel{6}}\, r\sqrt{1-r^2}\,dr=$

$=-\frac{3\pi}{2}\int_0^1(-2r)(1-r^2)^{\frac{1}{2}}\,dr=-\frac{3\pi}{2}\cdot\frac{(1-r^2)^{\frac{3}{2}}}{\frac{3}{2}}\Big|_0^1=\frac{3\pi}{2}\cdot\frac{2}{3}=\pi$