Tutoriat III

$\sum_{n=1}^{\infty} a_n = \lim_{N \to \infty} \sum_{n=1}^{N} a_n \qquad S_N = \sum_{n=1}^{N} a_n \to$ șirul sumelor parțiale

Criteriul Cauchy

1) Șiruri: $(x_n)_n$ convergent $\Leftrightarrow \forall \varepsilon > 0, \exists N_\varepsilon \in \mathbb{N}$ a.î. $\forall m \geq n \geq N_\varepsilon$,

$|x_m - x_n| < \varepsilon$.

2) $\sum a_n$ conv. $\Leftrightarrow (S_N)_N$ conv $\Leftrightarrow \forall \varepsilon > 0, \exists N_\varepsilon \in \mathbb{N}$ a.î. $\forall m \geq n \geq$ $\geq N_\varepsilon$, avem $|S_m - S_n| < \varepsilon \Leftrightarrow \left|\sum_{k=n+1}^{m} a_k\right| < \varepsilon$

$\sum_{k=1}^{m} a_k - \sum_{k=1}^{n} a_k = \sum_{k=n+1}^{m} a_k$

(1) $\sum_{n=1}^{\infty} \frac{1}{n}$ divergent

Vom nega criteriul Cauchy:

$\exists \varepsilon > 0, \forall N_\varepsilon \in \mathbb{N}$ și $\exists m, n \in \mathbb{N}$,

$m \geq n \geq N_\varepsilon$ a.î. $\left|\sum_{k=n+1}^{m} a_k\right| > \varepsilon$.

Fie $\varepsilon = 1$. Fie N_ε fixat. $n+1 = 2^k$

$m+1 = 2^{k+1}$.

$$\left|\sum_{k=n+1}^{m} a_k\right| = \left|\sum_{k=n+1}^{m} \frac{1}{n}\right| = \frac{1}{n+1} + \frac{1}{n+2} + \dots + \frac{1}{m} = \frac{1}{2^k} + \underbrace{\frac{1}{2^k+1}}_{> \frac{1}{2^k}} + \dots + \underbrace{\frac{1}{2^{k+1}}}_{> \frac{1}{2^k}} > 2^k \cdot \frac{1}{2^k} = 1$$

2) a) $\sum_{n=1}^{\infty} \frac{1}{n^2+3n+2} = \sum_{n=1}^{\infty} \frac{1}{(n+1)(n+2)} = \sum_{n=1}^{\infty} \left(\frac{1}{n+1} - \frac{1}{n+2}\right) = \lim_{N\to\infty} \sum_{n=1}^{N} \left(\frac{1}{n+1} - \frac{1}{n+2}\right)$

$= \lim_{N\to\infty} \left(\frac{1}{2} - \cancel{\frac{1}{3}} + \cancel{\frac{1}{3}} - \cancel{\frac{1}{4}} + \cancel{\dots} + \cancel{\frac{1}{N+1}} - \frac{1}{N+2} \right) = \lim_{N\to\infty} \left(\frac{1}{2} - \frac{1}{N+2} \right)$

$= \frac{1}{2}$

b) $\sum_{n=1}^{\infty} \frac{n^2+(-1)^n}{n^2}$

$\lim_{n\to\infty} \frac{n^2+(-1)^n}{n^2} = \lim_{n\to\infty} \left(\frac{n^2}{n^2} + \underbrace{\frac{(-1)^n}{n^2}}_{\to 0} \right) =$

$= \lim_{n\to\infty} (1+0) = 1 \Rightarrow \sum_{n=1}^{\infty} \frac{n^2+(-1)^n}{n^2}$ divergentă

$\sum_{n=1}^{\infty} a_n$

Dacă $\lim_{n\to\infty} a_n \neq 0$

$\Rightarrow \sum_{n=1}^{\infty} a_n$ divergentă.

c) $\sum_{n=1}^{\infty} \frac{\cos(\pi n)}{n}$.

$-\frac{1}{n} \leq \frac{\cos(\pi n)}{n} \leq \frac{1}{n} \overset{\text{CLEȘTE}}{\Rightarrow} \lim_{n\to\infty} \frac{\cos(\pi n)}{n} = 0 \Rightarrow$ nu putem trage o concluzie.

Criteriul Abel Dirichlet

Fie $(a_n)_n$, $(b_n)_n \subseteq \mathbb{R}$ a.î. se aplică I sau II

I $\begin{cases} (a_n)_n \searrow \text{ și } \lim_{n\to\infty} a_n = 0 \\ \left(\sum_{k=1}^{n} b_k\right)_n \text{ mărginită} \end{cases}$

II $\begin{cases} (a_n)_n \text{ monoton și mărginit} \\ \sum b_n \text{ convergentă} \end{cases}$

Atunci $\sum a_n b_n$ convergentă.

Criteriul lui Leibnitz

Fie $(a_n)_n \subseteq \mathbb{R}$ a.î. $\lim\limits_{n\to\infty} a_n = 0$ și $(a_n)_n \searrow$. Atunci $\sum\limits_{n=1}^{\infty} (-1)^n a_n$ conv.

c) $$\left.\begin{aligned} &\sum_{n=1}^{\infty} \frac{\cos(n\pi)}{n} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n} \\ &a_n = \frac{1}{n} \searrow 0 \end{aligned}\right\} \Rightarrow \sum_{n=1}^{\infty} \frac{\cos(n\pi)}{n} \text{ conv.}$$

d) $$\sum_{n=1}^{\infty} \frac{\cos(n)}{n}$$

$b_n = \cos(n)$

$a_n = \frac{1}{n}$

Mai trb. să arătăm că $\sum\limits_{n=1}^{N} \cos(n)$ mărginită $\forall n \in \mathbb{N}$

De Moivre: $(\cos x + i\sin x)^n = \cos nx + i\sin nx$, $x \in \mathbb{R}$ și $n \in \mathbb{N}$

$$\sum_{n=1}^{N} \cos(n) = \cos(1) + \cos(2) + \ldots + \cos(N) = \mathrm{Re}\Big(\sum_{n=1}^{N} (\underbrace{\cos(1) + i\sin(1)}_{\text{Not. } a \text{ - constant}})^n\Big)$$

$$\left.\begin{aligned} &= \mathrm{Re}\Big(\sum_{n=1}^{\infty} a^n\Big) \\ &\sum_{n=1}^{N} a^n = a \cdot \frac{1-a^{n+1}}{1-a} \end{aligned}\right\} = \mathrm{Re}\left(a \cdot \frac{1-a^{N+1}}{1-a}\right) \leq \left|a \cdot \frac{1-a^{N+1}}{1-a}\right| =$$

$$= |a| \cdot \left|\frac{1-a^{N+1}}{1-a}\right| = 1 \cdot \frac{|1| + |-a^{N+1}|}{|1| + |a|} =$$

$$= 1 \cdot \frac{1+1}{1+1} = 1 \Rightarrow \sum_{n=1}^{N} \cos(n) \text{ este mărginită}$$

Deci, $\sum\limits_{n=1}^{\infty} \frac{\cos(n)}{n}$ este convergentă.

$$S_N = \sum_{n=k}^{N} a^n = a^k + a^{k+1} + \dots + a^N \Big| \cdot a$$

$$aS_N = a^{k+1} + a^{k+2} + \dots + a^N + a^{N+1}$$

$\ominus$

$$(1-a)S_n = a^k - a^{N+1} \Big| : a \; (\text{pt. } a \neq 1) \Rightarrow S_N = \frac{a^k - a^{N+1}}{1-a}$$

I $a \in (-1,1) \Rightarrow a^N \xrightarrow{N \to \infty} 0$

II $a \in \mathbb{R} \setminus (-1,1) \Rightarrow a^N$ nu e conv.

Deci, pentru $a \in (-1,1)$ avem că $\displaystyle\sum_{n=k}^{\infty} a_n = \frac{a^k}{1-a}$

e) $\displaystyle\sum_{n=1}^{\infty} (-1)^n \sin\left(\frac{1}{n}\right)$

Vom aplica criteriul Leibniz:

Trebuie să arătăm că $\lim\limits_{n \to \infty} b_n = 0$, unde $b_n = \sin\left(\frac{1}{n}\right)$

$$\lim_{n \to \infty} \sin\left(\frac{1}{n}\right) = \sin\left(\lim_{n \to \infty} \frac{1}{n}\right) = \sin(0) = 0$$

$(b_n)_n$ desc: fie $f(x) = \sin\left(\frac{1}{x}\right)$

$f'(x) = -\frac{1}{x^2}\cos\left(\frac{1}{x}\right)$

pentru $x \in [1, \infty) \Rightarrow 0 < \frac{1}{x} < 1 \Rightarrow \cos\left(\frac{1}{x}\right) > 0 \Rightarrow$

$\Rightarrow f'(x) < 0 \Rightarrow b_n \searrow$

Deci din Leibniz avem că $\displaystyle\sum_{n=1}^{\infty} (-1)^n \sin\left(\frac{1}{n}\right)$ e conv.

f) $\sum_{n=1}^{\infty}\left(\frac{n+1}{2n}\right)^n = \sum_{n=1}^{\infty}\left(\frac{1}{2}+\frac{1}{2n}\right)^n = \sum_{n=1}^{\infty}\frac{1}{2^n}\left(1+\frac{1}{n}\right)^n.$

$$\sum_{n=1}^{\infty}\left(\frac{1}{2}\right)^n = \frac{1}{2}\cdot\frac{1}{1-\frac{1}{2}} = 1$$

$\sum_{n=0}^{\infty} a^n = \frac{1}{1-a}$, cu $|a|<1$

Vrem să arătăm că $\left(1+\frac{1}{n}\right)^n$ este monoton și mărginit.

Știm că $\left(1+\frac{1}{n}\right)^n \xrightarrow[n\to\infty]{} e$ și șirul este monoton

Deci, din Abel-Dirichlet $\Rightarrow \sum_{n=1}^{\infty}\left(\frac{n+1}{2n}\right)^n$ este monoton

! $\left(1+\frac{1}{n}\right)^n \nearrow e \swarrow \left(1+\frac{1}{n}\right)^{n+1}$

2\. a) $A=\left\{\frac{m+n}{1+mn} \mid m,n\in\mathbb{N}^*\right\}$

$A\subseteq\mathbb{R}$

$\sup A=\alpha \iff \forall x\in A,\ x\le\alpha$ și $\exists (x_n)_n\subseteq A$ a.î. $x_n \xrightarrow{n\to\infty} \alpha$

$\forall\ a\in A,\ a\ge 0$

$m=n:\ \frac{2n}{1+n^2},\ n\in\mathbb{N} \in A$

fie $x_n=\frac{2n}{1+n^2},\ n\ge 1$ $\Rightarrow \inf(A)=0$

$x_n \xrightarrow{n\to\infty} 0$

$m=1,\ n=1:\ 1$
$m=2,\ n=1:\ 1$
$m=3,\ n=2:\ \frac{5}{7}$
$m=4,\ n=3:\ \frac{7}{13}$

Vrem să arătăm că $\sup(A) = 1$

Este suficient să arătăm că $a \leq 1, \forall a \in A$.

Fie $m, n \in \mathbb{N}^*$

$$\frac{m+n}{1+mn} \leq 1 \iff m+n \leq 1+mn \iff m(1-n) + n-1 \leq 0$$

$$\iff (m-1)(1-n) \leq 0 \iff (m-1)(n-1) \geq 0 \overset{m,n \in \mathbb{N}^*}{\iff}$$

$m, n \geq 1 \Rightarrow \sup(A) = 1$

Pentru că $1 \in A \Rightarrow \max(A) = \sup(A) = 1$

b) $B = \left\{ \sum_{k=n}^{m} \left(\frac{1}{2}\right)^k \mid m \geq n \in \mathbb{N} \right\}$

$$= \left(\frac{1}{2}\right)^n \cdot \frac{\left(\frac{1}{2}\right)^{m+1} - 1}{\frac{1}{2} - 1} = \frac{1}{2^m}.$$

$m = n = k \in \mathbb{N} \Rightarrow \frac{1}{2^k} \xrightarrow{k \to \infty} 0$

$$\Rightarrow \sum_{k=n}^{m} \left(\frac{1}{2}\right)^k \geq 0$$

Fie $x_t = \frac{1}{2^t} = \sum_{k=t}^{t} \frac{1}{2^k} \in B, \forall t \in \mathbb{N} \Rightarrow (x_t)_t \subseteq B$ $\Bigg| \Rightarrow$

$$\lim_{t \to \infty} x_t = 0$$

$\Rightarrow \inf(B) = 0$

$$\sum_{k=n}^{n}\left(\frac{1}{2}\right)^k \leq \sum_{k=0}^{\infty}\left(\frac{1}{2}\right)^k = 2$$

Fie $y_t = \sum_{k=0}^{t}\left(\frac{1}{2}\right)^k \in B$, $\forall t \in \mathbb{N} \Rightarrow (y_t)_t \subseteq B$

$$\lim_{t\to\infty} y_t = \sum_{k=0}^{\infty}\left(\frac{1}{2}\right)^k = 2$$

$\Rightarrow$

$$\Rightarrow \sup(B) = 2$$

c) $C = \{ n \ln(1 + \frac{1}{n}) \mid n \in \mathbb{N}^* \}$

$n = 1$: $\ln(2)$
$n = 2$: $2\ln(\frac{3}{2})$

fie $a_n = n \ln(1 + \frac{1}{n})$

Vom dem. că a_n este descrescător

$n \ln(1+\frac{1}{n}) = \ln(1+\frac{1}{n})^n \Rightarrow a_n \nearrow e$

$C = \{a_1, a_2, \dots\}$
$a_1 < a_2 < a_3 < \dots$

$\Rightarrow \inf(C) = \min(C) = \ln 2$

$$\lim_{n\to\infty} n\ln(1+\frac{1}{n}) = \lim_{n\to\infty} \frac{\ln(1+\frac{1}{n})}{\frac{1}{n}} = 1$$

Deci, $\sup(C) = 1$

$\mathcal{L}((x_n)_n) = \{ x \in \mathbb{R} \mid \exists (x_{k(n)})_n$ subșir al lui x_n a.î. $\lim\limits_{n\to\infty} x_n = x\} \subseteq \overline{\mathbb{R}}$

Dacă avem $(x_{k_p(n)})_n, \dots, (x_{k_\ell(n)})_n$ subșiruri ale lui $(x_n)_p$ a.î. $(x_n)_n = \bigcup\limits_{i=1}^{\ell} (x_{k_i}(n))_n$, atunci $\mathcal{L}((x_n)_n) = \bigcup\limits_{i=1}^{\ell} \mathcal{L}((x_{k_i}(n))_n)$

$(-1)^n$:

$$\left.\begin{array}{l} x_{2n} = 1 \Rightarrow \mathcal{L}((x_{2n})_n) = \{1\} \\ x_{2n+1} = -1 \Rightarrow \mathcal{L}((x_{2n+1})_n) = \{-1\} \\ (x_{2n})_n \cup (x_{2n+1})_n = (x_n)_n \end{array}\right\} \Rightarrow (\mathcal{L}(x_n)_n) = \{\pm 1\}$$

$$\limsup_{n\to\infty} x_n = \max(\mathcal{L}((x_n)_n))$$

$$\liminf_{n\to\infty} x_n = \min(\mathcal{L}((x_n)_n))$$

$$\limsup_{n\to\infty} x_n = \lim_{n\to\infty} \sup_{k\geq n} x_k$$

$$\liminf_{n\to\infty} x_n = \lim_{n\to\infty} \inf_{k\geq n} x_k$$