SEMINAR III - 19.10.2023 Sisteme. Determinanți

1. $\sum_{i=1}^{k} (1+i)x_i + \sum_{i=1}^{4-k} i x_{i+k} = 0, \quad \forall k = \overline{1,3}$

$$\begin{cases} 2x_1 + 1\cdot x_2 + 2x_3 + 3x_4 = 0 \\ 2x_1 + 3x_2 + x_3 + 2x_4 = 0 \\ 2x_1 + 3x_2 + 4x_3 + x_4 = 0 \end{cases} \qquad A = \left(\begin{array}{cccc|c} 2 & 1 & 2 & 3 & 0 \\ 2 & 3 & 1 & 2 & 0 \\ 2 & 3 & 4 & 1 & 0 \end{array}\right)$$

$$\Delta_1 = \begin{vmatrix} 2 & 1 & 2 \\ 2 & 3 & 1 \\ 2 & 3 & 4 \end{vmatrix} \underset{L_3' = L_3 - L_2}{=} \begin{vmatrix} 2 & 1 & 2 \\ 2 & 3 & 1 \\ 0 & 0 & 3 \end{vmatrix} = 3\cdot(-1)^{3+3} \begin{vmatrix} 2 & 1 \\ 2 & 3 \end{vmatrix} =$$

$= 3\cdot(6-2) = 3\cdot 4 = 12 \neq 0 \Rightarrow \text{rang } A = 3$

$\text{rang } A = \text{rang } \bar{A} = 3 \Rightarrow$ S.C.S.N $\quad x_1, x_2, x_3$ - variabile principale

$x_4 = \alpha$ variabila secundară

$$\begin{cases} 2x_1 + x_2 + 2x_3 = -3\alpha & (1) \\ 2x_1 + 3x_2 + x_3 = -2\alpha & (2) \\ 2x_1 + 3x_2 + 4x_3 = -\alpha & (3) \end{cases}$$

$(3) - (2) \Rightarrow 3x_3 = \alpha \Rightarrow \boxed{x_3 = \frac{\alpha}{3}}$

$$\begin{cases} 2x_1 + x_2 = -3\alpha - \frac{2\alpha}{3} \\ 2x_1 + 3x_2 = -2\alpha - \frac{\alpha}{3} \end{cases} \quad (-)$$

$2x_2 = \alpha + \frac{\alpha}{3} \Rightarrow \boxed{x_2 = \frac{2\alpha}{3}}$

$2x_1 = -3\alpha - \frac{2\alpha}{3} - \frac{2\alpha}{3} = \frac{-(9-2-2)\alpha}{3} = \frac{-13\alpha}{3}$

$\Rightarrow \boxed{x_1 = \frac{-13\alpha}{6}}$

$(x_1, x_2, x_3, x_4) \in \left\{ \left(-\frac{13\alpha}{6}, \frac{2\alpha}{3}, \frac{\alpha}{3}, \alpha\right) \middle| \alpha \in \mathbb{R} \right\}$

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2. $\sum_{j=1}^{4} a_{ij}x_j = 4^{i-1}, \ \forall i=\overline{1,4}$, unde $a_{ij} = j^{i-1}, \ \forall i,j=\overline{1,4}$

$$A = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \\ 1 & 2^2 & 3^2 & 4^2 \\ 1 & 2^3 & 3^3 & 4^3 \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 1 \\ 4 \\ 4^2 \\ 4^3 \end{pmatrix}$$

$$\Delta = \det A = (4-3)(4-2)(4-1)(3-2)(3-1)(2-1) =$$
$$= 1 \cdot 2 \cdot 3 \cdot 1 \cdot 2 \cdot 1 = 12 \neq 0 \Rightarrow S.C.D$$

$$\Delta x_1 = \begin{vmatrix} 1 & 1 & 1 & 1 \\ 4 & 2 & 3 & 4 \\ 4^2 & 2^2 & 3^2 & 4^2 \\ 4^3 & 2^3 & 3^3 & 4^3 \end{vmatrix} = 0 = \Delta x_2 = \Delta x_3$$

$\Delta x_4 = \Delta$

$(x_1, x_2, x_3, x_4) = (0, 0, 0, 1)$

3. $A = \begin{pmatrix} 1 & 1 & 2 & 3 \\ 1 & 1 & 3 & 4 \\ 2 & 5 & 0 & 1 \\ 4 & 1 & 0 & 2 \end{pmatrix}$

det $A = ?$ (Teorema Laplace)

$p = 2$.

a) l_2, l_3 fixate

b) c_1, c_2 fixate

$$\det A = (-1)^{2+3+1+2} \begin{vmatrix} 1 & 1 \\ 2 & 5 \end{vmatrix} \begin{vmatrix} 2 & 3 \\ 0 & 2 \end{vmatrix} + (-1)^{2+3+1+3} \begin{vmatrix} 1 & 3 \\ 2 & 0 \end{vmatrix} \begin{vmatrix} 1 & 3 \\ 1 & 2 \end{vmatrix} +$$
$$+ (-1)^{2+3+1+4} \begin{vmatrix} 1 & 4 \\ 2 & 1 \end{vmatrix} \begin{vmatrix} 1 & 2 \\ 1 & 0 \end{vmatrix} + (-1)^{2+3+2+3} \begin{vmatrix} 1 & 3 \\ 5 & 0 \end{vmatrix} \begin{vmatrix} 1 & 3 \\ 4 & 2 \end{vmatrix} +$$
$$+ (-1)^{2+3+2+4} \begin{vmatrix} 1 & 4 \\ 5 & 1 \end{vmatrix} \begin{vmatrix} 1 & 2 \\ 4 & 0 \end{vmatrix} + (-1)^{2+3+3+4} \begin{vmatrix} 3 & 4 \\ 0 & 1 \end{vmatrix} \begin{vmatrix} 1 & 1 \\ 4 & 1 \end{vmatrix} =$$

$$\Rightarrow \det A = 3 \cdot 4 - (-6) \cdot (-1) + (-7) \cdot (-2) + (-15) \cdot (-10) - (-19) \cdot (-8) + 3 \cdot (-3)$$

$= 12 - 6 + 14 + 150 - 152 - 9 = 9$

7. $x^3 + px + q = 0$

$$\begin{vmatrix} 1 & 1 & 1 \\ x_1 & x_2 & x_3 \\ x_1^2 & x_2^2 & x_3^2 \end{vmatrix}^2 = ?$$

$$A = \begin{pmatrix} 1 & 1 & 1 \\ x_1 & x_2 & x_3 \\ x_1^2 & x_2^2 & x_3^2 \end{pmatrix}$$

$$A^t = \begin{pmatrix} 1 & x_1 & x_1^2 \\ 1 & x_2 & x_2^2 \\ 1 & x_3 & x_3^2 \end{pmatrix}$$

$$A \cdot A^t = \begin{pmatrix} 1 & 1 & 1 \\ x_1 & x_2 & x_3 \\ x_1^2 & x_2^2 & x_3^2 \end{pmatrix} \cdot \begin{pmatrix} 1 & x_1 & x_1^2 \\ 1 & x_2 & x_2^2 \\ 1 & x_3 & x_3^2 \end{pmatrix} =$$

$$= \begin{pmatrix} 3 & x_1 + x_2 + x_3 & x_1^2 + x_2^2 + x_3^2 \\ x_1 + x_2 + x_3 & x_1^2 + x_2^2 + x_3^2 & x_1^3 + x_2^3 + x_3^3 \\ x_1^2 + x_2^2 + x_3^2 & x_1^3 + x_2^3 + x_3^3 & x_1^4 + x_2^4 + x_3^4 \end{pmatrix} = \begin{pmatrix} 3 & S_1 & S_2 \\ S_1 & S_2 & S_3 \\ S_2 & S_3 & S_4 \end{pmatrix}$$

$s_1 = x_1 + x_2 + x_3 = -\frac{b}{a} = 0$

$s_2 = x_1x_2 + x_1x_3 + x_2x_3 = \frac{c}{a} = p$

$s_3 = x_1 x_2 x_3 = -\frac{d}{a} = -q$

$S_1 = s_1 = 0$

$S_2 = s_1^2 - 2s_2 = 0 - 2p = -2p$

$S_3 = -3q$

$$\begin{cases} x_1^3 + px_1 + qx_1 = 0 \\ x_2^3 + px_2 + qx_2 = 0 \\ x_3^3 + px_3 + qx_3 = 0 \end{cases} (+)$$

$S_3 + ps_1 + 3q = 0$

$\Rightarrow S_3 = -3q$

$$\begin{cases} x_1^4 + px_1^2 + q = 0 \\ x_2^4 + px_2^2 + q = 0 \\ x_3^4 + px_3^2 + q = 0 \end{cases} (+)$$

$S_4 + pS_2 - 3qS_1 = 0$

$\Rightarrow S_4 = 2p^2$

$$\Rightarrow A \cdot A^t = \begin{pmatrix} 3 & 0 & -2p \\ 0 & -2 & -3q \\ -2p & -3q & 2p^2 \end{pmatrix}$$

$$\det(A \cdot A^t) = \det A \cdot \det A^t = (\det A)^2$$

8. $A, B \in \mathcal{M}_n(\mathbb{C})$ - inversabile

$\text{rang}(A^{-1} + B^{-1}) = \text{rang}(A+B)$

PROPRIETATE: Dacă $B \in \mathcal{M}_m(\mathbb{C})$, $C \in \mathcal{M}_n(\mathbb{C})$ sunt inversabile

$\Rightarrow \text{rang}(BAC) = \text{rang}\, A, \ \forall A \in \mathcal{M}_{m,n}(\mathbb{C})$

$A(A^{-1} + B^{-1}) \cdot B = A \cdot A^{-1} \cdot B + A \cdot B^{-1} \cdot B = B + A \ | \text{rang}()$

$\Rightarrow \text{rang}(A(A^{-1}+B^{-1})B) = \text{rang}(A+B)$

$\text{rang}(A^{-1} + B^{-1}) = \text{rang}(A+B)$ conform propoziţiei

10. $A = \begin{pmatrix} 1 & 0 \\ 7 & 4 \end{pmatrix}$ $\quad A^n = ?$

$$A^2 - \underbrace{(\text{Tr } A)}_{5} \cdot A + \underbrace{(\det A)}_{4} I_2 = O_2$$

$A^2 = 5A - 4I_2$

$A^n = x_n A + y_n I_2$

$A^{n+1} = A^n \cdot A = x_{n+1} A + y_{n+1} I_2 = x_n \cdot A^2 + y_n \cdot A =$

$= x_n(5A - 4I_2) + y_n A = (5x_n + y_n)A - 4x_n I_2$

$x_{n+1} = 5x_n + y_n$

$y_{n+1} = -4x_n \Rightarrow y_n = -4x_{n-1}$

$x_{n+1} = 5x_n - 4x_{n-1} \Rightarrow x_{n+1} - 5x_n + 4x_{n-1} = 0, \ \forall n \geq 2$

$x_1 = 1 \qquad x_2 = 5 \qquad t^2 - 5t + 4 = 0 \begin{cases} t_1 = 1 \\ t_2 = 4 \end{cases}$

$A = 1 \cdot A + 0 \cdot I_2$

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$x_n = c_1 t_1^n + c_2 t_1^n$

$n=1 \Rightarrow 1 = c_1 + 4c_2$

$n=2 \Rightarrow 5 = c_1 + 6c_2$

$4 = 12c_2 \Rightarrow c_2 = \frac{1}{3}, \; c_1 = -\frac{1}{3}$

$x_n = -\frac{1}{3} + \frac{1}{3} \cdot 4^n, \; \forall \; n \geq 1$

$y_n = -4\left(-\frac{1}{3} + \frac{1}{3} \cdot 4^{n-1}\right) = \frac{4}{3} - \frac{1}{3} \cdot 4^n \quad \forall \; n \geq 1$

11. $X^{2024} = A = \begin{pmatrix} 2 & -4 \\ -1 & 2 \end{pmatrix}$

a) $(\det X)^{2024} = \det A = 0 \Rightarrow \det X = 0$

$X^2 - Tr X \cdot X + 0 \cdot I_2 = O_2$

$X^2 = Tr X \cdot X$

PROPOZIȚIE: Dacă $X^2 = \alpha X$ atunci $X^n = \alpha^{n-1} X, \; \forall \; n \geq 2$ (inducție)

$X^{2024} = (Tr X)^{2023} \cdot X = A \;|\; Tr \; ()$

PROPRIETATE: $Tr(\alpha \cdot A) = \alpha \, Tr A$

$(Tr X)^{2024} = 4$

ecuația are două soluții reale $Tr X = \pm \sqrt[2024]{4} = \pm \sqrt[1012]{2}$

$X = \left(\frac{1}{\pm\sqrt[1012]{2}}\right)^{2023} \cdot A$

14. $A, B \in \mathcal{M}_n(\mathbb{R})$

$AB = BA$

? $\det(A^2 + B^2) \geq 0$

$A^2 + B^2 = A^2 - i^2 B^2 = (A - iB)(A + iB)$

$\det(A^2 + B^2) = \det(A - iB)\det(A + iB) = |\det(A + iB)|^2 \geq 0$

PROP.: $z \cdot \bar{z} = |z^2| = |z|^2 \geq 0$

$$\Delta(x) = \begin{vmatrix} 1 & 1 & 1 & 1 \\ 1 & x & -1 & 2 \\ 1 & x^3 & -1 & 8 \\ 1 & x^2 & 1 & 4 \end{vmatrix} \begin{matrix} C_2' = C_2 - C_1 \\ C_3' = C_3 - C_1 \\ C_4' = C_4 - C_1 \end{matrix} = \begin{vmatrix} 1 & 0 & 0 & 0 \\ 1 & x-1 & -2 & 1 \\ 1 & x^3-1 & -2 & 7 \\ 1 & x^2-1 & 0 & 3 \end{vmatrix} =$$

$$= (-1)^{1+1} \begin{vmatrix} x-1 & -2 & 1 \\ x^3-1 & -2 & 7 \\ x^2-1 & 0 & 3 \end{vmatrix} = (x-1) \begin{vmatrix} 1 & -2 & 1 \\ x^2+x+1 & -2 & 7 \\ x+1 & 0 & 3 \end{vmatrix} =$$

$$= (x-1) \begin{vmatrix} 1 & -2 & 1 \\ x^2+x & 0 & 6 \\ x+1 & 0 & 3 \end{vmatrix} = (x-1)(-2)\cdot(-1)^{1+2} \begin{vmatrix} x^2+x & 6 \\ x+1 & 3 \end{vmatrix} =$$

$$= (x-1)\cdot 2(x+1) \begin{vmatrix} x & 6 \\ 1 & 3 \end{vmatrix} = (x-1)\cdot 2\cdot(x+1)\cdot 3 \begin{vmatrix} x & 2 \\ 1 & 1 \end{vmatrix} =$$

$$= 6(x+1)(x-1)(x-2) = 0 \Rightarrow x_1 = -1$$

$$x_2 = 1$$

$$x_3 = 2$$

4. $\begin{cases} 3x + 2y + 5z + 4t = -1 \\ 2x + y + 3z + 3t = 0 \\ x + 2y + 3z = -3 \end{cases}$

$$\bar{A} = \left(\begin{array}{cccc|c} 3 & 2 & 5 & 4 & -1 \\ 2 & 1 & 3 & 3 & 0 \\ 1 & 2 & 3 & 0 & -3 \end{array}\right) \sim \left(\begin{array}{cccc|c} 1 & 2 & 3 & 0 & -3 \\ 2 & 1 & 3 & 3 & 0 \\ 3 & 2 & 5 & 4 & -1 \end{array}\right) \underset{\substack{L_2' = L_2 - 2L_1 \\ L_3' = L_3 - 3L_1}}{\sim}$$

$$\left(\begin{array}{cccc|c} 1 & 2 & 3 & 0 & -3 \\ 0 & -3 & -3 & 3 & 6 \\ 0 & -4 & -4 & 4 & 8 \end{array}\right) \underset{\substack{L_2' = -\frac{1}{3}L_2 \\ L_3' = -\frac{1}{4}L_3}}{\sim} \left(\begin{array}{cccc|c} 1 & 2 & 3 & 0 & -3 \\ 0 & 1 & 1 & -1 & -2 \\ 0 & 1 & 1 & -1 & -2 \end{array}\right) \underset{L_3' = L_3 - L_2}{\sim}$$

$$\left(\begin{array}{cccc|c} \boxed{1} & 0 & 1 & 2 & 1 \\ 0 & \boxed{1} & 1 & -1 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

2 pivoți $\Rightarrow$ rangul $= 2$

$z = \alpha$, $t = \beta$

$$\begin{cases} y = -2 - \alpha + \beta \\ x = 1 - \alpha - 2\beta, \ \alpha, \beta \in \mathbb{R} \end{cases}$$

$\Rightarrow$ Sistem compatibil dublu nedeterminat

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