$$\sum_{n=1}^{\infty}\frac{(-1)^n x^n}{n}$$

$$\begin{cases} x_0 = 0 \\ a_n = \frac{(-1)^n}{n}, n\in\mathbb{N}^* \\ a_0 = 0 \end{cases}$$

$$R = \frac{1}{\rho}$$

$$\rho = \overline{\lim}\sqrt[n]{|a_n|}, \text{sau } \lim_{n\to\infty}\frac{|a_{n+1}|}{|a_n|}(\text{daca } \exists)$$

$$\rho = \lim_{n\to\infty}\frac{\left|\frac{(-1)^{n+1}}{n+1}\right|}{\left|\frac{(-1)^n}{n}\right|} = \lim_{n\to\infty}\left|\frac{-1}{1+\frac{1}{n}}\right| = 1$$

$$R = \frac{1}{\rho} = 1$$

$$A\leq\mathbb{R}$$

$$(x_0 - R, x_0 + R)\leq A\leq[x_0 - R, x_0 + R] = \begin{cases} A\leq\mathbb{R} \\ (-1,1)\leq\mathbb{R}\leq[-1,1] \end{cases}$$

$$-1\in A?\quad \sum_{n=1}^{\infty}\frac{(-1)^n(-1)^n}{n} = \sum_{n=1}^{\infty}\frac{(-1)^{2n}}{n} = \sum_{n=1}^{\infty}\frac{1}{n}\Rightarrow\text{serie armonica cu } \alpha\leq 1\Rightarrow\text{suma este divergenta}\Rightarrow -1\notin A$$

$$1\in A?\quad \sum_{n=1}^{\infty}\frac{(-1)^n(1)^n}{n} = \sum_{n=1}^{\infty}\frac{(-1)^n}{n}$$

$$\lim_{n\to\infty}\frac{1}{n} = 0(1)$$

$$\frac{1}{n+1}\leq\frac{1}{n}(2)$$

$$\Rightarrow\text{conf. crit. lui Leibniz}\Rightarrow\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\text{conv. }\Rightarrow 1\in A$$

$$A = (-1, 1]$$

$$f : A(= (-1,1]) \to \mathbb{R}, f(x) = \sum_{n=1}^{\infty}\frac{(-1)^n x^n}{n}$$

$$f'(x) = \sum_{n=1}^{\infty}\left(\frac{(-1)^n x^n}{n}\right)' = \sum_{n=1}^{\infty}\frac{n(-1)^n x^{n-1}}{n} = -\sum_{n=1}^{\infty}(-1)^{n-1}x^{n-1}\forall x\in(-1,1]$$

$$f'(x) = -\sum_{n=1}^{\infty}(-1)^{n-1}x^{n-1}, \forall x\in(-1,1]$$

$$\underline{\underline{n = m+1}} - \sum_{m=0}^{\infty}(-1)^m x^m = -\frac{1}{1+x}\forall x\in(-1,1)\Big|\int$$

$$f(x) = -\int\frac{1}{1+x}\,dx = -\ln(1+x) + C, \forall x\in(-1,1)$$

$$f(x_0) = a_0\Leftrightarrow f(0) = 0\Leftrightarrow -\ln(1+0) + C = 0\Leftrightarrow C = 0$$

$$f(x) = -\ln(1+x), \forall x\in(-1,1)$$

f cont în $1 \Rightarrow f(1) = \lim\limits_{\substack{x\to1\\x<1}} f(x)$

$f(1) = -\ln 2$

$\lim\limits_{\substack{x\to1\\x<1}} f(x) = -\ln 2$

$\Rightarrow f(x) = -\ln(1+x)$

Demonstrație pentru două din seriile de puteri speciale:

$$S = \sum_{n=0}^{\infty} x^n \Leftrightarrow S = \lim_{n\to\infty} 1 + x + x^2 + ... + x^n \Leftrightarrow xS = \lim_{n\to\infty} x + x^2 + x^3 + ... + x^{n+1} \Leftrightarrow$$

$$\Leftrightarrow xS - S = \lim_{n\to\infty} x + x^2 + x^3 + ... + x^{n+1} - 1 - x - x^2 - ... - x^n \Leftrightarrow S(x-1) = \lim_{n\to\infty} x^{n+1} - 1 \Leftrightarrow$$

$$\Leftrightarrow S = \lim_{n\to\infty} \frac{x^{n+1}-1}{x-1}, \text{ dar } x\in(-1,1) \Rightarrow S = \frac{1}{1-x}$$

$f : \mathbb{R} \to \mathbb{R},\ f(x) = \sin x$

$f(x) = \sin x, \forall x \in \mathbb{R} \Rightarrow f(0) = 0$

$f'(x) = \cos x, \forall x \in \mathbb{R} \Rightarrow f'(0) = 1$

$f''(x) = -\sin x, \forall x \in \mathbb{R} \Rightarrow f''(0) = 0$

$f'''(x) = -\cos x, \forall x \in \mathbb{R} \Rightarrow f'''(0) = -1$

$f^{(4)}(x) = \sin x, \forall x \in \mathbb{R} \Rightarrow f^{(4)}(0) = 0$

$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$

conf. Formulei lui Taylor cu restul sub forma lui Lagrange, $\forall\, x \in \mathbb{R}^* (i.e.\ x \neq 0)$,

$\exists\ c$ intre 0 si x $(i.e.$ ¸$\in (0,x)$ sau $c \in (x,0))$ $a.i.$

$$f(x) = f(0) + \frac{f'(0)}{1!}(x-0) + \frac{f''(0)}{2!}(x-0)^2 + ... + \frac{f^{(n)}(0)}{n!}(x-0)^n + \underbrace{\frac{f^{(n+1)}(c)}{(n+1)!}(x-0)^{n+1}}_{R_n(x)}$$

$$R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-0)^{n+1}$$

$$|R_n(x)| = \frac{\left|f^{(n+1)}(c)\right|}{n+1}|x|^{n+1} \leq \frac{1}{(n+1)!}|x|^{n+1} = \frac{|x|^{n+1}}{(n+1)!}.$$

Fie $x_n = \dfrac{|x|^{n+1}}{(n+1)!}, \forall n \in \mathbb{N}$.

$$\lim_{n\to\infty} \frac{x_{n+1}}{x_n} = \lim_{n\to\infty} \frac{\frac{|x|^{n+2}}{(n+1)!}}{\frac{|x|^{n+1}}{(n+1)!}} = \lim_{n\to\infty} \frac{|x|}{n+2} = 0 \leq 1.$$

Conform Crit. raportului (de notat ca crit. raportului are doua def, una pt. limita si una pt. conv.)

$\lim\limits_{n\to\infty} x_n = 0.$

Avem

$$\begin{tikzcd} 0 \arrow[dr] & \leq |R_n(x)| \leq \arrow[d, dashed] & x_n \arrow[dl] \\ & 0 & \end{tikzcd}$$

Deci $\lim\limits_{n\to\infty}|R_n(x)|=0; i.e. \lim\limits_{n\to\infty}R_n(x)=0.$

Prin urmare $f(x)=\sum\limits_{n=0}^{\infty}\frac{f^n(0)}{n!}(x-0)^n=0+\frac{1}{1!}x+0-\frac{1}{3!}x^3+0+\frac{1}{5!}x^5+...=\sum\limits_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}x^{2n+1}$

$$\left.\begin{array}{r} f(0)=\sin 0=0 \\ \sum\limits_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}0^{2n+1}=0 \end{array}\right| \Rightarrow f(0)=\sum\limits_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}0^{2n+1}$$

Deci $f(x)=\sum\limits_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}x^{2n+1}, \forall x\in\mathbb{R}$

Restul seriilor de puteri ori serezolvă în still asemănător cu acestea ori sunt consecințe ale lor

$$\sum_{n=0}^{\infty}(n+2)(n+1)x^n$$

$$\begin{cases} x_0=0 \\ a_n=(n+2)(n+1), n\in\mathbb{N}^* \\ a_0=0 \end{cases}$$

$R=\frac{1}{\rho}$

$\rho=\overline{\lim}\sqrt[n]{|a_n|}$, sau $\lim\limits_{n\to\infty}\frac{|a_{n+1}|}{|a_n|}$(daca $\exists$)

$\rho=\lim\limits_{n\to\infty}\sqrt[n]{(n+2)(n+1)}=1$

$R=\frac{1}{\rho}=1$

$A\leq\mathbb{R}$

$(x_0-R,x_0+R)\leq A\leq[x_0-R,x_0+R]=\begin{cases} A\leq\mathbb{R} \\ (-1,1)\leq\mathbb{R}\leq[-1,1] \end{cases}$

$-1\in A?\quad \sum\limits_{n=0}^{\infty}(n+2)(n+1)(-1)^n$

$\lim\limits_{n\to\infty}(n+2)(n+1)=\infty\Rightarrow$suma este div $\Rightarrow -1\notin A$

$1\in A?\quad \sum\limits_{n=0}^{\infty}(n+2)(n+1)$

$\lim\limits_{n\to\infty}(n+2)(n+1)=\infty\Rightarrow$suma este div $\Rightarrow 1\notin A$

$A=(-1,1)$

$$f : A(= (-1,1)) \to \mathbb{R}, f(x) = \sum_{n=1}^{\infty}(n+2)(n+1)x^n$$

$$\int f(x)\,dx = \int \sum_{n=0}^{\infty}(n+2)(n+1)x^n\,dx = \sum_{n=0}^{\infty}\int (n+2)(n+1)x^n\,dx = \sum_{n=1}^{\infty}(n+2)x^{n+1} + C, \forall x \in (-1,1)$$

$$\int\int f(x) + C\,dx = \int \sum_{n=0}^{\infty}(n+2)x^(n+1) + C\,dx = \sum_{n=0}^{\infty}\int (n+2)x^{n+1} + C\,dx = \sum_{n=1}^{\infty}x^{n+2} + C, \forall x \in (-1,1)$$

$$\int\int f(x)\,dx\,dx = \sum_{n=1}^{\infty}x^{n+2} + C, \forall x \in (-1,1)$$

$$\underline{\underline{n = m-2}}\sum_{m=3}^{\infty}x^m + C = \sum_{m=0}^{\infty}x^m + C - x^0 - x^1 - x^2 = \frac{1}{1-x} - 1 - x - x^2 + C, \forall x \in (-1,1)\Big|''$$

$$f(x) = \left(\frac{1}{1+x} - 1 - x - x^2 + C\right)'' = \left(-\frac{1}{(1+x)^2} - 1 - 2x\right)' = \frac{2}{(1+x)^3} - 2, \forall x \in (-1,1)$$

$$f(x) = \frac{2}{(1+x)^3} - 2, \forall x \in (-1,1)$$

$$\sum_{n=1}^{\infty}\frac{(-2)^n}{\sqrt[3]{n+1}\sqrt[4]{n+2}}(x+2)^n$$

$$\begin{cases} x_0 = 2 \\ a_n = \frac{(-2)^n}{\sqrt[3]{n+1}\sqrt[4]{n+2}} \forall n \in \mathbb{N}^* \\ a_0 = 0 \end{cases}$$

$$R = \frac{1}{\rho}$$

$$\rho = \overline{\lim}\sqrt[n]{|a_n|}, \text{sau } \lim_{n\to\infty}\frac{|a_{n+1}|}{|a_n|}(\text{daca } \exists)$$

$$\rho = \lim_{n\to\infty}\frac{\left|\frac{(-2)^n}{\sqrt[3]{n+1}\sqrt[4]{n+2}}\right|}{\left|\frac{(-2)^n}{\sqrt[3]{n+1}\sqrt[4]{n+2}}\right|} = \lim_{n\to\infty}\left|-2\sqrt[3]{\frac{n+1}{n+2}}\sqrt[4]{\frac{n+2}{n+3}}\right| = 2$$

$$R = \frac{1}{\rho} = \frac{1}{2}$$

$$A \leq \mathbb{R}$$

$$(x_0 - R, x_0 + R) \leq A \leq [x_0 - R, x_0 + R] = \begin{cases} A \leq \mathbb{R} \\ \left(\frac{3}{2}, \frac{5}{2}\right) \leq \mathbb{R} \leq \left[\frac{3}{2}, \frac{5}{2}\right] \end{cases}$$

$$\frac{3}{2} \in A? \quad \sum_{n=1}^{\infty}\frac{(-2)^n}{\sqrt[3]{n+1}\sqrt[4]{n+2}}\left(\frac{7}{2}\right)^n$$

$$\lim_{n\to\infty}\frac{\left(\frac{7}{2}\right)^n}{\sqrt[3]{n+1}\sqrt[4]{n+2}} \overset{\frac{\infty}{\infty}}{=} \infty$$

$$\sqrt[3]{n+1}\sqrt[4]{n+2} << \left(\frac{7}{2}\right)^n$$, exponențiala crește mult mai rapid decât polinomiala

$\infty \Rightarrow$ suma este div $\Rightarrow \frac{3}{2} \notin A$

$\frac{5}{2} \in A?$ $\sum_{n=1}^{\infty} \frac{(-2)^n}{\sqrt[3]{n+1}\sqrt[4]{n+2}} \left(\frac{9}{2}\right)^n$

$\lim_{n\to\infty} \frac{\left(\frac{9}{2}\right)^n}{\sqrt[3]{n+1}\sqrt[4]{n+2}} \overset{\frac{\infty}{\infty}}{=} \infty$

$\sqrt[3]{n+1}\sqrt[4]{n+2} << \left(\frac{9}{2}\right)^n$, exponențiala crește mult mai rapid decât polinomiala

$\infty \Rightarrow$ suma este div $\Rightarrow \frac{5}{2} \notin A$

$A = \left(\frac{3}{2}, \frac{5}{2}\right)$