Curs 14

Teoremă (teorema de permutare a limitei cu integrala) - cazul multidimensional

• Fie $p \in \mathbb{N}^*$, $\emptyset \neq A \in \mathcal{J}(\mathbb{R}^p)$, $f_n, f: A \to \mathbb{R}$ $\forall n \in \mathbb{N}$ a.î.:

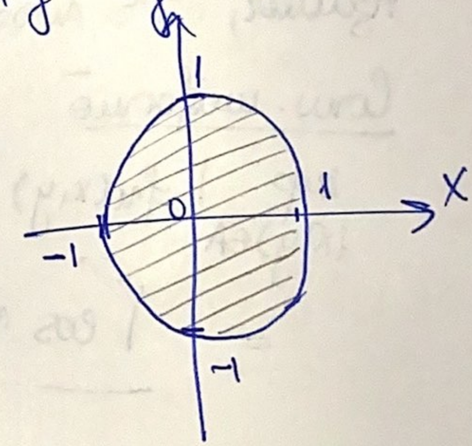
1) f_n integrab R şi mărg pe A
2) $f_n \xrightarrow[n\to\infty]{u} f$

Atunci f este integrab R şi mărginită şi
$$\lim_{n\to\infty} \int_A f_n(x)dx = \int_A f(x)dx$$

Ex: Fie $A = B[(0,0),1] = \bar{B}((0,0),1)$
$= \{(x,y) \in \mathbb{R}^2 \mid x^2+y^2 \leq 1\}$

Det $\displaystyle\lim_{n\to\infty} \iint_A \frac{\cos(n(x+y)) + 2(x^2+y^2)}{n^2+nx^2+y^2}dxdy$

Sol



A convexă (oricare 2 pct distincte unite => segm $\in A$)
şi mărg $\Rightarrow A \in \mathcal{J}(\mathbb{R}^2)$

A compactă (închisă + mărg)

Fie $f_n: A \to \mathbb{R}$ ~~$f_n(x+y)$~~

$$f_n(x,y) = \frac{\cos n(x+y) + 2(x^2+y^2)}{n^2+nx^2+y^2} \quad \forall n \in \mathbb{N}^*$$

f_n continuă pe compactă $\Rightarrow f_n$ mărg.

$\left.\begin{array}{l} f_n \text{ cont } \forall n \in \mathbb{N}^* \\ A \in \mathcal{J}(\mathbb{R}^2) \text{ şi } A \text{ compactă} \end{array}\right| \Rightarrow f_n$ integr R şi f_n mărg $\forall n \in \mathbb{N}^*$

Conv simplă

Fie $(x,y) \in A$

$$0 \le |f_n(x,y)| = \frac{|\cos n(x+y) + 2(x^2+y^2)|}{\underbrace{n^2 + nx^2 + y^2}_{}} \le$$

e poz, nu mai e nevoie de mod

$$\frac{|\cos n(x+y)| + |2(x^2+y^2)|}{n^2+nx^2+y^2} \le \frac{1+2}{n^2} = \frac{3}{n^2} \quad \forall n \in \mathbb{N}^*$$

Avem $0 \le |f_n(x,y)| \le \frac{3}{n^2} \quad \forall n \in \mathbb{N}^*$

$\to 0 \leftarrow$

Deci: $\lim_{n\to\infty} |f_n(x,y)| = 0 \Rightarrow$ Prin urmare, $\lim_{n\to\infty} f_n(x,y) = 0$

Așadar, $f_n \xrightarrow[n\to\infty]{s} f$, unde $f: A \to \mathbb{R}$ $f(x) = 0$

Conv. uniformă

$$\sup_{(x,y)\in A} |f_n(x,y) - f(x)| = \sup_{(x,y)\in A} \left| \frac{\cos n(x+y) + 2(x^2+y^2)}{n^2+nx^2+y^2} - 0 \right|$$

$$= \frac{|\cos n(x+y) + 2(x^2+y^2)|}{n^2+nx^2+y^2} \le \frac{3}{n^2} \xrightarrow[n\to\infty]{} 0$$

Deci, $f_n \xrightarrow[n\to\infty]{u} f$

Conform teoremei de permutare a limitei cu integrala - Cazul multidim. avem că f e integr. R și mărg și $\lim_{n\to\infty} \iint_A f_n(x,y)\,dx\,dy = \iint_A f(x,y)\,dx\,dy$

$= \iint_A 0 \, dx\,dy = 0$

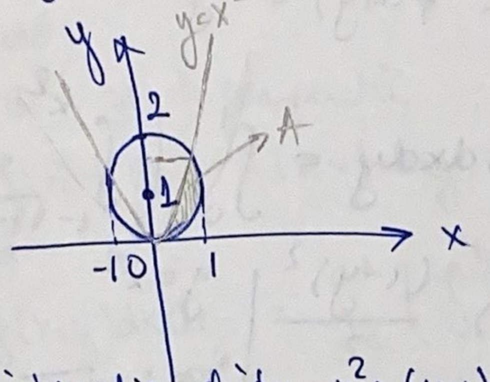
$\square$

centru

1. Determinați:

a) $\iint_A (1-y)\,dx\,dy$, unde $A=\{(x,y)\in\mathbb{R}^2 \mid x^2+(y-1)^2\le 1,\ y\le x^2,\ x\ge 0\}$

Sol:



Det pct. de intersecție dintre $x^2+(y-1)^2=1$ și $y=x^2$

$$\begin{cases} x^2+(y-1)^2=1 \\ y=x^2 \end{cases} \Rightarrow x^2+(x^2-1)^2-1=0$$

$$x^2+x^4-2x^2+\cancel{1}-\cancel{1}=0$$

$$x^4-x^2=0$$

$$x^2(x^2-1)=0$$

$$x^2(x-1)(x+1)=0 \Rightarrow x\in\{-1,0,1\}$$

$x_1=-1 \to y_1=1$

$x_2=0 \to y_2=0$

$x_3=1 \to y_3=1$

$$x^2+(y-1)^2\le 1$$

$$(y-1)^2\le 1-x^2$$

$$-\sqrt{1-x^2}\le y-1\le\sqrt{1-x^2}$$

$$\Rightarrow \boxed{\underbrace{1-\sqrt{1-x^2}}_{\text{îl iau în jos}}\le y\le 1+\sqrt{1-x^2}}$$

$A=\{(x,y)\in\mathbb{R}^2 \mid x\in[0;1],\ 1-\sqrt{1-x^2}\le y\le x^2\}$

Fie $\alpha,\beta:[0,1]\to\mathbb{R}$ $\begin{cases}\alpha(x)=1-\sqrt{1-x^2}\\ \beta(x)=x^2\end{cases}$

α, β cont

$A \in \mathcal{J}(\mathbb{R}^2)$ şi A compactă

Fie $f: A \to \mathbb{R}$ $f(x,y) = 1-y$

f cont

$$\iint_A f(x,y)\,dx\,dy = \int_0^1 \left(\int_{1-\sqrt{1-x^2}}^{x^2} (1-y)\,dy \right) dx$$

$$= \int_0^1 (-1) \frac{(1-y)^2}{2} \Big|_{y=1-\sqrt{1-x^2}}^{y=x^2} \Big) dx$$

$$= -\frac{1}{2} \left(\int_0^1 \left[(1-x^2)^2 - (\cancel{1}-\cancel{1}+\sqrt{1-x^2})^2 \, dx \right] \right)$$

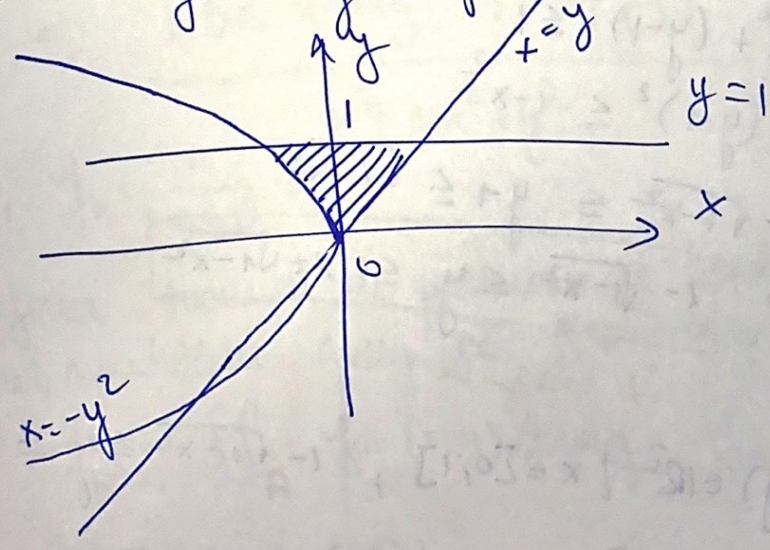
$$= -\frac{1}{2} \int_0^1 (\cancel{1}+x^4-2x^2-\cancel{1}+x^2)\,dx$$

$$= -\frac{1}{2} \int_0^1 (x^4-x^2)\,dx = -\frac{1}{2}\left(\frac{1}{5}-\frac{1}{3}\right) = -\frac{1}{\cancel{2}} \cdot \left(-\frac{\cancel{2}}{15}\right)$$

$$= \frac{1}{15}$$

b) $\iint_A y\,dx\,dy$ unde A e mulț plană mărg

de $x = -y^2$ $x = y$ şi $y = 1$



Sol $A = \{ (x,y) \in \mathbb{R}^2 \mid y \in [0;1], -y^2 \le x \le y \}$

Fie $\varphi, \psi : [0,1] \to \mathbb{R}$, $\varphi(y) = -y^2$, $\psi(y) = y$

φ, ψ cont

$A \in \mathcal{J}(\mathbb{R}^2)$ şi A compactă

Fie $f : A \to \mathbb{R}$ $f(x,y) = y$

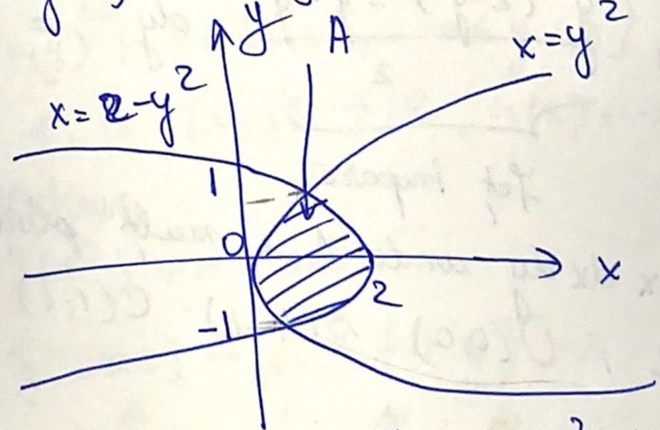
f cont

$$\iint_A f(x,y)\,dx\,dy = \int_0^1 \left(\int_{-y^2}^{y} y\,dx \right) dy$$

$$= \int_0^1 \left(yx \Big|_{x=-y^2}^{x=y} \right) dy = \int_0^1 y(y+y^2)\,dy$$

$$= \int_0^1 y^2 + y^3\,dy = \frac{1}{3} + \frac{1}{4} = \frac{7}{12} \quad \boxtimes$$

c) $\iint_A xy\,dx\,dy$, unde A e mult, plană limitată de $x = y^2$ şi $x = 2 - y^2$



Det. pct de intersecție dintre $x = y^2$ şi $x = 2 - y^2$

$$\begin{cases} x = y^2 \\ x = 2 - y^2 \end{cases} \iff y^2 = 2 - y^2$$

$$2 - 2y^2 = 0$$

$$2(1-y)(1+y) = 0$$

$$\Rightarrow y \in \{\pm 1\} \Rightarrow x \in \{+1\}$$

$A = \{ (x,y) \in \mathbb{R}^2 \mid y \in [-1,1],\ y^2 \le x \le 2 - y^2 \}$

Fie $\varphi, \psi : [-1,1] \to \mathbb{R}$ $\varphi(y) = y^2$, $\psi(y) = 2 - y^2$

φ, ψ continue

$A \in \mathcal{J}(\mathbb{R}^2)$, A compactă

Fie $f : A \to \mathbb{R}$ $f(x,y) = xy$

f continuă

$$\iint_A f(x,y)\,dx\,dy = \int_{-1}^{1} \left(\int_{y^2}^{2-y^2} xy\,dx \right) dy$$

$$= \int_{-1}^{1} \left(\frac{x^2 y}{2} \Big|_{x=y^2}^{x=2-y^2} \right) dy$$

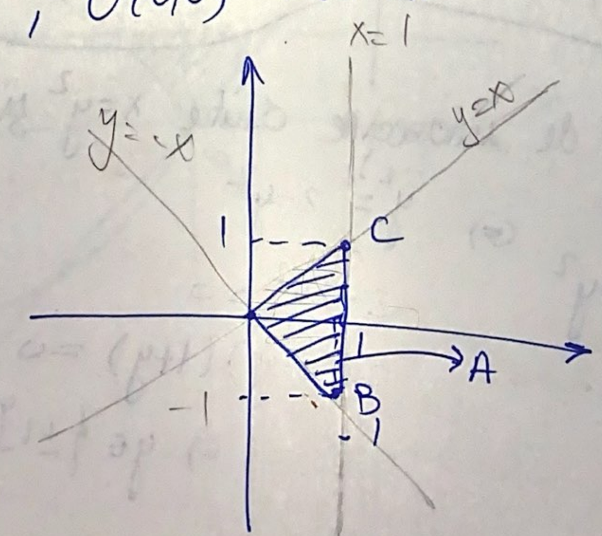
$$= \cancel{\int_{-1}^{1} \frac{y}{2} \left(2-y^2 - y^2 \right)}$$

$$= \int_{-1}^{1} \underbrace{\frac{y\,(2-y^2)^2 - y \cdot y^4}{2}}_{\text{fct impară}}\, dy = 0$$

$\square$

d) $\iint_A x\,dx\,dy$ unde A e mulț plană mărg de $\triangle OBC$, $O(0,0)$ $B(1,-1)$ $C(1,1)$

Sol



Ox ✓

Oy rupă în 2.

Sol

Scriem ec. dreptelor OB, OC și BC

OB: $\dfrac{y-y_0}{y_B-y_0} = \dfrac{x-x_0}{x_B-x_0} \Leftrightarrow \dfrac{y}{-1} = \dfrac{x}{1}$

$\Rightarrow \boxed{y=-x}$

OC: $\dfrac{y-y_0}{y_C-y_0} = \dfrac{x-x_0}{x_C-x_0} \Leftrightarrow \dfrac{y}{1} = \dfrac{x}{1}$

$\Rightarrow \boxed{y=x}$

BC: $\dfrac{y_B-y_C}{y_B-y_C} = \dfrac{x-x_C}{x_B-x_C} \Leftrightarrow \dfrac{y-1}{-1-1} = \dfrac{x-1}{1-1}$

$\Rightarrow x-1=0 \Rightarrow \boxed{x=1}$

$A = \{(x,y) \in \mathbb{R}^2 \mid x \in [0;1],\ -x \le y \le x\}$

Fie $\alpha, \beta : [0;1] \to \mathbb{R}$, $\alpha(x) = -x$ $\beta(x) = x$

α, β continue

$A \in \mathcal{J}(\mathbb{R}^2)$ + A compactă

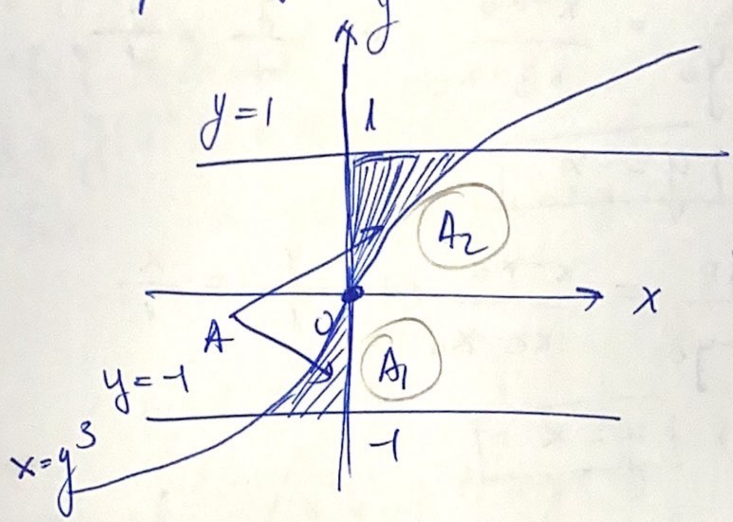
Fie $f : A \to \mathbb{R}$ $f(x,y) = x$

$$\iint_A f(x,y)\,dx\,dy = \int_0^1 \left(\int_{-x}^{x} x\,dy \right) dx$$

$$= \int_0^1 \left(xy \Big|_{y=-x}^{y=x} \right) dx$$

$$= \int_0^1 2x^2\,dx = 2\,\frac{x^3}{3} \Big|_{x=0}^{x=1} = \frac{2}{3}$$

1) $\iint_A e^{y^4}\,dx\,dy$, unde A mulț plană mărg de $x=y^3$, $y=1$, $y=-1$ și $x=0$



$A = A_1 \cup A_2$ unde $A_1 = \{(x,y)\in\mathbb{R}^2 \mid y\in[-1,0]$, $y^3 \le x \le 0\}$

și $A_2 = \{(x,y)\in\mathbb{R}^2 \mid y\in[0,1]$, $0\le x \le y^3\}$

Fie $\varphi_1, \psi_1 : [-1,0] \to \mathbb{R}$

$\varphi_1(y) = y^3$, $\psi_1(y) = 0$

φ_1, ψ_1 continue

$A_1 \in \mathcal{J}(\mathbb{R}^2)$, A_1 compactă

Fie $\varphi_2, \psi_2 : [0,1] \to \mathbb{R}^2$

$\varphi_2(y) = 0$, $\psi_2(y) = y^3$

φ_2, ψ_2 cont

$A_2 \in \mathcal{J}(\mathbb{R}^2)$ A_2 compactă

Deci $A = A_1 \cup A_2 \in \mathcal{J}(\mathbb{R}^2)$ și A compactă

$A_1 \cap A_2 = \{(0,0)\} = \{0\}\times\{0\} \Rightarrow \mu(A_1\cap A_2) = 0\cdot 0 = 0$

Fie $f: A \to \mathbb{R}$ $f(x,y) = e^{y^4}$

f cont

$$\iint_A f(x,y)\,dx\,dy = \iint_{A_1} f(x,y)\,dx\,dy + \iint_{A_2} f(x,y)\,dx\,dy$$

- $$\iint_{A_1} f(x,y)\,dx\,dy = \int_{-1}^{0}\left(\int_{y^3}^{0} e^{y^4}\,dx\right)dy$$

$$= \int_{-1}^{0} e^{y^4}\cdot x \Big|_{x=y^3}^{x=0} = \int_{-1}^{0} e^{y^4}(0-y^3)\,dy$$

$$= -\frac{1}{4}\int_{-1}^{0} 4y^3 e^{y^4}\,dy = -\frac{1}{4} e^{y^4}\Big|_{y=-1}^{y=0}$$

$$= -\frac{1}{4}(1-e) = \frac{e-1}{4}$$

- $$\iint_{A_2} f(x,y)\,dx\,dy = \int_0^1\left(\int_0^{y^3} e^{y^4}\,dx\right)dy$$

$$= \int_0^1 x\cdot e^{y^4}\,dy\Big|_{x=0}^{x=y^3} = \int_0^1 y^3\cdot e^{y^4}\,dy$$

$$= \frac{1}{4}\int_0^1 4\cdot y^3 e^{y^4}\,dy = \frac{1}{4}\cdot e^{y^4}\Big|_{y=0}^{y=1}$$

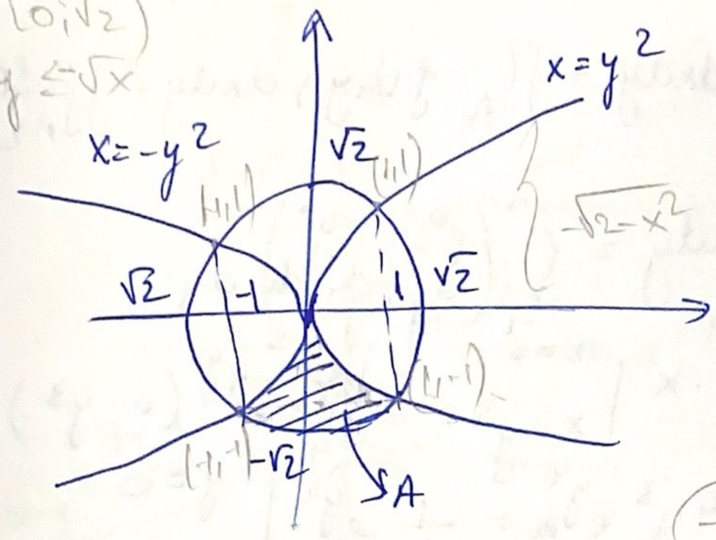
$$= \frac{1}{4}(e-1)$$

$$\iint_A f(x,y)\,dx\,dy = \frac{e-1}{4} + \frac{e-1}{4} = \frac{e-1}{2} \quad \square$$

f) $\iint_A y\,dx\,dy$, unde $A = \{(x,y) \in \mathbb{R}^2 \mid x^2+y^2 \le 2,\ x \le y^2,\ x \ge -y^2,\ y \le 0\}$

$x \in [0;\sqrt{2})$

$-\sqrt{2-x^2} \le y \le \sqrt{x}$



$x \in [-\sqrt{2};0]$

$-\sqrt{2-x^2} < y \le \sqrt{-x}$

$y^2 \le 2-x^2$

Ox

$-\sqrt{2-x^2} \le y \le \sqrt{2-x^2}$

Det. pct de intersecție dintre $x = y^2$ și $x^2+y^2=2$

$$\left.\begin{array}{l} x=y^2 \\ x^2+y^2=2 \end{array}\right| \Rightarrow x^2+x=2 \Rightarrow x^2+x-2=0$$

$\cancel{x(x+1)=2}$

$(x+2)(x-1)=0$

$x_{1,2} \to -2$ ✗ $(x=y^2)$; $\to 1$

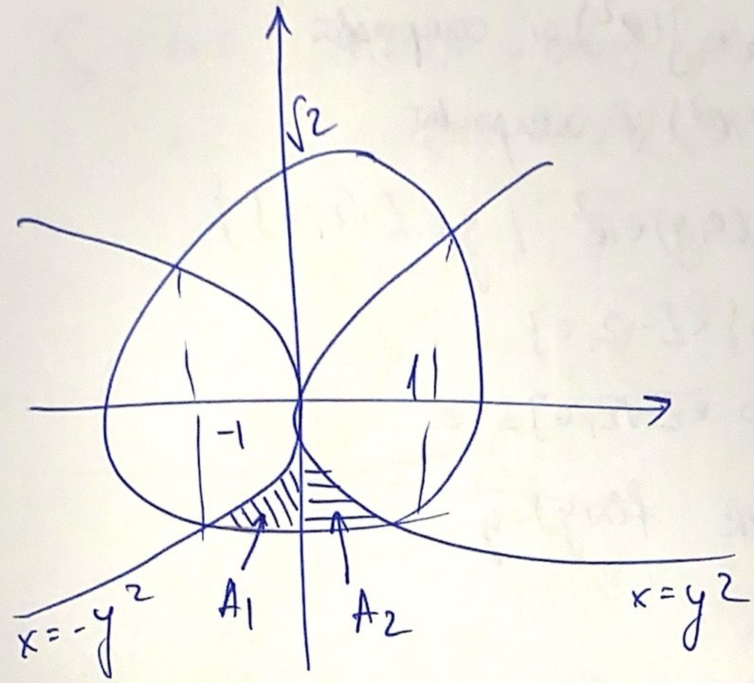
$x=1 \Rightarrow y=\pm 1$

Det. pct de intersecție dintre $x=-y^2$ și $x^2+y^2=2$

$$\begin{cases} x=-y^2 \\ x^2+y^2=2 \end{cases} \Rightarrow x^2-x-2=0$$

$x_{1,2} \to 2$; $\to -1$

$x=-1 \Rightarrow y=\pm 1$



$x^2+y^2 \le 2 \Rightarrow \cancel{x^2 \le 2-y} \quad y^2 \le 2-x^2$

$-\sqrt{2-x^2} \le y \le \sqrt{2-x^2}$

$x \le y^2 \Rightarrow y^2 \ge x \Rightarrow y \le -\sqrt{x}$ sau $y \ge \sqrt{x}$

$x \ge -y^2 \Rightarrow -y^2 \le x \Rightarrow y^2 \ge -x \Rightarrow y \le -\sqrt{-x}$ sau $y \ge \sqrt{-x}$

$A = A_1 \cup A_2$, unde $A_1 = \{ (x,y) \in \mathbb{R}^2 \mid x \in [-1,0]$

$-\sqrt{2-x^2} \le y \le -\sqrt{-x} \}$

şi $A_2 = \{ (x,y) \in \mathbb{R}^2 \mid x \in [0,1], -\sqrt{2-x^2} \le y \le \sqrt{x} \}$

Fie $\alpha_1, \beta_1 : [-1,0] \to \mathbb{R}$ $\begin{cases} \alpha_1(x) = -\sqrt{2-x^2} \\ \beta_1(x) = -\sqrt{-x} \end{cases}$

α_1, β_1 cont, $A_1 \in \mathcal{J}(\mathbb{R}^2)$ şi compactă

Fie $\alpha_2, \beta_2 : [0,1] \to \mathbb{R}$ $\begin{cases} \alpha_2(x) = -\sqrt{2-x^2} \\ \beta_2(x) = \sqrt{x} \end{cases}$

α_2, β_2 cont $A_2 \in J(\mathbb{R}^2)$ şi compactă

$A = A_1 \cup A_2 \in J(\mathbb{R}^2)$ şi compactă

$A_1 \cap A_2 = \{ (0,y) \in \mathbb{R}^2 \mid y \in [-\sqrt{2}, 0] \}$

$= \{0\} \times [-\sqrt{2}, 0]$

$\mu(A_1 \cap A_2) = 0 \times [-\sqrt{2}, 0] = 0$

Fie $f: A \to \mathbb{R}$ $f(x,y) = y$

f cont

$$\iint_A f(x,y)\,dx\,dy = \iint_{A_1} f(x,y)\,dx\,dy + \iint_{A_2} f(x,y)\,dx\,dy$$

$$\iint_{A_1} f(x,y)\,dx\,dy = \int_{-1}^{0} \left(\int_{-\sqrt{2-x^2}}^{-\sqrt{-x}} y\,dy \right) dx$$

$$= \int_{-1}^{0} \left(\left(\frac{1}{2} y^2\right) \Big|_{y=-\sqrt{2-x^2}}^{y=-\sqrt{-x}} \right) dx$$

$$= \int_{-1}^{0} \frac{1}{2}(-x - 2 + x^2)\,dx = --- = -\frac{7}{12}$$

$$\iint_{A_2} f(x,y)\,dx\,dy = \int_0^1 \left(\int_{-\sqrt{2-x^2}}^{-\sqrt{x}} y\,dy \right) dx$$

$$= \int_0^1 \frac{1}{2}(x - 2 + x^2)\,dx = --- = -\frac{7}{12}$$

Deci, $\iint_A f(x,y)\,dx\,dy = -\frac{7}{12} - \frac{7}{12} = -\frac{7}{6}$ □

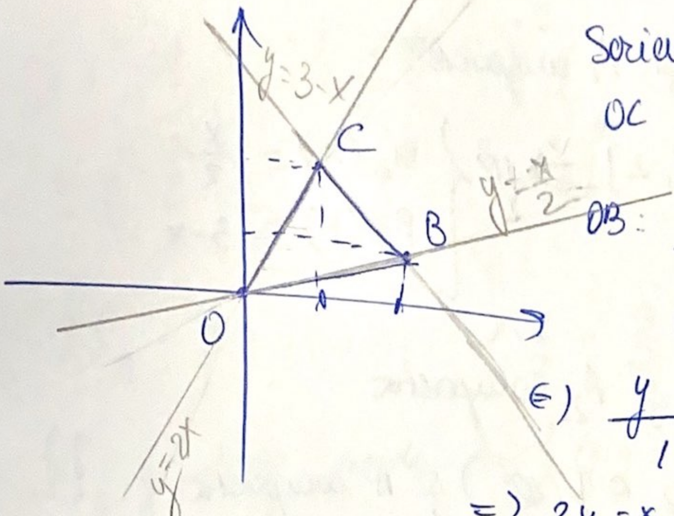
Dacă Δ n-are nicio latură paralelă cu axele

tb rupt în 2 bucăţi

Dacă are o lat paralelă, merge dintr-o bucată

şi facem proiecţie pe cealaltă axă

g) $\iint_A x \, dx\,dy$, unde A e mult plană limitată de ΔOBC, $O(0,0)$ $B(2;1)$ $C(1;2)$



Scriem ec. dreptelor OB, OC, BC.

OB: $\dfrac{y-y_0}{y_B-y_0} = \dfrac{x-x_0}{x_B-x_0}$

$\Leftrightarrow \dfrac{y}{1} = \dfrac{x}{2}$

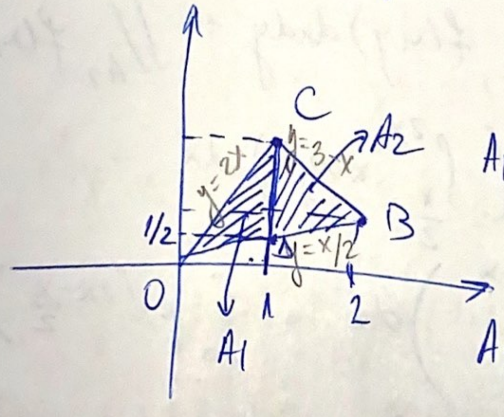
$\Rightarrow 2y = x \Rightarrow y = \dfrac{x}{2}$

OC: $\dfrac{y-y_0}{y_C-y_0} = \dfrac{x-x_0}{x_C-x_0} \Leftrightarrow \dfrac{y}{2} = \dfrac{x}{1}$

$\Rightarrow y = 2x$

BC: $\dfrac{y-y_B}{y_C-y_B} = \dfrac{x-x_B}{x_C-x_B} \Leftrightarrow \dfrac{y-1}{2-1} = \dfrac{x-2}{-1}$

$\Rightarrow y-1 = 2-x \Rightarrow y = 3-x$



$A = A_1 \cup A_2$, unde

$A_1 = \{(x,y) \in \mathbb{R}^2 / x \in [0;1], \frac{x}{2} \le y \le 2x\}$

$A_2 = \{(x,y) \in \mathbb{R}^2 / x \in [1;2], \frac{x}{2} \le y \le 3-x\}$

Fie $\alpha_1, \beta_1 : [0,1] \to \mathbb{R}$ $\begin{cases} \alpha_1(x) = \frac{x}{2} \\ \beta_1(x) = 2x \end{cases}$

α_1, β_1 cont

$A_1 \in \mathcal{J}(\mathbb{R}^2)$ și A_1 compactă

Fie $\alpha_2, \beta_2 : [1,2] \to \mathbb{R}$ $\begin{cases} \alpha_2(x) = \frac{x}{2} \\ \beta_2(x) = 3-x \end{cases}$

α_2, β_2 cont

$A_2 \in \mathcal{J}(\mathbb{R}^2)$ și A_2 compactă

$A = A_1 \cup A_2 \in \mathcal{J}(\mathbb{R}^2)$ și A compactă

Fie $D(x_D, y_D) \in OB$ a.î. $x_D = 1$

Avem $y_D = \frac{x_D}{2} = \frac{1}{2}$

$A_1 \cap A_2 = [CD] = \{1\} \times [\frac{1}{2}; 2] \Rightarrow \mu(A_1 \cap A_2) = 0.$

$(2 - \frac{1}{2}) \cdot 0$

Fie $f : A \to \mathbb{R}$ $f(x,y) = x$

f cont.

$$\iint_A f(x,y)\,dx\,dy = \iint_{A_1} f(x,y)\,dx\,dy + \iint_{A_2} f(x,y)\,dx\,dy$$

$$\iint_{A_1} f(x,y)\,dx\,dy = \int_0^1 \left(\int_{\frac{x}{2}}^{2x} x\,dy \right) dx$$

$$= \int_0^1 \left(xy \Big|_{y=\frac{x}{2}}^{y=2x} \right) dx = \int_0^1 x \left(2x - \frac{x}{2}\right) dx$$

$$= \int_0^1 x \cdot \frac{3x}{2} = \frac{\cancel{3}}{2} \frac{x^3}{\cancel{3}} \Big|_0^1 = \frac{1}{2}$$

$$\iint_{A_2} f(x,y)\,dxdy = \int_1^2 \left(\int_{x/2}^{3-x} x\,dy \right) dx = \int_1^2 \left(xy \Big|_{y=\frac{x}{2}}^{y=3-x} \right) dx$$

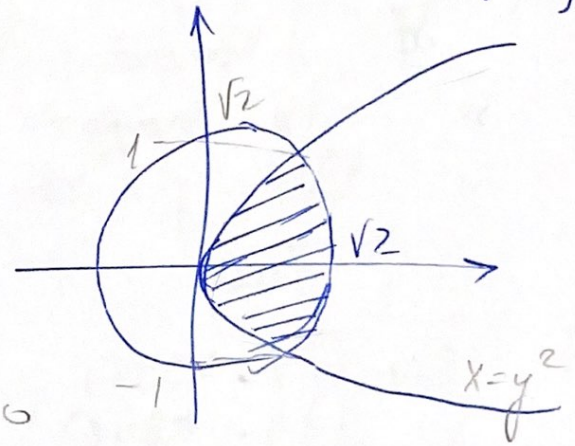
$$= \int_1^2 x\left(3-x-\frac{x}{2}\right) dx = \int_1^2 \left(3x - x^2 - \frac{x^2}{2}\right) dx$$

$$= 3\frac{x^2}{2} - \frac{x^3}{3} - \frac{x^3}{6} \Big|_1^2 = 1$$

$$\Rightarrow \iint_A f(x,y)\,dxdy = 1+\frac{1}{2} = \frac{3}{2}$$

b) $\iint_A y\,dxdy$, unde $A = \{(x,y)\in\mathbb{R}^2 \mid x \geq y^2,\ x^2+y^2 \leq 2\}$

Sol Det pct de intersecție dintre $x \geq y^2$ și $x^2+y^2 \leq 2$



$\begin{cases} x = y^2 \\ x^2+y^2 = 2 \end{cases} \Rightarrow x^2+x-2=0$

$(x+2)(x-1)=0$

$\Rightarrow x_1 = -2$ (X)

$x_2 = 1$

$\Rightarrow y = \pm 1$

$x^2+y^2 \leq 2 \Rightarrow x^2 \leq 2-y^2 \Rightarrow -\sqrt{2-y^2} \leq x \leq \sqrt{2-y^2}$

$A = \{(x,y)\in\mathbb{R}^2 \mid y \in (-1,1),\ y^2 \leq x \leq \sqrt{2-y^2}\}$

Fie $\varphi,\psi: A \to \mathbb{R}$ $\varphi(y) = y^2$, $\psi(y) = \sqrt{2-y^2}$

φ,ψ cont

$A \in J(\mathbb{R}^2)$ și compactă

Fie $f : A \to \mathbb{R}$, $f(x) = y$ cont

$$\iint_A f(x,y)\,dx\,dy = \int_{-1}^{1}\left(\int_{y^2}^{\sqrt{2-y^2}} y\,dx\right)dy$$

$$= \int_{-1}^{1} yx \Big|_{x=y^2}^{x=\sqrt{2-y^2}} dy = \int_{-1}^{1} \underbrace{y\left(\sqrt{2-y^2} - y^2\right)}_{\text{impară}} dy = 0$$

2. Det.

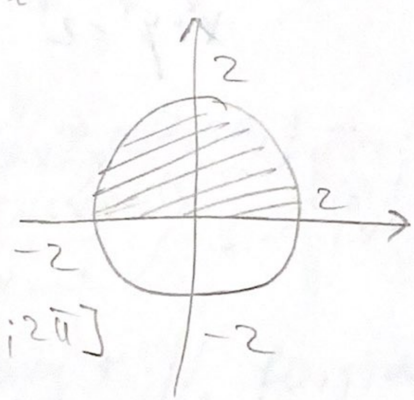
a) $\iint_A e^{-x^2-y^2}\,dx\,dy$, $A = \{(x,y) \in \mathbb{R}^2 \mid x^2+y^2 \le 4, y \ge 0\}$

Sol A convexă și mărg $\Rightarrow A \in J(\mathbb{R}^2)$

A compactă

Fie $f : A \to \mathbb{R}$ $f(x,y) = e^{-x^2-y^2}$

f cont



S.V: $\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$,

$r \in [0,\infty)$, $\theta \in [0; 2\pi]$

$(x,y) \in A \Leftrightarrow \begin{cases} x^2+y^2 \le 4 \\ y \ge 0 \end{cases} \Rightarrow \begin{cases} r^2\cos^2\theta + r^2\sin^2\theta \le 4 \\ r\sin\theta \ge 0 \end{cases} \Rightarrow$

$\begin{cases} r^2(\cos^2\theta + \sin^2\theta) \le 4 \\ r\sin\theta \ge 0 \end{cases} \Rightarrow \begin{cases} r^2 \le 4 \\ r\sin\theta \ge 0 \end{cases} \Rightarrow \begin{cases} r \in [0;2] \\ \theta \in [0;\pi] \end{cases}$

Fie $B = [0;2] \times [0;\pi]$

$$\Rightarrow \begin{cases} r^2 \le 1 \\ 3r\cos\theta \ge 0 \\ 2r\sin\theta \ge 0 \end{cases} \Rightarrow \begin{cases} r \in [0;1] \\ \theta \in [0; \frac{\pi}{2}] \end{cases}$$

Fie $B = [0;1] \times [0; \frac{\pi}{2}]$

$$\iint_A f(x,y)\,dx\,dy = \iint_B 3 \cdot 2 \cdot r\, f(3r\cos\theta, 2r\sin\theta)\,dr\,d\theta$$

$$= \int_0^1 \left(\int_0^{\frac{\pi}{2}} 6r\sqrt{1-r^2}\,d\theta \right) dr$$

$$= \int_0^1 \frac{\pi}{2} \cdot 6r\sqrt{1-r^2}\,dr = -\frac{3\pi}{2}\int_0^1 (-2r)(1-r^2)^{\frac{1}{2}}\,dr$$

$$= -\frac{3\pi}{\cancel{2}} \frac{(1-r^2)^{\frac{3}{2}}}{\frac{\cancel{3}}{2}} \Bigg|_{r=0}^{r=1} = \pi \qquad \square$$