Consultații

Friday, February 26, 2021 2:55 PM

1. R inel (com și 1) și $I_1, I_2 \trianglelefteq R$.

$I_1 \cup I_2$ este ideal $\iff I_1 \subset I_2$ sau $I_2 \subset I_1$.

"$\Leftarrow$" clar

"$\Rightarrow$" Pp că $I_1 \not\subset I_2$ și $I_2 \not\subset I_1 \Rightarrow \exists\, x \in I_2 \setminus I_1$

$\exists\, y \in I_1 \setminus I_2$.

$x, y \in I_1 \cup I_2 \Rightarrow x+y \in I_1 \cup I_2 \Rightarrow x+y \in I_1$; dar și $y \in I_1 \Rightarrow x \in I_1$ ↯

sau

$x+y \in I_2$; — $x \in I_2 \Rightarrow y \in I_2$

2. Dacă $R_1, \dots R_n$ inele. Idealele lui $R_1 \times \dots \times R_n$ sunt

$I_1 \times \dots \times I_n$, unde $I_j \trianglelefteq R_j$.

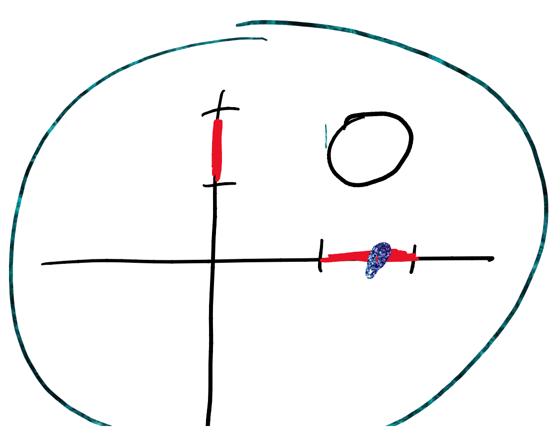
<u>Dem</u> Fie $I \trianglelefteq R_1 \times \dots \times R_n$. Fie $p_j : R_1 \times \dots \times R_n \to R_j$

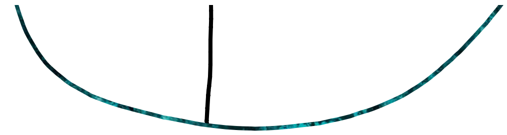
$p_j(x_1, \dots x_n) = x_j \in R_j$.

$I_j := p_j(I)$.

- $I_j \trianglelefteq R_j$ (pt că p_j surjectiv)
- $I = I_1 \times I_2 \times \dots \times I_n$

"$\subset$" clar





„⊂" clar

„⊃" Fie $x=(x_1,x_2,\dots,x_n)$ cu $x_j\in I_j \Rightarrow x\in I$

$$x=(x_1,0,0,\dots,0)+(0,x_2,0,\dots,0)+\dots+(0,0,\dots,0,x_n)$$

$x_1\in I_1 = p_1(I) \Rightarrow \exists\ (x_1,?,?,\dots,?)\in I$ (ai $p_1(x_1,?,\dots,?)=x_1$)

$$\underbrace{(1_{R_1},0_{R_2},0_{R_3},\dots,0_{R_n})}_{\in R_1\times R_2\times\dots\times R_n}\cdot\underbrace{(x_1,?,?,\dots,?)}_{\in I \ \Rightarrow}=\underbrace{(x_1,0_{R_2},0_{R_3},\dots,0_{R_n})}_{\in I}$$

2. R, $\underline{\underline{I,J\trianglelefteq R}}$.

$I:J=\{r\in R \mid rJ\subset I\}$.

$rj\in I,\ \forall j\in J$

$rJ=\{rj \mid j\in J\}$

a) $I:J\trianglelefteq R$:

- $(I:J,+)$ subgrup: Fie $r_1,r_2\in I:J$. $(r_1-r_2)J\subset I$ (pt că I e subgrup)
- $\forall a\in R,\ \forall x\in I:J \Rightarrow ax\in I:J$

$a\underbrace{xJ}\subset I$

$xJ\subset I$ (pt că $x\in I:J$)

$\subset I$ (pt că I ideal)

$I:J=\{r\in R\mid rJ\subset I\}$

$\subset I$ (pt că I ideal)

b) $\underline{I : \mathcal{Y} \supseteq I}$

$x\mathcal{Y} \subset I$ dacă $x \in I$

($xy \in I$ dacă $x \in I$)
$\forall y \in \mathcal{Y}$

c) $\mathcal{Y} \subset I \iff I : \mathcal{Y} = R$

Dem „$\Rightarrow$" $\mathcal{Y} \subset I$

$\Rightarrow x\mathcal{Y} \subset xI \subset I, \forall x \in R$

„$\Leftarrow$" $1 \in I : \mathcal{Y} \Rightarrow 1\mathcal{Y} = \mathcal{Y} \subset I.$

Def – Un ideal $I \trianglelefteq R$ s.n. principal dacă e generat de un singur element.

– Dacă toate idealele unui inel sunt principale, acel inel s.n. inel principal

Ex $\mathbb{Z}$ e inel principal: $n\mathbb{Z}$ toate idealele sale.

$\mathbb{Z}_n$ e inel principal: $m\mathbb{Z}_n$

De fapt, **Prop** Dacă R e inel principal $\Rightarrow R/I$ e inel principal, $\forall I \trianglelefteq R$ (Th de corespondență a idealelor)

Ex $R = \mathbb{Z}[X]$, $I = (2, X)$.

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$(2, X) = \{ P \in \mathbb{Z}[X] \mid P$ are termen liber par $\}$

$P = 2f + Xg$

Pp că $\boxed{I = (2, X) = (f) \Rightarrow \begin{matrix} f \mid 2 \\ f \mid X \end{matrix}}$ $\xRightarrow{\mathbb{Z} \text{ domeniu}}$ $\deg(f) = 0$

$\Rightarrow f = c \in \mathbb{Z}$ și în plus $c \mid 2 \Rightarrow c = 1$ sau $c = 2$

$c = 1 \Rightarrow (2, X) = (1) = \mathbb{Z}[X]$ ✗

$c = 2 \Rightarrow 2 \mid X$ ✗

$\left(X = (0, 1, 0, 0, \ldots) \right)$

Exemplu de inel și ideal care nu e finit generat

În $\mathbb{Z}[[X]]$, $I = (p_i X^i)$

$p_1, p_2, \ldots$ șir de nr prime

~~$J \subset I$ și I finit generat $\Rightarrow J$ finit generat~~

Ex Fie R un inel. U.A.S.E.:

Exc Fie R un inel. U.A.S.E.:

$R = (1)$

(i) Orice ideal al lui R e finit generat.

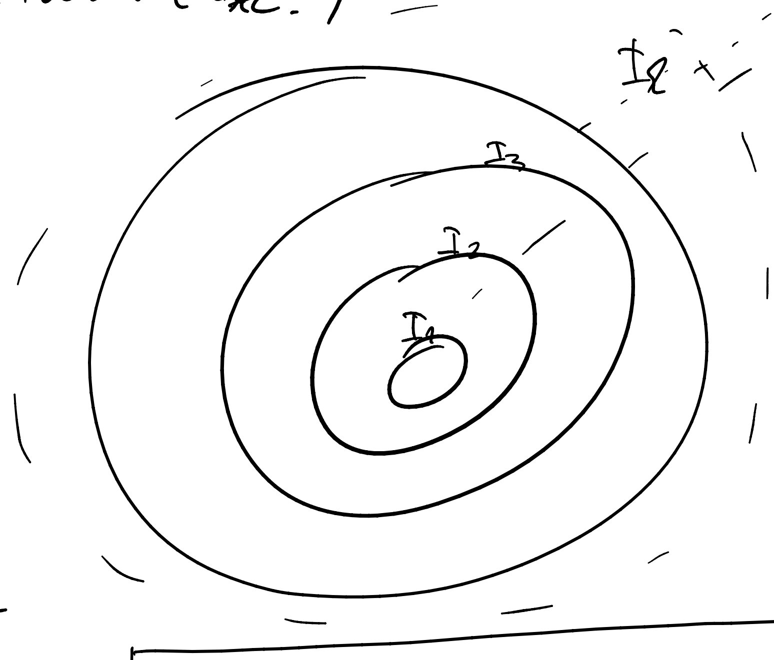
(ii) Orice șir ascendent de ideale este staționar:

$$I_1 \subseteq I_2 \subseteq I_3 \subseteq \dots \subseteq I_\ell \subseteq \dots$$

$$\Rightarrow \exists\, m \text{ a.î. } I_m = I_{m+1} = I_{m+2} = \dots$$

Dem (i) $\Rightarrow$ (ii) Fie că am $I_1 \subseteq I_2 \subseteq \dots \subseteq I_\ell \subseteq \dots$

$$I = \bigcup_{\ell \geq 1} I_\ell \leftarrow \text{e ideal (Exc!)}$$



$\Rightarrow I = (x_1, \dots, x_n)$.

$x_1, x_2, \dots, x_n \in I = \bigcup_{\ell \geq 1} I_\ell$

$x_1 \in I_{i_1} \Rightarrow x_1 \in I_m, \forall m \geq i_1$

$x_2 \in I_{i_2} \Rightarrow x_2 \in I_m, \forall m \geq i_2$

$\vdots$

$\Rightarrow x_1, x_2, \dots, x_n \in I_{\max(i_1, i_2, \dots, i_n) = i}$

$\Rightarrow \boxed{I = I_i.}$

$R[X_1, X_2, \dots, X_n, \dots]$

$(X_1) \subset (X_1, X_2) \subset (X_1, X_2, X_3) \subset \dots$

(ii) $\Rightarrow$ (i) Fie I un ideal. Pp că I nu e finit generat

Alegem $x_1 \in I \Rightarrow I \neq (x_1) \Rightarrow \exists\, x_2 \in I \setminus (x_1)$

$I \neq (x_1, x_2) \Rightarrow \exists\, x_3 \in I \setminus (x_1, x_2)$ — — —

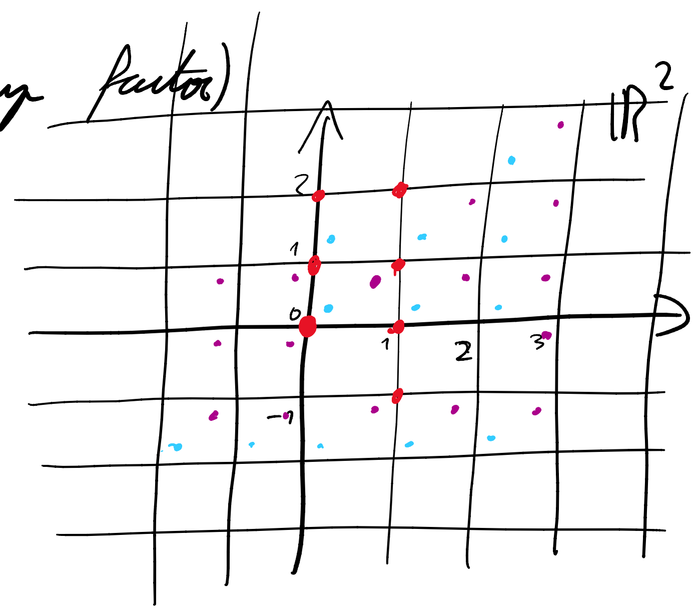
Construim un șir $x_1, x_2, \ldots, x_n, \ldots$ a.î.

$$(x_1) \subsetneq (x_1, x_2) \subsetneq (x_1, x_2, x_3) \subsetneq \ldots \subsetneq (x_1 \ldots x_n) \subsetneq \ldots$$

șir care nu e staționar ↯

Inelul factor

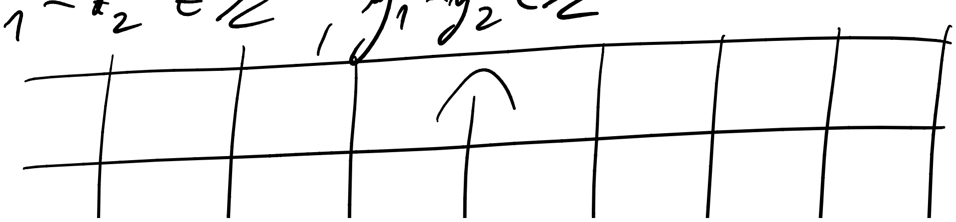
$\mathbb{R}\times\mathbb{R} \,/\, \mathbb{Z}\times\mathbb{Z} = ?$ (ca grup factor)

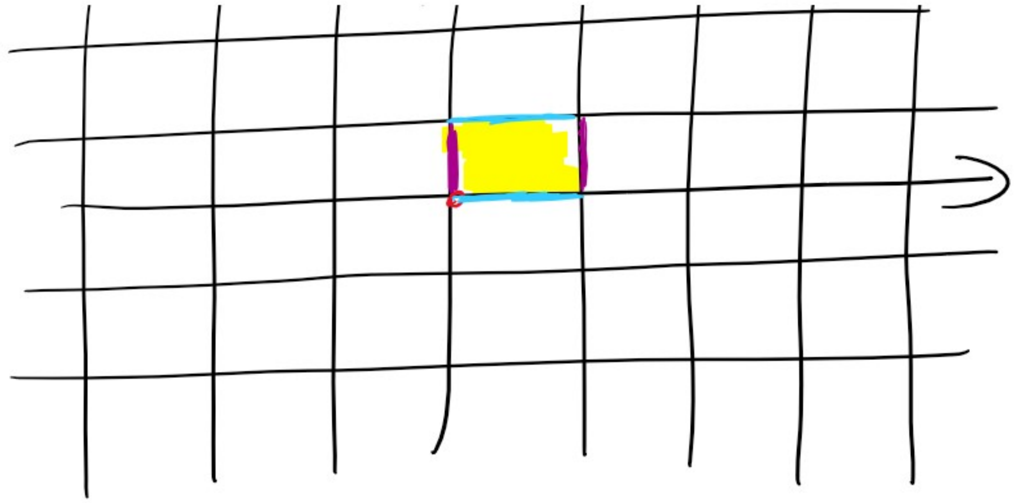


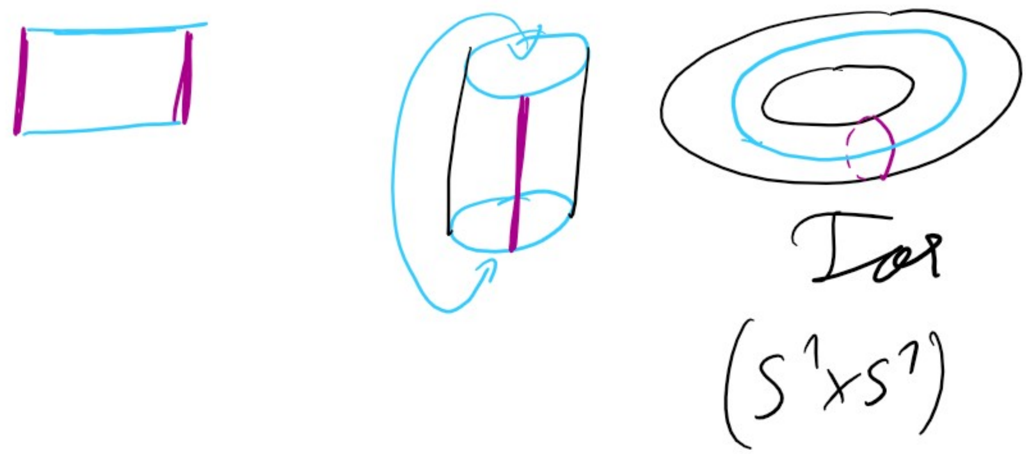
$(x_1, y_1) \sim (x_2, y_2)$

$\Leftrightarrow (x_1, y_1) - (x_2, y_2) \in \mathbb{Z}\times\mathbb{Z}$

$\Leftrightarrow x_1 - x_2 \in \mathbb{Z},\ y_1 - y_2 \in \mathbb{Z}$







Def $I \trianglelefteq R$, $R/I = \{\hat{x} \mid x \in R\}$, $\hat{x} = \hat{y} \iff x - y \in I$

R/I are structură de inel: $\hat{x} + \hat{y} = \widehat{x+y}$

$$\hat{x} \cdot \hat{y} = \widehat{x \cdot y}$$

De ex., înmulțirea e bine definită:

Fie $\hat{x} = \hat{x'}$, $\hat{y} = \hat{y'}$. Vreau $\widehat{xy} = \widehat{x'y'}$

$\Leftrightarrow \exists\, a \in I$ a.î. $x - x' = a \Leftrightarrow x = x' + a$

$\exists\, b \in I$ a.î. $y - y' = b \Leftrightarrow y = y' + b$

$$xy = (x'+a)(y'+b) = x'y' + \overbrace{\underset{I}{\underset{\cap}{x'b}} + \underset{I}{\underset{\cap}{ay'}} + \underset{I}{\underset{\cap}{ab}}}^{\in I} \Rightarrow \widehat{xy} = \widehat{x'y'}$$

Teorema fundamentală de izomorfism (la inele)

Fie $f : R \to S$ morfism de inele.

Atunci $\boxed{R/\mathrm{Ker} f \simeq \mathrm{Im} f}$.

Exc a) $R[X]/(X-a)$, pt R inel, $a \in R$

b) $\mathbb{Z}[X]/(n)$.

Dem a) $\boxed{R[X]/(X-a) \simeq R}$

Fie $\varphi : R[X] \to R$, $\varphi(P) = P(a)$.

Fie $\varphi: R[X] \longrightarrow R$, $\varphi(P) = P(a)$.

- φ e morfism de inele — evident.

  $\hookrightarrow$
  - $\varphi(PQ) = \varphi(P)\cdot\varphi(Q) \Leftrightarrow (P\cdot Q)(a) = P(a)\cdot Q(a)$
  - $\varphi(P+Q) = \varphi(P) + \varphi(Q) \Leftrightarrow (P+Q)(a) = P(a) + Q(a)$
  - $\varphi(1) = 1 \Leftrightarrow 1(a) = 1$.

- φ surjectivă: $\forall r \in R$, alegem $P = r\,(\cancel{a}+1-\cancel{a}) = r \in$ poli de grad 0

  sau $P = X - a + r$.

- $\underline{\operatorname{Ker} \varphi = (X-a)}$

"$\supseteq$" evident: Fie $P \in (X-a) \Rightarrow P = (X-a)\cdot Q$, pt un $Q \in R[X]$

$\Rightarrow P(a) = (a-a)Q(a) = 0$

"$\subseteq$" Dacă $P \in R[X]$ cu $P(a) = 0$.

Teoremă Bezout $\Rightarrow (X-a) \mid P$.

$P = c_n X^n + c_{n-1}X^{n-1} + \ldots + c_1 X + c_0$

$P(a) = 0 \Leftrightarrow c_n a^n + c_{n-1} a^{n-1} + \ldots + c_1 a + c_0 = 0$

$$P = c_n X^n + \dots + c_1 X + c_0 - c_n a^n - \dots - c_1 a - c_0$$

$$= c_n (X^n - a^n) + c_{n-1}(X^{n-1} - a^{n-1}) + \dots + c_1 (X - a)$$

$$= (X-a)\left(c_n (X^{n-1} + aX^{n-2} + \dots + a^{n-1}) + c_{n-1}(\dots) + \dots + c_1 \right)$$

$$\xRightarrow{TFI} \quad R[X] / (X-a) \simeq R.$$

b) $\mathbb{Z}[X] / (n) \simeq \mathbb{Z}_n [X].$

Fie $\varphi : \mathbb{Z}[X] \to \mathbb{Z}_n[X]$

$$\varphi(a_n X^n + a_{n-1} X^{n-1} + \dots + a_1 X + a_0) = \hat{a}_n X^n + \hat{a}_{n-1} X^{n-1} + \dots + \hat{a}_1 X + \hat{a}_0 \in \mathbb{Z}_n[X]$$

• φ e morfism de inele

• φ e surjectiv: $\hat{5} X^6 - \hat{7} X^3 + \hat{2} X + \hat{1} \in \mathbb{Z}_{11}[X]$

$$\uparrow \varphi$$

$$5X^6 - 7X^3 + 2X + 1 \in \mathbb{Z}[X]$$

• $\ker \varphi = \{ P \in \mathbb{Z}[X] \mid$ toți coef sunt divizibili cu $n\}$

$= \{ P \in \mathbb{Z}[X] \mid P = nQ,\ Q \in \mathbb{Z}[X] \}$

$= (n)$.

$$\Rightarrow \mathbb{Z}[X]/(n) \simeq \mathbb{Z}_n[X].$$

<u>Ex</u> Dem că $\mathbb{Z}[X]/(X^2)$ nu e domeniu de integritate.

$$\hat{X} \cdot \hat{X} = \hat{X}^2 = \hat{0} \qquad \square$$

$$\boxed{\frac{R/I}{J/I}} \simeq R/J$$

$$\mathbb{Z}[X]/(2,X) \simeq \frac{\mathbb{Z}[X]/(2)}{(2,X)/(2)} = \frac{\mathbb{Z}_2[X]}{(X)} \simeq \mathbb{Z}_2$$

<u>Recap</u> $J/I \trianglelefteq R/I$, $J \trianglelefteq R$, $J \supset I$

$$J/I \overset{\text{not}}{=} \{ \hat{x} \mid x \in J \}$$

$$m\mathbb{Z}/n\mathbb{Z} = m\mathbb{Z}_n$$

$$\mathbb{Z}[X]/(X^2-2) \simeq \mathbb{Z}[\sqrt{2}]$$

$$\mathbb{Z}[X]/(X^2+1) \simeq \mathbb{Z}[i]$$

$$\mathbb{Z}[X]/(2X-1) \simeq \mathbb{Z}[\tfrac{1}{2}]$$

$$\mathbb{R}[X]/(X^2+1) \simeq \mathbb{C}$$

<u>Obs fundamentală</u> Dacă $P \in R[X]$, chiar și dacă P nu are rădăcini în R, are rădăcini în $R[X]/(P)$.

<u>Explicație</u>

$$P = a_n X^n + a_{n-1}X^{n-1} + \dots + aX + a_0.$$

Au sens, în $R[X]/(P)$: $\hat{a}_n, \hat{a}_{n-1}, \dots \hat{a}_0 \in R[X]/(P)$,

deci are sens polinomul $\tilde{P} = \hat{a}_n Y^n + \hat{a}_{n-1} Y^{n-1} + \dots + \hat{a}_1 Y + \hat{a}_0$
$$\in \left(R[X]/(P)\right)[Y]$$

$\tilde{P}$ are rădăcini în $R[X]$, anume $\hat{X}$.

$\tilde{P}$ are o rădăcină în $R[X]/(P)$, anume $\underbrace{\hat{X}}$.

$$\mathbb{Z}[X]/(X^2+X+1) \simeq \{a+b\varepsilon \mid a,b\in\mathbb{Z}\},\quad \varepsilon^3=1,\ \varepsilon\neq 1.$$

$$\mathbb{Z}[i]/(7+1\cdot i) \simeq \mathbb{Z}_{\underset{7^2+1^2}{\overset{50}{\|}}}.$$

Ex $\mathbb{Z}[i]/(7+i)$ nu e domeniu: $\underset{\neq \hat{0}}{\hat{2}} \cdot \underset{\neq\hat{0}}{\widehat{25}} = \widehat{(7+i)}\widehat{(7-i)} = \hat{0}$