SEMINAR VI - 9. 11. 2023

Subspații vectoriale. Aplicații liniare

! $(\mathbb{R}^3, +, \cdot)_{/\mathbb{R}}$ $S = \{(1,2,3), (-1,1,5)\}$

$S' = \{(1,5,11), (2,1,-2), (3,6,9)\}$

a) $<S> = <S'> \overset{Not.}{=} V'$

b) Să se descrie V' printr-un sistem de ecuații liniare

c) $V'' = ?$ a.î. $\mathbb{R}^3 = V' \oplus V''$

$rg\begin{pmatrix} 1 & -1 \\ 2 & 1 \\ 3 & 5 \end{pmatrix} = 2 = \text{maxim} \Rightarrow S$ este SLi

$\det\begin{pmatrix} 1 & 2 & 3 \\ 5 & 1 & 6 \\ 11 & -2 & 9 \end{pmatrix} = 0$ $(C_3 = C_1 + C_2)$ $S'' = S \setminus (3,6,9)$ este SLi maximal

$\exists\, a, b \in \mathbb{R}$ a.î. :

$(1,2,3) = a(1,5,11) + b(2,1,-2) \Rightarrow (1,2,3) \in <S'>$

$$* \begin{cases} a + 2b = 1 \\ 5a + b = 2 \\ 11a - 2b = 3 \end{cases} \qquad \left(\begin{array}{cc|c} 1 & 2 & 1 \\ 5 & 1 & 2 \\ 11 & -2 & 3 \end{array}\right)$$

$*$ este sist. compatibil $\Leftrightarrow \Delta c = 0$

$$\Delta c = \begin{vmatrix} 1 & 2 & 1 \\ 5 & 1 & 2 \\ 11 & -2 & 3 \end{vmatrix} = 3 - 10 + 44 - 11 + 4 - 30 = 0$$

$\exists\ c, d \in \mathbb{R}$ a.î. $(-1, 1, 5)$

$(-1, 1, 5) = c(1, 5, 11) + d(2, 1, -2)$

$$** \begin{cases} c + 2d = -1 \\ 5c + d = 1 \\ 11c - 2d = 5 \end{cases}$$

** este sist. compatibil $\Leftrightarrow \Delta c = 0$

$$\det ** = \begin{vmatrix} 1 & 2 & -1 \\ 5 & 1 & 1 \\ 11 & -2 & 5 \end{vmatrix} = 5 + 10 + 22 + 11 + 2 - 50 = 0$$

$$\left.\begin{array}{l} \langle S \rangle \subset \langle S' \rangle \\ \dim \langle S \rangle = \dim \langle S' \rangle = 2 \end{array}\right\} \Rightarrow \langle S \rangle = \langle S' \rangle = \langle S'' \rangle$$

b) $V' = \langle (1, 2, 3), (-1, 1, 5) \rangle = \{ a(1, 2, 3) + b(-1, 1, 5) \mid a, b \in \mathbb{R} \}$

$\forall\ x = (x_1, x_2, x_3) \in V'\ \exists\ a, b$ a.î.:

$$*** \begin{cases} a - b = x_1 \\ 2a + b = x_2 \\ 3a + 5b = x_3 \end{cases}$$

*** este S.C. $\Leftrightarrow \Delta c = 0$

$$\Delta c = \begin{vmatrix} 1 & -1 & x_1 \\ 2 & 1 & x_2 \\ 3 & 5 & x_3 \end{vmatrix} = x_3 + 10x_1 - 3x_2 - 3x_1 - 5x_2 + 2x_3 = 7x_1 - 8x_2 + 3x_3 = 0$$

$$V' = \{ x \in \mathbb{R} \mid 7x_1 - 8x_2 + 3x_3 = 0 \}$$

c) $V' = \langle S \rangle$ extindem S la un reper în $\mathbb{R}^2$

$$\operatorname{rg} \left(\begin{array}{cc|c} 1 & -1 & 1 \\ 2 & 1 & 0 \\ 3 & 5 & 0 \end{array} \right) = 3$$

$V'' = \langle \{(1,0,0)\} \rangle$

Matricea asociată unei aplicații liniare

$f : V_1 \to V_2$ - aplicație liniară

$R_1 = \{ e_1, \dots, e_n \} \to R_2 \{ \bar{e}_1, \dots, \bar{e}_m \}$

reper în V_1 reper în V_2

$$f(e_i) = \sum_{j=1}^{m} a_j \bar{e}_j, \ \forall i = \overline{1, n}, \ A \in \mathcal{M}_{m,n}(\mathbb{K})$$

1. $f(x) = y \Leftrightarrow AX = Y$

$f : \mathbb{R}^2 \to \mathbb{R}^2$

$f(x_1, x_2) = (x_1 + x_2, -x_2)$

? $f \in Aut(\mathbb{R}^2)$

$R_0 = \{ e_1 = (1,0), e_2 = (0,1) \}$

$A = [f]_{R_0, R_0}$

Metoda 1:

$f(e_1) = f(1,0) = (1,0) = 1 \cdot e_1 + 0 \cdot e_2$

$$A = \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix}$$

$f(e_2) = f(0,1) = (1,-1) = e_1 - e_2$

Metoda 2:

$$\underbrace{\begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix}}_{A} \cdot \underbrace{\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}}_{X} = \underbrace{\begin{pmatrix} x_1 + x_2 \\ -x_2 \end{pmatrix}}_{Y} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$$

$f(x) = y$

f inj $\Leftrightarrow$ $Ker(f) = \{0_{\mathbb{R}^2}\}$

$Ker(f) = \{X \in \mathbb{R}^2 \mid f(X) = \{0_{\mathbb{R}^2}\}\}$ $\quad AX = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ (*) $\quad \det A = -1$

$A = \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix}$ $\quad$ (*) SLO $\Rightarrow Ker = \{0_{\mathbb{R}^2}\}$

$\dim(Ker f) = 2 - rg\, A = 0 \Rightarrow f.$ inj

Teorema dimensiunii

$\left.\begin{array}{l} \dim \mathbb{R}^2 = \dim Ker + \dim Im f = 0 + \dim Im f \\ Im f \subset \mathbb{R}^2 \end{array}\right\} \Rightarrow$

$\Rightarrow Im f = \mathbb{R}^2 \Rightarrow f.$ surj

$\Rightarrow f$ bijectiv + f liniară $\Rightarrow$ f automorfism

3. $f : \mathbb{R}^2 \to \mathbb{R}^3$, $f(x) = (3x_1 - 2x_2, 2x_1 - x_2, -x_1 + x_2)$

a) f liniară, $A = [f]_{R_0, R_0'}$

b) f - inj

c) $Im f$

$$\underbrace{\begin{pmatrix} 3 & -2 \\ 2 & -1 \\ -1 & 1 \end{pmatrix}}_{A \in \mathcal{M}_{3,2}(\mathbb{R})} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 3x_1 - 2x_2 \\ 2x_1 - x_2 \\ -x_1 + x_2 \end{pmatrix} = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}$$

$f(x) = y \in \mathbb{R}^3 \Leftrightarrow AX = Y \Rightarrow f.$ liniară

b) $Ker f = \{x \in \mathbb{R}^2 \mid f(x) = 0_{\mathbb{R}^3}\} = S(A)$

$$AX = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

rang $A = 2$

$\dim \mathrm{Ker} f = 0 \Rightarrow \mathrm{Ker} f = \{0_{\mathbb{R}^2}\} \Rightarrow$

$\Rightarrow f$ inj $\Leftrightarrow$ rg $A = \max = \dim V_1$

$f: V_1 \to V_2$

c) Teorema dimensiunii

$\dim \mathbb{R}^2 = \dim \mathrm{Ker} f + \dim \mathrm{Im} f \Rightarrow \dim \mathrm{Im} f = 2$

$R_0 = \{e_1, e_2\}$ - reper în $\mathbb{R}^2$

$f(e_1) = f(1,0) = (3, 2, -1)$

$f(e_2) = f(0,1) = (-2, -1, 1)$

$\{f(1,0), f(0,1)\}$ - reper în $\mathrm{Im} f$

<u>sau</u>

$\mathrm{Im} f = \{y \in \mathbb{R}^3 \mid \exists x \in \mathbb{R}^2, f(x) = y\}$

$$A = \left(\begin{array}{cc|c} 3 & -2 & y_1 \\ 2 & -1 & y_2 \\ 1 & 1 & y_3 \end{array}\right)$$

$\Delta c = 0$

$$\begin{vmatrix} 3 & -2 & y_1 \\ 2 & -1 & y_2 \\ 1 & 1 & y_3 \end{vmatrix} = y_1(-1)^{1+3} \begin{vmatrix} 2 & -1 \\ -1 & 1 \end{vmatrix} + y_2(-1)^{2+3} \begin{vmatrix} 3 & -2 \\ -1 & 1 \end{vmatrix} + y_3(-1)^{3+3} \begin{vmatrix} 3 & -2 \\ 2 & -1 \end{vmatrix}$$

$= y_1 - y_2 + y_3 = 0$

$\mathrm{Im} f = \{y \in \mathbb{R}^3 \mid y_1 - y_2 + y_3 = 0\}$

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$f: \mathbb{R}_3[X] \to \mathbb{R}_2[X] \qquad f(P) = P'$

a) $[f]_{R_0, R_0'} = A = ?$ $\qquad R_0, R_0'$ repere canonice în $\mathbb{R}_3[X]$, respectiv $\mathbb{R}_2[X]$

b) dim Ker f, dim Im f

a) $P = a_0 + a_1 X + a_2 X^2 + a_3 X^3 \equiv (a_0, a_1, a_2, a_3)$

$P' = a_1 + 2a_2 X + 3a_3 X^2 = (a_1, 2a_2, 3a_3)$

$R_0 = \{1, X, X^2, X^3\} \to R_0' = \{1, X, X^2\}$

reper în $\mathbb{R}_3[X]$ $\qquad$ reper în $\mathbb{R}_2[X]$

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix}$$

$f(1) = 0$

$f(X) = 1$

$f(X^2) = 2X = 0 \cdot 1 + 2 \cdot X + 0 \cdot X^2$

$f(X^3) = 3X^2 = 0 \cdot 1 + 0 \cdot X + 3 \cdot X^2$

$f(a) = (a_1, 2a_2, 3a_3)$

$\parallel$

$(a_0, a_1, a_2, a_3) \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{pmatrix} = \begin{pmatrix} a_1 \\ 2a_2 \\ 3a_3 \end{pmatrix}$

b) $\text{Ker} f = \{P \in \mathbb{R}_3[X] \mid f(P) = 0\} = \mathbb{R}$

$\Rightarrow \dim \text{Ker} f = 1$

Teorema dimensiunii

$\dim \mathbb{R}_3[X] = 4 = \dim \text{Ker} f + \dim \text{Im} f \Rightarrow \dim \text{Im} f = 3$

• $f \in End(V)$ a.î. $f^2 = 0$ $(f \circ f = 0)$

Să se arate că $g = id_V + f \in Aut(V)$

R reper în V

$A \overset{not.}{=} [f]_{R,R}$

$A^2 = O_n \overset{?}{\Rightarrow} I_n + A \in GL(n, \mathbb{R})$

$I_n^2 - A^2 = (I_n - A)(I_n + A) = (I_n + A)(I_n - A) = I_n$

$(I_n + A)^{-1} = I_n - A$