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Seminarul 7

Aplicatii liniare. Vectori proprii

Document (S6)

(5) $f:\mathbb{R}^3\to\mathbb{R}^3$, $f(x_1,x_2,x_3)=(x_1+2x_2+x_3, -x_1-2x_2-x_3, x_1+x_2+x_3)$

a) $[f]_{R_0,R_0}=A=?$

b) $\dim \ker f$, $\dim \operatorname{Im} f=?$

c) $V'=\left\{(x_1,x_2,x_3)\in\mathbb{R}^3 \;\middle|\; \begin{cases} x_1-x_2+x_3=0 \\ x_1+2x_2-x_3=0 \end{cases}\right\}$

$f(V')=?$

Obs: $f_{|V'}:V'\to\mathbb{R}^3$

Teorema dimensiunii

$\dim V'=\dim\ker(f_{|V'})+\dim[\operatorname{Im} f(V')] \Rightarrow \dim f(V')\le \dim V'$

c) $f(V')=?$

$$A=\left(\begin{array}{cc|c} 1 & -1 & 1 \\ 1 & 2 & -1 \end{array}\right)\left|\begin{array}{c} 0 \\ 0 \end{array}\right.$$

$x_3=\alpha$

$$\begin{cases} x_1-x_2=-\alpha \\ x_1+2x_2=+\alpha \end{cases} \quad (-)$$

$-3x_2=-2\alpha \Rightarrow x_2=\frac{2}{3}\alpha$

$x_1=-\alpha+\frac{2}{3}\alpha=-\frac{1}{3}\alpha$

$V'=\left\{\left(-\frac{1}{3}\alpha, \frac{2}{3}\alpha, \alpha\right) \;\middle|\; \alpha\in\mathbb{R}\right\}$

$$\left(-\frac{1}{3}\alpha, \frac{2}{3}\alpha, \alpha\right) = \frac{\alpha}{3}(-1, 2, 3)$$

$V' = <\{(-1,2,3)\}>$

$\dim V' = 1 \Rightarrow$ dreapta care trece prin origine

$f(V') =$ (înlocuim cu sol obținute după rezolvarea sistemului) $=$

$= f((-1,2,3)) = (-1+4+3, 1-4-3, -1+5) = (6,-6,4) =$

$= 2(3,-3,2)$

$f(V') = <\{(3,-3,2)\}>$

⑥ $(\mathbb{R}^2, +, \cdot)$ $\quad R_0 = \{e_1 = (1,0), e_2 = (0,1)\} \xrightarrow{C} R' = \{e_1' = (e_1 - e_2),$
$e_2' = (e_1 + 2e_2)\}$

$((\mathbb{R}^2)^*, +, \cdot)_{/\mathbb{R}}$

$(\mathbb{R}^2)^* = \{f : \mathbb{R}^2 \to \mathbb{R} \mid f \text{ liniar}\}$ (spațiul dual lui V)
$V = \mathbb{R}^2$

$R^* = \{e_1^*, e_2^*\} \xrightarrow{C} R' = \{e_1'^*, e_2'^*\}$

$e_i^* : \mathbb{R}^2 \to \mathbb{R}$

$e_i^*(e_j) = \delta_{ij} = \begin{cases} 1, & i = j \\ 0, & i \neq j \end{cases}$ $\qquad e_i'^*(e_j') = \delta_{ij}$

legătura dintre C și D, C, D = ?

$e_1' = e_1 - e_2$
$e_2' = e_1 + 2e_2$
$\qquad C = \begin{pmatrix} 1 & 1 \\ -1 & 2 \end{pmatrix}$

$e_1'^* = a e_1^* + b e_2^*$

$1 = \underline{e_1'^*}(\underline{\underline{e_1'}}) = (a e_1^* + b e_2^*)(e_1 - e_2) = a \overbrace{e_1^*(e_1)}^{1} + b \overbrace{e_2^*(e_1)}^{0} -$

$- a \underbrace{e_1^*(e_2)}_{0} - b \underbrace{e_2^*(e_2)}_{1} = \boxed{a - b = 1}$

$f(ax+b) = af(x) + bf(y)$

$(f+g)(e_1-e_2)=(f+g)e_1-(f+g)e_2$

$0=e_1'^*(e_2')=(ae_1^*+be_2^*)(e_1+2e_2)=$

$=ae_1^*(e_1)+2ae_1^*(e_2)+be_2^*(e_1)+be_2^*2(e_2)=$

$=\boxed{a+2b=0}$

$$\begin{cases} a-b=1 \\ a+2b=0 \end{cases} (-)$$

$-3b=1 \Rightarrow b=-\frac{1}{3}$

$a=1-\frac{1}{3}=\frac{2}{3}.$

$a=\frac{2}{3};\ b=-\frac{1}{3}$

a, b pt matricea D repr prima coloană

$$D=\begin{pmatrix} \frac{2}{3} & \frac{1}{3} \\ -\frac{1}{3} & \frac{1}{3} \end{pmatrix}$$

$e_2'^*=ce_1^*+de_2^*$

$0=e_2'^*(e_1')=(c\,e_1^*+de_2^*)=c\underbrace{e_1^*(e_1)}_{1}+d\underbrace{e_2^*(e_1)}_{0}-c\underbrace{e_1^*(e_2)}_{0}-$

$-d\underbrace{e_2^*(e_2)}_{1}=c-d$

$1=e_2'^*(e_2')=(ce_1^*+de_2^*)(e_1+2e_2)=c\underbrace{e_1^*(e_1)}_{1}+2c\underbrace{e_1^*(e_2)}_{0}+$

$+d\underbrace{e_2^*(e_1)}_{0}+d\underbrace{e_2^*(e_2)}_{1}=c+2d$

$d=\frac{1}{3}$

$$\begin{cases} c-d=0 \\ c+2d=1 \end{cases} \Rightarrow c=d=\frac{1}{3}$$

$\det C = 3$

$$C^t = \begin{pmatrix} 1 & -1 \\ 1 & 2 \end{pmatrix}$$

$$C^* = \begin{pmatrix} 2 & -1 \\ +1 & 2 \end{pmatrix}$$

$$C^{-1} = \frac{1}{3}\begin{pmatrix} 2 & -1 \\ 1 & 2 \end{pmatrix} = \begin{pmatrix} \frac{2}{3} & -\frac{1}{3} \\ \frac{1}{3} & \frac{2}{3} \end{pmatrix}$$

$$D = (C^{-1})^t = \begin{pmatrix} \frac{2}{3} & \frac{1}{3} \\ -\frac{1}{3} & \frac{2}{3} \end{pmatrix}$$

9) $f: \mathbb{R}^3 \to \mathbb{R}^3$, liniară, $f(v_i) = u_i$, $i = \overline{1,3}$

$R = \{v_1 = (-1,1,1), v_2 = (1,1,1), v_3 = (0,2,1)\}$

$R' = \{u_1 = 2v_1 + 3v_2 - v_3, u_2 = v_1 + 3v_2 + v_3, u_3 = v_3\}$

a) $[f]_{R,R'}$

b) $[f]_{R,R_0}$

c) $g = ?$

d) $\operatorname{Ker}(f), \operatorname{Im} f = ?$

$f(v_1) = u_1 = 2v_1 + 3v_2 - v_3$

$f(v_2) = u_2 = v_1 + 3v_2 + v_3$

$f(v_3) = u_3 = v_3$

$$\Rightarrow [f]_{R,R'} = \begin{pmatrix} 2 & 1 & 0 \\ 3 & 3 & 0 \\ -1 & 1 & 1 \end{pmatrix}$$

$R_0 = \{e_1 = (1,0,0),\ e_2 = (0,1,0),\ e_3 = (0,0,1)\}$

$$f(v_1) = f(-e_1+e_2+e_3) \overset{\text{liniară}}{=} -f(e_1)+f(e_2)+f(e_3) =$$
$$= 2(-1,1,1)+3(1,1,1)-(0,2,1) = (1,3,4) =$$
$$= e_1+3e_2+4e_3$$

$$\Rightarrow \boxed{-f(e_1)+f(e_2)+f(e_3) = e_1+3e_2+4e_3}$$

$$f(v_2) = u_2 \Rightarrow f(v_2) = f(e_1+e_2+e_3) = (-1,1,1)+3(1,1,1)+(0$$

$$\Rightarrow \boxed{f(e_1)+f(e_2)+f(e_3) = (2,6,5) = 2e_1+6e_2+5e_3}$$

$$f(v_3) = u_3 \Rightarrow f(v_3) = f(2e_2+e_3) = (0,2,1) \xrightarrow{\text{lin.}}$$

$$\Rightarrow \boxed{2f(e_2)+f(e_3) = 2e_2+e_3}$$

$$\begin{cases} -f(e_1)+f(e_2)+f(e_3) = e_1+3e_2+4e_3 & (1) \\ f(e_1)+f(e_2)+f(e_3) = 2e_1+6e_2+5e_3 & (2) \\ 2f(e_2)+f(e_3) = 2e_3+e_3 & (3) \ \mid(-1) \end{cases}$$

Vrem să exprimăm $f(e_i)$ în fct de e_i, $i=\overline{1,3}$

$(1)+(2)$ $\quad 2f(e_2)+2f(e_3) = 3e_1+9e_2+9e_3$

$(2)+(-1)\cdot(3) \Rightarrow -2f(e_2)-f(e_3) = -2e_2-e_3 \quad (+)$

$$\boxed{f(e_3) = 3e_1+7e_2+8e_3}$$

$$\Rightarrow f(e_2) = \frac{2e_2+e_3-3e_1-7e_2-8e_3}{2}$$

$$\boxed{f(e_2) = -\frac{3}{2}e_1 - \frac{5}{2}e_2 - \frac{7}{2}e_3}$$

(2), $f(e_1) = 2e_1 + 6e_2 + 5e_3 - 3e_1 - 7e_2 - 8e_3 + \frac{3}{2}e_1 + \frac{5}{2}e_2 + \frac{7}{2}e_3 =$

$= \frac{1}{2}e_1 + \frac{3}{2}e_2 + \frac{1}{2}e_3$

$$\rightarrow [f]_{R_0,R_0} = \begin{pmatrix} \frac{1}{2} & -\frac{3}{2} & 3 \\ \frac{3}{2} & -\frac{5}{2} & 7 \\ \frac{1}{3} & -\frac{7}{2} & 8 \end{pmatrix}$$

Obs:

$$\begin{tikzcd} R_0 = \{e_1, e_2, e_3\} \arrow[r, "A = {[f]_{R_0,R_0}}"] \arrow[d, "C"', "{[f]_{R,R'}}"] & R_0 = \{e_1, e_2, e_3\} \arrow[d, "D"] \\ R = \{v_1, v_2, v_3\} \arrow[r, "A' = {[f]_{R,R'}}"] & R' = \{w_1, w_2, w_3\} \end{tikzcd}$$

$v_1 = -e_1 + e_2 + e_3$, $v_2 = e_1 + e_2 + e_3$, $v_3 = 2e_2 + e_3$

$w_1 = 2v_1 + 3v_2 - v_3 = e_1 + 3e_2 + 4e_3$, $w_2 = v_1 + 3v_2 + v_3 = 2e_1 + 6e_2 + 5e_3$, $w_3 = v_3 = 2e_2 + e_3$

$$C = \begin{pmatrix} -1 & 1 & 0 \\ 1 & 1 & 2 \\ 1 & 1 & 1 \end{pmatrix}$$

$$D = \begin{pmatrix} 1 & 2 & 0 \\ 3 & 6 & 2 \\ 4 & 5 & 1 \end{pmatrix}$$

$A' = D^{-1} A C$

$\boxed{A = D A' C^{-1}}$

(13) $f: \mathbb{R}_1[X] \to \mathbb{R}_1[X]$ liniară

Să se afle expresia analitică pt f dacă:

a) $[f]_{R_0,R_0} = A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$

b) $[f]_{R,R} = A' = \begin{pmatrix} 1 & 2 \\ 0 & -1 \end{pmatrix}$ R este $\{x-1, 2x\}$

$R_0 = \{1, x\}$ $f(1) = 1 \cdot 1 + 3 \cdot x$

$f(x) = 2 \cdot 1 + 4 \cdot x$

$f(a+bx) = af(1) + bf(x) = a(1+3x) + b(2+4x) =$
$= (a+2b) + (3a+4b)x$

Obs: Dacă $f: \underset{\mathbb{R}^2}{\underset{\parallel}{\mathbb{R}_1[x]}} \to \mathbb{R}_1[x]$

$A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ $\begin{cases} f(x_1, x_2) = (x_1 + 2x_2, 3x_1 + 4x_2) \\ f(x_1 + x_2 X) = x_1 + 2x_2 + (3x_1 + 4x_2)X \end{cases}$

b) (I)

$$\begin{array}{ccc} R_0 & \xrightarrow{[f]_{R_0,R_0}} & R_0 \\ C \downarrow & & \downarrow C \\ R & \xrightarrow[{[f]_{R,R}}]{} & R \end{array}$$

$R_0 = \{1, x\}$

$R = \{\underbrace{-1+x}_{(-1)\cdot 1 + 1 \cdot x}, 2+2x\}$

$C = \begin{pmatrix} -1 & 2 \\ 1 & 2 \end{pmatrix}$

$A' = C^{-1} A C$

$A = C A' C^{-1}$ Analog cu a)

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$\underline{II}$ $f(x-1) = 1\cdot(x-1) + 0(2x-2)$

$f(x-1) = x-1$

$f(2x+2) = 2(x-1) + (-1)(2x+2) =$

$f(2x+2) = 2x-2-2x-2 = 4$

$$\begin{cases} -f(1) + f(x) = -1+x \quad |\cdot 2 \\ 2f(1) + 2f(x) = -4 \end{cases}$$
$(+)$

$4f(x) = -6+2x \Rightarrow f(x) = -\frac{3}{2} + \frac{1}{2}x$

$\Rightarrow f(1) = -\frac{3}{2} + \frac{1}{2}x + 1 - x = -\frac{1}{2} - \frac{1}{2}x = f(1)$

$f(a+bx) = af(1) + bf(x) = a\left(-\frac{1}{2} - \frac{1}{2}x\right) + b\left(-\frac{3}{2} + \frac{1}{2}x\right) =$

$= -\frac{a}{2} - \frac{3b}{2} - \left(\frac{a}{2} - \frac{b}{2}\right)x$

(14) $\boxed{\text{! Lucrare !}}$

$f: \mathbb{R}^3 \to \mathbb{R}^3, f(x) = (x_1 - 2x_2 + 5x_3, mx_1 + 3x_2 - x_3, x_2 - 3x_3)$

a) $m = ?$ f inj

b) $m = ?$ f surj

$[f]_{R_0, R_0} = \begin{pmatrix} & & \\ & & \\ & & \end{pmatrix}$

$$\begin{pmatrix} 1 & -2 & 5 \\ m & 3 & -1 \\ 0 & 1 & -3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} x_1 - 2x_2 + 5x_3 \\ mx_1 + 3x_2 - x_3 \\ x_2 - 3x_3 \end{pmatrix}$$

$$[f]_{R_0, R_0} = A$$

Obs: $f : V \to W$

a) f inj $\Leftrightarrow$ $\dim V = \operatorname{rg} A$

b) f surj $\Leftrightarrow$ $\dim W = \operatorname{rg} A$

$$\det A = \begin{vmatrix} 1 & -2 & 5 \\ m & 3 & 1 \\ 0 & 1 & -3 \end{vmatrix} \xrightarrow{C_3 + 3C_2} \begin{vmatrix} 1 & 2 & -1 \\ m & 3 & 8 \\ 0 & 1 & 0 \end{vmatrix} =$$

$$= (-1)^{3+2} \begin{vmatrix} 1 & -1 \\ m & 8 \end{vmatrix} = -(8+m) = -m-8$$

$m \neq -8 \Leftrightarrow \operatorname{rg} A = 3 = \dim \mathbb{R}^3 \Leftrightarrow f$ inj $\Leftrightarrow f$ surj