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3.10.2024.

Matrice. Determinanți. Sisteme
(seminar 1)

① a) $A = \begin{pmatrix} a & b & c \\ b & c & a \\ c & a & b \end{pmatrix}$

$\det A = ?$

sau $L_1' = L_1 + L_2 + L_3$

$$\det(A) = \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} \xrightarrow{L_3 + L_2 + L_1} \begin{vmatrix} a+b+c & a+b+c & a+b+c \\ b & c & a \\ c & a & b \end{vmatrix} =$$

$$= (a+b+c) \begin{vmatrix} 1 & 1 & 1 \\ b & c & a \\ c & a & b \end{vmatrix} \xrightarrow[C_2' = C_2 - C_1]{C_3' = C_3 - C_1} (a+b+c) \begin{vmatrix} 1 & 0 & 0 \\ b & c-b & a-b \\ c & a-c & b-c \end{vmatrix} =$$

$$= 1 \cdot (-1)^{1+1} (a+b+c) \begin{vmatrix} c-b & a-b \\ a-c & b-c \end{vmatrix} = (a+b+c)[(c-b)(b-c) - (a-b)(a-$$

$$\Rightarrow \Delta = (a+b+c)(2bc - b^2 - c^2 - a^2 + ac + ab - bc) =$$

$$= (a+b+c)(-a^2 - b^2 - c^2 + ab + ac + bc) = (a+b+c)(a^2+b^2+c^2+ab+ac+$$

$$= -\frac{1}{2}(a+b+c)[(a^2 - 2ab + b^2) + (a^2 - 2ac + c^2) + (b^2 - 2bc + c^2)] =$$

$$= -\frac{1}{2}(a+b+c)[(a-b)^2 + (a-c)^2 + (b-c)^2]$$

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(2) $A = \begin{pmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{pmatrix} \in \mathcal{M}_3(\mathbb{R})$

$\det(A) = V(a,b,c)$

$$\det(A) = \begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{vmatrix} \underset{=}{\overset{C_3-C_1}{C_2-C_1}} \begin{vmatrix} 1 & 0 & 0 \\ a & b-a & c-a \\ a^2 & b^2-a^2 & c^2-a^2 \end{vmatrix} =$$

$$= \begin{vmatrix} b-a & c-a \\ b^2-a^2 & c^2-a^2 \end{vmatrix} = (b-a)(c-a) \begin{vmatrix} 1 & 1 \\ b+a & c+a \end{vmatrix} =$$

$$= (b-a)(c-a)(c+a-b-a) = (b-a)(c-a)(c-b) = (c-b)(c-a)(b-a)$$

(determinant Vandermonde)

(3) $A = \begin{pmatrix} 1+a^2 & ba & ca \\ ba & 1+b^2 & cb \\ ca & bc & 1+c^2 \end{pmatrix}$

$\det(A^*) = ?$

$\det(\alpha \cdot A) = \alpha^n \cdot \det(A)$

$A^{-1} = \frac{1}{\det A} \cdot A^* \quad | \det$

$A \cdot A^{-1} = I_n$

$\det(A) \cdot \det(A^{-1}) = 1 \Rightarrow$

$\det(A^{-1}) = \left(\frac{1}{\det A}\right)^n \cdot \det(A^*)$

$\Rightarrow \det(A^{-1}) = \frac{1}{\det(A)}$

$\Rightarrow \boxed{\det(A^*) = [\det(A)]^{n-1}, \ n \geq 2}$

$\det A^* = (\det A)^2$

$$\det(A) = \begin{vmatrix} 1+a^2 & ba & ca \\ ba & 1+b^2 & cb \\ ca & bc & 1+c^2 \end{vmatrix} = \begin{array}{ccc} C_1 \quad C_1' & C_2 \quad C_2' & C_3 \quad C_3' \\ \left| \begin{matrix} 1+a^2 \\ 0+ba \\ 0+ca \end{matrix} \right. & \begin{matrix} 0+ba \\ 1+b^2 \\ 0+bc \end{matrix} & \left. \begin{matrix} 0+ca \\ 0+cb \\ 1+c^2 \end{matrix} \right| \end{array} =$$

$$= |C_1 \; C_2 \; C_3| + |C_1 \; C_2 \; C_3'| + |C_1 \; C_2' \; C_3| +$$

$$+ \underbrace{\boxed{|C_1 \; C_2' \; C_3'|}}_{0} + |C_1' \; C_2 \; C_3| + \underbrace{\boxed{|C_1' \; C_2 \; C_3'|}}_{0} +$$

$$+ |C_1' \; C_2' \; C_3| + \underbrace{\boxed{|C_1' \; C_2' \; C_3'|}}_{0}$$

$$|C_1 + C_2 + C_3'| = \begin{vmatrix} 1 & 0 & ca \\ 0 & 1 & bc \\ 0 & 0 & c^2 \end{vmatrix} = c^2$$

$$|C_1 C_2' C_3| = \begin{vmatrix} 1 & ba & 0 \\ 0 & b^2 & 0 \\ 0 & bc & 1 \end{vmatrix} = b^2$$

$$|C_1' \; C_2 \; C_3| = \begin{vmatrix} a^2 & 0 & 0 \\ ab & 1 & 0 \\ ac & 0 & 1 \end{vmatrix} = a^2$$

$$|C_1 \; C_2 \; C_3| = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = 1$$

$\Rightarrow \det(A) = a^2 + b^2 + c^2 + 1$

$\det(A^*) = (\Delta A)^2 = (1 + a^2 + b^2 + c^2)^2$

4) $A = \begin{pmatrix} 2 & -1 & 3m+4 \\ 1 & m & 1 \\ -1 & -1 & 0 \end{pmatrix} \in \mathcal{M}_3(\mathbb{Z})$

a) Det m a.î. $A^{-1} \in \mathcal{M}_3(\mathbb{Z})$

b) Calc $A^{-1} = ?$

$A^{-1}, A \in \mathcal{M}_3(\mathbb{Z}) \Rightarrow \det(A) \in \mathbb{Z}; \det(A^{-1}) \in \mathbb{Z}$

$\det(A) \cdot \det(A^{-1}) \overset{def}{=} I_2 \Rightarrow \det(A^{-1}) = \dfrac{1}{\det(A)}$ $\Big| \Rightarrow$

$\Rightarrow \det(A) \in \mathcal{D}_1 = \{\pm 1\}$

$\det(A) = \begin{vmatrix} 2 & -1 & 3m+4 \\ 1 & m & 1 \\ -1 & -1 & 0 \end{vmatrix} \underline{\underline{-C_1+C_2}} \begin{vmatrix} 2 & -3 & 3m+4 \\ 1 & m-1 & 1 \\ -1 & 0 & 0 \end{vmatrix} =$

$= (-1) \cdot (-1)^{3+1} \begin{vmatrix} -3 & 3m+4 \\ m-1 & 1 \end{vmatrix} = (-1)\left[-3 - (m-1)(3m+4)\right]$

$= 3 + (m-1)(3m+4) = 3 + 3m^2 + 4m - 3m - 4 =$

$= 3 + 3m^2 + m - 4 = 3m^2 + m - 1$

I Dacă $\det(A) = -1 \Rightarrow 3m^2 + m - 1 = -1 \Rightarrow 3m^2 + m = 0 \Rightarrow$

$\Rightarrow m(3m+1) = 0 \Rightarrow m = 0 \in \mathbb{Z}$

$3m+1 = 0 \Rightarrow m = -\frac{1}{3} \notin \mathbb{Z}$ (F)

II Dacă $\det(A) = 1 \Rightarrow 3m^2 + m - 1 = 1 \Rightarrow 3m^2 + m - 2 = 0$

$\Delta = 1 - 4(-2) \cdot 3 = 25$

$m_{1,2} = \dfrac{-1 \pm 5}{6} \Rightarrow m_1 = -1$

$m_2 = \frac{2}{3} \notin \mathbb{Z}$ (F)

Obs.: $m_1 = -1$

$$m_1 m_2 = \frac{c}{a} = \frac{-2}{3} \Rightarrow m_2 = +\frac{2}{3}$$

În concluzie, $A^{-1} \in \mathcal{M}_3(\mathbb{Z}) \Leftrightarrow m \in \{-1, 0\}$

(10) $A = \begin{pmatrix} a & 1 & 2 \\ 1 & 1 & 1 \\ -1 & 1 & 1-a \end{pmatrix} \in \mathcal{M}_3(\mathbb{R})$

$\operatorname{rg} A = ?$ Discuție

$$\det(A) = \begin{vmatrix} a & 1 & 2 \\ 1 & 1 & 1 \\ -1 & 1 & 1-a \end{vmatrix} \overset{\substack{L_3 - L_1 \\ L_2 - L_1}}{=} \begin{vmatrix} a & 1 & 2 \\ 1-a & 0 & -1 \\ -1-a & 0 & -1-a \end{vmatrix} =$$

$$= (-1)^{1+2} \begin{vmatrix} 1-a & -1 \\ -1-a & -1-a \end{vmatrix} = (-1)(-1-a) \begin{vmatrix} 1-a & -1 \\ 1 & 1 \end{vmatrix} =$$

$$= (1+a)(1-a+1) = (1+a)(2-a)$$

I Dacă $(1+a)(2-a) \neq 0 \Leftrightarrow a \in \mathbb{R} \setminus \{-1, 2\} \Rightarrow \operatorname{rg}(A) = 3$

II Dacă $\Delta = 0 \Rightarrow a \in \{-1, 2\}$

a) $a = -1$

$$A = \begin{pmatrix} -1 & 1 & 2 \\ 1 & 1 & 1 \\ -1 & 1 & 2 \end{pmatrix}$$

$$\begin{vmatrix} -1 & 1 \\ 1 & 1 \end{vmatrix} = -1-1 = -2 \neq 0 \Rightarrow \operatorname{rg}(A) = 2.$$

b) $a=2 \Rightarrow A = \begin{pmatrix} 2 & 1 & 2 \\ 1 & 1 & 1 \\ -1 & 1 & -1 \end{pmatrix}$

$\begin{vmatrix} 2 & 1 \\ 1 & 1 \end{vmatrix} = 2-1 = 1 \neq 0 \Rightarrow rg(A)=2.$

11) $A = \begin{pmatrix} \boxed{1} & 2 & 3 & 1 \\ 2 & 0 & a & 1 \\ 0 & 1 & 3 & b \end{pmatrix} \in \mathcal{M}_{3,4}(\mathbb{R})$

$a, b = ?$ a.î. $rg\,A = 2$

$|1| \neq 0 \Rightarrow rg\,A \geq 1$

$\begin{vmatrix} 1 & 2 \\ 2 & 0 \end{vmatrix} = -4 \neq 0 \Rightarrow rg\,A \geq 2$

Din ip $\Rightarrow rg\,A = 2$

$\Rightarrow \Delta_1 = \begin{vmatrix} 1 & 2 & 3 \\ 2 & 0 & a \\ 0 & 1 & 3 \end{vmatrix} = 0 \Leftrightarrow \overset{L_2' = L_2 - 2L_1}{} \begin{vmatrix} 1 & 2 & 3 \\ 0 & -4 & a-6 \\ 0 & 1 & 3 \end{vmatrix} = 0 \Leftrightarrow$

$\Leftrightarrow \begin{vmatrix} -4 & a-6 \\ 1 & 3 \end{vmatrix} = 0 \Leftrightarrow -12 - a + 6 = 0 \Leftrightarrow -a - 6 = 0 \Leftrightarrow$

$\Leftrightarrow \boxed{a = -6}$

$\Delta_2 = \begin{vmatrix} 1 & 2 & 1 \\ 2 & 0 & 1 \\ 0 & 1 & b \end{vmatrix} = 0 \; (\underline{L_2' = L_2 - 2L_1}) \begin{vmatrix} 1 & 2 & 1 \\ 0 & -4 & -1 \\ 0 & 1 & b \end{vmatrix} = 0 \Rightarrow$

$$\Rightarrow \begin{vmatrix} -4 & -1 \\ 1 & b \end{vmatrix} = 0 \Leftrightarrow -4b+1=0 \Rightarrow \boxed{b = \frac{1}{4}}$$

12) $A \in \mathcal{M}_3(\mathbb{R}),\ A^{2023} - 2023A - I_3 = O_3$

a) $rg\, A = ?$

b) $rg\,(2023A + I_3) = ?$

a) $A^{2023} - 2023A = I_3 \Rightarrow A(A^{2022} - 2023 I_3) = I_3$

$\det(A) \cdot \det(A^{2022} - 2023 I_3) = 1$

$\hookrightarrow \det(A) \neq 0 \Rightarrow rg(A) = 0$

b) $A^{2023} = 2023A + I_3$ $\boxed{\det(A^n) = [\det(A)]^n}$

$\det(A^{2023}) = \det(2023A + I_3)$

$\underbrace{[\det(A)]}_{\neq 0}{}^{2023} = \det(2023A + I_3) \Rightarrow \det(2023A + I_3) \neq 0 \Rightarrow$

$\Rightarrow rg(2023A + I_3) = 3$

5) $A \in \mathcal{M}_2(\mathbb{C})$

Dacă $(\exists)\ k \in \mathbb{N},\ k \geq 2$ a.î. $A^k = O_2$, atunci $A^2 = O_2$.

Teorema Hamilton-Cayley

$$\boxed{A^2 - Tr(A) \cdot A + \det(A) \cdot I_2 = O_2}$$

$A^k = O_2 \mid \det \Rightarrow (\det(A))^k = 0 \Rightarrow \det(A) = 0$

$\xRightarrow{T.H.C} A^2 = Tr(A) \cdot A$ (*)

Obs: $\boxed{A^2 = \alpha \cdot A \Rightarrow A^n = \alpha^{n-1} \cdot A,\ (\forall)\ n \geq 2}$

$A^k = \underbrace{(Tr(A))}_{\alpha}{}^{k-1} \cdot A = O_2 \mid Tr$

Obs!: $\boxed{Tr(\alpha \cdot A) = \alpha \cdot Tr(A)}$

$(Tr(A))^{k-1} \cdot Tr(A) = 0 \Rightarrow (Tr(A))^{k} = 0 \Rightarrow Tr(A) = 0$

* $A^2 = O_2$

(6) $f : \mathcal{M}_n(\mathbb{R}) \to \mathbb{R}, f(X) = \det(X)$

Precizați dacă f este inj, resp. surjectivă.

INJECTIVITATE:

$f(O_n) = f(X) = 0$

$X = \begin{pmatrix} \boxed{0 \quad 0 \quad 0 \dots 0} \\ a_1 \quad a_2 \quad a_3 \dots a_n \\ \dots \end{pmatrix}, O_n \neq X$

$a_1 \neq 0$

$\Rightarrow$ f nu e injectivă

SURJECTIVITATE

f surj $\Leftrightarrow$ $Imf = \mathbb{R}$

$(\forall) a \in \mathbb{R}, (\exists) X = \begin{pmatrix} a & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ 0 & 0 & 0 & \dots & 1 \end{pmatrix}$

$\det(X) = a \in \mathbb{R} \Rightarrow f$ surj

f nu e inj, dar este surj

T - 7, 13 (de făcut o-uri), ex1 /pag3 (de adunat toate coloanele)

ex6 /pag3,