CURS#5

8. Factorizarea LU cu pivotare (PLU): teorema de existenţă; algoritm; legătura cu MEG cu pivotare parţială (scalată).
9. Factorizarea Cholesky: proprietăţi ale matricelor simetrice şi pozitiv definite (SPD); condiţii necesare şi suficiente; teorema de existenţă şi unicitate; algoritm; exemplu.

PROBLEME

1) Fie $\mathbf{A} = \big(a_{ij}\big)_{i,j=\overline{1,n}} \in \mathscr{M}_n(\mathbb{R})$ o matrice SPD. Atunci:

   (i) $\max\limits_{1\leq k,j\leq n} |a_{kj}| \leq \max\limits_{1\leq i\leq n} |a_{ii}|$;

   (ii) $\big(a_{ij}\big)^2 < a_{ii}\, a_{jj}, \ \forall\ 1 \leq i < j \leq n$.

3.3. FACTORIZAREA LU CU PIVOTARE (PLU)

<u>**TEOREMA #4**</u> (existența factorizării LU cu pivotare, ie PLU):

Fie $A \in \mathcal{M}_n(\mathbb{R})$ inversabilă.

Atunci A admite <u>o factorizare LU cu pivotare</u> (<u>factorizare PLU</u>),

$\exists\ P \in \mathcal{M}_n(\mathbb{R})$ matrice permutare;

$\exists\ L = (l_{ij})_{i,j=\overline{1,n}} \in \mathcal{M}_n(\mathbb{R})$:

- inferior triunghiulară;
- $l_{ii} = 1,\ i = \overline{1,n}$;

$\exists\ U = (u_{ij})_{i,j=\overline{1,n}} \in \mathcal{M}_n(\mathbb{R})$:

- superior triunghiulară;

(• $u_{ii} = a_{ii}\ , i = \overline{1,n}$);

astfel încât $\boxed{A = PLU}$

OBS: Factorizarea LU cu pivotare (PLU) nu este unică!

Dem:

Constructivă, prin inducție după $n \geq 1$.

$n=1$: $A = (a_{11}) \in \mathcal{M}_1(\mathbb{R})$ inv

$\Rightarrow$ $A = PLU$, unde

$P = 1$, $L = 1$, $U = a_{11} \neq 0$

$n-1 \longmapsto n$:

Presupunem că orice $\tilde{A} \in \mathcal{M}_{n-1}(\mathbb{R})$ inversabilă admite o factorizare LU fără pivotare, i.e. $\tilde{A} = PLU$ cu

- $P \in \mathcal{M}_{n-1}(\mathbb{R})$ matrice permutare;
- $L = (l_{ij})_{i,j=\overline{1,n-1}} \in \mathcal{M}_{n-1}(\mathbb{R})$ inf triunghiulară cu $l_{ii} = 1$, $i = \overline{1,n-1}$;
- $U = (u_{ij})_{i,j=\overline{1,n-1}} \in \mathcal{M}_{n-1}(\mathbb{R})$ sup triung.

Fie $A \in \mathcal{M}_n(\mathbb{R})$ inversabilă $\Rightarrow$

$\exists\, i \in \overline{1,n}$ a.î. $a_{i1} \neq 0$ (în caz contrar, $a_{i1} = 0$, $\forall\, i = \overline{1,n} \Rightarrow \det A = 0$ ∦)

Fie $P_1 \in \mathcal{M}_n(\mathbb{R})$ o matrice permutare simplă care interschimbă liniile 1 și i ale matricei A, unde $i \in \overline{1,n}$ se alege a.î. $|a_{i1}| = \max\limits_{j=\overline{1,n}} |a_{j1}|$.

Mai exact, avem:

$$P_1 = \mathrm{col}\left[\underline{e}^{(i)}\ \underline{e}^{(2)} \dots \underline{e}^{(i-1)}\ \underline{e}^{(1)}\ \underline{e}^{(i+1)} \dots \underline{e}^{(n)}\right]$$

și $P_1^T = P_1^{-1} = P_1$, i.e. $P_1^2 = I_n$.

Fie $\tilde{A} := P_1^T A = \begin{bmatrix} \tilde{a}_{11} & \underline{\tilde{A}}_{12}^T \\ \underline{\tilde{A}}_{21} & \tilde{A}_{22} \end{bmatrix}$, unde

$\underline{\tilde{A}}_{12}^T \in \mathcal{M}_{1,n-1}(\mathbb{R})$

$\underline{\tilde{A}}_{21} \in \mathcal{M}_{n-1,1}(\mathbb{R}) \equiv \mathbb{R}^{n-1}$

$\tilde{A}_{22} \in \mathcal{M}_{n-1}(\mathbb{R})$

Cum A inversabilă și $\tilde{a}_{11} \neq 0$, rezultă că

$$S := \tilde{A}_{22} - \frac{1}{\tilde{a}_{11}} \tilde{\underline{A}}_{21} \tilde{\underline{A}}_{12}^T \in \mathcal{M}_{n-1}(\mathbb{R})$$

complementul Schur asociat lui $\tilde{a}_{11}$ este matrice inversabilă. Ex!

Cf ipotezei de inducție, S admite o factorizare LU cu pivotare, ie

$$S = P_2 L_{22} U_{22}$$

unde:

- $P_2 \in \mathcal{M}_{n-1}(\mathbb{R})$ matrice permutare;
- $L_{22} \in \mathcal{M}_{n-1}(\mathbb{R})$ inf. triunghiulară și $(L_{22})_{ii} = 1$, $i = \overline{1, n-1}$;
- $U_{22} \in \mathcal{M}_{n-1}(\mathbb{R})$ sup. triunghiulară.

Atunci:

$$A = P_1 \tilde{A} = P_1 \begin{bmatrix} \tilde{a}_{11} & \tilde{\underline{A}}_{12}^T \\ \tilde{\underline{A}}_{21} & A_{22} \end{bmatrix} =$$

$$= P_1 \begin{bmatrix} \tilde{a}_{11} & \underline{\tilde{A}}_{12}^T \\ P_2\left(P_2^T \underline{\tilde{A}}_{21}\right) & P_2\left(P_2^T \tilde{A}_{22}\right) \end{bmatrix}$$

$$= P_1 \begin{bmatrix} 1 & \underline{0}_{1,n-1} \\ \underline{0}_{n-1,1} & P_2 \end{bmatrix} \begin{bmatrix} \tilde{a}_{11} & \underline{\tilde{A}}_{12}^T \\ P_2^T \underline{\tilde{A}}_{21} & P_2^T \tilde{A}_{22} \end{bmatrix}$$

$$= P_1 \begin{bmatrix} 1 & \underline{0}_{1,n-1} \\ \underline{0}_{n-1,1} & P_2 \end{bmatrix} \begin{bmatrix} \tilde{a}_{11} & \underline{\tilde{A}}_{12}^T \\ P_2^T \underline{\tilde{A}}_{21} & P_2^T\left(\frac{1}{\tilde{a}_{11}} \underline{\tilde{A}}_{21} \underline{\tilde{A}}_{12}^T + P_2 L_{22} U_{22}\right) \end{bmatrix}$$

$$= P_1 \begin{bmatrix} 1 & \underline{0}_{1,n-1} \\ \underline{0}_{n-1,1} & P_2 \end{bmatrix} \begin{bmatrix} \tilde{a}_{11} & \underline{\tilde{A}}_{12}^T \\ P_2^T \underline{\tilde{A}}_{21} & \left(\frac{1}{\tilde{a}_{11}} P_2^T \underline{\tilde{A}}_{21}\right) \underline{\tilde{A}}_{12}^T + L_{22} U_{22} \end{bmatrix}$$

$$= \underbrace{P_1 \begin{bmatrix} 1 & \underline{0}_{1,n-1} \\ \underline{0}_{n-1,1} & P_2 \end{bmatrix}}_{=:\, P \text{ permutare}} \underbrace{\begin{bmatrix} 1 & \underline{0}_{1,n-1} \\ \frac{1}{\tilde{a}_{11}} P_2^T \underline{\tilde{A}}_{21}^T & L_{22} \end{bmatrix}}_{=:\, L \text{ inferior triunghiulară}}$$

$$\underbrace{\begin{bmatrix} \tilde{a}_{11} & \tilde{A}_{12}^{T} \\ \underline{0}_{n-1,1} & U_{22} \end{bmatrix}}_{:= U \text{ superior triunghiulară}} = PLU$$

$\square$

OBSERVAȚII:

1) Factorizarea LU fără pivotare este un caz particular al factorizării LU cu pivotare, cu $P \equiv I_n$.

2) Pentru a se obține rezultate <u>stabile</u> în cazul factorizării LU cu pivotare, trebuie ca pivotarea să se facă în conformitate cu MEGPP sau cu MEGPPS (de alegerea pivotului).

ALGORITM (factorizarea PLU):

$$\underbrace{\begin{bmatrix} a_{11}^{(1)} & a_{12}^{(1)} & \cdots & a_{1n}^{(1)} \\ a_{21}^{(1)} & a_{22}^{(1)} & \cdots & a_{22}^{(1)} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1}^{(1)} & a_{n2}^{(1)} & \cdots & a_{nn}^{(1)} \end{bmatrix}}_{A \equiv A^{(1)}} \longrightarrow \underbrace{\begin{bmatrix} a_{11}^{(2)} & a_{12}^{(2)} & \cdots & a_{1n}^{(2)} \\ a_{21}^{(2)} & a_{22}^{(2)} & \cdots & a_{2n}^{(2)} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1}^{(2)} & a_{n2}^{(2)} & \cdots & a_{nn}^{(2)} \end{bmatrix}}_{A^{(2)}}$$

$$A^{(1)} \xrightarrow{P^{(1)}} \begin{bmatrix} a_{i1}^{(1)} & a_{i2}^{(1)} & \cdots & a_{in}^{(1)} \\ \vdots & \vdots & & \vdots \\ a_{11}^{(1)} & a_{12}^{(1)} & \cdots & a_{1n}^{(1)} \\ \vdots & \vdots & & \vdots \\ a_{n1}^{(1)} & a_{n2}^{(1)} & \cdots & a_{nn}^{(1)} \end{bmatrix} \xrightarrow{M^{(1)}} A^{(2)}$$

interschimb

$$P^{(1)} := \text{col}\left[\underline{e}^{(i)} \ \underline{e}^{(2)} \ \ldots \ \underline{e}^{(i-1)} \ \underline{e}^{(1)} \ \underline{e}^{(i+1)} \ \ldots \ \underline{e}^{(n)}\right]$$

$$M^{(1)} := I_n - \underline{m}^{(1)} \left(\underline{e}^{(1)}\right)^T$$

$$\underline{m}^{(1)} := \left(0 \ \ m_{21}^{(1)} \ \ldots \ m_{n1}^{(1)}\right)^T \in \mathbb{R}^n$$

$$m_{i1}^{(1)} := a_{i1}^{(1)} / a_{11}^{(1)}, \quad i = \overline{2,n}$$

Obţinem

$$A \equiv A^{(1)} \longmapsto A^{(2)}: \quad \boxed{A^{(2)} = M^{(1)} P^{(1)} A^{(1)}}$$

• Procedând analog la fiecare pas, obţinem:

$$U \equiv A^{(n)} = M^{(n-1)} P^{(n-1)} A^{(n-1)}$$

$$= M^{(n-1)} P^{(n-1)} M^{(n-2)} P^{(n-2)} A^{(n-2)}$$

$$= \dots =$$

$$= \underbrace{\left(M^{(n-1)} P^{(n-1)} M^{(n-2)} P^{(n-2)} \dots M^{(1)} P^{(1)}\right)}_{=: M} \underbrace{A^{(1)}}_{\equiv A} \Rightarrow$$

$$U = MA = (MP^{-1}) PA$$

$$M := M^{(n-1)} P^{(n-1)} M^{(n-2)} P^{(n-2)} \dots M^{(1)} P^{(1)} \Rightarrow$$

$$P := P^{(n-1)} P^{(n-2)} \dots P^{(1)} \text{ permutare}$$

$$\boxed{\begin{array}{l} (PM^{-1})\, U = PA \\ M := M^{(n-1)} P^{(n-1)} M^{(n-2)} P^{(n-2)} \dots M^{(1)} P^{(1)} \\ P := P^{(n-1)} P^{(n-2)} \dots P^{(1)} \text{ permutare} \end{array}}$$

• Evident, $P \in \mathcal{M}_n(\mathbb{R})$ este o matrice <u>permutare</u>, fiind produs de permutări simple $P^{(k)}$, $k=\overline{1,n-1}$.

• Mai trebuie arătat că $L = PM^{-1}$ sau $L^{-1} = MP^{-1}$ este o matrice <u>inferior triunghiulară</u>.

//

$$L^{-1} := MP^{-1}$$

$$= \left(M^{(n-1)} P^{(n-1)} M^{(n-2)} P^{(n-2)} \dots M^{(1)} P^{(1)}\right) \left(P^{(n-1)} P^{(n-2)} \dots P^{(1)}\right)^{-1}$$

$$= M^{(n-1)} P^{(n-1)} \dots M^{(2)} P^{(2)} M^{(1)} \underbrace{P^{(1)} \left(P^{(1)}\right)^{-1}}_{=I_n} \left(P^{(2)}\right)^{-1} \dots \left(P^{(n-1)}\right)^{-1}$$

$$= M^{(n-1)} P^{(n-1)} \dots M^{(2)} \left[P^{(2)} M^{(1)} \left(P^{(2)}\right)^{-1}\right] \left(P^{(3)}\right)^{-1} \dots \left(P^{(n-1)}\right)^{-1}$$

$$= M^{(n-1)} P^{(n-1)} \dots M^{(2)} \left(P^{(2)} M^{(1)} P^{(2)}\right) P^{(3)} \dots P^{(n-1)}$$

<u>Ex</u>: $P^{(2)} M^{(1)} P^{(2)}$ rămâne o matrice inferior triunghiulară și are

aceeasi structură ca $M^{(1)}$, ie

$$\boxed{\begin{aligned} P^{(2)} M^{(1)} P^{(2)} &= I_n - \underline{\tilde{m}}^{(1)} (\underline{e}^{(1)})^T \\ \underline{\tilde{m}}^{(1)} &:= (0 \ \tilde{m}^{(1)}_{21} \dots \tilde{m}^{(1)}_{n1})^T \in \mathbb{R}^n, \end{aligned}} \quad (a)$$

- Prin urmare, rezultă:

$$\begin{aligned} N^{(2)} &:= M^{(2)} \left(P^{(2)} M^{(1)} P^{(2)}\right) \\ &= \left[I_n - \underline{m}^{(2)} (\underline{e}^{(2)})^T\right] \left[I_n - \underline{\tilde{m}}^{(1)} (\underline{e}^{(1)})^T\right] \\ &= I_n - \underline{\tilde{m}}^{(1)} (\underline{e}^{(1)})^T - \underline{m}^{(2)} (\underline{e}^{(2)})^T \\ &\quad + \underline{m}^{(2)} \underbrace{(\underline{e}^{(2)})^T \underline{\tilde{m}}^{(1)}}_{= \tilde{m}^{(1)}_{21} \in \mathbb{R}} (\underline{e}^{(1)})^T \Longrightarrow \end{aligned}$$

Obtinem:

$$\boxed{\begin{aligned} N^{(2)} &:= M^{(2)} \left(P^{(2)} M^{(1)} P^{(2)}\right) \\ &= I_n + \left(\tilde{m}^{(1)}_{21} \underline{m}^{(2)} - \underline{\tilde{m}}^{(1)}\right) (\underline{e}^{(1)})^T \\ &\qquad - \underline{m}^{(2)} (\underline{e}^{(2)})^T \end{aligned}} \quad (b)$$

ie $N^{(2)} := M^{(2)} \left(P^{(2)} M^{(1)} P^{(2)}\right)$ este o matrice inferior triunghiulară cu elemente nenule pe coloanele 1 şi 2.

• Prin generalizarea raționamentelor de la (a) și (b), obținem că $L^{-1} = MP^{-1}$ este o matrice inferior triunghiulară, deci și $L = PM^{-1}$ este o matrice inferior triung:

$$\boxed{\begin{aligned} &LU = PA \\ &L = PM^{-1} \\ &P := P^{(n-1)} P^{(n-2)} \dots P^{(1)} \\ &M := M^{(n-1)} P^{(n-1)} M^{(n-2)} P^{(n-2)} \dots M^{(1)} P^{(1)} \end{aligned}}$$

3.4. FACTORIZAREA CHOLESKY

DEFINIȚIE:

$A \in \mathcal{M}_n(\mathbb{R})$ s.n. matrice simetrică și pozitiv definită (SPD) dacă:

(i) $A^T = A$ (A simetrică);

(ii) $\underline{x}^T A \underline{x} > 0$, $\forall \underline{x} \in \mathbb{R}^n \setminus \{\underline{0}\}$

(A pozitiv definită).

TEOREMA #1 (proprietăți - SPD):

Fie $A = (a_{ij})_{i,j=\overline{1,n}} \in \mathcal{M}_n(\mathbb{R})$ SPD. Atunci:

(i) A inversabilă;

(ii) $a_{ii} > 0$, $i = \overline{1,n}$;

(iii) matricele $A^{(k)} = (a_{ij})_{i,j=\overline{1,k}} \in \mathcal{M}_k(\mathbb{R})$, $k = \overline{1, n-1}$, sunt SPD;

(iv) complementul Schur asociat lui a_{11} este o matrice SPD.

Dem:

(i) Presupunem că $A \in \mathcal{M}_n(\mathbb{R})$ nu este inversabilă $\Rightarrow$

$\exists\, \underline{x}_0 \in \mathbb{R}^n \setminus \{\underline{0}\} : A\underline{x}_0 = \underline{0} \Rightarrow$

$\exists\, \underline{x}_0 \in \mathbb{R}^n \setminus \{\underline{0}\} : \underline{x}_0^T A \underline{x}_0 = \underline{x}_0^T \underline{0} = 0$

contradicție!

(ii) Fie $\underline{x}^{(i)} = (\delta_{ji})_{j=\overline{1,n}} \in \mathbb{R}^n, i=\overline{1,n} \Rightarrow$

$\underline{x}^{(i)} \neq \underline{0}, i=\overline{1,n}$ & $(\underline{x}^{(i)})^T A \underline{x}^{(i)} = a_{ii} > 0, i=\overline{1,n}$

(iii) • $A = A^T \Rightarrow A^{(k)} = (A^{(k)})^T, k=\overline{1,n-1} \Rightarrow$

$A^{(k)}$ simetrică, $k=\overline{1,n-1}$

• Fie $k \in \overline{1,n-1}$ arbitrar fixat.

Fie $\underline{z}^{(k)} = [z_1^{(k)}\ z_2^{(k)} \ldots z_k^{(k)}]^T \in \mathbb{R}^k \setminus \{\underline{0}_k\}$

Definim $\underline{x} = [z_1^{(k)}\ z_2^{(k)} \ldots z_k^{(k)}\ 0 \ldots 0]^T \in \mathbb{R}^n$

și $\underline{x} \neq \underline{0}_n$. $\Rightarrow$

$0 < \underline{x}^T A \underline{x} = (\underline{z}^{(k)})^T A^{(k)} \underline{z}^{(k)} \Rightarrow$

$A^{(k)}$ este PD, $\forall k=\overline{1,n-1}$

(iv) $A = \begin{bmatrix} a_{11} & \underline{A}_{12}^T \\ \underline{A}_{21} & A_{22} \end{bmatrix} \Rightarrow$

Complementul Schur asociat lui a_{11}

$$S := A_{22} - \frac{1}{a_{11}} \underline{A}_{21} \underline{A}_{12}^T$$

- S <u>simetrică</u>:

A simetrică $\Rightarrow \begin{cases} A_{22}^T = A_{22} \\ \underline{A}_{21}^T = \underline{A}_{12}^T \end{cases} \Rightarrow$

$$(\underline{A}_{21} \underline{A}_{12}^T)^T = (\underline{A}_{12}^T)^T \underline{A}_{21}^T = \underline{A}_{21} \underline{A}_{12}^T \Rightarrow$$

$S^T = S$, ie S <u>simetrică</u>

- S <u>PD</u>:

Fie $\underline{x} = [x_1\ x_2 \ldots x_n]^T \in \mathbb{R}^n \setminus \{\underline{0}_n\}$

$$= \begin{bmatrix} x_1 \\ \underline{v} \end{bmatrix}, \quad x_1 \in \mathbb{R}$$

$\underline{v} := [x_2\ x_3 \ldots, x_n]^T \in \mathbb{R}^{n-1}$

$$S = A_{22} - \frac{1}{a_{11}} \underline{A}_{21} \underline{A}_{12}^T \implies$$

$$A_{22} = S + \frac{1}{a_{11}} \underline{A}_{21} \underline{A}_{12}^T = S + \frac{1}{a_{11}} \underline{A}_{21} \underline{A}_{21}^T$$

$A \in \mathcal{M}_n(\mathbb{R})$ este PD $\implies$

$$0 < \underline{x}^T A \underline{x} = \begin{bmatrix} x_1 & \underline{v}^T \end{bmatrix} \begin{bmatrix} a_{11} & \underline{A}_{21}^T \\ \underline{A}_{21} & A_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ \underline{v} \end{bmatrix}$$

$$= \begin{bmatrix} x_1 & \underline{v}^T \end{bmatrix} \begin{bmatrix} a_{11} x_1 + \underline{A}_{21}^T \underline{v} \\ \underline{A}_{21} x_1 + A_{22} \underline{v} \end{bmatrix}$$

$$= x_1 \left(a_{11} x_1 + \underline{A}_{21}^T \underline{v} \right) + \underline{v}^T \left(\underline{A}_{21} x_1 + A_{22} \underline{v} \right)$$

$$= x_1 \left(a_{11} x_1 + \underline{A}_{21}^T \underline{v} \right) + \underline{v}^T \underline{A}_{21} x_1$$
$$+ \underline{v}^T \left(S + \frac{1}{a_{11}} \underline{A}_{21} \underline{A}_{21}^T \right) \underline{v}$$

$$= x_1 \left(a_{11} x_1 + \underline{A}_{21}^T \underline{v} \right) + \underline{v}^T \underline{A}_{21} \left(x_1 + \frac{1}{a_{11}} \underline{A}_{21}^T \underline{v} \right)$$
$$+ \underline{v}^T S \underline{v}$$

$$= x_1 \left(a_{11} x_1 + \underline{A}_{21}^T \underline{v} \right) + \frac{1}{a_{11}} \underline{v}^T \underline{A}_{21} \left(a_{11} x_1 + \underline{A}_{21}^T \underline{v} \right)$$
$$+ \underline{v}^T S \underline{v}$$

$$= \left(x_1 + \frac{1}{a_{11}} \underline{v}^T \underline{A}_{21}\right)\left(a_{11}x_1 + \underline{A}_{21}^T \underline{v}\right)$$

$$+ \underline{v}^T S \underline{v}$$

$$= \frac{1}{a_{11}} \underbrace{\left(a_{11}x_1 + \underline{v}^T \underline{A}_{21}\right)}_{= \left(a_{11}x_1 + \underline{A}_{21}^T \underline{v}\right)^T} \left(a_{11}x_1 + \underline{A}_{21}^T \underline{v}\right)$$

$$+ \underline{v}^T S \underline{v}$$

$$= \frac{1}{a_{11}} \left(a_{11}x_1 + \underline{A}_{21}^T \underline{v}\right)^T \left(a_{11}x_1 + \underline{A}_{21}^T \underline{v}\right)$$

$$+ \underline{v}^T S \underline{v}$$

Fie $x_1 := -\frac{1}{a_{11}} \underline{A}_{21}^T \underline{v}$, $\underline{v} \in \mathbb{R}^{n-1} \setminus \{\underline{0}_{n-1}\}$ arbitrar

Atunci:

$0 < \underline{x}^T A \underline{x} = \underline{v}^T S \underline{v} \Rightarrow S$ este PD

În concluzie, complementul Schur asociat lui a_{11}, S, este SPD.

$\square$

TEOREMA #2 (factorizarea Cholesky):

Fie $A \in \mathcal{M}_n(\mathbb{R})$ o matrice SPD.

Atunci a admite o unică factorizare Cholesky

$$A = L L^T$$

unde $L = (l_{ij})_{i,j=\overline{1,n}} \in \mathcal{M}_n(\mathbb{R})$ inferior triunghiulară, cu $l_{ii} > 0$, $i = \overline{1,n}$.

Dem:

EXISTENȚA: Inducție după $n \geq 1$.

$n = 1$: $A = a_{11} \in \mathcal{M}_1(\mathbb{R})$ SPD $\Rightarrow$

$\Rightarrow a_{11} > 0$.

Alegem $L = \sqrt{a_{11}} > 0$ și $A = L L^T$.

$n-1 \mapsto n$: Presupunem că orice matrice SPD, $S \in \mathcal{M}_{n-1}(\mathbb{R})$, admite o factorizare Cholesky.

Fie $A = \begin{bmatrix} a_{11} & \underline{A}_{21}^T \\ \underline{A}_{21} & A_{22} \end{bmatrix} \in \mathcal{M}_n(\mathbb{R})$ SPD.

Vrem să determinăm

$$L = \begin{bmatrix} l_{11} & \underline{0}_{1,n-1} \\ \underline{L}_{21} & L_{22} \end{bmatrix} \in \mathcal{M}_n(\mathbb{R}) \quad \text{a.î.}$$

(i) $l_{11} > 0$;

(ii) $\underline{0}_{1,n-1} \in \mathcal{M}_{1,n-1}(\mathbb{R})$;

(iii) $\underline{L}_{21} \in \mathcal{M}_{n-1,1}(\mathbb{R}) \equiv \mathbb{R}^{n-1}$;

(iv) $L_{22} \in \mathcal{M}_{n-1}(\mathbb{R})$ inferior triunghiulară, cu $l_{ii} > 0$, $i = \overline{2,n}$;

și $A = LL^T$, i.e.

$$A = \begin{bmatrix} a_{11} & \underline{A}_{21}^T \\ \underline{A}_{21} & A_{22} \end{bmatrix} = \underbrace{\begin{bmatrix} l_{11} & \underline{0}_{1,n-1} \\ \underline{L}_{21} & L_{22} \end{bmatrix}}_{=L} \underbrace{\begin{bmatrix} l_{11} & \underline{L}_{21}^T \\ \underline{0}_{n-1,1} & L_{22}^T \end{bmatrix}}_{=L^T}$$

$$= \begin{bmatrix} l_{11}^2 & l_{11}\underline{L}_{21}^T \\ \underline{L}_{21} l_{11} & \underline{L}_{21}\underline{L}_{21}^T + L_{22}L_{22}^T \end{bmatrix} = LL^T \Longrightarrow$$

- $l_{11}^2 = a_{11} > 0 \Rightarrow \boxed{l_{11} = \sqrt{a_{11}} > 0}$

- $\begin{cases} l_{11} \underline{L}_{21}^T = \underline{A}_{21}^T \\ \underline{L}_{21} l_{11} = \underline{A}_{21} \end{cases} \Rightarrow \boxed{\underline{L}_{21} = \frac{1}{\sqrt{a_{11}}} \underline{A}_{21}}$

- $\underline{L}_{21} \underline{L}_{21}^T + L_{22} L_{22}^T = A_{22} \Rightarrow$

$$L_{22} L_{22}^T = A_{22} - \underline{L}_{21} \underline{L}_{21}^T$$

$$= A_{22} - \frac{1}{a_{11}} \underline{A}_{21} \underline{A}_{21}^T =: S$$

complementul Schur asociat lui a_{11}

Cum $S := A_{22} - \frac{1}{a_{11}} \underline{A}_{21} \underline{A}_{21}^T$ este SPD

(cf Teoremei #1), din ipoteza de inducție rezultă că

$\exists\, L_{22} \in \mathcal{M}_{n-1}(\mathbb{R})$ inferior triunghiulară

$l_{ii} > 0$, $i = \overline{2,n}$, a.î. $S = L_{22} L_{22}^T$

UNICITATEA:

Presupunem că există două factorizări Cholesky distincte ale lui $A \in \mathcal{M}_n(\mathbb{R})$ SPD, ie

$$A = L^{(1)} (L^{(1)})^T = L^{(2)} (L^{(2)})^T$$

$L^{(k)} = (l_{ij}^{(k)})_{i,j=\overline{1,n}} \in \mathcal{M}_n(\mathbb{R})$, $k=1,2$, inferior triunghiulare cu $l_{ii}^{(k)} > 0$, $i=\overline{1,n}$, $k=1,2$.

Atunci:

$$\underbrace{(L^{(2)})^{-1} L^{(1)}}_{\text{inferior triung.}} = \underbrace{(L^{(2)})^T (L^{(1)})^{-T}}_{\text{superior triung}} \Longrightarrow$$

$$(L^{(2)})^{-1} L^{(1)} = (L^{(2)})^T (L^{(1)})^{-T} = D \Longrightarrow$$

diagonală

$$\begin{cases} L^{(1)} = L^{(2)} D \\ (L^{(2)})^T = D (L^{(1)})^T \end{cases} \Longrightarrow$$

Pentru termenii diagonali, obținem:

$$\begin{cases} l_{ii}^{(1)} = l_{ii}^{(2)} d_i, & i=\overline{1,n} \\ l_{ii}^{(2)} = l_{ii}^{(1)} d_i, & i=\overline{1,n} \end{cases} \Rightarrow$$

$$l_{ii}^{(1)} = l_{ii}^{(1)} d_i^2, \quad i=\overline{1,n} \Rightarrow$$

$$d_i^2 = 1, \quad i=\overline{1,n} \Rightarrow d_i = \pm 1, \quad i=\overline{1,n}$$

Cum $l_{ii}^{(k)} > 0$, $i=\overline{1,n}$, $k=1,2$, rezultă

$$d_i = 1, \quad i=\overline{1,n}$$

ceea ce implică $D = I_n$.

Obținem $L^{(1)} = L^{(2)}$, ie contradicție.

$\square$

TEOREMA #3 (factorizarea Cholesky - cond. necesare și suficiente):

Fie $A \in \mathcal{M}_n(\mathbb{R})$ a.î. $A = A^T$. Echivalente:

(i) A este PD;

(ii) $\det A^{(k)} > 0$, $k = \overline{1,n}$, unde
$A^{(k)} := (a_{ij})_{i,j=\overline{1,k}} \in \mathcal{M}_k(\mathbb{R})$, $k = \overline{1,n}$
(Criteriul lui Sylvester);

(iii) A admite MEGFP și toți pivoții sunt strict pozitivi;

(iv) A admite factorizarea LDL^T,
$L = (l_{ij})_{i,j=\overline{1,n}} \in \mathcal{M}_n(\mathbb{R})$ inferior triunghiulară, cu $l_{ii} > 0$, $i = \overline{1,n}$, și
$D = \mathrm{diag}(d_1, d_2, \dots, d_n) \in \mathcal{M}_n(\mathbb{R})$, cu $d_i > 0$, $i = \overline{1,n}$;

(v) A admite factorizarea Cholesky LL^T, $L = (l_{ij})_{i,j=\overline{1,n}} \in \mathcal{M}_n(\mathbb{R})$ inferior triunghiulară, cu $l_{ii} > 0$, $i = \overline{1,n}$.

ALGORITM (factorizarea Cholesky):

Date: $A = (a_{ij})_{i,j=\overline{1,n}}$ SPD

$L = 0 \in \mathcal{M}_n(\mathbb{R})$

$k = \overline{1,n}$:

$\quad l_{kk} = \sqrt{a_{kk}}$

$\quad i = \overline{k+1,n}$:

$\quad\quad l_{ik} = a_{ik} / l_{kk}$

$\quad$ end

$\quad i = \overline{k+1,n}$:

$\quad\quad j = \overline{k+1,n}$:

$\quad\quad\quad a_{ij} = a_{ij} - l_{ik} l_{jk}$

$\quad\quad$ end

$\quad$ end

end

Output: $L = (l_{ij})_{i,j=\overline{1,n}}$

OBS: Rezolvarea sistemelor de ecuații liniare $A\underline{x} = \underline{b}$, unde $A \in \mathcal{M}_n(\mathbb{R})$ SPD și $\underline{b} \in \mathbb{R}^n$:

1) Factorizarea Cholesky a matricei $A = LL^T$
2) Rezolvarea sistemului inferior triunghiular $L\underline{y} = \underline{b}$ prin metoda substituției ascendente, ie $\underline{y} := L^{-1}\underline{b} \in \mathbb{R}^n$
3) Rezolvarea sistemului superior triunghiular $L^T\underline{x} = \underline{y}$ prin metoda substituției descendente, ie $\underline{x} := L^{-T}\underline{y}$

OBS: Analog se rezolvă sistemele de ecuații liniare $A\underline{x} = \underline{b}$, unde $A \in \mathcal{M}_n(\mathbb{R})$ inversabilă și $\underline{b} \in \mathbb{R}^n$ (PLU).