Seminar 2

1. Det $\underline{\lim}\, x_n$, $\overline{\lim}\, x_n$ și precizați dacă $\exists \lim\limits_{n\to\infty} x_n$ unde

$$a)\quad x_n = \left(1+\frac{1}{n}\right)^n \sin\frac{n\pi}{3} \quad \forall\ n\in\mathbb{N}^*$$

$\left(1+\frac{1}{n}\right)^n \to e$

$$b)\quad x_n = \frac{n\cos\frac{n\pi}{2}}{n^2+1} \quad \forall\ n\in\mathbb{N}$$

sau luăm $4k$, $4k+1$, $4k+2$, $4k+3$

Sol

a)

- $n = 3k$

$$x_n = \left(1+\frac{1}{3k}\right)^{3k} \sin\frac{\cancel{3}k\pi}{\cancel{3}}$$

$$= \left(1+\frac{1}{3k}\right)^{3k} \sin k\pi$$

$$= \left(1+\frac{1}{3k}\right)^{3k}\cdot 0 = 0 \xrightarrow[k\to\infty]{} \boxed{0}$$

- $n = 3k+1$

$$x_n = \left(1+\frac{1}{3k+1}\right)^{3k+1}\cdot \sin(3k+1)\frac{\pi}{3}$$

$$= \left(1+\frac{1}{3k+1}\right)^{3k+1}\cdot \sin\left(k\pi+\frac{\pi}{3}\right)$$

$$= \left(1+\frac{1}{3k+1}\right)^{3k+1}\cdot\left(\underbrace{\sin k\pi}_{0}\cos\frac{\pi}{3}+\underbrace{\cos k\pi}_{(-1)^k}\sin\frac{\pi}{3}\right)$$

$$= \left(1+\frac{1}{3k+1}\right)^{3k+1}\cdot(-1)^k\cdot\frac{\sqrt{3}}{2}$$

$k=2p$

$$* \quad x_{3\cdot 2p+1} = x_{6p+1} = \left(1+\frac{1}{6p+1}\right)^{6p+1}\cdot +\frac{\sqrt{3}}{2}$$

$\xrightarrow[p\to\infty]{} e\frac{\sqrt{3}}{2}\, e$ $\quad \sin\frac{2\pi}{3} = 2\sin\frac{\pi}{3}\cos\frac{\pi}{3} = \cancel{2}\cdot\frac{1}{\cancel{2}}\cdot\frac{\sqrt{3}}{2}$

$*\ K = 2p+1$

$$x_{3(2p+1)+1} = x_{6p+4} = \left(1+\frac{1}{6p+4}\right)^{6p+4}\cdot(-1)\frac{\sqrt{3}}{2}$$

$$\xrightarrow[p\to\infty]{} -\frac{\sqrt{3}}{2}\, e$$

• $n = 3K+2$

$$x_n = x_{3K+2} = \left(1+\frac{1}{3K+2}\right)^{3K+2}\cdot\sin\frac{(3K+2)\pi}{3}$$

$$= \left(1+\frac{1}{3K+2}\right)^{3K+2}\cdot\sin\left(K\pi+2\frac{\pi}{3}\right)$$

$$= \left(1+\frac{1}{3K+2}\right)^{3K+2}\cdot\left(\underbrace{\sin K\pi}_{0}\cos\frac{2\pi}{3}+\underbrace{\cos K\pi}_{(-1)^K}\sin\frac{2\pi}{3}\right)$$

$$= \left(1+\frac{1}{3K+2}\right)^{3K+2}\cdot(-1)^K\frac{\sqrt{3}}{2}$$

$*\ K = 2p$

$$x_{6p+2} = \left(1+\frac{1}{6p+2}\right)^{6p+2}\cdot(-1)^{2p}\frac{\sqrt{3}}{2}$$

$$\xrightarrow[p\to\infty]{} e\cdot\frac{\sqrt{3}}{2}$$

$*\ K = 2p+1$

$$x_{6p+5} = \left(1+\frac{1}{6p+5}\right)^{6p+5}\cdot(-1)^{2p+1}\frac{\sqrt{3}}{2}$$

$$\xrightarrow[p\to\infty]{} -e\frac{\sqrt{3}}{2}$$

fatto lo qui e complet argum

$$\mathbb{N}^* = (3\mathbb{N}^*)\cup(6\mathbb{N}+1)\cup(6\mathbb{N}+4)\cup(6\mathbb{N}+2)\cup(6\mathbb{N}+5)$$

$\mathcal{L}((x_n)_n) = \{\pm \frac{\sqrt{3}}{2}e ; 0\}$

$\underline{\lim}\, x_n = -\frac{\sqrt{3}}{2}e$

$\overline{\lim}\, x_n = \frac{\sqrt{3}}{2}e$

Cum $\underline{\lim}\, x_n \neq \overline{\lim}\, x_n \rightarrow \nexists \lim_{n\to\infty} x_n$

b) $-1 \leq \cos \frac{n\pi}{2} \leq 1 \quad | \cdot \frac{n}{n^2+1}$

$-\frac{n}{n^2+1} \leq x_n \leq \frac{n}{n^2+1} \quad | \lim_{n\to\infty}$

$0 \leq \lim_{n\to\infty} x_n \leq 0$

Cf. crit. cleşt. $\lim_{n\to\infty} x_n = 0$, deci $\overline{\lim}\, x_n =$ $\square$

$\underline{\lim}\, x_n = 0$

2. Det suma seriei $\sum_{n=1}^{\infty} \frac{n}{(n+1)!}$ şi precizaţi dacă e converg.

$x_n = \frac{n}{(n+1)!}$

Sol: $s_n = x_1 + x_2 + \ldots + x_n$

$= \frac{1}{2!} + \frac{2}{3!} + \frac{3}{4!} + \ldots + \frac{n}{(n+1)!}$

$= \frac{2-1}{2!} + \frac{3-1}{3!} + \frac{4-1}{4!} + \ldots + \frac{n+1-1}{(n+1)!}$

$= 1 - \cancel{\frac{1}{2!}} + \cancel{\frac{1}{2!}} - \cancel{\frac{1}{3!}} + \cancel{\frac{1}{3!}} - \cancel{\frac{1}{4!}} + \ldots + \cancel{\frac{1}{n!}} - \frac{1}{(n+1)!}$

$= 1 - \frac{1}{(n+1)!}$

$$\lim_{n\to\infty} s_n = \lim_{n\to\infty} 1 - \underbrace{\frac{1}{(n+1)!}}_{0} = \underline{1}$$

$$\Rightarrow \sum_{n=1}^{\infty} x_n = 1, \text{ deci } \sum_{n=1}^{\infty} x_n \text{ e converg.}$$

3. Studiați convergența / natura următoarelor serii

a) $\sum_{n=1}^{\infty} \frac{\sqrt{n+1}}{n^2}$

b) $\sum_{n=1}^{\infty} \left(\frac{an^2+3n+4}{n^2+n+1}\right)^n$, $a>0$

c) $\sum_{n=1}^{\infty} \frac{a^n}{\sqrt[n]{n}}$, $a>0$

d) $\sum_{n=2}^{\infty} \frac{1}{n \ln n}$

$\cancel{\frac{a^{n+1}}{\sqrt[n+1]{n+1}} \cdot \frac{\sqrt[n]{n}}{a^n}}$ $\cancel{1} a$

$a \in (1, \infty)$

$\frac{\sqrt[n]{n}}{\sqrt[n]{n+1}} - 1$

$\cancel{\frac{n^2+(a-1)n^2+3n+4}{n^2+n+1}}$

$\cancel{n^2+n+1}+(a-1)$

$\cancel{\frac{a^n}{n!}} <$

$\frac{a^n}{\sqrt[n]{n}} \leq a^n$

$a^n$ conv pt $a \in (0,1)$

$\Rightarrow \frac{a^n}{\sqrt[n]{n}}$ converg

$$\boxed{\sum_{n=1}^{\infty} \frac{\sqrt{n}}{n^2} = \sum_{n=1}^{\infty} \frac{n^{\frac{1}{2}}}{n^2} = \sum_{n=1}^{\infty} \frac{1}{n^{3/2}} - \boxed{\text{conv.}}}$$

$\frac{\sqrt{n+1}}{n^2} < \underbrace{\frac{\sqrt{n}}{n^2}}_{\text{conv}}$

$\Rightarrow$ conv.

$\lim_{n\to\infty} \sqrt[n]{n} = 1$

Sol

a) Fie $x_n = \frac{\sqrt{n-1}}{n^2}$, $\forall n \in \mathbb{N}^*$

$y_n = \frac{\sqrt{n}}{n^2}$

$x_n < y_n \quad \forall n \in \mathbb{N}^*$

$$\sum_n y_n = \sum_n \frac{\sqrt{n}}{n^2} = \sum_n \frac{n^{1/2}}{n^2} = \sum_n \frac{1}{n^{3/2}} \text{ conv}$$

(serie armonică generalizată cu $\alpha = \frac{3}{2}$)

$\underset{\text{comp. cu ineg}}{\overset{\text{cf crit}}{\Rightarrow}} \sum_n x_n$ e converg $\square$

3) Aplicăm criteriul radicalului

Fie $x_n = \left(\frac{an^2+3n+4}{n^2+n+1}\right)^n \quad \forall\, n \in \mathbb{N}^*$

Aplicăm criteriul rad.

$$\lim_{n\to\infty} \sqrt[n]{x_n} = \lim_{n\to\infty} \sqrt[n]{\left(\frac{an^2+3n+4}{n^2+n+1}\right)^n} =$$

$$\lim_{n\to\infty} \frac{an^2+3n+4}{n^2+n+1} = a$$

- pt $a > 1 \Rightarrow \sum_n x_n$ diverg
- pt $a < 1 \Rightarrow \sum_n x_n$ converg
- pt $a = 1$ crit nu decide

Fie $a = 1$

$$x_n = \left(\frac{n^2+3n+4}{n^2+n+1}\right)^n$$

$$\lim_{n\to\infty} \left(\frac{n^2+n+1}{n^2+n+1} + \frac{2n+3}{n^2+n+1}\right)^n$$

$$\lim_{n\to\infty} \left[\left(1 + \frac{2n+3}{n^2+n+1}\right)^{\frac{n^2+n+1}{2n+3}}\right]^{\frac{2n+3}{n^2+n+1} \cdot n} = e^2 \neq 0$$

Cum $\lim\limits_{n\to\infty} x_n \neq 0 \Rightarrow \sum\limits_n x_n$ este diverg.

$$\sum_{n=1}^{\infty} x_n \begin{cases} \nearrow \text{conv} & a\in(0,1) \\ \searrow \text{diverg.} & a\in[1,+\infty) \end{cases}$$

c) Fie $x_n = \dfrac{a^n}{\sqrt[n]{n}}$ $\forall\, n\in\mathbb{N}^*$

$$\lim_{n\to\infty} \frac{x_{n+1}}{x_n} = \lim_{n\to\infty} \frac{a^{n+1}}{\underbrace{\sqrt[n+1]{n+1}}_{\to 1}} \cdot \frac{\overbrace{\sqrt[n]{n}}^{\to 1}}{a^n}$$

$$= \cancel{\tfrac{1}{a}}\ a$$

(i.e. $a\in(\cancel{1},\cancel{\infty})$)

- pt $a>1 \Rightarrow \cancel{\tfrac{1}{a}<1} \Rightarrow \sum\limits_n x_n$ e ~~converg~~ diverg
  i.e. $a\in(0,\cancel{1})$
- pt $a<1 \Rightarrow \cancel{\tfrac{1}{a}>1} \Rightarrow \sum\limits_n x_n$ e ~~divergent~~ converg
- pt $a=1 \Rightarrow$ crit nu decide

$$x_n = \frac{1}{\sqrt[n]{n}},\ \forall\, n\in\mathbb{N}^*$$

~~$\lim\limits_{n\to\infty}$~~

$$\lim_{n\to\infty} x_n = \frac{1}{1} = 1 \neq 0$$

Cf crit suf. de diverg, avem că $\sum\limits_n x_n$ e diverg $\square$

d) Fie $x_n = \dfrac{1}{n\cdot \ln n}$ $\forall\, n\geq 2$

$$x_{2^n} = \frac{1}{2^n\cdot \ln 2^n} = \frac{1}{n\cdot 2^n\cdot \ln 2}$$

$(x_n)_n$ e descresc

Cf. crit condensării avem că: $\sum_{n=2}^{\infty} x_n \sim \sum_{n=2}^{\infty} 2^n \cdot x_{2^n}$

$y_n = 2^n \cdot x_{2^n} = \cancel{2^n} \cdot \frac{1}{n \cdot \cancel{2^n} \cdot \ln 2} = \frac{1}{n \cdot \ln 2}$

$\cancel{y_2 + y_3 + \dots + y_n = \frac{1}{2\ln 2} + \frac{1}{3\ln 2} + \dots + \frac{1}{n \ln 2}}$

$=$

$\sum_{n=2}^{\infty} 2^n \cdot x_{2^n} = \sum_{n=2}^{\infty} \frac{1}{n \cdot \ln 2} = \sum_{n=2}^{\infty} \frac{1}{\ln 2} \cdot \frac{1}{n}$ → const

Cum $\sum_{n=2}^{\infty} \frac{1}{n}$ e diverg $\Rightarrow \sum_{n=2}^{\infty} \frac{1}{\ln 2} \cdot \frac{1}{n}$ e diverg

$\Rightarrow \sum_{n=2}^{\infty} x_n$ e diverg.

$6(n+1)+1 = 6n+7$

$5(n+1)+3 = 5n+8$

$\sum_{n=2}^{\infty} \frac{1}{\ln n}$ → merge cu condensare

l) $\sum_{n=1}^{\infty} \frac{7 \cdot 13 \cdot 19 \cdot \dots \cdot (6n+1)}{8 \cdot 13 \cdot 18 \cdot \dots \cdot (5n+3)} \cdot x^n$, $x > 0$ → $(x_n)$

Sol

$\lim_{n\to\infty} \frac{x_{n+1}}{x_n} = \lim_{n\to\infty} \frac{\cancel{7 \cdot 13 \cdot 19 \dots (6n+1)} \cdot (6n+7)}{\cancel{8 \cdot 13 \cdot 18 \cdot \dots \cdot (5n+3)}(5n+8)} \cdot x^{\cancel{n+1}} \cdot \frac{\cancel{8 \cdot 13 \cdot 18 \dots (5n+3)}}{\cancel{7 \cdot 13 \cdot 19 \cdot \dots \cdot (6n+1)} \cdot \cancel{x^n}}$

$\lim_{n\to\infty} \frac{6n+7}{5n+8} \cdot x = \frac{6}{5} x$

Cum $x > 0 \Rightarrow \begin{cases} \frac{6}{5}x = 1 & \text{pt } x = \frac{5}{6} \\ \frac{6}{5}x > 1, & \text{pt } x > \frac{5}{6} \Rightarrow \sum_n x_n \text{ diverg} \\ \frac{6}{5}x < 1, & \text{pt } x < \frac{5}{6} \Rightarrow x \in (0, \frac{5}{6}) \end{cases}$

$\Rightarrow \sum_n x_n$ e conv

- pt $x \cancel{>} \frac{5}{6}$

- pt $x = \frac{5}{6}$

$$\lim_{n\to\infty} n\left(\frac{x_n}{x_{n+1}} - 1\right) = \lim_{n\to\infty} n\left(\frac{5n+8}{6n+7} \cdot \frac{6}{5} - 1\right)$$

$$= \lim_{n\to\infty} n \cdot \left(\frac{\cancel{30n} + 48 - \cancel{30n} - 35}{30n+35}\right)$$

$$= \lim_{n\to\infty} n \cdot \frac{13}{30n+35} = \frac{13}{30} < 1$$

$\Rightarrow \sum_n x_n$ e divergentă

## Curs 3

## Topologie

**Def** Fie $X \neq \emptyset$ (oarecare, mulțimi) TOPOLOGIC

O mulțime $\tau$ (tau) $\subset \mathcal{P}(X)$ ($\tau$ conține mulțimi: $x \in X$, $\{x\} \in \tau$) se numește **topologie pe $X$** dacă:

1) $\emptyset, X \in \tau$

2) $\forall D_1, D_2 \in \tau$ avem $D_1 \cap D_2 \in \tau$