Aplicație: Considerăm următoarele matrici pe blocuri:

1) $A = \left(\begin{array}{c|c} M_{m} & N_{m\times p} \\ \hline O & P_p \end{array}\right) \Rightarrow \det A = \det M \cdot \det P$

$A \in M_{m\times p}$

(matrice superior triunghiulară) $\rightarrow$ elementele de sub diagonala principală sunt 0 $a_{ij}=0,\ j<i,\ i,j=\overline{1,n}$

2) $A = \left(\begin{array}{c|c} M_{m} & O \\ \hline N_{p\times m} & P_p \end{array}\right) \Rightarrow \det A = \det M \cdot \det P$

(matrice inferior triunghiulară) elementele de deasupra diag. princ. sunt 0 i.e. $a_{ij}=0,\ j>i,\ i,j=\overline{1,n}$

3) $A = \left(\begin{array}{c|c} N_{m\times p} & M_{m} \\ \hline P_p & O \end{array}\right)$

$\Rightarrow \det A = (-1)^{mp} \det M \cdot \det P$

4) $A = \left(\begin{array}{c|c} O & M_{m} \\ \hline P_p & N_{p\times m} \end{array}\right) \Rightarrow \det A = (-1)^{mp} \cdot \det M \cdot \det P$

(la examen nu mai trebuie demonstrate)

EXAMEN 2022 (EX 1 este temă!)

2. $A_n \in M_n(\mathbb{R}),\ n \in \mathbb{N},\ n \geq 3,\ \Delta_n = \det A_n$

- $-3 \rightarrow (i,j),\ i \leq j$: $a_{11}, a_{12}, a_{13} \dots a_{1n}$; $a_{22}, a_{33} \dots a_{nn}$; $a_{23}, a_{24} \dots$; $a_{34}, a_{35} \dots$ ciornă
- $1 \rightarrow (i, i-1),\ 2 \leq i \leq n$: $a_{21}, a_{32}, a_{43}, \dots a_{n,n-1}$
- $0 \rightarrow$ în rest

a) Să se calc. $\det(A_3)$ și $\det(A_4)$

$$\det(A_3) = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = \begin{vmatrix} -3 & -3 & -3 \\ 1 & -3 & -3 \\ 0 & 1 & -3 \end{vmatrix} \xlongequal{L_1 \to L_1 - L_2} \begin{vmatrix} -4 & 0 & 0 \\ 1 & -3 & -3 \\ 0 & 1 & -3 \end{vmatrix} \xlongequal{L_2 \to L_2 - L_3} \begin{vmatrix} -4 & 0 & 0 \\ 1 & -4 & 0 \\ 0 & 1 & -3 \end{vmatrix} \overset{2)}{=}$$

$$= (-4)\cdot(-4)\cdot(-3) = (-4)^2 \cdot (-3) = -48$$

$$\det(A_4) = \begin{vmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{vmatrix} = \begin{vmatrix} -3 & -3 & -3 & -3 \\ 1 & -3 & -3 & -3 \\ 0 & 1 & -3 & -3 \\ 0 & 0 & 1 & -3 \end{vmatrix} \xlongequal[\substack{L_2 \to L_2 - L_3 \\ L_3 \to L_3 - L_4}]{L_1 \to L_1 - L_2} \begin{vmatrix} -4 & 0 & 0 & 0 \\ 1 & -4 & 0 & 0 \\ 0 & 1 & -4 & 0 \\ 0 & 0 & 1 & -3 \end{vmatrix} \overset{2)}{=}$$

$$= (-4)\cdot(-4)\cdot(-4)\cdot(-3) = (-4)^3 \cdot (-3) = -64 \cdot 3 = -192$$

b) Să se arate că A_3 e inversabilă și să se calculeze A_3^{-1}

A_3 e inversabilă $\Leftrightarrow \det A_3 \neq 0$

$\det A_3 = -48 \neq 0 \Rightarrow A$ inversabilă

A_3^{-1} $\xrightarrow{\text{met I}}$ $A_3^{-1} = \frac{1}{\det A} \cdot A^*$ (liceu)

$\xrightarrow{\text{met II}}$ $(A_3 | I_3) \rightsquigarrow (I_3 | A_3^{-1})$

OBS! Se poate folosi orice metodă!

Met I

$$A_3 = \begin{pmatrix} -3 & -3 & -3 \\ 1 & -3 & -3 \\ 0 & 1 & -3 \end{pmatrix}$$

$$A_3^{-1} = \frac{1}{\det A} \cdot A^*$$

$a_{31} = (-1)^{3+1} \cdot \begin{vmatrix} 1 & 0 \\ -3 & 1 \end{vmatrix} = 1$

$a_{32} = (-1)^{3+2} \cdot \begin{vmatrix} -3 & 0 \\ -3 & 1 \end{vmatrix} = +3$

$a_{33} = (-1)^{3+3} \cdot \begin{vmatrix} -3 & 1 \\ -3 & -3 \end{vmatrix} = 9+3 = 12$

$$A^t = \begin{pmatrix} -3 & 1 & 0 \\ -3 & -3 & 1 \\ -3 & -3 & -3 \end{pmatrix}$$

$$A^* = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

$a_{11} = (-1)^{1+1} \cdot \begin{vmatrix} -3 & 1 \\ -3 & -3 \end{vmatrix} = 9+3 = 12$

$a_{12} = (-1)^{1+2} \cdot \begin{vmatrix} -3 & 1 \\ -3 & -3 \end{vmatrix} = -(9+3) = -12$

$a_{13} = (-1)^{1+3} \cdot \begin{vmatrix} -3 & -3 \\ -3 & -3 \end{vmatrix} = 9 - 9 = 0 \in \mathbb{Z}$

$a_{21} = (-1)^{2+1} \cdot \begin{vmatrix} 1 & 0 \\ -3 & -3 \end{vmatrix} = 3$

$a_{22} = (-1)^{2+2} \cdot \begin{vmatrix} -3 & 0 \\ -3 & -3 \end{vmatrix} = 9$

$a_{23} = (-1)^{2+3} \cdot \begin{vmatrix} -3 & 1 \\ -3 & -3 \end{vmatrix} = -12$

Metoda a II-a

$$\left(\begin{array}{ccc|ccc} -3 & -3 & -3 & 1 & 0 & 0 \\ 1 & -3 & -3 & 0 & 1 & 0 \\ 0 & 1 & -3 & 0 & 0 & 0 \end{array}\right) \xleftrightarrow{L_1 \leftrightarrow L_2} \left(\begin{array}{ccc|ccc} 1 & -3 & -3 & 0 & 1 & 0 \\ -3 & -3 & -3 & 1 & 0 & 1 \\ 0 & 1 & -3 & 0 & 0 & 0 \end{array}\right) \xrightarrow{L_2 \to L_2 + 3L_1} \left(\begin{array}{ccc|ccc} 1 & -3 & -3 & 0 & 1 & 0 \\ 0 & -12 & -12 & 1 & 3 & 0 \\ 0 & 1 & -3 & 0 & 0 & 0 \end{array}\right) \xrightarrow{L_2 \leftrightarrow L_3}$$

$$\sim \left(\begin{array}{ccc|ccc} 1 & -3 & -3 & 0 & 1 & 0 \\ 0 & 1 & -3 & 0 & 0 & 0 \\ 0 & -12 & -12 & 1 & 3 & 0 \end{array}\right) \xrightarrow[L_3 \to L_3 + 12L_2]{L_1 \to L_1 + 3L_2} \left(\begin{array}{ccc|ccc} 1 & 0 & -12 & 0 & 1 & 3 \\ 0 & 1 & -3 & 0 & 0 & 0 \\ 0 & 0 & -48 & 1 & 3 & 12 \end{array}\right) \xrightarrow{L_3 \to -\frac{1}{48}L_3} \left(\begin{array}{ccc|ccc} 1 & 0 & -12 & 0 & 1 & 3 \\ 0 & 1 & -3 & 0 & 0 & 0 \\ 0 & 0 & 1 & -\frac{1}{48} & -\frac{1}{16} & -\frac{1}{4} \end{array}\right)$$

$-9-27-12=-36+2$

$$\xrightarrow[L_2 \to L_2 + 3L_3]{L_1 \to L_1 + 12L_3} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -\frac{1}{4} & \frac{1}{4} & 0 \\ 0 & 1 & 0 & -\frac{1}{16} & -\frac{3}{16} & \frac{1}{4} \\ 0 & 0 & 1 & -\frac{1}{48} & -\frac{1}{16} & -\frac{1}{4} \end{array}\right)$$

I_3 $\quad$ A_3^{-1}

$$A_3^{-1} = \frac{1}{\det A} \cdot A^* \Rightarrow -\frac{1}{48}\begin{pmatrix} 12 & -12 & 0 \\ 3 & 9 & -12 \\ 1 & 3 & 12 \end{pmatrix}$$

Ne verificăm:

$\det A_3 \neq 0 \Rightarrow \exists A_3^{-1}$ a.î. $A \cdot A_3^{-1} = A_3^{-1} A = I_3$

$$-\frac{1}{48}\begin{pmatrix} 12 & -12 & 0 \\ 3 & 9 & -12 \\ 1 & 3 & 12 \end{pmatrix} \cdot \begin{pmatrix} -3 & -3 & -3 \\ 1 & -3 & -3 \\ 0 & 1 & -3 \end{pmatrix} = -\frac{1}{48}\begin{pmatrix} -48 & 0 & 0 \\ 0 & -48 & 0 \\ 0 & 0 & -48 \end{pmatrix} = I_3 \checkmark$$

c) Să se calculeze $\det(A_n)$, $\forall\, n \geq 3$

$$\det(A_n) = \begin{vmatrix} -3 & -3 & -3 & -3 & \dots & -3 \\ 1 & -3 & -3 & -3 & \dots & -3 \\ 0 & 1 & -3 & -3 & \dots & -3 \\ 0 & 0 \dots 1 & & -3 & \dots & -3 \\ \vdots & \vdots & & & \ddots & \vdots \\ 0 & 0 & \dots & & 1 & -3 \end{vmatrix} \quad \begin{array}{l} L_1 \to L_1 - L_2 \\ L_2 \to L_2 - L_3 \\ \vdots \\ L_{n-1} \to L_{n-1} - L_n \end{array} \quad \begin{vmatrix} -4 & 0 & 0 & \dots & 0 \\ 1 & -4 & & & 0 \\ 0 & 1 & -4 & & \vdots \\ 0 & 0 & \ddots & \ddots & \vdots \\ \vdots & \vdots & & 1 & -4 \\ 0 & 0 & \dots & 1 & -3 \end{vmatrix}$$

$$= \underbrace{(-4)(-4)\cdot \dots \cdot (-4)}_{(-4)^?} \cdot (-3)$$

$\longrightarrow \det(A_3) = (-4)^2 \cdot (-3)$; $2 = 3-1$

$\det(A_4) = (-4)^3 \cdot (-3)$; $3 = 4-1$

$\vdots$

$$\Rightarrow \det(A_n) = (-4)^{n-1} \cdot (-3),\ \forall\, n \geq 3$$

d) Să se arate că pt $\forall\, \underline{n \geq 3}$, A_n e inversabilă și inversa ei are cel puțin un element nr întreg

A_n inv $\Leftrightarrow \det A_n \neq 0 \Rightarrow (-3)\cdot(-4)^{n-1} \neq 0$ (A) pt că $-3 \neq 0$

$n \geq 3 \Rightarrow n-1 \geq 2 \Rightarrow (-4)^{n-1} \neq 0 \Big\} \Rightarrow$

$\Rightarrow (-3)\cdot(-4)^{n-1} \neq 0 \Rightarrow A_n$ inversabilă

$$A_3^{-1} = \begin{pmatrix} -\frac{1}{4} & -\frac{1}{4} & 0 \\ -\frac{1}{16} & -\frac{3}{16} & \frac{1}{4} \\ -\frac{1}{48} & -\frac{1}{16} & -\frac{1}{4} \end{pmatrix} \Rightarrow$$ a_{13} singurul element nr întreg

$$A^T = \begin{vmatrix} \cancel{-3} & \cancel{1} & \cancel{0} & \dots & \cancel{0} \\ -3 & -3 & 1 & \dots & 0 \\ -3 & -3 & -3 & \dots & 0 \\ -3 & -3 & -3 & -3 & 1 \\ -3 & -3 & -3 & \dots & -3 \end{vmatrix}$$

$$A_n^{-1} = \frac{1}{\det A_n} \cdot A^*$$

$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$

$A^t = \begin{pmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \end{pmatrix}$

$A^* = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

$$A^* = \begin{pmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \dots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \dots & a_{3n} \\ \vdots & \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \dots & a_{nn} \end{pmatrix}$$

$a = (-1)^{1+1} \cdot a_{22}$
$b = (-1)^{1+2} \cdot a_{12}$
$c = (-1)^{2+1} \cdot a_{21}$
$d = (-1)^{2+2} \cdot a_{11}$

$$a_{1n} = (-1)^{1+n} \cdot \underbrace{\begin{vmatrix} -3 & -3 & 1 & \dots & 0 \\ -3 & -3 & 1 & \dots & 0 \\ -3 & & \ddots & 1 & 0 \\ \vdots & & & & \\ -3 & \dots & & & -3 \\ -3 & -3 & \dots & & -3 \end{vmatrix}}_{0} = 0 \in \mathbb{Z} \Rightarrow \text{concluzie dorită}$$

$(L_n = L_{n-1})$