Seminar 1 - Alg. lin. - 2.10.2023

Teorie:

1) notație: k - corp, $k \subset \mathbb{C}$ ex: $\begin{cases} (\mathbb{R},+,\cdot) \\ (\mathbb{Q},+,\cdot) \\ (\mathbb{C},+,\cdot) \\ (\mathbb{Z}_p,+,\cdot), \ p \text{ prim} \end{cases}$

2) $\mathbb{Q}(\sqrt{2}) = \left(\dfrac{\mathbb{Q}[X]}{X^2-2} \right)$

verif: $(a+b\sqrt{2})(a-b\sqrt{2}) = a^2-2b^2 \neq 0.$

$$(a+b\sqrt{2})^{-1} = \frac{a-b\sqrt{2}}{a^2-2b^2}$$

3) $A \in M_{m,n}(k)$, $A = \begin{pmatrix} a_{11} & \dots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \dots & a_{mn} \end{pmatrix}$

Notă: A_{ij} = elem de pe $\begin{cases} \text{lin } i \\ \text{col } j \end{cases}$

4) op. cu matrici
- adunarea: $(A+B)_{ij} = A_{ij} + B_{ij}$
- înmulțirea $\underset{m\times n}{A} \cdot \underset{n\times p}{B} \to \underset{m\times p}{AB}.$

$$(AB)_{ij} = \sum_{k=1}^{n} A_{ik} \cdot B_{kj}$$

Dem înmulțirea mat. e asoc: $\underset{m\times n}{A}$, $\underset{n\times p}{B}$, $\underset{p\times q}{C}$.

$$[(A\cdot B)C]_{ij} = \sum_{k=1}^{p} (AB)_{ik} \cdot B_{kj}$$

$$[(AB)C]_{ij} = \sum_{k=1}^{p} \sum_{l=1}^{m} A_{il} \cdot B_{lk} \cdot C_{kj}$$

$$= \sum_{l=1}^{m} \sum_{k=1}^{p} A_{il} \cdot B_{lk} \cdot C_{kj} = \sum_{l=1}^{m} A_{il} \cdot (BC)_{lj}$$

$$= [A(BC)]_{ij}.$$

<u>Exerciții:</u>

<u>Pb1:</u> $\begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}^n$, $\begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}^n$ + generalizare.

inductiv $\begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}^n = \begin{pmatrix} 1 & na \\ 0 & 1 \end{pmatrix}$

Se obs că $A^3 = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}^3 = O_3 \Rightarrow A^n = O_3,\ n \geq 3.$

<u>Generalizare:</u> $\forall$ matrice sup. triunghiulară la $n = O_n$.

$$\begin{pmatrix} 0 & a & \cdots & \cdots \\ 0 & 0 & a & \cdots \\ - & - & - & - \\ 0 & 0 & \cdots & 0 \end{pmatrix}^n = O_n.$$

$\star$ $\left(\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 1 & 0 \\ \hline 0 & 0 & 2 \end{array}\right)^n = ?$

$$\left(\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 1 & 0 \\ \hline 0 & 0 & 2 \end{array}\right)^n = \left(\begin{array}{cc|c} \multicolumn{2}{c|}{\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}^n} & \begin{matrix} 0 \\ 0 \end{matrix} \\ \hline 0 & 0 & 2^n \end{array}\right) = \left(\begin{array}{cc|c} 1 & n & 0 \\ 0 & 1 & 0 \\ \hline 0 & 0 & 2^n \end{array}\right).$$

$\star\ A \in M_{m\times n}(k) \Rightarrow {}^tA \in M_{n\times m}(k)$

Pb5: ${}^t(AB) = {}^tB \cdot {}^tA$.

$${}^t(AB)_{ij} = (AB)_{ji} = \sum_{k=1}^{m} a_{jk} \cdot b_{ki} = ({}^tB \cdot {}^tA)_{ij}.$$

Pb7: Fie $A = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$. Arătați că $A^3 = 0$ și $(I-A)(I+A+A^2) = ?$

$A^2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$; $A^3 = O_3$.

$(I-A)(I+A+A^2) = I + \cancel{A} + \cancel{A^2} - \cancel{A} - \cancel{A^2} - A^3 = I$.

Generalizare: $A^n = O_n \Rightarrow (I-A)(I+A+A^2+\dots+A^{n-1}) = I$.

$$\frac{1}{1-x} = 1 + x + x^2 + \dots$$

Pb8: a) Mat. $\begin{cases} \text{sim} : A = {}^tA. \\ \text{antisim} : {}^tA = -A. \end{cases}$

${}^t(A + {}^tA) = {}^tA + {}^t({}^tA) = {}^tA + A \Rightarrow \underline{A + {}^tA \text{ sim}}$.

${}^t(A - {}^tA) = {}^tA - {}^t({}^tA) = {}^tA - A \Rightarrow \underline{A - {}^tA \text{ antisim}}$

b) $\forall A \in M_n(k) \Rightarrow \exists!\ B, C \in M_n(k)$ $\begin{cases} B \text{ sim} \\ C \text{ antisim} \end{cases}$ a.î. $A = B + C$.

Evident $A = \frac{1}{2}\left((A + {}^tA) + (A - {}^tA)\right)$.

Unicitatea: $A = B + C = B' + C'$ cu $\begin{cases} B, B' \text{ sim} \\ C, C' \text{ antisim}. \end{cases}$

$$\left.\begin{array}{l} A = B + C \\ {}^tA = B - C \end{array}\right\} \Rightarrow B = \frac{A + {}^tA}{2}, \quad C = \frac{A - {}^tA}{2}.$$

$$\boxed{M_n(K) = Sym \oplus Asym}$$

Pb10: $A \in M_{m \times n}(K)$. $CA = I_n$, $AD = I_m \overset{?}{\Rightarrow} m = n$, $C = D$.

$$\left.\begin{array}{l} (CA)D = I_n \cdot D = D. \\ C(AD) = C \cdot I_m = C \end{array}\right\} \Rightarrow C = D,\ n = m.$$

★ Definim: $D_i(u) = \begin{array}{c|c} & \begin{array}{c} \\ j \end{array} \\ \hline i & \begin{pmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & \vdots \\ & & & u & \\ 0 & 0 & \dots & 0 & 1 \end{pmatrix} \end{array}$ = I_n care are pe poz i,i elem. u.

$P_{ij} = \begin{array}{c|c} & \begin{array}{cc} i & j \end{array} \\ \hline \begin{array}{c} \\ \\ i \\ j \\ \\ \end{array} & \begin{pmatrix} 1 & 0 & \dots & \dots & 0 \\ 0 & 1 & \dots & \dots & 0 \\ 0 & 0 \dots 0 & \dots & 1 & 0 \dots \\ & 1 & \dots & 0 & \\ 0 & 0 & \dots & 0 & 1 \end{pmatrix} \end{array}$ = I_n în care se permută lin i cu lin j.

$T_{ij}(a) = \begin{array}{c|c} & j \\ \hline i & \begin{pmatrix} 1 & & & \\ & 1 & a & 0 \\ & 0 & \ddots & \\ & & & 1 \end{pmatrix} \end{array}$ = I_n în care pe poz ij se află a.

Ce se întâmplă când înmulțim o mat. A la st/dr. cu o mat $D_i(u)$, P_{ij}, $T_{ij}(a)$?

la stg.

$D_i(u)\cdot A \Rightarrow A$ în care $L_i \leftarrow u\cdot L_i$

$P_{ij}\cdot A \Rightarrow A$ în care se permută $L_i \leftrightarrow L_j$

$T_{ij}(a)A \Rightarrow A$ în care pe linia i este $L_i + aL_j$.

la dr.

$A\cdot D_i(u) \Rightarrow A$ în care $C_i \leftarrow u\, C_i$

$A\cdot P_{ij}(A) \Rightarrow A$ în care se permută $C_i \leftrightarrow C_j$

$A\cdot T_{ij}(A) \Rightarrow A$ în care pe col j este $C_j + a\, C_i$

Obs: $D_i^{-1}(u) = D_i\left(\frac{1}{u}\right)$

$P_{ij}^{-1} = P_{ij}$

$T_{ij}^{-1}(a) = T_{ij}(-a)$

Pb12e) Caut A a.î. $AB = BA$ $\forall\, B \in M_n(K)$. $A \in M_n(K)$.

$D_i(u)A = A\cdot D_i(u) \Rightarrow u\cdot a_{ij} = a_{ij}$ $\forall\, j \neq i \Rightarrow a_{ij} = 0$.

$u\cdot a_{ji} = a_{ji}$ $\forall\, j\neq i \Rightarrow a_{ji} = 0$

$\Rightarrow A$ e mat. diag.

$T_{ij}(1)\cdot A = A\cdot T_{ij}(1) \Rightarrow \lambda_i = \lambda_j \Rightarrow \boxed{A = \lambda\cdot I_n}$

Seminar 2 - Alg. lin - 9.10.2023

Pb. săpt. trecută:

R - inel.

x s.n. $\begin{cases} \text{div. al lui } 0 \text{ dc. } \exists y \neq 0 \text{ a.î. } xy = 0 \\ \text{inversabil dc. } \exists z \text{ a.î. } xz = zx = 1_R \end{cases}$

Obs: x inv $\Rightarrow$ nu e div. al lui 0.

$\cancel{x}z = 1.$

$yx = 0 \Rightarrow y = y(xz) = (yx)z = 0$

$\boxed{Q}$ $\begin{cases} A \in M_n(\mathbb{C}) \text{ nu e inv} \Rightarrow \text{e diviz. al lui } 0. \\ \text{dați ex. de } R \text{ a.î. } \exists \text{ elem. care nu e inv + nu e diviz al lui } 0. \end{cases}$

Teorie:

1) A nu e inv $\Leftrightarrow$ rk. $A \leq n-1$

$A = (c_1, c_2, \ldots, c_{n-1}, f(c_1, \ldots, c_{n-1}))$

2) rk A = nr. de pivoți din forma eșalon redusă.

Exerciții:

Pb1 $\begin{pmatrix} 1 & 2 & 1 & 2 & 1 \\ 2 & 1 & 2 & 1 & 2 \\ 0 & 1 & 0 & 1 & 0 \end{pmatrix} \xrightarrow{L_2 \leftarrow L_2 - 2L_1} \begin{pmatrix} \boxed{1} & 2 & 1 & 2 & 1 \\ 0 & -3 & 0 & -3 & 0 \\ 0 & 1 & 0 & 1 & 0 \end{pmatrix} \xrightarrow{L_2 \leftrightarrow L_3}$

$$\rightarrow \begin{pmatrix} \boxed{1} & 2 & 1 & 2 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & -3 & 0 & -3 & 0 \end{pmatrix} \xrightarrow[L_3 \leftarrow L_3 + 3L_2]{L_1 \leftarrow L_1 - 2L_2} \begin{pmatrix} \boxed{1} & 0 & 1 & 0 & 1 \\ 0 & \boxed{1} & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Pb 2 a) $\begin{cases} x_2 + 4x_3 = 5 \\ x_1 + 3x_2 + 5x_3 = -2 \\ 3x_1 + 7x_2 + 7x_3 = 6 \end{cases}$

$$\begin{pmatrix} 0 & 1 & 4 & 5 \\ 1 & 3 & 5 & -2 \\ 3 & 7 & 7 & 6 \end{pmatrix} \xrightarrow{L_1 \leftrightarrow L_2} \begin{pmatrix} 1 & 3 & 5 & -2 \\ 0 & 1 & 4 & 5 \\ 3 & 7 & 7 & 6 \end{pmatrix} \xrightarrow{L_3 \leftarrow L_3 - 3L_1}$$

$$\rightarrow \begin{pmatrix} \boxed{1} & 3 & 5 & -2 \\ 0 & 1 & 4 & 5 \\ 0 & -2 & -8 & 12 \end{pmatrix} \xrightarrow[L_3 \leftarrow L_1 + 2L_2]{L_1 \leftarrow L_1 - 3L_2} \begin{pmatrix} \boxed{1} & 0 & -7 & -17 \\ 0 & \boxed{1} & 4 & 5 \\ 0 & 0 & 0 & 22 \end{pmatrix} \xrightarrow[\substack{L_1 \leftarrow L_1 + 17L_3 \\ L_2 \leftarrow L_2 - 5L_1}]{L_3 \leftarrow \frac{1}{22} L_3}$$

$$\rightarrow \begin{pmatrix} \boxed{1} & 0 & -7 & 0 \\ 0 & \boxed{1} & 4 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \Rightarrow \text{sist. incompatibil}$$

! $A \in M_n(k)$ are forma eșalon redusă $I_n \Leftrightarrow A$ inv.

Dem: $\text{rk} A = n \Leftrightarrow A$ în forma eșalon redusă are n pivoți $\Leftrightarrow$ este I_n.

! A, B au ac. formă eșalon redusă $\Leftrightarrow \exists P \in M_n(k)$ inv. a.î. $PA = B$ $(A \sim B)$

$A = UF$; $U, F \in GL_n(k) \Leftrightarrow B = VF = (V \cdot U^{-1}) A$.
$B = VF$

! $(A|B) \longrightarrow (C|D) \Rightarrow A \sim C$

$$(A|B) = P\cdot(C|D) = (PC|PD) \Rightarrow A \sim C.$$

! $(A|I_n) \longrightarrow (I_n|A^{-1})$

$$(A|I_n) = P\cdot(I_n, B) \Rightarrow \left.\begin{array}{l} A = P\cdot I_n \\ I_n = P\cdot B \end{array}\right\} \Rightarrow \underline{B = A^{-1}}.$$

Seminar 3 - Alg. lin - 16.10.2023

Teorie:

k - corp comutativ.
$(V,+)$ gr. abelian
op. ext: „ · ": $k \times V \to V$ a.î.
$$\begin{cases} \alpha(v+w) = \alpha v + \alpha w \\ (\alpha+\beta) v = \alpha v + \beta v \\ \alpha(\beta v) = (\alpha\beta) v \\ 1 \cdot v = v. \end{cases}$$

Exerciții:

Pb1: k^n sp. vect.

$(x_1, \dots, x_n) + (y_1, \dots, y_n) \stackrel{def}{=} (x_1+y_1, \dots, x_n+y_n)$ gr. ab.

$\alpha(x_1, \dots, x_n) \stackrel{def}{=} (\alpha x_1, \dots, \alpha x_n)$.

verif 1: $\alpha[(x_1, \dots x_n) + (y_1, \dots y_n)] = (\alpha(x_1+y_1), \dots, \alpha(x_n+y_n))$
$= (\alpha x_1, \dots, \alpha x_n) + (\alpha y_1, \dots \alpha y_n) = \alpha(x_1 \dots x_n) + \alpha(y_1, \dots, y_n)$.

Analog pt. restul.

Pb2: a) $\mathbb{R}$ e $\mathbb{R}$ sp. vect.

Da, cu $+$ și $\cdot$ normale.

b) $\mathbb{Z}$ e $\mathbb{R}$ sp. vect.

Nu: $\frac{1}{2} \in \mathbb{R}$, $1 \in \mathbb{Z}$, $\frac{1}{2} \cdot 1 \notin \mathbb{Z}$.

<u>Pb3</u>: a) $0 \cdot v = 0 \; \forall v \in V$.

$0 \cdot v = (0+0)v = 0v + 0v \Rightarrow 0v = 0$.

b) $\alpha \cdot 0 = 0$

$\alpha \cdot 0 = \alpha(0+0) = \alpha 0 + \alpha 0 \Rightarrow \alpha 0 = 0$.

c) $a v = 0 \Rightarrow a = 0$ sau $v = 0$

Dc. $a \neq 0 \Rightarrow a$ inv $\Rightarrow 0 = a^{-1} 0 = a^{-1} \cdot a v = v$

d) $(-\alpha)v = \alpha(-v) = -\alpha v$.

$0 = (\alpha - \alpha)v = \alpha v + (-\alpha)v \Rightarrow (-\alpha)v = -\alpha v$

$0 = \alpha(v - v) = \alpha v + \alpha \cdot (-v) \Rightarrow -\alpha v = \alpha \cdot (-v)$.

! Dc. în def. la sp. vect. înlocuim K corp cu inel $\Rightarrow V$ este R-modul.

$\mathbb{Z}_4^2$ este $\mathbb{Z}_4$ modul.

$\hat{2} \cdot (\hat{2}, \hat{2}) = (\hat{0}, \hat{0})$ <u>are torsiune</u>

! $|\langle v \rangle_K| = |\{\alpha v \mid \alpha \in K\}| = |K|$

* a) $\mathbb{Z}$ nu e $\mathbb{Q}$ sp. vect.
  b) $\mathbb{Z}_p$ nu e $\mathbb{Q}$ sp. vect.

a) Pp. R.A $\exists \odot : \mathbb{Q} \times \mathbb{Z} \to \mathbb{Z}$ op. ext.

$1 \odot m = m$.

$2 \odot m = \cancel{2m}\ 2m$.

inductiv $q \odot m = q\, m$, $q \in \mathbb{N}$.

$$n = \left(q \cdot \frac{1}{q}\right) \odot n = q \odot \left(\frac{1}{q} \odot n\right) = q\left(\frac{1}{q} \odot n\right) \Rightarrow$$

$$\Rightarrow \frac{1}{q} \odot n = \frac{1}{q} n \Rightarrow m \odot n = mn \quad \forall m \in \mathbb{Q}.$$

$$\Rightarrow \underbrace{\frac{1}{2} \odot 1}_{\in \mathbb{Z}} = \frac{1}{2} \quad (F).$$

! $(V,+)$ gr. abelian. Atunci V are str. de $\mathbb{Q}$ sp. vect.

$\Leftrightarrow$ 1) este divizibil $\begin{cases} \forall v \in V \\ \forall n \geq 1 \end{cases} \Rightarrow \exists w \in V$ a.î. $n \cdot w = v$

2) nu are torsiune $\begin{cases} \forall x \in V \\ \forall n \geq 1 \end{cases}$ $nx = 0 \Rightarrow x = 0$.

Dem „$\Leftarrow$" $\frac{m}{n} \odot v = ?$

$m \odot v = \underbrace{v + \ldots + v}_{m \text{ ori}} \quad \forall m \in \mathbb{Z}.$

$\frac{1}{n} \odot v = w$ unde $nw = v$.

Dc. w_1, w_2 au prop. că $nw_1 = nw_2 \Rightarrow n(w_1 - w_2) = 0$
$\Rightarrow w_1 = w_2$.

$\Rightarrow$ op. e bine def.

! $\mathbb{Z}$ - nu are torsiune, nu e divizibil.
$\mathbb{Z}_p$ - e divizibil, are torsiune, p prim.
$\mathbb{Z}_n$ - nu e divizibil, nu are torsiune, $n \neq$ prim.

$\Rightarrow$ nu sunt $\mathbb{Q}$ sp. vect.

Pb5 : k corp şi $L \leq k$.
V e k-sp. vect $\overset{?}{\Rightarrow}$ V e L-sp. vect.

$V = k \Rightarrow k$ e L sp. vect.

$L \subset k$ def. $\cdot_L : L \times V \to V$

$$\cdot_L = \cdot_k /_{L \times V}$$

★ $\mathbb{R}$ nu e $\mathbb{Z}_p$ sp. vect.

$\alpha x = \beta x \Rightarrow (\alpha - \beta) x \vdots p$.

Fie „$\circ$" op. ext $\Rightarrow \hat{1} \circ x = x$, $x \neq 0$ | + de p ori.

$0 = px$. (F).

Dc. $(G, +)$ gr. abeliană are o sh. de $\mathbb{Z}_p$ sp. vect. $\Rightarrow$

$\Rightarrow G$ are p torsiune ($\underbrace{x + \ldots + x}_{p \text{ ori}} = 0$, $\forall x \in G$).

Pb7: $\mathbb{R} \overset{\exp}{\simeq} \mathbb{R}^*_+$

$(\mathbb{R}, +)$ e $\mathbb{R}$ sp. vect $\Rightarrow (\mathbb{R}, \oplus)$ e $\mathbb{R}$ sp. vect.

! $\mathbb{C}$ e $\mathbb{R}$ sp. vect. $\left\{\begin{array}{l} 1) \text{ se verifică} \\ 2) \ \mathbb{C} \simeq \mathbb{R}^2 \\ 3) \ \mathbb{R} \leq \mathbb{C}. \end{array}\right.$

$(\mathbb{R}, +)$ are o sh. de $\mathbb{C}$ sp. vect.
$(\mathbb{Q}, +)$ nu are sh. de $\left\{\begin{array}{l} \mathbb{C} \text{ sp. vect.} \\ \mathbb{R} \text{ sp. vect.} \end{array}\right.$

Seminar 5 - Alg. lin. - 30.10.2023

Subsp. vect. Baze - Teorie:

1) S.n. sist. lin. indep (SLI) o mulțime $\{v_1, \dots, v_m\}$ a.î. $\alpha_1 v_1 + \dots + \alpha_m v_m \Rightarrow \alpha_i = 0 \ \forall i = \overline{1,m}$

2) S.n. sist. de gen. (SDG) o mulțime $S = \{v_1, \dots, v_k\} \subset V$ a.î. $\langle S \rangle = V$.

3) S.n. bază: $B \subset V$ dc. e $\begin{cases} \text{S.L.I} \\ \text{S.D.G.} \end{cases} \Rightarrow \dim_k V = |B|$.

<u>Obs</u>: $B = \{v_1, \dots, v_k\} \subset V$ și $\dim_k V = n$. Urm. afirm. sunt echivalente:

- B e sist. de gen $+ |B| = n$
- B e sist. lin. indep $+ |B| = n$
- B e bază

$\boxed{Q}$ În k^n cum recunoaștem S.D.G, SL.I și bazele?

$$S = \left\{ v_1 = \begin{pmatrix} a_{11} \\ \vdots \\ a_{n1} \end{pmatrix}, \dots, v_m = \begin{pmatrix} a_{1m} \\ \vdots \\ a_{nm} \end{pmatrix} \right\} \text{ și}$$

$$A = \begin{pmatrix} a_{11} & \dots & a_{1m} \\ \vdots & & \vdots \\ a_{n1} & \dots & a_{nm} \end{pmatrix} = \begin{pmatrix} | & & | \\ v_1 & \dots & v_m \\ | & & | \end{pmatrix} \in M_{n \times m}$$

! S e S.L.I $\Rightarrow \boxed{\text{rk } A = m}$

$\alpha_1 v_1 + \dots + \alpha_m v_m = 0 \Rightarrow \alpha_i = 0 \Rightarrow A\begin{pmatrix}\alpha_1\\ \vdots \\ \alpha_m\end{pmatrix} = \begin{pmatrix}0\\ \vdots\\ 0\end{pmatrix}$ are

doar sol nulă $\Rightarrow \operatorname{rk} A = m$.

! S este S.D.G $\Rightarrow \boxed{\operatorname{rk} A = n}$, $A = \begin{pmatrix} | & & | \\ v_1 & \dots & v_m \\ | & & | \end{pmatrix}$

S este bază $\Rightarrow \boxed{n = m \text{ și } \det\begin{pmatrix} | & & | \\ v_1 & \dots & v_m \\ | & & | \end{pmatrix} \neq 0}$

Exerciții:

Pb1: a) $\left\{\begin{pmatrix}1\\3\\0\end{pmatrix}, \begin{pmatrix}0\\-1\\3\end{pmatrix}\right\} \subset \mathbb{R}^3$ e S.L.I?

Da: $\operatorname{rk}\begin{pmatrix}1 & 0\\ 3 & -1\\ 0 & 3\end{pmatrix} = 2 \Rightarrow$ bază: $\mathcal{B} = \left\{\begin{pmatrix}1\\3\\0\end{pmatrix}, \begin{pmatrix}0\\-1\\3\end{pmatrix}, \begin{pmatrix}0\\0\\1\end{pmatrix}\right\}$

c) $\left\{\begin{pmatrix}1\\-2\\4\end{pmatrix}\right\} \subset \mathbb{R}^3$ e S.L.I?

Da: $\begin{pmatrix}1\\-2\\4\end{pmatrix} \neq 0 \Rightarrow$ bază: $\mathcal{B} = \left\{\begin{pmatrix}1\\-2\\4\end{pmatrix}, \begin{pmatrix}0\\1\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\end{pmatrix}\right\}$

Pb2: b) $\left\{\begin{pmatrix}2\\-3\\0\end{pmatrix}, \begin{pmatrix}-4\\6\\3\end{pmatrix}\right\} \subset \mathbb{R}^3$ e S.D.G?

Nu: $\operatorname{rk}\begin{pmatrix}2 & -4\\ -3 & 6\\ 0 & 3\end{pmatrix} \leq 2 < 3 = \dim_{\mathbb{R}} \mathbb{R}^3$

c) $\left\{\begin{pmatrix}0\\-2\\3\end{pmatrix}, \begin{pmatrix}-2\\0\\3\end{pmatrix}, \cancel{\begin{pmatrix}0\\-4\\-3\end{pmatrix}}, \begin{pmatrix}1\\1\\1\end{pmatrix}\right\} \subset \mathbb{R}^3$ e S.D.G?

Da: $\begin{vmatrix}0 & -2 & 1\\ -2 & 0 & 1\\ 3 & 3 & 1\end{vmatrix} = -6 - 6 - 4 \neq 0$.

Pb3: d) $W = \left\{ \begin{pmatrix} 3a+6b-c \\ 6a-2b-2c \\ -5a+5b+3c \\ -3a+b+c \end{pmatrix} \middle| a,b,c \in \mathbb{R} \right\}$

$$W = \left\{ a \cdot \begin{pmatrix} 3 \\ 6 \\ -5 \\ -3 \end{pmatrix} + b \begin{pmatrix} 6 \\ -2 \\ 5 \\ 1 \end{pmatrix} + c \begin{pmatrix} -1 \\ -2 \\ 3 \\ 1 \end{pmatrix} \right\} =$$

$$= \left\langle \begin{pmatrix} 3 \\ 6 \\ -5 \\ -3 \end{pmatrix}, \begin{pmatrix} 6 \\ -2 \\ 5 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ -2 \\ 3 \\ 1 \end{pmatrix} \right\rangle \Rightarrow \dim_{\mathbb{R}} W = 2, \text{ baza: } \left\{ \begin{pmatrix} 3 \\ 6 \\ -5 \\ -3 \end{pmatrix}, \begin{pmatrix} 6 \\ -2 \\ 5 \\ 1 \end{pmatrix} \right\}$$

$\times 3.$

Pb4: $W = \left\{ (x_1, \dots, x_4) \in \mathbb{R}^4 \middle| \begin{cases} -3x_1 + 6x_2 - x_3 + x_4 = 0 \\ x_1 - 2x_2 + 2x_3 + 3x_4 = 0 \\ 2x_1 - 4x_2 + 5x_3 + 8x_4 = 0 \end{cases} \right\}$

a) $$\underbrace{\begin{pmatrix} -3 & 6 & -1 & 1 \\ 1 & -2 & 2 & 3 \\ 2 & -4 & 5 & 8 \end{pmatrix}}_{A} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow W \leq_{\mathbb{R}} \mathbb{R}^4.$$

b) $\text{rk } A = 2.$, $x_2 = \alpha$, $x_3 = \beta.$ $\begin{cases} -3x_1 + 6\alpha - \beta + x_4 = 0. \\ x_1 - 2\alpha + 2\beta + 3x_4 = 0. \end{cases}$

$$-3x_1 + 6\alpha - \beta + x_4 = 0$$
$$3x_1 - 6\alpha + 6\beta + 9x_4 = 0 \quad (+)$$

$$5\beta + 10x_4 = 0 \Rightarrow x_4 = \frac{\beta}{2} \Rightarrow x_1 = 2\alpha + \frac{7\beta}{2}$$

$$\Rightarrow W = \left\langle (1, 1, 0, 0), \left(\frac{7}{2}, 0, 1, \frac{1}{2}\right) \right\rangle \Rightarrow \dim_{\mathbb{R}} W = 2.$$

Pb6: $W_1 = \left\langle \underbrace{\begin{pmatrix}1\\2\\0\\1\end{pmatrix}}_{v_1}, \underbrace{\begin{pmatrix}1\\1\\1\\0\end{pmatrix}}_{w_1} \right\rangle$, $W_2 = \left\langle \underbrace{\begin{pmatrix}1\\0\\1\\0\end{pmatrix}}_{v_2}, \underbrace{\begin{pmatrix}1\\3\\0\\1\end{pmatrix}}_{w_2} \right\rangle$. baze pt. W_1, W_2, W_1+W_2, $W_1 \cap W_2$.

! Th. Grassmann: $\boxed{\dim(W_1+W_2) = \dim W_1 + \dim W_2 - \dim W_1 \cap W_2}$

$\dim W_1 = 2$, $W_1 + W_2 = \langle v_1, v_2, w_1, w_2 \rangle$.
$\dim W_2 = 2$.

$$\begin{vmatrix} 1 & 1 & 1 & 1 \\ 2 & 1 & 0 & 3 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{vmatrix} = 0 \quad \Rightarrow \dim(W_1+W_2) = 3 \Rightarrow \dim W_1 \cap W_2 = 1.$$

Obs: $\underbrace{v_1 + w_1 = v_2 + w_2}_{\subset W_1 \cap W_2} \Rightarrow$

$$B_{W_1 \cap W_2} = \{ v_1 + w_1 \} = \begin{pmatrix}2\\3\\1\\1\end{pmatrix}$$

Pb10: $V = \{ f: \mathbb{R} \to \mathbb{R} \mid f \text{ cont} \}$. $\lambda_1 \dots \lambda_n \in \mathbb{R}$ dif. Dem. $\{ e^{\lambda_1 x}, \dots, e^{\lambda_n x} \}$ s.l.i.

Fie α_i a.î. $\sum_{i=1}^{n} \alpha_i e^{\lambda_i x} = 0$ eg. de funcții $\Rightarrow$

$\Rightarrow x = 0 \Rightarrow \sum_{i=1}^{n} \alpha_i = 0$.

$x = 1 \Rightarrow \sum_{i=1}^{n} \alpha_i e^{\lambda_i} = 0$

$\vdots$

$x = n-1 \Rightarrow \sum_{i=1}^{n} \alpha_i e^{\lambda_i (n-1)} = 0$.

$$\Rightarrow \begin{vmatrix} 1 & \dots & 1 \\ e^{\lambda_1} & & e^{\lambda_n} \\ \vdots & & \vdots \\ e^{\lambda_1 (n-1)} & & e^{\lambda_n (n-1)} \end{vmatrix} = \underbrace{\prod_{i<j} (e^{\lambda_i} - e^{\lambda_j})}_{\neq 0} \neq 0 \Rightarrow \text{are doar sol } (0, \dots, 0)$$

$\Rightarrow$ L s.l.i $\Rightarrow \dim_{\mathbb{R}} V = \infty$.

! V - k sp. vect și $\dim_k V = n \Rightarrow V \simeq k^n$.

Aleg $(v_1, \dots, v_n)$ o bază $\Rightarrow v = \alpha_1 v_1 + \dots + \alpha_n v_n$.

$f(v) \to (\alpha_1, \dots, \alpha_n) \in k^n$ bij. coord. lui v în k^n.

Pb11: Fie F un corp cu q elem. și V un F-sp. vect. de dim n. Calc. nr. subsp. vect $W \leq_F V$ de dim k. $\leftarrow g(n,k)$.

$g(n,0) = 1$

$g(n,n) = 1$.

$\boxed{\text{Pt. } k=1} \Rightarrow W = \langle v \rangle, \ v \neq 0$

$\bar{v} = \bar{w} \Rightarrow \exists \lambda \in F \setminus \{0\}$ a.î. $v = \lambda w$.

$$g(n,1) = \frac{q^n - 1}{q - 1}.$$

$\boxed{\text{Pt } k}$ ~~$W \Rightarrow$~~ $W = \langle v_1, v_2, \dots, v_k \rangle$

$q^n - 1 \quad q^n - q \quad q^n - q^{k-1}$.

$$g(n,k) = \frac{(q^n-1)(q^n-q)\dots(q^n-q^{k-1})}{(q^k-1)(q^k-q)\dots(q^k-q^{k-1})}$$ $\leftarrow$ modurile de a alege o bază.

! Temă: $\{1, \sqrt{p_1}, \sqrt{p_2}, \dots, \sqrt{p_m}\}$ s.l.i peste $\mathbb{Q}$.

$p_i \leftarrow$ al i-lea nr. prim

Seminar 6-7 - Alg. lin. - 6.11.2023

S6: **Pb1:** $B = \{ b_1 = (1,1,1,1), b_2 = (0,1,1,1), b_3 = (0,0,1,1), b_4 = (0,0,0,1) \}$ e bază în $\mathbb{R}^4$ + coord $(9,-3,2,4)$ în B.

$\det \begin{pmatrix} | & & | \\ b_1 & \cdots & b_4 \\ | & & | \end{pmatrix} \neq 0 \Leftrightarrow B$ bază.

$$\begin{vmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{vmatrix} = 1 \neq 0$$

$(9,-3,2,4) = \alpha_1 b_1 + \dots + \alpha_4 b_4$

$= (\alpha_1, \alpha_1 + \alpha_2, \alpha_1 + \alpha_2 + \alpha_3, \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4)$

$\Rightarrow (9, -12, 5, 2)$.

Pb2 a) $\dim_k M_{m \times n}(k) = m \times n$.

$B_0 = \{ E_{ij} \mid \begin{smallmatrix} i = \overline{1,m} \\ j = \overline{1,n} \end{smallmatrix} \}$, $|B_0| = m \times n$.

$M_{m \times n}(k) \sim k^{mn}$.

b) $M_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, $M_2 = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$, $M_3 = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$, $M_4 = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ bază $M_2(k)$

e bază evident din Ex1

! $U_1 \oplus U_2 = V \leftarrow$ sumă directă $\Leftrightarrow \begin{cases} U_1 + U_2 = V \\ U_1 \cap U_2 = \{0\} \end{cases}$

Pb3: $U_1 = \{ (t, 2t, 3t) \mid t \in \mathbb{R} \}$

$U_2 = \{ (x,y,z) \mid x + y + z = 0 \}$.

$U_1 \oplus U_2 = \mathbb{R}^3$

Grassmann: $\underbrace{\dim_{\mathbb{R}}(U_1+U_2)}_{\overset{\shortparallel}{3}} + \underbrace{\dim_{\mathbb{R}}(U_1 \cap U_2)}_{\overset{\shortparallel}{0}} = \underbrace{\dim_{\mathbb{R}} U_1}_{\overset{\shortparallel}{1}} + \underbrace{\dim_{\mathbb{R}} U_2}_{\overset{\shortparallel}{2}}$

$U_1 = \langle (1,2,3) \rangle \quad \Rightarrow \dim_{\mathbb{R}} U_1 = 1$

$U_2 = \langle (1,0,-1), (0,1,-1) \rangle \ \dim_{\mathbb{R}} U_2 = 2.$

$v \in U_1 \cap U_2 \Rightarrow \left.\begin{array}{l} v = \alpha \cdot (1,2,3) \\ v = (x, y, -x-y) \end{array}\right\} \Rightarrow 6\alpha = 0 \Rightarrow \alpha = 0 \Leftrightarrow v = 0$

$\Rightarrow U_1 \cap U_2 = \{0\} \Rightarrow \dim_{\mathbb{R}}(U_1 + U_2) = 3 \Rightarrow U_1 + U_2 = \mathbb{R}^3$

$U_1 \cap U_2 = \{0\} \Rightarrow U_1 \oplus U_2 = \mathbb{R}^3$

Pb 4: $Sym = \{A \in M_n(\mathbb{R}) \mid A = {}^tA\}$

$ASym = \{A \in M_n(\mathbb{R}) \mid -A = {}^tA\}$

a) $Sym, ASym \leq_{\mathbb{R}} M_n(\mathbb{R})$ + dim.

$\forall X, Y \in Sym$, $\forall \alpha, \beta \in \mathbb{R} \Rightarrow {}^t(\alpha X + \beta Y) = \alpha\, {}^tX + \beta\, {}^tY = \alpha X + \beta Y \in Sym.$

$\mathbb{B}_{Sym} = \{E_{ii} \mid i = \overline{1,n}\} \cup \{E_{ij} + E_{ji} \mid i<j\}$

$\Rightarrow \dim_{\mathbb{R}} Sym = \dfrac{n(n+1)}{2}$

$\mathbb{B}_{ASym} = \{E_{ij} - E_{ji} \mid i<j\} \Rightarrow \dim_{\mathbb{R}} ASym = \dfrac{n(n-1)}{2}$

b) $Sym \oplus ASym = M_n(\mathbb{R}).$

Grassman. $\underbrace{\dim(Sym + Asym)}_{\overset{\shortparallel}{n^2}} + \underbrace{\dim(Sym \cap Asym)}_{\overset{\shortparallel}{0}} = \underbrace{\dim Sym}_{\overset{\shortparallel}{\frac{n(n+1)}{2}}} + \underbrace{\dim Asym}_{\overset{\shortparallel}{\frac{n(n-1)}{2}}}$

$X \in Sym \cap ASym \Rightarrow \left.\begin{array}{l} X = {}^tX \\ -X = {}^tX \end{array}\right\} \Rightarrow X = O_n$

2/10

4/5

Aplicații liniare:

Pb 5: Decideți dc urm. apl. sunt liniare:

0) $f: \mathbb{C}^n \to \mathbb{C}^m$, $f(v) = A \cdot v$, $A \in M_{m\times n}$

Da: $\alpha, \beta \in \mathbb{C}$
$u, v \in \mathbb{C}^n \Rightarrow f(\alpha u + \beta v) = A(\alpha u + \beta v) = \alpha A u + \beta A v$
$= \alpha f(u) + \beta f(v)$.

a) $f: \mathbb{R}^2 \to \mathbb{R}$, $f(x,y) = (5x - y)$

Da: $f(x,y) = (5, -1)\begin{pmatrix} x \\ y \end{pmatrix} \Rightarrow$ apl. lin.

b) $f: \mathbb{R}^2 \to \mathbb{R}$, $f(x,y) = 5xy$

Nu: $f(2(x,y)) = 20xy = 4f(x,y)$.

c) $f: \mathbb{C}[X]_n \to \mathbb{C}[X]_n$, $f(P(x)) = P'(x)$.

Da: $f(\alpha P + \beta Q) = \alpha P' + \beta Q' = \alpha f(P) + \beta f(Q)$.

Pb 6: $f: \mathbb{R}^4 \to \mathbb{R}^3$, $f(x,y,z,t) = (3x + 2y,\ x + z - 2t,\ 5x + 2y + 2z - 4t)$

a) f apl. lin.

$$f(x,y,z,t) = \begin{pmatrix} 3 & 2 & 0 & 0 \\ 1 & 0 & 1 & -2 \\ 5 & 2 & 2 & -4 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix} \Rightarrow \text{apl. lin}$$

b) baze ker f, Im f.

$$\begin{pmatrix} 3 & 2 & 0 & 0 \\ 1 & 0 & 1 & -2 \\ 5 & 2 & 2 & -4 \end{pmatrix} \xrightarrow{L_1 \leftrightarrow L_2} \begin{pmatrix} 1 & 0 & 1 & -2 \\ 3 & 2 & 0 & 0 \\ 5 & 2 & 2 & -4 \end{pmatrix} \xrightarrow[L_3 \leftarrow L_3 - 5L_1]{L_2 \leftarrow L_2 - 3L_1} \begin{pmatrix} \boxed{1} & 0 & 1 & -2 \\ 0 & 2 & -3 & 6 \\ 0 & 2 & -3 & 6 \end{pmatrix} \xrightarrow[L_2 \leftarrow \frac{1}{2}L_2]{L_3 \leftarrow L_3 - L_2}$$

$$\to \begin{pmatrix} \boxed{1} & 0 & 1 & -2 \\ 0 & \boxed{1} & -\frac{3}{2} & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix} \Rightarrow \begin{matrix} z = \alpha \\ t = \beta \end{matrix} \Rightarrow \begin{cases} x + \alpha - 2\beta = 0 \\ y - \frac{3}{2}\alpha + 3\beta = 0 \end{cases}$$

$$\Rightarrow \ker f = \{(-\alpha + 2\beta,\ \tfrac{3}{2}\alpha - 3\beta,\ \alpha,\ \beta) \mid \alpha, \beta \in \mathbb{R}\}$$

$$= \langle (-1, \tfrac{3}{2}, 1, 0), (2, -3, 0, 1) \rangle.$$

$$\underbrace{\dim_{\mathbb{R}} \mathbb{R}^4}_{=4} = \underbrace{\dim_{\mathbb{R}} \ker f}_{=2} + \dim_{\mathbb{R}} \operatorname{Im} f \Rightarrow \boxed{\dim_{\mathbb{R}} \operatorname{Im} f = 2}$$

$$l_3 = l_1 + 2 l_2$$

$$\operatorname{Im} f = \langle f(1,0,0,0), \dots, f(0,0,0,1) \rangle$$

$$= \langle \underbrace{\begin{pmatrix} 3 \\ 1 \\ 5 \end{pmatrix}, \begin{pmatrix} 2 \\ 0 \\ 2 \end{pmatrix}}_{\text{bază}}, \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 0 \\ -2 \\ -4 \end{pmatrix} \rangle$$

Pb 7: $f: \mathbb{R}^3 \to \mathbb{R}^3$, $f(x,y,z) = (3z - x + 3y,\ x - y,\ 2x + y + z)$

izo lin + $f^{-1} = ?$

$$f(x,y,z) = \underbrace{\begin{pmatrix} -1 & 3 & 3 \\ 1 & -1 & 0 \\ 2 & 1 & 1 \end{pmatrix}}_{A} \begin{pmatrix} x \\ y \\ z \end{pmatrix} \quad f \text{ izo} \Leftrightarrow \begin{cases} f \text{ inj} \Leftrightarrow \ker f = \{0\} \\ f \text{ surj} \Leftrightarrow \operatorname{Im} f = \mathbb{R}^3 \end{cases} \Leftrightarrow \boxed{\det A \neq 0}.$$

$$f^{-1}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = A^{-1} \begin{pmatrix} x \\ y \\ z \end{pmatrix}.$$

Pb 8: a) $\operatorname{tr}: M_n(\mathbb{R}) \to \mathbb{R}$ morfism lin. (evident)

b) $sl_n(\mathbb{R}) = \{A \in M_n(\mathbb{R}) \mid \operatorname{tr} A = 0\}. \leq_{\mathbb{R}} M_n(\mathbb{R})$

$sl_n = \ker(\operatorname{tr}) \leq_{\mathbb{R}} M_n(\mathbb{R})$

c) dim + bază $sl_n(\mathbb{R})$.

$$\dim \ker tr + \underset{\substack{\parallel \\ 1}}{\dim tr} = n^2 \Rightarrow \dim \ker(tr) = n^2-1$$

$\mathcal{B} = \{E_{ij} \mid i \neq j\} \cup \{E_{11} - E_{ii} \mid i = \overline{2,n}\}$ bază ker tr.

Pb10: V, W - k sp. vect.

$Lin(V,W) = \{f : V \to W \mid f \text{ apl. lin}\}$.

a) $Lin(V,W)$ are str. canonică de k sp. vect.

pt. un $v \in k$ ⇒ definesc $(f+g)(v) = f(v) + g(v)$

$(\lambda f)(v) = \lambda \cdot f(v)$

$\left\{ \begin{array}{l} f, g \in Lin(V,W) \\ \lambda \in k \end{array} \right.$

b) $\dim_k V = n$ și $\dim_k W = m \overset{?}{\Rightarrow} \dim_k Lin(V,W) = ?$

$$\begin{array}{ccc} V & \xrightarrow{f} & W \\ \downarrow\varphi \ \wr\!\wr & & \wr\!\wr \ \downarrow\psi \\ k^n & \xrightarrow[\tilde{f}]{} & k^m \end{array}$$

$\tilde{f} = \psi \circ f \circ \varphi^{-1}$

$\parallel$

înmulțirea cu o matrice $M_{m\times n}(k)$.

$\Rightarrow \boxed{\dim_k Lin(V,W) = m \cdot n}$

! $V^* = Lin(V, k)$ s.n. dualul lui V.

S7: Teorie: Fie V un k sp. vect. și $H \leq_k V$ s.n. hiperplan dacă $\boxed{\dim_k H = \dim_k V - 1}$

Pb1 a) $H_1, H_2 \leq V$ hiperplane $\Rightarrow H_1 = H_2$ sau $H_1 + H_2 = V$.

$\boxed{I\,2020}$

Pp $H_1 \neq H_2 \Rightarrow$ vreau $H_1 + H_2 = V$.

Grassmann $\Rightarrow \dim(H_1 + H_2) + \dim(H_1 \cap H_2) = \underset{n-1}{\dim H_1} + \underset{n-1}{\dim H_2}$.

$\dim(H_1 \cap H_2) \leq \dim H_1 = n-1$.

dc. $\dim(H_1 \cap H_2) = n-1 \Rightarrow H_1 \cap H_2 = H_1 = H_2$ F.

$\Rightarrow \dim(H_1 \cap H_2) \leq n-2 \Rightarrow \dim(H_1 + H_2) \geq n \Rightarrow$

$\Rightarrow \dim(H_1 + H_2) = n \Rightarrow \boxed{H_1 + H_2 = V.}$

b) $\forall\, W \leq V$ subsp. de dim k se poate scrie ca $\cap$ de $n-k$ hiperplane.

Iau $\{b_1, \dots, b_k\}$ bază în W.

$B = \{b_1, \dots, b_k, e_1, \dots, e_{n-k}\}$ bază în V.

$H_i = \langle B \setminus \{e_i\} \rangle \Rightarrow W \leq H_i \Rightarrow W \leq \cap H_i$.

$v \in \cap H_i$

$v = \sum_1^k \alpha_i b_i + \sum_1^{n-k} \beta_i e_i$.

$v \in \cap H_i \Rightarrow v \in H_i \quad \forall i \Rightarrow$ ~~$v = a_i b_1 + \dots +$~~

$\Rightarrow v = \sum_{j=1}^{k} a_{ij} b_j + \sum_{\substack{j=1 \\ j \neq i}}^{n-k} c_{ij} e_j \overset{SL.I}{\Rightarrow} v = \underbrace{\sum_1^k a_{ij} b_j}_{\in W} \Rightarrow$

$\Rightarrow \boxed{\cap H_i = W.}$

Pb2 V un $\mathbb{R}$ - sp. vect. de dim $n \geq 1$. V nu se poate scrie
[2023] ca reuniune finită de subsp. proprii.

Dem: Pp. $\exists\, W_1, \dots, W_k \leq V$ proprii a.î.

$V = W_1 \cup \dots \cup W_k$ cu k min.

$$\left.\begin{aligned} &w_1 \in W_1 \\ &w_2 \in V \setminus W_1 \end{aligned}\right\} \Rightarrow w_1 + a w_2 \in \underbrace{V \setminus W_1}_{W_2 \cup \dots \cup W_k} \quad \forall a \in \mathbb{R}.$$

$$\Rightarrow w_1 + a w_2 \in W_{i_a},\ i_a \geq 2 \Rightarrow \exists\, a > b \text{ a.î. } \begin{matrix} w_1 + a w_2 \\ w_1 + b w_2 \end{matrix} \in W_i,\ i \geq 2.$$

$$\Rightarrow (a-b) w_2 \in W_i \Rightarrow w_2 \in W_i \Rightarrow w_1 \in W_i \Rightarrow W_1 \subseteq W_2 \cup \dots \cup W_k \quad \text{(F)}.$$

Pb6: $|k| < \infty$ corp $\Rightarrow |k| = p^n$, p prim

! k corp $\Rightarrow$ fie $\mathbb{Z}_p \hookrightarrow k$ (char $k \neq 0$)
fie $\mathbb{Q} \hookrightarrow k$ (char $k = 0$) ca subcorp.

$\varphi : \mathbb{Z} \to k$, $\varphi(n) = \underbrace{1 + \dots + 1}_{n \text{ ori}}$ morfism

$\ker \varphi \trianglelefteq \mathbb{Z}$

$k \subset L \Rightarrow L$ e k sp. vect.

$\mathbb{Z}_p \subset k \Rightarrow |k| = |\mathbb{Z}_p|^{\dim_{\mathbb{Z}_p} k} = p^n$.

Pb3 W subsp. vect. V a.î. pt. $\forall\, U$ subsp. de dim p,
$\dim(W \cap U) \geq 1$. Să se arate că $\dim V - \dim W \leq p-1$

Pp. $\dim V - \dim W \geq p$

pt. $W \leq V \Rightarrow \exists W'$ a.î. $\boxed{W \oplus W' = V}$
complement pt. W.

$\dim W' = \dim V - \dim W \geq p \Rightarrow \dim(W \cap W') \geq 1$ (F).

Pbs: $S \leq M_m(\mathbb{R})$, $S = \langle AB - BA \mid A, B \in M_m(k) \rangle$

$\overset{?}{\Rightarrow} \dim S = m^2 - 1$.

$$X \in S \Rightarrow X = \sum_{fin} (AB - BA) \Rightarrow \operatorname{tr} X = 0 \Rightarrow \dim S \leq m^2 - 1$$

! $E_{ij} \cdot E_{kl} = \begin{cases} E_{il}, & \text{pt. } j = k. \\ 0 & \text{altfel.} \end{cases}$

$A = E_{i1}$, $B = E_{1j}$ $\Rightarrow AB - BA = E_{ij}$, $i \neq j$.
$\Rightarrow \langle E_{ij}, i \neq j \rangle \subseteq S$.
$A = E_{ij}$, $B = E_{ji}$, $i \neq j$
$AB - BA = E_{ii} - E_{jj} \Rightarrow \langle E_{ii} - E_{jj} \rangle \subseteq S$

$\mathcal{A} = \{E_{ij} \mid i \neq j\} \cup \{E_{ii} - E_{jj} \mid i \neq j\} \subseteq S \Rightarrow \dim S = m^2 - 1$.

! $M_m(k)/S \simeq k$.

Teorie dual:

1) $V^* = \{ f : V \to k \mid f \text{ apl. lin} \}$, V k-sp. vect.

2) $\dim_k V = m \Rightarrow \dim_k V^* = m$

3) $(V^*)^* = V^{**} \leftarrow$ bidual $\Rightarrow \exists\, i : V \to V^{**}$ morfism canonic $\xleftarrow{}$ nu dep. de alegere
inj.

a) alegem $\mathbb{B} = \{b_1, \dots, b_n\}$ bază în V.

! pt. $v = a_1 b_1 + \dots + a_n b_n$ definesc $b_j^*(v) = a_j$

$\Rightarrow b_j^*(b_i) = \delta_{ij} = \begin{cases} 1, & i=j \\ 0, & \text{altfel.} \end{cases}$

$\alpha_1 b_1^* + \dots + \alpha_n b_n^* = 0$ în V^* (ca funcții) $\Rightarrow$

$(\alpha_1 b_1^* + \dots + \alpha_n b_n^*)(v) = 0 \ \forall v \in V.$

$v = b_j \Rightarrow \underbrace{(\alpha_1 b_1^* + \dots + \alpha_n b_n^*)(b_j)}_{\alpha_j} = 0 \Rightarrow$ sist. lin indep. în V^*

Aleg $w^* \in V^* \Rightarrow$ notez $w^*(b_i) = \beta_i$

$w^*(v) = w^*(a_1 b_1 + \dots + a_n b_n) = a_1 \beta_1 + \dots + a_n \beta_n$

$= \beta_1 b_1^* + \dots + \beta_n b_n^* \in \langle b_1^*, \dots b_n^* \rangle$

$\Rightarrow \{b_1^*, \dots, b_n^*\}$ bază pt. V^* $\Rightarrow$ $\underline{\dim V = n}$ $\Rightarrow$ $\underline{\dim V^* = n}$.

b) ! definesc $i: V \to V^{**}$

$(i(v))(v^*) = v^*(v)$ apl. lin.

$i(v) \in V^{**}$ funcție evaluată în $v^* \in V^*$

$v \in V$.

Arăt că i e inj. iau $v \in \ker i \Rightarrow i(v) = 0_{V^{**}}$

$\Rightarrow i(v)(v^*) = 0$

$ \ \ \ \ \ \| $

$ v^*(v)$

Dc. $v \neq 0 \Rightarrow \exists \mathcal{B}$ bază cu $v \in \mathcal{B}$.

Iau $v^* \in V^*$ a.î. $\begin{cases} v^*(v) = 1. \\ v^*(x) = 0 \ \forall x \in \mathcal{B} \setminus \{v\} \end{cases}$ $\Rightarrow \exists v^*$ a.î. $v^*(v) \neq 0$.

! Dc. V este finit dimensional $\Rightarrow$ i e izo canonic.

$$V \simeq V^{**}$$

Pbs $V \leq K$, $\dim V = n$. și $\{v_1, \dots, v_n\} \subset V$

2021 $\{f_1, \dots, f_n\} \subset V^*$

a.î. $f_1(x) v_1 + \dots + f_n(x) v_n = x \;\forall\, x \in V$.

$\{v_1, \dots, v_n\}$ bază V și $\{f_1, \dots, f_n\}$ bază pt. V^*

Din ipoteză $\{v_1, \dots, v_n\}$ sist. de gen $V \Rightarrow$ bază.

$x = v_j \Rightarrow f_1(v_j) v_1 + \dots + f_n(v_j) v_n = v_j \Rightarrow$

$\Rightarrow f_1(v_j) v_1 + \dots + (f_1(v_j) - 1) v_j + \dots + f_n(v_j) v_n = 0 \underset{\text{bază}}{\overset{\{v_1, \dots, v_n\}}{\Rightarrow}}$

$\Rightarrow f_i(v_j) = \delta_{ij} \Rightarrow f_i$ baza canonică V^*.

Ex 10 $\mathbb{Q}$ nu e $\mathbb{C}$ sp. vect.

$\mathbb{R}$ e $\mathbb{C}$ sp. vect.

Dem: $\dim_{\mathbb{Q}} \mathbb{R} = |\mathbb{R}| = c$

$\dim_{\mathbb{Q}} \mathbb{C} = c \Rightarrow \mathbb{R} \underset{\mathbb{Q}}{\simeq} \mathbb{C}$ ($\mathbb{Q}$ sp. vect).

$(\mathbb{R}, +) \simeq (\mathbb{C}, +)$ ca grupuri.

$\mathbb{C}$ sp. vect.

$\Rightarrow \mathbb{R}$ devine $\mathbb{C}$ sp. vect.

Seminar 8 - Alg. lin. - 13.11.2023

Teorie:

1) Fie V cu B, B' baze. $\forall v \in V \Rightarrow [v]_B \leftarrow$ coord lui v în B.

$M_{B,B'} \in M_n(K)$ a.î. $[v]_{B'} = M_{B,B'} \cdot [v]_B$

$\uparrow$
$GL_n(K)$

$$\boxed{M_{B,B'} = \begin{pmatrix} | & & | \\ [b_1]_{B'} & \dots & [b_n]_{B'} \\ | & & | \end{pmatrix}}$$

2) $f : V \to W$ apl. lin. și $\begin{cases} B \text{ - bază } V, & |B| = n \\ C \text{ - bază } W, & |C| = m \end{cases}$

$M_{B,C}(f) \in M_{m \times n}(K)$ a.î. $M_{B,C}(f) \cdot [v]_B = [f(v)]_C$

$$\boxed{M_{B,C}(f) = \begin{pmatrix} | & & | \\ [f(b_1)]_C & \dots & [f(b_n)]_C \\ | & & | \end{pmatrix}}$$

$$\begin{tikzcd} V \arrow[r, "f"] \arrow[d, "B \wr"'] & W \arrow[d, "\wr C"] \\ K^n \arrow[r, dashed, "M_{B,C}(f)"'] & K^m \end{tikzcd}$$

3) $M_{B,B'} = M_{B,B'}(id)$

4) $\underset{B,B'}{V} \xrightarrow{f} \underset{C,C'}{W} \quad M_{B'C'} = M_{CC'} \cdot M_{B C}(f) \cdot M_{B'B}$

5) $f : K^n \to K^m$

$\uparrow \qquad\quad \uparrow$

$B_0^n \qquad B_0^m \leftarrow$ baze canonice

$B \qquad\quad C \leftarrow$ alte baze.

$M_{BC}(f) = M_{B_0^m, C}\, M_{B_0^n B_0^m} \cdot M_{B, B_0^n}$ ← simplu de scris

Exerciții:

Pb 2: $f:\mathbb{R}^4 \to \mathbb{R}^2$, $f(x,y,z,t) = (2x - z - t,\ x + y - 3t)$

a) f apl. lin.

$$f(x,y,z,t) = \begin{pmatrix} 2 & 0 & -1 & -1 \\ 1 & 1 & 0 & -3 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix} \Rightarrow \text{apl. lin.}$$

b) baze ker f. Im f.

$$\underbrace{\dim_{\mathbb{R}} \mathbb{R}^4}_{4} = \dim_{\mathbb{R}} \ker f + \underbrace{\dim_{\mathbb{R}} \operatorname{Im} f}_{2 = \operatorname{rg} A} \Rightarrow \dim_{\mathbb{R}} \ker f = 2.$$

c) $B = \left\{ \begin{pmatrix} 1 \\ 2 \\ 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 5 \\ -1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 5 \\ 2 \\ -1 \end{pmatrix}, \begin{pmatrix} 3 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\}$ și $C = \left\{ \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 2 \\ 5 \end{pmatrix} \right\}$

$$M_{B,C}(f) = \underbrace{M_{B_0^2 C}}_{M^{-1}_{C B_0^2}} \cdot M_{B_0^4, B_0^2}(f) \cdot M_{B, B_0^4}$$

$$M_{B,C}(f) = \begin{pmatrix} 1 & 2 \\ 2 & 5 \end{pmatrix}^{-1} \cdot \begin{pmatrix} 2 & 0 & -1 & -1 \\ 1 & 1 & 0 & -3 \end{pmatrix} \cdot \begin{pmatrix} 1 & 5 & 1 & 3 \\ 2 & -1 & 5 & 0 \\ 1 & 0 & 2 & 0 \\ 2 & 0 & -1 & 1 \end{pmatrix}.$$

Pb 4: $W = \left\langle \begin{pmatrix} 1 \\ 2 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 2 \\ 3 \\ 1 \\ 1 \end{pmatrix} \right\rangle$. caut f a.î. $W = \ker f$.

$$\underbrace{\dim \ker f}_{\dim W = 2} + \underbrace{\dim \operatorname{Im} f}_{2} = 4. \Rightarrow \text{aleg } \underline{m = 2}.$$

$$v \in W \Rightarrow (x,y,z,t) = \alpha \begin{pmatrix} 1 \\ 2 \\ 0 \\ -1 \end{pmatrix} + \beta \begin{pmatrix} 2 \\ 3 \\ 1 \\ 1 \end{pmatrix} \Rightarrow \begin{cases} \alpha + 2\beta = x \\ 2\alpha + 3\beta = y \\ \beta = z \\ -\alpha + \beta = t \end{cases} \text{compatibil}$$

$$\Rightarrow rk \begin{pmatrix} 1 & 2 & x \\ 2 & 3 & y \\ 0 & 1 & z \\ -1 & 1 & t \end{pmatrix} = 2 \Rightarrow \begin{vmatrix} 1 & 2 & x \\ 2 & 3 & y \\ 0 & 1 & z \end{vmatrix} = 0 \text{ și } \begin{vmatrix} 1 & 2 & x \\ 2 & 3 & y \\ -1 & 1 & t \end{vmatrix} = 0$$

$$\Rightarrow \begin{cases} x - 3z + t = 0 \\ y - 5z + 2t = 0 \end{cases} \Rightarrow \text{def } f(x,y,z,t) = (x-3z+t,\ y-5z+2t)$$

Pb6 $\varphi(P) = X^2P'' - 6XP' + 12P$.

a) φ apl. lin (evident)

b) bază pt. ker φ, Im φ.

$\varphi(X^4) = 12X^4 - 24X^4 + 12X^4 = 0$.

$\varphi(X^3) = 6X^3 - 18X^3 + 12X^3 = 0$.

$\varphi(X^2) = P$.

$\varphi(X) = Q$.

$\varphi(1) = R$.

$\Rightarrow \text{Im } \varphi = \langle P, Q, R \rangle \Rightarrow \dim \text{Im } \varphi \Rightarrow \dim \ker \varphi \ldots$

Pb8: V, W, k-sp. vect., $f: V \to W$ apl. lin.

a) $\exists$ o apl. nat $f^*: W^* \to V^*$.

$V^* = Lin(V, k) \leftarrow$ sp. dual.

$f^*(w^*) = w^*(f)$

! $f: V \to W \Rightarrow f^*: W^* \to V^*$

$f^*(w^*)(v) = w^*(f(v))\ \forall v \in V$.

b) $(1_V)^* = 1_{V^*}$ evident

c) $f: V \to W$, $g: W \to U \Rightarrow (g\circ f)^* = f^* \circ g^*$

$$\begin{tikzcd} V \arrow[r, "f"] \arrow[rr, bend right, "g\circ f"'] & W \arrow[r, "g"] & U \end{tikzcd} \quad \text{și} \quad \begin{tikzcd} V^* & W^* \arrow[l, "f^*"'] & U^* \arrow[l, "g^*"'] \arrow[ll, bend left, "f^*\circ g^*"] \end{tikzcd}$$

$$(g\circ f)^*(u^*) = u^*(g\circ f) = (u^*\circ g)\circ f = f^*(u^* \circ g)$$
$$= f^*\circ g^*.$$

d) $f: V \to W$ inj $\Leftrightarrow f^*: W^* \to V^*$ surj.

$f: V \to W$ inj $\Leftrightarrow \exists g: W \to V$ apl. lin a.î. $g\circ f = 1_V$.

$\{b_1, \dots, b_n\}$ bază în $V \Rightarrow \{\underset{w_1}{f(b_1)}, \dots, \underset{w_n}{f(b_n)}\}$ ~~bază în W~~ S.L.I W.

$\Rightarrow$ îl completez la o bază $\{w_1, \dots, w_m\}$ în W.

Definesc g a.î. $\begin{array}{ccc} w_1, \dots, & w_n, & \underbrace{w_{n+1}, \dots, w_m} \\ \downarrow & \downarrow & \downarrow \\ b_1 & b_n & \text{orice} \end{array}$

$(g\circ f)^* = (id_V)^* \Rightarrow f^*\circ g^* = id_{V^*} \Rightarrow g^*$ surj.

Analog pt. e), f)

g) Det. mat. lui f^* în raport cu bazele duale în funcție de matricea lui f.

$f: V \to W$ apl. lin.

$\quad\ \uparrow \qquad \uparrow$

$\quad\ \mathcal{B} \qquad \mathcal{C}$ baze $\quad \mathcal{B} = \{v_1, \dots, v_n\}$

$\qquad\qquad\qquad\qquad \mathcal{C} = \{w_1, \dots, w_m\}$.

$\Rightarrow \mathcal{B}^*$ bază V^*, $\mathcal{B}^* = \{v_1^*, \dots, v_n^*\}$

$\mathcal{C}^*$ bază W^*, $\mathcal{C}^* = \{w_1^*, \dots, w_m^*\}$.

$v_i^*(v_j) = \delta_{ij}$

$$M_{\mathcal{B},\mathcal{C}}(f) \longrightarrow M_{\mathcal{B}^*,\mathcal{C}^*}(f^*) = (\alpha_{ij})$$

$$\hookrightarrow f^*(w_j^*) = \sum_{i=1}^{m} \alpha_{ij} v_i^*$$

$$\| $$

$$\begin{pmatrix} | & & | \\ [f(v_1)]_{\mathcal{C}} & \cdots & [f(v_n)]_{\mathcal{C}} \\ | & & | \end{pmatrix} = (a_{ij})$$

$$\hookrightarrow w_j^*(f) = \sum_{1} \alpha_{ij} v_i^* \quad | (v_k).$$

$$w_j^*(f(v_k)) = \sum_{1}^{m} \alpha_{ij} v_i^*(v_k) = \alpha_{kj}$$

$$w_j^*\left(\sum_{1}^{m} a_{ik} w_i\right) = \alpha_{kj}$$

! $V = V_1 \oplus V_2 \Rightarrow \exists! \begin{cases} v_1 \in V_1 \\ v_2 \in V_2 \end{cases}$ a.î. $v_1 + v_2 = v$.

$$\left.\begin{array}{l} pr_{V_1}^{V_2}(v) = v_1 \\ pr_{V_1}^{V_2}: V \to V_1 \end{array}\right. \Rightarrow \left\{\begin{array}{l} \operatorname{Ker} pr_{V_1}^{V_2} = V_2. \\ \operatorname{Im} pr_{V_1}^{V_2} = V_1. \end{array}\right.$$

<u>Pb 7</u>: $f: V \to V$, $f \circ f = f \Leftrightarrow f$ pr. pe un subsp. vect.

„$\Leftarrow$" evident (se verifică)

„$\Rightarrow$" f pr $\Leftrightarrow \operatorname{Ker} f \oplus \operatorname{Im} f = V$.

Fie $v \in \mathrm{Im} f \cap \ker f \Rightarrow f(v) = 0$. (din ker)

$\Rightarrow \cancel{f} \; v = f(u)$. (din Im f).

$0 = f(v) = f(f(u)) = f(u) = v \Rightarrow \mathrm{Im} f \cap \ker f = \{0\}$.

$\Rightarrow$ Grassmann $\mathrm{Im} f \oplus \ker f$.

Fie $v \in V \Rightarrow v = \underset{\in \ker}{v_1} + \underset{\in \mathrm{Im} f}{v_2}$.

$f(v) = f(v_1) + f(v_2) = f(v_2)$.

$v_2 \in \mathrm{Im} f \Rightarrow v_2 = f(u) \Rightarrow f(v_2) = f(f(u)) = f(u) = v_2 \Rightarrow$

$\Rightarrow \boxed{f(v) = v_2} \Rightarrow f$ e proiecție.

Seminar 5 - Alg. lin. - 20.11.2023

Teorie:

1) $(i_1, i_2, \dots, i_k)$ perm. care duce $i_1 \to i_2 \to i_3 \to \cdots \to i_1$

2) $\varepsilon : S_n \to \mathbb{Z}_2 = \{\pm 1\}$ morfism de gr.

$(1,2,3,4) = (1,2)\cdot(2,3)\cdot(3,4)$.

$\varepsilon(1,2,3,4) = \varepsilon(1,2)\cdot\varepsilon(2,3)\cdot\varepsilon(3,4) = (-1)^3 = -1$.

4) $\varepsilon(\text{ciclu de lg } k) = (-1)^{k-1}$

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 1 & 3 & 4 & 2 & 6 & 8 & 5 & 7 \end{pmatrix} = (2,3,4)\cdot(5,6,8,7)$$

$\varepsilon(\sigma) = (-1)^2\cdot(-1)^3 = -1$.

Exerciții:

Pb1: Fie $A = (a_{ij})_{1 \le i,j \le 4}$. Cu ce semn apare $a_{12}\cdot a_{34}\cdot a_{23}\cdot a_{41}$ în det A?

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \end{pmatrix} = (1,2,3,4) \Rightarrow \text{apare cu } -1.$$

Pb2: a)
$$\begin{vmatrix} 1 & 2 & 3 & 4 \\ -2 & 6 & 0 & 1 \\ 0 & 1 & -1 & 2 \\ 0 & 11 & 4 & 11 \end{vmatrix} \overset{\text{dezv.}}{\underset{\text{linii } 1,2}{=\!=}} \begin{vmatrix} 1 & 2 \\ -2 & 6 \end{vmatrix}\cdot(-1)^{10}\cdot\begin{vmatrix} -1 & 2 \\ 4 & 11 \end{vmatrix} -$$

$$- \begin{vmatrix} 1 & 3 \\ -2 & 0 \end{vmatrix}\cdot\begin{vmatrix} 1 & 2 \\ 11 & 11 \end{vmatrix} + \begin{vmatrix} 1 & 4 \\ -2 & 1 \end{vmatrix}\cdot\begin{vmatrix} 1 & -1 \\ 11 & 4 \end{vmatrix} + \begin{vmatrix} 2 & 3 \\ 6 & 0 \end{vmatrix}\cdot\begin{vmatrix} 0 & 2 \\ 0 & 11 \end{vmatrix} -$$

$$- \begin{vmatrix} 2 & 4 \\ 6 & 1 \end{vmatrix}\cdot\begin{vmatrix} 0 & -1 \\ 0 & 4 \end{vmatrix} + \begin{vmatrix} 3 & 4 \\ 0 & 1 \end{vmatrix}\cdot\begin{vmatrix} 0 & 1 \\ 0 & 11 \end{vmatrix}$$

Pb3: $A_m = (a_{ij}) = \begin{cases} j \ ; & i<j \\ -j \ ; & i>j \\ 1 \ ; & i=j=1 \\ 0 \ ; & i=j\neq 1. \end{cases}$

a) $\det A_3$, $\det A_4$.

$$|A_3| = \begin{vmatrix} 1 & 2 & 3 \\ -1 & 0 & 3 \\ -1 & -2 & 0 \end{vmatrix} = \cancel{6} - \cancel{6} + 6 = 6 \neq 0.$$

$$|A_4| = \begin{vmatrix} 1 & 2 & 3 & 4 \\ -1 & 0 & 3 & 4 \\ -1 & -2 & 0 & 4 \\ -1 & -2 & -3 & 0 \end{vmatrix} \xlongequal{L_4 \leftarrow L_4 + L_1} \begin{vmatrix} 1 & 2 & 3 & 4 \\ -1 & 0 & 3 & 4 \\ -1 & 2 & 0 & 4 \\ 0 & 0 & 0 & 4 \end{vmatrix} = 4\,|A_3| = 24$$

c) $\det A_m = ?$

$$|A_m| = \begin{vmatrix} 1 & 2 & \dots & m \\ -1 & 0 & \dots & m \\ \vdots & & & \\ -1 & -2 & \dots & 0 \end{vmatrix} \xlongequal{L_m \leftarrow L_m + L_1} \begin{vmatrix} 1 & 2 & \dots & m \\ -1 & 0 & \dots & m \\ \vdots & & & \\ 0 & 0 & \dots & m \end{vmatrix} = m\,|A_{m-1}|$$

$\Rightarrow |A_m| = m!$

Pb5: $A = (a_{ij}) \in M_4(k)$. Rotim A cu $90°$ în sens trigo în jurul „centrului".

a) $\det A' = ?$

b) generalizare

$$A' = \begin{pmatrix} a_{14} & a_{24} & a_{34} & a_{44} \\ a_{13} & a_{23} & a_{33} & a_{43} \\ a_{12} & a_{22} & a_{32} & a_{42} \\ a_{11} & a_{21} & a_{31} & a_{41} \end{pmatrix} \Rightarrow {}^tA' = \begin{pmatrix} a_{41} & a_{42} & a_{43} & a_{44} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{11} & a_{21} & a_{13} & a_{14} \end{pmatrix}$$

$L_1 \leftrightarrow L_4$ și $L_2 \leftrightarrow L_3 \Rightarrow \det A' = \det A$.

Generalizare
$$\left.\begin{array}{l} n = 4k \\ n = 4k+1 \end{array}\right\} \det A = \det A'$$
$$\left.\begin{array}{l} n = 4k+2 \\ n = 4k+3 \end{array}\right\} \det A = -\det A'$$

Pbs a) $A = \begin{vmatrix} M & * \\ 0 & P \end{vmatrix} \Rightarrow \det A = \det M \cdot \det P$.

Laplace pt. liniile $\{l_1, \dots l_m\} = I$.

$$|A| = \sum_{|J| = m} (-1)^{\frac{m(m+1)}{2} + \Sigma j} \cdot A_{IJ} \cdot A_{\bar{I}\bar{J}}$$

Pt. $J \neq \{1, \dots, m\} \Rightarrow A_{\bar{I}\bar{J}}$ va avea o col $\begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \Rightarrow \det 0$.

$\Rightarrow |A| = (-1)^{m(m+1)} \cdot \det M \cdot \det P = \det M \cdot \det P$.

c) $A = \begin{vmatrix} * & M \\ P & 0 \end{vmatrix} \Rightarrow \det A = (-1)^{mp} \cdot \det M$.

Laplace pt. liniile $\{1, \dots, m\} = I$.

$$|A| = \sum_{|J| = m} (-1)^{\frac{m(m+1)}{2} + \Sigma j} \cdot A_{IJ} \cdot A_{\bar{I}\bar{J}}$$

Pt. $J \neq \{p+1, \dots, p+m\} \Rightarrow A_{\bar{I}\bar{J}}$ are o col $\begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix} \Rightarrow$

$$\Rightarrow |A| = \sum_{|J|=m} (-1)^{\frac{m(m+1)}{2} + mp + \frac{m(m+1)}{2}} \cdot \det M \cdot \det P.$$

$|A| = (-1)^{mp} \cdot \det M \cdot \det P$.

Pb 7 $\begin{vmatrix} 1 & \dots & 1 \\ x_1 & & x_m \\ \vdots & & \vdots \\ x_1^{m-1} & \dots & x_m^{m-1} \end{vmatrix} = \prod_{i<j} (x_j - x_i)$

n

$\boxed{M1}$ $V(x_1, \dots, x_m)$

$\left.\begin{array}{l} \deg(V(x_1, \dots, x_{m-1}, x)) = m-1 \\ \text{si } V(x_1, \dots, x_{m-1}, x_i) = 0 \quad \forall i = \overline{1, m-1} \end{array}\right\} \Rightarrow$

$V(x_1, \dots, x_{m-1}, x) = a \prod_{i=1}^{m-1} (x - x_i)$

$a = V(x_1, \dots, x_{m-1}) \Rightarrow$ recursiv $V(x_1, \dots, x_m) = \prod_{j>i} (x_j - x_i)$

$\boxed{M2}$

$V(x_1, \dots, x_m) \xlongequal{C_i - C_1} \begin{vmatrix} 1 & 0 & \dots & 0 \\ x_1 & x_2 - x_1 & \dots & x_m - x_1 \\ x_1^2 & x_2^2 - x_1^2 & \dots & \vdots \\ \vdots & \vdots & & \vdots \\ x_1^{m-1} & x_2^{m-1} - x_1^{m-1} & & x_m^{m-1} - x_1^{m-1} \end{vmatrix} =$

$= \begin{vmatrix} x_2 - x_1 & \dots & x_m - x_1 \\ \vdots & & \\ x_2^{m-1} - x_1^{m-1} & \dots & x_m^{m-1} - x_1^{m-1} \end{vmatrix} = (x_2 - x_1) \dots (x_m - x_1) \begin{vmatrix} 1 & \dots & 1 \\ x_1 + x_2 & \dots & x_1 + x_m \\ \vdots & & \\ x_2^{m-2} + \dots + x_1^{m-2} & \dots & x_1^{m-2} + \dots + x_m^{m-2} \end{vmatrix}$

$= (x_2 - x_1) \dots (x_m - x_1) \begin{vmatrix} 1 & - & - & - \\ x_2 + x_1 & - & - & - \\ x_2^2 + x_1(x_2 + x_1) & - & & - \\ \vdots & & & \\ x_2^{m-2} + x_1(x_2^{m-3} + \dots + x_1^{m-3}) & - & - & \end{vmatrix} \xrightarrow{L_{i+1} \leftarrow L_{i+1} - x_1 L_i}$

$= (x_2 - x_1) \dots (x_m - x_1) \cdot V(x_2, \dots, x_m) \Rightarrow$ inductie.

7/5

Pb 8: $f: V \to V$ apl. lin, $\dim V = n$ și $A = M_B(f)$.

$\overset{?}{\Rightarrow}$ $\det A$, $\operatorname{tr} A$ etc nu depind de alegerea lui B.

$$M_B(f) = \underbrace{M_{B'B}}_{P^{-1}} \cdot M_{B'}(f) \cdot \underbrace{M_{B,B'}}_{P} = P^{-1} \cdot M_{B'}(f) \cdot P.$$

$\det M_B(f) = \det P^{-1} M_{B'}(f) P = \det M_{B'}(f)$.

~~det~~ $\operatorname{tr}(M_B(f)) = \operatorname{tr}(P^{-1} M_{B'}(f) P) = \operatorname{tr} M_{B'}(f)$.

! Dc. $A = M_B(f)$, $\det A$, $\operatorname{tr} A$ nu depind de baza aleasă

$\Rightarrow$ are sens să vb. de. $\det f$ și $\operatorname{tr} f$.

Pb 9: $f: \mathbb{R}^n \to \mathbb{R}^n$ apl. lin. $\lambda \in \mathbb{R}$. $\exists v \in \mathbb{R}^n$ a.î. $f(v) = \lambda v$

$\Leftrightarrow \det(f - \lambda \, id_{\mathbb{R}^n}) = 0$.

$\exists v \in \mathbb{R}^n$ a.î. ~~det f~~ $f(v) = \lambda v \Leftrightarrow (f - \lambda id)(v) = 0 \Leftrightarrow$

$\Leftrightarrow \ker(f - \lambda id) \neq \{0\} \Leftrightarrow f - \lambda id$ nu e inj $\Leftrightarrow \det(f - \lambda id) = 0$.

Pb 10 $\det AB \overset{?}{=} \det BA$, $A \in M_{m \times n}(k)$, $B \in M_{n \times m}(k)$.

$m < n \Rightarrow \det BA = 0$, pe când AB nu e neapărat 0.

$\operatorname{rg} BA \leq \min\{\operatorname{rg} A, \operatorname{rg} B\} \leq m < n$

Seminar 10 - Alg. liniară - 29.11.2023

Teorie:

1) $\left.\begin{array}{l} AX = b \\ A \in M_{m\times n}(k) \end{array}\right\}$ compatibil $\iff$ $rk\, A = rk\, A^e$

Dem: $X = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \Rightarrow AX = (x_1 C_1 + \dots + x_n C_n) = b$ compatibil

$\iff b \in Im\, f_A \iff b \in \langle C_1, \dots, C_n \rangle \iff \underbrace{\dim \langle C_1, \dots C_n \rangle}_{rk\, A} = \underbrace{\dim \langle C_1, \dots, C_n, b \rangle}_{rk\, A^e}$

Pb2: $\begin{cases} x_1 + 4x_2 + 2x_3 - 3x_5 = 0 \\ 2x_1 + 9x_2 + 5x_3 + 2x_4 + x_5 = 0 \\ x_1 + 3x_2 + x_3 - 2x_4 - 9x_5 = 0 \end{cases}$

$A = \begin{pmatrix} 1 & 4 & 2 & 0 & -3 \\ 2 & 9 & 5 & 2 & 1 \\ 1 & 3 & 1 & -2 & -9 \end{pmatrix}$, $A^e = \begin{pmatrix} 1 & 4 & 2 & 0 & -3 & 0 \\ 2 & 9 & 5 & 2 & 1 & 0 \\ 1 & 3 & 1 & -2 & -9 & 0 \end{pmatrix} \Rightarrow$ compatibil.

. . .

Pb4:

1 - F	6 - A	10 - A	14 - F
2 - A	7 - A	11 - A	15 - A
3 - F	8 - F	12 - A	16 - A
4 - A	9 - A	13 - F	17 - F $(m < n)$
5 - A			

Pt. 8 - contraex $m < n$.

Pt. 11 - Aleg $Y_0 \in \mathbb{R}^m$ a.î. $AY_0 = b$ (dc. există). $AX = 0$

$\varphi(X) = X + Y_0$.

$A\varphi(X) = A(X + Y_0) = b$

$\boxed{Pb}$ Caut B a.î. $M_B(f)$ să fie cât mai simplă

$V = k^n$, B_0 baza canonică. Caut $M_B(f)$ simplă $\Leftrightarrow$

caut P a.î. PAP^{-1} simplă.

! Th. $\forall$ matrice $\exists$ o bază B a.î. $M_B(f) = \begin{pmatrix} \lambda_1 & & \star \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}$ în $\mathbb{C}$

$\{b_1, \dots, b_n\} = B \Rightarrow M_B(f) = \begin{pmatrix} | & & | \\ [f(b_1)]_B & \cdots & [f(b_n)]_B \\ | & & | \end{pmatrix}$

Vreau $f(b_i) = \lambda_i b_i \Leftrightarrow \det(f - \lambda_i \, id) = 0$

! În general dat $\lambda \in k$, $\exists v \neq 0$ a.î. $f(v) = \lambda v \Leftrightarrow$

λ e sol pt. polinomul $\det(f - \lambda I_n) = 0$

$$\underset{(-1)^n \lambda^n + \dots + \det f}{\overset{\shortparallel}{\det(f-\lambda I_n)}}$$

<u>Dem prin inducție după n:</u>

$\boxed{n=1} \Rightarrow f(x) = \lambda x$ pt. $\lambda \in \mathbb{C} \Rightarrow M_B(f) = (\lambda)$ ✓.

$f: \mathbb{C}^n \to \mathbb{C}^n$

$\boxed{n-1 \Rightarrow n}$. $\det(f - \lambda \, id) = 0$ are o răd ($\mathbb{C}$ algebric închis)

$\Rightarrow \exists b_1$ a.î. $f(b_1) = \lambda_1 b_1$.

Completez la o bază $\mathbb{B} = \{b_1, \dots b_n\} \Rightarrow M_{\mathbb{B}}(f) = \begin{pmatrix} \lambda_1 & \\ 0 & \bigcirc \\ \vdots & \\ 0 & \end{pmatrix}$

Fie $V_1 = \langle b_1 \rangle$ și V_2 a.î. $V_1 \oplus V_2 = \mathbb{C}^n$

$f(b_1) = \lambda_1 b_1$.

$f/_{V_1} : \cancel{V}_2 \to \cancel{V}_2$, $p^{V_1}_{V_2} : \mathbb{C}^n \to V_2$

$\mathcal{B}' = \{b_2, \dots, b_m\}$ bază pt. $V_2 \Rightarrow M_\mathcal{B}(f) = \left(\begin{array}{c|c} \lambda_1 & * \\ \hline 0 & \\ 0 & pr^{V_1}_{V_2} \circ f \\ \vdots & \shortparallel \\ 0 & M_{\mathcal{B}'}(f) \end{array}\right)$

Din inducție, $M_{\mathcal{B}'}(f)$ este echivalentă cu o mat. sup. Δ.

! $M = \begin{pmatrix} \lambda_1 & & * \\ & \lambda_2 & \\ 0 & & \ddots & \\ & & & \lambda_m \end{pmatrix} \Rightarrow \lambda_i$ val proprii, $\det(f - \lambda I_m) = 0$.

$\det(M - \lambda I_m) = (\lambda_1 - \lambda) \dots (\lambda_m - \lambda)$

f nilpotentă $\Rightarrow f^m = 0$.

$M^k = \begin{pmatrix} \lambda_1^k & & * \\ & \lambda_2^k & \\ 0 & & \ddots & \lambda_m^k \end{pmatrix}$

Atenție, nu merge neapărat și peste $\mathbb{R}$: $A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \Rightarrow$

$\det(A - \lambda I_2) = \lambda^2 + 1$ nu are răd în $\mathbb{R}$.

$A \sim \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$

<u>Pb1</u> V un k-sp. vect. cu $\dim V = m$.

$S \subset V \Rightarrow Ann(S) = \{f \in V^* \mid \underset{\forall v \in S}{f(v) = 0}\}$

a) $Ann(S) \leq_k V^*$ (cu def)

b) $\dim(Ann(W)) = \dim V - \dim W$.

$W \leq V$

$\{b_1, \dots, b_m\}$ bază în $V \Rightarrow \{b_1^*, \dots, b_m^*\}$ bază în V^*

$\{b_1, \dots, b_t\}$ bază $W \overset{?}{\Rightarrow} \{b_{t+1}^*, \dots, b_m^*\}$ bază $Ann(W)$

$f \in \text{Ann}(W) \Rightarrow f = \sum_1^n \alpha_i b_i^*$

$0 = f(b_j) = \sum_1^n \alpha_i b_i^*(b_j) = \alpha_j \; ; \; j = \overline{1,t}$
$f \in \text{Ann}$

$\Rightarrow f \in \langle b_{t+1}^*, \dots, b_n^* \rangle$. + s.l.i. $\Rightarrow$ bază.

b) $W, W_1, W_2 \leq_K V \Rightarrow \begin{cases} 1) \text{Ann}(W_1 + W_2) = \text{Ann}\, W_1 \cap \text{Ann}\, W_2 \\ 2) \text{Ann}(W_1 \cap W_2) = \text{Ann}\, W_1 + \text{Ann}\, W_2. \end{cases}$

$W_1 \subset W_2 \Rightarrow \text{Ann}\, W_1 \supset \text{Ann}\, W_2$.

1) „$\subseteq$"
$W_1 + W_2 \supset W_1, W_2 \Rightarrow \text{Ann}(W_1 + W_2) \subseteq \text{Ann}\, W_1 \cap \text{Ann}\, W_2$.

„$\supseteq$" evident.

2) $W_1 \cap W_2 \subset W_1, W_2 \Rightarrow \text{Ann}(W_1 \cap W_2) \supset \text{Ann}\, W_1 + \text{Ann}\, W_2$.

$\dim(\text{Ann}\, W_1 + \text{Ann}\, W_2) = \underbrace{\dim \text{Ann}\, W_1}_{n - \dim W_1} + \underbrace{\dim \text{Ann}\, W_2}_{n - \dim W_2} -$

$- \underbrace{\dim(\text{Ann}\, W_1 \cap \text{Ann}\, W_2)}_{\overset{1)}{=} \dim(\text{Ann}\, W_1 + W_2)}$

$\underbrace{\dim(\text{Ann}\, W_1 + W_2)}_{}$

~~dim~~ $n - \dim(W_1 + W_2) = n - \dim W_1 - \dim W_2 + \dim(W_1 \cap W_2)$

$\Rightarrow \dim(\text{Ann}\, W_1 + \text{Ann}\, W_2) = n - \dim(W_1 \cap W_2)$
$= \cancel{\text{An}} \dim \text{Ann}\, W_1 \cap W_2$.

Seminar 11 - Alg. lin - 4.12.2023

Pb1: d) $Ann(Ann\,W) = i(W)$, $i: V \to V^{**}$ izo canonic

$\dim(Ann(Ann\,W)) = \dim V^* - \dim Ann\,W =$

$\dim V^* - \dim V + \dim W = \dim W = \dim i(W)$.

$i(w)$, $w \in W$.

$i(w)(w^*) = w^*(w) = 0 \ \forall w^* \in Ann\,W$

$\Rightarrow i(W) \subset Ann(Ann(W))$.

e) $W_1 = W_2 \Leftrightarrow Ann\,W_1 = Ann\,W_2$

$Ann\,W_1 = Ann\,W_2 \mid Ann$

$Ann(Ann\,W_1) = Ann(Ann\,W_2)$

$\underset{i(W_1)}{\shortparallel} = \underset{i(W_2)}{\shortparallel} \overset{i\ inj}{\Rightarrow} W_1 = W_2$.

f) $Ann(Ann\,S) = i(\langle S \rangle) \overset{?}{=} Ann(Ann\langle S \rangle)$.

$\underset{\langle i(S) \rangle}{\shortparallel}$.

„$\supseteq$" evidentă.

$Ann\,S \underset{\supset}{=} Ann\langle S \rangle$

$f(v) = 0 \ \forall v \in S$.

$f(\sum a_i v_i) = \cancel{a_i \sum f} \sum a_i f(v_i) = 0$

Pb2: V un k-sp. vect. de dim n. Fie $f_1, \dots, f_k \in V^*$

a.î. $\bigcap \ker f_i \subseteq \ker f$.

$\Leftrightarrow$

$f \in$ comb. lin de $f_1, \dots, f_k$

$\boxed{M1}$ „$\Leftarrow$" evident.

„$\Rightarrow$" Inducție după k.

$\boxed{k=1}$ $\ker g \subseteq \ker f \overset{?}{\Rightarrow} f = \lambda g$.

$g \neq 0 \Rightarrow \underbrace{\dim \operatorname{Im} g}_{\leq 1} + \underbrace{\dim \ker g}_{\leq n-1} = n. \Rightarrow$

$\Rightarrow \dim \operatorname{Im} g = 1$

$\dim \ker g = n-1 \Rightarrow e \notin \ker g \Rightarrow \ker g + \langle e \rangle = V.$

$g(x) = g(\underset{\ker g \subseteq \ker f.}{\underset{\cap}{y}} + \alpha e) = \alpha g(e)$

$f(x) = f(x + \alpha e) = \alpha f(e)$

$f = 0 \Rightarrow f = 0 \cdot g.$

$f \neq 0 \Rightarrow \dim \ker f = n-1 \Rightarrow f(e) \neq 0. \Rightarrow f(e) = \beta g(e).$

$\Rightarrow f(x) = \beta g(x).$

Inducție $\boxed{k-1 \Rightarrow k}$

$V' = \ker f$

$f' = f/_{V'}$ și $f_i' = f_i/_{V'} \Rightarrow \ker f_i' = \ker f_i \cap V'$

$\Rightarrow \bigcap_{i=1}^{k-1} \ker f_i' = \bigcap_{i=1}^{k-1} \ker(f_i \cap V') = \bigcap_{1}^{k-1} \ker f_i \cap V' \overset{\text{inducție}}{\subseteq} \ker f \cap V'$

$= \ker f'$

$\Rightarrow f' = \sum \alpha_i f_i' \Rightarrow f(x) = \sum \alpha_i f(x) \quad \forall x \in V'$

$h = f - \sum \alpha_i f_i$ se anulează pe $\underset{\overset{\|}{\ker f}}{V'} < \ker h$

$\Rightarrow f - \sum_1^{t-1} \alpha_i f_i = \alpha_t f_t \Rightarrow f = \sum_1^t \alpha_i f_i$

$\boxed{M2}$ $\bigcap_1^t \ker f_i < \ker f \overset{?}{\Rightarrow} f \in \langle f_1, \dots, f_t \rangle$.

$W = \langle f_1, \dots f_t \rangle \leq V^*$ și vreau $f \in W$.

$\operatorname{Ann} W \leq V^{**} = i(V)$.

$\operatorname{Ann} W \leq \operatorname{Ann}(\langle f \rangle)$

$\overset{\|}{i}(\bigcap \ker f_j) \leq i(\ker f)$

$\boxed{M3}$ $\{e_1, \dots, e_n\}$ bază V. Mă uit la sistemul.

$f(e_i) = \alpha_1 f_1(e_i) + \dots + \alpha_t f_t(e_i)$

$$\left|\begin{matrix} f_1(e_1) & \dots & f_t(e_1) \\ \vdots & & \vdots \\ f_1(e_t) & \dots & f_t(e_t) \end{matrix}\right| \quad \left|\begin{matrix} f(e_1) \\ \vdots \\ f(e_t) \end{matrix}\right|$$

$\Rightarrow f_i(a_1 e_1 + \dots + a_n e_n) = 0$.

$f(\ \downarrow\ ) = 0$.

! $V_\lambda = \{ v \in V \mid f(v) = \lambda v \} = \ker(f - \lambda\, id)$

$\mathrm{Spec}(f) = \{ \lambda \in k \mid \lambda$ val proprie $\}$

$\lambda = 0$ val. proprie $\Leftrightarrow f$ nu e izo.

ex: $\mathrm{Spec}(f) = ?$, $f: \mathbb{R}^2 \to \mathbb{R}^2$, $f(x,y) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$

$\det(f - \lambda\, id) = \lambda^2 + 1 \Rightarrow \mathrm{Spec}(f) = \{ \pm i \}$.

Pb 1: a) $A^2 = I_n \Rightarrow \lambda^2 = 1 \ \forall \lambda$ val proprie

$A \cdot v = \lambda v \Rightarrow \underset{v}{\underset{\parallel}{A^2 v}} = \lambda A v = \lambda^2 v \Rightarrow \lambda^2 = 1$.

b) $\underset{\text{pol}}{\underset{\uparrow}{P}}(A) = 0 \Rightarrow P(\lambda) = 0$.

$P(A) = 0 \ \mid \cdot v \Rightarrow P(A) \cdot v = 0 = P(\lambda) v \Rightarrow P(\lambda) = 0$

Pb 2: a) A și ${}^tA$ au ac. val proprii

$f: V \to V$ mat. A în $\mathcal{B}$ bază $\Rightarrow f^*: V^* \to V^*$ are mat. ${}^tA$ în $\mathcal{B}^*$

$f^*(v^*) = v^*(f)$

Pb 4: $f(A) = {}^tA$ $\overset{?}{\Rightarrow}$ val proprii.

$(f \circ f)(A) = f({}^tA) = A \Rightarrow f^2 = id \Rightarrow \lambda^2 = 1$

$V_1 = \{ A \mid A = {}^tA \}$ și $V_{-1} = \{ A \mid -A = {}^tA \}$.

$M_n(k) = \bigoplus_{\lambda \in \mathrm{Spec}(f)} V_\lambda$.

Pb 5: a) B inv $\Rightarrow \exists c$ a.î. $A + cB$ neinv.

$(AB^{-1} + c I_n) \cdot B$ iau $c = -\lambda_{AB^{-1}} \Rightarrow \det(AB^{-1} + c I_n) = 0$.

b) B neinv $\Rightarrow$ ex. de A și B a.î. $A + cB$ inversabilă $\forall c$.

$A = I_n$, $B = O_n$.

Seminar 12 - Alg. lin - 8.01.2024

Teorie - Diagonalizare:

1) $f: k^n \to k^n$ apl. lin. $\Rightarrow A = M_{B_0}(f) \leftarrow$ matr. lui f în baza canonică

2) Dc. $\exists\, v \neq 0$ a.î. $f(v) = \lambda v \Rightarrow \lambda$ s.n. val. proprie

3) $Spec(f) = \{\lambda \mid \lambda$ val proprie$\}$

$\uparrow$ răd. lui $P_f = \det(f - \lambda\, id) \leftarrow$ pol. caracteristic

4) $m_a(\lambda)$ - ord. lui λ ca răd al lui P_f

$\uparrow$ multiplicitate alg.

ex: $P_f(\lambda) = (\lambda-2)^2(\lambda-3) \Rightarrow \begin{cases} m_a(2) = 2 \\ m_a(3) = 1 \end{cases}$

5) Dc. toate val proprii sunt în $k \Rightarrow \sum\limits_{\lambda \in Spec(f)} m_a(\lambda) = n$.

6) $V_\lambda = \{v \in k^n \mid f(v) = \lambda v\}$ sp. vectorilor proprii de val. proprie λ

7) $m_g(\lambda) = \dim_k V_\lambda \leftarrow$ multiplicitate geometrică

8) Urm. afirm sunt echiv: $\exists\, B$ bază în care

$M_B(f) = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix} \Leftrightarrow k^n = V = \bigoplus\limits_{\lambda \in Spec(f)} V_\lambda \Leftrightarrow \sum\limits_{\lambda \in Spec(f)} m_g(\lambda) = n$

$\Leftrightarrow \begin{cases} \text{toate } \lambda \in k \\ m_g(\lambda) = m_a(\lambda) \end{cases}$

! Oricum suma e directă $\Leftrightarrow V_\lambda \cap \Big(+_{\mu \in \operatorname{Spec} f \setminus \{\lambda\}} V_\mu \Big) = \{0\}$

~~Pp.~~ $\exists \mu \neq \lambda$ cu $V_\lambda \cap V_\mu \neq \emptyset$.

$0 \neq v_i$ vect. propriu $\lambda_i \Rightarrow$

$$\alpha_1 v_1 + \dots + \alpha_m v_m = 0 \mid f$$

$$\alpha_1 \lambda_1 v_1 + \dots \alpha_m \lambda_m v_m = 0.$$

$\lambda_1 \cdot$ $\quad \lambda_1 \alpha_1 v_1 + \dots + \alpha_m \lambda_1 v_m = 0$

(−) + inducție

$\boxed{\text{„}\Leftarrow\text{"}}$ $k^n = \bigoplus V_\lambda$. Aleg B bază reuniune de baze din V_λ

$$\Rightarrow M_B(f) = \begin{pmatrix} \boxed{\begin{matrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_1 \end{matrix}} & & & \\ & \boxed{\begin{matrix} \lambda_2 & & \\ & \ddots & \\ & & \lambda_2 \end{matrix}} & & \\ & & \ddots & \\ & & & \boxed{} \end{pmatrix}$$

9) $W_1, W_2 \leq_k V$; $W_1 \cap W_2 = \{0\} \Rightarrow$

$$W_1 \oplus W_2 = V \Leftrightarrow \dim W_1 + \dim W_2 = \dim V$$

10) $m_g(\lambda) \leq m_a(\lambda)$

Pb1: $f(x,y,z) = \begin{pmatrix} -1 & 4 & 2 \\ -3 & 4 & 0 \\ -3 & 1 & 3 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$

$$P_f(\lambda) = \det(A - \lambda I_3) = \begin{vmatrix} -1-\lambda & 4 & -2 \\ -3 & 4-\lambda & 0 \\ -3 & 1 & 3-\lambda \end{vmatrix} = -\lambda^3 + 6\lambda^2 - 11\lambda + 6$$

$P_f(\lambda) = -(\lambda-1)(\lambda-2)(\lambda-3) \Rightarrow Spec(f) = \{\underset{\lambda_1}{\overset{\shortparallel}{1}}, \underset{\lambda_2}{\overset{\shortparallel}{2}}, \underset{\lambda_3}{\overset{\shortparallel}{3}}\}.$

! val proprii în k + distincte ⇒ diagonalizabilă

Calculez baze în V_{λ_i}:

$\boxed{\lambda_1}$ $v \in \mathbb{R}^3$ cu $f(v) = v \Rightarrow \begin{cases} -x + 4y - 2z = x \\ -3x + 4y = y. \\ -3x + y + 3z = z \end{cases} \Rightarrow V_{\lambda_1} = \langle (1,1,1) \rangle.$

$\boxed{\lambda_2}$ $v \in \mathbb{R}^3$ cu $f(v) = 2v \Rightarrow \begin{cases} -x + 4y - 2z = 2x \\ -3x + 4y = 2y \\ -3x + y + 3z = 2z \end{cases} \Rightarrow V_{\lambda_2} = \langle (2,3,3) \rangle.$

$\boxed{\lambda_3}$ $v \in \mathbb{R}^3$ cu $f(v) = 3v \Rightarrow \begin{cases} -x + 4y - 2z = 3x \\ -3x + 4y = 3y \\ -3x + y + 3z = 3z \end{cases} \Rightarrow V_{\lambda_3} = \langle (1,3,4) \rangle.$

$\Rightarrow$ în baza $B = \left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 3 \\ 3 \end{pmatrix}, \begin{pmatrix} 1 \\ 3 \\ 4 \end{pmatrix} \right\}$ mat. lui f e $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix} = \Delta$

$$A = M_B(f) = \underset{P}{\underset{\shortparallel}{M_{B,B_0}}} \cdot \underset{\Delta}{\underset{\shortparallel}{M_{B_0}(f)}} \cdot \underset{P^{-1}}{\underset{\shortparallel}{M_{B_0,B}}}$$

$$P = \begin{pmatrix} 1 & 2 & 1 \\ 1 & 3 & 3 \\ 1 & 3 & 4 \end{pmatrix}$$

T. Hamilton - Cayley: $P_A(A) = 0$

a) Dc. A e diag $\Rightarrow A = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}$; $\lambda_i \in \mathbb{C} \Rightarrow P_A(A) = \begin{pmatrix} P_A(\lambda_1) & & 0 \\ & \ddots & \\ 0 & & P_A(\lambda_n) \end{pmatrix}$

b) Dc. A e diagonalizabilă $\Rightarrow A = PDP^{-1}$, D diagonală.

$P_A(A) = P_A(PDP^{-1}) = PP_A(D)P^{-1} = PP_D(D)P^{-1} = 0.$

c) $\mathcal{D} = \{A \in M_n(\mathbb{C}) \mid A$ diagonalizabilă$\}$ e densă în $M_n(\mathbb{C})$

$\Leftrightarrow \overline{\mathcal{D}} = M_n(\mathbb{C})$.

Pe $\mathbb{C} \simeq \mathbb{R}^2$ ∃ metrică $d(z, w) = |z - w|$.

Pe $M_n(\mathbb{C}) \simeq \mathbb{C}^{n^2}$, Pe $\mathbb{C}^k$: $d((z_1, \dots, z_k), (w_1, \dots w_k)) = \sqrt{\sum |z_i - w_i|^2}$

$(z_1^n, \dots z_2^n) \xrightarrow{n \to \infty} (z_1, \dots, z_n) \Leftrightarrow z_i^n \to z_i \ \forall i$.

Înapoi la pb c) $\Leftrightarrow \forall A \in M_n(\mathbb{C}) \ \exists \underset{\in \mathcal{D}}{A_m} \xrightarrow{m \to \infty} A$.

$A = P \begin{pmatrix} \lambda_1 & & * \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix} P^{-1}$. Putem să modificăm elem. de pe diag ca să fie dif $\Rightarrow$ matr. diagonală.

ex: $\begin{pmatrix} 5 & * \\ 0 & 5 \\ & & 1 \end{pmatrix} \to \begin{pmatrix} 5{,}05 & * \\ 0 & 5 \\ & & 1 \end{pmatrix}$

d) $\left.\begin{array}{l} A \xmapsto{\varphi} P_A(A) \text{ cont.} \\ \varphi|_{\mathcal{D}} = 0 \end{array}\right\} \Rightarrow \boxed{\varphi = 0}$

- $e^x = \sum_{m \geq 0} \frac{1}{m!} x^m \ \forall x \in \mathbb{C} \Rightarrow$ def. $\exp(A) = \sum_{m \geq 0} \frac{1}{m!} A^m$
- $\boxed{\text{Th}}$: $\det(\exp(A)) = e^{\operatorname{tr}(A)} \ \forall A \in M_n(\mathbb{C})$

Seminarii 13-14 - Alg. lin. - 15.01.2023

Teorie

1) V - un k sp. vect. $h: V\times V \to k$ s.n. biliniară dc. e lin. în ambele arg:
$$\begin{cases} h(\alpha x+\beta y, z) = \alpha h(x,z) + \beta h(y,z) \\ h(x, \alpha y+\beta z) = \alpha h(x,y) + \beta h(x,z) \end{cases}$$

ex: $h: \mathbb{R}^3 \to \mathbb{R}$, $h(x,y) = 6x_1y_3 - 2x_1y_1 + 3x_2y_2 - 5x_2y_3$

2) Fie $\mathcal{B} = \{b_1, \dots b_m\}$ bază V. $\Rightarrow \exists [h]_{\mathcal{B}} \in M_m(k)$ a.î.

$$\boxed{h(x,y) = {}^t[x]_{\mathcal{B}} \cdot [h]_{\mathcal{B}} \cdot [y]_{\mathcal{B}}}$$

$\Rightarrow ([h]_{\mathcal{B}})_{ij} = h(b_i, b_j)$

în ex: $[h]_{\mathcal{B}_0} = \begin{pmatrix} -2 & 0 & 6 \\ 0 & 3 & -5 \\ 0 & 0 & 0 \end{pmatrix}$
$\mathcal{B}_0$ baza canonică

3) h simetrică $\Leftrightarrow h(x,y) = h(y,x)$

4) h antisim $\Leftrightarrow h(x,y) = -h(y,x)$

5) h nedegenerată $\Leftrightarrow \begin{cases} \forall x \neq 0 \Rightarrow \exists y \neq 0 \text{ cu } h(x,y) \neq 0. \\ \forall y \neq 0 \Rightarrow \exists x \neq 0 \text{ cu } h(x,y) \neq 0. \end{cases}$

6) $\boxed{k = \mathbb{R}}$ h e poz. def. (neg) : dc. $h(x,x) \geq 0$ (≤ 0) cu eg $\Leftrightarrow x = 0$.

7) h bilin + $\mathcal{B}$ bază, $A = [h]_{\mathcal{B}}$ mat. ei. h sim $\Leftrightarrow A$ sim.

Dem $\boxed{\text{"}\Rightarrow\text{"}}$ $h(b_i, b_j) = a_{ij}$
$\quad\quad \shortparallel \quad\quad\quad\quad \Rightarrow A = {}^tA.$
$h(b_j, b_i) = a_{ji}$

$\boxed{"\Leftarrow"}$ $h(b_i, b_j) = h(b_j, b_i)\ \forall\ i,j$ $\boxed{A = {}^tA}$.

$h(x,y) = \underbrace{{}^t[x]_{\mathcal{B}} \cdot A \cdot [y]_{\mathcal{B}}}_{\in k} = {}^t[y]_{\mathcal{B}} \cdot A [x]_{\mathcal{B}} = h(y,x)$

8) h nedeg. $\Leftrightarrow$ A inversabilă

Dem: $h(x,y) = {}^t[x] \underbrace{A \cdot [y]_{\mathcal{B}}}_{\text{inj + surj}}$.

Fie $x \neq 0 \Rightarrow \exists\, i_0$ cu $x_{i_0} \neq 0$. Aleg $y_0 \in V$ a.î. $A[y_0]_{\mathcal{B}} = e_{i_0}$ ← surj

$\Rightarrow h(x, y_0) = x_{i_0} \neq 0$.

9) $Q : V \to k$ s.n. formă pătratică dc. $Q(x) = h(x,x)$ pt. h bilin. sim

ex: $h(x,y) = 2x_1y_2 + 2x_2y_1 - 4x_1y_1 + 6x_2y_2 + x_2y_3 + x_3y_2$.

$\Rightarrow Q(x) = 4x_1x_2 - 4x_1^2 + 6x_2^2 + 2x_2x_3$.

10) matr. lui Q e matr. formei h din care provine.

$\boxed{\text{Pb 1}}$ a) $h: \mathbb{R}^3 \to \mathbb{R}$, $h(x,y) = x_1y_1 - 2x_1y_2 + 6x_2y_3 + 3x_3y_3$

$[h]_{\mathcal{B}_0} = \begin{pmatrix} 1 & -2 & 0 \\ 0 & 0 & 6 \\ 0 & 0 & 3 \end{pmatrix} = A$. Obs că $h(x,y) = {}^txAy \Rightarrow h$ biliniară

$\mathcal{B} = \left\{ \underset{v_1}{\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}}, \underset{v_2}{\begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix}}, \underset{v_3}{\begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix}} \right\}$

$h(v_1, v_1) = 1 - 0 - 0 + 3 = 4$

$h(v_1, v_2) = -1$

$h(v_1, v_3) = h((1,0,1),(0,2,1)) = -1$

$\Rightarrow [h]_{\mathcal{B}} = \begin{pmatrix} 4 & -1 & 1 \\ - & - & - \\ - & - & - \end{pmatrix}$

b) forma biliniară h^s asociată matr $\frac{1}{2}\left([h]_{\mathbb{B}_0} + {}^t[h]_{\mathbb{B}_0}\right)$

$$= \begin{pmatrix} 1 & -1 & 0 \\ -1 & 0 & 3 \\ 0 & 3 & 3 \end{pmatrix} \Rightarrow h^s(x,y) = x_1y_1 + 3x_3y_3 - x_1y_2 - x_2y_1 + 3x_2y_3 + 3x_3y_2$$

Clar $\underline{h(x,x) = h^s(x,x)}$

Obs: Q formă pătratică, $h(x,x) = Q(x)$ h bilin. sim $\Rightarrow$
$\Rightarrow h$ e unică

Dem: $h(x+y, x+y) = h(x,x) + 2h(x,y) + h(y,y)$

$$\boxed{h(x,y) = \frac{1}{2}\left(h(x+y,x+y) - h(x,x) - h(y,y)\right)}$$ $\leftarrow$ formula de polarizare

Q $h: V\times V \to k$ bilin. și $\mathbb{B}_1, \mathbb{B}_2$ baze. Vreau rel A_1, A_2

$[h]_{\mathbb{B}_1} = A_1 \quad A_2 = [h]_{\mathbb{B}_2}$

$h: \langle -,- \rangle$ pe $\mathbb{R}^2$, $[h]_{\mathbb{B}_0} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

~~Iau~~ $\mathbb{B} = \{2e_1, e_2\} \Rightarrow [h]_{\mathbb{B}} = \begin{pmatrix} 4 & 0 \\ 0 & 1 \end{pmatrix}$

Fie $P = M_{\mathbb{B}_1, \mathbb{B}_2}$

$[x]_{\mathbb{B}_2} = P \cdot [x]_{\mathbb{B}_1}$

$$h(x,y) = {}^t[x]_{\mathbb{B}_2} A_2 [y]_{\mathbb{B}_2}$$
$$= {}^t(P[x]_{\mathbb{B}_1}) A_2 \cdot P[y]_{\mathbb{B}_1}$$
$$= {}^t[x]_{\mathbb{B}_1} ({}^tP A_2 P) [y]_{\mathbb{B}_1}$$

$\boxed{A_1 = {}^tPAP}$; $P = M_{\mathbb{B}_1, \mathbb{B}_2}$

Teoremă (Aducerea la formă normală): Fie $Q : V \to k$ formă pătratică. $\Rightarrow \exists\ \mathcal{B}$ bază în care $Q(x) \overset{\text{în baza } \mathcal{B}}{=} \sum_{1}^{r} \lambda_i x_i^2$. În plus:

$\boxed{k = \mathbb{R}} \Rightarrow Q(x) = \sum_{1}^{p} x_i^2 - \sum_{p+1}^{r} x_j^2$ $\begin{cases} p \text{ plusuri} \\ r-p \text{ minusuri} \end{cases}$

$\boxed{k = \mathbb{C}} \Rightarrow Q(x) = \sum_{1}^{r} x_i^2$

signatura lui $Q = p - (r - p)$

<u>Dem</u>: Gauss

$\boxed{\text{Pb1 d)}}$ $Q(x) = x_1^2 - 2x_1x_2 + 6x_2x_3 + 3x_3^2$

$Q(x) = (x_1 - x_2)^2 - x_2^2 + 6x_2x_3 + 3x_3^2$

$Q(x) = y_1^2 - y_2^2 + 6y_2y_3 + 3y_3^2$

$Q(x) = y_1^2 - (y_2 - 3y_3)^2 + 12y_3^2$

$Q(x) = z_1^2 - z_2^2 + z_3^2 \Rightarrow sgn(Q) = 2 - 1 = 1.$

Q nedegenerată (apar toate 3)

! O formă bilin $\begin{cases} \oplus \text{ def} \Leftrightarrow \text{toate val proprii } \oplus. \\ \ominus \text{ def} \Leftrightarrow \text{—//— } \ominus. \end{cases}$

$$\begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} = \underbrace{\begin{pmatrix} 1 & -1 & 0 \\ 0 & 0 & \sqrt{12} \\ 0 & 1 & -3 \end{pmatrix}}_{A} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

$\boxed{\text{Pb2}}$ d) $Q(x) = 2x_2^2 + 2x_1x_2 + 4x_1x_3 - 2x_2x_3.$

$A = \begin{pmatrix} 0 & 1 & 2 \\ 1 & 2 & -1 \\ 2 & -1 & 0 \end{pmatrix} \overset{\text{Gauss}}{\Rightarrow} Q(x) = z_1^2 + z_2^2 - z_3^2,\ sgn(Q) = 1.$

$\boxed{Pb\,5}$ $h: V\times V \to k$ nedeg., V fin. gen $\Rightarrow h$ = izo între V și V^*

$\varphi: V \to V^*$, $\varphi(v)(w) = h(v,w)$

$\varphi(v) = h(v, \_) \in V^*$ pt. că e lin în arg 2.

φ lin pt. că $h(\_, w)$ e lin în arg 1

$\forall x \neq 0 \Rightarrow \exists y$ cu $h(x,y) \neq 0$ (h nedeg) $\Leftrightarrow \varphi$ inj $\overset{V \text{ fin}}{\underset{\dim}{\Rightarrow}} \varphi$ izo

$\boxed{Ex\,2}$ Dc. V admite $h: V\times V \to \mathbb{R}$ formă antisim nedeg $\Rightarrow$ $\dim V = 2k$.

$(V, h) \leftarrow$ sp. vectorial simplectic

Dem: $\mathcal{B}$ bază $\Rightarrow A = [b]_{\mathcal{B}}$

$\begin{cases} \det A \neq 0. \\ {}^tA = -A \Rightarrow \det A = (-1)^n \det A \neq 0 \Rightarrow \underline{n \text{ par}}. \end{cases}$

Spații vectoriale euclidiene

1) V ~~...~~ în $\mathbb{R}$ sp. vect, $h: \langle -, - \rangle$ s.n. produs scalar dc. e o formă bilin. $\begin{cases} \text{sim} \\ \text{nedeg.} \\ \text{poz. def.} \end{cases}$ $\Rightarrow \|x\| = \sqrt{\langle x, x\rangle} \leftarrow$ norma

$\|x\|^2 = Q(x)$.

2) orice sp. vect. real are un prod. scalar.

3) $\mathcal{B} = \{f_1, \dots f_n\}$ bază s.n. $\begin{cases} \text{ortogonală dc. } \langle f_i, f_j\rangle = 0, \ i \neq j \\ \text{ortonormală dc. } \langle f_i, f_j \rangle = \delta_{ij} \end{cases}$

ex: $\mathcal{B}_0$ bază ortonormală pt. $\langle x, y \rangle_0 = \sum x_i y_i$

Gram-Schmidt: Fie $(V, \langle -,- \rangle)$ sp. euclid și $\mathcal{B} = \{b_1, \dots, b_n\}$.

Atunci $\exists$ o bază ortonormală $\{f_1, \dots f_n\}$ a.î. $\langle f_1, \dots f_k \rangle = \langle b_1, \dots b_k \rangle$ $\forall k$.

$$\boxed{f_i = \frac{f_i'}{\|f_i'\|}, \quad f_i' = b_k - \sum_{1}^{k-1} \frac{\langle b_k, f_i' \rangle}{\|f_i\|^2}}$$

Dem (Ca la curs)

Pb1 $\mathcal{B} = \left\{ \begin{pmatrix} \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{pmatrix}, \begin{pmatrix} \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \end{pmatrix}, \begin{pmatrix} \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{6}} \\ \frac{-\sqrt{6}}{3} \end{pmatrix} \right\}$ bază ortonormală

$\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \alpha_1 f_1 + \alpha_2 f_2 + \alpha_3 f_3 \Rightarrow \alpha_i = \langle v, f_i \rangle$
$= v$

$v = \sum \langle v, f_i \rangle f_i$

$\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \frac{6}{\sqrt{3}} f_1 + \frac{1}{\sqrt{2}} f_2 + 0 f_3.$

Pb2 a) $W = \left\langle \begin{pmatrix} 3 \\ 6 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix} \right\rangle$ bază ortonormală W.

$f_1' = \begin{pmatrix} 3 \\ 6 \\ 0 \end{pmatrix} \Rightarrow f_1 = \frac{1}{3\sqrt{5}} \begin{pmatrix} 3 \\ 6 \\ 0 \end{pmatrix}$

$f_2' = \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix} - \frac{5}{3} \cdot \begin{pmatrix} 3 \\ 6 \\ 0 \end{pmatrix} = \begin{pmatrix} -4 \\ -8 \\ 2 \end{pmatrix} \Rightarrow f_2 = \frac{1}{82} \begin{pmatrix} -4 \\ -8 \\ 2 \end{pmatrix}$

Pb3 $f(x,y,z) = (2x - y + z, -4x + 2y - 2z)$.

$\ker f, \operatorname{Im} f$

$f(x,y,z) = \begin{pmatrix} 2 & -1 & 1 \\ -4 & 2 & -2 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$

$v \in \ker f \Rightarrow \begin{cases} 2x - y + z = 0 \\ -2x + y - z = 0 \end{cases} \Rightarrow$ bază $\ker f$: $\left\{ \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \right\}$

$f_1 = \frac{1}{\sqrt{5}} \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}, f_2' = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} - \frac{2}{2} \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} = \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix}$

$f_2 = \frac{1}{\sqrt{3}} \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix}$

$\dim \operatorname{Im} f = 1 \Rightarrow \{(1, -2)\}$ bază $\operatorname{Im} f$.

$e = \frac{1}{\sqrt{5}} \begin{pmatrix} 1 \\ -2 \end{pmatrix}$

Def: $W \leq (V, \langle -, - \rangle)$

$W^\perp = \{ v \in V \mid \langle v, w \rangle = 0 \ \forall w \in W \}$.

$\boxed{\text{Pbc}}$ a) $\dim W^\perp = \dim V - \dim W$

$V = W \oplus W^\perp$

b) $(W_1 + W_2)^\perp = W_1^\perp \cap W_2^\perp$

$(W_1 \cap W_2)^\perp = W_1^\perp + W_2^\perp$

c) $(W^\perp)^\perp = W$.

d) $W_1 = W_2 \Leftrightarrow W_1^\perp = W_2^\perp$

Mai demult, $W \leq V \to \operatorname{Ann} W = \{ f \in V^* \mid f|_W \equiv 0 \}$

Ora trecută: $\varphi: V \to V^*$ izo cu $\varphi(v) = \langle v, - \rangle$ F)

$\Rightarrow \varphi(W^\perp) = \operatorname{Ann} W$

$V^* = \{ \langle v, - \rangle \mid v \in V \}$.

$f \in \operatorname{Ann} W$, $f = \langle v, - \rangle \Leftrightarrow f|_W \equiv 0 \Leftrightarrow \langle v, w \rangle = 0$

$\forall w \in W \Rightarrow v \in W^\perp$

Știm de la anulatori tot, mai puțin $V = W \oplus W^\perp$.

$\dim W + \dim W^\perp = \dim V$.

$v \in W \cap W^\perp = \{0\}$

$v \in W^\perp \Rightarrow \langle v, w \rangle = 0 \ \forall w \in W$

$v \in W \Rightarrow \langle v, v \rangle = 0 \Rightarrow v = 0$.

! Pt. infinit dim. $\mathcal{C}([0,1]) = \{f : [0,1] \to \mathbb{R} \text{ cont}\}$.

$\langle f, g \rangle = \int_0^1 f(t) g(t) dt$

$\mathcal{C}^\infty([0,1]) \leq \mathcal{C}([0,1])$

$(\mathcal{C}^\infty)^\perp = \{0\}$ pt. că e densă

1) $W = \left\{ \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 1 \\ 3 \\ -1 \end{pmatrix} \right\}$, $W^\perp = \left\{ (x,y,z) \in \mathbb{R}^3 \,\middle|\, \begin{matrix} \langle (x,y,z), (1,2,3) \rangle = 0 \\ \langle (x,y,z), (1,3,-1) \rangle = 0 \end{matrix} \right\}$

$= \left\{ (x,y,z) \in \mathbb{R}^3 \,\middle|\, \begin{cases} x + 2y + 3z = 0 \\ x + 3y - z = 0 \end{cases} \right\}$.

$W \leq (V, \langle -, - \rangle)$

proiecția $pr_W^{W^\perp} : V \to V$ $\quad pr_W^{W^\perp}(\underset{\in W}{v'} + \underset{\in W^\perp}{v''}) = v'$

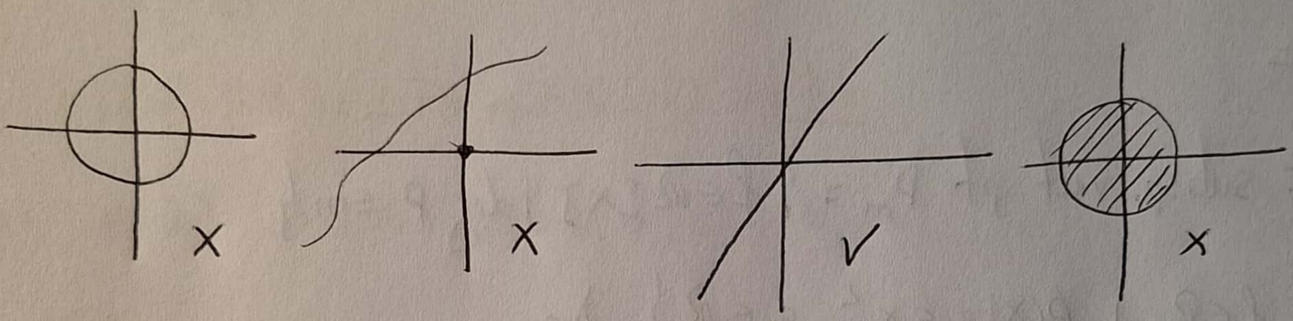
ortogonală

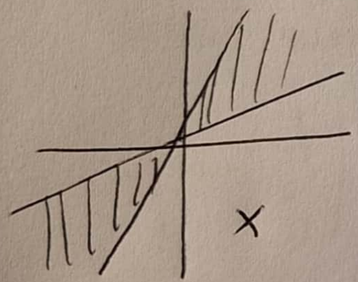
Seminar 4 - Alg. lin - 23.10.2023

Teorie - Subsp. vect:

V un k-sp. vect.

1) $W \subset V$ s.n. subsp. vect ($W \leq_k V$) dc. $ax + by \in W$ $\begin{cases} \forall a, b \in k \\ \forall x, y \in W \end{cases}$





Subsp. vect $\mathbb{R}^2/\mathbb{R}$.

2) $\{0\}$, V sunt subsp. vect pt. V.

3) W subsp $\Rightarrow 0 \in W$.

Exerciții:

Pb 1: $M_n(\mathbb{R})/\mathbb{R}$ sp. vect.

a) $U(M_n(\mathbb{R})) = \{A \in M_n(\mathbb{R}) \mid A \text{ inv}\}$

Nu $\quad I_n \in U(M_n(\mathbb{R}))$

$\quad I_n - I_n = O_n \notin U(M_n(\mathbb{R}))$

b) $\mathfrak{sl}_n(\mathbb{R}) = \{A \in M_n(\mathbb{R}) \mid \operatorname{tr} A = 0\}$

$$\operatorname{tr}(\alpha A + \beta B) = \alpha \operatorname{tr} A + \beta \operatorname{tr} B = 0 \quad \forall A, B \in \mathfrak{sl}_n(\mathbb{R}), \ \forall \alpha, \beta \in \mathbb{R}.$$

c) $SL_m(\mathbb{R}) = \{A \in M_m(\mathbb{R}) \mid \det A = 1\}$

<u>Nu</u> $O_m \notin SL_m(\mathbb{R})$

d) $\underline{so}_m(\mathbb{R}) = \{A \in M_m(\mathbb{R}) \mid A = {}^tA\}$

<u>Da</u> ${}^t(\alpha A + \beta B) = \alpha\, {}^tA + \beta\, {}^tB = \alpha A + \beta B \quad \forall A, B \in \underline{so}_m$
$\forall \alpha, \beta \in \mathbb{R}$.

e) $\Delta = \{A \in M_m(\mathbb{R}) \mid A \text{ diag}\}$

<u>Da</u>

<u>Pb2</u>: subsp. vect pt $P_m = \{f \in \mathbb{R}[X] \mid \deg P \leq m\}$

a) $\{f \in P_m \mid f(X) = aX^2,\ a \in \mathbb{R}\}$ <u>Da</u>.

b) $\{p \in P_m \mid p(X) = a + X^2,\ a \in \mathbb{R}\}$ <u>Nu</u>

c) $p(0) = 0$ <u>Da</u>

d) $\{p \in P_m \mid p$ cu coef întregi$\}$ nu e $\mathbb{R}$ sp. vect, e $\mathbb{Q}$ sp. vect.

<u>Pb3</u>: $\mathcal{F} = \{f : [0,1] \to \mathbb{R},\ f \text{ cont}\}$ $\mathbb{R}$ sp. vect.

$W = \{f \in \mathcal{F} \mid f \text{ derivab}\} \leq_{\mathbb{R}} \mathcal{F}$

$\{x^n \mid n \geq 0\} \not\leq_{\mathbb{R}} \mathcal{F}$

<u>Pb5</u>: $V = \mathbb{R}^3$, $W = \left\{ (x,y,z) \in \mathbb{R}^3 \,\middle|\, \begin{cases} x - 2y + z = 0 \\ y + 2z = 0 \\ -x + y + z = 0 \end{cases} \right\} \leq_{\mathbb{R}} V$

<u>Dem</u>: $A = \begin{pmatrix} 1 & -2 & 1 \\ 0 & 1 & 2 \\ -1 & 1 & 1 \end{pmatrix} \Rightarrow A\begin{pmatrix} x \\ y \\ z \end{pmatrix} = O_{31}$

$u, v \in W \Rightarrow A(\alpha u + \beta v) = \alpha Au + \beta Av = O_{31} \Rightarrow \alpha u + \beta v \in W$

! $W = \{v \in \mathbb{R}^n \mid Av = b\} \leq \mathbb{R}^n$

$b = 0 \Rightarrow W \leq \mathbb{R}^n$ ✓

$b \neq 0 \Rightarrow A(\alpha v + \beta w) = \alpha A v + \beta A w = (\alpha + \beta) b = b \Rightarrow \alpha + \beta = 1$

$\Rightarrow$ nu e subsp. vect.

<u>Pb 5</u>: $F \in M_{3,2}(\mathbb{R})$ fixată și $W = \{A \in M_{2,5}(\mathbb{R}) \mid FA = 0\}$

$W \overset{?}{\leq}_{\mathbb{R}} M_{2,5}(\mathbb{R})$.

$F(\alpha A + \beta B) = \alpha FA + \beta FB = 0 \Rightarrow$ <u>Da</u>.

<u>Pb 7</u>: $V = \mathbb{R}^4 \Rightarrow (a-3b,\ b-a,\ a,\ b) \leq_{\mathbb{R}} V$.

Vectorii sunt de forma $a \cdot \begin{pmatrix} 1 \\ -1 \\ 1 \\ 0 \end{pmatrix} + b \begin{pmatrix} -3 \\ 1 \\ 0 \\ 1 \end{pmatrix}$

$$\alpha\left(a \cdot \begin{pmatrix} 1 \\ -1 \\ 1 \\ 0 \end{pmatrix} + b\begin{pmatrix} -3 \\ 1 \\ 0 \\ 1 \end{pmatrix}\right) + \beta\left(x\begin{pmatrix} 1 \\ -1 \\ 1 \\ 0 \end{pmatrix} + y\begin{pmatrix} -3 \\ 1 \\ 0 \\ 1 \end{pmatrix}\right) = (\alpha a + \beta x)\begin{pmatrix} 1 \\ -1 \\ 1 \\ 0 \end{pmatrix} + (\alpha b + \beta y)\begin{pmatrix} -3 \\ 1 \\ 0 \\ 1 \end{pmatrix}$$

<u>Pb 8</u>: $(3a+b,\ 4-b,\ a-5b) \overset{?}{\leq}_{\mathbb{R}} \mathbb{R}^3$

Dc. da $\Rightarrow (0,0,0) = (3a+b,\ 4-b,\ a-5b) \Rightarrow 4-b=0 \Rightarrow b=4$

$\Rightarrow 3a+b=0 \Rightarrow a = -\frac{4}{3}$

$\Rightarrow a-5b=0$ (F)

<u>Pb 12</u>: V - k-sp. vect., $x,y,z \in V$ a.î. $\alpha x + \beta y + \gamma z = 0$ cu $\alpha\gamma \neq 0$.

$\Rightarrow \langle x, y \rangle = \langle y, z \rangle$.

$x = -\frac{\beta}{\alpha} y - \frac{\gamma}{\alpha} z \Rightarrow \left.\begin{matrix} x \in \langle y, z\rangle \\ y \in \langle y, z \rangle \end{matrix}\right\} \Rightarrow \langle x, y\rangle \subseteq \langle y, z \rangle$

$z = -\frac{\alpha}{\gamma} x - \frac{\beta}{\gamma} y \Rightarrow \left.\begin{matrix} z \in \langle x, y\rangle \\ y \in \langle x, y \rangle \end{matrix}\right\} \Rightarrow \langle y, z\rangle \subseteq \langle x, y \rangle$

$\Rightarrow \langle x, y \rangle = \langle y, z \rangle$

Pb13: $v_1 = (2,-1,1)$ $\Rightarrow u \in \langle v_1, v_2 \rangle$

$v_2 = (-3,1,4)$ $u = 2v_1 + v_2.$

$*$ $v_1 = (2,-1,1,3)$

$v_2 = (1,0,-6,5)$

$u = (1,1,1,1) \in \langle v_1, v_2 \rangle \Leftrightarrow \exists \alpha, \beta$ a.î $\alpha v_1 + \beta v_2 = 0.$

$$\Leftrightarrow \begin{cases} 2\alpha + \beta = 1 \\ -\alpha = 1 \\ \alpha - 6\beta = 1 \\ 3\alpha + 5\beta = 1 \end{cases} \quad \dots$$

Pb14: $v_1 = (1,-1,2,3)$

$v_2 = (-1,2,-1,-1)$

$v_3 = (2,-1,1,3)$

$\alpha = ?$ a.î $(2,3,1,\alpha) \in \langle v_1, v_2, v_3 \rangle$

$$av_1 + bv_2 + cv_3 = 0 \Rightarrow \begin{cases} a - b + 2c = 2 \\ -a + 2b - c = 3 \\ 2a - b + c = 1 \\ 3a - b + 3c = \alpha \end{cases} \Rightarrow \begin{cases} a = 0 \\ b = 3 \\ c = 3 \end{cases} \Rightarrow$$

$$\Rightarrow \boxed{\alpha = -12}$$

Pb15: $\mathbb{R}[x]$ nu e $\mathbb{R}$ sp. vect. fin. gen.

Pp. $\mathbb{R}[x] = \langle P_1(x), \dots, P_n(x) \rangle$.

$N = \max \deg P(x) \Rightarrow \deg(\sum \alpha_i P_i(x)) \leq N$ abs.

! $\mathbb{R}$ e $\mathbb{Q}$ sp. vect.

Pp. $\mathbb{R} = \langle x_1, \dots, x_m \rangle_{\mathbb{Q}} \Rightarrow \forall x \in \mathbb{R}\ \exists q_1, \dots, q_n \in \mathbb{Q}$ a.î $x = \sum q_i x_i$

$\Rightarrow f: \mathbb{Q}^m \to \mathbb{R}$, $f(q_1, \dots, q_m) = \sum q_i x_i$ esurj $\Rightarrow |\mathbb{Q}^m| \geq |\mathbb{R}|$ fals