Tutoriat nu știu nr.

Tuesday, 28 October 2025 10:24

Ex 2. Examen 2023

2. 1 pe toate poz (i,j) cu $i+j \le n+1$

1 pe toate poz (i,i) cu $1 \le i \le n$

0 pe toate celelalte poziții

a) Să se calculeze $\det(A_3)$ și $\det(A_4)$

$$A_3 = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$$

pt. A_3, $n=3$, $i+j \le 4$

1 pe poz $(1,1), (1,2), (1,3), (2,1), (2,2)$
$(3,1)$, $(3,3)$ (Nu uităm de diag principală
1 pe poz (i,i))

$$\det A_3 = \begin{vmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{vmatrix} = 1+0+0-1-1=-1.$$

$$A_4 = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}$$

pt A_4 $n=4$ $i+j \le 5$

1 pe poz $(1,1), (1,2), (1,3), (1,4), (2,1), (2,2)$,
$(2,3)$, $(3,1)$, $(3,2)$, $(3,3)$, $(4,1)$,
$(4,4)$.

$$\det A_4 = \begin{vmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{vmatrix} \overset{L_2=L_3}{=} 0.$$

b) Să se arate că A_3 inversabilă și să se calculeze inversa ei

din a) $\det A_3 = -1 \ne 0 \Rightarrow A_3$ inversabilă

$(A_3 \mid I_3) \to (I_3 \mid A_3^{-1})$

$$\left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 \end{array}\right) \xrightarrow[L_3 \to L_3 - L_1]{L_2 \to L_2 - L_1} \left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & -1 & -1 & 1 & 0 \\ 0 & -1 & 0 & -1 & 0 & 1 \end{array}\right) \xrightarrow{L_2 \leftrightarrow L_3} \left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & -1 & 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & -1 & 1 & 0 \end{array}\right) \xrightarrow[L_3(-1)]{L_2(-1)} \left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 & -1 & 0 \end{array}\right)$$

$$\xrightarrow{L_1 \to L_1 - L_2} \left(\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 & -1 & 0 \end{array}\right) \xrightarrow{L_1 \to L_1 - L_3} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 & -1 & 0 \end{array}\right) \Rightarrow A_3^{-1} = \begin{pmatrix} -1 & 1 & 1 \\ 1 & 0 & -1 \\ 1 & -1 & 0 \end{pmatrix}$$

c) Să se arate că pt orice $n \ge 2$, A_{2n} are liniile n și $n+1$ egale.

1 pe poz (i,j) cu $i+j \le 2n+1$

Avem 1 pe L_n pe poz: $(n,1), (n,2) \ldots (n,n), (n,n+1)$

de la $(n,n+2) \ldots (n,2n)$ avem 0.

Avem 1 pe L_{n+1} pe poz: $(n+1,1) \ldots (n+1,n), (n+1,n+1)$ (Nu uităm de diag pp)

0 în rest.

$$\begin{array}{l} C_n \quad C_{n+1} \quad C_{n+2} \quad \ldots \quad C_{2n} \\ L_n = (1 \;\; 1 \;\; \ldots \;\; 1 \;\; 1 \;\; 0 \;\; \ldots \;\; 0) \\ L_{n+1} = (1 \;\; 1 \;\; \ldots \;\; 1 \;\; 1 \;\; 0 \;\; \ldots \;\; 0) \end{array} \Bigg\} \Rightarrow L_n = L_{n+1} \quad \forall n \ge 2$$

C_n C_{n+1} C_{n+2} C_{2n}
(pică pe diagonala pp).

d) Să se calculeze $\det A_n$ $\forall n \ge 3$

$(\forall)\, n \ge 2$, $n \in \mathbb{N}$ din c) știm A_{2n} are $L_n = L_{n+1} \Rightarrow \det A_{2n} = 0$

Pt. A_{2n+1} $\forall n \in \mathbb{N}$ $n \ge 1$.

$$A_{2n+1} = \begin{pmatrix} 1 & 1 & 1 & \cdots & 1 & 1 & 1 \\ 1 & 1 & 1 & \cdots & 1 & 1 & 0 \\ 1 & 1 & 1 & \cdots & 1 & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots \\ 1 & 1 & 0 & \cdots & 0 & 1 & 0 \\ 1 & 0 & 0 & \cdots & 0 & 0 & 1 \end{pmatrix} \begin{array}{l} \to L_1 \\ \to L_2 \\ \to L_3 \\ \\ \to L_{2n} \\ \to L_{2n+1} \end{array}$$

$i+j \le 2n+2$

$$\det A_{2n+1} = \begin{array}{c} C_n \;\; C_{n+1} \;\; C_{n+2} \\ \begin{vmatrix} 1 & 1 & 1 \ldots 1 & 1 & 1 \ldots 1 & 1 & 1 \\ 1 & 1 & 1 \ldots 1 & 1 & 1 \ldots 1 & 1 & 0 \\ 1 & 1 & 1 \ldots 1 & 1 & 1 \ldots 1 & 0 & 0 \\ 1 & 1 & 1 \ldots 1 & 1 & 1 \ldots 0 & 0 & 0 \\ \vdots & & & & & & \\ 1 & 1 & 1 \ldots 1 & 1 & 1 \ldots 0 & 0 & 0 \\ 1 & 1 & 1 \ldots 1 & 1 & 0 \ldots 0 & 0 & 0 \\ 1 & 1 & 1 \ldots 1 & 0 & 1 \ldots 0 & 0 & 0 \\ \vdots & & & & & & \\ 1 & 1 & 0 \ldots 0 & 0 & 0 \ldots 0 & 1 & 0 \\ 1 & 0 & 0 \ldots 0 & 0 & 0 \ldots 0 & 0 & 1 \end{vmatrix} \end{array} \begin{array}{l} \to L_1 \\ \to L_2 \\ \to L_3 \\ \to L_4 \\ \\ \to L_n \\ \to L_{n+1} \\ \to L_{n+2} \\ \\ \to L_{2n} \\ \\ \end{array}$$

$L_1 \to L_1 - L_2$
$L_2 \to L_2 - L_3$
$L_3 \to L_3 - L_4$
$\vdots$
$L_n \to L_n - L_{n+1}$

$L_1 - L_2 : L_1$ $\quad L_1 = (0\,0\,0 \ldots 0\,0\,0 \ldots 0\,0\,1)\ (= e_{2n+1})$

$L_2 = (0\,0\,0 \ldots 0\,0\,0 \ldots 0\,1\,0)\ (= e_{2n})$

$L_3 = (0\,0\,0 \ldots 0\,0\,0 \ldots 1\,0\,0)$

$L_n \to L_n - L_{n+1}$ $\quad L_n = (0\,0\,0 \ldots 0\,0\,\underset{C_{n+2}}{1} \ldots 0\,0\,0)\ (= e_{n+2})$

$$= \begin{array}{c} C_{n+2} \\ \begin{vmatrix} 0 & 0 & 0 \ldots 0\,0\,0 \ldots 0 & 0 & 1 \\ 0 & 0 & 0 \ldots 0\,0\,0 \ldots 0 & 1 & 0 \\ 0 & 0 & 0 \ldots 0\,0\,0 \ldots 1 & 0 & 0 \\ \vdots & & & & \\ 0 & 0 & 0 \ldots 0\,0\,1 \ldots 0 & 0 & 0 \\ 1 & 1 & 1 \ldots 1\,1\,0 \ldots 0 & 0 & 0 \\ 1 & 1 & 1 \ldots 1\,0\,1 \ldots 0 & 0 & 0 \\ \vdots & & & & \\ 1 & 1 & 0 \ldots 0\,0\,0 \ldots 0 & 1 & 0 \\ 1 & 0 & 0 \ldots 0\,0\,0 \ldots 0 & 0 & 1 \end{vmatrix} \end{array} \begin{array}{l} \to L_1 \\ \to L_2 \\ \to L_3 \\ \\ \to L_n \\ \to L_{n+1} \\ \to L_{n+2} \\ \\ \to L_{2n} \\ \to L_{2n+1} \end{array}$$

$$\underset{\text{după } L_1}{\overset{\text{dezvoltăm succesiv}}{=}} (-1)^{1+2n+1} (-1)^{1+2n} (-1)^{1+2n-1} \cdots (-1)^{1+n+2}.$$

$$\underbrace{\begin{vmatrix} 1 & 1 & 1 & \ldots & 1 & 1 & 1 \\ 1 & 1 & 1 & \ldots & 1 & 1 & 0 \\ 1 & 1 & 1 & \ldots & 1 & 0 & 0 \\ \vdots & & & & & & \\ 1 & 1 & 1 & \ldots & 0 & 0 & 0 \\ 1 & 1 & 0 & \ldots & 0 & 0 & 0 \\ 1 & 0 & 0 & \ldots & 0 & 0 & 0 \end{vmatrix}}_{(n+1)\times(n+1)}$$

$$\overset{\text{dezv.}}{=} (-1)^{2n+2+2n+1+\ldots+n+3} (-1)^{n+1+1} (-1)^{n+1} + \ldots (-1)^{3+1} (-1)^{2+1} \cdot 1$$

$$= (-1)^{\frac{(2n+2)(2n+3)}{2} - 3} = (-1)^{(n+1)(2n+3)-6} = (-1)^{(n+1)} \cdot \underbrace{(-1)^{2n+3}}_{=-1} \underbrace{(-1)^{-6}}_{=1} = (-1)(-1)^{n+1} = (-1)\cdot\underbrace{(-1)\cdot(-1)}_{=1}{}^{n} = (-1)^n$$

$\to$ impar

$2n^2 + 3n + 2n - 3$

$2n^2 + 5n - 3$

$\Delta = 25 + 3 \cdot 4 \cdot 2 = 25 + 24 = 49$

$\sqrt{\Delta} = 7$

$n_1 = \frac{-5+7}{4} = \frac{2}{4} = \frac{1}{2}$

$n_2 = \frac{-5-7}{4} = \frac{-12}{4} = -3$

Observăm că sub diagonala secundară avem numai 0-uri deci putem dezvolta după ultima linie succesiv. (sau știm că $\det(A_n) = (-1)^{\frac{n(n-1)}{2}}$, dacă avem 0 sub diag secundară)

formulă de la seminar probabil

$$\det(A_n) = \begin{cases} 0, & \text{dacă } n = 2k \\ (-1)^k & \text{dacă } n = 2k+1. \end{cases}$$

$k = 2p$, $k = 2p+1$

$$\det(A_n) = \begin{cases} 0, & \text{dacă } n = 4p \\ 1, & \text{dacă } n = 4p+1 \\ -1, & \text{dacă } n = 4p+3 \\ 0, & \text{dacă } n = 4p+2 \end{cases}$$