$\sigma_1 = Tr(A) = 3$

$$\sigma_2 = \begin{vmatrix} 1 & 0 \\ 2 & 1 \end{vmatrix} + \begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix} + \begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix} = 1$$

$\sigma_3 = \det(A) = 1$

$$(-1)^3 \left[A^3 - 3A^2 + A - I_3\right] = O_3 \quad | \cdot A^{-1}$$

$$A^2 - 3A + I_3 - A^{-1} = O_3 \Rightarrow A^{-1} = A^2 - 3A + I_3 =$$

$$= \begin{pmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & -1 & 1 \end{pmatrix}$$

10.10.2024.

Algebră și Geometrie (seminar 2)

Forme eșalon. Teorema Hamilton-Cayley

1) $A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}$

$A^{-1} = ?$ → a) $A^{-1} = \frac{1}{\det A} A^*$

→ b) cu Th. H-C

→ c) Algoritmul Gauss-Jordan (forme eșalon)

$$\det A = \begin{vmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 1 \end{vmatrix} = 1 \cdot (-1)^{1+1} \begin{vmatrix} 0 & 1 \\ 1 & 1 \end{vmatrix} = -1 \neq 0 \Rightarrow$$

$\Rightarrow$ A inversabilă

c) Gauss - Jordan

$$(A|J_3) = \left(\begin{array}{ccc|ccc} \boxed{1} & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right) \sim$$

$$(J_3 | A^{-1})$$

$$\sim \left(\begin{array}{ccc|ccc} \boxed{1} & 1 & 0 & 1 & 0 & 0 \\ 0 & \boxed{1} & 1 & 0 & 0 & 1 \\ 0 & 0 & \boxed{1} & 0 & 1 & 0 \end{array}\right) \underset{L_2' = L_2 - L_3}{\sim}$$

Forme échelon

$$\sim \left(\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 \end{array}\right) \underset{L_1' = L_1 - L_2}{\sim}$$

$$\sim \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 1 & -1 \\ 0 & 1 & 0 & 0 & -1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 \end{array}\right)$$

$$\underbrace{\hspace{3cm}}_{J_3} \qquad \underbrace{\hspace{3cm}}_{A^{-1}}$$

$$A^{-1} = \begin{pmatrix} 1 & 1 & -1 \\ 0 & -1 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

a) $A^{-1} = \frac{1}{\det A} A^*$

$$A^t = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix} \qquad A^* = \begin{pmatrix} -1 & -1 & 1 \\ 0 & 1 & -1 \\ 0 & -1 & 0 \end{pmatrix}$$

$$A^*_{11} = (-1)^{1+1} \begin{vmatrix} 0 & 1 \\ 1 & 1 \end{vmatrix} = -1$$

$$A^*_{12} = (-1)^{1+2} \begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix} = -1$$

$$A^*_{13} = (-1)^{1+3} \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1$$

$$A^*_{21} = (-1)^{2+1} \begin{vmatrix} 0 & 0 \\ 1 & 1 \end{vmatrix} = 0$$

$$A^*_{22} = (-1)^{2+2} \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1$$

$$A^*_{23} = (-1)^{2+3} \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = -1$$

$$A^*_{31} = (-1)^{3+1} \begin{vmatrix} 0 & 0 \\ 0 & 1 \end{vmatrix} = 0$$

$$A^*_{32} = (-1)^{3+2} \begin{vmatrix} 1 & 0 \\ 1 & 1 \end{vmatrix} = -1$$

$$A^*_{33} = (-1)^{3+3} \begin{vmatrix} 1 & 0 \\ 1 & 0 \end{vmatrix} = 0$$

$\det A = -1 \Rightarrow A^{-1} = -A^{*} = \begin{pmatrix} 1 & 1 & -1 \\ 0 & -1 & 1 \\ 0 & 1 & 0 \end{pmatrix}$

b) Hamilton - Cayley

$$\boxed{A^3 - \sigma_1 A^2 + \sigma_2 A - \sigma_3 I_3 = O_3}$$

$\sigma_1 = Tr(A) = 2$

$\sigma_2 = (\det L_1C_1;\ L_2C_2;\ L_3C_3) = \begin{vmatrix} 0 & 1 \\ 1 & 1 \end{vmatrix} + \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} + \begin{vmatrix} 1 & 1 \\ 0 & 0 \end{vmatrix} = -1 + 1 = 0$

$\sigma_3 = \det A = -1$

$\Rightarrow A^3 - 2A^2 + I_3 = O_3 \quad \Big| \cdot A^{-1}$

$A^2 - 2 \cdot A + A^{-1} = O_3 \Rightarrow A^{-1} = 2A - A^2$

$A^2 = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 2 \end{pmatrix}$

$A^{-1} = \begin{pmatrix} 2 & 2 & 0 \\ 0 & 0 & 2 \\ 0 & 2 & 2 \end{pmatrix} - \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 1 & -1 \\ 0 & -1 & 1 \\ 0 & 1 & 0 \end{pmatrix}$

2) $A=\begin{pmatrix} 3 & 1 & 2 \\ 0 & 4 & 1 \\ 1 & 1 & 0 \end{pmatrix}$

a) forma eșalon ; forma eșalon redusă

b) rang $A = ?$

$$A=\begin{pmatrix} 3 & 1 & 2 \\ 0 & 4 & 1 \\ 1 & 1 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & 0 \\ 0 & 4 & 1 \\ 3 & 1 & 2 \end{pmatrix} \underset{L_3 = L_3 - 3L_1}{\sim}$$

$$\sim \begin{pmatrix} 1 & 1 & 0 \\ 0 & 4 & 1 \\ 0 & -2 & 2 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & 0 \\ 0 & -2 & 2 \\ 0 & 4 & 1 \end{pmatrix} \underset{L_2 \cdot \left(-\frac{1}{2}\right)}{\sim}$$

$$\sim \begin{pmatrix} \boxed{1} & 1 & 0 \\ 0 & \boxed{1} & -1 \\ 0 & 4 & 1 \end{pmatrix} \underset{L_3 - 4L_2}{\sim} \begin{pmatrix} \boxed{1} & 1 & 0 \\ 0 & \boxed{1} & -1 \\ 0 & 0 & \boxed{5} \end{pmatrix} \underset{\frac{1}{5}L_3}{\sim}$$

forma eșalon

$$\sim \begin{pmatrix} \boxed{1} & 1 & 0 \\ 0 & \boxed{1} & -1 \\ 0 & 0 & \boxed{1} \end{pmatrix} \underset{L_2 + L_3}{\sim} \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \underset{L_1 - L_2}{\sim} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

forma eșalon redusă

$\Rightarrow$ rang $A = 3$.

3) $A = \begin{pmatrix} 0 & 1 & 1 & 0 \\ -1 & 2 & 0 & 1 \\ -1 & 0 & -2 & 1 \\ 0 & -1 & -1 & 0 \end{pmatrix}$

a) Să se scrie polinomul caracteristic

b) $A^{100} = ?$, utilizând Th H-C.

a) $P_A(x) = \det(A - xI_4) = \begin{vmatrix} -x & 1 & 1 & 0 \\ -1 & 2-x & 0 & 1 \\ -1 & 0 & -2-x & 1 \\ 0 & -1 & -1 & -x \end{vmatrix} =$

$= (-1)^4 \cdot x^4 - \sigma_1 \cdot x^3 + \sigma_2 x^2 - \sigma_3 x + \sigma_4$

$\sigma_1 = \operatorname{Tr} A = 0$

$\sigma_2 = \begin{array}{c} \begin{matrix} 11 & 12 \end{matrix} \\ \begin{vmatrix} 0 & 1 \\ -1 & 2 \end{vmatrix} \\ \begin{matrix} 21 & 22 \end{matrix} \end{array} + \begin{vmatrix} 0 & 1 \\ -1 & -2 \end{vmatrix} + \begin{vmatrix} 0 & 0 \\ 0 & 0 \end{vmatrix} + \begin{vmatrix} 2 & 0 \\ 0 & -2 \end{vmatrix} +$

$+ \begin{vmatrix} 2 & 1 \\ -1 & 0 \end{vmatrix} + \begin{vmatrix} -2 & 1 \\ -1 & 0 \end{vmatrix} = 1 + 1 + 0 - 4 + 1 + 1 = 0$

$\sigma_3 = \begin{vmatrix} 2 & 0 & 1 \\ 0 & -2 & 1 \\ -1 & -1 & 0 \end{vmatrix} + \underbrace{\begin{vmatrix} 0 & 1 & 0 \\ -1 & -2 & 1 \\ 0 & -1 & 0 \end{vmatrix}}_{0} + \underbrace{\begin{vmatrix} 0 & 1 & 0 \\ -1 & 2 & 1 \\ 0 & -1 & 0 \end{vmatrix}}_{0} +$

$+ \begin{vmatrix} 0 & 1 & 1 \\ -1 & -2 & 0 \\ -1 & 0 & -2 \end{vmatrix}$

$$\sigma_3 = \begin{vmatrix} 0 & 0 & 1 \\ -1 & 2 & 0 \\ -1 & 2 & -2 \end{vmatrix} + \begin{vmatrix} 2 & 0 & 1 \\ -2 & -2 & 0 \\ -1 & -1 & 0 \end{vmatrix} = 0$$

$\sigma_4 = \det A = 0 \Rightarrow P_A(X) = X^4$

$P_A(A) = O_4 \Rightarrow A^4 = O_4 \;\Big|^{25} \Rightarrow A^{100} = O_4$

4) $A = \begin{pmatrix} 1 & -1 \\ 2 & 0 \end{pmatrix}$; $B = A^4 - 3A^3 + 3A^2 - 2A + 8 I_2$

$a, b = ?$; $a, b \in \mathbb{R}$ a.î. $B = aA + bI_2$

Th H-C $\quad A^2 - Tr(A) \cdot A + \det(A) \cdot I_2 = O_2$

$Tr(A) = 1$; $\det(A) = 2$

$A^2 - A + 2I_2 = O_2$

$P_A(X) = X^2 - X + 2$ (polinomul caracteristic)

$Q(X) = X^4 - 3X^3 + 3X^2 - 2X + 8$

$Q(X) = C(X) \cdot P_A(X) + R$ (TÎR)

$$\begin{array}{rrrrr|l}
X^4 & -3X^3 & +3X^2 & -2X & +8 & X^2 - X + 2 \\
\cline{6-6}
-X^4 & +X^3 & -2X^2 & & & X^2 - 2X - 1 \\
\hline
/ & -2X^3 & +X^2 & -2X & +8 & \\
 & 2X^3 & -2X^2 & +4X & & \\
\hline
 & / & -X^2 & +2X & +8 & \\
 & & X^2 & -X & +2 & \\
\hline
\end{array}$$

$B = (A^2 - 2A - I_2)\underbrace{(A^2 - A + 2I_2)}_{O_2} + A + 10I_2 =$

$= A + 10I_2 \Rightarrow a = 1$

$b = 10$

sau

$M_2)$ $A^2 = A - 2I_2$

$A^3 = A^2 - 2A = A - 2I_2 - 2A = -A - 2I_2$

$A^4 = A^2 - 2A = -A + 2I_2 - 2A = -3A + 2I_2$

$B = -3A + 2I_2 - 3(-A - 2I_2) + 3(A - 2I_2) - 2A + 8I_2 =$

$= A + 10I_2 \Rightarrow a = 1$

$b = 10$

11

$X^{2024} = A = \begin{pmatrix} 2 & -4 \\ -1 & 2 \end{pmatrix}$; $X \in M_2(\mathbb{R})$

a) Precizați nr de sol

b) Dacă $X \in M_2(\mathbb{C})$, care este nr de sol

a) $\det A = 0 \Rightarrow \det(X^{2024}) = [\det(X)]^{2024} = 0 \Rightarrow \det(X) = 0$

$X^2 - Tr(X) \cdot X + \det(X) \cdot I_2 = O_2$

$X^2 = Tr(X) \cdot X$

$\Rightarrow X^{2024} = Tr^{2023}(X) \cdot X$

$A = Tr^{2024}(X) \cdot X \mid Tr$

$Tr(A) = Tr^{2024}(X) = 4$

$\Rightarrow Tr(X) = \pm \sqrt[2024]{4}$

$X = \dfrac{1}{\pm \sqrt[2024]{4^{2023}}} A \Rightarrow$ ect are 2 sol

Obs: • $X^2 = \alpha \cdot X$

$X^n = \alpha^{n-1} \cdot X$

• $Tr(\alpha \cdot X) = \alpha \cdot Tr(X)$

• $z^{2n} = 1$ 2 sol reale ± 1

• $z^{2n+1} = 1$ 1 sol reale 1

b) $\mathrm{Tr}^{2024}(X)=4$ are 2024 răd complexe

$\Rightarrow$ ec $X^{2024}=A$ are 2024 răd complexe

(14) $A, B \in \mathcal{M}_n(\mathbb{R})$ și $AB=BA$

Dem că $\det(A^2+B^2)\geq 0$

$\det(A^2+B^2)=\det(A^2-i^2B^2)=\det[(A-iB)(A+iB)]=$

$=\underbrace{\det(A-iB)}_{\bar{z}}\underbrace{\det(A+iB)}_{z}=|z|^2\geq 0$

(7) x_1, x_2, x_3 răd $x^3+px+q=0$

$$\Delta=\begin{vmatrix} 1 & 1 & 1 \\ x_1 & x_2 & x_3 \\ x_1^2 & x_2^2 & x_3^2 \end{vmatrix}^2=?$$

$$A=\begin{pmatrix} 1 & 1 & 1 \\ x_1 & x_2 & x_3 \\ x_1^2 & x_2^2 & x_3^2 \end{pmatrix}$$

$$\Delta = \det(A\cdot A^t) = \det(A)\cdot\det(A^t) = \det A^2$$

$$A\cdot A^t = \begin{pmatrix} 1 & 1 & 1 \\ x_1 & x_2 & x_3 \\ x_1^2 & x_2^2 & x_3^2 \end{pmatrix}\begin{pmatrix} 1 & x_1 & x_1^2 \\ 1 & x_2 & x_2^2 \\ 1 & x_3 & x_3^2 \end{pmatrix} =$$

<u>Not</u>

$S_k = x_1^k + x_2^k + x_3^k$

$$= \begin{pmatrix} S_0 & S_1 & S_2 \\ S_1 & S_2 & S_3 \\ S_2 & S_3 & S_4 \end{pmatrix}$$

$$\begin{cases} \Delta_1 = x_1 + x_2 + x_3 = 0 \\ \Delta_2 = x_1x_2 + x_1x_3 + x_2x_3 = p \\ \Delta_3 = x_1x_2x_3 = -q \end{cases}$$

$S_0 = 3$

$S_1 = \Delta_1 = 0$

$S_2 = \Delta_1^2 - 2\Delta_2 = -2p$

$S_3 = -3q$

$S_4 = 2p^2$

$$\begin{cases} x_1^3 + px_1 + q = 0 \\ x_2^3 + px_2 + q = 0 \\ x_3^3 + px_3 + q = 0 \end{cases} \quad (+)$$

$$S_3 + pS_1 + 3q = 0 \Rightarrow S_3 = -p\underbrace{S_1}_{0} - 3q \Rightarrow S_3 = -3q$$

$$S_4 + pS_2 + qS_1 = 0 \Rightarrow S_4 = 2p^2$$

$$\det(A\cdot A^t)=\begin{vmatrix}3 & 0 & -2p\\ 0 & -2p & -3q\\ -2p & -3q & 2p^2\end{vmatrix}=$$

$$=-12p^3+8p^3-27q^2=-4p^3-27q^2$$

9) $A=\begin{pmatrix}a & b\\ -b & a\end{pmatrix}$; $B=\begin{pmatrix}c & d\\ -d & c\end{pmatrix}$

Arătați că $(a^2+b^2)(c^2+d^2)=(ac-bd)^2+(ad+bc)^2$ (*)

$\det A\cdot\det B=\det(AB)$ (1)

$$AB=\begin{pmatrix}a & b\\ -b & a\end{pmatrix}\begin{pmatrix}c & d\\ -d & c\end{pmatrix}=\begin{pmatrix}ac-bd & ad+bc\\ -bc-ad & -bd+ac\end{pmatrix}$$

$$\det(AB)=(ac-bd)^2-(ad+bc)(-bc-ad)=$$

$$=(ac-bd)^2+(ad+bc)^2 \quad (2)$$

Din (1), (2) $\Rightarrow$ (*)

<u>Listă probleme</u> → 6) după curs

8)

10) după curs

12) simplu - factor comun + 0-uri

13) Indicații

$$\Delta(x)=\begin{vmatrix}f(x) & g(x)\\ h(x) & l(x)\end{vmatrix}$$

$$\Delta'(x)=\begin{vmatrix}f'(x) & g'(x)\\ h(x) & l(x)\end{vmatrix}+\begin{vmatrix}f(x) & g(x)\\ h'(x) & l'(x)\end{vmatrix}$$

14) Hamilton - Cayley