SEMINAR VIII - 23. 11. 2023

Endomorfisme. Diagonalizare. Forme biliniare

1. $f \in End(\mathbb{R}^3)$, $R_0 = \{e_1, e_2, e_3\}$ reper canonic în $\mathbb{R}^3$

$f(e_1) = e_3 = 0 \cdot e_1 + 0 \cdot e_2 + 1 \cdot e_3$

$f(e_2) = e_2 = 0 \cdot e_1 + 1 \cdot e_2 + 0 \cdot e_3$

$f(e_3) = e_1 = 1 \cdot e_1 + 0 \cdot e_2 + 0 \cdot e_3$

a) Determinați valorile proprii și subspațiile proprii

b) determinați R reper în $\mathbb{R}^3$ a. î. $[f]_{R,R}$ este diagonală

c) să se determine A

a) Etapa 1: $A = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$

$A = [f]_{R_0, R_0}$

OBSERVAȚIE: $f: \mathbb{R}^3 \to \mathbb{R}^3$, $f(x) = y$ $\quad X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$

$Y = AX$

$f(x) = (x_3, x_2, x_1)$

Etapa 2: $p(\lambda) = \det(A - \lambda I_3) = \begin{vmatrix} -\lambda & 0 & 1 \\ 0 & 1-\lambda & 0 \\ 1 & 0 & -\lambda \end{vmatrix} = 0 \Rightarrow$

$\Rightarrow (1-\lambda) \begin{vmatrix} -\lambda & 1 \\ 1 & -\lambda \end{vmatrix} = (1-\lambda)(\lambda^2 - 1)$

$(1-\lambda)(\lambda-1)(\lambda+1) = -(\lambda-1)^2(\lambda+1) = 0$

$\lambda_1 = 1$, $m_1 = 2$

$\lambda_2 = -1$, $m_2 = 1$

Etapa 3. $V_{\lambda_1} = \{x \in \mathbb{R}^3 \mid f(x) = \lambda_1 x\} = \{(x_1, x_2, x_1) \mid x_1, x_2 \in \mathbb{R}\}$

$$AX = 1 \cdot X = (A - i_3)X = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow \begin{pmatrix} -1 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & -1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$\Rightarrow -x_1 + x_3 = 0$

$x_1 - x_3 = 0$ $\Rightarrow x_1 - x_3$

$$V_{\lambda_1} = \underbrace{\mathcal{L}\{(1,0,1), (0,1,0)\}}_{R_1 \text{ reper în } V_{\lambda_1}}$$

$V_{\lambda_2} = \{x \in \mathbb{R}^3 \mid f(x) = -1 \cdot x\} = \{(-x_3, 0, x_3) \mid x_3 \in \mathbb{R}\}$

$$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$= <\underbrace{\{(-1,0,1)\}}_{R_2 - \text{reper în } V_{\lambda_2}}>$

$\dim V_{\lambda_2} = 1 = m_2$

$$AX = -X \Rightarrow (A + i_3)X = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{cases} x_1 + x_3 = 0 \Rightarrow x_1 = -x_3 \\ 2x_2 = 0 \Rightarrow x_2 = 0 \\ x_1 + x_3 = 0 \end{cases}$$

b) $\mathbb{R}^3 = V_{\lambda_1} \oplus V_{\lambda_2}$, $R = R_1 \cup R_2 = \{(1,0,1), (0,1,0), (-1,0,1)\}$ reper în $\mathbb{R}^3$

$$A' = [f]_{R,R} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

c) $R_0 \xrightarrow{C} R$

$e_1' = e_1 + e_3$

$e_2' = e_2$

$e_3 = -e_1 + e_3$

$$C = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$$

$A' = C^{-1} A C \rightarrow A = C A' C^{-1}$

$A^n = (C A' C^{-1})(C A' C^{-1}) \dots (C A' C^{-1}) = C \cdot A'^n \cdot C^{-1} =$

$= C \cdot \begin{pmatrix} 1^n & 0 & 0 \\ 0 & 1^n & 0 \\ 0 & 0 & (-1)^n \end{pmatrix}$

5. $f \in End(\mathbb{R}^3)$, $\lambda_1 = 3$, $\lambda_2 = -2$, $\lambda_3 = 1$ sunt valorile proprii

$\underbrace{v_1 = (-3, 2, 1),\ v_2 = (-2, 1, 0),\ v_3 = (-6, 3, 1)}_{\text{vectorii proprii corespunzători}}$

$A = [f]_{R_0, R_0} = ?$

x vector propriu corespunzător valorii proprii λ

$x \neq 0_{\mathbb{R}^3}$ $\quad f(x) = \lambda x$

$f(v_1) = \lambda_1 v_1$

$f(v_2) = \lambda_2 v_2$

$f(v_3) = \lambda_3 v_3$

$R = \{v_1, v_2, v_3\}$ reper în $\mathbb{R}^3$ a.î.

$$A' = [f]_{R,R} = \begin{pmatrix} 3 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

v_1, v_2, v_3 vectori proprii corespunzători la valori proprii distincte $\xrightarrow{\text{teorie}}$ SLi

$\left.\begin{array}{l} \dim \mathbb{R}^3 = 3 = \text{card } R \\ + \text{SLi} \end{array}\right\} \Rightarrow R$ reper în $\mathbb{R}^3$

$R_0 \xrightarrow{C} R \quad C = \begin{pmatrix} -3 & -2 & -6 \\ 2 & 1 & 3 \\ 1 & 0 & 1 \end{pmatrix}$

6. $f: \mathbb{R}^3 \to \mathbb{R}^3$, $f(x) = (4x_1 + x_2 + x_3, x_1 + 4x_2 + x_3, x_1 + x_2 + 4x_3)$

Precizați dacă $\exists$ un reper în R în raport cu care $[f]_{R,R}$ este diagonală

$$A = \begin{pmatrix} 4 & 1 & 1 \\ 1 & 4 & 1 \\ 1 & 1 & 4 \end{pmatrix}$$

$$P(\lambda) = \det(A - \lambda I_3) = \begin{vmatrix} 4-\lambda & 1 & 1 \\ 1 & 4-\lambda & 1 \\ 1 & 1 & 4-\lambda \end{vmatrix} \overset{L_1' = L_1+L_2+L_3}{=}$$

$$= \begin{vmatrix} 6-\lambda & 6-\lambda & 6-\lambda \\ 1 & 4-\lambda & 1 \\ 1 & 1 & 4-\lambda \end{vmatrix} = (6-\lambda) \begin{vmatrix} 1 & 1 & 1 \\ 1 & 4-\lambda & 1 \\ 1 & 1 & 4-\lambda \end{vmatrix} \underset{C_2' = C_2 - C_1}{\overset{C_3' = C_3 - C_1}{=}}$$

$$= (6-\lambda) \begin{vmatrix} 1 & 0 & 0 \\ 1 & 3-\lambda & 0 \\ 1 & 0 & 3-\lambda \end{vmatrix} = (6-\lambda)(3-\lambda)^2$$

$\lambda_1 = 6, m_1 = 1$

$\lambda_2 = 3, m_2 = 2$

$V_{\lambda_1} = \{ X \in \mathbb{R}^3 \mid f(x) = \lambda_1 x \}$

$AX = 6X$

$(A - 6I_3)X = 0$

din $V_{\lambda_2} = 3 - rg(A - 6I_2) = 3 - 2 = 1 = m_1$ (1)

$V_{\lambda_2} = \{x \in \mathbb{R}^3 \mid f(x) = \lambda_2 x\}$

$AX = 3X$

$(A - 3I_2)X = 0_{3,1}$

$$A - 3I_3 = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$$

$\dim V_{\lambda_2} = 3 - 1 = 2 = m_2$ (2)

Din (1),(2) $\Rightarrow \exists$ reper pt $\mathbb{R}^3$ a.î. $[f]_{R,R}$ diag

8. $g : \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}$, $g(x,y) = x_1y_1 - x_2y_2 - x_1y_3 - x_3y_1 + 2x_2y_3 + 2x_3y_2$

a) $g \in L^1(\mathbb{R}^3, \mathbb{R}^3; \mathbb{R})$ (formă biliniară simetrică)

b) Precizați matricea G asociată lui g în raport cu $R_0 = \{e_1, e_2, e_3\}$

c) $\operatorname{Ker} g = ?$ Este g nedegenerată ?

d) Să se afle matricea G' asociată lui g în raport cu reperul $R' = \{e_1' = (1,1,1), e_2' = (1,2,1), e_3' = (0,0,1)\}$

a) $\left.\begin{array}{l} g(ax + bz, y) = a g(x,y) + b g(z,y) \\ g(x, ay + bz) = a g(x,y) + b g(x,z) \end{array}\right\}$ g biliniar

g simetrică $\Leftrightarrow g(x,y) = g(y,x) \Leftrightarrow G = G^t$

$R_0 = \{e_1, e_2, e_3\}$ reper canonic în $\mathbb{R}^3$

$g_{i,j} = g(e_i, e_j)$, $i,j = \overline{1,n}$

$g(x,y) = g\left(\sum_{i=1}^{3} x_i e_i, \sum_{j=1}^{3} y_j e_j\right) = \sum_{i,j=1}^{3} x_i y_j \overbrace{g(e_i, e_j)}^{g_{ij}}$

$= X^T G Y = (x_1, x_2, x_3) \begin{pmatrix} g_{11} & g_{12} & g_{13} \\ g_{21} & g_{22} & g_{23} \\ g_{31} & g_{32} & g_{33} \end{pmatrix}$

$$G = \begin{pmatrix} 1 & 0 & -1 \\ 0 & -1 & 2 \\ -1 & 2 & 0 \end{pmatrix} = G^t$$

g nedegenerata $\Leftrightarrow Ker\, g = \{0_{\mathbb{R}^3}\} \Leftrightarrow \det G \neq 0$ (G matrice nedegenerată)

$Ker\, g = \{x \in \mathbb{R}^3 \mid g(x,y)=0,\ x,y \in \mathbb{R}\}$

$g(x, e_1) = 0 \Rightarrow x_1 \cdot 1 - 1 \cdot x_3 = 0$

$g(x, e_2) = 0 \Rightarrow -1 \cdot x_2 + 2x_3 = 0$ $\quad G = \begin{pmatrix} 1 & 0 & -1 \\ 0 & -1 & 2 \\ -1 & 2 & 0 \end{pmatrix}$

$g(x, e_3) = 0 \Rightarrow -1 \cdot x_1 + 2x_2 = 0$

$$\Rightarrow \det G = \begin{vmatrix} 1 & 0 & 0 \\ 0 & -1 & 2 \\ -1 & 2 & -1 \end{vmatrix} = 1 - 4 = -3 \neq 0$$

$\Rightarrow Ker\, g = \{0_{\mathbb{R}^3}\}$

$R_0 : G$ $\quad C = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 2 & 0 \\ 1 & 1 & 1 \end{pmatrix}$

$R' : G'$

$R_0 \xrightarrow{C} R'$ $\quad C^t = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 1 \\ 0 & 0 & 1 \end{pmatrix}$

$G' = C^t G C$

$$G' = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & -1 \\ 0 & -1 & 2 \\ -1 & 2 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 & 0 \\ 1 & 2 & 0 \\ 1 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 3 \\ -1 & 2 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 & 0 \\ 1 & 2 & 0 \\ 1 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 3 & 1 \\ 3 & 3 & 3 \\ 1 & 3 & 0 \end{pmatrix} = G'$$

sau

$G = (g'_{ij})_{ij} = g'_{ij} = g(e_i', e_j')$

$g'_{11} = g(e_1', e_1') = g((1,1,1),(1,1,1)) = 2$

$\vdots$

$g'_{33} =$