Seminarul 6

Wednesday, March 24, 2021 8:49 AM

Inele de polinoame. Inel factor

Reamintire R inel (comutativ), $I \subset J$

Atunci $J/I = \{ \hat{x} \mid x \in J \} \trianglelefteq R/I$ și

$\hat{x}$ modulo I

$$\frac{R/I}{J/I} \simeq R/J$$

(Th I de izomorfism)

2.3. a) $\mathbb{Z}[X]/(2, X)$ ← data trecută

b) $\mathbb{Z}[X]/(7, X-2)$

c) $\mathbb{Z}[X]/(X+5, X-2)$

} Temă

c) $\mathbb{Z}[X]/(X+5, X-2)$

Var 1 $\boxed{(X+5, X-2) = (7, X-2)}$

$7 = (X+5) - (X-2)$ $X+5 = (X-2) + 7$

$$= \mathbb{Z}[X]/ \nearrow \simeq \frac{\mathbb{Z}[X]/(7)}{(7, X-2)/} \simeq \frac{\mathbb{Z}_7[X]}{(\varphi(7), \varphi(X-2))} \simeq$$

$\mathbb{Z}[X]/(7) \xrightarrow[\sim]{\varphi} \mathbb{Z}_7[X]$

$$= \frac{\mathbb{Z}[X]}{(7, X-2)} \nearrow \simeq \frac{\mathbb{Z}[X]/(7)}{(7, X-2)/(7)} \simeq \frac{\ \ }{(\varphi(7), \varphi(X-2))} \simeq$$

$$\frac{R[X]}{(X-a)} \simeq R \qquad \varphi(a_n X^n + \dots + a_1 X + a_0) = \hat{a}_n X^n + \dots + \hat{a}_1 X + \hat{a}_0$$

$$\simeq \frac{\mathbb{Z}_7[X]}{(\hat{0}, X-\hat{2})} \simeq \frac{\mathbb{Z}_7[X]}{(X-\hat{2})} \simeq \mathbb{Z}_7 .$$

$\boxed{\begin{array}{l} f : R \to S, \ \ I \trianglelefteq R, I = (x_1, \dots, x_n) \\ \text{surjectivă} \\ \Rightarrow f(I) = (f(x_1), \dots, f(x_n)) \ ! \end{array}}$

$$\frac{R[X]}{(X-a)} \simeq R$$

$$\searrow \frac{\mathbb{Z}[X]/(X-2)}{(7, X-2)/(X-2)} \simeq \frac{\mathbb{Z}}{(\varphi(7), \varphi(X-2))} = \frac{\mathbb{Z}}{(7, 0)} = \mathbb{Z}/(7) \simeq \mathbb{Z}_7$$

$$\mathbb{Z}[X]/(X-2) \xrightarrow{\varphi} \mathbb{Z}$$

$$\varphi(\hat{P}) = P(2)$$

<u>Var 2</u>

$$\frac{\mathbb{Z}[X]}{(X-2, X+5)} \overset{\text{TI Iz}}{\simeq} \frac{\mathbb{Z}[X]/(X-2)}{(X-2, X+5)/(X-2)} \simeq \frac{\mathbb{Z}}{(0, 7)}$$

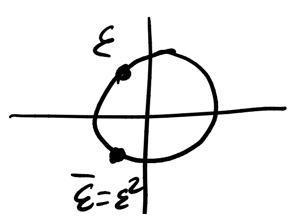
$$\simeq \mathbb{Z}_7 .$$

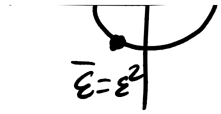
d) $\dfrac{\mathbb{Z}[X]}{(X^2+X+1)} \simeq \mathbb{Z}[\varepsilon] = \{a + b\varepsilon \mid a, b \in \mathbb{Z}\}$

$\varepsilon^2 = 1 - 1 \cdot \varepsilon$

$\Rightarrow X^2 + X + 1$ are rădăcinile ε, $\varepsilon^2 = \bar{\varepsilon}$ a.î. $\varepsilon^3 = 1$

şi $\varepsilon^2 + \varepsilon + 1 = 0$





<u>Dem</u> Fie $\varphi: \mathbb{Z}[X] \longrightarrow \mathbb{Z}[\varepsilon]$, $\varphi(P)=P(\varepsilon)$.

- morfism de inele (din proprietatea de universalitate)
- φ surjectiv : $a+b\varepsilon = \varphi(a+bX)$, $\forall a,b\in\mathbb{Z}$
- $\operatorname{Ker}\varphi = (X^2+X+1)$

"$\supset$" clar

"$\subset$" TIR împărțirii cu rest: Fie P a.î. $P(\varepsilon)=0$.

$$P = Q(X^2+X+1) + a+bX \quad, a,b\in\mathbb{Z} \text{ (unice!)}$$

$$\overset{\text{în } \varepsilon}{\Longrightarrow} 0=P(\varepsilon) = Q(\varepsilon)\underbrace{(\varepsilon^2+\varepsilon+1)}_{0} + a+b\varepsilon \iff a+b\varepsilon=0 \overset{\varepsilon\notin\mathbb{R}}{\iff} a=b=0$$

$$\Longrightarrow P=Q(X^2+X+1) \iff P\in(X^2+X+1)$$

$$\text{TIR F.I.} \Longrightarrow \mathbb{Z}[X]/(X^2+X+1) \simeq \mathbb{Z}[\varepsilon].$$

$$d')\quad \mathbb{Z}[X]/(X^3-1) = \mathbb{Z}[X]/(X-1)(X^2+X+1) \overset{LCR}{\simeq} \cancel{\mathbb{Z}[X]/(X-1) \times \mathbb{Z}[X]/(X^2+X+1)}$$

$$(X-1)+(X^2+X+1) = \mathbb{Z}[X] \;?$$

Dar regis nu e domeniu! $-X(X-1)+X^2+X+1 = 2X+1$

$$2(X-1)-(2X+1) = -3$$

[illegible]

$$2(x-1)-(2x+1)=-3$$

e) $$\mathbb{Z}[x]/(7,x^2+x+1) \overset{\text{Th I Izo}}{\simeq} \frac{\mathbb{Z}[x]/(7)}{(7,x^2+x+1)/(7)}$$

$$\simeq \frac{\mathbb{Z}_7[x]}{(x^2+x+\hat{1})}$$

$$(x^2+x+\hat{1})=(x-\hat{2})(x+\hat{3})$$

$$\simeq \mathbb{Z}_7[x]/(x-\hat{2})(x+\hat{3}) \overset{\text{L.C.R.}}{\simeq} \mathbb{Z}_7\times\mathbb{Z}_7 .$$

$$\hat{1}\cdot(x-\hat{2})-\hat{1}(x+\hat{3})=\hat{2}\in U(\mathbb{Z}_7)$$

$$\mathbb{Z}[x]/(n) \simeq \mathbb{Z}_n[x], \quad \mathbb{Z}[x]/(x-b)\simeq\mathbb{Z}$$

$\hat{p} \longmapsto$ coeficienții modulo n $\qquad \hat{p}\longmapsto P(b)$

f) $$\mathbb{Z}[i]/(7+i) \simeq \frac{\mathbb{Z}[x]/(x^2+1)}{(7+x)} \;\times\; \frac{\mathbb{Z}[x]/(x^2+1)}{(7+x)/(x^2+1)}$$

nu are sens

$$\varphi: \mathbb{Z}[x]/(x^2+1) \simeq \mathbb{Z}[i], \quad \varphi(\hat{p})=P(i).$$

Ce $\hat{p}$ are $\varphi(\hat{p})=(7+i)$

$$(\widehat{7+x}) \trianglelefteq \mathbb{Z}[x]/(x^2+1) \Rightarrow (\widehat{7+x}) = \mathcal{J}/(x^2+1) \quad \text{pt}$$

un ideal $\mathcal{J} \trianglelefteq \mathbb{Z}[x]$

$\mathcal{J} \supset (x^2+1)$

$$\frac{\mathbb{Z}[x]/(x^2+1)}{(\widehat{7+x})} = \frac{\mathbb{Z}[x]/(x^2+1)}{(7+x,\, x^2+1)/(x^2+1)}$$

$$(\widehat{7+x}) = (7+x,\, x^2+1)/(x^2+1) \overset{def}{=} \{\hat{f} \mid f \in (7+x,\, x^2+1)\}$$

$$= (\widehat{7+x},\, \widehat{x^2+1}) = (\widehat{7+x})$$

Deci $\boxed{\mathbb{Z}[i]/(7+i) \cong \frac{\mathbb{Z}[x]/(x^2+1)}{(7+x,\, x^2+1)/(x^2+1)}} \overset{\text{III Iso}}{\cong} \frac{\mathbb{Z}[x]}{(7+x,\, x^2+1)}$

$\Big($ Idee: $(7+x,\, x^2+1) = (7+x,\, 50)$? (Alg. lui Euclid?) $\Big)$

$$R[x]/(x-a) \cong R$$

$$\overset{\text{III Iso}}{\cong} \frac{\mathbb{Z}[x]/(7+x)}{(7+x,\, x^2+1)} \cong \frac{\mathbb{Z}}{(0,\, 50)} \cong \mathbb{Z}_{50}$$

$$\simeq \frac{\cdots/(7+x)}{(7+x,\ x^2+1)/(7+x)} \simeq \frac{\cdots}{(0,\ 50)} \simeq \mathbb{Z}_{50}$$

g) $$\mathbb{Z}[i]/(1+2i) \overset{\text{la fel}}{\simeq} \frac{\mathbb{Z}[x]/(x^2+1)}{(1+2x,\ x^2+1)/(x^2+1)} \overset{\text{T I izo}}{\simeq} \frac{\mathbb{Z}[x]}{(1+2x,\ x^2+1)} = (*)$$

$$\simeq \frac{\mathbb{Z}[x]/(1+2x)}{(1+2x,\ x^2+1)/(1+2x)} \simeq \frac{\mathbb{Z}[\frac{1}{2}]}{(\frac{5}{4})} \simeq \cdots$$

$\mathbb{Z}[x]/(1+2x) \simeq \mathbb{Z}[\frac{1}{2}]$

$$\boxed{(1+2x,\ x^2+1) = (x-2,\ x^2+1)}$$

$x-2 = \boxed{x(1+2x)} - 2(x^2+1)$ $\qquad$ $1+2x = x^2+1 - x(x-2) = 2x+1$

$$(*) = \frac{\mathbb{Z}[x]}{(x-2,\ x^2+1)} \simeq \frac{\mathbb{Z}[x]/(x-2)}{(x-2,\ x^2+1)/(x-2)} \simeq \mathbb{Z}_5.$$

Altfel: $(1+2i) = (i-2)$: $\boxed{i(1+2i)} = i-2$

$i \in U(\mathbb{Z}[i])$

2.4. $I=(2,x^2+1) \trianglelefteq \mathbb{Z}[X]$.

a) Nu este principal

b) $\mathbb{Z}[X]/I$ este inel cu 4 elemente, neizomorf cu $\mathbb{Z}_2\times\mathbb{Z}_2$.

a) <u>Var 1</u> Pp că $(2,x^2+1)=(f)$, $f\in\mathbb{Z}[X]$

$$\Rightarrow f \mid 2, \quad f \mid x^2+1$$

$$f=\pm 1 \text{ sau } \cancel{f=\pm 2} \quad \Rightarrow f=\pm 1$$

$$\Rightarrow (2,x^2+1)=\mathbb{Z}[X]$$

Pp că $\exists\, g,h\in\mathbb{Z}[X]$ a.î. $1=2g+(x^2+1)h$.

$\xRightarrow{\text{evaluat în } 1}$ $1=2g(1)+2h(1) \;\vdots\; 2$ abs

$\xRightarrow{\text{evaluat în } i}$ $1=2\underset{\mathbb{Z}[i]}{\underset{\cap}{g(i)}}$ abs

b) $$\mathbb{Z}[X]/I = \mathbb{Z}[X]/(2,X^2+1) \overset{\text{T. I izo}}{\simeq} \frac{\mathbb{Z}[X]/(2)}{(2,X^2+1)/(2)}$$

$$\simeq \frac{\mathbb{Z}_2[X]}{(X^2+\hat{1})} = \frac{\mathbb{Z}_2[X]}{(X+\hat{1})^2}$$

- 4 elemente: $R[X]/(f) = \{\, \overline{a_0+a_1X+\dots+a_{n-1}X^{n-1}} \mid a_i \in K\}$

  f monic de grad n

$$\Rightarrow \mathbb{Z}_2[X]/(X^2+\hat{1}) = \{\, \overline{\hat{a}+\hat{b}X} \mid \hat{a},\hat{b} \in \mathbb{Z}_2\} \Leftarrow \text{4 elemente}$$

clasa în $\mathbb{Z}_2[X]/(X^2+\hat{1})$

- $\mathbb{Z}_2[X]/(X+\hat{1})^2 \not\simeq \mathbb{Z}_2 \times \mathbb{Z}_2$

$\cup$

$\overline{\hat{1}}, \overline{X}$ au proprietatea că $\overline{\hat{1}}^2 = \overline{\hat{1}}$

$\overline{X}^2 = \overline{X^2} = \overline{-\hat{1}} = \overline{\hat{1}}$

$\boxed{\text{Doar } (\hat{1},\hat{1}) \text{ are proprietatea că } (\hat{1},\hat{1})^2 = (\hat{1},\hat{1})}$

Sau: în $\mathbb{Z}_2[X]/(X+\hat{1})^2$, $\overline{X+\hat{1}}$ este nilpotent!

Pe când în $\mathbb{Z}_2\times\mathbb{Z}_2$, nu avem decât nilpotentul trivial $(\hat{0},\hat{0})$!

Revenind la a) : Var 2 $(2, x^2+1) = (f)$, f monic

$$\Rightarrow \mathbb{Z}[X]/(f) = \{\hat{g} \mid \deg g < \deg f\}$$ → o infinitate de elemente

→ 1 element, de $f = 1$

2.5. b) $\mathbb{Z}[\sqrt{7}]/(6+\sqrt{7})$ $\underline{\text{corp cu 29 de elemente}}$

$$\varphi(\hat{P}) = P(\sqrt{7})$$

$$\mathbb{Z}[\sqrt{7}]/(6+\sqrt{7}) \overset{\text{Exc de}}{\underset{\text{data trecuta (pt } 7 \mapsto 2)}{\simeq}} \frac{\mathbb{Z}[X]/(X^2-7)}{(6+X, X^2-7)/(X^2-7)} \overset{\text{T.I.}_{20}}{\simeq} \boxed{\frac{\mathbb{Z}[X]}{(6+X, X^2-7)}}$$

J/I, $J \supset I$

$$\overset{\text{T.I.}_{20}}{\simeq} \frac{\mathbb{Z}[X]/(6+X)}{(6+X, X^2-7)/(6+X)} \overset{\hat{X} \mapsto -6}{\simeq} \frac{\mathbb{Z}}{(36-7)} = \mathbb{Z}_{29}$$ corp cu 29 de elemente.

a) r, q prime , $\mathbb{Z}[X]/(X^2-r) \simeq \mathbb{Z}[X]/(X^2-q) \Leftrightarrow r = q$.

(x^2-p) — (x^2-q)

(Temă!)

2.7. $f = x^2$, $g = \boxed{2}x$ în $\mathbb{Z}[X]$.

Merge Th de împărțire cu rest?

Dacă da, $\exists!\ q, r \in \mathbb{Z}[X]$ a.î. $f = qg + r$, $\deg r < \deg g$

$$\Leftrightarrow \boxed{x^2 = \boxed{2xq} + a}, \quad a \in \mathbb{Z}$$

$\Rightarrow \deg q = 1$ — termenul de grad 2 $\Rightarrow$ are coef par după

$$x^2 \,\big|\, \underline{2x}$$
$$\quad \tfrac{1}{2}x$$

2.8. Fie $a, b, c \in \mathbb{R}$, nenule a.î. $a+b+c \neq 0$ și

$$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} = \frac{1}{a+b+c}.$$

Dem că $\boxed{\forall n \text{ impar}, \quad \frac{1}{a^n} + \frac{1}{b^n} + \frac{1}{c^n} = \frac{1}{a^n+b^n+c^n}.}$

Dem $\dfrac{ab+ac+bc}{abc} = \dfrac{1}{a+b+c} \Leftrightarrow (a+b+c)(ab+ac+bc) = abc.$

Dem $\frac{ab+ac+bc}{abc} = \frac{1}{a+b+c} \Leftrightarrow \underbrace{(a+b+c)}_{s_1}\underbrace{(ab+ac+bc)}_{s_2}=\underbrace{abc}_{s_3}.$

$\Rightarrow s_3 = s_1 \cdot s_2.$

a, b, c sunt rădăcinile lui $X^3 - s_1X^2 + s_2X - s_3 =$

$= X^3 - s_1X^2 + s_2X - s_1s_2 = X^2(X-s_1) + s_2(X-s_1) = \underbrace{(X-s_1)(X^2+s_2)}_{\text{are răd } a, b, c}$

a, b, c reale $\Rightarrow s_2 \le 0$

De exemplu, $a = s_1$, $b = \sqrt{-s_2}$, $c = -\sqrt{-s_2}$ $\Rightarrow \boxed{b=-c}$

$\Downarrow$

$\forall n$ impar, $\frac{1}{a^n} + \cancel{\frac{1}{b^n}} + \cancel{\frac{1}{c^n}} = \frac{1}{a^n + \cancel{b^n} + \cancel{c^n}}$

2.9. $P \in \mathbb{R}[X]$ de grad n a.î. $P(k) = \frac{k}{k+1}$, $\forall k = 0, 1, \dots n$.

$P(n+1) = ?$

Fie $Q = (X+1)P - X \in \mathbb{R}[X]$. $Q(k) = 0$, $\forall k = 0, \dots n$

$\deg Q = n+1 \Rightarrow 0, 1, \dots n$ sunt rădăcinile lui Q.

$\Longrightarrow Q = aX(X-1)\dots(X-n)$ pt un $a \in \mathbb{R}$

$\Longrightarrow Q = aX(X-1)\ldots(X-n)$ pt un $a \in \mathbb{K}$

$(X+1)P - X$

Deci aflu $a \Rightarrow$ aflu $P(n+1)$!

$$(X+1)P - X = aX(X-1)\ldots(X-n).$$

În -1: $1 = a(-1)(-2)\ldots(-n-1) = a(-1)^{n+1}\cdot(n+1)!$

$$\Rightarrow a = \frac{(-1)^{n+1}}{(n+1)!}$$

$$\Rightarrow (X+1)P - X = \frac{(-1)^{n+1}}{(n+1)!} X(X-1)\ldots(X-n).$$

În $n+1$: $(n+2)P(n+1) - (n+1) = \frac{(-1)^{n+1}}{\cancel{(n+1)!}}\cdot\cancel{(n+1)!}$

$$\Rightarrow \boxed{P(n+1) = \frac{(-1)^{n+1}+(n+1)}{n+2}}$$

2.10. Fie p prim (Th Euler pt p prim)

a) $\mathbb{Z}_p \setminus \{\hat{0}\}$ e grup cu înmulțirea $\Rightarrow \hat{a}^{p-1} = \hat{1}$, $\forall \hat{a} \neq \hat{0}$.

$$|\mathbb{Z}_p \setminus \{\hat{0}\}| = p-1.$$

G grup (comutativ) finit $\Rightarrow$ $\underbrace{\text{ord } x \mid \text{ord } G}$ $= |G|$

$$\{e, x, x^2, \ldots\} = \langle x \rangle$$

$$a \sim b \iff a\langle x\rangle = b\langle x\rangle$$

echivalență (la stânga) relativ la $\langle x \rangle$

$$\Rightarrow G = \bigcup_{\hat{a} \in {}_{\langle x\rangle}\backslash G} \underbrace{a\langle x\rangle}_{\text{ord } x \text{ elemente}}$$

$$\Rightarrow |G| = |{}_{\langle x\rangle}\backslash G| \cdot |\langle x \rangle|$$

b) $f = X^{p-1} - \hat{1} \in \mathbb{Z}_p[X]$.

are rădăcinile $\hat{1}, \hat{2}, \ldots \widehat{p-1}$

c) Demonstrați Th lui Wilson: $(p-1)! \equiv -1 \pmod{p}$

Viete pt f: $\underbrace{\hat{1} \cdot \hat{2} \cdot \ldots \cdot \widehat{(p-1)}}_{\widehat{(p-1)!}} = (-1)^{p-1} \cdot (-\hat{1}) = \begin{cases} -\hat{1} & p \neq 2 \\ \hat{1} = -\hat{1}. & p = 2 \end{cases}$

$(p-1)$.

$1 = \ldots$

$\uparrow$

în $\mathbb{Z}_2$

d) <u>Def</u> $a \in \mathbb{Z}$ e <u>rest pătratic modulo p</u> $\leftarrow$ prim

$\Leftrightarrow$ ecuația $X^2 = a$ are soluții modulo p

$\Leftrightarrow \exists\, b \in \mathbb{Z}$ a.î. $b^2 \equiv a \pmod p$

$$\left(\frac{a}{p}\right) = \begin{cases} 1, & a \text{ e rest pătratic} \\ -1, & a \text{ \underline{nu} e rest pătratic} \\ (0, & a=0) \end{cases}.$$

Dem că -1 e rest pătratic modulo $p \Leftrightarrow p=2$ sau $p \equiv 1 \pmod 4$.

„$\Rightarrow$" Pp $p \neq 2$: $\exists\, \hat{a} \in \mathbb{Z}_p$ a.î. $\hat{a}^2 = -\hat{1} \underset{\hat{a}\neq\hat{0}}{\Rightarrow} \hat{a}^4 = \hat{1} \Rightarrow 4 \mid p-1$

$\Rightarrow p \equiv 1 \pmod 4$

„$\Leftarrow$" $p=2$: $1^2 = -1 \pmod 2$

$p \equiv 1 \pmod 4$: Știm că $(p-1)! \equiv -1 \pmod p$

$$\overset{p\neq 2}{\Leftrightarrow} \hat{1}\cdot\hat{2}\cdot\hat{3}\cdot\hat{4}\cdots\widehat{\tfrac{p-1}{2}}\cdot\widehat{\tfrac{p+1}{2}}\cdots\widehat{(p-4)}\,\widehat{(p-3)}\,\widehat{(p-2)}\,\widehat{(p-1)} = -\hat{1} \quad (\text{în } \mathbb{Z}_p)$$

$\parallel \quad \parallel \quad \parallel \quad \parallel$

$\Leftrightarrow 1\cdot 2\cdot 3\cdot 4 \cdots \underset{-\hat{4}}{(p-4)}\,\underset{-\hat{3}}{(p-3)}\,\underset{-\hat{2}}{(p-2)}\,\underset{-\hat{1}}{(p-1)} = -1 \pmod p$

$$\Leftrightarrow \underbrace{\hat{1}\cdot(-\hat{1})}\cdot\underbrace{\hat{2}\cdot(-\hat{2})}\cdots\underbrace{\widehat{\tfrac{p-1}{2}}\cdot\widehat{\tfrac{p+1}{2}}} = -\hat{1}$$

$$\Leftrightarrow (-1)^{\frac{p-1}{2}}\cdot \widehat{\left(\frac{p-1}{2}\right)!}^{\,2} = -\hat{1} \Rightarrow \left(\left(\frac{p-1}{2}\right)!\right)^2 = -1 \pmod p$$

$$(-1)^{\frac{p-1}{2}} \overset{p\equiv 1 \,(\mathrm{mod}\, 4)}{=\!=} 1$$

Temă: 2.11. $I=(3, X^3-X^2+2X+1) \trianglelefteq \mathbb{Z}[X]$. Arătați că I nu e principal și că $\mathbb{Z}[X]/I$ nu e corp.

Inele de polinoame în mai multe nedeterminate

2.12. $P\in\mathbb{Z}[X_1,X_2,X_3]$,

$$P = 3X_1^2X_2^2X_3 + 7X_1^3X_2^2X_3 + 5X_1^6 - 3X_1X_2X_3 - 25X_3^3+1.$$

Scrieți P ca element în $(\mathbb{Z}[X_1,X_2])[X_3]$

și

$(\mathbb{Z}[X_2,X_3])[X_1]$

și

cu sumă de componente omogene.

Dem $P = -25X_3^3 + (3X_1^2X_2^2 + 7X_1^3X_2^2 - 3X_1X_2)X_3 + (5X_1^6 + 1)$

$\deg_{X_3} = 3$ $\qquad$ $\deg_{X_3} = 1$ $\qquad$ $\deg_{X_3} = 0$

$P = 5X_1^6 + (7X_2^2X_3)\cdot X_1^3 + (3X_2^2X_3)\cdot X_1^2 - (3X_2X_3)X_1 + (-25X_3^3 + 1)$

$$P = \underbrace{(7X_1^3X_2^2X_3 + 5X_1^6)}_{\deg = 6} + \underbrace{3X_1^2X_2^2X_3}_{\deg 5} + \underbrace{(-25X_3^3 - 3X_1X_2X_3)}_{\deg 3} + \underbrace{1}_{\deg 0}$$

2.13. R comutativ

$$R[X,Y]/(X-Y) \simeq R[X]$$

Dem Direct: $\varphi : R[X,Y] \to R[X]$, $\quad \varphi(P) \overset{Y \mapsto X}{=} P(X,X)$.

$$\boxed{\begin{array}{c} a \in S \\ S[Y]/(Y-a) \simeq S \\ \varphi(P) = P(a) \end{array}}$$

$$R[X,Y]/(X-Y) \simeq (R[X])[Y]/\underbrace{(Y-X)}_{\deg 1} \simeq R[X]$$

Generalizați!

$$\frac{R[X,Y]}{(Y^2-X)} \simeq R[X]$$

$$\| $$

$$\frac{(R[Y])[X]}{(X-Y^2)}$$

doar o rotație

$$\frac{R[X,Y]}{(X-P)} \simeq R[Y] \quad \left(\simeq R[X]\right)$$

<u>Orice $P \in R[Y]$</u> !

2.14. K corp sau doar domeniu. Demonstrați că $\frac{K[X,Y]}{(Y^2-X)}$ și $\frac{K[X,Y]}{(Y^2-X^2)}$ nu sunt izomorfe.

$$\frac{K[X,Y]}{(Y^2-X)} \simeq K[Y] \text{ domeniu}$$

Dar $(\widehat{Y-X}) \cdot (\widehat{Y+X}) = \hat{0}$ în $\frac{K[X,Y]}{(Y^2-X^2)}$

<u>Ex 2.15.</u> R inel (com.), $a_1, a_2, \dots a_n \in R$.

$$\frac{R[X_1, X_2, \dots X_n]}{(X_1-a_1, X_2-a_2, \dots X_n-a_n)} \simeq R$$

$$\Big/ (X_1-a_1, X_2-a_2, \dots X_n-a_n)$$

$$\frac{R[X_1,\dots X_n]}{(X_1-a_1,\dots X_n-a_n)} \simeq \frac{R[X_1,\dots X_{n-1}][X_n] \big/ (X_n-a_n)}{(X_1-a_1,\dots X_n-a_n) \big/ (X_n-a_n)} \xrightarrow{X_n \mapsto a_n} \simeq$$

$$\simeq \frac{R[X_1,\dots X_{n-1}]}{(X_1-a_1,\dots X_{n-1}-a_{n-1})} \overset{\text{inducție}}{\simeq} R.$$

Temă: "Altă" demonstrație?

În particular, dacă $R=K$ corp, atunci

$I=(X_1-a_1, X_2-a_2, \dots X_n-a_n)$ este maximal

(pt că $K[X_1,\dots X_n] \big/ I \simeq K$ corp)

b) Exemple de corpuri K cu alte ideale maximale în $K[X_1,\dots X_n]$?

$K[X_1,X_2]$ $\qquad$ $\mathbb{R}[X] \big/ (X^2+1) \simeq \mathbb{C}$!

$$K[X_1, X_2] \qquad \mathbb{R}[\ldots] / (X^2+1) \simeq \mathbb{C}.$$

$$\Rightarrow \quad \mathbb{R}[X,Y] / (X^2+1,\ Y-1) \overset{\text{TI iz.}}{\simeq} \mathbb{R}[X] / (X^2+1) \simeq \mathbb{C}$$

Teoremă (weak Nullstellensatz, Hilbert)

Dacă K e algebric închis $\Rightarrow$ $(X_1-a_1, \ldots, X_n-a_n)$ sunt singurele ideale maximale în $K[X_1, \ldots, X_n]$.

$$\Longrightarrow \quad I \trianglelefteq K[X_1, \ldots, X_n], \quad Z(I) = \{(x_1, \ldots, x_n) \in K^n \mid f(x_1, \ldots, x_n) = 0, \ \forall f \in I\}$$

$$\neq \emptyset \quad \text{dc} \quad I \neq K[X_1, \ldots, X_n]$$

$$Z((X_1-a_1, \ldots, X_n-a_n)) \ni (a_1, \ldots, a_n)$$