Geometrie analitică euclidiană

1) $(\mathbb{R}^3, (\mathbb{R}^3, g_0), \varphi)$ $\quad A(3,-1,3)$, $B(5,1,-1)$

$u = (-3, 5, 6)$

a) ec. dreptei D a.î. $A \in D$, $V_D = \langle \{u\} \rangle$

b) ec. dr AB

c) pct. de int ale lui D cu planele de coord

a) $D: \dfrac{x_1-3}{-3} = \dfrac{x_2+1}{5} = \dfrac{x_3-3}{-6} = t \Rightarrow \begin{cases} x_1 = -3t+3 \\ x_2 = 5t-1 \\ x_3 = -6t+3 \end{cases}$

b) $\overrightarrow{AB} = (5-3, 1+1, -1-3) = (2, 2, -4) = 2(1,1,-2)$

$AB: \dfrac{x_1-3}{1} = \dfrac{x_2+1}{1} = \dfrac{x_3-3}{-2} = t \Rightarrow (*) \begin{cases} x_1 = t+3 \\ x_2 = t-1, \quad t \in \mathbb{R} \\ x_3 = -2t+3 \end{cases}$

$\forall M \in AB, \exists\, t \in \mathbb{R}$ a.î. $\overrightarrow{AM} = t\overrightarrow{AB}$

c) $Ox_1x_2 \ni O$, $N = (0,0,1)$

$\left.\begin{array}{l} 0 \cdot x_1 + 0 \cdot x_2 + 1 \cdot x_3 + d = 0 \\ O \in Ox_1x_2 \end{array}\right\} \Rightarrow x_3 = 0$

1) $D \cap Ox_1x_2 : x_3 = 0$

$\overset{(*)}{\Rightarrow} -6t+3 = 0 \Rightarrow t = \dfrac{1}{2}$

$M\left(-3 \cdot \frac{1}{2} + 3, 5 \cdot \frac{1}{2} - 1, 0\right)$

$\Leftrightarrow M\left(\frac{3}{2}, \frac{3}{2}, 0\right)$

2) $D \cap Ox_2x_3 \quad x_1 = 0$

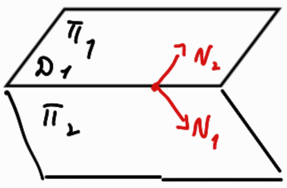
$\overset{(*)}{\Rightarrow} -3t+3 = 0 \Rightarrow t = 1$

$P(0, 4, -3)$

3) $D \cap Ox_1x_3 \qquad x_2 = 0$

$\circledast \Rightarrow 5t - 1 = 0 \Rightarrow t = \frac{1}{5}$

$Q\left(-\frac{3}{5}+3, 0, -\frac{6}{5}+3\right) \Leftrightarrow Q\left(\frac{12}{5}, 0, \frac{9}{5}\right)$



2. $D': \begin{cases} 2x_1 - x_2 + 3x_3 + 1 = 0 \\ 5x_1 + 4x_2 - x_1 + 1 = 0 \end{cases} \qquad \left(\boxed{\begin{matrix} 2 & -1 \\ 5 & 4 \end{matrix}} \begin{matrix} 3 \\ -1 \end{matrix} \middle| \begin{matrix} -1 \\ -1 \end{matrix}\right)$

$D = ?$ a.î. $D \parallel D'$ și $A(2,-5,3) \in D$

$\begin{cases} 2x_1 - x_2 = -3x_3 - 1 \quad | \cdot 4 \\ 5x_1 + 4x_2 = x_3 - 1 \end{cases}$

$\begin{cases} 8x_1 - 4x_2 = -12x_3 - 4 \\ 5x_1 + 4x_2 = x_3 - 1 \end{cases}$ (+)

$13x_1 = -11x_3 - 5$

$x_1 = \frac{-11x_3}{13} - \frac{5}{13}$

$x_2 = \frac{-22x_3}{13} - \frac{10}{13} + 3x_3 + 1 = \frac{17x_3}{13} + \frac{3}{13}$

$x_3 = t \Rightarrow D_1: \begin{cases} x_1 = \frac{11}{13}t - \frac{5}{13} \\ x_2 = \frac{17}{13}t + \frac{3}{13}, \ t \in \mathbb{R} \\ x_3 = t \end{cases}$

$U_1 = \left(-\frac{11}{13}, \frac{17}{13}, 1\right) = \frac{1}{13}(-11, 17, 13)$

Metoda 2:

$N_1 = (2, -1, 3)$

$N_2 = (5, 4, -1)$

$$N_1 \times N_2 = \begin{vmatrix} e_1 & e_2 & e_3 \\ 2 & -1 & 3 \\ 5 & 4 & -1 \end{vmatrix} = e_1 \cdot \begin{vmatrix} -1 & 3 \\ 4 & -1 \end{vmatrix} - e_2 \begin{vmatrix} 2 & 3 \\ 5 & -1 \end{vmatrix} + e_3 \begin{vmatrix} 2 & -1 \\ 5 & 4 \end{vmatrix} =$$

$$= -11 e_1 + 17 e_2 + 13 e_3 = (-11, 17, 13)$$

$$D: \frac{x_1 - 2}{-11} = \frac{x_2 + 5}{17} = \frac{x_3 - 3}{13}$$

$\mathcal{D} \parallel \mathcal{D}_1$

$V_{\mathcal{D}} = V_{\mathcal{D}_1} = \langle \{ N_1 \times N_2 \} \rangle$

$$\Rightarrow \begin{cases} x_1 = -11t + 2 \\ x_2 = 17t - 5 \\ x_3 = 13t + 3 \end{cases}, \ t \in \mathbb{R}$$

$\boxed{\text{EX 2}}$

$\pi: x_1 + x_2 + x_3 = 1$

$M(1, 2, -1)$

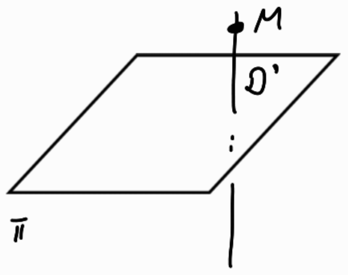
$$D: \frac{x_1 - 1}{2} = \frac{x_2 - 1}{-1} = \frac{x_3}{3}$$

a) $D' = ?$ a.î. $M \in D'$ și $D' \perp \pi$

b) -||- planul π' a.î. $M \in \pi'$ și $\pi' \perp D$

c) -||- planul π'' a.î. $M \in \pi''$ și $D \subset \pi''$

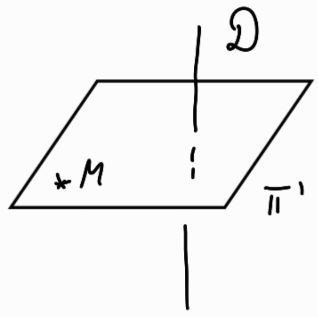
SOL:



a) $D' \perp \pi \Rightarrow N_{D'} = N_{\pi} = (1, 1, 1)$

$M(1, 2, -1) \in D'$

$$D': \frac{x_1 - 1}{1} = \frac{x_2 - 2}{1} = \frac{x_3 + 1}{1} = t \Rightarrow \begin{cases} x_1 = t + 1 \\ x_2 = t + 2 \\ x_3 = t - 1 \end{cases}$$

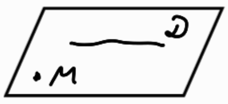
b)  $U_D = N_{\pi'} = (2, -1, 3)$

$\pi' : 2x_1 - x_2 + 3x_3 + d = 0$

$M(1, 2, -1) \in \pi' \Leftrightarrow$

$\Leftrightarrow 2-2-3+d=0 \Rightarrow d=3$

c) $D \subset \pi''$, $M \in \pi''$



$U_D = (2, -1, 3)$

$A(1, 1, 0) \in D$

$\vec{MA} = (0, -1, 1)$

$$\pi' : \begin{vmatrix} x_1 - 1 & 2 & 0 \\ x_2 - 2 & -1 & -1 \\ x_3 + 1 & 3 & 1 \end{vmatrix} = 0$$

sau $N_{\pi''} = U_D \times \vec{MA} = \begin{vmatrix} e_1 & e_2 & e_3 \\ 2 & -1 & 3 \\ 0 & -1 & 1 \end{vmatrix} = (2, -2, -2) = 2(1, -1, -1)$

d) $\pi' \perp D$, $M \in \pi'$, $\pi' : 2x_1 - x_2 + 3x_3 - 3 = 0$

$P = D \cap \pi'$

$$D : \begin{cases} x_1 = 2t+1 \\ x_2 = -t+1 \\ x_3 = 3t \end{cases} \Rightarrow$$

$2(2t+1) + t - 1 + 9t - 3 = 0$

$4t + 2 + 10t - 4 = 0 \Rightarrow 14t = 2 \Rightarrow t = \frac{2}{14} = \frac{1}{7}$

$P\left(\frac{9}{7}, \frac{6}{7}, \frac{3}{7}\right)$

e) $pr_{\pi} M = ?$

$D \perp \pi$, $M \in D' \Rightarrow Q = \pi \cap D'$

$\pi = x_1 + x_2 + x_3 - 1 = 0$

$\Leftrightarrow t + 1 + t + 2 + t - 1 - 1 = 0$

$3t + 1 = 0 \Rightarrow t = \frac{-1}{3}$

$Q\left(\frac{2}{3}, \frac{5}{3}, \frac{-4}{3}\right)$

$Q = \pi \cap D'$

(4) $D_1 : \frac{x_1 - 1}{1} = \frac{x_2 - 2}{-1} = \frac{x_3 + 2}{2}$

$D_2 : \begin{cases} 2x_1 - x_3 - 1 = 0 \\ 2x_2 + x_3 + 3 = 0 \end{cases}$

a) D_1, D_2 coplanare

b) ec. plan det de D_1, D_2

c) $d(D_1, D_2) = ?$

Rez

$N_1 = (2, 0, -1)$

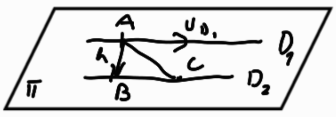
$N_2 = (0, 2, 1)$

$U_{D_2} = N_1 \times N_2 = \begin{vmatrix} e_1 & e_2 & e_3 \\ 2 & 0 & -1 \\ 0 & 2 & 1 \end{vmatrix} = 2e_1 - 2e_2 + 4e_3 =$

$= 2(1, -1, 2) = 2U_{D_1}$

$\Rightarrow D_1 \parallel D_2 \Rightarrow D_1, D_2$ coplanare

$U_{D_1} = (1, -1, 2)$



$\vec{AB} = (0, -4, 3)$

$N_\pi = U_{D_1} \times \vec{AB} = \begin{vmatrix} e_1 & e_2 & e_3 \\ 1 & -1 & 2 \\ 0 & -4 & 3 \end{vmatrix} =$

$A(1, 2, -2)$

$B(1, -2, 1)$

$= 5e_1 - 3e_2 - 4e_3 = (5, -3, -4)$

$\pi : 5x_1 - 3x_2 - 4x_3 + d = 0$

$A(1, 2, -2) \in \pi$

$5 - 6 + 8 + d = 0 \Rightarrow d = (-7)$

$\Pi: -5x_1 - 3x_2 - 4x_3 - 7 = 0$

$$A_{\triangle ABC} = \frac{1}{2} \|\vec{AB} \times \vec{AC}\|$$

$$= \frac{h \cdot \|\vec{BC}\|}{2} \Rightarrow h = \frac{\|\vec{AB} \times \vec{AC}\|}{\|\vec{BC}\|}$$

c) $C(0, -1, -1)$

$\vec{AC} = (-1, -3, 1)$

$\vec{BC} = (-1, 1, -2)$

$$\vec{AB} \times \vec{AC} = \begin{vmatrix} e_1 & e_2 & e_3 \\ 0 & -4 & 3 \\ -1 & -3 & 1 \end{vmatrix} = 5e_1 - 3e_2 - 4e_3$$

$$\|\vec{AB} \times \vec{AC}\| = \sqrt{25 + 9 + 16} = 5\sqrt{2}$$

$$\|\vec{BC}\| = \sqrt{1+1+4} = \sqrt{6}$$

$$h = \frac{5\sqrt{2}}{\sqrt{6}} = \frac{5}{\sqrt{3}} = \frac{5\sqrt{3}}{2}$$

$A(1,3,0)$, $B(3,-2,1)$, $C(\alpha, 1, -3)$, $D(7,-2,3)$

$\alpha = ?$ A, B, C, D coplanare

$$\Delta = \begin{vmatrix} 1 & 3 & 0 & 1 \\ 3 & -2 & 1 & 1 \\ \alpha & 1 & -3 & 1 \\ 7 & -2 & 3 & 1 \end{vmatrix} = 0 \Leftrightarrow \begin{vmatrix} 0 & 0 & 0 & 1 \\ 2 & -5 & 1 & 1 \\ \alpha-1 & -2 & -3 & 1 \\ 6 & -5 & 3 & 1 \end{vmatrix} = 0$$

$$\Leftrightarrow \begin{vmatrix} 2 & -5 & 1 \\ \alpha-1 & -2 & -3 \\ 6 & -5 & 3 \end{vmatrix} = 0 \Leftrightarrow \begin{vmatrix} 2 & -5 & 1 \\ \alpha+5 & -17 & 0 \\ 0 & 10 & 0 \end{vmatrix} = 0$$

$\Leftrightarrow -10(\alpha+5) = \alpha = (-5)$

OBS Tetraedru $ABCD$ Vol $= \frac{|\Delta|}{6}$

II Aducerea la forma canonică a conicelor cu centru unic ($\delta \neq 0$)

$\Gamma : f(x_1, x_2) = 5x_1^2 + 8x_1x_2 + 5x_2^2 - 18x_1 - 18x_2 + 9 = 0$

a) să se aducă la o formă canonică efectuând izometrii

b) rep. graf.

Rz:

$A = \begin{pmatrix} 5 & 4 \\ 4 & 5 \end{pmatrix}$ $B = (-9, -9)$, $c = 9$

$\tilde{A} = \begin{pmatrix} 5 & 4 & -9 \\ 4 & 5 & -9 \\ -9 & -9 & 9 \end{pmatrix}$ $\delta = \det(A) = 9 \neq 0 \Rightarrow$ centru unic (elipsă, hip), $\delta > 0 \Rightarrow$ elips

$\Delta = \det(\tilde{A}) = 1 \Rightarrow$ nedegenerată

$\begin{vmatrix} 5 & 4 & -9 \\ 4 & 5 & -9 \\ -9 & -9 & 9 \end{vmatrix} = \begin{vmatrix} 0 & 0 & -9 \\ 4 & 5 & -9 \\ -9 & -9 & 9 \end{vmatrix} = -9 \cdot 9 = (-81)$

$\mathcal{R} = \{O; e_1, e_2\} \xrightarrow{\theta} \mathcal{R}' = \{P_0; e_1, e_2\}$

$\theta : X = X' + X_0$

$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_1' \\ x_2' \end{pmatrix} + \begin{pmatrix} x_{1,0} \\ x_{2,0} \end{pmatrix}$

centru: $\frac{\partial f}{\partial x_1} = 0$, $\frac{\partial f}{\partial x_2} = 0$

$\Rightarrow 10x_1 + 8x_2 - 18 = 0$

$8x_1 + 10x_2 - 18 = 0 \quad \Big| \rightarrow 2x_1 - 2x_2 = 0 \Rightarrow x_1 = x_2 = 1$

$\theta(\Gamma): \; 5x_1'^2 + 8x_1'x_2' + 5x_2'^2 + \dfrac{\Delta}{\delta} = 0$ (teorie)

$\dfrac{\Delta}{\delta} = (-9)$

$Q: \mathbb{R}^2 \rightarrow \mathbb{R}$

$Q(x) = 5x_1'^2 + 8x_1'x_2' + 5x_2'^2$

Aducem la o formă canonică utilizând metoda val proprii

$P(\lambda) = \lambda^2 - 10\lambda + 9 = 0$

$(\lambda - 1)(\lambda - 9) = 0 \Rightarrow \lambda_1 = 1, \; \lambda_2 = 9$

$A \begin{pmatrix} 5 & 4 \\ 4 & 5 \end{pmatrix}$

$V_{\lambda_1} = \{x \in \mathbb{R}^2 \mid Ax = x\}$

$(A - I_2)X = 0_{2,1}$

$\begin{pmatrix} 4 & 4 \\ 4 & 4 \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$\Rightarrow x_2 = -x_1$

$\Rightarrow V_{\lambda_1} = \{(x_1, -x_1) \mid x_1 \in \mathbb{R}\} = \langle\{(1, -1)\}\rangle$

$e_1' = \dfrac{1}{\sqrt{2}}(1, -1)$

$V_{\lambda_2} = \{x \in \mathbb{R}^2 \mid Ax = 9x\} \Rightarrow (A - 9I_2)X = 0_{2,1}$

$\Rightarrow \begin{pmatrix} -4 & 4 \\ 4 & -4 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0 \Rightarrow x_2 = x_1 \Rightarrow V_{\lambda_2} = \{(x_1, x_1) \mid x_1 \in \mathbb{R}\}$

$= \langle\{(1,1)\}\rangle \Rightarrow e_2' = \dfrac{1}{\sqrt{2}}(1, 1)$

$$\mathcal{R}' \xrightarrow{\tau} \mathcal{R}'' = \{P_0;\ e_1',\ e_2'\}$$

$P_0(1,1)$

$$\tau : X' = R\,X''$$

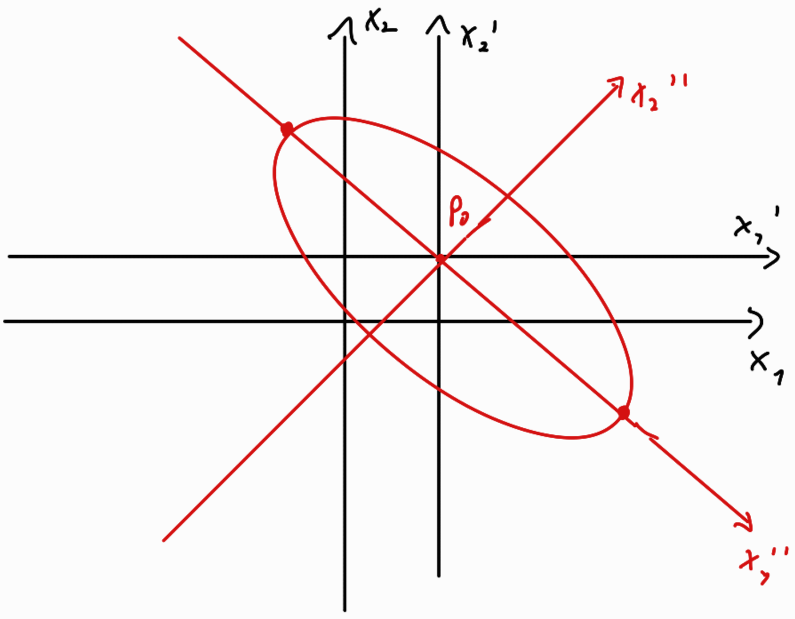
$$R = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \in SO(2)$$

$$Q(x) = 1 \cdot {x_1''}^2 + 9 \cdot {x_2''}^2$$

$$\tau(\theta(T)) : 1 \cdot {x_1''}^2 + 9{x_2''}^2 - 9 = 0$$

$$\Rightarrow \frac{{x_1''}^2}{9} + {x_2''}^2 = 1$$

$$\Rightarrow a = 3,\quad b = 1 \quad (\text{Elipsă})$$



INFO (Alegere sens)

Sensul lui x_2'' și x_1'' prin e_1', e_2'

e.g. $e_1' = \frac{1}{\sqrt{2}}(1,-1)$

$e_2' = \frac{1}{\sqrt{2}}(1,1)$

Punem $(1,-1)$ și $(1,1)$ și trasăm liniile de la P_0 la $(1,-1)$ și $(1,1)$