Seminar 1 - GAL

01.03.2024

1) Să se rezolve sistemul

$$\begin{cases} x-y+z=1 \\ y-z=2 \\ z=1 \end{cases} \Leftrightarrow \begin{cases} x-y=0 \\ y=3 \\ z=1 \end{cases} \Rightarrow \begin{cases} x=3 \\ y=3 \\ z=1 \end{cases}$$

A_i

2) $$\begin{cases} x-2y+2z=1 \\ 2x-y-z=2 \\ x+y+z=3 \end{cases} \xRightarrow{L_2=L_2-2L_1} \begin{cases} x-2y+2z=1 \\ 3y-5z=0 \\ 3y-z=2 \end{cases} \xRightarrow{L_2=\frac{L_2}{3}} \begin{cases} x-2y+2z=1 \\ y-\frac{5}{3}z=0 \\ 3y-z=2 \end{cases} \Rightarrow$$

A_f

$$\xRightarrow[L_3=L_3-3L_1]{L_1=L_1+2L_2} \begin{cases} x-\frac{4}{3}z=1 \\ y-\frac{5}{3}z=0 \\ 4z=2 \end{cases} \xRightarrow{L_3=\frac{L_3}{4}} \begin{cases} x-\frac{4}{3}z=1 \\ y-\frac{5}{3}z=0 \\ z=\frac{1}{2} \end{cases} \Rightarrow \begin{cases} x=\frac{5}{3} \\ y=\frac{5}{6} \\ z=\frac{1}{2} \end{cases}$$

$$A_i=\begin{pmatrix} 1 & -2 & 2 \\ 2 & -1 & -1 \\ 1 & 1 & 1 \end{pmatrix} \qquad A_f=\begin{pmatrix} 1 & 0 & -\frac{4}{3} \\ 0 & 1 & -\frac{5}{3} \\ 0 & 0 & 4 \end{pmatrix}$$

$\det A_i = 3 \det A_f = 12$

$$\left(\begin{array}{ccc|c} 1 & -2 & 2 & 1 \\ 2 & -1 & -1 & 2 \\ 1 & 1 & 1 & 3 \end{array}\right) \xrightarrow[L_3=L_3-L_1]{L_2=L_2-2L_1} \left(\begin{array}{ccc|c} 1 & -2 & 2 & 1 \\ 0 & 3 & -5 & 0 \\ 0 & 3 & -1 & 2 \end{array}\right) \rightarrow$$

$$\xrightarrow{L_2=\frac{L_2}{3}} \left(\begin{array}{ccc|c} 1 & -2 & 2 & 1 \\ 0 & 1 & -\frac{5}{3} & 0 \\ 0 & 3 & -1 & 2 \end{array}\right) \xrightarrow{L_3=L_3-3L_2} \left(\begin{array}{ccc|c} 1 & 0 & -\frac{4}{3} & 1 \\ 0 & 1 & -\frac{5}{3} & 0 \\ 0 & 0 & 4 & 2 \end{array}\right) \xrightarrow{L_3=\frac{L_3}{4}}$$

$$\left(\begin{array}{ccc|c} 1 & 0 & -\frac{1}{3} & 1 \\ 0 & 1 & -\frac{5}{3} & 0 \\ 0 & 0 & 1 & \frac{1}{2} \end{array}\right) \xrightarrow[L_2 = L_2 + \frac{5}{3}L_3]{L_1 = L_1 + \frac{1}{3}L_3} \left(\begin{array}{ccc|c} 1 & 0 & 0 & \frac{5}{3} \\ 0 & 1 & 0 & +\frac{5}{6} \\ 0 & 0 & 1 & \frac{1}{2} \end{array}\right)$$

3) $\begin{cases} x - y - z = 2 \\ 2x - 2y - z = 3 \\ 3x - 3y - 2z = 5 \end{cases}$

$$\left(\begin{array}{ccc|c} 1 & -1 & -1 & 2 \\ 2 & -2 & -1 & 3 \\ 3 & -3 & -2 & 5 \end{array}\right) \xrightarrow[L_3 = L_3 - 3L_1]{L_2 = L_2 - 2L_1} \left(\begin{array}{ccc|c} 1 & -1 & -1 & 2 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & -1 \end{array}\right) \xrightarrow{L_3 = L_3 - L_2} \left(\begin{array}{ccc|c} 1 & -1 & -1 & 2 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$$\xrightarrow{L_1 = L_1 + L_2} \left(\begin{array}{ccc|c} 1 & -1 & 0 & 1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$x = 1 + y$

$z = -1$

$$S_3 = \{(1+y, y, -1)^t \mid y \in \mathbb{R}\} = \left\{ \begin{pmatrix} 1+y \\ y \\ -1 \end{pmatrix} ; y \in \mathbb{R} \right\}$$

4) $\begin{cases} x - y - z = 0 \\ 2x - 2y - z = 0 \\ 3x - 3y - 2z = 0 \end{cases}$ Sistem omogen

$$\begin{pmatrix} 1 & -1 & -1 \\ 2 & -2 & -1 \\ 3 & -3 & -2 \end{pmatrix} \xrightarrow{\text{(ca mai sus)}} \cdots \longrightarrow \begin{pmatrix} \boxed{1} & -1 & \boxed{0} \\ \boxed{0} & 0 & \boxed{1} \end{pmatrix} \Rightarrow \begin{cases} x = y \\ z = 0 \end{cases}$$

$$S_4 = \left\{ \begin{pmatrix} y \\ y \\ \end{pmatrix} ; y \in \mathbb{R} \right\} = \left\{ \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} + \vec{x} ; \vec{x} \in S_3 \right\}$$

5) $\begin{cases} x+y+z+t=1 \\ x+y \quad\quad +t=1 \end{cases}$

$$\left(\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 \end{array}\right) \xrightarrow{L_2=L_2-L_1} \left(\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & -1 & 0 & 0 \end{array}\right) \xrightarrow[L_2=L_2\cdot(-1)]{L_1=L_1+L_2}$$

$$\rightarrow \left(\begin{array}{cccc|c} 1 & 1 & 0 & 1 & 1 \\ 0 & 0 & +1 & 0 & 0 \end{array}\right)$$

$$S_5 = \left\{ \begin{pmatrix} 1-y-t \\ y \\ 0 \\ t \end{pmatrix} ; (y,t) \in \mathbb{R}^2 \right\}$$

6) $\begin{cases} x+y+z+t=\hat{1} \\ x+y \quad\quad +t=\hat{1} \end{cases}$ în $\mathbb{Z}_2$

$$\left(\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 \end{array}\right) \xrightarrow{L_2=L_2+L_1} \left(\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 \end{array}\right) \xrightarrow{L_1=L_1+L_2} \left(\begin{array}{cccc|c} \hat{1} & \hat{1} & \hat{0} & \hat{1} & \hat{1} \\ \hat{0} & \hat{0} & \hat{1} & \hat{0} & \hat{0} \end{array}\right)$$

$$S_6 = \left\{ \begin{pmatrix} \hat{1}+y+t \\ y \\ \hat{0} \\ t \end{pmatrix} ; (y,t) \in \mathbb{Z}_2^2 \right\}$$

$$\text{card } S_6 = \text{card}(\mathbb{Z}_2 \times \mathbb{Z}_2) = 2^2 = 4$$

7) Acelaşi ca la (6) dar în $\mathbb{Z}_3$

$$S_7 = \left\{ \begin{pmatrix} 1+\hat{2}y+\hat{2}t \\ y \\ \hat{0} \\ t \end{pmatrix} ; (y,t) \in \mathbb{Z}_3^2 \right\} \quad \text{card}_{\mathbb{Z}_3} S_7 = 9 = 3^2$$

2. Fie sistemul $Ax = b$

$A \in \mathcal{M}_{m,n}(\mathbb{Z}_p)$

$x = \mathcal{M}_{n,1}(\mathbb{Z}_p)$

$b = \mathcal{M}_{m,1}(\mathbb{Z}_p)$

$S = \{x \in \mathcal{M}_{n,1}(\mathbb{Z}_p) \mid Ax = b\}$

$\boxed{|S| = p^{n - \operatorname{rang} A}}$

$A \in \mathcal{M}_n$ (inversabila) GL_n

$Ax = b \Rightarrow x = A^{-1} \cdot b$

Fie $A = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \\ a_{n1} & \cdots & a_{nn} \end{pmatrix} = \operatorname{col}(c_1, \dots, c_n)$

$c_1 = \begin{pmatrix} a_{11} \\ \vdots \\ a_{n1} \end{pmatrix}$

$c_n = \begin{pmatrix} a_{1n} \\ \vdots \\ a_{nn} \end{pmatrix}$

$A = (a_{ij}), \forall i,j = \overline{1,n}$

$B = \operatorname{col}(b_1, \dots, b_n)$

$A \cdot B = \operatorname{col}(Ab_1, \dots, A \cdot b_n)$

8) $A=\begin{pmatrix} 1 & 0 & 1 \\ 2 & -1 & 1 \\ 3 & 1 & 0 \end{pmatrix}$

$B=\begin{pmatrix} 2 & 2 & 2 \\ 0 & 1 & 1 \\ 3 & -2 & 3 \end{pmatrix}$

$A\cdot B=\begin{pmatrix} 1 & 0 & 1 \\ 2 & -1 & 1 \\ 3 & 1 & 0 \end{pmatrix}\begin{pmatrix} 2 & 2 & 2 \\ 0 & 1 & 1 \\ 3 & -2 & 3 \end{pmatrix} = \begin{pmatrix} 5 & 0 & 5 \\ 7 & 1 & 6 \\ 6 & 7 & 7 \end{pmatrix}$

$A\cdot B = I_n$

$col(A\,b_1, \dots, A\cdot b_n) = col\left(\begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{pmatrix}, \dots, \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix}\right)$

Aflarea lui B $\Leftrightarrow$ rezolvarea sistemelor $Ab_1=\begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$ $A\cdot b_2=\begin{pmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{pmatrix}, \dots, A\cdot b_n=\begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}$

9) $A=\begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}$, $A^{-1}=?$

$\left(\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right) \xrightarrow{L_2=L_2-L_1} \left(\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & -1 & 1 & -1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right) \longrightarrow$

$\xrightarrow[L_1=L_1+L_2]{L_3=L_3+L_2} \left(\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & -1 & 1 & -1 & 1 & 0 \\ 0 & 0 & 2 & -1 & 1 & 1 \end{array}\right) \xrightarrow{L_3=\frac{L_3}{2}} \left(\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 1 & -1 & 0 \\ 0 & 0 & 1 & -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{array}\right) \xrightarrow[L_2=L_2+L_3]{L_1=L_1-L_3}$

$\longrightarrow \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ 0 & 1 & 0 & \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 1 & -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{array}\right)$

I_3 — A^{-1}

10) $A \in \mathcal{M}_3(\mathbb{Z}_2)$

$$A = \begin{pmatrix} \hat{1} & \hat{1} & \hat{0} \\ \hat{1} & \hat{0} & \hat{1} \\ \hat{0} & \hat{1} & \hat{1} \end{pmatrix}$$

11) $\det A = ?$

$$A = \begin{pmatrix} 1 & 1 & 0 \\ 2 & -1 & 1 \\ 1 & -1 & 3 \end{pmatrix} \xrightarrow[L_3 = L_3 - L_1]{L_2 = L_2 - 2L_1} \begin{pmatrix} 1 & 1 & 0 \\ 0 & -3 & 1 \\ 0 & -2 & 3 \end{pmatrix} \xrightarrow[L_3 = L_3 - \frac{2}{3}L_2]{} \begin{pmatrix} 1 & 1 & 0 \\ 0 & -3 & 1 \\ 0 & 0 & \frac{7}{3} \end{pmatrix} =$$

$$\det A = \begin{vmatrix} 1 & 1 & 0 \\ 0 & -3 & 1 \\ 0 & 0 & \frac{7}{3} \end{vmatrix} = 1 \cdot (-3) \cdot \frac{7}{3} = -7$$

12) Fie

$$A = \begin{pmatrix} 1 & 1 & 3 & 2 & 1 & 5 \\ 2 & -1 & 4 & -3 & 2 & -3 \\ 3 & 0 & 7 & -1 & 3 & 2 \\ 1 & -2 & 1 & -5 & 1 & -8 \\ 2 & 2 & 6 & 4 & 2 & 10 \end{pmatrix} \xrightarrow[\substack{L_3 = L_3 - 3L_1 \\ L_4 = L_4 - L_1 \\ L_5 = L_5 - 2L_1}]{L_2 = L_2 - 2L_1} \begin{pmatrix} 1 & 1 & 3 & 2 & 1 & 5 \\ 0 & -3 & -2 & -7 & 0 & -13 \\ 0 & -3 & -2 & -7 & 0 & -13 \\ 0 & -3 & -2 & -7 & 0 & -13 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \longrightarrow$$

$$\longrightarrow \begin{pmatrix} 1 & 1 & 3 & 2 & 1 & 5 \\ 0 & -3 & -2 & -7 & 0 & -13 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Temă: Scrieți un program care calculează forma eșalon și determinantul unei matrice pătratice

pt bonus

Seminar 2 - GAL - Halunga

1) Care dintre următoarele ~~sunt~~ sunt K - sp. vectoriale?

a) K^n / K

b) $K[x] / K$

c) $\{P(x) \in K[x] \mid \text{grad } P(x) = n\} / K$

d) $\{P(x) \in K[x] \mid \text{grad } P(x) \leq n\} / K$

e) $\mathbb{R}[x] / \mathbb{C}$

f) $\mathbb{C}[x] / \mathbb{R}$

V sp. vectorial peste K dacă

$(V, +)$ grup comutativ

$K \times V \to V \quad a, b \in K, \ x, y \in V$

$a \cdot x \in V$

$(a+b)x = ax + bx$

$a \cdot (x+y) = ax + ay$

$1 \cdot x = x$

$(ab)x = a \cdot (bx)$

a) $K^n = \{(x_1, \dots, x_n)^t \mid x_1, \dots, x_n \in K\}$

$x, y \in K^n$

$x + y = \begin{pmatrix} x_1 + y_1 \\ x_2 + y_2 \\ \vdots \\ x_n + y_n \end{pmatrix}$

$a \in K$

$a \cdot \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} a x_1 \\ a x_2 \\ \vdots \\ a x_n \end{pmatrix}$

$(K^n, +)$ parte stabilă

$(K^n, \cdot)$ „parte stabilă"

$$x+y = \begin{pmatrix} x_1+y_1 \\ \vdots \\ x_n+y_n \end{pmatrix} = \begin{pmatrix} y_1+x_1 \\ \vdots \\ y_n+x_n \end{pmatrix} = y+x$$

Analog asoc, elem neutru $0 = (0, \dots, 0)$

$-x = (-x_1, \dots, -x_n)$ invers

$$(a+b)\cdot x = (a+b)\cdot(x_1, \dots x_n)^t = ((a+b)x_1, \dots, (a+b)x_n)^t = (ax_1+bx_1, \dots, ax_n+bx_n)^t$$

$= ax+bx$

Verificați restul axiomelor

SG

$$x = (x_1, \dots, x_n)^t = x_1(1,0,\dots 0)^t + x_2(0,1,\dots 0)^t + \dots + x_n(0, \dots 0, 1)^t$$

$$K^n = \langle \{(1,0,\dots 0)^t, \dots, (0,0,\dots 1)^t\} \rangle$$

5) $P(x) = a_0 + a_1 x + \dots + a_n x^n \to (a_0, \dots, a_n)$

SG pt $K[x] = \{1, x, \dots, x^n, x^{n+1}, \dots\}$

$K[x] = \langle \{x^n \mid n \in \mathbb{N}\} \rangle$ nu e finit generat

c) Nu este pt că $0 \notin V$

$\overline{V} = V \cup \{0\}$ este sp vect?

$P(x), Q(x) \in V$ grad $(P(x)+Q(x)) \leq n$

Ex $P(x) = x^n + x + 1$

$Q(x) = -x^n$ $\quad P(x) + Q(x) = x+1$

Grad $= 1$ $\quad \Rightarrow (\overline{V}, +)$ nu e parte stabilă

$\overline{V}$ s.v. con

d) Este sp. vect.

$P(x), Q(x) \in K_n[x] \Rightarrow grad(P(x)+Q(x)) \leq n \Rightarrow P(x)+Q(x) \in K_n[x]$

$K_n[x] = <\{1, x, \dots, x^n\}>$

e) $(R[x], +)$ grup abelian Nu e sp. vect.

$P(x) \in R[x], \; i \cdot P(x) \notin R[x]$

nu e parte stabilă la înmulțire

f) $\mathbb{C}[x]$ este R - sp. vect.

$\mathbb{C}[x] = <\{1, x, x^2, \dots, x^n\}>$ peste $\mathbb{C}$

SG al lui $\mathbb{C}[x]/_R$ $\{1, i, x, ix, \dots, x^n, ix^n, \dots\}$

$L \subset K$ corpuri

K este L - sp. vect.

2) Care din următoarele sunt subspații vectoriale

a) $A = (1, 3, 2)$ $V = \{X \in R^3 \mid A \cdot X = (0, 0, \dots, 0)^t\} \subset R^3$

b) ——"—— $V = \{X \in R^3 \mid A \cdot X = (1, \dots, 1)^t\} \subset R^3$

c) ——"—— $V_1 = \{P(x) \in K[x] \mid P(-x) = P(x)\}$

$V_2 = \{P(x) \in K[x] \mid P(-x) = -P(x)\}$

d) $V_1 = \{A \in \mathcal{M}_n(K) \mid A^t = A\}$

$V_2 = \{A \in \mathcal{M}_n(K) \mid A^t = -A\}$

$$\boxed{\text{Teorie: } W \subset V \text{ subsp.} \Leftrightarrow \forall \alpha, \beta \in K, \forall x, y \in W \Rightarrow \alpha x + \beta y \in W}$$

CamScanner

a) Fie $x, y \in V$, $\alpha, \beta \in \mathbb{R}$

$A(\alpha x + \beta y) = A \cdot \alpha \cdot x + A(\beta y) = \alpha \cdot Ax + \beta \cdot Ay = 0 \Rightarrow$

$\Rightarrow \alpha x + \beta y \in V$

SG in V?

$x \in V \Leftrightarrow (1,3,2)\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0 \Leftrightarrow x_1 + 3x_2 + 2x_3 = 0 \Leftrightarrow x_1 = -3x_2 - 2x_3$

$V = \left\{ \begin{pmatrix} -3x_2 - 2x_3 \\ x_2 \\ x_3 \end{pmatrix} \middle| x_1, x_2, x_3 \in \mathbb{R} \right\} \subset \mathbb{R}^3$

$x = (-3x_2 - 2x_3, x_2, x_3) = (-3x_2, x_2, 0)^t + (-2x_3, 0, x_3)^t = x_2(-3, 1, 0)^t + x_3(-2, 0, 1)^t$

$V = \langle \{ (-3, 1, 0)^t ; (-2, 0, 1)^t \} \rangle$

b) $(0,0,0)^t \notin V \Rightarrow V$ nu e subsp vect.

c) $P(x), Q(x) \in V_1$

$(\alpha P + \beta Q)(-x) = \alpha P(-x) + \beta Q(-x) = \alpha P(x) + \beta Q(x) = (\alpha P + \beta Q)(x)$

$\alpha P(x) + \beta Q(x) \in V_1$

V_1 subsp.

$P(x), Q(x) \in V_1$

$(\alpha P + \beta Q)(-x) = \alpha P(-x) + \beta Q(-x) = -\alpha P(x) - \beta Q(x) =$

$= -(\alpha P + \beta Q)(x) \Rightarrow -(\alpha P + \beta Q)(x) \in V_2 \Rightarrow V_2$ subsp.

1) $\cdot V_1 \cap V_2 = \{0\}$

$P(x) = P(-x) = -P(x) \Rightarrow P(x) = 0$

2) $\cdot V_1 \oplus V_2 = K[x]$

$\forall P(x) \in K[x]\ \exists P_1(x) \in V_1, P_2(x) \in V$ a.î.

$P(x) = P_1(x) + P_2(x)$

$P(-x) = P_1(x) - P_2(x)$ (+)

$P(x) + P(-x) = 2P_1(x)$

$P_1(x) = \frac{P(x) + P(-x)}{2} \Rightarrow P_2(x) = \frac{P(x) - P(-x)}{2}$

S6

$V_1 = \langle \{1, x^2, x^4, \dots x^{2n} \dots\} \rangle$

$V_2 = \langle \{x, x^3, x^5, \dots, x^{2n+1} \dots\} \rangle$

2) $V_1 = \{A \in M_n(K) \mid A = A^t\}$, $V_2 = \{A \in M_n(K) \mid A = -A^t\}$

Fie $A, B \in V_1$, $\alpha, \beta \in K$

$(\alpha A + \beta B)^t = \alpha A^t + \beta \cdot B^t = \alpha A + \beta B \Rightarrow \alpha A + \beta B \in V_1$

Fie $A, B \in V_2$, $\alpha, \beta \in K$

$(\alpha A + \beta B)^t = -\alpha A - \beta B = -(\alpha A + \beta B) \Rightarrow \alpha A + \beta B \in V_2$

$V_1 \oplus V_2 = M_n(K)$

$V_1 \cap V_2 = \{0\}$

$A \in V_1 \cap V_2 \Rightarrow A = A^t = -A \Rightarrow A = 0$

$A = A_1 + A_2$ $\qquad A_1 = \frac{1}{2}(A + A^t) \in V_1$

$A^t = A_1^t + A_2^t = A_1 - A_2$ $\qquad A_2 = \frac{1}{2}(A - A^t) \in V_2$

SG pt V_1 şi V_2

$V_1 =$ $n = 3$

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{12} & a_{22} & a_{23} \\ a_{13} & a_{23} & a_{33} \end{pmatrix}$$

$V_1 = \langle \{ E_{ij} \mid 1 \le i \le j \le n \} \rangle$

$E_{ij} = (a_{uv})_{\overline{u,v} = 1,n}$ $\qquad a_{uv} = \begin{cases} 1 & u = i \ v = j \text{ sau} \\ & u = j \ v = i \\ 0 & \text{rest} \end{cases}$

$|\{ E_{ij} \mid 1 \le i \le j \le n \}| = 1 + 2 + \dots + n = \frac{n(n+1)}{2}$

V_2, $n = 3$

$$A = \begin{pmatrix} 0 & a_{12} & a_{13} \\ -a_{12} & 0 & a_{23} \\ -a_{13} & -a_{23} & 0 \end{pmatrix}$$

$F_{ij} = (b_{uv})_{\overline{u,v} = 1,n}$ $\qquad b_{uv} = \begin{cases} 1 & u = i, v = j \\ -1 & u = j, v = i \\ 0 & \text{rest} \end{cases}$

$V_2 = \langle \{ F_{ij} \mid 1 \le i < j \le n \} \rangle$

Seminar 3 - GAL

$B = \{e_1, \dots, e_k\} \subset V$, B bază

~~BSG~~

B este SG și B este SLI

$a_1 e_1 + a_2 e_2 + \dots + a_k e_k = 0 \Rightarrow a_1 = \dots = a_k = 0$

$x \in V$, $x = x_1 e_1 + \dots + x_k e_k$ $(x_1, \dots, x_k)$ unice

Ex 1. Fie $L = \langle (1,1,1,1), (2,3,1,0), (1,2,0,-1) \rangle$

a) Arătați că $(4,6,2,0) \in L$

b) Det o bază în L

a) $\left(\begin{array}{ccc|c} 1 & 2 & 1 & 4 \\ 1 & 3 & 2 & 6 \\ 1 & 1 & 0 & 2 \\ 1 & 0 & -1 & 0 \end{array}\right) \xrightarrow[L_4 = L_4 - L_1]{L_2 = L_2 - L_1,\ L_3 = L_3 - L_1} \left(\begin{array}{ccc|c} 1 & 2 & 1 & 4 \\ 0 & 1 & 1 & 2 \\ 0 & -1 & -1 & -2 \\ 0 & -2 & -2 & -4 \end{array}\right) \longrightarrow \begin{array}{cccc} x_1 & x_2 & x_3 & \\ \end{array} \left(\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$

$$\begin{cases} x_1 = x_3 \\ x_2 = 2 - x_3 \end{cases}$$

$(4,6,2,0) = x_3(1,1,1,1) + (2 - x_3)(2,3,1,0) + x_3(1,2,0,-1)\ \forall x_3 \in \mathbb{R} \Rightarrow$

$\Rightarrow \langle (1,1,1,1), (2,3,1,0), (1,2,0,-1) \rangle$ este SG, dar nu bază

b) $\{(1,1,1,1), (2,3,1,0)\}$ este SLI $\begin{array}{cc} x_1 & x_2 \end{array} \begin{pmatrix} 1 & 2 \\ 1 & 3 \\ 1 & 1 \\ 1 & 0 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} \Rightarrow$ SLI

$a \cdot (1,1,1,1) + b(2,3,1,0) = (0,0,0,0) \Rightarrow a = b = 0 \Rightarrow \dim L = 2$

2) Fie $B=\{(1,1,2),(2,1,3),(1,2,1)\}$

a) Arătați că B este bază în $\mathbb{R}^3$

b) Det coordonatele lui $(3,3,1)$ în raport cu B

$$B=\begin{pmatrix}1&2&1\\1&1&2\\2&3&1\end{pmatrix}\xrightarrow[L_3=L_3-2L_1]{L_2=L_2-L_1}\begin{pmatrix}1&2&1\\0&-1&1\\0&-1&-1\end{pmatrix}\xrightarrow{L_1=L_1+2L_2}\begin{pmatrix}1&0&3\\0&-1&-1\\0&0&-1\end{pmatrix}\longrightarrow\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}\Rightarrow B \text{ este bază}$$

b) ca la a) în plus $(3,3,1)$

$$\left(\begin{array}{ccc|c}1&2&1&3\\1&1&2&3\\2&3&1&1\end{array}\right)\longrightarrow\left(\begin{array}{ccc|c}1&2&1&3\\0&-1&1&0\\0&-1&-1&-5\end{array}\right)\longrightarrow\left(\begin{array}{ccc|c}1&0&3&3\\0&-1&1&0\\0&0&-2&-5\end{array}\right)\longrightarrow\left(\begin{array}{ccc|c}1&0&3&3\\0&1&-1&0\\0&0&1&\frac{5}{2}\end{array}\right)\longrightarrow$$

$$\longrightarrow\left(\begin{array}{ccc|c}1&0&0&-\frac{9}{2}\\0&1&0&\frac{5}{2}\\0&0&1&\frac{5}{2}\end{array}\right)\Rightarrow\begin{cases}x_1=-\frac{9}{2}\\x_2=\frac{5}{2}\\x_3=\frac{5}{2}\end{cases}$$

$$\left(-\frac{9}{2}\right)\cdot(1,1,2)+\frac{5}{2}(2,1,3)+\frac{5}{2}(1,2,1)=(3,3,1)\Rightarrow B \text{ bază}$$

3

3) Fie $A=\begin{pmatrix}1&2&1&3&4\\1&4&3&5&9\\3&3&0&6&6\end{pmatrix}$

$c_1,\dots,c_5$ coloanele lui A

$l_1,\dots,l_3$ liniile lui A

$V_1=\langle c_1,\dots,c_5\rangle\subset\mathbb{R}^3$

$V_2=\langle l_1,l_2,l_3\rangle\subset\mathbb{R}^5$

Calculați $\dim V_1$ și $\dim V_2$

pt col

$$A=\begin{pmatrix}1&2&1&3&4\\0&2&2&2&5\\0&-3&-3&-3&-6\end{pmatrix}\longrightarrow\begin{pmatrix}1&0&0&0&0\\0&1&0&0&0\\0&0&0&0&0\end{pmatrix}\Rightarrow\{c_1,c_2\} \text{ bază în } V_1\Rightarrow\dim V_1=2$$

pt linii

$$A^t=\begin{pmatrix}1&1&3\\2&4&3\\1&3&0\\3&5&6\end{pmatrix}\longrightarrow\begin{pmatrix}1&0&0\\0&1&0\\0&0&0\\0&0&0\\0&0&0\end{pmatrix}\Rightarrow\{l_1,l_2\} \text{ bază în } V_2\Rightarrow\dim V_2=2$$

14

$A \in M_{mn}(K)$ cu coloanele $\{c_1, \dots, c_n\}$ și liniile $\{l_1, \dots, l_m\}$, atunci

$$\dim <\{c_1, \dots, c_n\}> = \dim <\{l_1, \dots, l_m\}> = \text{rang } A$$

$$\cap \qquad\qquad \cap$$
$$K^m \qquad\qquad K^n$$

4) Fie $V = \{x \in \mathbb{R}^5 \mid A \cdot x = 0\}$

$$A = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 3 & 2 & 1 & 1 & -3 \\ 0 & 1 & 2 & 2 & 6 \\ 5 & 4 & 3 & 3 & -1 \end{pmatrix}$$

Să se calculeze dim din V

$$A = \begin{array}{l} \\ L_2 = L_2 - 3L_1 \\ \\ L_4 = L_4 - 5L_1 \end{array} \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & -1 & -2 & -2 & -6 \\ 0 & 1 & 2 & 2 & 6 \\ 0 & -1 & -2 & -2 & -6 \end{pmatrix} \xrightarrow[L_3 = L_3 + L_2]{} \begin{pmatrix} \cancel{1 \; 1 \; 1 \; 1 \; 1} \\ \\ \\ \\ \end{pmatrix} \begin{pmatrix} 1 & 0 & -1 & -1 & -5 \\ 0 & 1 & 2 & 2 & 6 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

x_1, x_2 principale

$x_1 = x_3 + x_4 + 3x_5$

$x_2 = -2x_3 - 2x_4 - 6x_5$

$$x \in V \Rightarrow x = (x_3 + x_4 + 5x_5, -2x_3 - 2x_4 - 6x_5, x_3, x_4, x_5) =$$

$$= (x_3, -2x_3, x_3, 0, 0) + (x_4, -2x_4, 0, x_4, 0) + (5x_5, -6x_5, 0, 0, x_5) =$$

$$= x_3(1, -2, 1, 0, 0) + x_4(1, -2, 0, 1, 0) + x_5(5, -6, 0, 0, 1)$$

$$<(1, -2, 1, 0, 0)^t, (1, -2, 0, 1, 0)^t; (5, -6, 0, 0, 1)^t> \text{ SG}$$

$$B = \begin{pmatrix} 1 & 1 & 5 \\ -2 & -2 & -6 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{Rang } B = 3 \Rightarrow \{(1, -2, 1, 0, 0)^t, (1, -2, 0, 1, 0)^t, (5, -6, 0, 0, 1)^t\}$$

bază

nr necunoscute − rang A

$\dim V = 3 = 5 - \text{rang } A = \cancel{5-2}$

baza canonică

Notăm $(x_3, x_4, x_5) = (1,0,0) \Rightarrow (1,-2,1,0,0)$

$(x_3, x_4, x_5) = (0,1,0) \Rightarrow (1,-2,0,1,0)$

$(x_3, x_4, x_5) = (0,0,1) \Rightarrow (5,-6,0,0,1)$

Teorie

$(a_{11}, \dots, a_{n1})^t, \dots, (a_{1k}, \dots, a_{nk})^t$

$A = \begin{pmatrix} a_{11} & \dots & a_{1k} \\ a_{n1} & \dots & a_{nk} \end{pmatrix}$ $\{c_1, \dots, c_k\}$ SG în $K^n \Leftrightarrow \operatorname{rang} A = n$

$\{c_1, \dots, c_k\}$ SG în $K^n \Leftrightarrow \operatorname{rang} A = n \Rightarrow (k \geq n)$

$\{c_1, \dots, c_k\}$ SLI $\Leftrightarrow \operatorname{rang} A = k \Rightarrow (k \leq n)$

V este K-sp. vect. $\dim V = n$ $\{e_1, \dots, e_n\}$ bază în V

$f : V \to K^n$ $f(x_1 e_1 + \dots + x_n e_n) = (x_1, \dots, x_n)$

f bine-definită $\Rightarrow x_1 e_1 + \dots + x_n e_n$ unic $\Rightarrow f$ inj.

(unicitate)

f surjectivă $\operatorname{Im} f = K^n$ $(x_1, \dots, x_n) \in K^n$

Deci f e bijectivă

Deci $V \simeq K^n$ $V = K e_1 \oplus \dots \oplus K e_n$

5) Dacă K este un corp finit $\Rightarrow |K| = p^{\alpha}$ cu p prim

$M = \{ n \in \mathbb{N}^* ; n \cdot 1 = 0 \}$

$M = \emptyset \Leftrightarrow$ „caracteristica" $K = 0$

$M \neq \emptyset \Leftrightarrow$ „caracteristica" $K = \min M$

~~Pp că $K > 0$~~

Pp că „caracteristica" $K > 0 \Rightarrow$ „caracteristica" K este un nr prim

$p =$ „caracteristica" lui K

$p = a \cdot b ;\ a, b > 1 \quad a, b < p$

$(p \cdot 1) = 0 \Rightarrow (a \cdot b) \cdot 1 = 0 \Rightarrow (a \cdot 1)(b \cdot 1) = 0$

$a \cdot 1 = 0$

sau

$b \cdot 1 = 0$

Contradicție cu minimalitatea lui p

$\forall x \in K \quad px = 0$

$K_p = \{0, 1, 1+1, \dots, (p-1) \cdot 1\}$

$(K_p, +, \cdot)$, $a, b < p$

$(a \cdot 1) + (b \cdot 1) = (a+b) \cdot 1 = (cp + r) \cdot 1 = r \cdot 1 \in K_p$ unde $0 \leq r \leq p$ (parte stabilă)

$a \cdot 1 \in K_p \Rightarrow -(a \cdot 1) = (p-a) \cdot 1 \in K_p$ (invers)

$\Rightarrow (K_p, +)$ subgrup

$(a \cdot 1)(b \cdot 1) = (a b) \cdot 1 = (c' p + r') \cdot 1 = r' \cdot 1$ cu $0 \leq r' < p$

$a \neq 0 \Rightarrow (a \cdot 1)^{-1} \in K_p$??

Aplicând Th. Euclid

$\exists x, y$ a.î. $ax + py = 1 \Rightarrow ax = 1 - py \Rightarrow ax = py - 1$

$x < 0$

$x > 0$, $a < p$

$\shortparallel$

b

$(x\,a)\,1 = (1 - py) \cdot 1$ $x > 0$, $x \leq p$

$(K_p, +, \cdot)$ subgrup a.î. $K_p \cong \mathbb{Z}_p$ (izomorf)

K este un K_p sp. vectorial $\Rightarrow n = \dim_{K_p} K$

$|K| = |K_p|^n = p^n$

Seminar 4 - GAL

$|K| = p^n \quad \forall K$ corp finit

$p=2 \quad n=2$

$\mathbb{C} \simeq \frac{\mathbb{R}[x]}{x^2+1} = \mathbb{R}[i], \; i^2+1=0$

$P(x) \in \mathbb{F}_2[x] \; (= \mathbb{Z}_2[x])$ grad $P(x)=2$

$P(x)$ ireductibil

$P(x) = \{x^2, x^2+1, x^2+x, x^2+x+1\}$

$x^2+1 = (x+1)^2$ (in $\mathbb{Z}_2[x]$)

$\hat{1}$ rd

$\mathbb{F}_4 \simeq \frac{\mathbb{F}_2[x]}{x^2+x+1} = \mathbb{F}_2[\alpha] \quad \alpha^2+\alpha+1=0$

$+$	0	1	α	$\alpha+1$
0	0	1	α	$\alpha+1$
1	1	0	$\alpha+1$	α
α	α	$\alpha+1$	0	1
$\alpha+1$	$\alpha+1$	α	1	0

$\cdot$	0	1	α	$\alpha+1$
0	0	0	0	0
1	0	1	α	$\alpha+1$
α	0	α	$\alpha+1$	1
$\alpha+1$	0	$\alpha+1$	1	α

$p=2, \; n=3 \qquad P(x) = x^3+x^2+1$

$\mathbb{F}_8 = \frac{\mathbb{F}_2[x]}{P(x)}$

Temă 1) Arătați că

$\mathbb{F}_2[x]/(x^3+x^2+1) \simeq \mathbb{F}_2[x]/(x^3+x+1)$

2) $K=\mathbb{Q}$ At. arătați că $\forall P(x) \in \mathbb{Q}[x]$ grad $P(x)=n$, $V = \frac{\mathbb{Q}[x]}{(P(x))}$ sp. vect. și

$\dim_{\mathbb{Q}} V = n$

V sp. vect. $B_1=\{e_1,\dots,e_n\}$, $B_2=\{f_1,\dots,f_n\}$ baze

Matricea de trecere de la B_1 la B_2

$$A=(a_{ij})_{i,j=\overline{1,n}}$$

$$e_1=a_{11}f_1+\dots+a_{n1}f_n$$

$$e_k=a_{1k}f_1+\dots+a_{nk}f_n \quad \forall k\in\overline{1,n}$$

$$x=x_1e_1+\dots+x_ne_n=\sum_{k=1}^{n}x_k\cdot e_k=\sum_{k=1}^{n}x_k\left(\sum_{j=1}^{n}a_{jk}f_j\right)=$$

$$=(a_{11}x_1+a_{12}x_2+\dots+a_{1n}x_n)\cdot f_1+\dots+(a_{n1}x_1+a_{n2}x_2+\dots+a_{nn}x_n)f_n$$

$$\begin{pmatrix}y_1\\ \vdots\\ y_n\end{pmatrix}=A\begin{pmatrix}x_1\\ \vdots\\ x_n\end{pmatrix}$$

Ex1 a) Fie $B=\{(1,1,1)^t,(1,2,2)^t,(2,1,3)^t\}$, B_0 baza canonică în $\mathbb{R}^3$

Det mat de trecere de la B la B_0 și de la B_0 la B

$$B\xrightarrow{L}B_0 \qquad B_0\xrightarrow{M}B$$

$$L=\begin{pmatrix}1&1&2\\1&2&1\\1&2&3\end{pmatrix}$$

$$M=L^{-1}=\left(\begin{array}{ccc|ccc}1&1&2&1&0&0\\1&2&1&0&1&0\\1&2&3&0&0&1\end{array}\right)\xrightarrow[L_3=L_3-L_1]{L_2=L_2-L_1}\left(\begin{array}{ccc|ccc}1&1&2&1&0&0\\0&1&-1&-1&1&0\\0&1&1&-1&0&1\end{array}\right)\to$$

$$\xrightarrow[L_3=L_1-L_3]{L_1=L_1+L_2}\left(\begin{array}{ccc|ccc}1&0&3&2&-1&0\\0&1&-1&-1&1&0\\0&0&2&0&-1&1\end{array}\right)\to\left(\begin{array}{ccc|ccc}1&0&3&2&-1&0\\0&1&-1&-1&1&0\\0&0&1&0&-\frac{1}{2}&\frac{1}{2}\end{array}\right)\xrightarrow[L_2=L_2+L_3]{L_1=L_1-3L_1}$$

$$\to\left(\begin{array}{ccc|ccc}1&0&0&2&\frac{1}{2}&-\frac{3}{2}\\0&1&0&-1&\frac{1}{2}&\frac{1}{3}\\0&0&1&0&-\frac{1}{2}&\frac{1}{2}\end{array}\right)\Rightarrow M=\begin{pmatrix}2&\frac{1}{2}&-\frac{3}{2}\\-1&\frac{1}{2}&\frac{1}{2}\\0&-\frac{1}{2}&\frac{1}{2}\end{pmatrix}$$

$(1,0,0) = 2\cdot(1,1,1) - (1,2,2)$ ✓ (P.S. Sunt alese pe coloane!!!)

$(0,1,0) = \frac{1}{2}(1,1,1) + \frac{1}{2}(1,2,2) - \frac{1}{2}(2,1,3)$ ✓

$(0,0,1) = -\frac{3}{2}(1,1,1) + \frac{1}{2}(1,2,2) + \frac{1}{2}(2,1,3)$ ✓

Să se det. coord. lui $x = (1,1,0)^T$ în raport cu B

$$\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = M\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 & \frac{1}{2} & -\frac{3}{2} \\ -1 & \frac{1}{2} & \frac{1}{2} \\ 0 & -\frac{1}{2} & \frac{1}{2} \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} \frac{3}{2} \\ -\frac{1}{2} \\ -\frac{1}{2} \end{pmatrix}$$

Ex2 $B = \{-2, x+1, x^2-x-2\} \subset \mathbb{R}_2[x]$

Det matricea de trecere de la Baza canonică la B și coordonatele lui $P(x) = 4x^2+5x+3$ în B

$\{1, x, x^2\}$

$$L = \left(\begin{array}{ccc|ccc} -2 & 1 & -2 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right) \xrightarrow{L_1 - L_2} \left(\begin{array}{ccc|ccc} -2 & 0 & -1 & 1 & -1 & 0 \\ 0 & 1 & -1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right) \longrightarrow$$

$$\longrightarrow \left(\begin{array}{ccc|ccc} -2 & 0 & 0 & 1 & -1 & 1 \\ 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right)$$

$$M = \begin{pmatrix} -\frac{1}{2} & +\frac{1}{2} & -\frac{1}{2} \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

Coordonatele $= \begin{pmatrix} -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 3 \\ 5 \\ 4 \end{pmatrix} = \begin{pmatrix} -1 \\ 9 \\ 4 \end{pmatrix}$ $4x^2+5x+3 = -(-2) + 9(x+2) + 4(x^2-x-2)$

V, W sp. vect   Teorie

$f: V \to W$ ap. liniară

$f(ax+by) = a \cdot f(x) + b \cdot f(y)$

Ex 3

$f: \mathbb{R}^3 \to \mathbb{R}^2$

$f(x_1, x_2, x_3) = (x_1 - x_2, x_2 - x_3)$

$B_1 = \{(1,1,1); (1,2,2); (2,1,3)\} \subset \mathbb{R}^3$

$B_2 = \{(1,3), (2,1)\} \subset \mathbb{R}^2$

Să se det matricea ~~...~~ $M_f^{B_1, B_2}$

$$f(x_1, x_2, x_3) = \underbrace{\begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \end{pmatrix}}_{A} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

Teorie

V, W sp vect

$\{e_1, \dots, e_n\}$ bază în V

$\{f_1, \dots, f_n\}$ bază în W

$f: V \to W$

$M_f = (a_{ij})$

$f(e_1) = a_{11} f_1 + \dots + a_{n1} f_n$

$f(e_k) = a_{k1} f_1 + \dots + a_{kn} f_n$

Pas 1 ~~f(1,1)~~

$f(1,1,1)=\begin{pmatrix}0\\0\end{pmatrix}$

$f(1,2,2)=\begin{pmatrix}-1\\0\end{pmatrix}$

$f(2,1,3)=\begin{pmatrix}1\\-2\end{pmatrix}$

$$M_f^{B_1,\text{canonică}}=\begin{pmatrix}0&-1&1\\0&0&-2\end{pmatrix}$$

Pas 2

ţa nu-l uităm

$$\left(\begin{array}{cc|ccc}1&2&0&-1&1\\3&1&0&0&-2\end{array}\right)\to\left(\begin{array}{cc|ccc}1&2&0&-1&1\\0&-5&0&5&-5\end{array}\right)\to\left(\begin{array}{cc|ccc}1&2&0&-1&1\\0&1&0&-\frac{3}{5}&1\end{array}\right)\to$$

$$L_1=\begin{pmatrix}1&1&2\\1&2&1\\1&2&3\end{pmatrix}\qquad L_2=\begin{pmatrix}1&2\\3&1\end{pmatrix}$$

$$\to\left(\begin{array}{cc|ccc}1&0&0&\frac{1}{5}&-1\\0&1&0&-\frac{3}{5}&1\end{array}\right)$$

$$M_f^{B_1,B_2}=\begin{pmatrix}0&\frac{1}{5}&-1\\0&-\frac{3}{5}&1\end{pmatrix}\qquad M=L_2^{-1}\cdot A\cdot L_1$$

Ex: $f:\mathbb{R}^3\to\mathbb{R}^3$ $f(x_1,x_2,x_3)=(x_1+x_2-2x_3,\ x_1+x_2-x_3,\ x_1-x_2)$

Mat lui f în raport cu baza $B=\{(1,1,1),(1,2,2),(2,1,3)\}$

$$f(x_1,x_2,x_3)=\begin{pmatrix}1&1&-2\\1&1&-1\\1&-1&0\end{pmatrix}\cdot\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}$$

A

B

1mod

P1

$$M=L_1^{-1}\cdot A\cdot L_1$$

Pas 1

$f(1,1,1)=\begin{pmatrix}0\\1\\0\end{pmatrix}$

$$B=\begin{pmatrix}0&-1&-3\\1&1&0\\0&-1&1\end{pmatrix}$$

$f(1,2,2)=\begin{pmatrix}-1\\1\\-1\end{pmatrix}$

$f(2,1,3)=\begin{pmatrix}-3\\0\end{pmatrix}$

$$M = L_1^{-1} \cdot B = \left(\begin{array}{ccc|ccc} 1 & 1 & 2 & 0 & -1 & -3 \\ 1 & 2 & 1 & 1 & 1 & 0 \\ 1 & 2 & 3 & 0 & -1 & 1 \end{array}\right) \to \text{Gauss}$$

$$M = L_1^{-1} \cdot B = \begin{pmatrix} 2 & \frac{1}{2} & -\frac{3}{2} \\ -1 & \frac{1}{2} & \frac{1}{2} \\ 0 & -\frac{1}{2} & \frac{1}{2} \end{pmatrix} \cdot \begin{pmatrix} 0 & -1 & -3 \\ 1 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & 0 & -\frac{15}{2} \\ \frac{1}{2} & 1 & \frac{5}{2} \\ -\frac{1}{2} & -1 & \frac{1}{2} \end{pmatrix} = M_f^B$$

29.03.2024

Seminar 5 - GAL

1) Fie

$f: \mathbb{R}^2 \to \mathbb{R}^3 \quad f(x_1, x_2) = (x_1+x_2, x_1+x_2, x_1+x_2)$

Să se det $\operatorname{Ker} f$, $\operatorname{Im} f$ şi f este inj, surj, bij?

$\operatorname{Ker} f = \{(x_1, x_2) \in \mathbb{R}^2 \mid f(x_1, x_2) = 0\}$

$\operatorname{Ker} f: \begin{cases} x_1+x_2=0 \\ x_1+x_2=0 \\ x_1+x_2=0 \end{cases} \Rightarrow \operatorname{Ker} f: x_1+x_2=0$

$\operatorname{Ker} f = \langle (1,-1) \rangle$

$\operatorname{Im} f \subset \mathbb{R}^3$

$\mathbb{R}^2 = \langle \{e_1, e_2\} \rangle \quad e_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad e_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$\operatorname{Im} f = \langle \{f(e_1), f(e_2)\} \rangle$

~~f(e_1)~~

$f(x_1, x_2) = \underbrace{\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}}_{f(e_1)} \underbrace{\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}}_{f(e_2)} \cdot \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \langle (1,1,1)^t, (1,1,1)^t \rangle = \langle (1,1,1)^t \rangle$

$\boxed{\text{Teorie} \quad f: \mathbb{R}^m \to \mathbb{R}^n \quad f(x) = Ax \Rightarrow \operatorname{Im} f = \langle \text{Coloanele lui } A \rangle}$

$B \in M_{2,3}(\mathbb{R})$ a.î. $\operatorname{Im} f = \{x \in \mathbb{R}^3 \mid B \cdot x = 0\}$

$\dim \operatorname{Im} f = 1 = 3 - 2 \Rightarrow \operatorname{rang} B = 2$

$$B=\begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{pmatrix} \quad B\cdot\begin{pmatrix}1\\1\\1\end{pmatrix}=0 \Leftrightarrow \begin{cases} a_1+b_1+c_1=0 \\ a_2+b_2+c_2=0 \end{cases}$$

$a_1=-b_1-c_1$

$b_1=1 \quad c_1=0 \Rightarrow (-1,1,0)$

$b_1=0 \quad c_1=-1 \Rightarrow (-1,0,1)$

$$\operatorname{Im} f: \begin{cases} x_1-x_2=0 \\ x_1-x_3=0 \end{cases}$$

$\dim \mathbb{R}^3 > \dim \mathbb{R}^2 \Rightarrow f$ nu poate fi surj

$\operatorname{Ker} f \neq 0 \Rightarrow f$ nu e injectivă

Ex2 $f: \mathbb{R}^3 \to \mathbb{R}^3$

$f(x_1,x_2,x_3)=(x_1+x_2-x_3, x_1-x_2+x_3, 2x_1)$

$\operatorname{Ker} f$? $\operatorname{Im} f$? f e inj? surj? bij?

~~Dacă f~~

$\mathbb{R}^3 \to \mathbb{R}^3$ deci dacă e inj $\Rightarrow$ surj $\Rightarrow$ bij

$$A=\begin{pmatrix} 1 & 1 & -1 \\ 1 & -1 & 1 \\ 2 & 0 & 0 \end{pmatrix}$$

$$x \in \operatorname{Ker} f \Leftrightarrow \begin{cases} x_1+x_2-x_3=0 \\ x_1-x_2+x_3=0 \\ 2x_1 = 0 \end{cases} \Rightarrow \begin{cases} x_1+x_2-x_3=0 \\ x_1-x_2+x_3=0 \end{cases} \Rightarrow \begin{cases} x_1=0 \\ x_2=x_3 \end{cases}$$

$\Downarrow$ $x_1=0$

$\operatorname{Ker} f = \langle (0,1,1)^t \rangle$

$\dim \operatorname{Im} f = \cancel{\mathbb{R}^3}\ 2$

$\mathrm{Im} f = \langle (1,1,2)^t, (1,-1,0)^t, (-1,1,0)^t \rangle = \langle \{(1,1,2)^t, (1,-1,0)^t\} \rangle$

$\boxed{\text{Nr de ec necesare este } \dim \mathbb{R}^3 - \dim \mathrm{Im} f}$

$x \in \mathrm{Im} f \Leftrightarrow a x_1 + b x_2 + c x_3 = 0$

$$\begin{cases} a+b+2c=0 & ((1,1,2) \in \mathrm{Im} f) \\ a-b=0 & ((1,-1,0) \in \mathrm{Im} f) \end{cases}$$

$$\begin{cases} a=b \\ 2a=-2c \end{cases} \Rightarrow \begin{cases} b=a \\ c=-a \end{cases}$$

$x \in \mathrm{Im} f \Leftrightarrow x_1 + x_2 - x_3 = 0$

$\dim \mathrm{Ker} f = 1$

$\dim \mathrm{Im} f = 2$

$\mathbb{R}^3 \stackrel{?}{=} \mathrm{Ker} f \oplus \mathrm{Im} f$? Fals (deși $\dim \mathbb{R}^3 = \dim \mathrm{Ker} f + \dim \mathrm{Im} f$)

Fals pt că $(0,1,1) \in \mathrm{Im} f$

$\mathrm{Ker} f \in \mathrm{Im} f$?

$\mathrm{Ker} f \neq 0 \Rightarrow f$ nu e inj

$(0,1,1) \in \mathrm{Im} f \Leftrightarrow \exists x \in \mathbb{R}^3$ a.î. $f(x) = (0,1,1)$

$x = ?$

$$\begin{cases} x_1 + x_2 - x_3 = 0 \\ x_1 - x_2 + x_3 = 1 \\ 2x_1 = 1 \end{cases} \Rightarrow \begin{cases} x_1 = \frac{1}{2} \\ \frac{1}{2} + x_2 - x_3 = 0 \\ \frac{1}{2} - x_2 + x_3 = 1 \end{cases} \Rightarrow \begin{cases} x_2 - x_3 = -\frac{1}{2} \\ x_2 = -\frac{1}{2} + x_3 \end{cases}$$

$\boxed{x = \left(\frac{1}{2}, -\frac{1}{2} + \alpha, \alpha\right)}$

$\Rightarrow f(x) = (0,1,1)$

Ex3 Fie $f: V \to V$ a.î. $f^2 = f$

$$\boxed{\begin{matrix} L_1 = \operatorname{Ker} f \\ L_2 = \operatorname{Im} f \end{matrix} \Rightarrow V = L_1 \oplus L_2}$$ Teorie

a) Dem că $L_1 \cap L_2 = \{0\}$

Fie $x \in L_1 \cap L_2$, $x = f(x_0)$ $(x \in \operatorname{Im} f)$

$f(x) = 0$ $(x \in \operatorname{Ker} f)$

$0 = f(x) = f^2(x_0) = f(x_0) = x \Rightarrow x = 0 \Rightarrow L_1 \cap L_2 = \{0\}$

$y \in L_2 = \operatorname{Im} f \Leftrightarrow f(y) = y$

$y \in \operatorname{Im} f \Rightarrow \exists x$ a.î. $y = f(x)$

$f(y) = f^2(x) = f(x) = y$

$L_1 \oplus L_2 = V$

$\forall x \in V \Rightarrow \exists x_1 \in L_1$ și $x_2 \in L_2$ a.î. $x = x_1 + x_2$

$f(x) = f(x_1) + f(x_2) = 0 + x_2 \Rightarrow f(x) = x_2$

$x_1 = x - f(x)$ $\quad f(x_1) = f(x) - f^2(x) = f(x) - f(x) = 0 \Rightarrow x_1 \in L_1$

f s.n. operator de proiecție,

Ex4 Fie $L_1 = \langle \{ (1,1,0,1)^t, (1,0,1,1)^t \} \rangle$

$L_2 = \langle \{ (0,1,1,1)^t, (1,1,1,0)^t \} \rangle$

a) Arătați că $L_1 \oplus L_2 = \mathbb{R}^4$

b) pt $x = (1,1,1,1)$ det descompunerea sa după $L_1 \oplus L_2$

c) Det $p_1 : \mathbb{R}^4 \to \mathbb{R}^4$ proiecția pe L_1

a) $\dim L_1 = \dim L_2 = 2$

$L_1 + L_2 = \langle (1,1,0,1), (1,0,1,1), (0,1,1,1), (1,1,1,0) \rangle$

Dacă B bază $\Rightarrow \dim(L_1 + L_2) = \dim L_1 + \dim L_2 - \dim(L_1 \cap L_2) = 4$

$\Downarrow$

$\dim L_1 \cap L_2 = 0$

a) $B = \begin{pmatrix} 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & 0 & 1 \\ 0 & -1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & -1 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & 0 & 1 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 1 & -1 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 2 & 1 \end{pmatrix}$

$\to \begin{pmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 3 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix} \to$ B bază $\Rightarrow \dim L_1 \cap L_2 = 0 \Rightarrow L_1 \oplus L_2 = \mathbb{R}^4$

b) $x = x_1 + x_2 \quad x_1 \in L_1 \quad x_2 \in L_2$

$$x = \underbrace{a \cdot (1,1,0,1) + b \cdot (1,0,1,1)}_{x_1} + \underbrace{c \cdot (0,1,1,1) + d \cdot (1,1,1,0)}_{x_2}$$

$\left(\begin{array}{cccc|cc} 1 & 1 & 0 & 1 & 1 & x_1 \\ 1 & 0 & 1 & 1 & 1 & x_2 \\ 0 & 1 & 1 & 1 & 1 & x_3 \\ 1 & 1 & 1 & 0 & 1 & x_4 \end{array}\right) \to \left(\begin{array}{cccc|cc} 1 & 1 & 0 & 1 & 1 & x_1 \\ 0 & -1 & 1 & 0 & 0 & x_2 - x_1 \\ 0 & 1 & 1 & 1 & 1 & x_3 \\ 0 & 0 & 1 & -1 & 0 & x_4 - x_1 \end{array}\right) \to$

$$\rightarrow \left(\begin{array}{cccc|c|c} 1 & 0 & 1 & 1 & 1 & x_2 \\ 0 & -1 & 1 & 0 & 0 & x_2-x_1 \\ 0 & 0 & 2 & 1 & 1 & x_3+x_2-x_1 \\ 0 & 0 & 1 & -1 & 0 & x_4-x_1 \end{array}\right) \rightarrow \left(\begin{array}{cccc|c|c} 1 & 0 & 1 & 1 & 1 & x_2 \\ 0 & -1 & 1 & 0 & 0 & x_2-x_1 \\ 0 & 0 & 1 & -1 & 0 & x_4-x_1 \\ 0 & 0 & 2 & 1 & 1 & x_3+x_2-x_1 \end{array}\right) \rightarrow$$

$$\rightarrow \left(\begin{array}{cccc|c|c} 1 & 0 & 0 & 2 & 1 & x_2+x_1-x_4 \\ 0 & -1 & 0 & 1 & 0 & x_2-x_4 \\ 0 & 0 & 1 & -1 & 0 & \cancel{x_1-x_2}\ x_4-x_1 \\ 0 & 0 & 0 & 3 & 1 & x_1+x_2+x_3-2x_4 \end{array}\right) \rightarrow$$

$$\rightarrow \left(\begin{array}{cccc|c|c} 1 & 0 & 0 & 2 & 1 & x_2+x_1-x_4 \\ 0 & -1 & 0 & 1 & 0 & x_2-x_4 \\ 0 & 0 & 1 & -1 & 0 & x_4-x_1 \\ 0 & 0 & 0 & 1 & \frac{1}{3} & \frac{x_1}{3}+\frac{x_2}{3}+\frac{x_3}{3}-\frac{2x_4}{3} \end{array}\right) \rightarrow \left(\begin{array}{cccc|c|c} 1 & 0 & 0 & 0 & \frac{1}{3} & \frac{1}{3}x_1+\frac{2}{3}x_1-\frac{2}{3}x_3+\frac{x_4}{3} \\ 0 & -1 & 0 & 0 & +\frac{1}{3} & +\frac{x_1}{3}+\frac{2x_2}{3}+\frac{x_3}{3}+\frac{x_4}{3} \\ 0 & 0 & 1 & 0 & \frac{1}{3} & \frac{-2x_1+x_2+x_3+x_4}{3} \\ 0 & 0 & 0 & 1 & \frac{1}{3} & \frac{x_1+x_2+x_3-2x_4}{3} \end{array}\right)$$

$$X = \frac{1}{3}\cdot(1,1,0,1)+\frac{1}{3}(1,0,1,1)+\frac{1}{3}\cdot(0,1,1,1)+\frac{1}{3}(1,1,1,0)$$

$$x_1 = \left(\frac{2}{3}; \frac{1}{3}; \frac{1}{3}; \frac{2}{3}\right)$$

$$x_2 = \left(\frac{1}{3}; \frac{2}{3}; \frac{2}{3}; \frac{1}{3}\right)$$

$$p: \mathbb{R}^4 \rightarrow \mathbb{R}^3$$

$$P_1(x_1,x_2,x_3,x_4) = \left(\frac{x_1+x_2-2x_3-x_4}{3}\right)\cdot(1,1,0,1)+\left(\frac{x_1}{3}-\frac{2x_2}{3}+\frac{x_3}{3}+\frac{x_4}{3}\right)\cdot(1,0,1,1) =$$

$$= \left(\frac{2x_1-x_2-x_3+2x_4}{3}, \frac{x_1+x_2-2x_3-x_4}{3}, \frac{x_1-2x_2+x_3+x_4}{3}, \frac{2x_1-x_2-x_3+2x_4}{3}\right)$$

$$M_{P_1} = \begin{pmatrix} \frac{2}{3} & -\frac{1}{3} & -\frac{1}{3} & \frac{2}{3} \\ \frac{1}{3} & \frac{1}{3} & -\frac{2}{3} & -\frac{1}{3} \\ \frac{1}{3} & -\frac{2}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{2}{3} & -\frac{1}{3} & -\frac{1}{3} & \frac{2}{3} \end{pmatrix} \qquad Rg\, M_{P_1} = 2$$

Să se calculeze cu formula lui Laplace

col 1,2

$$\begin{vmatrix}1&2&3&4\\-2&6&0&1\\0&1&-1&2\\0&11&4&11\end{vmatrix} = (-1)^6 \begin{vmatrix}1&2\\-2&6\end{vmatrix}\cdot\begin{vmatrix}-1&2\\4&11\end{vmatrix} + (-1)^7\begin{vmatrix}1&2\\0&1\end{vmatrix}\cdot\begin{vmatrix}0&1\\4&11\end{vmatrix} + (-1)^8 \begin{vmatrix}1&2\\0&11\end{vmatrix}\cdot\begin{vmatrix}0&1\\-1&2\end{vmatrix} + (-1)^8\cdot\begin{vmatrix}-2&6\\0&1\end{vmatrix}\cdot\begin{vmatrix}3&4\\4&11\end{vmatrix} + (-1)^9\cdot$$

$$\begin{vmatrix}-2&1\\0&11\end{vmatrix}\cdot\begin{vmatrix}3&4\\-1&2\end{vmatrix} + (-1)^{10}\cdot\begin{vmatrix}0&1\\0&11\end{vmatrix}\cdot\begin{vmatrix}3&4\\0&1\end{vmatrix} =$$

$$= 10\cdot(-19) - 1\cdot(-4) + 11\cdot 1 + (-2)\cdot 17 - (-22)\cdot 10 =$$

$$= 235 - 224 = 11$$

$$\begin{vmatrix}1&2&3&4\\-2&6&0&1\\0&1&-1&2\\0&11&4&11\end{vmatrix} = \begin{vmatrix}1&2&3&4\\0&10&6&9\\0&1&-1&2\\0&11&4&11\end{vmatrix} = (-1)^{1+1}\begin{vmatrix}10&6&9\\1&-1&2\\11&4&11\end{vmatrix} = \begin{vmatrix}10&16&-11\\1&0&0\\11&15&-11\end{vmatrix} =$$

$$= (-1)^{1+2}\cdot\begin{vmatrix}16&-11\\15&-11\end{vmatrix} = +11$$

col(1,2) col 1,3

liniile 2,4

$$\begin{vmatrix}1&2&3&4\\2&3&4&1\\3&4&1&2\\4&1&2&3\end{vmatrix} = (-1)^9\cdot\begin{vmatrix}2&3\\4&1\end{vmatrix}\cdot\begin{vmatrix}3&4\\1&2\end{vmatrix} + (-1)^{10}\begin{vmatrix}2&4\\4&2\end{vmatrix}\cdot\begin{vmatrix}2&4\\4&2\end{vmatrix} +$$

$$(-1)^{11}\begin{vmatrix}2&1\\4&3\end{vmatrix}\cdot\begin{vmatrix}2&3\\4&1\end{vmatrix} + (-1)^{11}\cdot\begin{vmatrix}3&4\\1&2\end{vmatrix}\cdot\begin{vmatrix}1&4\\3&2\end{vmatrix} + (-1)^{12}\begin{vmatrix}3&1\\1&3\end{vmatrix}\cdot\begin{vmatrix}1&3\\3&1\end{vmatrix}$$

$$+(-1)^{13}\cdot\begin{vmatrix}4&1\\2&3\end{vmatrix}\cdot\begin{vmatrix}1&2\\3&4\end{vmatrix} = \cancel{(-14)} - (-10)\cdot(2) + (-12)^2 - 2\cdot(-10) - 2\cdot(40)$$

$$+8\cdot(-8) - 10(-2) = 80 + 144 - 64 = 160$$

Să se det câte o bază în L_1+L_2 şi $L_1\cap L_2$ unde

$L_1 = \langle (2,1,0),(1,2,3),(-5,-2,1)\rangle = \langle\{(1,2,3),(2,1,0)\}\rangle$

$L_2 = \langle (1,1,2),(-1,3,0),(2,0,3)\rangle = \langle\{(1,1,2),(-1,3,0)\}\rangle$

$$\dim L_1 = Rg\begin{pmatrix}1&2&-5\\2&1&-2\\3&0&1\end{pmatrix} = Rg\begin{pmatrix}1&2&-5\\0&-3&8\\0&-6&-14\end{pmatrix} = Rg\begin{pmatrix}1&2&-5\\0&-3&8\\0&0&0\end{pmatrix} = 2$$

$$\dim L_2 = Rg\begin{pmatrix}1&-1&2\\1&3&0\\2&0&3\end{pmatrix} = Rg\begin{pmatrix}1&-1&2\\0&4&-2\\0&2&-1\end{pmatrix} = 2$$

$L_1+L_2 = \langle\{(1,2,3),(2,1,0),(1,1,2),(-1,3,0)\}\rangle$

$$\dim L_1+L_2 = Rg\begin{pmatrix}1&2&1&-1\\2&1&1&3\\3&0&2&0\end{pmatrix} = Rg\begin{pmatrix}1&2&1&-1\\0&-3&-1&5\\0&-6&-1&3\end{pmatrix} = Rg\left(\begin{array}{ccc|c}1&2&1&-1\\0&-3&-1&5\\0&0&1&-7\end{array}\right) = 3$$

$L_1+L_2 = \langle (1,2,3),(2,1,0),(1,1,2)\rangle$

Ş, $\dim L_1+L_2 = \dim L_1 + \dim L_2 - \dim(L_1\cap L_2)$

$3 = 2+2-\dim(L_1\cap L_2) \Rightarrow \dim(L_1\cap L_2) = 1$

$x\in L_1\cap L_2$

$x = x_1(1,2,3) + x_2(2,1,0) = y_1(1,1,2) + y_2(-1,3,0)$

$$\begin{cases} x_1 + 2x_2 - y_1 + y_2 = 0 \\ 2x_1 + x_2 - y_1 - 3y_2 = 0 \\ 3x_1 - 2y_1 = 0 \end{cases}$$

$rg\begin{pmatrix} 1 & 2 & -1 & 1 \\ 2 & 1 & -1 & -3 \\ 3 & 0 & -2 & 0 \end{pmatrix} = 3$ cu minorul principal $\begin{vmatrix} 1 & 2 & -1 \\ 2 & 1 & -1 \\ 3 & 0 & -2 \end{vmatrix} \neq 0 \Rightarrow y_2$ nec secundară

$$\begin{cases} x_1 + 2x_2 - y_1 = -y_2 \\ 2x_1 + x_2 - y_1 = 3y_2 \\ 3x_1 + 2y_2 = 0 \end{cases} \Rightarrow \begin{cases} x_1 + 2x_2 - y_1 = -y_2 \\ -3x_2 + y_1 = 5y_2 \\ -6x_2 + y_1 = 3y_2 \end{cases} \Rightarrow \begin{cases} x_1 + 2x_2 - y_1 = y_2 \\ -3x_2 + y_1 = 5y_2 \\ -3y_1 = -7y_2 \end{cases} \Rightarrow$$

$$\Rightarrow \begin{cases} y_1 = \frac{7}{3}y_2 \\ -3x_2 = 5y_2 - \frac{14}{3}y_2 \end{cases}$$

$-3x_2 = 5y_2 - 14y_2 = -9y_2 \Rightarrow$

$\Rightarrow x_2 = 3y_2$

$x_1 = \cancel{-y_2 + \frac{2}{9}y_2 + \frac{7}{3}y_2} = 0$

$-y_2 - 6y_2 + 7y_2 = 0$

$x_1(1,2,3) + x_2(2,1,0) = y_1(1,1,2) + y_2(-1,3,0)$

$\cancel{\frac{14}{9}y_2(1,2,3) - \frac{1}{9}y_2(2,1,0) = \frac{7}{3}y_2(1,1,2) + y_2(-1,3,0)} \Leftrightarrow$

$\cancel{\Leftrightarrow 14(1,2,3) - (2,1,0) = 21(1,1,2) + 9(-1,3,0)}$

$\cancel{-y_2 \quad \frac{7}{3}y_2 + \frac{2}{9}y_2}$

$z = 1$
t la întâmplare)

$7(1,1,2) + (-1,3,0) = (6,10,14)$

$3(1,2,3)$