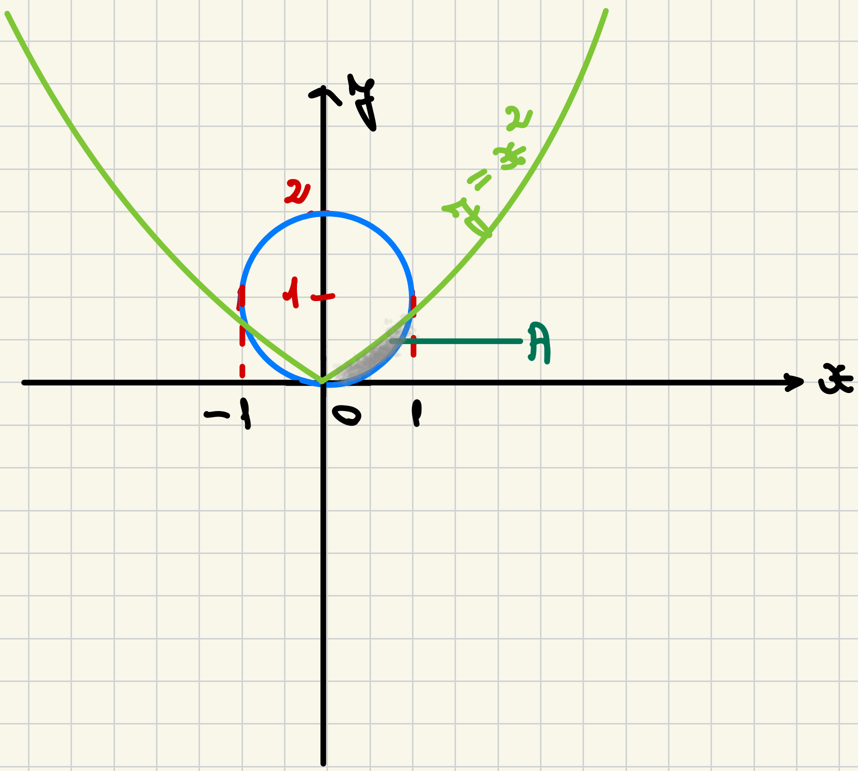
1. Determinați:

a) $\iint_A (1-y)\,dx\,dy$, unde $A=\{(x,y)\in\mathbb{R}^2 \mid x^2+(y-1)^2\le 1,$
$y\le x^2,\ x\ge 0\}$

Sol.:



Determinăm punctele de intersecție dintre
$x^2+(y-1)^2=1$ și $y=x^2$.

$$\begin{cases} x^2+(y-1)^2=1 \\ y=x^2 \end{cases} \Rightarrow y+(y-1)^2=1 \Rightarrow$$

$$\Rightarrow y^2-y+\cancel{1}=\cancel{1} \Rightarrow y(y-1)=0 \Rightarrow$$

$$\Rightarrow y\in\{0,1\}$$

$x^2=0 \Rightarrow x=0$

$x^2=1 \Rightarrow x=\pm 1$

$$x^2+(y-1)^2 \le 1 \Rightarrow (y-1)^2 \le 1-x^2 \Rightarrow$$

$$\Rightarrow -\sqrt{1-x^2} \le y-1 \le \sqrt{1-x^2} \Rightarrow 1-\sqrt{1-x^2} \le y \le 1+\sqrt{1-x^2}$$

$$A=\{(x,y)\in\mathbb{R}^2 \mid x\in[0,1],\ 1-\sqrt{1-x^2} \le y \le x^2\}$$

Fie $\alpha,\beta:[0,1]\to\mathbb{R}$, $\alpha(x)=1-\sqrt{1-x^2}$, $\beta(x)=x^2$

α,β continue

$A\in\mathcal{J}(\mathbb{R}^2)$ şi A compactă

Fie $f: A\to\mathbb{R}$, $f(x,y)=1-y$

f continuă

$$\iint_A f(x,y)\,dx\,dy=\int_0^1\left(\int_{1-\sqrt{1-x^2}}^{x^2}(1-y)\,dy\right)dx=$$

$$=-\int_0^1\left[\frac{(1-y)^2}{2}\Bigg|_{y=1-\sqrt{1-x^2}}^{y=x^2}\right]dx=$$

$$=-\frac{1}{2}\int_0^1\left[(1-x^2)^2-1+x^2\right]dx=$$

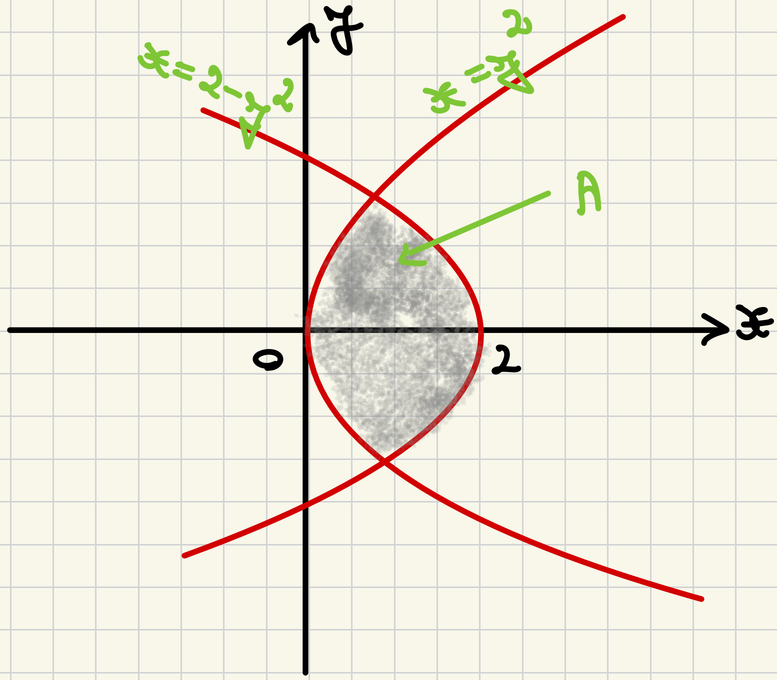
$$=-\frac{1}{2}\int_0^1(\cancel{1}+x^4-2x^2\cancel{-1}+x^2)\,dx$$

$$=-\frac{1}{2}\cdot\frac{x^5}{5}\Bigg|_{x=0}^{x=1}+\frac{1}{2}\cdot\frac{x^3}{3}\Bigg|_{x=0}^{x=1}=$$

$$=-\frac{1}{2}\left(\frac{1}{5}-\frac{1}{3}\right)=-\frac{1}{\cancel{2}}\cdot\frac{(-\cancel{2})}{15}=\frac{1}{15}\ \square$$

2) $\iint_A xy\,dx\,dy$, unde A este mulțimea plană mărginită de $x=y^2$ și $x=2-y^2$.

Sol.:



Determinăm punctele de intersecție dintre $x=2-y^2$ și $x=y^2$.

$$\begin{cases} x=2-y^2 \\ x=y^2 \end{cases} \Rightarrow y^2=2-y^2 \Rightarrow \cancel{2}y^2=\cancel{2} \Rightarrow y\in\{-1,1\}$$

$y=-1 \Rightarrow x=1$

$y=1 \Rightarrow x=1$

$A=\{(x,y)\in\mathbb{R}^2 \mid y\in[-1,1],\ y^2\le x\le 2-y^2\}$

Fie $\varphi,\psi:[-1,1]\to\mathbb{R}$, $\varphi(y)=y^2$, $\psi(y)=2-y^2$

φ,ψ continue

$A\in\mathcal{J}(\mathbb{R}^2)$ și A compactă

Fie $f:A\to\mathbb{R}$, $f(x,y)=xy$

f continuă

$$\iint_A f(x,y)\,dx\,dy = \int_{-1}^{1}\left(\int_{y^2}^{2-y^2} xy\,dx\right)dy$$

$$= \int_{-1}^{1}\left(\frac{y}{2}\,x^2\Big|_{x=y^2}^{x=2-y^2}\right)dy =$$

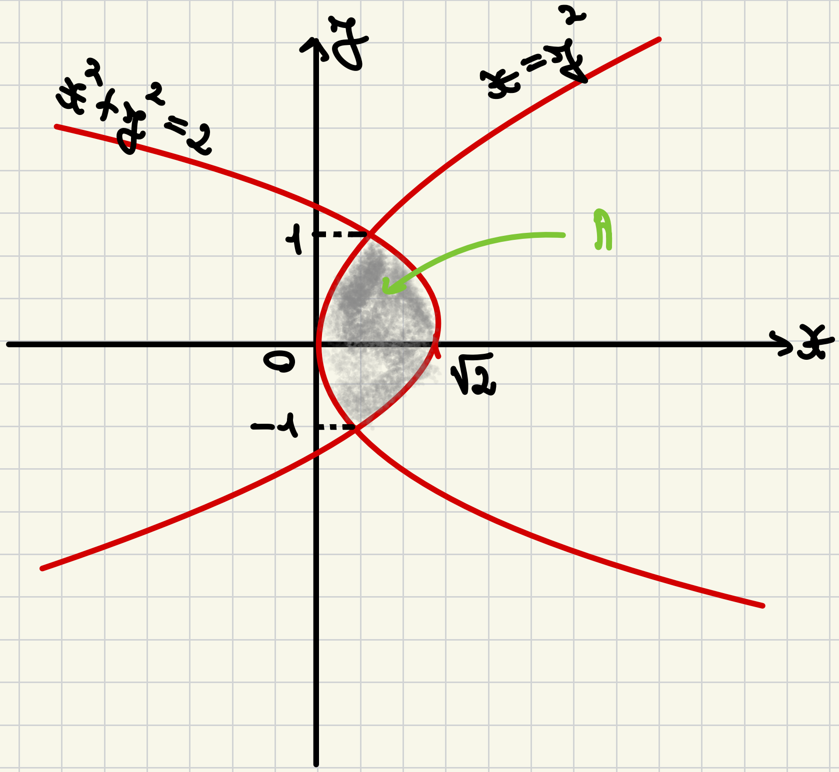
$$= \frac{1}{2}\int_{-1}^{1} y\left[(2-y^2)^2-y^4\right]dy = 0$$

$$\underset{[-1,1]}{\overset{\in}{y}} \longmapsto \underset{\mathbb{R}}{\overset{\in}{y\left[(2-y^2)^2-y^4\right]}} \text{ impară}$$

$\square$

c) $\iint_A y\,dx\,dy$, unde $A=\{(x,y)\in\mathbb{R}^2 \mid x \geq y^2,$ $x^2+y^2 \leq 2\}$

Sol.:



Determinăm punctele de intersecție dintre $x^2+y^2=2$ și $x=y^2$.

$$\begin{cases} x^2+y^2=2 \\ x=y^2 \end{cases} \Rightarrow \quad x^2+x-2=0$$

$\Delta = 1+8=9$, $\sqrt{\Delta}=3$

$x_1 = \frac{-1+3}{2} = 1$

$x_2 = \frac{-1-3}{2} = -2$

$x = y^2 \Rightarrow x \geq 0$

$y^2 = 1 \Rightarrow y = \pm 1$

$x^2+y^2 \leq 2 \Rightarrow x^2 \leq 2-y^2 \Rightarrow -\sqrt{2-y^2} \leq x \leq \sqrt{2-y^2}$

$A = \{ (x,y) \in \mathbb{R}^2 \mid y \in [-1,1] , y^2 \leq x \leq \sqrt{2-y^2} \}$

Fie $\varphi, \psi : [-1,1] \to \mathbb{R}$, $\varphi(y) = y^2$, $\psi(y) = \sqrt{2-y^2}$

φ, ψ continue

$A \in \mathcal{J}(\mathbb{R}^2)$ și A compactă

Fie $f : A \to \mathbb{R}$, $f(x,y) = y$

f continuă

$$\iint_A f(x,y)\,dx\,dy = \int_{-1}^{1} \left(\int_{y^2}^{\sqrt{2-y^2}} y\,dx \right) dy =$$

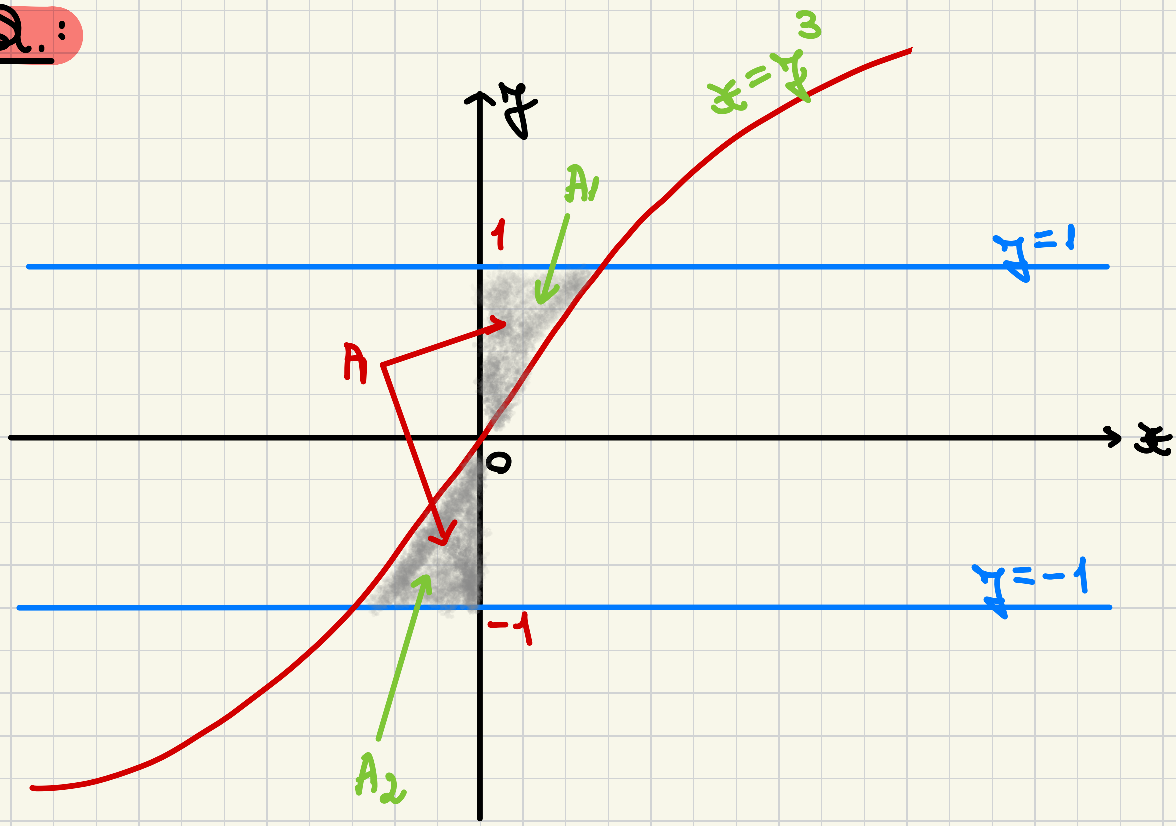
$$= \int_{-1}^{1} \left(yx \Big|_{x=y^2}^{x=\sqrt{2-y^2}} \right) dy = \int_{-1}^{1} y\left(\sqrt{2-y^2} - y^2\right) dy = 0$$

$[-1,1] \ni y \longmapsto y(\sqrt{2-y^2} - y^2)$ impară

$\square$

d) $\iint_A e^{y^4}\,dx\,dy$, unde A este mulțimea plană limitată de $x=y^3$, $x=0$, $y=1$, $y=-1$.

Sol.:



$A = A_1 \cup A_2$, unde $A_1 = \{(x,y)\in\mathbb{R}^2 \mid y\in[0,1],\ 0\le x\le y^3\}$ și $A_2 = \{(x,y)\in\mathbb{R}^2 \mid y\in[-1,0],\ y^3\le x\le 0\}$

Fie $\varphi_1, \psi_1 : [0,1]\to\mathbb{R}$, $\varphi_1(y)=0$, $\psi_1(y)=y^3$

φ_1, ψ_1 continue

$A_1 \in \mathcal{J}(\mathbb{R}^2)$ și A_1 compactă

Fie $\varphi_2, \psi_2 : [-1,0]\to\mathbb{R}$, $\varphi_2(y)=y^3$, $\psi_2(y)=0$

$A_2 \in \mathcal{J}(\mathbb{R}^2)$ și A_2 compactă

Deci $A\in\mathcal{J}(\mathbb{R}^2)$ și A compactă

Fie $f: A\to\mathbb{R}$, $f(x,y)=e^{y^4}$

f continuă

$A_1\cap A_2 = \{(0,0)\} \Rightarrow \mu(A_1\cap A_2)=0$

Deci $\iint_A f(x,y)\,dx\,dy = \iint_{A_1} f(x,y)\,dx\,dy +$

$\iint_{A_2} f(x,y)\,dx\,dy$

$$\iint_{A_1} f(x,y)\,dx\,dy = \int_0^1 \left(\int_0^{y^3} e^{y^4}\,dx \right) dy =$$

$$= \int_0^1 \left(e^{y^4} \cdot x \Big|_{x=0}^{x=y^3} \right) dy = \int_0^1 e^{y^4}(y^3 - 0)\,dy =$$

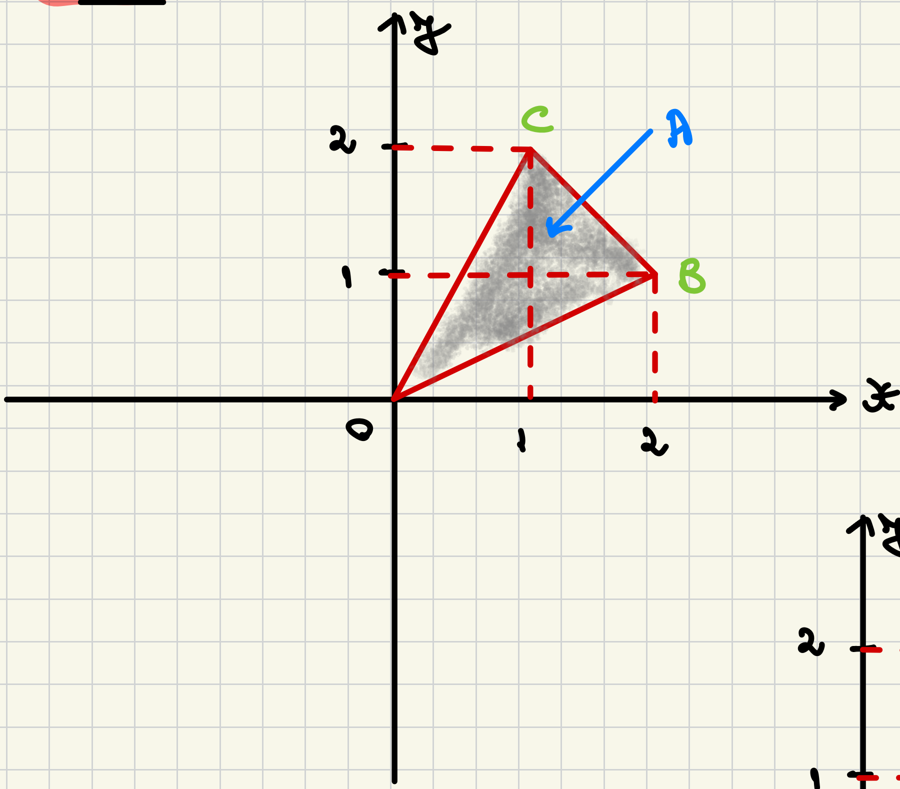
$$= \frac{1}{4} \int_0^1 4y^3 e^{y^4}\,dy = \frac{1}{4}\, e^{y^4} \Big|_{y=0}^{y=1} = \frac{1}{4}(e-1)$$

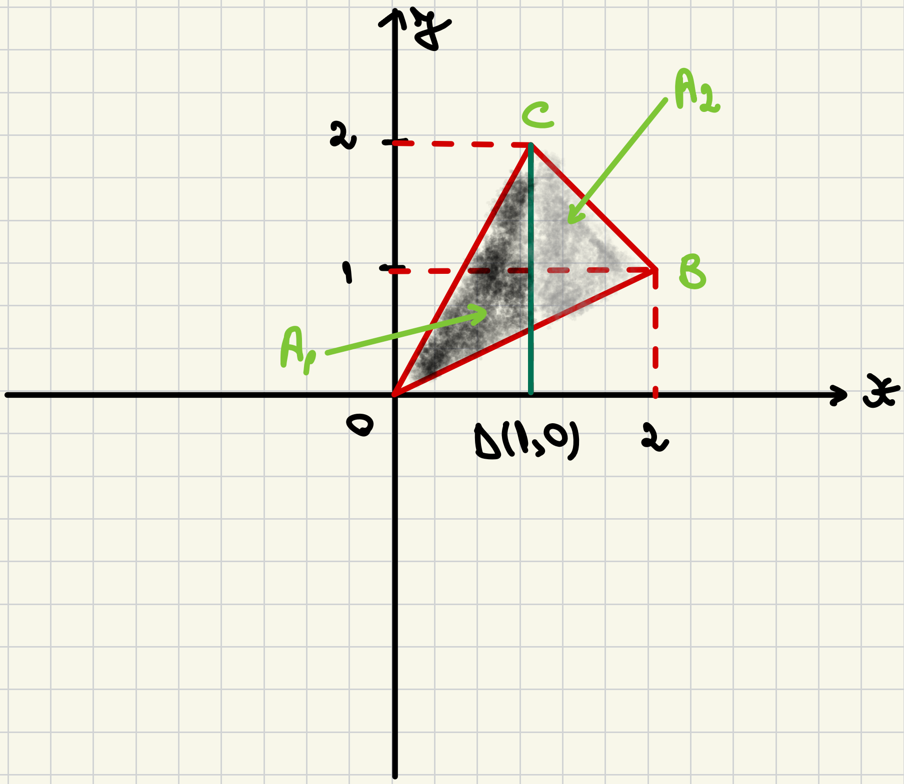
Deci $\iint_A f(x,y)\,dx\,dy = \frac{1}{4}(e-1) + \frac{1}{4}(e-1) = \frac{1}{2}(e-1)$ $\square$

2. Determinați:

a) $\iint_A x\,dx\,dy$, unde A este mulțimea plană mărginită de $\triangle OBC$, $O(0,0)$, $B(2,1)$, $C(1,2)$.

Sol.:





$OB: \dfrac{y-y_0}{y_B-y_0} = \dfrac{x-x_0}{x_B-x_0} \iff OB: \dfrac{y-0}{1-0} = \dfrac{x-0}{2-0} \iff$

$\iff \boxed{OB: y = \dfrac{x}{2}}$

$OC: \frac{y-y_O}{y_C-y_O} = \frac{x-x_O}{x_C-x_O} \Leftrightarrow \frac{y-0}{2-0} = \frac{x-0}{1-0} \Leftrightarrow$

$\Leftrightarrow \boxed{OC: y = 2x}$

$BC: \frac{y-y_B}{y_C-y_B} = \frac{x-x_B}{x_C-x_B} \Leftrightarrow BC: \frac{y-1}{2-1} = \frac{x-2}{1-2} \Leftrightarrow$

$\Leftrightarrow BC: y-1 = 2-x \Leftrightarrow \boxed{BC: y = 3-x}$

Avem $A = A_1 \cup A_2$, unde

$A_1 = \{(x,y) \in \mathbb{R}^2 \mid x \in [0,1], \frac{x}{2} \le y \le 2x\}$ și
$A_2 = \{(x,y) \in \mathbb{R}^2 \mid x \in [1,2], \frac{x}{2} \le y \le 3-x\}$

Fie $\alpha_1, \beta_1 : [0,1] \to \mathbb{R}$, $\alpha_1(x) = \frac{x}{2}$, $\beta_1(x) = 2x$

α_1, β_1 continue

$A_1 \in \mathcal{J}(\mathbb{R}^2)$ și A_1 compactă

Fie $\alpha_2, \beta_2 : [1,2] \to \mathbb{R}$, $\alpha_2(x) = \frac{x}{2}$, $\beta_2(x) = 3-x$

α_2, β_2 continue

$A_2 \in \mathcal{J}(\mathbb{R}^2)$ și A_2 compactă

Deci $A \in \mathcal{J}(\mathbb{R}^2)$ și A compactă

Fie $f : A \to \mathbb{R}$, $f(x,y) = x$

f continuă

Considerăm $D(1,0)$.

$$A_1 \cap A_2 \subset [CD] = \{1\} = \{1\} \times [0,2] \Rightarrow \mu(A_1 \cap A_2) =$$
$$\leq \mu(\{1\} \times [0,2]) = \underbrace{\mu(\{1\})}_{0} \cdot \underbrace{\mu([0,2])}_{2} = 0$$

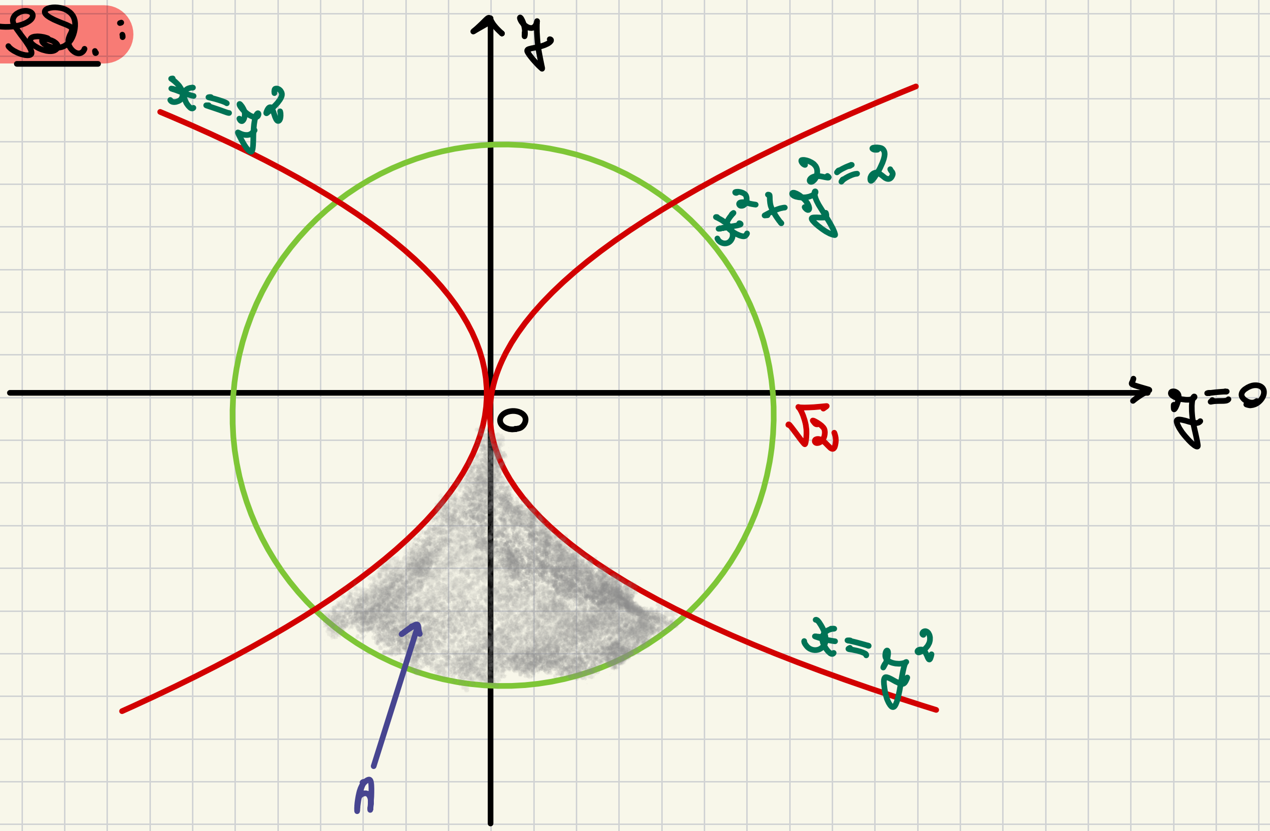
Deci $\iint_A f(x,y)\,dx\,dy = \iint_{A_1} f(x,y)\,dx\,dy +$
$\iint_{A_2} f(x,y)\,dx\,dy$

$$\iint_{A_1} f(x,y)\,dx\,dy = \int_0^1 \left(\int_{\frac{x}{2}}^{2x} x\,dy \right) dx =$$
$$= \int_0^1 \left(xy \Big|_{y=\frac{x}{2}}^{y=2x} \right) dx = \int_0^1 x\left(2x - \frac{x}{2}\right) dx =$$
$$= \int_0^1 x \cdot \frac{3x}{2}\,dx = \frac{\cancel{3}}{2} \cdot \frac{x^3}{\cancel{3}} \Big|_{x=0}^{x=1} = \frac{1}{2}$$

$$\iint_{A_2} f(x,y)\,dx\,dy = \int_1^2 \left(\int_{\frac{x}{2}}^{3-x} x\,dy \right) dx =$$
$$= \int_1^2 \left(xy \Big|_{y=\frac{x}{2}}^{y=3-x} \right) dx = \int_1^2 x\left(3 - x - \frac{x}{2}\right) dx =$$
$$= 3 \cdot \frac{x^2}{2} \Big|_{x=1}^{x=2} - \frac{\cancel{3}}{2} \cdot \frac{x^3}{\cancel{3}} \Big|_{x=1}^{x=2} = 3\left(\frac{4}{2} - \frac{1}{2}\right) - \frac{8}{2} + \frac{1}{2} =$$
$$= \frac{9}{2} - \frac{7}{2} = \frac{2}{2} = 1 \;\square$$

2) $\iint_A y\,dx\,dy$, unde $A=\{(x,y)\in\mathbb{R}^2 \mid x^2+y^2\le 2,$ $x\ge -y^2,\ x\le y^2,\ y\le 0\}$

Sol.:



Determinăm punctele de intersecție dintre $x^2+y^2=2$ și $x=-y^2$.

$$\begin{cases} x^2+y^2=2 \\ x=-y^2 \end{cases} \Rightarrow \begin{cases} x^2+y^2=2 \\ y^2=-x \end{cases}$$

$$\Rightarrow x^2-x=2 \Rightarrow x^2-x-2=0$$

$$\Delta = 1-4(-2)=1+8=9$$

$$x_{1,2}=\frac{1\pm 3}{2} \quad \begin{matrix} x_1=2 \quad \times \\ x_2=-1 \quad \checkmark \end{matrix}$$

Deci $x=-1$.

$y^2 = -x = 1 \Rightarrow y = \pm 1$

Determinăm intersecția dintre $x^2+y^2=2$ și $x=y^2$.

$$\begin{cases} x^2+y^2=2 \\ x=y^2 \end{cases} \Rightarrow x^2+x=2 \Rightarrow x^2+x-2=0$$

$$\Delta = 1-4(-2)=9$$

$$x_{1,2} = \frac{-1\pm 3}{2} \left\langle \begin{array}{l} x_1 = 1 \checkmark \\ x_2 = -2 \times \end{array} \right.$$

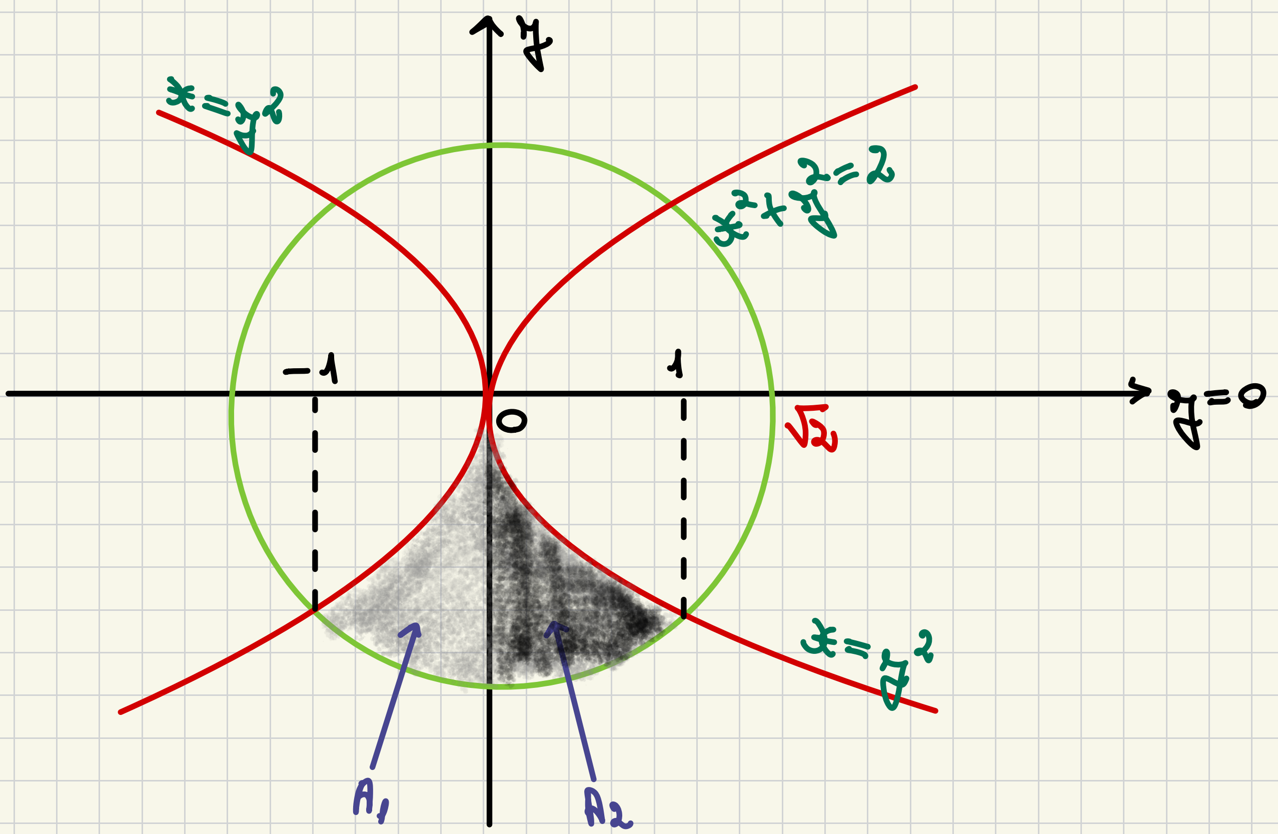
Deci $x=1$.

$y^2 = 1 \Rightarrow y = \pm 1$

$x^2+y^2 \leq 2 \Rightarrow y^2 \leq 2-x^2 \Rightarrow -\sqrt{2-x^2} \leq y \leq \sqrt{1-x^2}$

$x \geq -y^2 \Rightarrow -x \leq y^2 \Rightarrow y \in (-\infty, -\sqrt{-x}\,] \cup [\sqrt{-x}, +\infty)$

$x \leq y^2 \Rightarrow y \in (-\infty, -\sqrt{x}\,] \cup [\sqrt{x}, +\infty)$



$A = A_1 \cup A_2$, unde $A_1 = \{(x,y) \in \mathbb{R}^2 \mid x \in [-1,0],$
$-\sqrt{2-x^2} \leq y \leq -\sqrt{-x}\}$ și

$A_2 = \{(x,y) \in \mathbb{R}^2 \mid x \in [0,1], -\sqrt{2-x^2} \leq y \leq -\sqrt{x}\}$

$A_1 \cap A_2 = \{0\} \times [-\sqrt{2}, 0]$

$\mu(A_1 \cap A_2) = \mu(\{0\} \times [-\sqrt{2}, 0]) = \mu(\{0\}) \cdot \mu([-\sqrt{2}, 0]) = 0$

Fie $\alpha_1, \beta_1 : [-1, 0) \to \mathbb{R}$,

$\alpha_1(x) = -\sqrt{2-x^2}$, $\beta_1(x) = -\sqrt{-x}$

$A_1 \in \mathcal{J}(\mathbb{R}^2)$, A_1 compactă

Fie $\alpha_2, \beta_2 : [0,1] \to \mathbb{R}$,

$\alpha_2(x) = \sqrt{2-x^2}$, $\beta_2(x) = -\sqrt{x}$

α_2, β_2 continue

$A_2 \in \mathcal{J}(\mathbb{R}^2)$, A_2 compactă

Deci $A \in \mathcal{J}(\mathbb{R}^2)$, A compactă

Fie $f : A \to \mathbb{R}$, $f(x,y) = y$, f continuă

$$\iint_A f(x,y)\,dx\,dy = \iint_{A_1} f(x,y)\,dx\,dy +$$
$$+ \iint_{A_2} f(x,y)\,dx\,dy$$

$$\iint_{A_1} f(x,y)\,dx\,dy = \int_{-1}^{0} \left(\int_{-\sqrt{2-x^2}}^{-\sqrt{-x}} y\,dy \right) dx =$$

$$= \int_{-1}^{0} \frac{y^2}{2} \Bigg|_{y=-\sqrt{2-x^2}}^{y=-\sqrt{-x}} dx = \int_{-1}^{0} \frac{-x-2+x^2}{2}\,dx =$$

$$= \frac{1}{2} \left(-\frac{x^2}{2} \Bigg|_{x=-1}^{x=0} - 2x \Bigg|_{x=-1}^{x=0} + \frac{x^3}{3} \Bigg|_{x=-1}^{x=0} \right) =$$

$$= \frac{1}{2}\left(-\frac{1}{2} + 2 + \frac{1}{3} \right) = \frac{1}{2} \cdot \frac{3-12+12}{6} = \frac{3-14}{12} = -\frac{7}{12}$$

$$\iint_{A_2} f(x,y)\,dx\,dy = \int_{0}^{1} \left(\int_{-\sqrt{2-x^2}}^{\sqrt{x}} y\,dy \right) dx =$$

$$= \int_{0}^{1} \left(\frac{y^2}{2} \Bigg|_{y=-\sqrt{2-x^2}}^{y=\sqrt{x}} \right) dx =$$

$$= \int_{0}^{1} \frac{x-2+x^2}{2}\,dx = \frac{1}{2} \left(\frac{x^2}{2} - 2x + \frac{x^3}{3} \right) \Bigg|_{0}^{1}$$

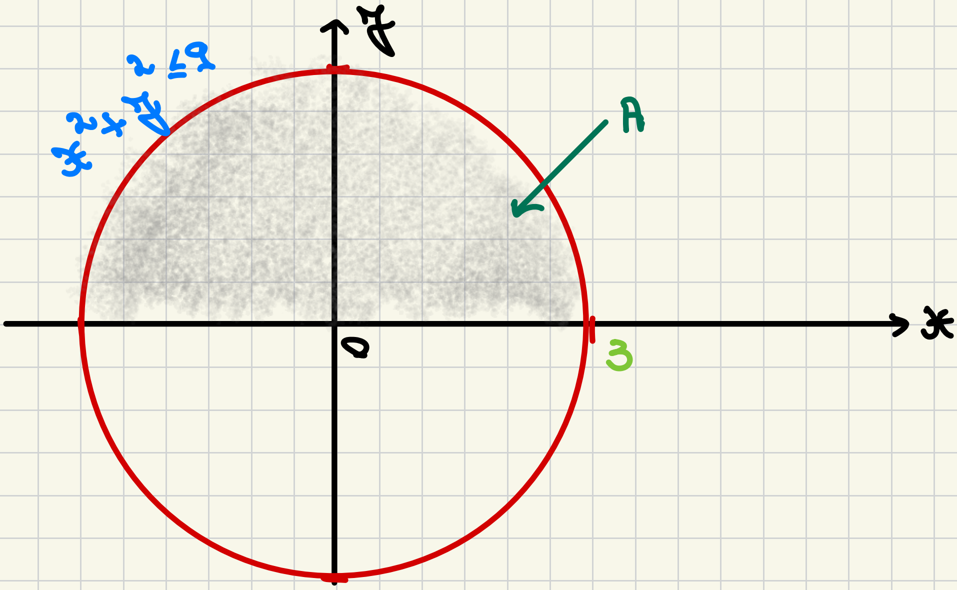
$$= \frac{1}{2}\left(\frac{1}{2} - 2 + \frac{1}{3} \right) = \frac{1}{2} \cdot \frac{3-12+2}{6} = -\frac{7}{12}$$

$$\iint_{A} f(x,y)\,dx\,dy = -\frac{7}{12} - \frac{7}{12} = -\frac{14}{12} = -\frac{7}{6} \quad \square$$

3. Determinați:

a) $\iint_A e^{-x^2-y^2}\,dx\,dy$, unde $A=\{(x,y)\in\mathbb{R}^2 \mid x^2+y^2\le 9,\ y\ge 0\}$

Sol.:



A convexă și mărginită $\Rightarrow A\in J(\mathbb{R}^2)$

A închisă și mărginită $\Rightarrow$ A compactă

Fie $f: A\to\mathbb{R}$, $f(x,y)=e^{-x^2-y^2}$

f continuă

S.V. $\begin{cases} x=r\cos\theta \\ y=r\sin\theta \end{cases}$, $r\in[0,\infty)$, $\theta\in[0,2\pi]$

$(x,y)\in A \Rightarrow \begin{cases} x^2+y^2\le 9 \\ y\ge 0 \end{cases} \Rightarrow \begin{cases} r^2\le 9 \\ r\sin\theta\ge 0 \end{cases} \Rightarrow \begin{cases} r\in[0,3] \\ \theta\in[0,\pi] \end{cases}$

Fie $B=[0,3]\times[0,\pi]$, $B\in J(\mathbb{R}^2)$, B compactă

S.V. $\iint_A f(x,y)\,dx\,dy = \iint_B r f(r\cos\theta, r\sin\theta)\,dr\,d\theta =$

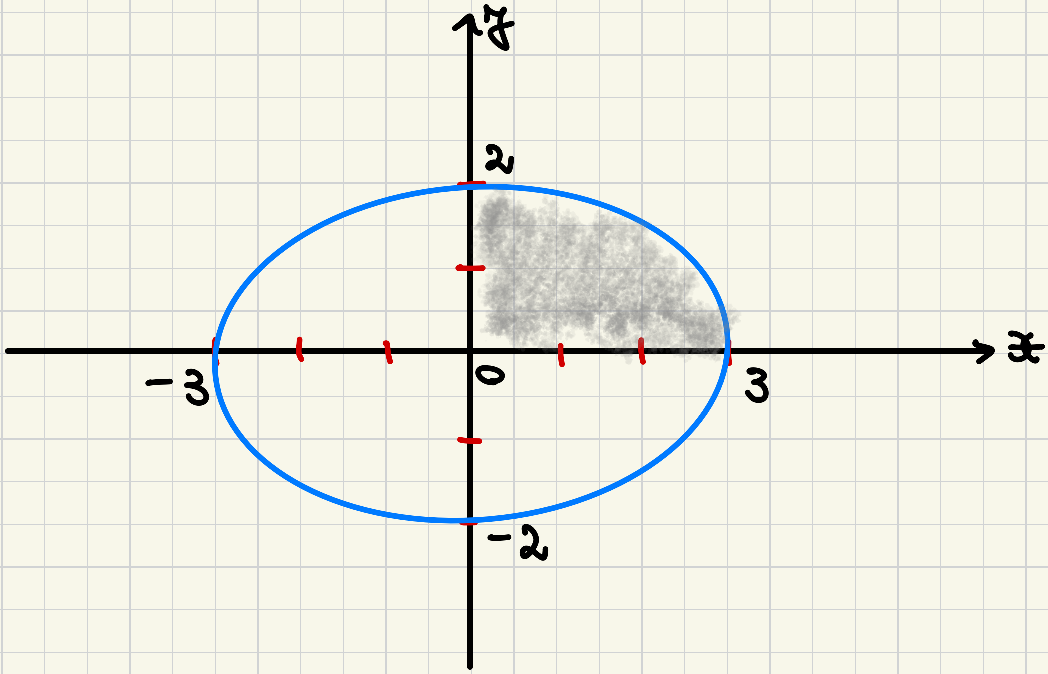
$$= \int_0^3 \left(\int_0^\pi (r\, e^{-r^2} d\theta) \right) dr = \int_0^3 \left(r\, e^{-r^2}\, \theta \Big|_{\theta=0}^{\theta=\pi} \right) dr$$

$$= \pi \int_0^3 r\, e^{-r^2} dr = -\frac{\pi}{2} \int_0^3 (-2r)\, e^{-r^2} dr =$$

$$= -\frac{\pi}{2}\, e^{-r^2} \Big|_{r=0}^{r=3} = -\frac{\pi}{2}\left(e^{-9} - e^{0}\right) = \frac{\pi}{2}\left(1 - e^{-9}\right) \; \square$$

b) $\iint_A \sqrt{1 - \frac{x^2}{9} - \frac{y^2}{4}}\, dx\,dy$, unde $A = \{(x,y) \in \mathbb{R}^2 \mid \frac{x^2}{9} + \frac{y^2}{4} \le 1,\ x \ge 0,\ y \ge 0\}$

Sol.:



A convexă și mărginită $\Rightarrow A \in J(\mathbb{R}^2)$

A închisă și mărginită $\Rightarrow$ A compactă

Fie $f: A \to \mathbb{R}$, $f(x,y) = \sqrt{1 - \frac{x^2}{9} - \frac{y^2}{4}}$

f continuă

S.V. $\begin{cases} x = 3r\cos\theta \\ y = 2r\sin\theta \end{cases}$, $r \in [0, +\infty)$, $\theta \in [0, 2\pi]$

$$(x,y) \in A \Rightarrow \begin{cases} \frac{x^2}{9} + \frac{y^2}{4} \le 1 \\ x \ge 0 \\ y \ge 0 \end{cases} \Rightarrow \begin{cases} \frac{\cancel{9}r^2\cos^2\theta}{\cancel{9}} + \frac{\cancel{4}r^2\sin^2\theta}{\cancel{4}} \le 1 \\ 3r\cos\theta \ge 0 \\ 2r\sin\theta \ge 0 \end{cases}$$

$$\begin{cases} r^2 \le 1 \\ 3r\cos\theta \ge 0 \\ 2r\sin\theta \ge 0 \end{cases} \Rightarrow \begin{cases} r \in [0,1] \\ \theta \in [0, \frac{\pi}{2}] \end{cases}$$

Fie $B = [0,1] \times [0, \frac{\pi}{2}]$

$B \in J(\mathbb{R}^2)$ și B compactă

$$\iint_A f(x,y)\,dx\,dy = \iint_B 3 \cdot 2r\, f(3r\cos\theta, 2r\sin\theta)\,dr\,d\theta =$$

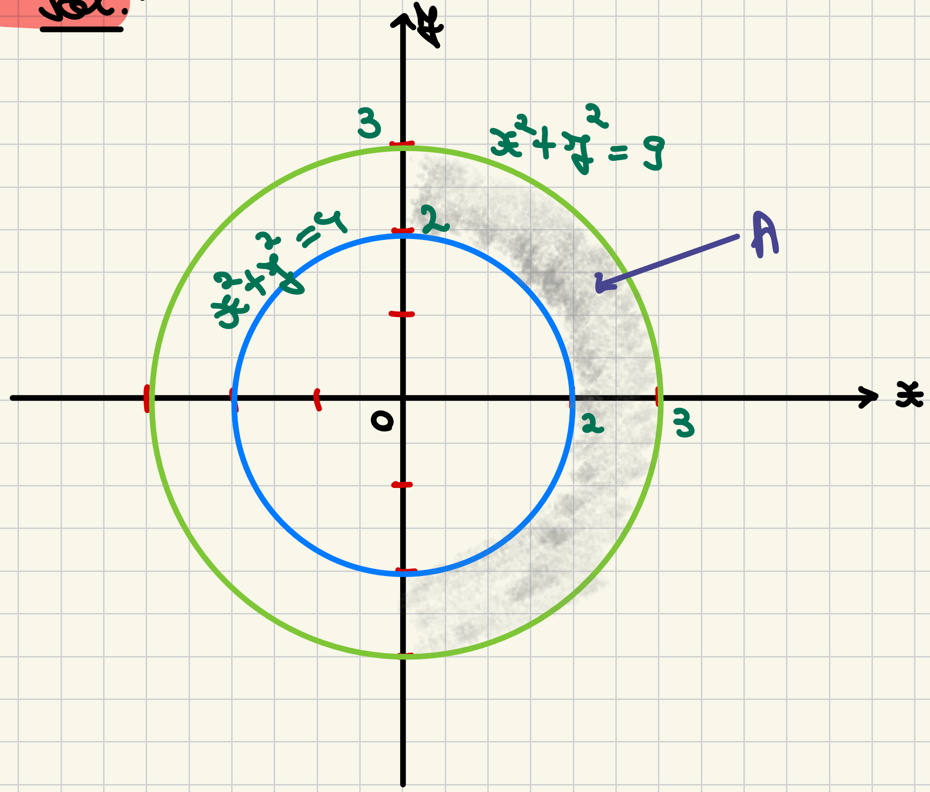
$$= \int_0^1 \left(\int_0^{\frac{\pi}{2}} 6r\sqrt{1-r^2}\, d\theta \right) dr = \int_0^1 \left[\left(6r\sqrt{1-r^2}\right) \cdot \theta \Big|_{\theta=0}^{\theta=\frac{\pi}{2}} \right] dr = -\frac{\pi}{2} \cdot 3 \int_0^1 (-2r)(1-r^2)^{\frac{1}{2}}\, dr =$$

$$= -\frac{3\pi}{2} \frac{(1-r^2)^{\frac{1}{2}+1}}{\frac{1}{2}+1} \Big|_{r=0}^{r=1} = -\frac{3\pi}{2}\left(0 - \frac{1}{\frac{3}{2}}\right) =$$

$$= -\frac{\cancel{3}\pi}{\cancel{2}}\left(-\frac{\cancel{2}}{\cancel{3}}\right) = \pi \quad \square$$

c) $\iint_A x \, dx \, dy$, unde $A = \{(x,y) \in \mathbb{R}^2 \mid 4 \le x^2+y^2 \le 9,\ x \ge 0\}$

Sol.:



$A = \underbrace{\{(x,y) \in \mathbb{R}^2 \mid x^2+y^2 \le 9,\ x \ge 0\}}$

convexă și mărginită $\Rightarrow$ măsurabilă J

$\setminus \underbrace{\{(x,y) \in \mathbb{R}^2 \mid x^2+y^2 < 4\}}$

convexă și mărginită $\Rightarrow$ măsurabilă J

Prin urmare, $A \in \mathcal{J}(\mathbb{R}^2)$.

A închisă și mărginită $\Rightarrow$ A compactă

Fie $f : A \to \mathbb{R}$, $f(x,y) = x$, f continuă

S.V. $\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$, $r \in [0, +\infty)$, $\theta \in [0, 2\pi]$

$$(x,y)\in A \Rightarrow \begin{cases} 4\le x^2+y^2\le 9 \\ x\ge 0\end{cases} \Rightarrow \begin{cases} 4\le r^2\le 9 \\ r\cos\theta\ge 0\end{cases} \Rightarrow$$

$$\Rightarrow \begin{cases} r\in[2,3] \\ \theta\in[0,\frac{\pi}{2}]\cup[\frac{3\pi}{2},2\pi]\end{cases}$$

Fie $B=[2,3]\times\left([0,\frac{\pi}{2}]\cup[\frac{3\pi}{2},2\pi]\right)$

$B\in\mathcal{J}(\mathbb{R}^2)$ şi B compactă

$$\iint_A f(x,y)\,dx\,dy = \iint_B r f(r\cos\theta, r\sin\theta)\,dr\,d\theta$$

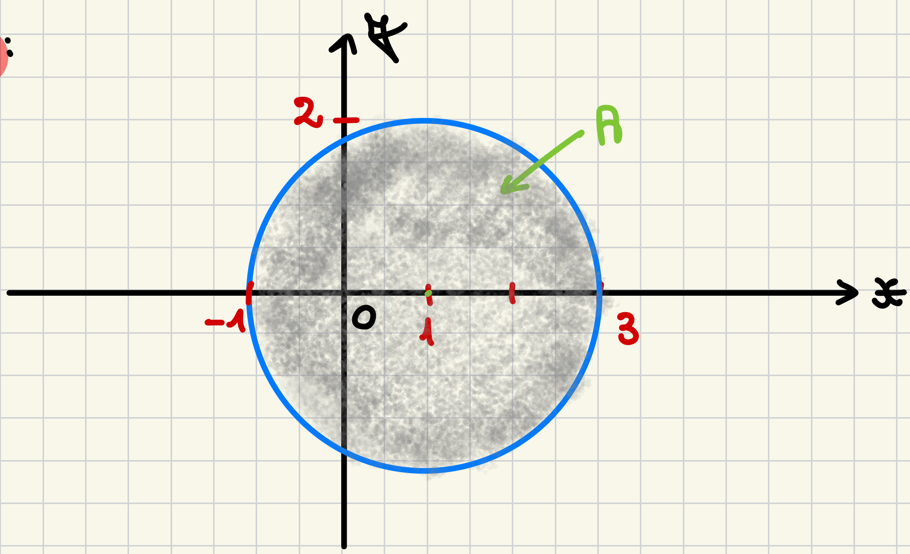
$$=\int_2^3\left(\int_0^{\frac{\pi}{2}} r r\cos\theta\,d\theta+\int_{\frac{\pi}{2}}^{2\pi} r r\cos\theta\,d\theta\right)dr=$$

$$=\int_2^3\left(r^2\sin\theta\Big|_{\theta=0}^{\theta=\frac{\pi}{2}}+r^2\sin\theta\Big|_{\theta=\frac{3\pi}{2}}^{\theta=2\pi}\right)dr=$$

$$=\int_2^r(r^2+r^2)\,dr=2\frac{r^3}{3}\Big|_{r=2}^{r=3}=\frac{2}{3}(27-8)=\frac{38}{3}\ \square$$

d) $\iint_A y\,dx\,dy$, unde $A=\{(x,y)\in\mathbb{R}^2 \mid (x-1)^2+y^2\le 4\}$

Sol.:



A închisă și mărginită $\Rightarrow$ A compactă

Fie $f: A \to \mathbb{R}$, $f(x,y) = y$

f continuă

S.V. $\begin{cases} x = 1 + r\cos\theta \\ y = r\sin\theta \end{cases}$, $r \in [0, +\infty)$, $\theta \in [0, 2\pi]$

$(x,y) \in A \Rightarrow (x-1)^2 + y^2 \le 4 \Rightarrow (1 + r\cos\theta - 1)^2 + (r\sin\theta)^2 \le 4 \Rightarrow$

$\Rightarrow r^2 \le 4 \Rightarrow \begin{cases} r \in [0,2] \\ \theta \in [0, 2\pi] \end{cases}$

Fie $B = [0,2] \times [0, 2\pi]$

$B \in \mathcal{J}(\mathbb{R}^2)$ și B compactă

$$\iint_A f(x,y)\,dx\,dy = \iint_B r f(1 + r\cos\theta, r\sin\theta)\,dr\,d\theta$$

$$= \int_0^2 \left(\int_0^{2\pi} r r \sin\theta \, d\theta \right) dr = \int_0^2 \left(r^2(-\cos\theta) \Big|_{\theta=0}^{\theta=2\pi} \right) dr$$

$$= \int_0^2 r^2(-1+1)\,dr = \int_0^2 0\,dr = 0 \quad \square$$

Completări Seminar 14
(Integrale triple)

Observație. Exercițiile în care se cere calculul unor integrale triple pe submulțimi din $\mathbb{R}^3$ ale căror reprezentări grafice necesită schițarea altor figuri geometrice din spațiu diferite de sfere și elipsoizi sunt formulate astfel încât să nu fie nevoie de reprezentările grafice ale mulțimilor de mai sus.

1. Determinați:

a) $\iiint_A (xyz+y^2)\,dx\,dy\,dz$, unde $A=[-1,1]\times[2,3]\times[0,1]$.

Soluție. $A=[-1,1]\times[2,3]\times[0,1] \Rightarrow A\in J(\mathbb{R}^3)$ și A compactă.

Fie $f:A\to\mathbb{R}$, $f(x,y,z)=xyz+y^2$.

f continuă.

$$\iiint_A f(x,y,z)\,dx\,dy\,dz=\int_{-1}^{1}\left(\int_2^3\left(\int_0^1 (xyz+y^2)\,dz\right)dy\right)dx=$$

$$=\int_{-1}^{1}\left(\int_2^3\left(xy\frac{z^2}{2}\Big|_{z=0}^{z=1}+y^2z\Big|_{z=0}^{z=1}\right)dy\right)dx=$$

$$=\int_{-1}^{1}\left(\int_2^3\left(\frac{xy}{2}+y^2\right)dy\right)dx=\int_{-1}^{1}\left(\frac{x}{2}\cdot\frac{y^2}{2}\Big|_{y=2}^{y=3}+\frac{y^3}{3}\Big|_{y=2}^{y=3}\right)dx=$$

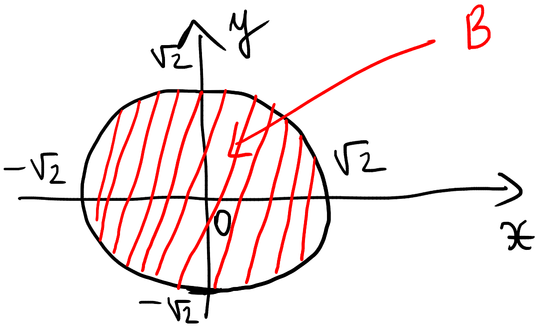
$$=\int_{-1}^{1}\left(\frac{x}{2}\cdot\frac{5}{2}+\frac{19}{3}\right)dx=\frac{5}{4}\cdot\frac{x^2}{2}\Big|_{x=-1}^{x=1}+\frac{19}{3}x\Big|_{x=-1}^{x=1}=0+\frac{38}{3}=\frac{38}{3}.\square$$

b) $\iiint_A x\,dx\,dy\,dz$, unde $A=[1,2]\times[0,1]\times[2,3]$.

Soluție. Rezolvați voi! $\square$

c) $\iiint_A (x^2+y^2)z\,dx\,dy\,dz$, unde $A=\{(x,y,z)\in\mathbb{R}^3 \mid (x,y)\in B,$
$x^2+y^2\le z\le\sqrt{6-x^2-y^2}\}$ și $B=\{(x,y)\in\mathbb{R}^2 \mid x^2+y^2\le 2\}$.

Soluție.



B convexă și mărginită $\Rightarrow B\in J(\mathbb{R}^2)$.

B închisă și mărginită $\Rightarrow$ B compactă.

Fie $\varphi,\psi:B\to\mathbb{R}$, $\varphi(x,y)=x^2+y^2$, $\psi(x,y)=\sqrt{6-x^2-y^2}$.

φ,ψ continue.

$A\in J(\mathbb{R}^3)$ și A compactă.

Fie $f:A\to\mathbb{R}$, $f(x,y,z)=(x^2+y^2)z$.

f continuă.

$$\iiint_A f(x,y,z)\,dx\,dy\,dz=\iint_B\left(\int_{x^2+y^2}^{\sqrt{6-x^2-y^2}}(x^2+y^2)z\,dz\right)dx\,dy =$$

$$=\iint_B\left[(x^2+y^2)\frac{z^2}{2}\Bigg|_{z=x^2+y^2}^{z=\sqrt{6-x^2-y^2}}\right]dx\,dy =$$

$$= \iint_B \frac{1}{2}(x^2+y^2)\left[6-x^2-y^2-(x^2+y^2)^2\right]dx\,dy .$$

Fie $g : B \to \mathbb{R}$, $g(x,y) = \frac{1}{2}(x^2+y^2)\left[6-x^2-y^2-(x^2+y^2)^2\right]$.

g continuă.

S.V. $\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$, $r \in [0,\infty)$, $\theta \in [0, 2\pi]$.

$(x,y) \in B \Rightarrow x^2+y^2 \leq 2 \Rightarrow r^2 \leq 2 \Rightarrow \begin{cases} r \in [0,\sqrt{2}] \\ \theta \in [0,2\pi]. \end{cases}$

Fie $C = [0,\sqrt{2}] \times [0,2\pi]$.

$C \in \mathcal{J}(\mathbb{R}^2)$ și C compactă.

$$\iint_B g(x,y)\,dx\,dy = \iint_C \frac{1}{2} r \cdot r^2 \left[6 - r^2 - (r^2)^2\right] dr\,d\theta =$$

$$= \frac{1}{2}\int_0^{\sqrt{2}} \left(\int_0^{2\pi} r^3(6-r^2-r^4)\,d\theta \right) dr = \frac{1}{2}\int_0^{\sqrt{2}} (6r^3 - r^5 - r^7)\theta \Big|_{\theta=0}^{\theta=2\pi} dr =$$

$$= \pi\left(\overset{3}{\cancel{6}}\, \frac{r^4}{\underset{2}{\cancel{4}}}\Big|_{r=0}^{r=\sqrt{2}} - \frac{r^6}{6}\Big|_{r=0}^{r=\sqrt{2}} - \frac{r^8}{8}\Big|_{r=0}^{r=\sqrt{2}} \right) =$$

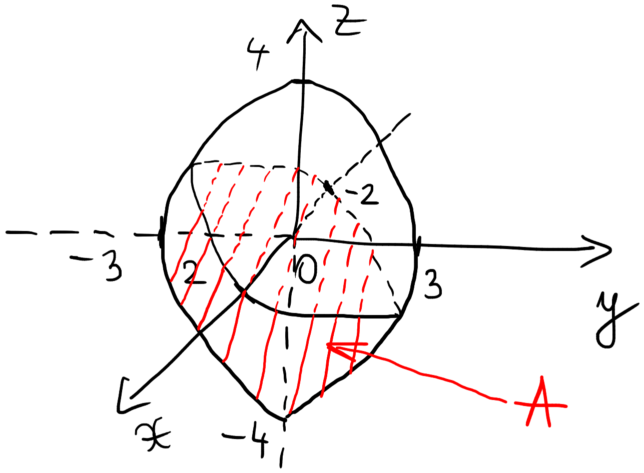
$$= \pi\left(3 \cdot \frac{\overset{2}{\cancel{4}}}{\cancel{2}} - \frac{\overset{4}{\cancel{8}}}{\underset{3}{\cancel{6}}} - \frac{\overset{2}{\cancel{16}}}{\cancel{8}}\right) = \pi\left(\overset{3}{} 4 - \frac{4}{3}\right) = \frac{8\pi}{3}. \quad \square$$

d) $\iiint_A xyz\,dx\,dy\,dz$, unde $A=\{(x,y,z)\in\mathbb{R}^3 \mid (x,y)\in B, \sqrt{x^2+y^2}\le z\le 5\}$ și $B=\{(x,y)\in\mathbb{R}^2 \mid 9\le x^2+y^2\le 25\}$.

Soluție. Rezolvați voi! $\square$

e) $\iiint_A \left(\frac{x^2}{4}+\frac{y^2}{9}+\frac{z^2}{16}\right)dx\,dy\,dz$, unde $A=\{(x,y,z)\in\mathbb{R}^3 \mid \frac{x^2}{4}+\frac{y^2}{9}+\frac{z^2}{16}\le 1,\ z\le 0\}$.

Soluție.



A convexă și mărginită $\Rightarrow A\in\mathcal{J}(\mathbb{R}^3)$.

A închisă și mărginită $\Rightarrow A$ compactă.

Fie $f:A\to\mathbb{R}$, $f(x,y,z)=\frac{x^2}{4}+\frac{y^2}{9}+\frac{z^2}{16}$.

f continuă.

S.V. $\begin{cases} x=2r\cos\theta\sin\varphi \\ y=3r\sin\theta\sin\varphi \\ z=4r\cos\varphi \end{cases}$, $r\in[0,\infty)$, $\theta\in[0,2\pi]$, $\varphi\in[0,\pi]$.

$$(x,y,z)\in A \Rightarrow \begin{cases} \frac{x^2}{4}+\frac{y^2}{9}+\frac{z^2}{16}\le 1 \\ z\le 0 \end{cases} \Rightarrow \begin{cases} r^2\le 1 \\ r\cos\varphi \le 0 \end{cases} \Rightarrow \begin{cases} r\in[0,1] \\ \theta\in[0,2\pi] \\ \varphi\in[\frac{\pi}{2},\pi]. \end{cases}$$

Fie $B=[0,1]\times[0,2\pi]\times[\frac{\pi}{2},\pi]$.

$$\iiint_A f(x,y,z)\,dx\,dy\,dz=$$

$$=\iiint_B (2\cdot 3\cdot 4r^2\sin\varphi)\, f(2r\cos\theta\sin\varphi,\ 3r\sin\theta\sin\varphi,\ 4r\cos\varphi)\,dr\,d\theta\,d\varphi=$$

$$=\int_0^1\left(\int_0^{2\pi}\left(\int_{\frac{\pi}{2}}^{\pi}24r^2\sin\varphi\cdot r^2\,d\varphi\right)d\theta\right)dr=\int_0^1\left(\int_0^{2\pi}\left[24r^4(-\cos\varphi)\Big|_{\varphi=\frac{\pi}{2}}^{\varphi=\pi}\right]d\theta\right)dr=$$

$$=\int_0^1\left(\int_0^{2\pi}24r^4(1+0)\,d\theta\right)dr=\int_0^1\left(24r^4\,\theta\Big|_{\theta=0}^{\theta=2\pi}\right)dr=$$

$$=48\pi\int_0^1 r^4\,dr=48\pi\cdot\frac{r^5}{5}\Big|_{r=0}^{r=1}=\frac{48\pi}{5}.\ \square$$

f) $\iiint_A x\,dx\,dy\,dz$, unde $A=\{(x,y,z)\in\mathbb{R}^3 \mid x^2+y^2+z^2\le 1,\ y\ge 0\}$.

<u>Soluție.</u> Rezolvați voi! $\square$