Seminar 13

I/ Aducerea la o formă canonică a conicelor cu $\delta = 0$

Conice - forma canonică ($\delta = 0$)

1) $(\mathbb{R}^2, (\mathbb{R}^2, g_0), \varphi)$

$$\Gamma : f(x) = \underline{3x_1^2 - 6x_1x_2 + 3x_2^2} + \underbrace{2x_1}_{2b_1x_1} + \underbrace{2x_2}_{2b_2x_2} - 2 = 0$$

Să se aducă la o formă canonică, efectuând izometrii.

$$A = \begin{pmatrix} 3 & -3 \\ -3 & 3 \end{pmatrix}$$

$$\delta = \begin{vmatrix} 3 & -3 \\ -3 & 3 \end{vmatrix} = 9 - 9 = 0 \ (\Gamma \text{ \underline{nu} are centru unic})$$

$$\tilde{A} = \begin{pmatrix} 3 & -3 & 1 = b_1 \\ -3 & 3 & 1 = b_2 \\ 1 & 1 & -2 \end{pmatrix}; \ \Delta = |\tilde{A}| = \begin{vmatrix} 3 & -3 & 1 \\ -3 & 3 & 1 \\ 1 & 1 & -2 \end{vmatrix} = \begin{vmatrix} 0 & -3 & 1 \\ 0 & 3 & 1 \\ 2 & 1 & -2 \end{vmatrix} =$$

$$= 2 \begin{vmatrix} -3 & 1 \\ 3 & 1 \end{vmatrix} = 2(-3-3) = -12 \neq 0.$$

$\Gamma$ - nedegenerată.

$$\mathcal{R} = \{O; e_1, e_2\} \xrightarrow[\tau]{\text{rotație}} \mathcal{R}' = \{O; e_1', e_2'\} \xrightarrow[\theta]{\text{translație}} \mathcal{R}'' \{P_0; e_1', e_2'\}.$$

$Q : \mathbb{R}^2 \to \mathbb{R}, \ Q(x) = 3x_1^2 - 6x_1x_2 + 3x_2^2.$

Aducem $Q$ la o formă canonică utilizând met. val. proprii

$\lambda^2 - \mathrm{tr}|A|\cdot\lambda + \det|A| = 0.$

$\lambda^2 - 6\lambda = 0 \Rightarrow \lambda_1 = 6$

$\lambda_2 = 0$

$V\lambda_1 = \{ X \in \mathbb{R}^2 / AX = 6X \} = \{(x_1, -x_1), x_1 \in \mathbb{R}\} =$

$(A - 6I_2)X = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$\begin{pmatrix} -3 & -3 \\ -3 & -3 \end{pmatrix} X = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow x_1 + x_2 = 0, x_2 = -x_1$

$= \langle \{(1, -1)\} \rangle.$

$$\boxed{\begin{array}{l} e_1' = \frac{1}{\sqrt{2}}(1, -1) \\ e_2' = \frac{1}{\sqrt{2}}(1, 1) \end{array}}$$

$V\lambda_2 = \{ X \in \mathbb{R}^2 / AX = 0 \cdot X \} \Rightarrow x_1 - x_2 = 0 \Rightarrow$

$\Rightarrow x_1 = x_2,\ V\lambda_2 = \{(x_1, x_1) / x_1 \in \mathbb{R}\} = \langle \{(1, 1)\} \rangle.$

$\Delta = -12.$

Fie rotația $T: X = R \cdot X'$

$R = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \quad \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} x_1' \\ x_2' \end{pmatrix} = \begin{cases} x_1 = \frac{1}{\sqrt{2}}(x_1' + x_2') \\ x_2 = \frac{1}{\sqrt{2}}(-x_1' + x_2') \end{cases}$

$Q(x) = 6x_1'^2$

$G(T): f(x) = 6x_1'^2 + \frac{2}{\sqrt{2}}(x_1' + x_2' - x_1' + x_2') - 2 = 0$

$f(x) = 6x_1'^2 + \frac{4}{\sqrt{2}} x_2' - 2 = 0.$

$= 6x_1'^2 + \frac{4}{\sqrt{2}}\left(x_2' - \frac{\sqrt{2}}{2}\right) = 0.$

Fie translația $\begin{cases} x_1'' = x_1' \\ x_2'' = x_2' - \frac{\sqrt{2}}{2} \end{cases}$

$$\begin{pmatrix} x_1' \\ x_2' \end{pmatrix} = \begin{pmatrix} x_1'' \\ x_2'' \end{pmatrix} + \begin{pmatrix} 0 \\ \frac{\sqrt{2}}{2} \end{pmatrix}$$

$\theta/T(+1): \; 6x_1''^2 + 2\sqrt{2}\,x_2'' = 0.$

$x_1''^2 = \frac{-\sqrt{2}}{3} x_2''.$

$X = RX'$

$X = R(X'' + X_0)$

$X = RX'' + \boxed{RX_0} \to P_0.$

$$\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} 0 \\ \frac{\sqrt{2}}{2} \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

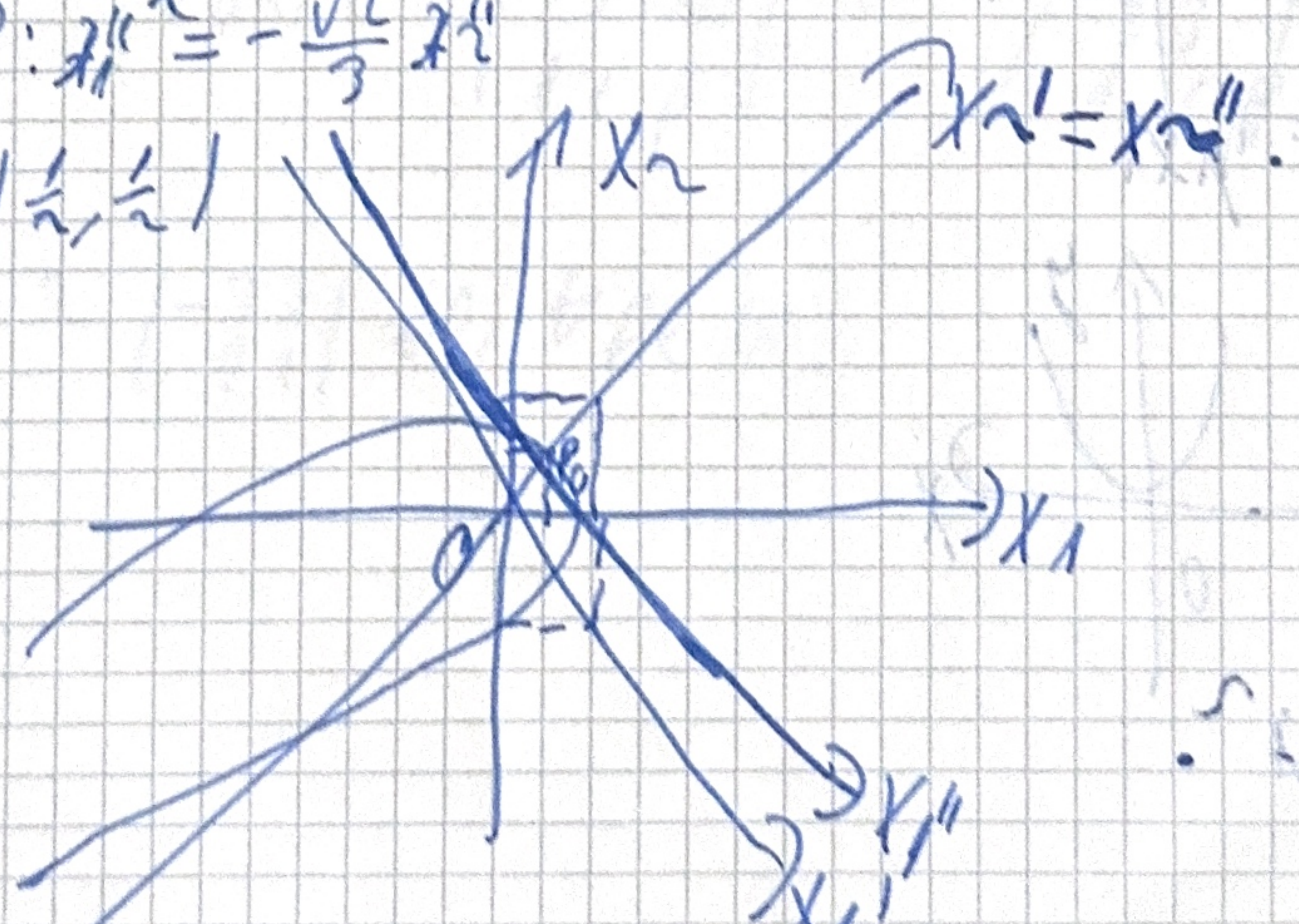
$P_0\left(\frac{1}{2}, \frac{1}{2}\right).$

$e_1' = \frac{1}{\sqrt{2}}(1, -1)$

$e_2' = \frac{1}{\sqrt{2}}(1, 1)$

$P: x_1''^2 = -\frac{\sqrt{2}}{3} x_2''$

$P_0\left(\frac{1}{2}, \frac{1}{2}\right)$



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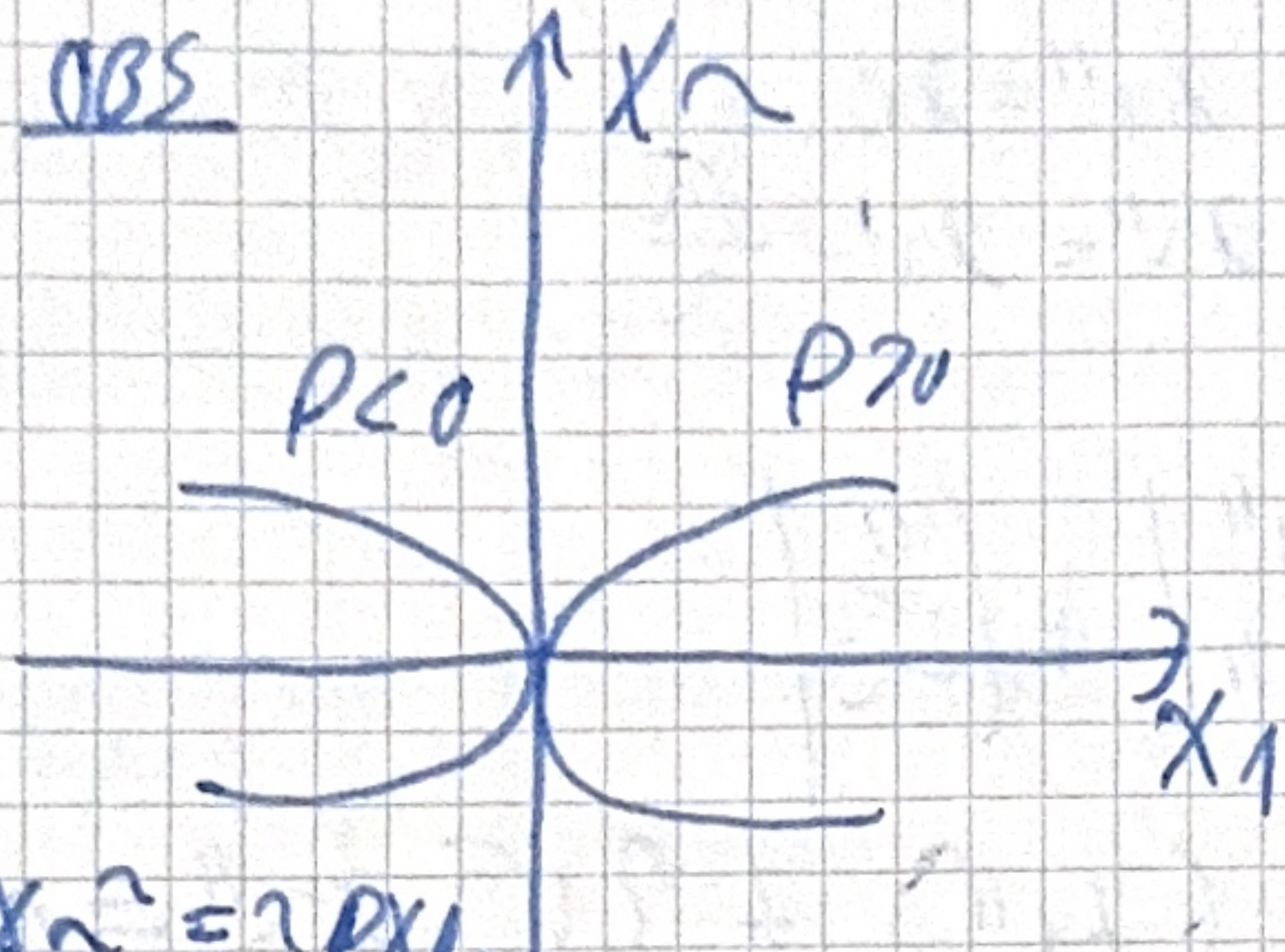
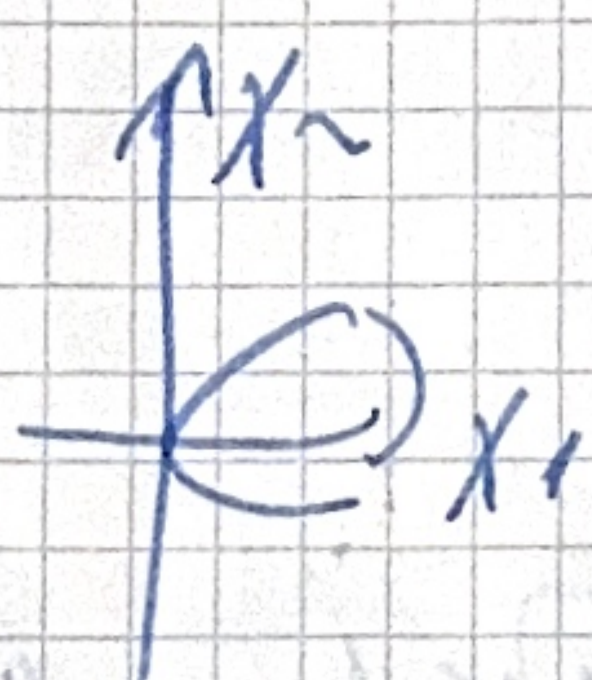



$X_2^2 = 2pX_1.$

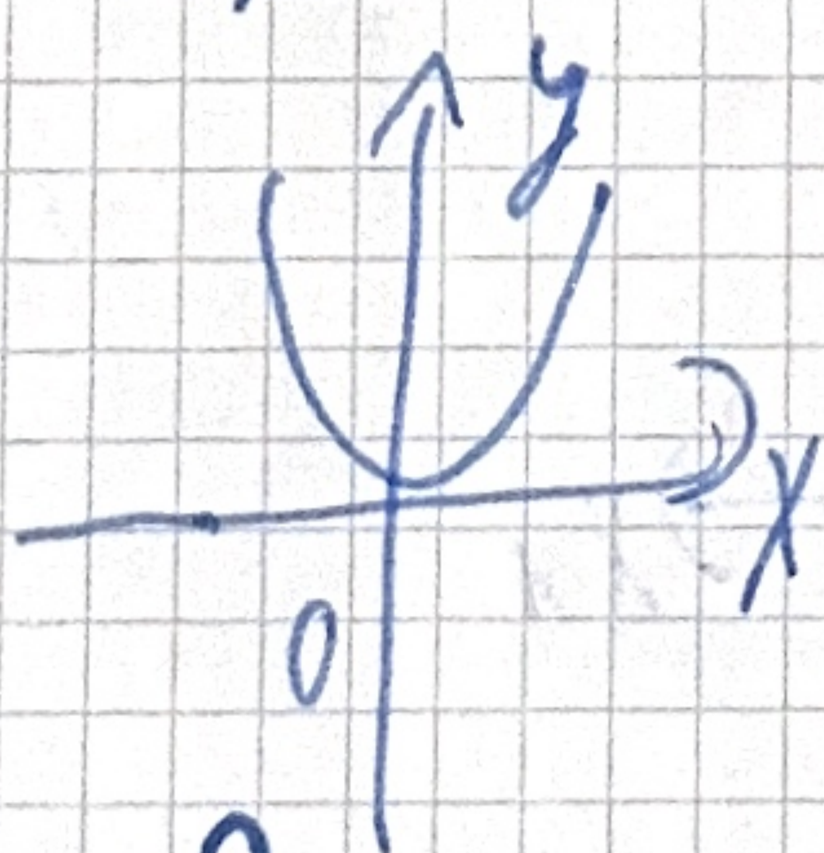
$$f(x) = \boxed{a_{11}x_1^2 + 2a_{12}x_1x_2 + a_{22}x_2^2} + 2b_1x_1 + 2b_2x_2 + c$$

$$\bar{A} = \begin{pmatrix} \boxed{A} & \begin{matrix} b_1 \\ b_2 \end{matrix} \\ b_1 \quad b_2 & c \end{pmatrix} \qquad f(x) = x^T A x + 2Bx + c = 0.$$

$$x = Rx' + Rx_0.$$




$X_2^2 = 2pX_1$




$y = x^2.$

$$f(x) = \underbrace{\boxed{a_{11}x_1^2 + 2a_{12}x_1x_2 + a_{22}x_2^2}}_{\lambda_1 x_1^2} + 2b_1x_1 + 2b_2x_2 + c.$$

$$\begin{pmatrix} \lambda & 0 \\ 0 & 0 \end{pmatrix}.$$

Ex Fie familia de conice $\Gamma_{\alpha,\beta}: f(x) = 3\alpha x_1^2 - 6x_1x_2 + 3x_2^2 + 2x_1 + 2x_2 + \beta = 0.$

a) $\alpha = ?$ a.î. conica nu are centru unic

$(\delta = 0)$

b) Pt $\alpha = 1$ aflați $\beta$ a.î. conica este nedegenerată

a) $\begin{vmatrix} 3\alpha & -3 \\ -3 & 3 \end{vmatrix} = 9\alpha - 9 = 9(\alpha - 1)$

$\alpha \in \{1\}$

b) $\Delta \neq 0$
$\forall \beta \in \mathbb{R}$

$$\begin{vmatrix} 3 & -3 & 1 \\ -3 & 3 & 1 \\ 1 & 1 & \beta \end{vmatrix} \neq 0 \Rightarrow \begin{vmatrix} 0 & 0 & 2 \\ -3 & 3 & 1 \\ 1 & 1 & \beta \end{vmatrix} =$$

$$= 2\begin{vmatrix} -3 & 3 \\ 1 & 1 \end{vmatrix} \neq 0, \forall \beta \in \mathbb{R}.$$

Ex. $u=(1,2,-1)$

$f \in End(\mathbb{R}^3)$

$f(x) = u \times x$.

a) $A = [f]_{R_0, R_0}$

b) $\dim Ker f = ?$; $\dim Im f = ?$

a) $f(x) = \begin{vmatrix} e_1 & e_2 & e_3 \\ 1 & 2 & -1 \\ x_1 & x_2 & x_3 \end{vmatrix} = (2x_3 + x_2, -x_1 - x_3, -2x_1 + x_2) =$
$= (x_2 + 2x_3, -x_1 - x_3, -2x_1 + x_2)$.

$$A = \begin{vmatrix} 0 & 1 & 2 \\ -1 & 0 & -1 \\ -2 & 1 & 0 \end{vmatrix}$$

$$|A| = \begin{vmatrix} \boxed{\begin{matrix} 0 & 1 \\ -1 & 0 \end{matrix}} & \begin{matrix} 2 \\ -1 \end{matrix} \\ \begin{matrix} -2 & 1 \end{matrix} & 0 \end{vmatrix} = \begin{vmatrix} 0 & 1 & 2 \\ -1 & 0 & 0 \\ -2 & 1 & 2 \end{vmatrix} = (-1)(-1) \begin{vmatrix} 1 & 2 \\ 1 & 2 \end{vmatrix} = 0.$$

$Ker f : AX = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

$Ker f = \{ x \in \mathbb{R}^3 \mid AX = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \}$.

$\dim Ker f = 3 - rg A = 3 - 2 = 1$.

$\Rightarrow \dim Im f = 1$.

$f : \mathbb{R}^3 \to \mathbb{R}^3$

$\dim \mathbb{R}^3 = \dim Ker f + \dim Im f$.

Ex $f \in End(\mathbb{R}^3)$ $(\mathbb{R}^3, g_0)$

$u = (2, 3, 2)$

$f(x) = u\, g_0(x, u)$

a) $f \in Sim(\mathbb{R}^3)$

b) $Q : \mathbb{R}^3 \to \mathbb{R}$ formă pătratică asociată

Să se aducă la formă canonică prin schimbare de rep. ortonormat

a) $f(x) = u\,(2x_1 + 3x_2 + 2x_3) =$
$= (4x_1 + 6x_2 + 4x_3,\ 6x_1 + 9x_2 + 6x_3,\ 4x_1 + 6x_2 + 4x_3)$.

$g_0(x, u) = 2x_1 + 3x_2 + 2x_3$.

$$A = \begin{pmatrix} 4 & 6 & 4 \\ 6 & 9 & 6 \\ 4 & 6 & 4 \end{pmatrix} = A^T \Rightarrow f \in Sim(\mathbb{R}^3)$$

b) $\det(A - \lambda I_3) = 0 \Leftrightarrow \lambda^3 - \underset{\substack{\parallel \\ Tr A = 17}}{\sigma_1} \lambda^2 + \sigma_2 \lambda - \underset{\substack{\parallel \\ \det A = 0}}{\sigma_3} = 0$

$\lambda^3 - 17\lambda^2 = 0 \Rightarrow \lambda_1 = 0 \quad m_1 = 2$
$\lambda_2 = 17 \quad m_2 = 1$

$$A = \begin{pmatrix} 0 & & \\ & 0 & \\ & & 17 \end{pmatrix}.$$

$V_{\lambda_1}$ reper arbitrar, Gram Schmitt.

$V_{\lambda_2}$ reper versor

$R_0 = \{e_1, e_2, e_3\}$

$\Downarrow C \in O(3)$

$R' = \{\underline{e_1', e_2'}, \underline{e_3'}\}$

$Q \in O(\mathbb{R}^3),\ C = [Q]_{R_0, R_0}$.

$e_1' = \frac{1}{\sqrt{2}}(1, 0, -1)$

$$C = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ -\frac{1}{\sqrt{2}} \end{pmatrix}$$

Ex $f: \mathbb{R}^3 \to \mathbb{R}^4,\ f(x_1, x_2, x_3) = (x_1 + x_2,\ x_1 - x_2,\ x_1, x_3)$.

a) $\operatorname{Ker}(f), \operatorname{Im}(f)$

$A = [f]_{R_0, R_0'}$

$$A = \begin{pmatrix} 1 & 1 & 0 \\ 1 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \in \mathcal{M}_{4,3}(\mathbb{R})$$

$\operatorname{Ker} f = \{x \in \mathbb{R}^3 / f(x) = 0\} \to \begin{cases} x_1 + x_2 = 0 \\ x_1 - x_2 = 0 \\ x_1 = 0 \\ x_3 = 0 \end{cases} \Rightarrow$

$\Rightarrow x_2 = 0 \Rightarrow \ker f = \{(0,0,0)\}$.

$\dim_{\mathbb{R}} \mathbb{R}^3 = \dim_{\mathbb{R}} \ker f + \dim_{\mathbb{R}} \operatorname{Im} f = 0 + \dim_{\mathbb{R}} \operatorname{Im} f =$
$= \dim_{\mathbb{R}} \operatorname{Im} f = 3 \Rightarrow \operatorname{Im} f = \mathbb{R}^3$.

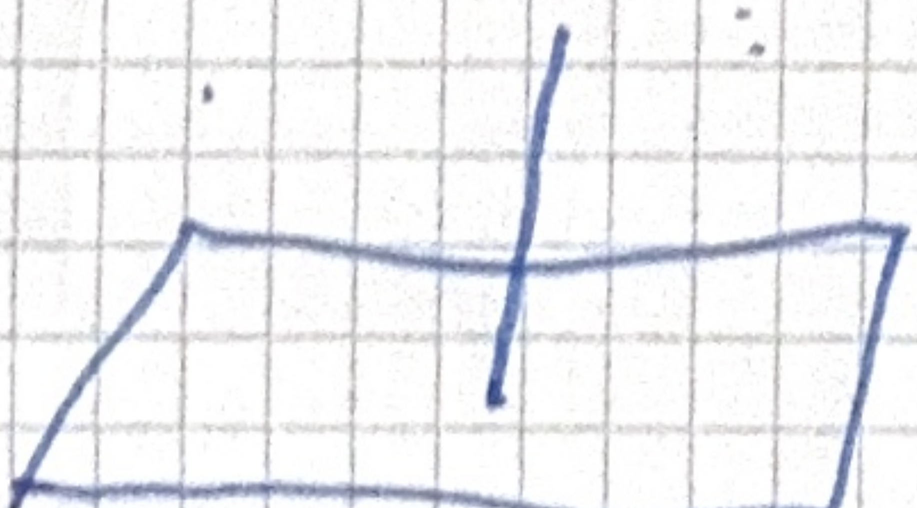
<u>Ex.</u> $(\mathbb{R}^3, +, \cdot)_{/\mathbb{R}}$

$(\mathbb{R}^3, g_0)$ $\quad g_0((x_1, x_2, x_3), (1, -1, 1))$.

$U = \{x \in \mathbb{R}^3 \mid x_1 - x_2 + x_3 = 0\}$.

a) Să se det $R = R_1 \cup R_2$ reper orton. în $\mathbb{R}^3$ a.î. $R_1$ rep. orton. în $U$ și $R_2$ rep. orton. $U^\perp$.

$U^\perp = \{u \in \mathbb{R}^3 \mid g_0(x, u) = 0, \forall x \in U\}$.



$U^\perp = \langle \{(1,-1,1)\} \rangle$.

$R_2 = \{\underbrace{\tfrac{1}{\sqrt{3}}(1,-1,1)}_{= e_3'}\}$.

$x_2 = x_1 + x_3 \Rightarrow U = \{(x_1, x_1 + x_3, x_3) \mid x_1, x_3 \in \mathbb{R}\} \Rightarrow$
$\Rightarrow U = \langle \{(1,1,0), (0,1,1)\} \rangle$.

$f_1 = (1,1,0)$

$f_2 = (0,1,1)$

$e_1 = f_1$

$e_1' = \frac{1}{\sqrt{2}}(1,1,0)$

$$e_2 = f_2 - \frac{g_0(f_2,e_1)}{g_0(e_1,e_1)} \cdot f_1 = (0,1,1) - \frac{1}{2}(1,1,0) =$$

$$= -\frac{1}{2}(1,-1,-2) = \frac{1}{2}(-1,1,2)$$

$e_2' = \frac{1}{\sqrt{6}}(-1,1,2)$

$B_1 = \{e_1', e_2'\}$

$\mathbb{R}^3 = U \oplus U^\perp$

$\Downarrow$

$x = \underset{\substack{\cap \\ U}}{\alpha} + \underset{\substack{\cap \\ U^\perp}}{\beta}$