Seminar 8.

1. Arătați că seria de funcții $\sum_{n=1}^{\infty} \text{arctg} \frac{2x}{x^2+n^4}$ converge uniform.

$\left| \text{arctg} \frac{2x}{x^2+n^4} \right|$

$\cancel{(x^2+n^4)^2 \geq 2n^2 x}$

$$\frac{2x}{x^2+n^4} \leq \left| \frac{2x}{x^2+n^4} \right|, \forall x \in \mathbb{R}, n \in \mathbb{N}^* \Big/ \text{arctg}(\,)$$

arctg - fct. cresc $\left(\text{arctg}' = \frac{1}{1+x^2} > 0 \right)$

$$\text{arctg} \frac{2x}{x^2+n^4} \leq \text{arctg} \left| \frac{2x}{x^2+n^4} \right| \forall x \in \mathbb{R}, n \in \mathbb{N}^*$$

Ineg. med.
$$\frac{x^2+n^4}{2} \geq \sqrt{x^2 \cdot n^4} = n^2 |x| \;\forall x \in \mathbb{R}, n \in \mathbb{N}^* \Leftrightarrow$$

$$|x^2+n^4| \geq 2n^2|x| \Big/ : |x^2+n^4| \Leftrightarrow \cancel{\frac{1}{|x^2+n^4|} \leq \frac{1}{2n^2|x|}} \Big/ \cdot 2|x|$$

$$1 \geq \frac{2n^2|x|}{|x^2+n^4|} \Big/ : n^2 \Rightarrow \frac{1}{n^2} \geq \frac{2|x|}{|x^2+n^4|} \Big/ \text{arctg}$$

$$\text{arctg} \frac{2|x|}{|x^2+n^4|} \leq \text{arctg} \frac{1}{n^2}$$

$$\text{arctg} \left| \frac{2x}{x^2+n^4} \right| \leq \text{arctg} \frac{1}{n^2}$$

Deci $\text{arctg} \left\{ \frac{2x}{x^2+n^4} \right\} \leq \text{arctg} \frac{1}{n^2}$

$$\left| \frac{2x}{x^2+n^4} \right| \leq \frac{1}{n^2} \Rightarrow -\frac{1}{n^2} \leq \frac{2x}{x^2+n^4} \leq \frac{1}{n^2}$$

$$\Rightarrow \operatorname{arctg}\left(-\frac{1}{n^2}\right) \le \operatorname{arctg}\frac{2x}{x^2+n^4} \le \operatorname{arctg}\frac{1}{n^2}$$

$$-\operatorname{arctg}\frac{1}{n^2}$$

$$\left|\operatorname{arctg}\frac{2x}{x^2+n^4}\right| \le \operatorname{arctg}\frac{1}{n^2}, \forall x\in\mathbb{R}, \forall n\in\mathbb{N}$$

Arătăm că $\operatorname{arctg} y \le y \quad \forall y\in[0,\infty)$

Fie $f:[0,\infty)\to\mathbb{R} \mid f(y)= ~~\operatorname{arctg} y~~ \, y - \operatorname{arctg} y$

$$f'(y) = -\frac{1}{y^2+1}+1$$

$$f'(y)>0 \quad \forall y\in[0,\infty)$$

y	0				∞
f'		+	+	+	+
f	0		$\nearrow$		

$\Rightarrow f(y)\ge 0 \ \forall y\in[0,\infty)$

$\Rightarrow \operatorname{arctg} y \le y \quad \forall y\in[0,\infty)$

$$\left|\cancel{\operatorname{arctg}\frac{2x}{x^2+n^4}}\right| \le \cancel{\frac{1}{n^2}}$$

$$\operatorname{arctg}\frac{1}{n^2} \le \frac{1}{n^2}$$

Alegem $\alpha_n = \frac{1}{n^2}$

Deoarece $\sum_n \frac{1}{n^2}$ econv avem, cf Crit Weiers

$\sum \operatorname{arctg} \frac{2x}{x^2+n^4}$ e uniform converg

2. Det mult de converg pt următ serii de puteri

a) $\sum_{n=1}^{\infty} \frac{1}{n \cdot 2^n} \cdot x^n$

$a_n = \frac{1}{n \cdot 2^n}$, $a_n > 0$

~~$\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \lim_{n\to\infty}$~~

$\lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} \sqrt[n]{\left|\frac{1}{n \cdot 2^n}\right|} = \lim_{n\to\infty} \frac{1}{2} \underbrace{\sqrt[n]{\frac{1}{n}}}_{\infty^0 = 1}$

$= \frac{1}{2}$

$R = \frac{1}{\frac{1}{2}} = 2$

$\Rightarrow (-2;2) \subset A \subset [-2;2]$

• Studiem convergența în 2

Seria devine: $\sum_{n=1}^{\infty} \frac{1}{n \cdot \cancel{2^n}} \cdot \cancel{2^n} \rightarrow$ divergentă (serie arm. gen cu $\alpha = 1$)

$\Rightarrow 2 \notin A$

• Stud. conv. în -2

Seria devine: $\sum_{n=1}^{\infty} \frac{1}{n \cdot \cancel{2^n}} \cdot (-1)^n \cdot \cancel{2^n}$

$\Rightarrow$ converg (crit lui Leibniz)

$\Rightarrow -2 \in A$

$A = [-2;2)$

b) $\sum_{n=1}^{\infty} \frac{(-1)^n n!}{(a+1)\dots(a+n)} x^n, a>1$

Sol Fie $a_n = \frac{(-1)^n \cdot n!}{(a+1)\dots(a+n)}$

$$\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \lim_{n\to\infty} \frac{(n+1)\cancel{!}}{\cancel{(a+1)}\dots\cancel{(a+n)}(a+n+1)} \cdot \frac{\cancel{(a+1)}\dots\cancel{(a+n)}}{\cancel{n!}}$$

$$\lim_{n\to\infty} \frac{n+1}{a+n+1} = 1$$

$$R = \frac{1}{1} = 1$$

$$(-1;1) \subseteq A \subseteq [-1;1]$$

- pt $x=-1$

$$\sum \frac{(-1)^n \cdot n! \, (-1)^n}{(a+1)\dots(a+n)} = \sum \frac{n!}{(a+1)\dots(a+n)}$$

$\downarrow (x_n)$

Aplicăm crit rap:

$$\lim_{n\to\infty} \frac{(n+1)\cancel{!}}{\cancel{(a+1)}\dots\cancel{(a+n)}(a+n+1)} \cdot \frac{\cancel{(a+1)}\dots\cancel{(a+n)}}{\cancel{n!}}$$

$\lim_{n\to\infty} \frac{n+1}{a+n+1} = 1 \Rightarrow$ nu putem spune nimic

$$\lim_{n\to\infty} \left[n \left(\frac{a+n+1}{n+1} - 1 \right) \right] = \lim_{n\to\infty} n \frac{a}{n+1} = a$$

$a>1$

>1

$\Rightarrow$ conv

$\Rightarrow -1 \in A$

- pt $x=1$

$$\sum \frac{(-1)^n \, n! \cdot 1}{(a+1)\dots(a+n)} = \sum \frac{(-1)^n \, n!}{(a+1)\dots(a+n)}$$

$\to x_n$

$\sum |x_n| = \sum \frac{n!}{(a+1)\dots(a+n)} \to$ conv (vezi mai sus)

$\Downarrow$

$\sum x_n$ absolut converg

Deci, $\sum x_n$ e și converg.

$1 \in A$

$A = [-1; 1]$

c) $\sum_{n=1}^{\infty} \frac{3^n}{\sqrt[3]{n}} (x+3)^n$

Notăm $x+3 = y$

$\sum_{n=1}^{\infty} \frac{3^n}{\sqrt[3]{n}} y^n$

$a_n = \frac{3^n}{\sqrt[3]{n}}$

$\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \lim_{n\to\infty} \frac{3^{n+1}}{\sqrt[3]{n+1}} \cdot \frac{\sqrt[3]{n}}{\cancel{3^n}} = 3$

$\Rightarrow R = \frac{1}{3}$ B = mult de conv $\sum_{n=1} \frac{3^n}{\sqrt[3]{n}} y^n$

$\left(-\frac{1}{3}; \frac{1}{3}\right) \subseteq B \subseteq \left[-\frac{1}{3}; \frac{1}{3}\right]$

• Pt $-\frac{1}{3}$

Seria devine $\sum_{n=1}^{\infty} \frac{\cancel{3^n}}{\sqrt[3]{n}} \cdot \frac{(-1)^n}{\cancel{3^n}} = \sum_{n=1}^{\infty} (-1)^n \frac{1}{\sqrt[3]{n}}$

conv $\Rightarrow -\frac{1}{3} \in B$

• Pt $\frac{1}{3}$

seria devine $\sum_{n=1}^{\infty} \frac{\cancel{3^n}}{\sqrt[3]{n}} \cdot \frac{1}{\cancel{3^n}} = \sum_{n=1}^{\infty} \frac{1}{n^{1/3}} \to$ diverg

$\Rightarrow +\frac{1}{3} \notin B$

$B = \left[-\frac{1}{3} ; \frac{1}{3}\right)$

A = mult de conv a lui $\sum_{n=1}^{\infty} \frac{3^n}{\sqrt[3]{n}} (x+3)^n$

$x+3 = y$

$y \in B = \left[-\frac{1}{3} ; +\frac{1}{3}\right) \Rightarrow -\frac{1}{3} \le x+3 < \frac{1}{3} \quad |(-3)$

$-\frac{10}{3} \le x < -\frac{8}{3}$

$\Rightarrow A = \left[-\frac{10}{3} ; -\frac{8}{3}\right)$

d) $\sum_{n=1}^{\infty} \frac{(-1)^n}{3n+1} (x-2)^n$

Sol : de rezolvat!

e) $\sum_{n=1}^{\infty} \frac{(-1)^n}{2n} x^{2n} = \frac{(-1)^1}{2\cdot 1} x^2 + \frac{(-1)^2}{2\cdot 2} x^4 + \frac{(-1)^3}{2\cdot 3} x^6$

Sol $\sum_{n=1}^{\infty} \frac{(-1)^n}{2n} x^{2n} = \sum_{n=1}^{\infty} a_k x^k$

$$a_k = \begin{cases} \frac{(-1)^n}{2n}, & k = 2n \quad (a_{2n}) \\ 0, & k = 2n-1 \quad (a_{2n-1}) \end{cases}$$

$R = \frac{1}{\overline{\lim} \sqrt[k]{|a_k|}}$

$\overline{\lim} \sqrt[k]{|a_k|} = ?$

$\lim_{n\to\infty} \sqrt[2n]{\left|\frac{(-1)^n}{2n}\right|} = \lim_{n\to\infty} \sqrt[2n]{\frac{1}{2n}} = 1$

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$\lim_{n\to\infty} \sqrt[2n+1]{0} = 0$

$\mathbb{N}^* = 2\mathbb{N}^* \cup (2\mathbb{N}^* - 1)$

$\Rightarrow \overline{\lim} \sqrt[k]{|a_k|} = 1$

$R = \frac{1}{1} = 1$

Fie A mulț. de conv. a seriei de puteri din enunț

Avem $(-1,1) \subset A \subset [-1,1]$

Studiem dacă $-1 \in A$ și $1 \in A$

Dacă $x = 1$ seria devine $\sum \frac{(-1)^n}{2n} \cdot 1^{2n}$

$= \sum (-1)^n \cdot \frac{1}{2n}$ conv (Crit. lui Leibniz)

$\Rightarrow 1 \in A$

Dacă $x = -1$ $\sum_{n=1}^{\infty} \frac{(-1)^n}{2n} \cdot (-1)^{2n}$

$= \sum_{n=1}^{\infty} \frac{(-1)^n}{2n}$ conv $\Rightarrow -1 \in A$

Așadar $A = [-1,1]$ $\square$

3. Să se dezvolte în serie de puteri ale lui x fct. de mai jos:

a) $f: \mathbb{R} \to \mathbb{R}$ $f(x) = \sin x$ $I = \mathbb{R}$

$a = 0 \in I$

$f \in C^\infty(I)$

$f'(0) = 1$ $f'(x) = +\cos x$ — $4k+1$

$f''(0) = 0$ $f''(x) = -\sin x$ — $4k+2$

$f'''(0) = -1$ $f'''(x) = -\cos x$ — $4k+3$

$f^{(4)}(0) = 0$ $f^{(4)}(x) = \sin x$ — $4k$

$$f^{(n)}(x) = \begin{cases} \sin x, & n = 4k \\ \cos x, & n = 4k+1 \\ -\sin x, & n = 4k+2 \\ -\cos x, & n = 4k+3 \end{cases}$$

Cf. formulei lui Taylor cu rest Lagrange $\forall\, x \in \mathbb{R},\ x \neq 0,\ \forall\, n \in \mathbb{N},\ \exists\, c$ între 0 şi x a.î.

$$f(x) = f(0) + \frac{f'(0)}{1!}(x-0) + \dots + \frac{f^{(n)}(0)}{n!}(x-0)^n + \frac{f^{(n+1)}(c)}{(n+1)!}(x-0)^{n+1}$$

$$R(x) = \frac{f^{(n+1)}(c)}{(n+1)!}\, x^{n+1}$$

Fie $x \in \mathbb{R}^*$

Ar. că $\lim\limits_{n \to \infty} R(x) = 0$

Este suf. să ar. că $\lim\limits_{n \to \infty} |R(x)| = 0$

$$|R(x)| = \left| \frac{f^{(n+1)}(c)\, x^{n+1}}{(n+1)!} \right| \qquad \left|\begin{smallmatrix} \sin \\ -\sin \end{smallmatrix}\right|, \left|\begin{smallmatrix} \cos \\ -\cos \end{smallmatrix}\right|$$

$$= \frac{|f^{(n+1)}(c)| \cdot |x|^{n+1}}{(n+1)!} \leq \frac{|x|^{n+1}}{(n+1)!}$$

$$\lim_{n \to \infty} \frac{|x|^{n+2}}{(n+2)!} \cdot \frac{\cancel{(n+1)!}}{\cancel{|x|^{n+1}}} = 0 < 1$$

cf $\Rightarrow$ crit rap $\frac{|x|^{n+1}}{(n+1)!}$ e conv.

Deci, $\lim_{n\to\infty} |R(x)| = 0$, așadar $\lim_{n\to\infty} R(x) = 0$

Prin urmare, $f(x) = \sum \frac{f^{(n)}(0)}{n!}(x-0)^n$

$= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1} \quad \forall x \in \mathbb{R}$

$f(0) = 0 \qquad f'''(0) = -1$

$f'(0) = 1 \qquad f^{(4)}(0) = 0$

$f''(0) = 0$

$f(0) = 0$

$\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} 0^{2n+1} = 0 \quad \Big| \Rightarrow f(0) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} 0^{2n+1}$

Deci, $f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(x)}{n!} x^n \quad \forall x \in \mathbb{R}$

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