Consultatii

Monday, April 26, 2021 5:11 PM

1. a) $\mathbb{Z}_2[X]/(X^2+X) \simeq \mathbb{Z}_2\times\mathbb{Z}_2$

$$\mathbb{Z}_2[X]/(X)\cdot(X+\hat{1}) \overset{LCR}{\simeq} \mathbb{Z}_2[X]/(X) \times \mathbb{Z}_2[X]/(X+\hat{1}) \simeq \mathbb{Z}_2\times\mathbb{Z}_2$$

$\hat{1}\cdot X + \hat{1}\cdot(X+\hat{1}) = \hat{1} \in U(\mathbb{Z}_2[X])$

$\Rightarrow (X)+(X+\hat{1}) = \mathbb{Z}_2[X]$

$\forall$ inel com R, $R[X]/(X-a) \simeq R$

b) $\mathbb{Z}_2[X]/(X^2+\hat{1}) = \{\overline{a+bX} \mid a,b\in\mathbb{Z}_2\}$ ← 4 elemente

Teorema de împărțire cu rest.

$$\mathbb{Z}_2[X]/(X^2+\hat{1}) \not\simeq \mathbb{Z}_2\times\mathbb{Z}_2$$

<u>Var 1</u> $\bar{X}^2 = \hat{1}$ în $\mathbb{Z}_2[X]/(X^2+\hat{1})$, deci $\bar{X}$ nu este idempotent

Dar <u>toate</u> elementele lui $\mathbb{Z}_2\times\mathbb{Z}_2$ sunt idempotente!

<u>Var 2</u> $(\overline{X+\hat{1}})^2 = \overline{X^2+\hat{1}} = \hat{0}$, deci $X+\hat{1}$ nilpotent nenul

Dar în $\mathbb{Z}_2\times\mathbb{Z}_2$, nu există nilpotent nenul.

<u>Ex 1.2.</u> $f\in\mathbb{Z}[X]$ a.î. $a,b\in\mathbb{Z}$ cu $f(a), f(b)$ și $a+b$ impare.

$\Rightarrow f$ nu are rădăcini întregi.

<u>Idem</u> Fie $\bar{f}$ polinomul f trecut în $\mathbb{Z}_2[X]$.

$$\left(\begin{tikzcd} a \arrow[r, mapsto] & \hat{a} & \\ \mathbb{Z} \arrow[r] \arrow[d, hook, bend right] & \mathbb{Z}_2[X] \ni & X \\ \mathbb{Z}[X] \ni X \arrow[ur] & & \bar{X} \arrow[u, mapsto] \end{tikzcd}\right)$$

$a+b$ impar $\Rightarrow \hat{a} \neq \hat{b}$ în $\mathbb{Z}_2$
$$\Updownarrow$$
$$\boxed{\begin{array}{ll} \bar{f}(\hat{a}) = \hat{1} & \{\hat{a}, \hat{b}\} = \{\hat{0}, \hat{1}\} \\ \bar{f}(\hat{b}) = \hat{1} & \end{array}}$$

$\bar{f}$ nu are rădăcini în $\mathbb{Z}_2[X]$

$$\Downarrow$$

f nu are rădăcini

<u>Var 2</u> $f(X) = g(X)(X-a) + f(a)$, $g \in \mathbb{Z}[X]$

$f(X) = h(X)(X-b) + f(b)$

Pp că $\exists\, \ell \in \mathbb{Z}$ rădăcină

$$\Rightarrow \underset{0}{f(\ell)} = g(\ell)(\ell - a) + f(a)$$

$$h(\ell)(\ell - b) + f(b)$$

$$f(X) = f_1(X)(X-\ell)$$

Exc 1.3. R inel $\quad I \trianglelefteq R, \quad \sqrt{I} = \{x \in R \mid \exists\, n \in \mathbb{N}^*,\ x^n \in I\}$

a) $\sqrt{I} \trianglelefteq R,\quad \sqrt{I} \supseteq I.\quad \left(\sqrt{A} = \{x \in R \mid \exists\, n \in \mathbb{N}^*,\ x^n \in A\}\right)$

Dem $\cdot\ x_1, x_2 \in \sqrt{I} \Rightarrow \begin{array}{l} x_1^{n_1} \in I \\ x_2^{n_2} \in I \end{array}$

$$(x_1 - x_2)^{n_1+n_2} = \sum_{\ell=0}^{n_1+n_2} C_{n_1+n_2}^{\ell}\, x_1^{\ell}\, x_2^{n_1+n_2-\ell}$$

$$= \sum_{\ell=0}^{n_1} C_{n_1+n_2}^{\ell}\, x_1^{\ell} \underbrace{x_2^{\overbrace{n_1+n_2-\ell}^{\geq n_2}}}_{\in I} + \sum_{\ell=n_1+1}^{n_1+n_2} C_{n_1+n_2}^{\ell} \underbrace{x_1^{\overbrace{\ell}^{\geq n_1}}}_{\in I} x_2^{n_1+n_2-\ell} \in I$$

$\forall\, r \in R,\ \forall\, x \in \sqrt{I}\ \left(x^n \in I \text{ pt } n \in \mathbb{N}^*\right),$

$$(rx)^n = r^n \underbrace{x^n}_{I} \in I$$

- $\sqrt{I} \supset I$ evident: $\quad \forall x \in I,\ x^1 \in I$

b) $\operatorname{Nil}(R) = \{x \in R \mid \exists\, n \in \mathbb{N}^*,\ x^n = 0\} = \sqrt{(0)} \trianglelefteq R$

c) $\sqrt{I}/I = \operatorname{Nil}(R/I)$

$$\sqrt{I}/I = \left\{ \hat{x} \in R/I \mid x \in \sqrt{I} \right\}$$

$\sqrt{I}/_{I} = \{ \ \ \hat{x} \in R/_{I} \ \ldots \}$

$$Nil(R/_{I}) = \{ \hat{x} \in R/_{I} \mid \underbrace{\exists\, n \in \mathbb{N}^* \text{ a.î. } \hat{x}^n = \hat{0} \iff x^n \in I}_{x \in \sqrt{I}} \}$$

d) Idealele lui $\mathbb{Z}$ pt care $\sqrt{I} = I$?

Fie $I \trianglelefteq \mathbb{Z} \Rightarrow I = n\mathbb{Z} = (n)$, pt un $n \in \mathbb{Z}$.

$$\sqrt{n\mathbb{Z}} = \{ a \in \mathbb{Z} \mid \underbrace{\exists\, n \in \mathbb{N}^* \text{ a.î. } a^n \in (n) \iff n \mid a^n}_{\iff \text{ orice factor prim al lui } n \text{ divide } a} \}$$

$n = p_1^{\alpha_1} p_2^{\alpha_2} \ldots p_k^{\alpha_k}$

$\Rightarrow \sqrt{n\mathbb{Z}} = (p_1 p_2 \ldots p_k)\mathbb{Z} \supset n\mathbb{Z}$

Ideale radicale $=$ $n\mathbb{Z}$ cu n liber de pătrate.

1.4. $A = \{ f \in \mathbb{Q}[X] \mid f \text{ nu are termen de grad } 1 \}$

$$\mathbb{Q}[X,Y]/_{(X^3 - Y^2)} \simeq A$$

Dem Idee: Să folosim T.F.I.:

Dem. Idee: Să folosim

$\varphi : \mathbb{Q}[X,Y] \longrightarrow A$ a.î. φ morfism surjectiv

și $\operatorname{Ker}\varphi = (X^3-Y^2)$

$$\varphi(P(X,Y)) = P(T^2,T^3) \in \mathbb{Q}[T].$$

- φ morfism (Prop. de universalitate) ✓
- φ corect definit și surjectiv:

Fie $f \in A$. $f = a_0 + a_2T^2 + a_3T^3 + \dots + a_nT^n$

$= a_0T^0 + a_3T^3 + a_6T^6 + \dots$ (0 mod 3)

$a_4T^4 + a_7T^7 + \dots$ (1 mod 3)

$a_2T^2 + a_5T^5 + a_8T^8 + \dots$ (2 mod 3)

$$= \varphi(a_0 + a_3Y + a_6Y^2 + \dots) +$$

$$+ \varphi(a_4X^2 + a_7X^2Y + a_{10}X^2Y^2 + \dots) +$$

$$+ \varphi(a_2X + a_5XY + a_8XY^2 + \dots)$$

- $\operatorname{Ker}\varphi = (X^3-Y^2)$,

"$\supseteq$" evident

"$\subseteq$" Fie $P \in \mathbb{Q}[X,Y]$, $P(T^2,T^3) = 0$

„$\subset$" Fie $P \in \mathbb{Q}[X,Y]$, ...

Privim $P\in(\mathbb{Q}[X])[Y]$ și scriem Th. de împărțire cu rest cu Y^2-X^3:

$$P(X,Y)= S(X,Y)\cdot(Y^2-X^3)+\underbrace{P_1(X)+P_2(X)\cdot Y}_{\text{polinom în } (\mathbb{Q}[X])[Y] \text{ de grad } <2.}$$

$$\varphi: 0=\varphi(P(X,Y)) = S(T^2,T^3)\cdot(T^6-T^6)+P_1(T^2)+P_2(T^2)\cdot T^3$$

Deci $\underbrace{P_1(T^2)}_{\text{puteri pare ale lui } T}+\underbrace{P_2(T^2)\cdot T^3}_{\text{puteri impare ale lui } T}=0$ în $\mathbb{Q}[T]$.

$$\Rightarrow P_1=P_2=0 \Longrightarrow P(X,Y)=S(X,Y)\,(X^3-Y^2).$$

1.5. Vedeți seminarul 9

1.6. Fie $R=\{f:[0,1]\to\mathbb{R},\ f \text{ continuă}\}$.

$\mathfrak{m}_c=\{f\in R \mid f(c)=0\}$. $\qquad I_0=\{f\in R \mid \operatorname{supp} f \text{ compact}\}=R$

Arătăm că $\mathfrak{m}_c$ e ideal maximal al lui R.

Fie $\varphi:R\to\mathbb{R}$, $\varphi(f)=f(c)$. φ evident morfism.

$$\Rightarrow R/_{\operatorname{Ker}\varphi}\simeq \operatorname{Im}\varphi$$

Dar $\mathrm{Ker}\,\varphi = \underline{m}_c \Rightarrow$ este ideal

$\mathrm{Im}\,\varphi = \mathbb{R}$ $\Big($ Fie $\alpha \in \mathbb{R}$. Atunci $\varphi(x-c+\alpha) = \alpha$ $\Big)$

sau $\varphi(\underbrace{\alpha \cdot \mathbb{1}_{[0,1]}}_{\text{fcția constantă } \alpha}) = \alpha$

corp!

$\Rightarrow \underline{m}_c$ este ideal maximal.

1.7. R aceluiași.

a) Orice ideal maximal al lui R este de tipul $\underline{m}_c$ pt un $c \in [0,1]$.

Fie $\underline{m}$ ideal maximal. Pp că $\underline{m} \neq \underline{m}_c$, $\forall c \in [0,1]$

$\overset{\underline{m} \text{ maximal}}{\Longleftrightarrow}$ $\underline{m} \not\subset \underline{m}_c$, $\forall c \in [0,1] \Longleftrightarrow \forall c \in [0,1]$, $\exists f_c \in \underline{m}$ a.î. $f_c(c) \neq 0$.

$\exists c \in [0,1]$ a.î. $\underline{m} \subset \underline{m}_c \overset{\underline{m} \text{ maximal}}{\Longleftrightarrow} \underline{m} = \underline{m}_c$

$\Updownarrow$ verificăm

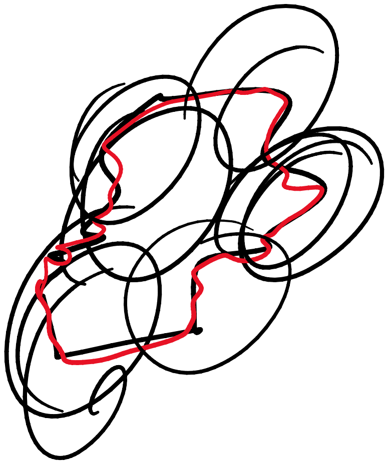
$\exists V_c \ni c$ a.î. $f_c(x) \neq 0$, $\forall x \in V_c$

$[0,1] \subset \bigcup_{c \in [0,1]} V_c \overset{[0,1] \text{ compactă}}{\Longrightarrow} \exists c_1, c_2, \dots c_n \in [0,1]$ a.î. $[0,1] \subset \bigcup_{i=1}^{n} V_{c_i}$.

$K \subset \mathbb{R}^n$ s.n. mulțime compactă dacă $\forall$ I mulțime și $(U_i)_{i \in I}$ acoperire a lui K: $K \subset \bigcup_{i \in I} U_i$, există

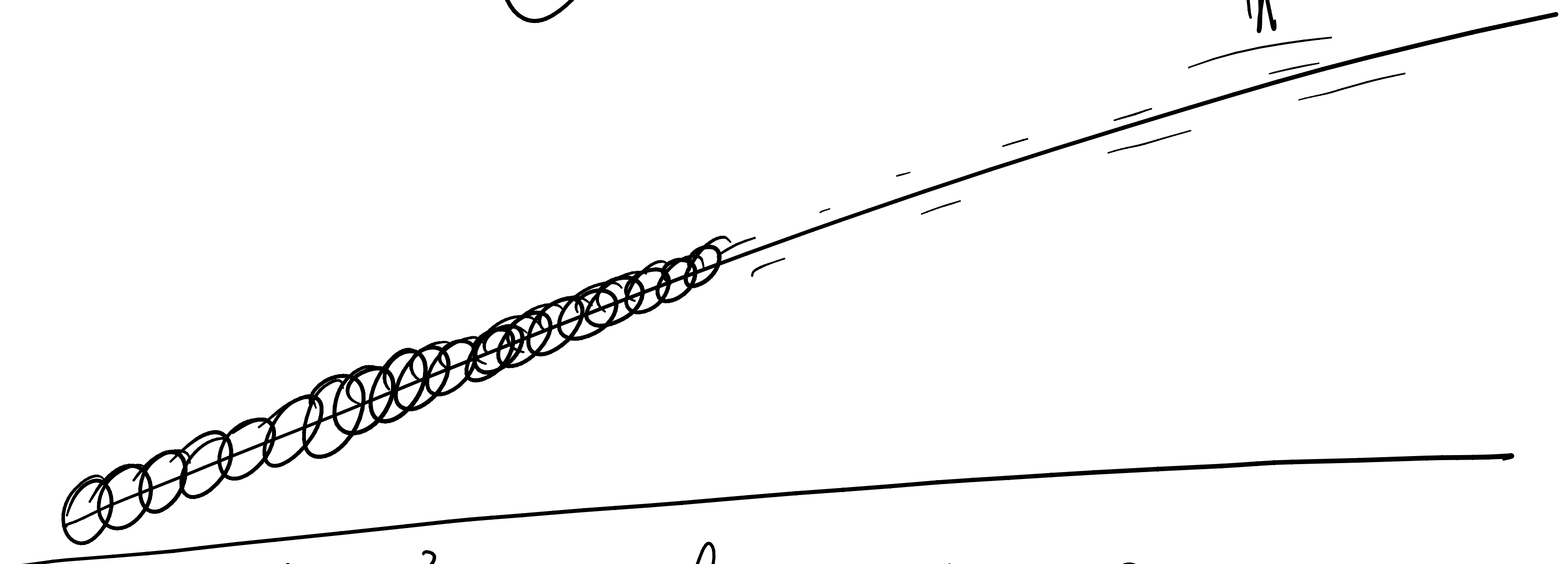
orice $(U_i)_{i\in I}$ acoperire a lui K: $K \subset \bigcup_{i\in I} U_i$, admite
cu deschiși

o subacoperire finită i.e. $\exists\, i_1, \dots, i_n \in I$ a.î. $K \subset \bigcup_{j=1}^{n} U_{i_j}$

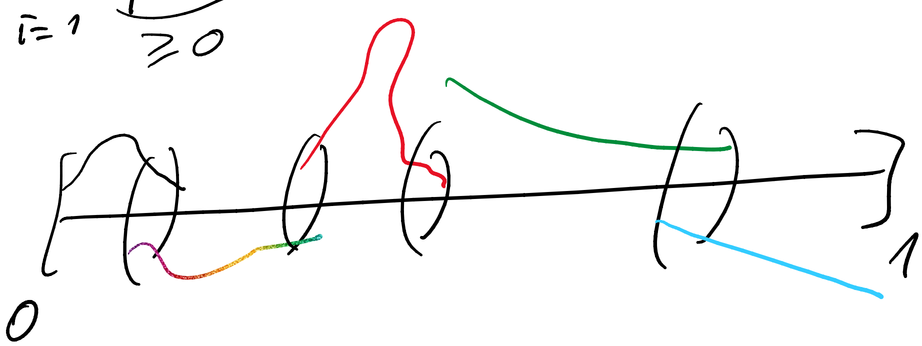


$\mathbb{R}^2$

$\mathbb{R}^2$



$$f = \sum_{i=1}^{n} \underbrace{f_{C_i}^{2}}_{\geq 0}, \quad f(x) \neq 0, \ \forall x \in [0,1]$$



$\Rightarrow \frac{1}{f} \in C(\mathbb{R})$. $f \in \underline{m}$ și $\underline{m}$ maximal $\Rightarrow \underline{m} \neq R$.

$\Rightarrow f \in U(\mathbb{R})$, $f \in \underline{m}$ abs $\underline{m}$ maximal $\Rightarrow \underline{m} \neq \mathbb{R}$.

b) $b \neq c \Rightarrow \underline{m}_b \neq \underline{m}_c$ evident

$\Downarrow$

$f(x) = x - b$

c) $\underline{m}_c \neq (x-c)$.

Fie $f = \sqrt{|x-c|} \in \underline{m}_c$. Pp că $f \in (x-c)$

$\Rightarrow \exists g : [0,1] \to \mathbb{R}$ continuă $f(x) = g(x)(x-c) \Leftrightarrow \sqrt{|x-c|} = g(x) \cdot (x-c)$

$$\Rightarrow \text{în } x \neq c, \quad \frac{\sqrt{|x-c|}}{x-c} = \begin{cases} \frac{1}{\sqrt{x-c}}, & \text{dc } x > c \\ -\frac{1}{\sqrt{x-c}}, & \text{dc } x < c \end{cases} = g(x)$$

$\Rightarrow \lim\limits_{x \searrow c} g(x) = \infty$ abs

$\lim\limits_{x \nearrow c} g(x) = -\infty$

d) $\underline{m}_c$ nu este finit generat

Pp că $\underline{m}_c = (f_1, f_2, \ldots f_n)$, $f_i : [0,1] \to \mathbb{R}$ continuă

$f_i(c) = 0$.

Fie $g = \sqrt{|f_1|} + \sqrt{|f_2|} + \dots + \sqrt{|f_n|} \in \underline{m}_c$

$g(c) = 0$ și $g(x) \neq 0, \ \forall x \neq c$ $\qquad \underline{m}_c = (f_1, \dots, f_n)$

(Dacă $g(d) = 0 \Rightarrow f_i(d) = 0 \Rightarrow \underline{m}_c \subset \underline{m}_d \overset{\max}{\Downarrow} \underline{m}_c = \underline{m}_d$ ↯ b)

$\underline{m}_c = (f_1, \dots, f_n) \Rightarrow g = \sqrt{|f_1|} + \dots + \sqrt{|f_n|} = g_1 f_1 + g_2 f_2 + \dots + g_n f_n.$

Fie $M = \max(|g_1|, \dots, |g_n|) > 0$

Th Weierstrass

def continuității

$\left.\begin{array}{l}\sqrt{|f_i|} \text{ continuă} \\ \in \underline{m}_c\end{array}\right\} \Rightarrow \exists\, V \ni c$ vecinătate a.î. $|f_i(x)| < \frac{1}{M}$, $\forall x \in V$

$|f_i(x)| < \frac{1}{M} \cdot \sqrt{|f_i|}$, $\forall i = \overline{1,n}$

$$\sum_{i=1}^{n} |f_i(x)| < \frac{1}{M} \sum_{i=1}^{n} \sqrt{|f_i(x)|}, \quad \forall x \in V \setminus \{c\}$$

$\forall x \in V \setminus \{c\}$,

$$|g(x)| = |g_1(x) f_1(x) + g_2(x) f_2(x) + \dots + g_n(x) f_n(x)|$$

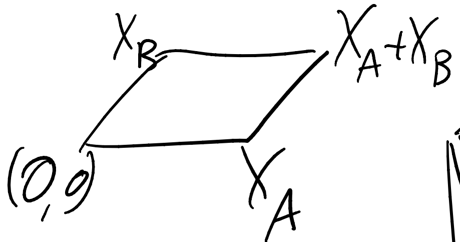
$$\leq M\left(|f_1(x)| + |f_2(x)| + \dots + |f_n(x)|\right)$$

$$< \sqrt{|f_1(x)|} + \sqrt{|f_2(x)|} + \dots + \sqrt{|f_n(x)|} = g(x)$$ db

$A, B, C, D \in \mathcal{C}$
paralelogram

$\mathcal{C}: {}^tXAX + 2b\cdot X + c = 0$

$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{12} & a_{22} \end{pmatrix}$, $b = (b_1 \ \ b_2)$



$(0,0) \in \mathcal{C} \iff c = 0$ $\qquad X_A = \begin{pmatrix} x_A \\ y_A \end{pmatrix}$

$X_A \in \mathcal{C}$.

$X_B \in \mathcal{C}$.

$X_A + X_B \in \mathcal{C}$. $\qquad \frac{X_A+X_B}{2}$

$\begin{pmatrix} x_p \\ y_p \end{pmatrix} = X_p \in \mathbb{R}^2$ e centru al conicei $\iff A X_p = -{}^tb$

Exc 1.8. Fie $R = \{ f : \mathbb{R} \to \mathbb{R} \mid f \text{ continuă} \}$. $m_c = \{ f \in R \mid f(c) = 0 \}$

$\forall f \in R$, $\quad supp(f) = \overline{\{ x \in \mathbb{R} \mid f(x) \neq 0 \}}$ $\iff$ mărginit

$I_0 = \{ f \in R \mid supp(f) \text{ compact} \}$

a) $I_0 \trianglelefteq R$

$supp(f+g) \subset (supp(f) \cup supp(g))$ ✓

$supp(f \cdot g) \subset (supp(f) \cap supp(g))$ ✓

$$\mathrm{supp}(f\cdot g) \subset (\mathrm{supp}(f) \cap \mathrm{supp}(g))$$ ✓

Deci, $\forall f \in I_0,\ \forall g \in R,\ \mathrm{supp}(fg) \subset \mathrm{supp}(f)$ mărginit

$\forall f, g \in I_0,\ \mathrm{supp}(f-g) \subset \mathrm{supp}\, f \cup \mathrm{supp}(g)$ mărginit

b) R/I_0 nu este domeniu de integritate

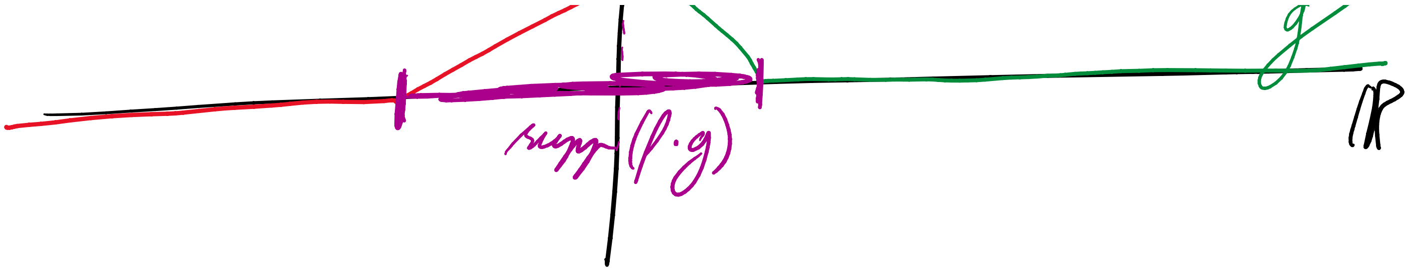
<u>**Obs**</u> R inel, $I \trianglelefteq R$

R/I domeniu $\iff \forall \hat{a}, \hat{b} \in R/I,\ \hat{a}\hat{b} = \hat{0} \Rightarrow \hat{a} = \hat{0}$ sau $\hat{b} = \hat{0}$

$$\iff \underbrace{\forall a, b \in R,\ ab \in I \Rightarrow a \in I \text{ sau } b \in I}_{I \text{ s.n. } \underline{\text{ideal prim}}}$$

$$\begin{array}{ccc} R/I \text{ domeniu} & \iff & I \text{ prim} \\ & & \Uparrow \\ R/I \text{ corp} & \iff & I \text{ maximal} \end{array}$$

Contraexemplu la R/I_0 domeniu:



c) Lema lui Krull: $\exists\ \underline{m}$ maximal a.î. $I_0 \subset \underline{m}$.

Arăt că $\underline{m} \neq \underline{m}_c$, $\forall c \in \mathbb{R}$.

Evident, nu toate funcțiile cu supp compact se anulează într-un punct comun!

Ex 1.9. K corp și $n \geq 1$.

Ideale în $K[X_1, \dots, X_n]$	Submulțimi în K^n
$I \longmapsto$	$Z(I) = \{x \in K^n \mid f(x) = 0, \forall f \in I\}$
$\{f \in K[X_1, \dots, X_n] \mid f(x) = 0\ \forall x \in A\} = \mathcal{J}(A) \longleftarrow$	A

a) $Z(\mathcal{J}(A)) \supset A$

$\forall x \in A$, $\forall f \in \mathcal{J}(A)$, $f(x) = 0 \Rightarrow x \in Z(\mathcal{J}(A))$.

b) $\mathcal{J}(Z(I)) \supset \sqrt{I}$

b) $\mathcal{I}(Z(I)) \supset \sqrt{I}$

Dem: Fie $f \in \sqrt{I} \Rightarrow f^n \in I$ pt un $n \in \mathbb{N}^*$.

Fie $x \in Z(I)$ | Vrem $f(x) = 0$.

Dar $f^n \in I$ } $\Rightarrow f^n(x) = 0 \overset{K \text{ corp}}{\Rightarrow}$

Teoremă (Strong Nullstellensatz, Hilbert)

K algebric închis ($\forall$ polinom $f \in K[X]$, $\deg f \geq 1$ are cel puțin o rădăcină în K)

$\Downarrow$

$\mathcal{I}(Z(I)) = \sqrt{I}$.

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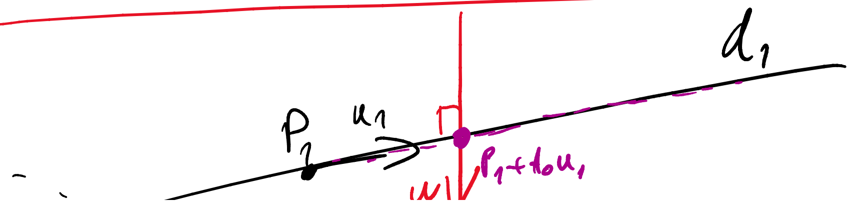
c) $\{Z(I) \mid I \trianglelefteq K[X_1, \ldots, X_n]\}$ sunt mulțimile închise într-o topologie

i.e. $Z(I) \cup Z(J) = Z(I \cdot J)$ $\qquad$ $I \subset J \Rightarrow Z(I) \supset Z(J)$

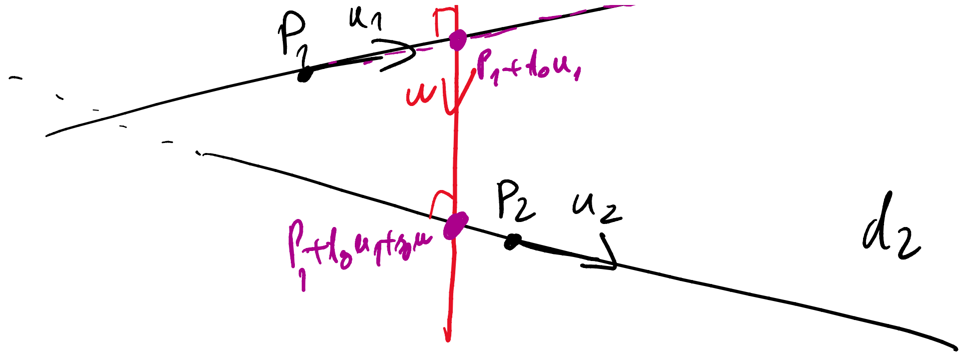
$$\bigcap_{j \in J} Z(I_j) = Z\Big(\sum_{j \in J} I_j\Big)$$

$$K^n = Z((0))$$

$$\emptyset = Z(K[X_1, \ldots, X_n])$$



$\mathbb{R}^3$



d are directie $u_1 \times u_2 \longrightarrow$

$P \in d_1: \quad P_1 + t u_1$