$$u \times v = \begin{vmatrix} e_1 & e_2 & e_3 \\ 2 & 3 & 1 \\ 4 & 1 & 3 \end{vmatrix} = (8, -2, -10)$$

$$\left. \begin{array}{l} 8x_1 - 2x_2 - 10x_3 + d = 0 \\ A(1,-1,2) \in \pi \end{array} \right| \Rightarrow 8 + 2 - 20 + d = 0$$

$$d = 10.$$

Seminar 12 Transformări ortogonale
Endomorfisme simetrice

② $(\mathbb{R}^3, g_0)$

$u = (1,1,0)$

a) $U^{\perp} = ?$ reper ortonormat?

b) $\varphi = \frac{\pi}{2}$ $f \in O(\mathbb{R}^3)$ spetă 1 = rep

Axa $= \langle u \rangle$

$U^{\perp} = \{x \in \mathbb{R}^3 \mid g_0(x,u) = 0\} = \{(-x_2, x_2, x_3) \mid x_2, x_3 \in \mathbb{R}\} =$

$\Rightarrow x_1 u_1 + x_2 u_2 + x_3 u_3 = 0$

$x_1 + x_2 = 0 \Rightarrow x_1 = -x_2$

$= \langle \{ \overset{f_2}{(-1,1,0)}, \underset{f_3}{(0,0,1)} \} \rangle$ SG

$\mathbb{R}^3 = U \oplus U^{\perp}$

$\Rightarrow \dim U^{\perp} = 2 \Rightarrow$ SLI

$\Rightarrow$ reper

Aplicăm GS

$e_2 = f_2 = (-1,1,0)$ $\qquad e_2' = \frac{(-1,1,0)}{\sqrt{2}}$

$e_3 = f_3 - \frac{\overset{=0}{\langle f_3, e_2 \rangle}}{\langle e_2, e_2 \rangle} e_2 = f_3$ $\qquad e_3' = (0,0,1)$

$\{e_2', e_3'\}$ reper ortonormat din $U^\perp$

$$e_1' = \frac{(1,1,0)}{\sqrt{2}}$$

b) $\mathcal{R} = \{e_1', e_2', e_3'\}$

$$A' = [f]_{\mathcal{R}',\mathcal{R}'} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\varphi & -\sin\varphi \\ 0 & \sin\varphi & \cos\varphi \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}$$

$$A = [f]_{\mathcal{R}_0,\mathcal{R}_0}$$

$$\mathcal{R}_0 \xrightarrow{C} \mathcal{R}'$$

$$C = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$A' = C^T A C$$

$$A = [f]_{\mathcal{R}_0\mathcal{R}_0} = CA'C^T = \frac{1}{2} \begin{pmatrix} 1 & -1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & \sqrt{2} \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} 1 & 1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & \sqrt{2} \end{pmatrix} =$$

$$= \frac{1}{2} \begin{pmatrix} 1 & 1 & \sqrt{2} \\ 1 & 1 & -\sqrt{2} \\ -\sqrt{2} & \sqrt{2} & 0 \end{pmatrix}$$

$f:\mathbb{R}^3\to\mathbb{R}^3$

$f(x)=\frac{1}{2}\left(x_1+x_2+\sqrt{2}x_3,\ x_1+x_2-\sqrt{2}x_3,\ \sqrt{2}x_1+\sqrt{2}x_2\right)$

• $(\mathbb{R}^3, g_0)$ $\quad f\in End(\mathbb{R}^3)$

$$A=[f]_{R_0,R_0}=\begin{pmatrix}1&0&1\\0&1&0\\1&0&1\end{pmatrix}$$

a) $f\in Sim(\mathbb{R}^3)$

b) Să se afle $Q:\mathbb{R}^3\to\mathbb{R}$ forma pătratică asociată lui f

c) Q la o formă canonică, efectuând o transf (schimb de reper ortonormat)

a) $A=A^t \Rightarrow f\in Sim(\mathbb{R}^3$

$f:\mathbb{R}^3\to\mathbb{R}^3 \quad f(x)=(x_1+x_3, x_2, x_1+x_3)$

b) $Q(x)=g_0(x,f(x))$

$Q(x)=x_1^2+x_2^2+x_3^2+2x_1x_3$

metoda valorilor proprii

$P_A(\lambda)=\det(A-\lambda I_3)$

$$\det(A-\lambda I_3)=\begin{vmatrix}1-\lambda&0&1\\0&1-\lambda&0\\1&0&1-\lambda\end{vmatrix}=(1-\lambda)\begin{vmatrix}1-\lambda&1\\1&1-\lambda\end{vmatrix}=$$

$$=(1-\lambda)[(1-\lambda)^2-1]=(1-\lambda)(1-\lambda-1)(1-\lambda+1)=$$

$$=(1-\lambda)\lambda(2-\lambda)=0$$

$\lambda_1=0 \quad m_1=1$

$\lambda_2=1 \quad m_2=1$

$\lambda_3=2 \quad m_2=1$

$$V_{\lambda_1} = \{ x \in \mathbb{R}^3 \mid \underbrace{AX = 0 \cdot X}_{\ker f} \}$$

$$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{cases} x_1 = -x_3 \\ x_2 = 0 \end{cases}$$

$$V_{\lambda_1} = \{ (-x_3, 0, x_3) \mid x_3 \in \mathbb{R} \} = \langle \{(-1, 0, 1)\} \rangle$$

$$e_1 = \frac{1}{\sqrt{2}} (-1, 0, 1)$$

$$V_{\lambda_2} = \{ x \in \mathbb{R}^3 \mid AX = X \} \qquad \mathbb{R}^3 = V_{\lambda_1} \oplus V_{\lambda_2} \oplus V_{\lambda_3}$$

$$(A - I_3) X = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{cases} x_3 = 0 \\ x_1 = 0 \end{cases} \qquad V_{\lambda_2} = \{ (0, x_2, 0) \mid x_2 \in \mathbb{R} \} = \langle \{(0, 1, 0)\} \rangle$$

$$e_2 = (0, 1, 0)$$

$$V_{\lambda_3} = \{ x \in \mathbb{R}^3 \mid AX = 2X \}$$

$$(A - 2I_3) X = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} -1 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{cases} -x_1+x_3=0 & \Rightarrow x_1=x_3 \\ -x_2=0 & \Rightarrow x_2=0 \\ x_1-x_3=0 \end{cases}$$

$$V_{\lambda_3}=\{(x_1,0,x_1) \mid x_1\in\mathbb{R}\}=\langle\{(1,0,1)\}\rangle$$

$$e_3=\frac{1}{\sqrt{2}}(1,0,1)$$

$$R_0=\{e_1^\circ,e_2^\circ,e_3^\circ\}\xrightarrow{C} R_1=\{e_1=\frac{1}{\sqrt{2}}(-1,0,1),\ e_2=(0,1,0),\ e_3=\frac{1}{\sqrt{2}}(1,0,1)\}$$

$C\in O(3)$

$$C=\frac{1}{\sqrt{2}}\begin{pmatrix}-1&0&1\\0&\sqrt{2}&0\\1&0&1\end{pmatrix} \qquad A'=[f]_{R,R}=\begin{pmatrix}0&0&0\\0&1&0\\0&0&2\end{pmatrix}$$

$h\in O(\mathbb{R}^3)$

$$h(x)=\frac{1}{\sqrt{2}}(-x_1+x_3,\ \sqrt{2}x_2,\ x_1+x_3)$$

sau $h\in O(\mathbb{R}^3)$, $h(e_i^\circ)=e_i$, $i=\overline{1,3}$

$$Q(x)=x_2'^2+2x_3'^2 \quad \text{sig } (2,0)$$

4) $(\mathbb{R}^3, g_0)$, $f:\mathbb{R}^3\to\mathbb{R}^3$

$f(x)=g_0(x,u)\,u$, $u=(1,-1,2)$

a) $f\in \mathrm{Sim}(\mathbb{R}^3)$, $f=?$

b) $Q=?$ (met val proprii)

a) $g_0(x,u)=x_1-x_2+2x_3$

$$\Rightarrow f(x)=(x_1-x_2+2x_3)\cdot(1,-1,2)=$$

$$=(x_1-x_2+2x_3,\ -x_1+x_2-2x_3,\ 2x_1-2x_2+4x_3)$$

$$[f]_{R_0,R_0} = A = \begin{pmatrix} 1 & -1 & 2 \\ -1 & 1 & -2 \\ 2 & -2 & 4 \end{pmatrix} = A^t \Rightarrow f \in Sim(\mathbb{R}^3)$$

b) $Q : \mathbb{R}^3 \to \mathbb{R}$, $Q(x) = x_1^2 + x_2^2 + 4x_3^2 - 2x_1x_2 + 4x_1x_3 - 4x_2x_3$

(vezi ex precedent)

- $f(x) = u \times x$, $u = (1,2,3)$

$$u \times x = \begin{vmatrix} e_1 & e_2 & e_3 \\ 1 & 2 & 3 \\ x_1 & x_2 & x_3 \end{vmatrix} = e_1(2x_3 - 3x_2) + e_2(x_3 - 3x_1) + e_3(x_2 - 2x_1)$$

$$\Rightarrow f(x) = (-3x_2 + 2x_3, 3x_1 - x_3, -2x_1 + x_2)$$

$$A = [f]_{R_0,R_0} = \begin{pmatrix} 0 & -2 & 2 \\ 3 & 0 & -1 \\ -2 & 1 & 0 \end{pmatrix}$$ f nu se poate diagonaliza (evident)

Geometrie analitică euclidiană

- $(\overset{\mathcal{A}}{\overset{\shortparallel}{\mathbb{R}^3}}, (\overset{V}{\overset{\shortparallel}{\mathbb{R}^3}}, g_0), \varphi)$ sp afin euclidian canonic

$g_0 : \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}$, $g_0(x,y) = x_1y_1 + x_2y_2 + x_3y_3$ (prod scalar canonic)

$\varphi : \underset{\mathcal{A}}{\underset{\shortparallel}{\mathbb{R}^3}} \times \underset{\mathcal{A}}{\underset{\shortparallel}{\mathbb{R}^3}} \to \underset{V}{\underset{\shortparallel}{\mathbb{R}^3}}$ $\varphi(u,v) = v - u$ struct afină canonică ex:

$A(3,-1,3)$, $B(5,1,-1)$, $u=(-3,5,-6)$

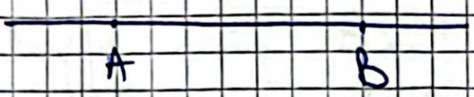
a) ec. dreptei $\mathcal{D}$ a.î. $A\in\mathcal{D}$ și $V_{\mathcal{D}}=\langle\{u\}\rangle$

b) ec. dr. AB

c) $\mathcal{D}\cap$ planele de coord

$\mathcal{D}$

$M(x_1,x_2,x_3)$, $(\forall)\, M\in\mathcal{D}$



$\mathcal{R}=\{O, e_1, e_2, e_3\}$ reper cartezian

$\vec{OA}=3e_1-e_2+3e_3$

$(\exists)\, t\in\mathbb{R}$ a.î. $\vec{AM}=tu=t(-3,5,-6)$

(x_1-3, x_2+1, x_3-3)

$$\mathcal{D}\begin{cases} x_1-3=3t \\ x_2+1=5t \\ x_3-3=-6t \end{cases}, \; t\in\mathbb{R} \text{ (ec. parametrice)}$$

$$\mathcal{D}: \frac{x_1-3}{-3}=\frac{x_2+1}{5}=\frac{x_3-3}{-6}=t$$

b) $u_{AB}=\vec{AB}=(5-3, 1+1, -1-3)=(2,2,-4)$

$$AB: \begin{cases} x_1-3=2t \\ x_2+1=2t \\ x_3-3=-4t \end{cases}$$

$$AB: \frac{x_1-3}{2}=\frac{x_2+1}{2}=\frac{x_3-3}{-4}=t$$

c) $\mathcal{D} \cap (Ox_1x_2)$

$x_3 = 0$

$\begin{cases} \pi : ax_1 + bx_2 + cx_3 + d = 0 \\ \pi \cap \mathcal{D} \\ \mathcal{D} \text{ cu ect parametrice} \end{cases}$

$\mathcal{D} : \begin{cases} x_1 = -3t + 3 \\ x_2 = 5t - 1 \\ x_3 = -6t + 3 \end{cases}$

$\Rightarrow 3 - 6t = 0 \Rightarrow t = \frac{1}{2}$

$P\left(3 - \frac{3}{2}, \frac{5}{2} - 1, 0\right)$

$P\left(\frac{3}{2}, \frac{3}{2}, 0\right)$

$\mathcal{D} \cap (Ox_1x_3)$

$x_2 = 0 \Rightarrow -1 + 5t = 0 \Rightarrow t = \frac{1}{5}$

$Q\left(3 - \frac{3}{5}, 0, 3 - \frac{6}{3}\right)$

$Q\left(\frac{12}{5}, 0, \frac{9}{3}\right)$

$\mathcal{D} \cap (Ox_2x_3)$

$x_1 = 0 \Rightarrow 3 - 3t = 0 \Rightarrow t = 1$

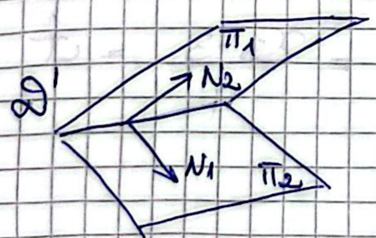
$S(0, -1+5, 3-6)$

$S(0, 4, -3)$

- $\mathcal{D} = ?$ a.î. $A(2, -5, 3) \in \mathcal{D}$ și $\mathcal{D} \parallel \mathcal{D}'$

$\mathcal{D}' : \begin{cases} 2x_1 - x_2 + 3x_3 + 1 = 0 & N_1(2, -1, 3) \\ 5x_1 + 4x_2 - x_3 + 1 = 0 & N_2(5, 4, -1) \end{cases}$

et



$u_{\mathcal{D}'} = N_1 \times N_2 =$

$= \begin{vmatrix} e_1 & e_2 & e_3 \\ 2 & -1 & 3 \\ 5 & 4 & -1 \end{vmatrix} =$

$$= e_1 \begin{vmatrix} -1 & 3 \\ 4 & -1 \end{vmatrix} - e_2 \begin{vmatrix} 2 & 3 \\ 5 & -1 \end{vmatrix} + e_3 \begin{vmatrix} 2 & -1 \\ 5 & 4 \end{vmatrix} =$$

$$= -11e_1 + 17e_2 + 13e_3 = (-11, 17, 13)$$

$$\mathcal{D} : \frac{x_1 - 2}{-11} = \frac{x_2 + 5}{17} = \frac{x_3 - 3}{13} = t$$

Met II

$$A = \left(\boxed{\begin{matrix} 2 & -1 \\ 5 & 4 \end{matrix}} \begin{matrix} 3 \\ -1 \end{matrix} \right) \begin{vmatrix} -1 \\ -1 \end{vmatrix}$$

$$x_3 = t \Rightarrow \begin{cases} 2x_1 - x_2 = -1 - 3t \mid 4 \quad (1) \\ 5x_1 + 4x_2 = -1 + t \end{cases} \quad (+)$$

$$13x_1 = -5 - 11t$$

$$x_1 = \frac{-5 - 11t}{13} = \frac{-5}{13} - \frac{11}{13} t \xrightarrow{(1)} 2\left(\frac{-5}{13} - \frac{11t}{13}\right) + 1 + 3t = x_2$$

$$\Rightarrow x_2 = t\left(\frac{-22 + 39}{13}\right) + \frac{-10 + 13}{13} = \frac{17}{13} t + \frac{3}{13}$$

$$\mathcal{D}' : \begin{cases} x_1 = \frac{-5}{13} - \frac{11}{13} t \\ x_2 = \frac{17}{13} t + \frac{3}{13} \\ x_3 = t \end{cases}$$

$$\left(\frac{-11}{13}, \frac{17}{13}, 1 \right) = \frac{1}{13} \underline{(-11, 17, 13)}$$

$$\mathcal{D}' : \frac{x_1 + \frac{5}{13}}{-\frac{11}{13}} = \frac{x_2 - \frac{3}{13}}{\frac{17}{13}} = \frac{x_3}{1} = t$$

$\pi \quad ax_1 + bx_2 + cx_3 + d = 0$

$N = (a, b, c)$ normala

Cursul 12

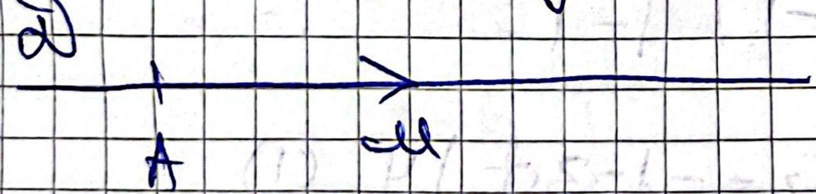
Geometrie analitică euclidiană

$(\mathbb{R}^n, (\mathbb{R}^n, g_0), \varphi)$ sp. afin euclidian

$g_0 : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R} \quad g_0(x, y) = \sum_{i=1}^{n} x_i y_i$

$\varphi : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n \quad \varphi(u, v) = v - u$

- Ec. unei drepte afine $\mathcal{D}$ din $\mathbb{R}^n$

a) 

$V_{\mathcal{D}} = \langle \{u\} \rangle$

$A(a_1, \ldots, a_n)$ și $u(u_1, \ldots, u_n)$

$(\forall) M(x_1, \ldots, x_n) \in \mathcal{D} \Rightarrow (\exists) t \in \mathbb{R}$ a.î. $\overline{AM} = t u$

$\mathcal{D} : x_i = a_i + t u_i, \ i = \overline{1, n}$