17 - 11 - 2025

Tutoriat III

1) **a)** $\sum_{n=1}^{\infty} \left(\frac{1}{n^2}+1\right)^2$

$$\lim_{n\to\infty}\left(1+\frac{1}{n^2}\right)^2 = 1 \neq 0 \Rightarrow \sum_{n=1}^{\infty}\left(1+\frac{1}{n^2}\right)^2 \text{ div.}$$

b) $\sum_{n=1}^{\infty} \frac{4n^2-n}{n^3+9}$

$$\frac{4n^2-n}{n^3+9} \approx \frac{4n^2}{n^3} = \frac{4}{n}$$

Aplicăm criteriul comparației $b_n = \frac{1}{n}$

$$\lim_{n\to\infty}\frac{\frac{4n^2-n}{n^3+9}}{\frac{1}{n}} = \lim_{n\to\infty}\frac{4n^3-n}{n^2+9} = 4 \in (0,\infty) \text{ Deci e div.}$$

c) $\sum_{n=1}^{\infty} \frac{\sqrt{2n^2+4n+1}}{n^3+9}$

Comparăm cu $b_n = \frac{1}{n^2}$

$$\lim_{n\to\infty}\frac{\frac{\sqrt{2n^2+4n+1}}{n^3+9}}{\frac{1}{n^2}} = \lim_{n\to\infty}\frac{n^2\sqrt{2n^2+4n+1}}{n^3+9} = \sqrt{2} \text{ Deci seria e div}$$

d) $\sum_{n=1}^{\infty} \frac{2^n\cos(5n)}{4^n+\sin^2(n)}$

$$\frac{2^n\cos(5n)}{4^n\sin^2(n)} \leq \frac{2^n}{4^n+\sin^2(n)} \leq \frac{2^n}{4^n} = \left(\frac{1}{2}\right)^n \Rightarrow \sum_{n=1}^{\infty}(\dots) \leq \sum\left(\frac{1}{2}\right)^n$$

$\Rightarrow \sum_{n=1}^{\infty}(\dots)$ este conv.

$$\left.\begin{cases} a_n, b_n \geq 0 \\ a_n \geq b_n \text{ și } \sum b_n \text{ este div} \end{cases}\right\} \Rightarrow \sum a_n \text{ div.}$$

e) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{(-1)^n(7+2n)} = \sum_{n=1}^{\infty} \frac{(-1)^n}{7+2n}$

Din Leibneiz $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{7+2n}$ este conv.

f) $\displaystyle\sum_{n=3}^{\infty} \frac{e^{4n}}{(n-2)!}$

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{e^{4n+4}}{(n-1)!} \cdot \frac{(n-2)!}{e^{4n}} = \lim_{n\to\infty} \frac{e^4}{n-1} = 0 < 1 \Rightarrow$$

$$\Rightarrow \sum_{n=3}^{\infty} \frac{e^{4n}}{(n-2)!} \text{ e conv!}$$

$\displaystyle\lim_{n\to\infty} \sqrt[n]{n} = 1$

g) $\displaystyle\sum_{n=1}^{\infty} \frac{n^{1-3n}}{4^{2n}}$

$$\lim_{n\to\infty} \left(\frac{n^{1-3n}}{4^{2n}}\right)^{\frac{1}{n}} = \lim_{n\to\infty} \left(n^{\frac{1}{n}} \cdot \frac{n^{-3}}{16}\right) = 0 < 1 \Rightarrow \sum_{n=1}^{\infty} \frac{n^{1-3n}}{4^{2n}} \text{ e conv.}$$

h) $\displaystyle\sum_{n=1}^{\infty} tg\left(\frac{n}{n^2+1}\right) arctg\left(\sqrt{\frac{n}{n+1}}\right)$

Comparăm cu $\displaystyle\frac{n}{n^2+1} \cdot \sqrt{\frac{n}{n+1}}$

$$\lim_{n\to\infty} \frac{tg\left(\frac{n}{n^2+1}\right) \cdot arctg\left(\sqrt{\frac{n}{n+1}}\right)}{\frac{n}{n^2+1} \cdot \sqrt{\frac{n}{n+1}}} = 1 \in (0, \infty)$$

Deoarece $\displaystyle\sum_{n=1}^{\infty} \frac{n}{n^2+1}\sqrt{\frac{n}{n+1}}$ este div. $\Rightarrow$ Seria este div

i) $\sum_{n=1}^{\infty} \left(\frac{2}{n}\right)^n n!$

$$\sum_{n=1}^{\infty} \left(\frac{2}{n+1}\right)^{n+1} (n+1)! \cdot \left(\frac{n}{2}\right)^n \frac{1}{n!} = \lim_{n\to\infty} 2\left(\frac{n}{n+1}\right)^n =$$

$$= 2\lim_{n\to\infty}\left(1+\frac{-1}{n+1}\right)^n = 2e^{-1} < 1 \Rightarrow \text{seria este conv.}$$

k) $\sum_{n=1}^{\infty} \frac{1}{n^{\alpha}(\ln(n))^{\beta}}$ $\alpha, \beta > 0$

Aplicăm criteriul condensării : $\sum_{n=1}^{\infty} 2^n \frac{1}{(\ln(2^n))^{\beta}} = \sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^{n(\alpha-1)} \cdot$

$\cdot \frac{1}{n^{\beta}(\ln(2))^{\beta}} = \frac{1}{(\ln(2))^{\beta}} \cdot \sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^{n(\alpha-1)} \cdot \frac{1}{n^{\beta}}$

Caz I : $\alpha > 1 \Rightarrow \alpha - 1 > 0$

$\sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^{n(\alpha-1)} \cdot \frac{1}{n^{\beta}} \leq \sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^{n(\alpha-1)}$ convergentă

. . . .

Caz II $\alpha < 1 \Rightarrow \alpha - 1 < 0$

$\sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^{n(\alpha-1)} \cdot \frac{1}{n^{\beta}}$

Aplicăm c. raport : $\lim_{n\to\infty} \frac{\left(\frac{1}{2}\right)^{(n+1)(\alpha-1)} \cdot \frac{1}{(n+1)^{\beta}}}{\left(\frac{1}{2}\right)^{n(\alpha-1)} \cdot \frac{1}{n^{\beta}}} = \ldots = \left(\frac{1}{2}\right)^{\alpha-1} =$

$= 2^{1-\alpha} > 1$

$\Rightarrow$ Seria este divergentă

Caz III $\alpha = 1$: $\sum_{n=1}^{\infty} \frac{1}{n^{\beta}(\ln(2))^{\beta}}$ ... $\begin{cases} \text{conv pt } \beta > 1 \\ \text{div. pt } \beta \leq 1 \end{cases}$

$$\sum_{n=1}^{\infty} \frac{\ln\left(\frac{n^2+1}{n^2}\right)}{\sin\left(\frac{1}{n}\right)}$$

Comparăm cu $\frac{\frac{1}{n^2}}{\frac{1}{n}} = \frac{1}{n}$

$$\lim_{n\to\infty} \frac{\ln\left(\frac{n^2+1}{n^2}\right)}{\sin\left(\frac{1}{n}\right)} \cdot n = \lim_{n\to\infty} \frac{\ln\left(\frac{n^2+1}{n^2}\right)}{\frac{1}{n^2}} = \frac{\frac{1}{n}}{\sin\left(\frac{1}{n}\right)} = 1$$

$f_n: A \to \mathbb{R}$, $f: A \to \mathbb{R}$

Spunem că f_n converge simplu (punctual) la f dacă ($f_n \xrightarrow{s} f$) dacă:

$\forall x \in A \ \forall \varepsilon > 0 \ \exists N \in \mathbb{N}$ a. î. pt. $n \geq N$ $|f_n(x) - f(x)| < \varepsilon$

$\iff \forall x \in A \ \lim_{n\to\infty} f_n(x) = f(x)$

Spunem că f_n conv. uniform la f ($f_n \xrightarrow{u} f$) dacă: (*) $\iff$

$\forall \varepsilon > 0 \ \exists N_\varepsilon \in \mathbb{N}$ a. î. pt. $n \geq N$ $|f_n(x) - f(x)| < \varepsilon \ \forall x \in A$.

(*) $\iff f_n \xrightarrow{u} f \iff \lim_{n\to\infty} \sup |f_n(x) - f(x)| = 0$

$$\left.\begin{array}{l} f_n \text{ sunt cnt.} \\ f_n \xrightarrow{u} f \end{array}\right\} \Rightarrow f \text{ este continuă}$$

$f_n \xrightarrow{u} f$. $\lim_{n\to\infty} \int f_n = \int \lim_{n\to\infty} f_n$.

2) $f_n(x) = \dfrac{nx}{1+nx^2}, \quad x \in (0,\infty)$

a) $\lim\limits_{n\to\infty} f_n(x) = ?$

$$\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} \frac{nx}{1+nx^2} = \frac{1}{x}$$

$$f_n(x) \xrightarrow{s} \frac{1}{x}$$

b) $\lim\limits_{n\to\infty} \sup\limits_{x\in(0,\infty)} \left| f_n - \frac{1}{x} \right| = \lim\limits_{n\to\infty} \sup \left| \frac{nx}{1+nx^2} - \frac{1}{x} \right| =$

$= \lim\limits_{n\to\infty} \sup\limits_{x\in(0,\infty)} \left| \frac{-1}{x+nx^3} \right| \geq \lim\limits_{n\to\infty} \left| \frac{n^2}{n+1} \right| \to \infty$

fie $x_n = \frac{1}{n}$: $\left| -\dfrac{1}{\frac{1}{n} + n\frac{1}{n^3}} \right| = \left| \dfrac{n^2}{n+1} \right| \xrightarrow[n\to\infty]{} \infty$

$\Rightarrow f_n \overset{u}{\not\to} \frac{1}{x}$

d) $x \in (1,\infty)$

$$\lim_{n\to\infty} \sup_{x\in(1,\infty)} \left| f_n - \frac{1}{x} \right| = \lim_{n\to\infty} \sup_{x\in(1,\infty)} \left| -\frac{1}{x+nx^3} \right| \leq \lim_{n\to\infty} \frac{1}{n} = 0$$

$x \in (1,\infty)$

$$\frac{1}{x+nx^3} \leq \frac{1}{n} \qquad \sup_{x\in(1,\infty)} \left(\frac{1}{x+nx^3} \right) \leq \sup\left(\frac{1}{n}\right) = \frac{1}{n}$$

$$\underbrace{0 \leq \lim_{n\to\infty} \sup \left| f_n - \frac{1}{x} \right|}_{0} \leq 0 \Rightarrow f_n \xrightarrow{u} \frac{1}{x}, \ x \in (1,\infty)$$

3. Fie $f_n(x) = \begin{cases} 1, & x = 1, \frac{1}{2}, \dots, \frac{1}{n} \\ 0, & \text{altfel} \end{cases}$

Fie $x \in \mathbb{R}$

$$\lim_{n \to \infty} f_n(x) = \begin{cases} 1, & \text{dacă } x = \frac{1}{n}, n \in \mathbb{N}^* \\ 0, & \text{altfel} \end{cases} = f$$

$f_n \xrightarrow{u} f$?

$$\lim_{n \to \infty} \sup_{x \in \mathbb{R}} |f_n(x) - f| \geq \lim_{n \to \infty} \left| f_n(n+1) - f\left(\frac{1}{n+1}\right) \right| = \lim_{n \to \infty} |0 - 1| = 1$$

nu este unif. conv.

3.1. $g_n(x) = \begin{cases} x, & x = 1, \frac{1}{2}, \dots, \frac{1}{n} \\ 0, & \text{altfel} \end{cases}$

pt $x \in \mathbb{R}$.

$$g(x) = \lim_{n \to \infty} g_n(x) = \begin{cases} x, & x = \frac{1}{n}, n \in \mathbb{N} \\ 0, & \text{altfel} \end{cases}$$

$g_n \xrightarrow{u} g$? $\lim_{n \to \infty} \sup_{x \in \mathbb{R}} |g_n(x) - g(x)| \leq$

Vrem să arătăm că $|g_n(x) - g(x)| < \frac{1}{n}$

Fie $x \in \mathbb{R}$.

Caz I: $x = \frac{1}{m}$

Dacă $m > n \Rightarrow g_n(x) = g_n\left(\frac{1}{m}\right) = 0$

$$\left| g_n\left(\frac{1}{m}\right) - g\left(\frac{1}{m}\right) \right| = \left|0 - \frac{1}{m}\right| < \frac{1}{n}$$

Dacă $m \le n$:

$$\left| g_n\left(\frac{1}{m}\right) - g\left(\frac{1}{m}\right) \right| = \left| \frac{1}{m} - \frac{1}{m} \right| < \frac{1}{n}$$

Caz II. $x \neq \frac{1}{m}$, $\forall m \in \mathbb{N}$

$$|g_n(x) - g(x)| = |0 - 0| = 0 < \frac{1}{n}$$

$$\lim_{n\to\infty} \sup_{x\in\mathbb{R}} |g_n(x) - g(x)| \le \lim_{n\to\infty} \frac{1}{n} = 0$$