$$\sum_{n=0}^{\infty} \frac{a^n n!}{n^n}$$

$$\lim_{n\to\infty} s_n = ?$$

$$\lim_{n\to\infty} \frac{x_{n+1}}{x_n} = \lim_{n\to\infty} \frac{\frac{a^{n+1}(n+1)!}{(n+1)^{n+1}}}{\frac{a^n n!}{n^n}} = \lim_{n\to\infty} \frac{a(n+1)n^n}{(n+1)^{n+1}} = \lim_{n\to\infty} \frac{an^n}{(n+1)^n} = \lim_{n\to\infty} \frac{an^n}{n^n(1+1/n)^n} = \lim_{n\to\infty} \frac{a}{(1+1/n)^n} = \frac{a}{e}$$

Discuție:

$l < 1 \Leftrightarrow \frac{a}{e} < 1 \Rightarrow \sum_{n=0}^{\infty} x_n$ este convergentă

$l > 1 \Leftrightarrow \frac{a}{e} > 1 \Rightarrow \sum_{n=0}^{\infty} x_n$ este divergentă

$l = 1 \Leftrightarrow \frac{a}{e} = 1 \Rightarrow$ nu putem determina cu criteriul raportului

pt. $\frac{a}{e} = 1 \Leftrightarrow a = e, x_n = \frac{e^n n!}{n^n}$

$$\lim_{n\to\infty} n\left(\frac{x_n}{x_{n+1}} - 1\right) = \lim_{n\to\infty} n\left(\frac{(1+1/n)^n}{e} - 1\right) = \underbrace{\lim_{n\to\infty} \frac{\frac{(1+1/n)^n}{e} - 1}{\frac{1}{n}}}_{\text{limită de serii}} = \underbrace{\lim_{x\to 0} \frac{\frac{(1+x)^{1/x}}{e} - 1}{x}}_{1/n=x\,\text{limită de funcție}} \overset{\frac{0}{0}}{\underset{\text{l'H}}{=}} \lim_{x\to 0} \frac{\frac{1}{e}\left((1+x)^{1/x}\right)'}{x'} =$$

$$\left((1+x)^{1/x}\right)' = \left(\left(e^{\ln(1+x)}\right)^{1/x}\right)' = \left(e^{\frac{\ln(1+x)}{x}}\right)' = e^{\frac{\ln(1+x)}{x}} \left(\frac{\ln(1+x)}{x}\right)' = (1+x)^{1/x} \frac{\frac{1}{1+x}x - \ln 1 + x}{x^2}$$

$$= \frac{1}{e} \lim_{x\to 0} \underbrace{(1+x)^{1/x}}_{e} \frac{\frac{x}{1+x} - \ln 1 + x}{x^2} = \lim_{x\to 0} \frac{x - (1+x)\ln(1+x)}{x^2\underbrace{(x+1)}_{1}} = \lim_{x\to 0} \frac{x - (1+x)\ln(1+x)}{x^2} \overset{\frac{0}{0}}{\underset{\text{l'H}}{=}} \lim_{x\to 0} \frac{1 - (\ln(1+x)+1)}{2x} \overset{\frac{0}{0}}{\underset{\text{l'H}}{=}}$$

$$= -\lim_{x\to 0} \frac{\frac{1}{1+x}}{2} = -\frac{1}{2} \Leftrightarrow \lim_{n\to 0} n\left(\frac{x_n}{x_{n+1}} - 1\right) = -\frac{1}{2} < 0 \Rightarrow \sum_{n=0}^{\infty} x_n \text{este divergentă}$$

Deci:

pt. $a < e, s_n$ este convergentă

pt. $a \geq e, s_n$ este divergentă

$f_n:\mathbb{R}\to\mathbb{R}, f_n(x)=\dfrac{2nx}{n^2+x^3}, \forall n\in\mathbb{N}$

$\lim\limits_{n\to\infty} f_n=\lim\limits_{n\to\infty}\dfrac{2nx}{n^2+x^3}=0$

$f:\mathbb{R}\to\mathbb{R}, f(x)=0, \forall x\in\mathbb{R}$

$f_n \xrightarrow[\mathbb{R}]{s} f$

$$\lim_{n\to\infty}\sup_{x\in\mathbb{R}}|f_n(x)-f(x)|=\lim_{n\to\infty}\sup_{x\in\mathbb{R}}\left|\frac{2nx}{n^2+x^3}-0\right|=\lim_{n\to\infty}\sup_{x\in\mathbb{R}}\underbrace{\left|\frac{2nx}{n^2+x^3}\right|}_{g(x)}$$

$$g'(x)=\left(\frac{2nx}{n^2+x^3}\right)'=\frac{(2nx)'(n^2+x^3)-(2nx)(n^2+x^3)'}{(n^2+x^3)^2}=\frac{2n^3-4nx^3}{(n^2+x^3)^2}=0\Rightarrow 2n^3-4nx^3=0\Rightarrow 4nx^3=2n^3\Rightarrow$$

$$\Rightarrow x^3=\frac{n^2}{2}\Rightarrow x=\sqrt[3]{\frac{n^2}{2}}$$

x	$-\infty$		$\sqrt[3]{\frac{n^2}{2}}$		∞
$g'(x)$		$+$	0	$-$	
$g(x)$	0	$\nearrow$	$\frac{2n\sqrt[3]{\frac{n^2}{2}}}{n^2+\frac{n^2}{2}}$	$\searrow$	0

$$\lim_{n\to\infty}\sup_{x\in\mathbb{R}}|f_n(x)-f(x)|=\lim_{n\to\infty}\frac{2n\sqrt[3]{\frac{n^2}{2}}}{n^2+\frac{n^2}{2}}=\lim_{n\to\infty}\frac{2n\sqrt[3]{\frac{n^2}{2}}}{n^2+\frac{n^2}{2}}=\lim_{n\to\infty}\frac{4n^{\frac{2}{3}+1}}{\sqrt[3]{2}(2n^2+n^2)}=\lim_{n\to\infty}\frac{4n^{\frac{5}{3}}}{3\sqrt[3]{2}n^2}=\lim_{n\to\infty}\frac{4}{3\sqrt[3]{2}n}=0\Rightarrow f_n \xrightarrow[(2,5]]{u} f$$

$f_n:[0,\infty)\to\mathbb{R}, f_n(x)=x^4e^{-nx+5}, \forall n\in\mathbb{N}$

$\lim\limits_{n\to\infty} f_n=\lim\limits_{n\to\infty} x^4e^{-nx+5}=0$

$f:[0,\infty)\to\mathbb{R}, f(x)=0, \forall x\in\mathbb{R}$

$f_n \xrightarrow[\mathbb{R}]{s} f$

$$\lim_{n\to\infty}\sup_{x\in\mathbb{R}}|f_n(x)-f(x)|=\lim_{n\to\infty}\sup_{x\in\mathbb{R}}\left|x^4e^{-nx+5}-0\right|=\lim_{n\to\infty}\sup_{x\in\mathbb{R}}\underbrace{\left|x^4e^{-nx+5}\right|}_{g(x)}$$

$$g'(x)=\left(x^4e^{-nx+5}\right)'=4x^3e^{-nx+5}+x^4e^{-nx+5}(-n)=x^3e^{-nx+5}(4-nx)=0 \quad\Rightarrow\quad \begin{cases} x^3=0\Leftrightarrow x=0\\ 4-nx=0\Leftrightarrow nx=4\Leftrightarrow x=\frac{4}{n}\in[0,\infty)\end{cases}$$

x	0		$\frac{4}{n}$		∞
$g'(x)$	0	$+$	0	$-$	
$g(x)$	0	$\nearrow$	$\left(\frac{4}{n}\right)^4e$	$\searrow$	0

$$\lim_{n\to\infty}\sup_{x\in\mathbb{R}}|f_n(x)-f(x)|=\lim_{n\to\infty}\left(\frac{4}{n}\right)^4e=0\Rightarrow f_n \xrightarrow[(2,5]]{u} f$$