Rezolvare exerciții tutoriat 1

1) a) - VEZI Exerciții limite → tutoriat 1

b) $$x_n = \frac{(-1)^n(2n^2+5)}{n^2+n+1}\operatorname{arctg}\left(\frac{(-1)^n}{\sqrt{3}}\right) + \frac{3n}{n+2}\cos\left(\frac{n\pi}{3}\right), \forall n \in \mathbb{N}^*$$

Sol:

$$\boxed{x_{6k}} = \frac{(-1)^{6k}(2\cdot(6k)^2+5)}{(6k)^2+(6k)+1}\operatorname{arctg}\left(\frac{(-1)^{6k}}{\sqrt{3}}\right) + \frac{3(6k)}{(6k)+2}\cos\left(\frac{6k\pi}{3}\right)$$

$$= \frac{\cancel{(6k)^2}\left(2+\frac{5}{(6k)^2}\right)}{\cancel{(6k)^2}\left(1+\frac{(6k)}{(6k)^2}+\frac{5}{(6k)^2}\right)}\underbrace{\operatorname{arctg}\left(\frac{1}{\sqrt{3}}\right)}_{=\operatorname{arctg}\left(\frac{\sqrt{3}}{3}\right)} + \frac{(6k)\cdot 3}{(6k)\left(1+\frac{2}{6k}\right)}\underbrace{\cos(2k\pi)}_{=1}$$

$$\xrightarrow[k\to\infty]{} = \frac{2}{1}\underbrace{\operatorname{arctg}\left(\frac{\sqrt{3}}{3}\right)}_{\frac{\pi}{6}} + \frac{3}{1}\cdot 1 = 2\cdot\frac{\pi}{6} + 3\cdot 1 = \boxed{\frac{\pi}{3}+3}$$

De acum voi scrie mai direct

$$\boxed{x_{6k+1}} = \frac{(-1)^{6k+1}(2\cdot(6k+1)^2+5)}{(6k+1)^2+(6k+1)+1}\operatorname{arctg}\left(\frac{(-1)^{6k+1}}{\sqrt{3}}\right) + \frac{3(6k+1)}{(6k+1)+2}\cos\left(\frac{6k\pi}{3}+\frac{\pi}{3}\right)$$

$$\xrightarrow[k\to\infty]{} = (-2)\underbrace{\operatorname{arctg}\left(-\frac{\sqrt{3}}{3}\right)}_{-\frac{\pi}{6}} + 3\underbrace{\cos\left(\frac{\pi}{3}\right)}_{\frac{1}{2}} \quad // \cos(2k\pi + x) = \cos x \,!!$$

$$= \boxed{\frac{\pi}{3}+\frac{3}{2}}$$

$$\boxed{x_{6k+2}} = \frac{(-1)^{6k+2}(2\cdot(6k+2)^2+5)}{(6k+2)^2+(6k+2)+1}\operatorname{arctg}\left(\frac{(-1)^{6k+2}}{\sqrt{3}}\right) + \frac{3(6k+2)}{(6k+2)+2}\cos\left(\frac{6k\pi}{3}+\frac{2\pi}{3}\right)$$

$$\xrightarrow[k\to\infty]{} = 2\operatorname{arctg}\left(\frac{\sqrt{3}}{3}\right) + 3\underbrace{\cos\left(\frac{2\pi}{3}\right)}_{=\left(-\frac{1}{2}\right)} = \boxed{\frac{\pi}{3}-\frac{3}{2}}$$

$$\boxed{x_{6k+3}} = \frac{(-1)^{6k+3}(2(6k+3)^2+5)}{(6k+3)^2+(6k+3)+1} \operatorname{arctg}\left(\frac{(-1)^{6k+3}}{\sqrt{3}}\right) + \frac{3(6k+3)}{(6k+3)+2} \cos\left(\frac{6k\pi}{3}+\frac{3\pi}{3}\right)$$

$$\xrightarrow[k\to\infty]{} = (-2)\operatorname{arctg}\left(-\frac{\sqrt{3}}{3}\right) + 3\underbrace{\cos(\pi)} = (-2)\cdot\left(-\frac{\pi}{6}\right) + 3(-1)$$

$$= \boxed{\frac{\pi}{3} - 3}$$

$$\boxed{x_{6k+4}} = \frac{(-1)^{6k+4}(2(6k+4)^2+5)}{(6k+4)^2+(6k+4)+1} \operatorname{arctg}\left(\frac{(-1)^{6k+4}}{\sqrt{3}}\right) + \frac{3(6k+4)}{(6k+4)+2} \cos\left(\frac{6k\pi}{3}+\frac{4\pi}{3}\right)$$

$$\xrightarrow[k\to\infty]{} = 2\underbrace{\operatorname{arctg}\left(\frac{\sqrt{3}}{3}\right)}_{=\frac{\pi}{6}} + 3\underbrace{\cos\left(\frac{4\pi}{3}\right)}_{=-\frac{1}{2}} = \boxed{\frac{\pi}{3} - \frac{3}{2}}$$

$$\boxed{x_{6k+5}} = \frac{(-1)^{6k+5}(2(6k+5)^2+5)}{(6k+5)^2+(6k+5)+1} \operatorname{arctg}\left(\frac{(-1)^{6k+5}}{\sqrt{3}}\right) + \frac{3(6k+5)}{(6k+5)+2} \cos\left(\frac{6k\pi}{3}+\frac{5\pi}{3}\right)$$

$$\xrightarrow[k\to\infty]{} (-2)\operatorname{arctg}\left(-\frac{\sqrt{3}}{3}\right) + 3\underbrace{\cos\left(\frac{5\pi}{3}\right)}_{=\frac{1}{2}} = \boxed{\frac{\pi}{3} + \frac{3}{2}}$$

$$\mathbb{N} = 6\mathbb{N} \cup (6\mathbb{N}+1) \cup (6\mathbb{N}+2) \cup (6\mathbb{N}+3) \cup (6\mathbb{N}+4) \cup (6\mathbb{N}+5)$$

$$\mathcal{L}((x_n)_n) = \left\{\frac{\pi}{3}-3;\ \frac{\pi}{3}-\frac{3}{2};\ \frac{\pi}{3}+\frac{3}{2};\ \frac{\pi}{3}+3\right\}$$

$$\Rightarrow \left.\begin{array}{l} \overline{\lim}\, x_n = \frac{\pi}{3}+3 \\ \underline{\lim}\, x_n = \frac{\pi}{3}-3 \end{array}\right\} \Rightarrow \overline{\lim}\, x_n \neq \underline{\lim}\, x_n$$

$$\Rightarrow \nexists \lim_{n\to\infty} x_n$$

$\square$

2) a) $\sum_{n=1}^{\infty} \frac{10\cdot 18\cdot \ldots \cdot (8n+2)}{10\cdot 19\cdot \ldots \cdot (9n+1)} \cdot x^n, \ x>0$

<u>Sol</u>: Folosim criteriul raportului:

Fie $x_n = \frac{10\cdot 18\cdot \ldots \cdot (8n+2)}{10\cdot 19\cdot \ldots \cdot (9n+1)} \cdot x^n$

$$\lim_{n\to\infty} \frac{x_{n+1}}{x_n} = \lim_{n\to\infty} \frac{\cancel{10\cdot 18\cdot \ldots \cdot (8n+2)}\cdot(8n+10)}{\cancel{10\cdot 19\cdot \ldots \cdot (9n+1)}\cdot(9n+10)} \cdot \frac{\cancel{10\cdot 19\cdot \ldots \cdot (9n+1)}}{\cancel{10\cdot 18\cdot \ldots \cdot (8n+2)}} \cdot \frac{\cancel{x^{n+1}}}{\cancel{x^n}}$$

$$= \lim_{n\to\infty} \frac{8n+10}{9n+10} \cdot x = \frac{8x}{9}$$

Conform criteriului raportului avem:

1) Dacă $\frac{8}{9}x < 1$ (i.e. $x \in \left(0, \frac{9}{8}\right)$), at. seria e conv.

2) Dacă $\frac{8}{9}x > 1$ (i.e. $x \in \left(\frac{9}{8}, \infty\right)$), at seria e div.

3) Dacă $\frac{8}{9}x = 1$ (i.e. $x = \frac{9}{8}$), at crit. nu decide

Luăm $x = \frac{9}{8}$, seria x_n devine:

$$x_n = \frac{10\cdot 18\cdot \ldots \cdot (8n+2)}{10\cdot 19\cdot \ldots \cdot (9n+1)} \cdot \left(\frac{9}{8}\right)^n$$

Folosim criteriul Raabe-Duhamel:

$$\lim_{n\to\infty} n\left(\frac{x_n}{x_{n+1}} - 1\right) = \lim_{n\to\infty} n\left(\frac{\cancel{\ldots}\cdot(8n+2)}{\cancel{\ldots}\cdot(9n+1)} \cdot \frac{\ldots \cdot \cancel{(9n+1)}(9n+10)}{\ldots \cdot \cancel{(8n+2)}(8n+10)} \cdot \left(\frac{9}{8}\right)^n \cdot \left(\frac{8}{9}\right)^{n+1} - 1\right)$$

$$= \lim_{n\to\infty} n\left(\frac{72n+80}{72n+90} - \frac{72n+90}{72n+90}\right) = \lim_{n\to\infty} n\left(\frac{-10}{72n+90}\right) = -\frac{10}{72} < 1$$

Conform criteriului Raabe-Duhamel x_n **div**

<u>Am obținut</u>:

$$\sum_{n=1}^{\infty} \frac{10\cdot 18\cdot \ldots \cdot (8n+2)}{10\cdot 19\cdot \ldots \cdot (9n+1)} \cdot x^n \begin{cases} \text{convergent pt } x \in \left(0, \frac{9}{8}\right) \\ \text{divergent pt } x \in \left[\frac{9}{8}, \infty\right) \end{cases} \square$$

REZ

Sol:

b) Fie $x_n = \frac{\ln n}{n^3}$ $\Big\}$ $\sum_{n=1}^{\infty} \frac{\ln n}{n^3}$

Folosesc Criteriul de comp. cu ineg.

Aleg $y_n = \frac{1}{n^2}$. Dem. că $x_n \le y_n$

$$\frac{\ln n}{n^3} \le \frac{1}{n^2} \Big/ \cdot n^3 \Leftrightarrow \ln n \le \frac{n^3}{n^2} \Leftrightarrow \ln n \le n \text{ (A)}, \forall n \in \mathbb{N}$$

Deci $\sum_n y_n = \sum_n \frac{1}{n^2} \to$ convergent, $\alpha > 1$, serie armonică generalizată

$\Rightarrow$ Conform Crit de comp cu ineg., $\sum_n \frac{\ln n}{n^3}$ convergentă $\square$

c) $\sum_{n=1}^{\infty} \left(a \cdot \frac{n^2+n+1}{n^2} \right)^n$, $a > 0$

Sol:

Fie $x_n = \left(a \cdot \frac{n^2+n+1}{n^2} \right)^n$. Folosesc Criteriul radicalului.

$$\lim_{n\to\infty} \sqrt[n]{x_n} = \lim_{n\to\infty} \sqrt[n]{\left(a \cdot \frac{n^2+n+1}{n^2} \right)^n} = \lim_{n\to\infty} a \cdot \frac{n^2+n+1}{n^2} = a$$

Conform Crit. radicalului:

1) Dacă $l < 1$ (i.e. $a < 1$), at $\sum_n x_n$ conv

2) Dacă $l > 1$ (i.e. $a > 1$), at $\sum_n x_n$ div

3) Dacă $l = 1$ (i.e. $a = 1$) at crit nu decide

Luăm $a = 1$. $\Rightarrow \sum_n x_n = \sum_n \left(\frac{n^2+n+1}{n^2} \right)^n = \sum_n \left(1 + \frac{n+1}{n^2} \right)^n$

! $\lim_{n\to\infty} \left(1 + \frac{1}{n} \right)^n = e \to$ forțăm asta în ecuația noastră și surplusul îl alăturăm exponentului deja existent

! $\frac{n+1}{n^2} \to 0$ când $n \to \infty \Rightarrow$ la exponent vom pune inversa fracției

) $\lim\limits_{n\to\infty} \left(1+\frac{n+1}{n^2}\right)^n = \lim\limits_{n\to\infty} \left[\underbrace{\left(1+\frac{n+1}{n^2}\right)^{\frac{n^2}{n+1}}}_{e}\right]^{\frac{n+1}{n^2}\cdot n \to 1}$

$= e^1 = e \neq 0 \Rightarrow$ divergent $\to$ de ce?? (*)

Am obținut

$$\sum_n \left(a\cdot\frac{n^2+n+1}{n^2}\right)^n$$ → convergentă pt $a\in(0,1)$

→ divergentă pt $a\in[1,\infty)$

(*)

~~Se aplică Condiția necesară de convergență divergență~~

Se aplică Criteriul suficient de divergență