Seminar 5

1. Fie $n \in \mathbb{N}^*$, $d_1, d_\infty : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$

$d_1(x,y) = \sum_{i=1}^{n} |x_i - y_i|$, $d_\infty(x,y) = \max\{|x_i - y_i| \; i = \overline{1,n}\}$

Ar. că $\exists\, a, b \in (0, \infty)$ aî $a\, d_1(x,y) \le d_\infty(x,y) \le b\, d_1(x,y)$, $\forall\, x, y \in \mathbb{R}^n$

Sol

$$d_1(x,y) = \sum_{i=1}^{n} |x_i - y_i| \le n \cdot \max\{|x_i - y_i| \mid i=\overline{1,n}\}$$

$$= n \cdot d_\infty(x,y), \quad (\forall)\, x,y \in \mathbb{R}^n$$

$$d_1(x,y) \le n\, d_\infty(x,y) \;/ :n$$

$$\frac{1}{n} d_1(x,y) \le d_\infty(x,y)$$

Alegem $a = \frac{1}{n}$

$$d_\infty(x,y) = \max\{|x_i - y_i| \mid i=\overline{1,n}\} \le \sum_{i=1}^{n} |x_i - y_i|$$

$$d_\infty(x,y) \le d_1(x,y), \ \forall\, x,y \in \mathbb{R}^n$$

Alegem $b = 1$

2. Faceți analiza topologică a mulțimilor

a) $A = (0,1) \cup \{2\}$

1. $\mathring{A} = (0,1)$ | se rezolvă ca în clasă
2. $\bar{A} = [0,1] \cup \{2\}$

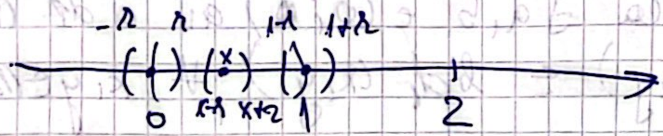
3. $\overset{*}{A}{}' = ?$

$x \in A' \iff \forall\, r > 0$ avem $(x-r, x+r) \cap A \setminus \{x\} \neq \emptyset$

$$A' \subset \bar{A} = [0,1] \cup \{2\}$$

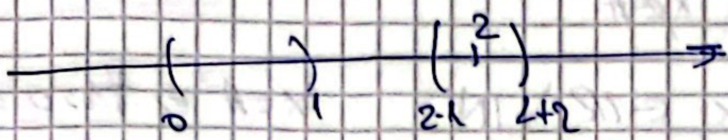
- fie $x \in [0,1]$

$x \in A' \iff \forall\, r > 0$ avem $(x-r, x+r) \cap A \setminus \{x\} \neq \emptyset$



Deci, $x \in A'$ i.e. $[0,1] \subset A'$

- $2 \in A' \Leftrightarrow \nexists r>0$ avem $(2-r, 2+r) \cap A \setminus \{2\} \neq \emptyset$



Deci, $2 \notin A'$

Aşadar, $A' = [0;1]$

4. $Fr_{\mathbb{R}}(A) = \partial A = \bar{A} \setminus \mathring{A} = [0,1] \cup \{2\} \setminus (0,1)$
$= \{0,1,2\}$

5. $Iz_{\mathbb{R}}(A) = {}^{\prime}A = \bar{A} \setminus A' = [0,1] \cup \{2\} \setminus [0,1]$
$= \{2\}$

b) $A = \mathbb{N} \subset (\mathbb{R}, d)$

1. $\mathring{A} = ?$

$x \in \mathring{A} \Leftrightarrow \exists r>0$ a.î. $(x-r, x+r) \subset A$

$(x-r, x+r) \subset \mathbb{N}$

Deci, $\mathring{A} = \emptyset$ deoarece orice interv. de forma $(x-r, x+r)$ $(r>0)$ conține o inf. de nr iraționale şi o inf. de nr raționale

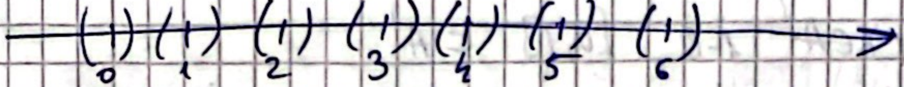
2. $A' = ?$

$x \in A' \Leftrightarrow \forall r>0$ avem $(x-r, x+r) \cap A \setminus \{x\} \neq \emptyset$

$(x-r, x+r) \cap \mathbb{N} \setminus \{x\} \neq \emptyset$

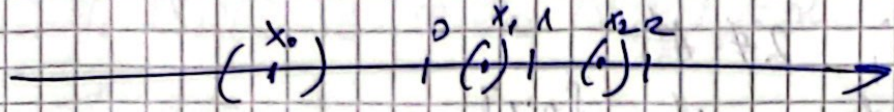
- fie $x \in \mathbb{N}$

$x \in A' \Leftrightarrow \forall r>0$ avem $(x-r, x+r) \cap A \setminus \{x\} \neq \emptyset$



Deci, $x \notin A'$

• Fie $x \in \mathbb{R} \setminus \mathbb{N}$ $\quad x \in A' \iff \forall r > 0$ avem $(x-r, x+r) \cap A \setminus \{x\} \neq$



Deci, $x \notin A'$

Așadar, $A' = \emptyset$

3. $\bar{A} = ?$

$\bar{A} = A \cup A' = \mathbb{N} \cup \emptyset = \mathbb{N}$

4. $Fr(A) = \partial A = \bar{A} \setminus \mathring{A} = \mathbb{N} \setminus \emptyset = \mathbb{N}$

5. $Iz(A) = \bar{A} \setminus A' = \mathbb{N} \setminus \emptyset = \mathbb{N}$

c) $A = \{\frac{1}{n} \mid n \in \mathbb{N}^*\} \subset (\mathbb{R}, d)$

$= \{1, \frac{1}{2}, \frac{1}{3}, \dots\}$

Sol

1) $\mathring{A} = ?$

$x \in \mathring{A} \iff \exists r > 0$ a.î. $\underbrace{B(x,r)}_{= (x-r,\, x+r)} \subset A$

$\mathring{A} = \emptyset$ deoarece orice interv. de forma $(x-r, x+r)$ $r>0$ conține o inf de nr raționale și o inf de nr iraționale

2) $\bar{A} = ?$ (pct acum. = pct limită)

$x \in \bar{A} \iff \exists (x_k)_k \subset A$ a.î. $\lim\limits_{k \to \infty} x_k = x$

Orice şir de elem din A poate avea drept limită un element al lui A (const. de la un rang încolo) sau 0

$\left(\lim_{k\to\infty} \frac{1}{k} = 0\right)$

$\emptyset$

$\bar{A} = A \cup \{0\}$

3) $A' = ?$

$x \in A' \Leftrightarrow \exists (x_k)_k \subset A \setminus \{x\}$ a.î. $\lim_{k\to\infty} x_k = x$

Cf. celor discutate la pct 2 avem $A' = \{0\}$

4. $Fr(A) = \partial A = \bar{A} \setminus \mathring{A} = A \cup \{0\} \setminus \emptyset = A \cup \{0\}$

5. $Izo(A) = {}^{i}A = \bar{A} \setminus A' = A \cup \{0\} \setminus \{0\}$
$= A$

Obs $(\mathbb{R}^2, \underset{d}{\overset{\shortparallel}{d_2}})$ e sp metric unde $d_2 : \mathbb{R}^2 \times \mathbb{R}^2$,

$d_2(x,y) = \sqrt{(x_1-y_1)^2 + (x_2-y_2)^2}$

Obs Fie $(x,y) \in \mathbb{R}^2$, şi $r > 0$

1) $B((x,y), r) = \{(z,t) \in \mathbb{R}^2 \mid d((x,y),(z,t)) < r\}$

$= \{(z,t) \in \mathbb{R}^2 \mid \sqrt{(x-z)^2 + (y-t)^2} < r\}$

$= \{(z,t) \in \mathbb{R}^2 \mid (z-x)^2 + (t-y)^2 < r^2\}$

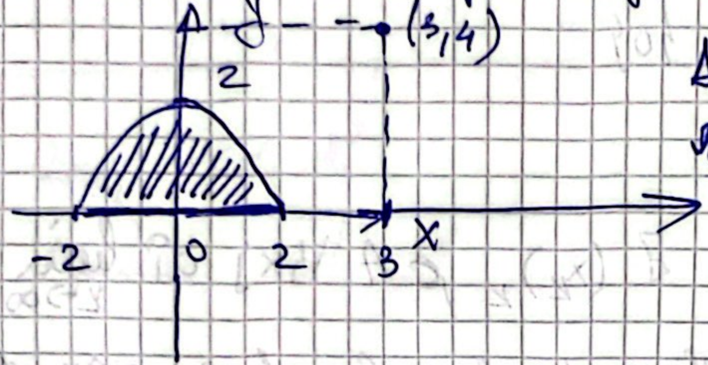
= discul deschis de centru (x,y) şi rază r

2) $B[(x,y), r] = \bar{B}((x,y), r) = \{(z,t) \in \mathbb{R}^2 \mid$
$(z-x)^2 + (t-y)^2 \le r^2\}$

= discul închis de centru (x,y) și rază r

3. Faceți analiza top. a mulțimii

$A = \{(x,y) \in \mathbb{R}^2 \mid x^2+y^2<4,\ y\geq 0\} \cup \{(3,4)\} \subset (\mathbb{R}^2, d)$



Deschi ineg → strict

Închisă ineg → egal

1) $\overset{\circ}{A} = ?$

$\overset{\circ}{A} \subset A$

$B = \{(x,y) \in \mathbb{R}^2 \mid x^2+y^2<4,\ y>0\} \subset A$

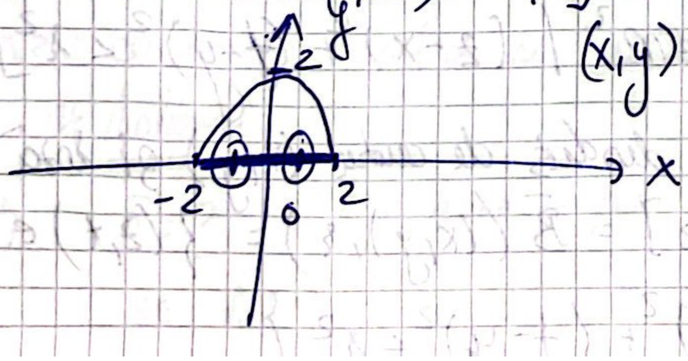
$\{(x,y) \in \mathbb{R}^2 \mid x^2+y^2<4,\ y>0\}$ - mulț deschisă

$\{(x,y) \in \mathbb{R}^2 \mid x^2+y^2<4,\ y>0\} \subset \overset{\circ}{A} \subset \{(x,y) \in \mathbb{R}^2 \mid x^2+y^2<4,\ y\geq 0\} \cup \{(3,4)\}$

• Studiem dacă $\{(x,y) \in \mathbb{R}^2 \mid \cancel{x^2+y^2<4}\ x \in (-2;2),\ y=0\} \cup \{(3,4)\} \subset \overset{\circ}{A}$

Fie $(x,y) \in \{(x,y) \in \mathbb{R}^2 \mid x \in (-2;2),\ y=0\}$

$= (-2;2) \times \{0\}$

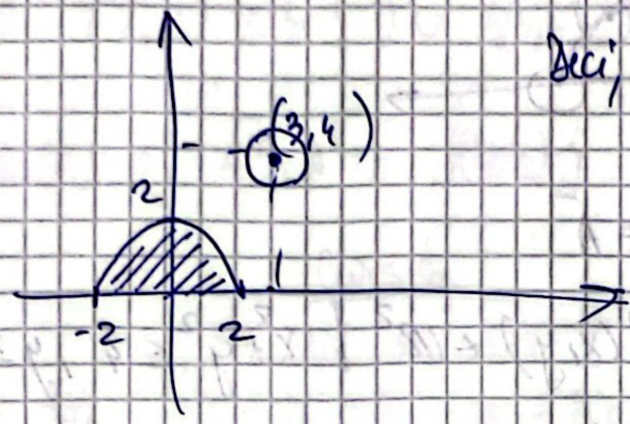


$(x,y) \in \overset{\circ}{A} \Leftrightarrow \exists r>0$ a.î. $B((x,y),r) \subset A$;

Deci $(x,y) \notin \mathring{A}$

- $(3,4)$

$(3,4) \in \mathring{A} \Leftrightarrow \exists\, r>0$ a.î. $B((3,4), r) \subset A$



Deci, $(3,4) \in \mathring{A}$

Aşadar, $\mathring{A} = \{(x,y) \in \mathbb{R}^2 \mid x^2+y^2<4, y>0\}$

2) $\bar{A} = ?$

$A \subset \bar{A}$

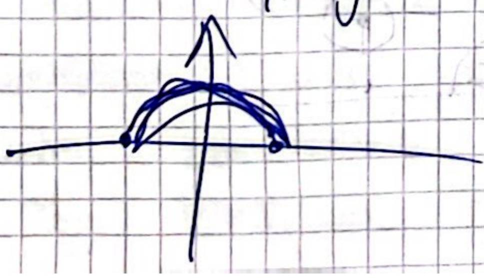
$A \subset \underbrace{\{(x,y) \in \mathbb{R}^2 \mid x^2+y^2 \leq 4, y \geq 0\} \cup \{(3,4)\}}_{\text{mulț închisă}}$

$\Rightarrow \bar{A} \subset \{(x,y) \in \mathbb{R}^2 \mid x^2+y^2 \leq 4, y \geq 0\} \cup \{(3,4)\}$

Avem

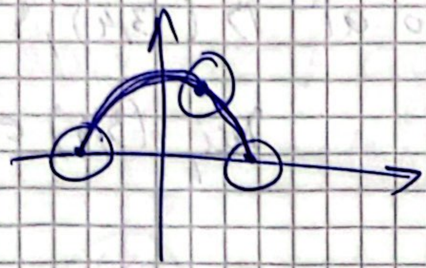
$\{(x,y) \in \mathbb{R}^2 \mid x^2+y^2<4, y \geq 0\} \cup \{(3,4)\} \subset \bar{A} \subset \{(x,y) \in \mathbb{R}^2 \mid$
$x^2+y^2 \leq 4, y \geq 0\} \cup \{(3,4)\}$

Studiem dacă $\{(x,y) \in \mathbb{R}^2 \mid x^2+y^2=4, y \geq 0\} \subset \bar{A}$



Fie $(x,y) \in \{(x,y) \in \overline{\mathbb{R}} \mid x^2+y^2=4,\ y=0\}$

$(x,y) \in \bar{A} \iff \forall r>0$ avem $B((x,y),r) \cap A \neq \emptyset$



Deci, $(x,y) \in \bar{A}$

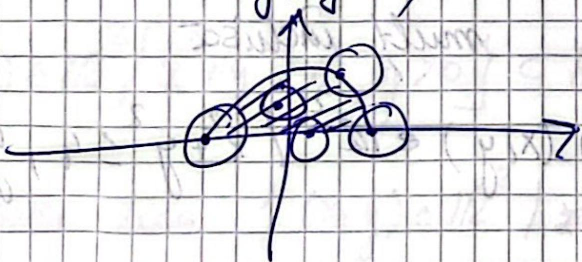
Deci, $\bar{A} = \{(x,y) \in \mathbb{R}^2 \mid x^2+y^2 \le 4,\ y \ge 0\} \cup \{(3,4)\}$

3) $A' = ?$

$A' \subset \bar{A} = \{(x,y) \in \mathbb{R}^2 \mid x^2+y^2 \le 4,\ y \ge 0\} \cup \{(3,4)\}$

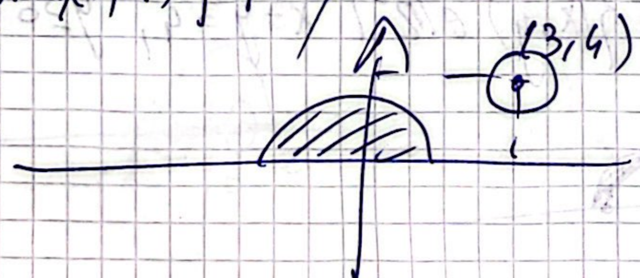
- Fie $(x,y) \in \{(x,y) \in \mathbb{R}^2 \mid x^2+y^2 \le 4,\ y \ge 0\}$

$(x,y) \in A' \iff \forall r>0$ avem $B((x,y),r) \cap A \setminus \{(x,y)\} \neq \emptyset$



Deci, $(x,y) \in A'$

- $(3,4) \in A' \iff \forall r>0$ avem $B((3,4),r) \cap A \setminus \{(3,4)\} \neq \emptyset$



Deci, $A' = \{(x,y) \in \mathbb{R}^2 \mid x^2+y^2 \leq 4,\ y \geq 0\}$

4) $Fr(A) = \bar{A} \setminus \mathring{A} = \{(x,y) \in \mathbb{R}^2 \mid x^2+y^2 = 4,\ y \geq 0\}$

$\cup \left((-2;2) \times \{0\}\right) \cup \{(3,4)\}$

5) $Izo(A) = \bar{A} \setminus A' = \{(3,4)\}$