SEMINAR XII - 11.01.2023 Geometrie analitică euclidiană

1) $(\mathbb{R}^3, (\mathbb{R}^3, g_0), \varphi)$

$g_0 : \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}, \ g_0(x,y) = x_1y_1 + x_2y_2 + x_3y_3$

$\varphi : \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}^3, \ \varphi(u,v) = v - u$

$A(3,-1,3), \ B(5,1,-1), \ u = (-3,5,-6)$

a) ecuația dreptei $\mathcal{D}$ a.î. $A \subset \mathcal{D}$, $\vec{\mathcal{D}} = \langle\{u\}\rangle$

b) ecuația dreptei AB

c) punctele de intersecție ale dreptei $\mathcal{D}$ cu planele de coordonate

a) $\mathcal{D} : \dfrac{x_1-3}{-3} = \dfrac{x_2+1}{5} = \dfrac{x_3-3}{-6} = t \Leftrightarrow \begin{cases} x_1 = -3t+3 \\ x_2 = 5t-1 \\ x_3 = -6t+3 \end{cases}, \ t \in \mathbb{R}$

b) $\vec{AB} = \varphi(A,B) = (5-3, 1+1, -1-3) = (2,2,-4) = 2(1,1,-2)$

$AB : \dfrac{x_1-3}{1} = \dfrac{x_2+1}{1} = \dfrac{x_3-3}{2} = t \Leftrightarrow \begin{cases} x_1 = t+3 \\ x_2 = t-1 \\ x_3 = 2t+3 \end{cases}$

c) $Ox_1x_2 : x_3 = 0$

$\mathcal{D} \cap Ox_1x_2 \Rightarrow x_3 = 0 \Rightarrow -6t+3 = 0 \quad t = \dfrac{-3}{-6} = \dfrac{1}{2}$

$M\left(-3\cdot\dfrac{1}{2}+3, 5\cdot\dfrac{1}{2}-1, 0\right) \Rightarrow M\left(\dfrac{3}{2}, \dfrac{3}{2}, 0\right)$

$\mathcal{D} \cap Ox_1x_3 \Rightarrow x_2 = 0 \Rightarrow 5t-1 = 0 \quad t = \dfrac{1}{5}$

$N\left(-\dfrac{3}{5}+3, 0, -\dfrac{6}{5}+3\right) \Rightarrow N\left(\dfrac{12}{5}, 0, \dfrac{9}{5}\right)$

$\mathcal{D} \cap Ox_2x_3 \Rightarrow x_1 = 0 \Rightarrow -3t+3 = 0 \Rightarrow t = 1$

$Q(0,4,-3)$

2. $\mathcal{D} = ?$ a.î. $A(2,-5,3) \in \mathcal{D}$, $\mathcal{D} \parallel \mathcal{D}'$

$$\mathcal{D}' \begin{cases} 2x_1 - x_2 + 3x_3 + 1 = 0 \\ 5x_1 + 4x_2 - x_3 + 1 = 0 \end{cases} \quad (*)$$

Metoda 1:

$N_1 = (2,-1,3) \qquad N_2 = (5,4,-1)$

$$u_{\mathcal{D}'} = N_1 \times N_2 = \begin{vmatrix} e_1 & e_2 & e_3 \\ 2 & -1 & 3 \\ 5 & 4 & -1 \end{vmatrix} = e_1 \begin{vmatrix} -1 & 3 \\ 4 & -1 \end{vmatrix} - e_2 \begin{vmatrix} 2 & 3 \\ 5 & -1 \end{vmatrix} + e_3 \begin{vmatrix} 2 & -1 \\ 5 & 4 \end{vmatrix}$$

$$= e_1(1-12) - e_2(-2-15) + e_3(8+5) = (-11, 17, 13)$$

$\mathcal{D} \parallel \mathcal{D}' \Rightarrow V_{\mathcal{D}} = V_{\mathcal{D}'} = <\{u_{\mathcal{D}'}\}>$

$$\mathcal{D}: \frac{x_1-2}{-11} = \frac{x_2+5}{17} = \frac{x_3-3}{13} = t \qquad \begin{cases} x_1 = -11t + 2 \\ x_2 = 17t + 5 \\ x_3 = 13t + 3 \end{cases}$$

Metoda 2:

$(*)$ sistem compatibil simplu nedeterminat

$x_3 = t$

$$\begin{cases} 2x_1 - x_2 = -1 - 3t \mid \cdot 4 \\ 5x_1 + 4x_2 = t - 1 \end{cases}$$

$$8x_1 - 4x_2 = -4 - 12t \quad \Bigg| \Rightarrow 13x_1 = -11t - 5$$

$$x_1 = \frac{-11t - 5}{13}$$

$$x_2 = \frac{2(-11t-5) + 13 + 39t}{13} = \frac{17t + 30}{13}$$

$$\mathcal{D}': \begin{cases} x_1 = \frac{-11}{13}t - \frac{5}{13} \\ x_2 = \frac{17t}{13} + \frac{3}{13} \\ x_3 = t \end{cases} \qquad w = \left(\frac{-11}{13}, \frac{17}{13}, 1\right) = \frac{1}{13}(-11, 17, 13)$$

$\mathcal{D}_1: \begin{cases} x_1 + x_3 = 0 \\ x_2 - x_3 - 1 = 0 \end{cases}$, $\mathcal{D}_2: \begin{cases} x_2 = 0 \\ x_3 = 0 \end{cases}$

a) $\mathcal{D}_1, \mathcal{D}_2$ - necoplanare

b) ecuația perpendicularelor comune a dreptelor $\mathcal{D}_1, \mathcal{D}_2$

c) dist $(\mathcal{D}_1, \mathcal{D}_2)$

a) $\mathcal{D}_1: \begin{cases} x_1 = -t \\ x_2 = t+1 \\ x_3 = t \end{cases} \Leftrightarrow \frac{x_1}{-1} = \frac{x_2 - 1}{1} = \frac{x_3}{1} = t$

$u_1 = (-1, 1, 1)$

$A(0, 1, 0) \in \mathcal{D}_1$

$\mathcal{D}_2: \begin{cases} x_1 = s \\ x_2 = 0 \\ x_3 = 0 \end{cases}$

$u_2 = (1, 0, 0)$

$B(0, 0, 0) \in \mathcal{D}_2$

$\overrightarrow{AB} = (0, -1, 0)$

$\Delta_2 = \begin{vmatrix} -1 & 1 & 0 \\ 1 & 0 & -1 \\ 1 & 0 & 0 \end{vmatrix} = \begin{vmatrix} 1 & 0 \\ 0 & -1 \end{vmatrix} = -1 \neq 0 \Rightarrow \mathcal{D}_1, \mathcal{D}_2$ necoplanare

Fie $P_1(-t, t+1, t) \in \mathcal{D}_1$, $P_2(1, 0, 0) \in \mathcal{D}_2$

$\overrightarrow{P_1P_2} = (1+t, -t-1, -t)$ $\begin{cases} g_0(\overrightarrow{P_1P_2}, u_1) = 0 \\ g_0(\overrightarrow{P_1P_2}, u_2) = 0 \end{cases} \Leftrightarrow$

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$\Leftrightarrow \begin{cases} -s-t-t-1-t=0 \\ s+t=0 \end{cases} \Leftrightarrow \begin{cases} -s-3t-1=0 \\ s+t=0 \end{cases}$

$-2t=1$

$t=-\frac{1}{2}$

$s=\frac{1}{2}$

$P_1\left(\frac{1}{2},\frac{1}{2},-\frac{1}{2}\right) \quad P_2\left(\frac{1}{2},0,0\right)$

$\overrightarrow{P_1P_2}=\left(0,-\frac{1}{2},\frac{1}{2}\right)$

$P_1P_2: \frac{x_1-\frac{1}{2}}{0}=\frac{x_2}{-1}=\frac{x_3}{1}=t \Leftrightarrow \begin{cases} x_1=\frac{1}{2} \\ x_2=-t \\ x_3=t \end{cases}$

$dist(D_1,D_2)=dist(P_1,P_2)=$

$\|\overrightarrow{P_1P_2}\|=\sqrt{0+\frac{1}{4}+\frac{1}{4}}=\frac{1}{\sqrt{2}}=\frac{\sqrt{2}}{2}$

4. $D_1: \frac{x_1-1}{1}=\frac{x_2-2}{-1}=\frac{x_3+2}{2}$

$D_2: \begin{cases} 2x_1-x_3-1=0 \\ 2x_2+x_3+3=0 \end{cases}$

a) D_1, D_2 coplanare

b) ecuația planului determinat de D_1, D_2

c) $dist(D_1, D_2)$

$N_1=(2,0,-1)$

$N_2=(0,2,1)$

$u_2=N_1\times N_2=\begin{vmatrix} e_1 & e_2 & e_3 \\ 2 & 0 & -1 \\ 0 & 2 & 1 \end{vmatrix}=e_1\begin{vmatrix} 0 & -1 \\ 2 & 1 \end{vmatrix}-e_2\begin{vmatrix} 2 & -1 \\ 0 & 1 \end{vmatrix}+e_3\begin{vmatrix} 2 & 0 \\ 0 & 2 \end{vmatrix}=$

$= 2e_1 - 2e_2 + 4e_3 = (2, -2, 4) = 2(1, -1, 2) = 2u_1 \Rightarrow \mathcal{D}_1 \parallel \mathcal{D}_2$

$A_1(1, 2, -2)$

$A_2\left(\frac{1}{2}, -\frac{3}{2}, 0\right)$

$\overline{A_1A_2} = \left(\frac{1}{2} - 1, -\frac{3}{2} - 2, 0 + 2\right) = \left(-\frac{1}{2}, -\frac{7}{2}, 2\right) = \frac{1}{2}(\underbrace{-1, -7, 4}_{w})$

$u_1 = (1, -1, 2) \qquad \pi = \pi(A_1, u_1, w)$

$\forall M \in \pi \qquad (x_1 - 1, x_2 - 2, x_3 + 2) = t(1, -1, 2) + s(-1, 7, 4)$

$\overline{AM} = t u_1 + s w$

$$\begin{cases} x_1 - 1 = t - s \\ x_2 - 2 = -t - 7s \quad , t, s \in \mathbb{R} \\ -x_3 + 2 = 2t + 4s \end{cases}$$

$$\pi . \begin{vmatrix} x_1 - 1 & 1 & -1 \\ x_2 - 2 & -1 & -7 \\ x_3 + 2 & 2 & 4 \end{vmatrix} = 0$$

$$(x_1 - 1)\begin{vmatrix} -1 & -7 \\ 2 & 4 \end{vmatrix} - (x - 2)\begin{vmatrix} 1 & -1 \\ 2 & 4 \end{vmatrix} + (x_3 + 2)\begin{vmatrix} 1 & -1 \\ -1 & -7 \end{vmatrix} = 0$$

$10(x_1 - 1) - 6(x - 2) - 6(x_3 + 2) = 0$

$\pi: 5x_1 - 3x_2 - 3x_3 - 7 = 0$

Metoda 2:

$$\pi: \begin{vmatrix} x_1 & x_2 & x_3 & 1 \\ x_1 A & x_2 A & x_3 A & 1 \\ x_1 B & x_2 B & x_3 B & 1 \\ x_1 C & x_2 C & x_3 C & 1 \end{vmatrix} = 0$$

- $A(1,3,0)$, $B(3,-2,1)$ $C(\alpha,1,-3)$, $D(7,-2,3)$

$\alpha = ?$ a.î. A, B, C, D coplanare

$$\begin{vmatrix} 1 & 3 & 0 & 1 \\ 3 & -2 & 1 & 1 \\ \alpha & 1 & -3 & 1 \\ 7 & -2 & 3 & 1 \end{vmatrix} = 0 \Leftrightarrow \begin{vmatrix} 1 & 3 & 0 & 1 \\ 2 & -5 & 1 & 0 \\ \alpha-1 & -2 & -3 & 0 \\ 6 & -5 & 3 & 0 \end{vmatrix} = 0$$

$$\Rightarrow \begin{vmatrix} 2 & -5 & 1 \\ \alpha-1 & -2 & -3 \\ 6 & -5 & 3 \end{vmatrix} = 0 \Rightarrow \begin{vmatrix} 2 & -5 & 1 \\ \alpha+5 & -17 & 0 \\ 0 & 10 & 0 \end{vmatrix} = 0 \Rightarrow$$

$\Rightarrow 10(\alpha+5) = 0 \Rightarrow \alpha = -5$

- $M(1,1,1)$, $\pi: x_1 + 2x_2 + 3x_3 - 1 = 0$ $\quad \mathcal{D}: \begin{cases} x_1 - x_2 + x_3 + 1 = 0 \\ x_1 - 2x_3 - 1 = 0 \end{cases}$

a) $\pi_1 - ?$ a.î. $M \in \pi_1$ și $\pi_1 \parallel \pi$

$\pi_1 \parallel \pi \Rightarrow N_1 = N = (1,2,3)$

$\pi_1: x_1 + 2x_2 + 3x_3 + d = 0$, $M \in \pi_1$, $M(1,1,1) \Rightarrow 1+2+3+d = 0$

$\Rightarrow d = -6$

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$$\begin{cases} x_1 = 2t+1 \\ x_2 = 3t+2 \\ x_3 = t \end{cases} \Rightarrow \mathcal{D} \cap \pi \Rightarrow 2t+1+6t+4+3t-1 = 0$$

$11t = -4 \Rightarrow t = -\frac{4}{11}$

$\mathcal{D}_1 \ni M$ $\quad u_{\mathcal{D}_1} = N = (1,2,3) \Rightarrow \mathcal{D}_1 \; \frac{x_1-1}{1} = \frac{x_2-1}{2} = \frac{x_3-1}{3} = t \Rightarrow$

$\mathcal{D}_1 \perp \pi$

$$\Rightarrow \begin{cases} x_1 = t+1 \\ x_2 = 2t+1 \\ x_3 = 3t+1 \end{cases}$$

$\text{dist}(P, \pi) = \text{dist}(P, Q) = \|\overrightarrow{PQ}\|$