Seminar 5

(9) $(\mathbb{R}^4, +, \cdot)/\mathbb{R}$, $V' = \langle\{u, v, w\}\rangle$, $V'' = \langle\{u', v', w'\}\rangle$

$u = (2, 3, 11, 5)$, $v = (1, 1, 5, 2)$, $w = (0, 1, 1, 1)$

$u' = (2, 1, 3, 2)$, $v' = (1, 1, 3, 4)$, $w' = (5, 2, 6, 2)$

a) ~~$V' \oplus V''$~~

$$V' : \operatorname{rg}\begin{pmatrix} 2 & 1 & 0 \\ 3 & 1 & 1 \\ 11 & 5 & 1 \\ 5 & 2 & 1 \end{pmatrix} = 2$$

minorii: $\det\begin{vmatrix} 2 & 1 & 0 \\ 3 & 1 & 1 \\ 11 & 5 & 1 \end{vmatrix} = 0$, $\begin{vmatrix} 2 & 1 & 0 \\ 3 & 1 & 1 \\ 5 & 2 & 1 \end{vmatrix} = 0$

$V' = \langle\{u, v\}\rangle$ $\dim V' = 2$

R' este SLI maximal

reper în V'

$$V'' : \operatorname{rg}\begin{pmatrix} 2 & 1 & 5 \\ 1 & 1 & 2 \\ 3 & 3 & 6 \\ 2 & 4 & 2 \end{pmatrix} = 2$$

$V'' = \langle\{u', v'\}\rangle = \langle R''\rangle$ R'' SLI maximal

$\dim(V' + V'') = \dim(V') + \dim(V'') - \dim(V' \cap V'')$

demonstrăm $V' \cap V'' = \{0_{\mathbb{R}^4}\}$

Fie $x \in V' \cap V'' \Rightarrow x \in V'$, $x \in V'' \Rightarrow x = au + bv$

$x = a' \cdot u' + b' v'$

$x = a(2, 3, 11, 5) + b(1, 1, 5, 2)$

$x = a'(2, 1, 3, 2) + b'(1, 1, 3, 4)$

$\Rightarrow$ ~~$2a + 3a$~~ $(2a + b, 3a + b, 11a + 5b, 5a + 2b) = (2a' + b', a' + b', 3(a' + b'), 2a' + 4b')$

$$\begin{cases} 2a + b = 2a' + b' \\ 3a + b = a' + b' \\ 11a + 5b = 3a' + 3b' \\ 5a + 2b = 2a' + 4b' \end{cases}$$

$$\Rightarrow \left(\begin{array}{cccc|c} 2 & 1 & -2 & -1 & 0 \\ 3 & 1 & -1 & -1 & 0 \\ 11 & 5 & -3 & -3 & 0 \\ 5 & 2 & -2 & -4 & 0 \end{array}\right)$$

$\det \neq 0$

$\Rightarrow$ (3) sol $(a, b, a', b') = (0, 0, 0, 0)$

$V' \oplus V''$

$\left.\begin{array}{l} \dim(V' \oplus V'') = 4 = \dim \mathbb{R}^4 \\ V' \oplus V'' \subset \mathbb{R}^4 \end{array}\right\} \Rightarrow \mathbb{R}^4 = V'_q \oplus V''_4$

3/4, S4

$S_1 = \{x \in \mathbb{R}^4 / x_2 + x_3 + x_4 = 0\}$

$S_2 = \{x \in \mathbb{R}^4 \mid \begin{cases} x_1 + x_2 = 0 \\ x_3 + 2x_4 = 0 \end{cases}\}$

a) $\dim S_1, S_2, S_1 + S_2, S_1 \cap S_2$

b) este un reper în fiecare?

$\dim S_1 = 4 - 1 = 3$

$\dim S_2 = 4 - 2 = 2$

$x_4 = -x_2 - x_3$

$S_1 = \{(x_1, x_2, x_3, -x_2 - x_3) \mid x_1, x_2, x_3 \in \mathbb{R}\}$

$= x_1(1, 0, 0, 0) + x_2(0, 1, 0, -1) + x_3(0, 0, 1, -1)$, u, v, w

$R_1 = \{u, v, w\}$

$\left.\begin{array}{l} \text{S.G. pt. } S_1 \\ \dim S_1 = 3 = |R_1| \end{array}\right\} \Rightarrow R_1 \text{ - reper}$

$$\operatorname{rg}\begin{pmatrix} 1 & \boxed{\begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix}} & 0 \\ 0 & & 2 \end{pmatrix} = 2$$

$x_2 = -x_1$

$x_3 = -2x_4$

$S_2 = \{(x_1, -x_1, -2x_4, x_4) \mid x_1, x_4 \in \mathbb{R}\}$

$= x_1(1, -1, 0, 0) + x_4(0, 0, -2, 1)$, u', v'

$R_2 = \{u', v'\}$

$\left.\begin{array}{l} R_2 \text{ - S.G. în } S_2 \\ \dim S_2 = 2 = |R_2| \end{array}\right\} \Rightarrow R_2 \text{ reper}$

$$S_1 \cap S_2 = \left\{x \in \mathbb{R}^4 \,\middle|\, \begin{cases} x_2 + x_3 + x_4 = 0 \\ x_1 + x_2 = 0 \\ x_3 + 2x_4 = 0 \end{cases}\right\}$$

$$A = \begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 2 \end{pmatrix} \quad \operatorname{rg}(A) = 3$$

$\Rightarrow \dim(S_1 \cap S_2) = 4 - 3 = 1$

$x_2 + x_3 = -x_4 \Rightarrow x_2 = -x_4 - x_3 = x_4 + 2x_4 = x_4$

$x_1 + x_2 = 0 \Rightarrow x_1 = -x_2$

$x_3 = -2x_4$

$S_1 \cap S_2 = \{(-x_4, x_4, -2x_4, x_4) \mid x_4 \in \mathbb{R}\}$

$x_4(-1, 1, -2, 1) \Rightarrow \{t\}$ reper în $S_1 \cap S_2$

t??

$\left.\begin{array}{l} \dim(S_1 + S_2) = 2 + 3 - 1 = 4 \\ \text{dar } S_1 + S_2 \subset \mathbb{R}^4 \end{array}\right\} \Rightarrow S_1 + S_2 = \mathbb{R}^4$

$R_0 = \{e_1, e_2, e_3, e_4\}$

Seminar 5

EXAMEN

(3) $(\mathbb{R}^3, +, \cdot) / \mathbb{R}$

$V_1 = \{(x, y, z) \in \mathbb{R}^3 \mid 2x - y + z = 0\}$

$V_2 = \langle \{(1, -1, 2), (3, 1, 0)\} \rangle$

a) $V_2 = a(1, -1, 2) + b(3, 1, 0)$

$= (a + 3b, -a + b, 2a)$

$\{u, v\}$ SLI ?

$\begin{pmatrix} 1 & 3 \\ -1 & 1 \\ 2 & 0 \end{pmatrix} = 2 \text{ (max)} \Rightarrow$ SLI

Fie $x \in V_2 \Rightarrow (\exists)\, a, b \in \mathbb{R}$

$\underline{x = au + bv}$

compatibil $\Leftrightarrow \Delta_c = 0$

$\begin{cases} a + 3b = x \\ -a + b = y \\ 2a = z \end{cases}$ $\quad A = \left(\begin{array}{cc|c} 1 & 3 & x \\ -1 & 1 & y \\ 2 & 0 & z \end{array}\right)$

$\Delta_c = 0 \Leftrightarrow z + 6y - 2x + 3z = 0$

$-2x + 6y + 4z = 0 \quad /:2$

$-x + 3y + 2z = 0$

$V_2 = \{(x, y, z) \in \mathbb{R}^3 \mid -x + 3y + 2z = 0\}$

b) $V_1 : z = -2x - y$

$\dim(V_1) = 3 - 1 = 2$

$V_1 = \{(x, y, -2x - y) \mid x, y \in \mathbb{R}\}$

$R_1 = x\underbrace{(1, 0, -2)}_{u'} + y\underbrace{(0, 1, -1)}_{v'}$

$V_1 \cap V_2 = \left\{(x, y, z) \in \mathbb{R}^3 \,\middle|\, \begin{array}{l} 2x + y + z = 0 \\ -x + 3y + 2z = 0 \end{array}\right\}$

$M_{V_1 \cap V_2} : \left(\begin{array}{ccc|c} \boxed{\begin{matrix} 2 & 1 \\ -1 & 3 \end{matrix}} & \begin{matrix} 1 \\ 1 \end{matrix} & \begin{matrix} 0 \\ 0 \end{matrix} \end{array}\right)$

$\dim(V_1 \cap V_2) = 3 - 2 = 1$

$2x + y = -z$

$-x + 3y = -2z \quad /\cdot 2$

$7y = -5z \Rightarrow y = -\frac{5}{7} z$

$x = -\frac{1}{7} z$

$V_1 \cap V_2 = \{ \frac{z}{7} (-1, -5, 7) \mid z \in \mathbb{R} \}$

$R = \{w\}$ - reper in $V_1 \cap V_2$

$\dim(V_1 + V_2) = 2 + 2 - 1 = 3$
$V_1 + V_2 \subseteq \mathbb{R}^3$ $\Big\} \Rightarrow V_1 + V_2 = \mathbb{R}^3$

nu e sumă directă $(V_1 \cap V_2 \neq 0_{\mathbb{R}^3})$

9. $(\mathbb{R}_4[X],+,\cdot)/\mathbb{R}$

$U=\langle\{1+2x+x^3,\ 1-x-x^2\}\rangle$

$V=\langle\{x+x^2-3x^3,\ 2+2x-2x^3\}\rangle$

a) o bază în $U\cap V$, $U+V$

b) Verif. Th. Grassmann

$U=\langle\{\underset{u}{(1,2,0,1,0)},\ \underset{v}{(1,-1,-1,0,0)}\}$

$V=\langle\{\underset{u'}{(0,1,1,-3,0)},\ \underset{v'}{(2,2,0,-2,0)}\}\rangle$

$$rg\begin{pmatrix}1&1\\2&1\\0&-1\\1&0\\0&0\end{pmatrix}=2$$

$\langle R\cup R'\rangle=U+V$

$$rg\begin{pmatrix}0&2\\1&2\\1&0\\-3&-2\\0&0\end{pmatrix}=2$$

$$rg\begin{pmatrix}1&1&0&2\\2&1&1&2\\0&-1&1&0\\1&0&-3&-2\\0&0&0&0\end{pmatrix}=3$$

$$\begin{vmatrix}1&1&0&2\\2&1&1&2\\0&-1&1&0\\1&0&-3&-2\end{vmatrix}=\begin{vmatrix}1&1&0&1\\2&0&1&1\\0&0&1&0\\1&-3&-3&-1\end{vmatrix}=\dots=0$$

$$\begin{vmatrix}1&1&2\\2&1&2\\0&-1&0\end{vmatrix}\neq0$$

$\{u,v,u'\}$ reper în $U+V$, $\dim(U+V)=3$

$\dim(U+V)=\dim U+\dim V-\dim U\cap V$

$3=2+2-\dim U\cap V=1$ $\quad\dim U\cap V=1$