Curs 2

Notez ^tA matricea transpusă

Pb1 A = inf Δ $\Rightarrow A = \begin{pmatrix} a_{11} & & & \\ a_{21} & a_{22} & & \\ \vdots & & \ddots & \\ a_{n1} & \dots & & a_{nn} \end{pmatrix}$ Vreau

A^{-1} = inf Δ.

$B = A^{-1} = \begin{pmatrix} b_{11} & \dots & b_{1n} \\ \vdots & & \\ b_{n1} & \dots & b_{nn} \end{pmatrix}$

$AB = BA = I_n$.

$(a_{11}b_{11} \quad a_{11}b_{12} \quad \dots \quad a_{11}b_{1n}) = L_1(AB) = (1, 0, \dots 0)$

$\Rightarrow a_{11}, b_{11} \neq 0, \quad a_{11}b_{1i} = 0, \; i \geq 2 \Rightarrow b_{1i} = 0$.

Inductiv: ~~ast pt.~~ presupun că am 0 peste diagonala pp. și peste linia $k-1$ în B.

$L_k(AB) = (\cancel{a_{kk}}b_k \; 0, 0, \dots, 0, a_{kk}b_{kk}, a_{kk}b_{k,k+1}, \dots a_{kk}b_{kn})$

$= (0, 0, \dots, 0, \underset{\substack{\uparrow \\ \text{poz } k}}{1}, 0, \dots 0)$

$\Rightarrow a_{kk}, b_{kk} \neq 0$.

$a_{kk}b_{kj} = 0, \; j \geq k+1 \Rightarrow b_{kj} = 0$ ✓.

$\Rightarrow B$ inf Δ.

<u>Obs</u>: inversa lui A e B.

inversa lui ${}^tA$ este C.

$${}^tA \cdot C = I_n \Rightarrow {}^tC \cdot A = \underset{\underset{B \cdot A}{\parallel}}{I_n} \Rightarrow {}^tC = B \Rightarrow \boxed{C = {}^tB}$$

Dacă A e ~~inf~~ sup. Δ ⇒ ${}^tA$ e inf Δ ⇒

⇒ inversa B a lui ${}^tA$ e inf Δ ⇒ inversa lui A

e ${}^tB$ = sup Δ.

<u>Pb2</u> A, B inf Δ $\overset{?}{\Rightarrow}$ AB inf Δ.

$$AB = \begin{pmatrix} a_{11} & & 0 \\ \vdots & \ddots & \\ a_{n1} & \cdots & a_{nn} \end{pmatrix} \cdot \begin{pmatrix} b_{11} & & 0 \\ \vdots & \ddots & \\ b_{n1} & \cdots & b_{nn} \end{pmatrix} =$$

$$= \begin{pmatrix} a_{11}b_{11} & 0 & 0 & \cdots & 0 \\ & & & & \\ & & & & \end{pmatrix}$$

Fie $(AB)_{ij} = L_i(A) \cdot C_j(B)$

$$= (a_{i1}, \ldots, a_{ii}, 0, \ldots, 0) \cdot \begin{pmatrix} 0 \\ 0 \\ b_{ij} \\ \vdots \\ b_{nj} \end{pmatrix} = 0 \quad \text{dc. } j > i$$

$$= \begin{pmatrix} & 0 \\ & \end{pmatrix} \Rightarrow AB \text{ inf } \Delta.$$

Dc. A și B sunt sup Δ ⇒ ${}^tA$, ${}^tB$ inf Δ.

${}^t(AB) = {}^tB\,{}^tA$ inf Δ ⇒ AB sup Δ.

Pb3 $M_k = I_n - m^{(k)}\,{}^t(e^{(k)})$

$$\left(I_n - m^{(k)}\,{}^t(e^{(k)})\right)\left(I_n + m^{(k)}\,{}^t(e^{(k)})\right) =$$

$$= I_n - m^{(k)}\,{}^t(e^{(k)}) \cdot \underbrace{\,\cdot\, m^{(k)}}_{m_k^{(k)} = 0}{}^t(e^{(k)}) = I_n.$$

$\Rightarrow I_n + m^{(k)}\,{}^t(e^{(k)}) =$ inversa lui M_k.

Obs $AB = I_n \Rightarrow BA = I_n$, $A, B \in M_n(\mathbb{R})$.

$\Downarrow$

A, B inv $\Rightarrow A^{-1} \mid AB = I_n \Rightarrow B = A^{-1}$

$BA = A^{-1}A = I_n$ ✓.

$\boxed{\text{Curs 3}}$

Pb1 $A = \left[\begin{array}{c|c} a_{11} & A_{12} \\ \hline A_{21} & A_{22} \end{array}\right]$, $S = A_{22} - \frac{1}{a_{11}} A_{21} A_{12}$ inversabile

$$S = \begin{bmatrix} a_{22} & \dots & a_{2n} \\ \vdots & & \\ a_{n2} & \dots & a_{nn} \end{bmatrix} - \frac{1}{a_{11}} \begin{bmatrix} a_{21}a_{12} & \dots & a_{21}a_{1n} \\ \vdots & & \\ a_{n1}a_{12} & \dots & a_{n1}a_{1n} \end{bmatrix}$$

$$S = \begin{bmatrix} a_{22} - \frac{a_{21}a_{12}}{a_{11}} & \dots & a_{2n} - \frac{a_{21}a_{1n}}{a_{11}} \\ \vdots & & \\ a_{n2} - \frac{a_{n1}a_{12}}{a_{11}} & \dots & a_{nn} - \frac{a_{n1}a_{1n}}{a_{11}} \end{bmatrix}$$

$$\det A \underset{0}{\neq} = \det \begin{vmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ \vdots & & & \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{vmatrix} \quad L_i \leftarrow L_i - \frac{a_{i1}}{a_{11}} L_1$$

$$= \left(\begin{array}{c|c} a_{11} & A_{12} \\ \hline O_{n-1\times 1} & S \end{array} \right) = \underbrace{a_{11} \cdot \det S}_{\neq 0} \underset{\neq 0}{} \Rightarrow \underline{\det S \neq 0}$$

$\boxed{\text{Curs 4}}$

<u>Pb1</u> $M^{(1)} = I_n - m^{(1)}\, {}^t(e^{(1)})$

$$P^{(2)} M^{(1)} P^{(2)} = P_{2l} \cdot [e^{(1)} - m^{(1)}, e^{(2)}, \dots e^{(n)}] P_{2l} =$$

schimbă liniile 2 și l

$$= \begin{array}{c} \\ 2 \rightarrow \\ \\ l \rightarrow \end{array} \begin{bmatrix} 1 & 0 & \dots & \\ -m_l & 0 & & \\ \vdots & & 1 & \\ -m_{l-1} & & & 1 \\ -m_2 & \dots & 0 & 1 \\ & & & & 0 \end{bmatrix} \cdot P_{2l} = \begin{bmatrix} 1 & 0 & & \\ -m_l & 1 & & \\ \vdots & & \ddots & \\ -m_{l-1} & & 1 & \\ -m_2 & & & 1 \\ \vdots & & & & \ddots \\ -m_n & & & & & 1 \end{bmatrix}$$

schimbă col 2 și l

$$= I_n - \tilde{m}\, {}^t(e^{(1)}) \quad \text{unde } \tilde{m}_i = \begin{cases} m_i^{(1)}, & i \neq 2, l \\ m_2, & i = l \\ m_l, & i = 2 \end{cases}$$

Pb2 $A \in M_n(\mathbb{R})$ admite MEGFP $\Leftrightarrow$ admite LU fără pivotare.

$\boxed{\Rightarrow}$ Știu că A admite MEGFP $\Rightarrow$

$\Rightarrow \exists M^{(i)}$ cu: ~~$M^{(1)} \dots M^{(n-1)}$~~

$\underbrace{M^{(n-1)} \dots M^{(1)}}_{\text{inf } \Delta + \text{inv.}} A = \underbrace{U}_{\text{sup } \Delta} \in M_n(\mathbb{R})$

$\overset{\text{def}}{=} L^{-1} \Rightarrow \boxed{A = \underbrace{L}_{\text{inf. } \Delta} U}$

$\boxed{\Leftarrow}$ Știu $A = LU$. Aș vrea ca $L^{-1} = M^{(n-1)} \dots M^{(1)}$

Dacă iau $m^{(k)} = (l_{ik})_{i=\overline{k+1,n}} \Rightarrow$

$\Rightarrow L = \underbrace{(I_n + m^{(1)})}_{[M^{(1)}]^{-1}} \dots \underbrace{(I_n + m^{(n-1)})}_{[M^{(n-1)}]^{-1}}$ q.e.d.

$\boxed{\text{Curs 5}}$

Pb1 $A \in M_n(\mathbb{R})$ SPD.

a) $\max\limits_{i,j} |a_{ij}| \leq \max\limits_{i} |a_{ii}|$

Fie $\max_{i,j} |a_{ij}| = |a_{pq}|$, $p \leq q$ (dc. $p > q$

iau ${}^tA$ în loc de A, ~~$\cancel{2}$~~ $\Rightarrow |a^t_{q,p}| = \max_{i,j} |a^t_{ij}|$)

A e S.P.D $\Rightarrow {}^tx A x \geq 0 \ \forall x \in \mathbb{R}^n$ ~~$\cancel{(\neq 0)}$~~

iau $x = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$ $\begin{matrix} \\ \\ \\ \leftarrow \text{poz } p \\ \\ \\ \leftarrow \text{poz } q \\ \\ \\ \\ \end{matrix}$ $\Rightarrow {}^tx A x = a_{pp} + a_{qq} + 2a_{pq} \geq 0$

(*)

iau $x = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \\ -1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$ $\begin{matrix} \\ \\ \\ \leftarrow \text{poz } p \\ \\ \\ \\ \leftarrow \text{poz } q \\ \\ \\ \\ \end{matrix}$ $\Rightarrow {}^tx A x = a_{pp} + a_{qq} - 2a_{pq} \geq 0$ (+).

$a_{pp} + a_{qq} \geq 0.$

$\Rightarrow \max\{a_{pp}, a_{qq}\} \geq 0.$

$\Rightarrow 2\max\{a_{pp}, a_{qq}\} \geq a_{pp} + a_{qq} \geq 2 \underset{\substack{\uparrow \\ \max |a_{ij}|}}{a_{pq}}$

$\Rightarrow a_{pq} = \max\{a_{pp}, a_{qq}\}$ sau $a_{pq} < 0$

ph. e gata

$\Downarrow$

$|a_{pq}| = -a_{pq}.$

(*)
$\Rightarrow |a_{pp} + a_{qq}| = a_{pp} + a_{qq} \geq -2a_{pq} = 2 \underset{\substack{\shortparallel \\ \max |a_{ij}|}}{|a_{pq}|}$

$\overset{\leq}{2\max\{|a_{pp}|, |a_{qq}|\}}$

$\Rightarrow |a_{pq}| \geq \max\{|a_{pp}|, |a_{qq}|\}$.

$\Rightarrow \max\limits_{1 \leq i,j \leq n} |a_{ij}| \leq \max\limits_{1 \leq i \leq n} |a_{ii}|$

b) $A \in$ S.P.D $\Rightarrow$ $\forall$ minor principal e >0.

iau minorii de tip. $\begin{vmatrix} a_{ii} & a_{ij} \\ a_{ji} & a_{jj} \end{vmatrix} = a_{ii} a_{jj} - a_{ij} a_{ji} > 0$

$A = {}^tA \Rightarrow a_{ij} = a_{ji} \Rightarrow a_{ii} a_{jj} > (a_{ij})^2$

$\boxed{\text{Curs 6}}$

$\boxed{\text{Pb1}}$ $A = [a_1, \dots, a_n] \in M_{m \times n}(\mathbb{R})$ cu $m \geq n$. U.A.S.E

1) A inv. la stânga.
2) $a_1 \dots a_n$ lin. indep.
3) ${}^tA \cdot A \in M_n(\mathbb{R})$ inversabilă.
4) $\text{rk} A = n$.
5) ${}^tA \cdot A \in$ SPD

2) $\Leftrightarrow$ 4)

$\text{rk} A = \dim \langle a_1, \dots a_n \rangle = n \Leftrightarrow a_1 \dots a_n$ lin. indep.

~~3) inv. la stg~~ 3) $\Rightarrow$ 1)

${}^tA \cdot A$ inversabilă $\Rightarrow \exists B \in M_n(\mathbb{R})$ a.î. $(B \cdot {}^tA) A = I$

$\Rightarrow A$ inv. la stânga.

$\boxed{3) \Rightarrow 4)}$

$n = rk({}^tAA) \le rk(A) \le n \Rightarrow rk(A) = n$.

$A = [a_1 \dots a_n]$.

$\boxed{1) \Rightarrow 2)}$ Iau $\underset{n \times m}{B}$ cu $B \cdot A = I_n$.

$\Rightarrow [B \cdot a_1 ; Ba_2 ; \dots ; Ba_n] = [e_1, e_2, \dots e_n]$

$\Rightarrow Ba_1, \dots Ba_n$ bază.

B) $\lambda_1 a_1 + \dots \lambda_n a_n = 0 \Rightarrow \lambda_1 Ba_1 + \dots + \lambda_n Ba_n = 0$

$\Rightarrow \lambda_1 = \dots = \lambda_n = 0 \Rightarrow a_1 \dots a_n$ lin. indep.

$\boxed{2) \Rightarrow 3)}$

$\ker A \subseteq \ker {}^tA \cdot A$

$X \in \ker {}^tA \cdot A \Rightarrow {}^tAAX = 0_n \Rightarrow {}^tX{}^tAAX = 0_n$

$\Rightarrow AX = 0_m \Rightarrow X \in \ker A \Rightarrow \ker {}^tA \cdot A \subseteq \ker A \Rightarrow$

$\Rightarrow \ker A = \ker {}^tA \cdot A$.

$n = \dim \operatorname{Im} {}^tA \cdot A + \dim \ker {}^tA \cdot A$

$n = \dim \operatorname{Im} A + \dim \ker A$.

$a_1 \dots a_n$ lin. indep $\Rightarrow \dim \operatorname{Im} A = n \Rightarrow \dim$

$\ker A = 0 \Rightarrow \dim \operatorname{Im} {}^tAA = n \Rightarrow {}^tA \cdot A$ inv.

$\boxed{4) \Rightarrow 5)}$

${}^tX({}^tAA)X = \|AX\|^2 \ge 0$

~~$m = rk\, {}^tA A = rk\, {}^tA \cdot A$~~ $\Rightarrow$ semipoz. def.

$rg\, A = n \Rightarrow {}^tA \cdot A$ inv

$\lambda \in Spec\; {}^tA \cdot A \Rightarrow \lambda \geq 0$ $\Big\} \Rightarrow \lambda > 0.$

${}^t({}^tA \cdot A) = {}^tA A \Rightarrow$ simetrică

$\boxed{S) \Rightarrow 1)}$ ${}^tA \cdot A$ SPD $\Rightarrow$ e inversabilă

$\Rightarrow \exists B$ cu $(B\,{}^tA) \cdot A = I_n \Rightarrow A$ inv. la stânga

Curs 7

Pb1 $Q \in M_{m\times n}(\mathbb{R})$, $m \geq n$ ortogonală.

1) ${}^t(QX)\cdot(QY) = {}^tX \underbrace{{}^tQ\,Q}_{I_n} Y = {}^tXY$.

2) $\|QX\|_2^2 = {}^t(QX)\cdot(QX) \overset{1)}{=} {}^tXX = \|X\|_2^2$

$\Rightarrow \|QX\|_2 = \|X\|_2$.

3) $QX = 0_m \;\Big|\; {}^tQ \Rightarrow \underbrace{{}^tQ\,Q}_{I_n} X = 0_n \Rightarrow X = 0_n$.

4) ${}^tQ\cdot Q = I_n \Rightarrow Q$ are inversa la stânga ${}^tQ$

5) $\|Q\|_2 = \sup\limits_{X\in\mathbb{R}^n\setminus\{0_n\}} \dfrac{\|QX\|_2}{\|X\|_2} \overset{2)}{=} \sup\limits_{X\in\mathbb{R}^n\setminus\{0_n\}} \dfrac{\|X\|_2}{\|X\|_2} = 1$.

6) $m = n \Rightarrow {}^tQ\cdot Q = I_n \Rightarrow \det {}^tQ \cdot \det Q = 1 \Rightarrow$

$\Rightarrow Q$ inversabilă $\quad {}^tQ\cdot Q = I_n \;\big|\cdot Q^{-1}$

${}^tQ = Q^{-1}$

7) $\lambda \in Spec(Q) \Rightarrow \exists\, v \in \mathbb{R}^n\setminus\{0_n\}$ a.î. $Q\cdot v = \lambda v$.

$\Rightarrow \|Qv\|_2 = \|\lambda v\|_2$

$\|v\|_2 = |\lambda|\cdot\|v\|_2 \qquad \Rightarrow \boxed{|\lambda| = 1}$

$\neq 0$

$\boxed{Pb2}$

1) Q mat. ortonormată $\Rightarrow {}^tQ \cdot Q = I_n$.

$${}^t(Q\,{}^tQ) = Q \cdot {}^tQ \Rightarrow \text{simetrică}$$
$$(Q\,{}^tQ)^2 = Q\,\underbrace{{}^tQ\,Q}_{I_n}\,{}^tQ = Q\,{}^tQ$$
$\Bigg\} \Rightarrow {}^tQQ$ mat. proiectie

Iau $X \in \mathbb{R}^n$ și vreau $Q \cdot {}^tQ\,X \in S = \langle g_1 \dots g_n \rangle$.

Completez s la o bază ortonormală (OV) în ~~Gram-Schmidt~~

$\mathbb{R}^m \Rightarrow \{g_1, \dots g_n, g_{n+1}, \dots, g_m\} \Rightarrow {}^tg_i \cdot g_j = \delta_{ij}$

$$\Rightarrow X = \sum_{i=1}^{m} \alpha_i g_i,\ \alpha_i \in \mathbb{R}.\ \forall i = \overline{1,m}.$$

$$Q\,{}^tQ\,X = [g_1 \dots g_n] \cdot \begin{bmatrix} {}^tg_1 \\ \vdots \\ {}^tg_n \end{bmatrix} \cdot \left(\sum_{i=1}^{m} \alpha_i g_i\right)$$

$$= [g_1 \dots g_n] \cdot \begin{bmatrix} \sum_{i=1}^{m} \alpha_i\,{}^tg_1 g_i \\ \vdots \\ \sum_{i=1}^{m} \alpha_i\,{}^tg_n g_i \end{bmatrix} =$$

$$= [g_1 \dots g_n] \cdot \begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{bmatrix} = \sum_{i=1}^{n} \alpha_i g_i \in \underline{S}.$$

2) $\mathrm{rk}\,A = n \Rightarrow A$ inv. la stânga $\Rightarrow {}^tA \cdot A$ inversabilă.

Vreau $A({}^tA \cdot A)^{-1}\,{}^tA$ mat. proiectie pe

$S = \langle a_1, \dots a_n \rangle$.

$$^t\left(\underbrace{A({}^tAA)^{-1}\,{}^tA}_{\text{not}\ M}\right) = A\,{}^t\left[({}^tAA)^{-1}\right]{}^tA$$

$$= A({}^tAA)^{-1}A \Rightarrow \text{simetrică}$$

$$M^2 = A(\cancel{{}^tAA})^{-1}\,\cancel{{}^tA\cdot A}({}^tAA)^{-1}A$$

$$= A({}^tAA)^{-1}\,{}^tA = M \Rightarrow$$

$\Rightarrow$ mat. proiectie

$S = \langle a_1, \dots a_n]$.

Știu $\mathbb{R}^m = S \oplus S^\perp \Rightarrow$ completez S

la o bază în $\mathbb{R}^m$ cu elem. din $S^\perp$.

$$B = \{a_1, \dots a_n, \underbrace{a_{n+1}, \dots, a_m}_{\in S^\perp}\} \Rightarrow {}^ta_i a_j = 0$$

pt. $i \le n$

$n+1 \le j \le m$

$$X \in \mathbb{R}^m \Rightarrow X = \sum_{i=1}^m \alpha_i a_i, \quad \alpha_i \in \mathbb{R},\ i = \overline{1,m}$$

$$\underset{n\times m}{{}^tA}\cdot \underset{m\times 1}{X} = \begin{bmatrix} {}^ta_1 \\ \vdots \\ {}^ta_n \end{bmatrix}\cdot\left[\sum_{i=1}^m \alpha_i a_i\right] = \begin{bmatrix} \sum_{i=1}^m \alpha_i\,{}^ta_1 a_i \\ \vdots \\ \sum_{i=1}^m \alpha_i\,{}^ta_n a_i \end{bmatrix}$$

$$= \begin{bmatrix} \sum_{i=1}^n \alpha_i\,{}^ta_1 a_i \\ \vdots \\ \sum_{i=1}^n \alpha_i\,{}^ta_n a_i \end{bmatrix} \Rightarrow \cancel{{}^tA\,X \in}$$

$$MX = [a_1 \dots a_n]\left(\begin{bmatrix} {}^ta_1 \\ \vdots \\ {}^ta_n \end{bmatrix}[a_1 \dots a_n]\right)^{-1}\cdot\begin{bmatrix} \sum_{i=1}^n \alpha_i\,{}^ta_1 a_i \\ \vdots \\ \sum_{i=1}^n \alpha_i\,{}^ta_n a_i \end{bmatrix}$$

$\in \langle a_1, \dots, a_n \rangle$.

3) ~~P mat. proi~~

P mat. proiecție ortog. pe S $\overset{?}{\Rightarrow} I_m - P$ proiecție ortog pe $S^\perp$.

$$ {}^t(I_m - P) = I_m - {}^tP \overset{{}^tP = P}{=} I_m - P, \text{ simetrică} $$

$$ (I_m - P)^2 = I_m - 2P + P^2 \overset{P^2 = P}{=\!=} I_m - P $$

$\Rightarrow I_m - P$ proiecție ortog.

Fie $\langle s_1, \ldots s_k \rangle$ bază în S. O completez la o bază în $\mathbb{R}^m$ $\langle \underbrace{s_1, \ldots s_k}_{\text{cu elem. din } S^\perp}, \underbrace{s_{k+1}, \ldots s_m}_{\in S^\perp} \rangle \Rightarrow$

$$ \Rightarrow {}^ts_i \cdot s_j = 0 \quad \text{pt.} \begin{cases} i \leq k \\ k+1 \leq j \leq m \end{cases} $$

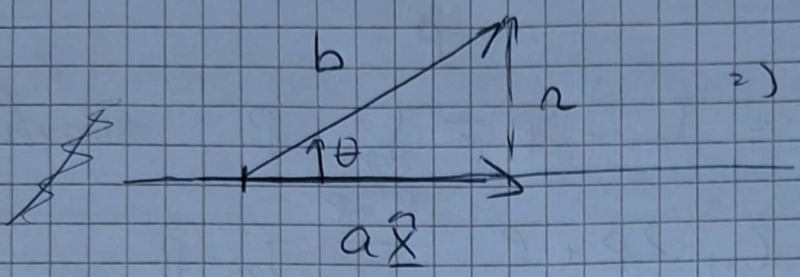
$$ X \in \mathbb{R}^m \Rightarrow X = \sum_{i=1}^{m} \alpha_i s_i \quad \alpha_i \in \mathbb{R}. $$

$$ (I_m - P)X = X - PX = \sum_{i=1}^{m} \alpha_i s_i - \sum_{i=1}^{k} \alpha_i s_i $$

$$ = \sum_{i=k+1}^{m} \alpha_i s_i \in \langle s_{k+1}, \ldots, s_m \rangle = S^\perp. $$

$\boxed{\text{Curs } 8}$

Pb 1 $\|r\|_2^2 = \|b - a\hat{x}\|_2^2$



$\Rightarrow \cancel{k}\ \|r\|_2 = \|b\|_2 \sin\theta$

$\Rightarrow \|r\|_2^2 = \|b\|_2^2 \sin^2\theta$.

Pb 2: $A \in M_{m\times n}(\mathbb{R})$, $m \geq n$, $rk\, A = n$

a) $\underline{{}^tA \cdot A}$ poz. def

$rk\, A = n \Rightarrow {}^tA\cdot A$ inversabilă $\Rightarrow \lambda \in Spec({}^tAA)$, $\lambda \neq 0$.

Am dem. că ${}^tA\cdot A$ e semipoz. def $(\lambda \geq 0)$

$\Rightarrow \boxed{\lambda > 0}$.

b) $\tilde{A} = \left(\begin{array}{c|c} I_m & \overset{m\times n}{A} \\ \hline {}^tA & O_n \end{array}\right)$ $\Rightarrow \det \tilde{A} = \det(A\cdot {}^tA)$.

$A \cdot {}^tA \in M_m(\mathbb{R})$

$m > n$.

$rk(A\cdot {}^tA) \leq rk(A) = n < m$

$\Rightarrow \det(A\cdot {}^tA) = 0 \Rightarrow \exists\, \lambda \in Spec(\tilde{A})$ cu $\lambda = 0$

$\Rightarrow \tilde{A}$ nu e poz. def.

Pb3 $Q = \left[\begin{array}{c|c} \underset{m\times m}{A} & \underset{m\times n}{B} \\ \hline \underset{n\times m}{0} & \underset{n\times n}{C} \end{array}\right] \in M_{m+n}(\mathbb{R})$.

$I_{m+n} = {}^tQ \cdot Q = \left[\begin{array}{c|c} {}^tA & 0_{m\times n} \\ \hline {}^tB & {}^tC \end{array}\right] \cdot \left[\begin{array}{c|c} A & B \\ \hline 0_{n\times m} & C \end{array}\right]$

$= \left[\begin{array}{c|c} {}^tA\cdot A & {}^tA\cdot B \\ \hline {}^tB\cdot A & {}^tB\cdot B + {}^tC\cdot C \end{array}\right] \Rightarrow {}^tA\cdot A = I_m \Rightarrow$ A ortog.

$\Rightarrow {}^tA\cdot B = 0_{m\times n}$. $\Rightarrow \| {}^tA\cdot B\|_2 = 0$, $\| {}^tA\cdot B\|_2 = \|B\|_2$ $\Big\} \Rightarrow B = 0_{m\times n}$.

$\Rightarrow \underbrace{{}^tB\cdot B}_{0_n} + {}^tC\cdot C = I_n \Rightarrow C$ ortog.

Pb4 $a \in M_{m,1}(\mathbb{R})$.

a) factorizare QR pt. a.

$a = q \cdot r$, $q \in \mathbb{R}^m$, $r \in \mathbb{R}$. $\Rightarrow$ ~~ortogonal~~ o

$\Rightarrow r = \|a\|_2$

$q = \frac{1}{r} \cdot a$, $\underline{a \neq 0_m}$

b) sol. PCMMP: $ax = b$.

$\hat{x} = ({}^ta\cdot a)^{-1} \cdot {}^ta \cdot b$.

$\hat{x} = (r \cdot {}^tq \cdot q \cdot r)^{-1} \cdot r \cdot {}^tq \cdot b$.

$$\hat{x} = \frac{1}{\cancel{a^2}} \cdot \cancel{a} \cdot \frac{1}{\cancel{a}} \cdot {}^t a \cdot b$$

$$\Rightarrow \hat{x} = \frac{1}{\|a\|_2^2} \, {}^t a \cdot b$$

<u>Pb5</u> Unicitatea factorizării QR ($rk A = n$) pornind de la Gram-Schmidt modif.

Demonstrez inductiv că elementele din Q și R sunt unic determinate de elem. anterior det.

La pasul <u>$k=1$</u>:

$$Q = \{ q_1 \dots q_n \}, \quad R = \begin{pmatrix} r_{11} & \dots & r_{1n} \\ & \ddots & \vdots \\ 0 & & r_{nn} \end{pmatrix}$$

$A = \{ a_1 \dots a_n \}$.

$a_1 = q_1 r_{11}$ | trec la normă

$\Rightarrow \|a_1\|_2 = r_{11} \Rightarrow q_1 = \frac{1}{\|a_1\|_2} a_1$

$\Rightarrow r_{11}$ și q_1 sunt unic determinate,

Pp. adevărat pt. pașii $x \leq k-1$ și dem. pt. $x = k$.

${}^t q_i \,|\, a_k = r_{1k} q_1 + \dots + r_{k-1,k} q_{k-1} + r_{kk} q_k \qquad i = \overline{1, k-1}$

${}^t q_i a_k = r_{ik}$ unic <u>det</u> $\Rightarrow$

$$\Rightarrow r_{kk} q_k = a_k - \sum_{i=1}^{k-1} r_{i,k} q_i = b_k$$

$\uparrow$ unic det.

trec la norme

$r_{kk} = \|b_k\|_2$ unic det.

$q_k = \frac{1}{r_{kk}} b_k$ unic det.

$\Rightarrow$ factorizarea este unică.

$\boxed{\text{Pb6}}$ $A = [a_1 \dots a_m]$, $\operatorname{rk} A = n$.

$A = QR$ cu $Q = [q_1 \dots q_m]$ și $Q_i = q_i {}^t q_i$

$Q_0 = O_{m \times m}$.

a) Gram-Schmidt clasică echivalentă cu

$$q_k r_{kk} = [I_m - (Q_0 + \dots + Q_{k-1})] a_k.$$

Dacă dem $\boxed{\text{„}\Rightarrow\text{"}}$ din unicitatea scrierii

rezultă și implicația $\boxed{\text{„}\Leftarrow\text{"}}$.

Dem $\boxed{\text{„}\Rightarrow\text{"}}$ ~~inductiv după k.~~

pt $k=1$ evident $r_{11} q_1 = a_1 = (I_m - Q_0) a_1$ ✓.

~~$k=1$ evident $r_{11} q_1 = a_1 = (I_m - Q_0) a_1$ ✓.~~

~~Pp. adevărat pt. pași $x \le k-1$ și dem pt. k.~~

pt $k \geq 2$

la un pas k am $a_k = r_{1k} q_1 + \dots + r_{k,k} q_k$

am văzut că $r_{i,k} = {}^t q_i a_k$, $i < k$

$${}_{\text{norma}}\, r_{i,k} \cdot q_i = q_i \cdot r_{i,k} = q_i {}^t q_i a_k = Q_i a_k$$

$\Rightarrow I_m a_k = Q_1 a_k + \dots + Q_{k-1} a_k + r_{kk} q_k.$

$r_{kk} q_k = (I_m - (Q_1 + \dots + Q_{k-1})) a_k.$

b) met. G-S. modif $\Leftrightarrow q_k r_{kk} = (I_m - Q_{k-1}) \dots (I_m - Q_1) a_k.$

$\boxed{k=1}$

$A = [a_1 \mid A_{12}]$

$Q = [q_1, Q_{12}] \quad R = \left[\begin{array}{c|c} r_{11} & R_{12} \\ \hline 0_{m-1} & R' \end{array}\right]$

$A = Q \cdot R = \cancel{\ldots} \; [q_1 \cdot r_{11} \mid q_1 R_{12} + Q_{12} R']$

$a_1 = q_1 r_{11} \qquad A_{12} = q_1 R_{12} + Q_{12} R'$

$\Downarrow \qquad\qquad \Downarrow$

$r_{11} = \|a_1\|_2 \qquad q_1^T A_{12} = R_{12} \Rightarrow q_1 R_{12} = Q_1 A_{12}$

$q_1 = \frac{1}{r_{11}} \|a_1\| \qquad Q_{12} R' = A_{12} - q_1 R_{12}$ det.

$Q_{12} R' = (I_m - Q_1) A_{12}.$

La urm. pas lucrez cu $Q_{12} R'$ pe post de A

și determin $q_2, R \dots$

Notez $A^{(k-1)}$ matricea ~~determinată~~ cu care lucrez la ~~pasul~~ la sf. pasului (k). Și arăt că e de forma:

$A^{(k-1)} = (I_m - Q_{k-1}) \dots (I_m - Q_0) \cdot A_k$

unde $A_k = [a_k, a_{k+1}, \dots a_m].$

Pt. $k=1$ ✓.

$$A^{(1)} = (I_m - Q_0) A_1 = A$$

Pp. k adevărat $\overset{?}{\Rightarrow} k+1$.

La pasul k am matricea $A^{(k)} = (I_m - Q_{k-1}) \dots (I_m - Q_0) A_k$

$$A^{(k)} = [a_1^{(k)} \mid A_0^{(k)}] = \underbrace{[g_k \mid Q']}_{\text{ortog.}} \begin{bmatrix} r_{kk} & R' \\ 0_{m-k,k} & R'' \end{bmatrix}$$

$$\Rightarrow r_{kk} g_k = a_1^{(k)}$$

$$^t g_k \mid g_k R' + Q' R'' = A_0^{(k)} \Rightarrow R' = {}^t g_k A_0^{(k)}$$

$$\Rightarrow Q' R'' = A_0^{(k)} - Q_k A_0^{(k)}$$

$$= (I_m - Q_k) A_0^{(k)}$$

$$= (I_m - Q_k) \underbrace{(C_2(A^{(k)}), \dots C_m(A^{(k)}))}_{A^{(k)}}$$

pas inducție

$$= (I_m - Q_k)(I_m - Q_{k-1}) \dots (I_m - Q_0)(C_2(A_k), \dots C_m(A_k))$$

$$= (I_m - Q_k) \dots (I_m - Q_0) A_{k+1}.$$

matricea pe care voi lucra la pasul $k+1$.

$\Rightarrow$ La un pas k lucrez pe $A^{(k)} = (I_m - Q_{k-1}) \dots (I_m - Q_0) A_k$

$$r_{kk} g_k = C_1((I_m - Q_{k-1}) \dots (I_m - Q_0) A_k)$$

$$= (I_m - Q_{k-1}) \dots (I_m - Q_0) a_k.$$

c) Arat forma a) <=> forma b).

$\cancel{(I_m - Q_{k-1})(I_m - Q_k)}$ $\qquad Q_i Q_j = O_m,\ i \neq j.$

$(I_m - Q_{k-1})(I_m - Q_{k-2})(\dots(I_m - Q_1) =$

$= (I_m - Q_{k-1} - Q_{k-2})(I_m - Q_{k-3})\dots(I_m - Q_1)$

$= (I_m - (Q_{k-1} + \dots + Q_1))$.

$[I_m - (Q_1 + \dots + Q_{k-1})]^2 = (I_m - Q_{k-1})^2 \dots (I_m - Q_1)^2$

$\cancel{(I_m - Q_i)^2 = \sum_{j=0}^{m} C_m^j (-1)^m Q_i^m}$

$\cancel{Q_i^2 = q_i\, {}^t q_i\, q_i\, {}^t Q_i =}$

$Q_i^2 = q_i \underbrace{{}^t q_i\, q_i}_{=1} {}^t q_i = q_i\, {}^t q_i = Q_i$ $\Big/ \Rightarrow$

${}^t Q_i = {}^t(q_i\, {}^t q_i) = q_i\, {}^t q_i = Q_i$

$\Rightarrow Q_i$ proiecție otog $\Rightarrow I_m - Q_i$ proiecție

otog $\Rightarrow (I_m - Q_i)^2 = I_m - Q_i$.

$\boxed{\text{Curs 8}}$

$\boxed{\text{Pb}_1}$ $A \in M_{m \times n}(\mathbb{R})$, $rk\,A = n$, $A = QR$, $r_{ii} > 0$

$^tA \cdot A = L\,^tL$, L inf Δ, $l_{ii} > 0$.

a) $^tA \cdot A = L \cdot {^tL}$.

$$\begin{array}{c} ^tA \cdot A \\ \parallel \\ ^tR \cdot {^tQ} \cdot Q \cdot R = {^tR} \cdot R. \end{array}$$

b) $^tA \cdot A$ inversabilă $\Rightarrow L, {^tL}, R, {^tR}$ inversabile.

$({^tR})^{-1} \,|\quad {^tR} \cdot R = L \cdot {^tL} \quad | \cdot R^{-1}$

$$I_n = \cancel{{^t}({^tL} \cdot R^{-1})} \cdot \underbrace{\left[({^tR})^{-1} \cdot L\right]}_{\text{inf } \Delta} \cdot \left[{^tL} \cdot R^{-1}\right]$$

$\Rightarrow \underbrace{{^tL} \cdot R^{-1}}_{M}$ ~~orto~~ ortogonală și sup Δ.

$$M = \begin{pmatrix} x_{11} & x_{12} & \cdots & x_{1n} \\ & x_{22} & \cdots & \vdots \\ & & \ddots & \vdots \\ & & & x_{nn} \end{pmatrix} \Rightarrow {^t\underline{X_i}} \cdot \underline{X_j} = 0, \; i \neq j.$$

inv. $\Downarrow$ $x_{ii} \neq 0$. $\quad \underline{X_1} \quad \underline{X_2} \; - \; \underline{X_n}$

${^tX_1} \cdot X_2 = 0 \Rightarrow x_{11} \cdot x_{12} = 0$

$\Rightarrow x_{12} = 0.$

${^tX_1}\, X_j = 0 \Rightarrow x_{11} \cdot x_{1j} = 0$, $j = \overline{2, n}$ $\Rightarrow x_{1j} = 0$.

Inductiv am $x_{2\,\cancel{k+1}}$ $\quad \begin{pmatrix} 0 \\ \vdots \\ 0 \\ x \\ 0 \\ \vdots \\ 0 \end{pmatrix}$

toate elem. de peste linia $k = 0$ mai puțin cele de pe diag.

$^tX_{k+1} X_j = 0 \Rightarrow \underset{\substack{\shortparallel \\ 0}}{X_{k+1,k+1}} X_{k+1,j} = 0 \Rightarrow X_{k+1,j} = 0.$

$j \geq k+2$

$\Rightarrow M = \begin{pmatrix} X_{11} & & 0 \\ & \ddots & \\ 0 & & X_{nn} \end{pmatrix}$ ortogonală $\Rightarrow M = I_n \Rightarrow$

$^tR \cdot L^{-1} = I_n \mid \cdot L$

$^tR = L$

$\boxed{Pb2}$ Fac factorizarea Cholsky $\Rightarrow$ aflu $^tA \cdot A = L \cdot {}^tL$

$\Rightarrow$ Iau $\boxed{R = {}^tL.}$

$Q = A \cdot R^{-1} = A \cdot ({}^tL)^{-1}.$

$\boxed{Pb3}$ $A = \underset{m \times n}{Q_1} \underset{n \times n}{R_1}$ și $A = \underset{m \times m}{Q_2} \begin{bmatrix} \underset{n \times n}{R_2} \\ \hline O_{m-n,n} \end{bmatrix}$

$Q_2 = [\underset{m \times n}{Q'} \mid \underset{m \times (m-n)}{Q''}]$

unicitate QR.

$\Rightarrow A = [Q' \mid Q''] \begin{bmatrix} R_2 \\ \hline O_{m-n,n} \end{bmatrix} = Q' R_2 \Rightarrow \boxed{R_2 = R_1}$

$\boxed{Q' = Q_1}$

$\Rightarrow Q_2 = \underbrace{[Q_1 \mid Q'']}_{\text{ortogonală.}}$

Curs 10

Pb1 $p \in \mathbb{N}^* \cup \{+\infty\}$

$$\|AB\|_p \le \|A\|_p \cdot \|B\|_p$$

Jau $X \in \mathbb{R}^n$ cu $\|X\|_p = 1$ a.î. $\|AB\|_p = \|ABX\|_p$

$$\|AB\|_p = \|ABX\|_p \le \|A\|_p \cdot \|BX\|_p \le \|A\|_p \|B\|_p \underbrace{\|X\|_p}_{1}$$

Pb2 B submatrice în A.

$$\|B\|_p \le \|A\|_p.$$

$$A = \begin{array}{cc} s\times r & s\times(n-r) \\ \left[\begin{array}{c|c} B & V \\ \hline U & T \end{array}\right] & \\ (m-s)r & (m-s)\times(n-r) \end{array}$$

Jau $X = \begin{pmatrix} X_1 \\ \hline 0 \end{pmatrix} \in \mathbb{R}^n$ cu $\|X\|_p = 1$

$(n-r)\times 1$

și $\|BX_1\|_p = \|B\|_p$, $\|X_1\| = 1$.

$$\|A\|_p \ge \|AX\|_p = \left\| \begin{matrix} BX_1 \\ UX_1 \end{matrix} \right\|_p \ge \|BX_1\|_p = \|B\|_p.$$

Pb3 $D = \mathrm{diag}(\mu_1, \ldots, \mu_k) \in M_{m,n}(\mathbb{R})$

$k = \min\{m, n\}$.

$$\|D\|_p = \max|\mu_i|$$

Pot presupune $m \ge n$ (altfel lucrez cu ${}^tD$.)

Fie $x \in \mathbb{R}^n$ cu $\|x\|_p = 1$. $\Rightarrow x_i \leq 1$, $i = \overline{1,n}$

$$\|\Delta x\|_p = \left\| \begin{pmatrix} x_1 \mu_1 \\ \vdots \\ x_n \mu_n \end{pmatrix} \right\|_p = \begin{cases} \max_{i=\overline{1,n}} |x_i \mu_i| \leq \max_{i=\overline{1,n}} |\mu_i|, & p = \infty \\ \sqrt[p]{x_1^p \mu_1^p + \dots + x_n^p \mu_n^p} \leq \max_{i=\overline{1,n}} |\mu_i| & p < \infty \end{cases}$$

cu egalitate pt $\boxed{e_{\arg\max |\mu_i|} = x}$.

Pb 4 a) $\|A\|_2 \leq \|A\|_F \leq \sqrt{n} \|A\|_2$. $\boxed{\text{tr } X = \text{urma mat. } X}$

$\|A\|_F = \sqrt{\text{tr}({}^tAA)}$

$\|A\|_2 = \sqrt{\lambda}$ unde λ = val proprie maximă a lui ${}^tAA$

$$\sqrt{\lambda} \leq \sqrt{\sum_{\lambda' \in \text{Spec}({}^tAA)} \underset{\geq 0}{\lambda'}} \leq \sqrt{n\lambda}.$$ ✓

b) $\|A\|_\Delta \leq \cancel{\max |a_{ij}|}\ \|A\|_2 \leq \sqrt{mn}\, \|A\|_\Delta$.