Seminar 3 Sisteme liniare. Spații vect.

SLI / SLD / SG / Bază

(1) $$\sum_{i=1}^{k} (1+i)x_i + \sum_{i=1}^{4-k} i\, x_{i+k} = 0, \ (\forall)\ k=\overline{1,3}$$

Să se rez sistemul.

$k=1$: $\quad 2x_1 + x_2 + 2x_3 + 3\cdot x_4 = 0$

$k=2$: $\quad 2x_1 + 3x_2 + x_3 + 2x_4 = 0$

$k=3$: $\quad 2x_1 + 3x_2 + 4x_3 + x_4 = 0$

$$\bar{A} = \left(\begin{array}{cccc|c} \boxed{2} & 1 & 2 & 3 & 0 \\ 2 & 3 & 1 & 2 & 0 \\ 2 & 3 & 4 & 1 & 0 \end{array}\right)$$

$$\begin{vmatrix} 2 & 1 & 2 \\ 2 & 3 & 1 \\ 2 & 3 & 4 \end{vmatrix} \underset{L_3' = L_3 - L_2}{=} \begin{vmatrix} 2 & 1 & 2 \\ 2 & 3 & 1 \\ 0 & 0 & 3 \end{vmatrix} = 3(6-2) = 3\cdot 4 = 12$$

$\Rightarrow \operatorname{rang} A = \operatorname{rang} \bar{A} = 3$

x_1, x_2, x_3 nec principale

$x_4 - \alpha$ nec sec

$$\begin{cases} 2x_1 + x_2 + 2x_3 = -3\alpha \\ 2x_1 + 3x_2 + x_3 = -2\alpha \\ 2x_1 + 3x_2 + 4x_3 = -\alpha \end{cases}$$

ec $(3)-(2) \Rightarrow 3x_3 = \alpha \Rightarrow x_3 = \frac{\alpha}{3}$

$$\begin{cases} 2x_1 + x_2 = -3\alpha - \frac{2}{3}\alpha \\ 2x_1 + 3x_2 = \cancel{-3\alpha} - 2\alpha - \frac{\alpha}{3} \end{cases} \quad (-)$$

$$2x_2 = \alpha + \frac{\alpha}{3} = \frac{4\alpha}{3}$$

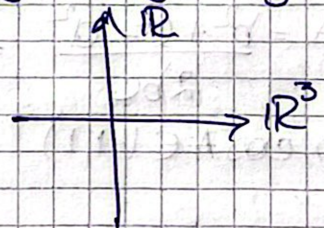
$$x_2 = \frac{2\alpha}{3}$$

$$2x_1 = -3\alpha - \frac{2}{3}\alpha - \frac{2}{3}\alpha = -\overset{3)}{3}\alpha - \frac{4}{3}\alpha = -\frac{13}{3}\alpha$$

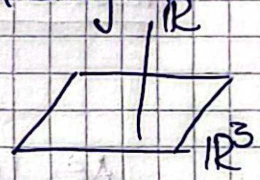
$$x_1 = -\frac{13}{6}\alpha$$

$$(x_1, x_2, x_3, x_4) \in \left\{\left(-\frac{13}{6}\alpha; \frac{2\alpha}{3}, \frac{\alpha}{3}, \alpha\right) \middle| \alpha \in \mathbb{R}\right\}$$

(pe $\mathbb{R}^4$ îl gândim:



sau



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(8) Fie ΔABC cu a, b, c lg laturilor

$$\begin{cases} ay + bx = c \\ cx + az = b \\ bz + cy = a \end{cases}$$

Să se arate că pt $(\forall)\ \Delta ABC$ sist are sol unică (x_0, y_0, z_0) și aceasta verifică $x_0, y_0, z_0 \in (-1, 1)$

$$\begin{cases} bx + ay = c \\ cx + + az = b \\ cy + bz = a \end{cases} \qquad A = \left(\begin{array}{ccc|c} b & a & 0 & c \\ c & 0 & a & b \\ 0 & c & b & a \end{array}\right)$$

$$\det A = \begin{vmatrix} b & a & 0 \\ c & 0 & a \\ 0 & c & b \end{vmatrix} = -abc - abc = -2abc \neq 0$$

$$x = \frac{\Delta x}{\Delta}$$

$$\Delta x = \begin{vmatrix} c & a & 0 \\ b & 0 & a \\ a & c & b \end{vmatrix} = a^3 - ac^2 - ab^2 = -a(b^2+c^2-a^2)$$

$$x = \frac{\cancel{-}\cancel{a}(b^2+c^2-a^2)}{\cancel{-}2\cancel{a}bc} = \frac{b^2+c^2-a^2}{2bc}$$

<u>Th. cosinusului</u>: $a^2 = b^2+c^2-2bc\cos A$

$$\cos A = \frac{b^2+c^2-a^2}{2bc} = x$$

$A \in (0,\pi) \Rightarrow \cos A \in (-1,1)$

Analog $y = \cos B$ și $z = \cos C$

$(x_0, y_0, z_0) = \{(\cos A, \cos B, \cos C)\}$

- $\begin{cases} x + 2y = m+1 \\ 2x - 3y = m-1 \\ mx + y = 3 \end{cases}$ $\quad A = \left(\begin{array}{cc|c} \boxed{1} & 2 & m+1 \\ 2 & -3 & m-1 \\ m & 1 & 3 \end{array}\right)$

$$\Delta p = \begin{vmatrix} 1 & 2 \\ 2 & -3 \end{vmatrix} = -7 \neq 0 \Rightarrow \text{rang } A = 2$$

Si $\Rightarrow \Delta car = \det \bar{A} \neq 0 \Rightarrow \begin{vmatrix} 1 & 2 & m+1 \\ 2 & -3 & m-1 \\ m & 1 & 3 \end{vmatrix} \neq 0$

$$\begin{array}{l} L_1-2L_3 \\ L_2+3L_3 \end{array} \begin{vmatrix} 1-2m & 0 & m-5 \\ 2+3m & 0 & m+8 \\ \cancel{m} & \cancel{1} & \cancel{3} \end{vmatrix} = (-1) \begin{vmatrix} 1-2m & m-5 \\ 2+3m & m+8 \end{vmatrix} =$$

$$= -(m+8-2m^2+16m+3m^2+15m-2m+10) =$$

$$= 5m^2+2m-18 \neq 0$$

$$\Delta = 4+4\cdot 5\cdot 18 = 4+20\cdot 18 = 4+360 = 364 = 4\cdot 91$$

$$m_{1,2} = \frac{-2\pm 2\sqrt{91}}{10} = \frac{-1\pm\sqrt{91}}{5}$$

$$\Rightarrow m \in \mathbb{R} \setminus \left\{ -\frac{1\pm\sqrt{91}}{5} \right\} \Rightarrow \text{sist incompatibil}$$

⑤ $A = \begin{pmatrix} 1 & 1 & 2 & 3 \\ 1 & 1 & 3 & 4 \\ 2 & 5 & 0 & 1 \\ 4 & 1 & 0 & 2 \end{pmatrix}$

Calc. det A, utilizând Th. Laplace pt $p=2$

l_2, l_3 fixate, resp. c_1, c_2 fixate.

$$\det A = (-1)^{1+2+1+2} \begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix} \cdot \begin{vmatrix} 0 & 1 \\ 0 & 2 \end{vmatrix} +$$

$$+ (-1)^{1+2+1+3} \begin{vmatrix} 1 & 1 \\ 2 & 5 \end{vmatrix} \cdot \begin{vmatrix} 3 & 4 \\ 0 & 2 \end{vmatrix} + (-1)^{1+2+1+4} \begin{vmatrix} 1 & 1 \\ 4 & 1 \end{vmatrix} \cdot \begin{vmatrix} 3 & 4 \\ 0 & 2 \end{vmatrix}$$

$$+ (-1)^{1+2+2+3} \begin{vmatrix} 1 & 1 \\ 2 & 5 \end{vmatrix} \cdot \begin{vmatrix} 2 & 3 \\ 0 & 2 \end{vmatrix} + (-1)^{1+2+2+4} \begin{vmatrix} 1 & 1 \\ 4 & 1 \end{vmatrix} \cdot \begin{vmatrix} 2 & 3 \\ 0 & 1 \end{vmatrix}$$

$$+ (-1)^{1+2+3+4} \begin{vmatrix} 2 & 5 \\ 4 & 1 \end{vmatrix} \cdot \begin{vmatrix} 2 & 3 \\ 3 & 4 \end{vmatrix} = -\cancel{18}-9+12+6+\cancel{18} =$$

$$= 9.$$

Spatii vectoriale. SLI / SG / Baze

① $(\mathbb{R}^3, +, \cdot)_{/\mathbb{R}}$, $S = \{u = (1,2,3);\ v = (2,3,1);\ w = (a+3, a+1, a+2)$
$a \in \mathbb{R}\}$

$a = ?$ a.î. a) S e SLI
b) S e SLD

a) S e SLI $\Leftrightarrow (\forall)\ b, c, d \in \mathbb{R}$ a.î. $bu + cv + dw = 0_{\mathbb{R}}$

$\Rightarrow b = c = d = 0$

$b(1,2,3) + c(2,3,1) + d(a+3, a+1, a+2) = (0,0,0)$

$\Rightarrow (b + 2c + d(a+3),\ 2b + 3c + d(a+1),\ 3b + c + d(a+2)) = (0,0,0)$

$$\begin{cases} b + 2c + d(a+3) = 0 \\ 2b + 3c + d(a+1) = 0 \\ 3b + c + d(a+2) = 0 \end{cases}$$

$$A = \left(\begin{array}{ccc|c} 1 & 2 & a+3 & 0 \\ 2 & 3 & a+1 & 0 \\ 3 & 1 & a+2 & 0 \end{array}\right)$$

S e SLI $\Leftrightarrow$ sist. trebuie să aibă sol. unică $\Rightarrow$

! $\Rightarrow \det A \neq 0 \Rightarrow \boxed{\text{rang } A = \text{maxim}}$

$$\det A = \begin{vmatrix} 1 & 2 & a+3 \\ 2 & 3 & a+1 \\ 3 & 1 & a+2 \end{vmatrix} \xrightarrow[L_2 - 2L_1]{L_3 - 3L_1} \begin{vmatrix} 1 & 2 & a+3 \\ 0 & -1 & -a-5 \\ 0 & -5 & -2a-7 \end{vmatrix}$$

$$= \begin{vmatrix} -1 & -a-5 \\ -5 & -2a-7 \end{vmatrix} = 2a+7+5(-a-5) = 2a+7-5a-25 =$$
$$= -3a-18 \neq 0 \Rightarrow \boxed{a \neq -6}$$

b) S e SLD $\Leftrightarrow \det(A) = 0 \Leftrightarrow a = -6$.

(2) $(\mathbb{R}^3, +, \cdot)$ $S''' = \{u_1 = (1,1,0), u_2 = (1,0,0), u_3 = (1,2,3), u_4 = (0,1,$

a) $S' = \{u_1, u_2\}$. Este SG? Este SLI?

b) $S'' = \{u_1, u_2, u_3\}$ — II —

c) $S''' = \{u_1, u_2, u_3, u_4\}$ — II —

a) Fie $a_1, a_2 \in \mathbb{R}$

$a_1(1,1,0) + a_2(1,0,0) = (0,0,0)$

$(a_1 + a_2, a_1, 0) = (0,0,0)$

$$\begin{cases} a_1 + a_2 = 0 \\ a_1 = 0 \end{cases} \qquad A = \left(\begin{array}{cc|c} 1 & 1 & 0 \\ 1 & 0 & 0 \end{array}\right)$$

$\det A = -1 \neq 0 \Rightarrow \text{rang} A = 2 = \max \Rightarrow$ SLI

sau $\text{rg}\begin{pmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 0 \end{pmatrix} = 2 = \max \Rightarrow$ SLI

SG

$\dim_{\mathbb{R}} \mathbb{R}^3 = 3$

$\left.\begin{array}{l} 3 = \text{nr min de vect care form. SG} \\ |S'| = 2 \end{array}\right| \Rightarrow S'$ nu e SG.

b) $A = \begin{array}{ccc} "u_1" & "u_2" & "u_3" \\ \downarrow & \downarrow & \downarrow \end{array}$
$$A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 2 \\ 0 & 0 & 3 \end{pmatrix}$$

$$\det A = \begin{vmatrix} 1 & 1 & 1 \\ 1 & 0 & 2 \\ 0 & 0 & 3 \end{vmatrix} = (-1)^{1+2} \begin{vmatrix} 1 & 2 \\ 0 & 3 \end{vmatrix} = -3 \neq 0$$

$\Rightarrow$ rang $A = 3 = \max \Rightarrow S''$ e SLI | prop de la curs $\Rightarrow$ S'' bază $\Rightarrow S''$ e SG.

card$(S'') = 3 = \dim \mathbb{R}^3$

c) $|S'''| = 4$
$3 = \dim \mathbb{R}^3 =$ nr max de SLI $\Big| \Rightarrow |S'''|$ e SLD ~~pentru că~~

S'' e SG
~~$S'' = S' \cup$~~ $S''' = S'' \cup \{u_4\}$ (supramulțime) $\Big| \Rightarrow$

$\Rightarrow S'''$ este SG.

(3) $(\mathbb{R}_2[X], +, \cdot)$ (sp. vect. al polinoamelor de grad cel mult 2)

a) $v_1 = 2X^2 - 3X = (0, -3, 2)$
$v_2 = X + 1 = (1, 1, 0)$
$v_3 = -X^2 + 4 = (4, 0, -1)$
$\{v_1, v_2, v_3\}$ bază în $\mathbb{R}_2[X]$

Obs: $\boxed{P = a_0 + a_1 X + a_2 X^2 \in \mathbb{R}_2[X] \\ \equiv (a_0, a_1, a_2) \in \mathbb{R}^3}$

dim $\mathbb{R}_2[X] = 3$ $\left(B_0 = \{(1, x, x^2)\} \right)$

$$\det A = \begin{vmatrix} 0 & 1 & 4 \\ -3 & 1 & 0 \\ 2 & 0 & -1 \end{vmatrix} \xrightarrow{L_1' = L_1 + 4L_3} \begin{vmatrix} 8 & 1 & 0 \\ -3 & 1 & 0 \\ 2 & 0 & -1 \end{vmatrix}$$

$$= (-1)^{3+3} \cdot (-1) \begin{vmatrix} 8 & 1 \\ -3 & 1 \end{vmatrix} = -(8+3) = -11 \neq 0 \Rightarrow \text{SLI}$$

$\Rightarrow$ rang$(A) = 3 = \max = \dim \mathbb{R}_2[X] \Rightarrow$ SLI) $\Rightarrow$ bază

→ de făcut restul exercițiilor pt examen

→ ap vect (3) $\Rightarrow$

$$A = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ u & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & y \\ y & 0 & \cancel{0} \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & z \\ 0 & 0 & 0 \\ 0 & z & 0 \end{pmatrix}$$

$$= u \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} + y \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + z \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$