7.11. 2024.

C6 AG 16.

Aplicații liniare. Teorema dimensiunii

Def $(V,+,\cdot)/\mathbb{K}$, $(W,+,\cdot)/\mathbb{K}$ sp vect

$f: V \to W$ s.n. aplicație liniară (sau morfism de spații vectoriale) $\Leftrightarrow \begin{cases} 1)\ f(x+y) = f(x)+f(y) \\ 2)\ f(ax) = a f(x) \end{cases} \Leftrightarrow$

$\Leftrightarrow f(ax+by) = a f(x) + b f(y)$, $\forall x,y \in V$, $\forall a,b \in \mathbb{K}$

Prop $\mathrm{Ker}(f) = \{x \in V \mid f(x) = 0_W\} \subseteq V$ (subsp. vect) nucleul lui f

$\mathrm{Im} f = \{y \in W \mid \exists x \in V$ ai $f(x) = y\} \subseteq W$ subsp vect imag. lui f

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Prop $f: V \to W$ aplicație liniară

a) f injectivă $\Leftrightarrow \mathrm{Ker} f = \{0_V\}$

b) f surjectivă $\Leftrightarrow \dim \mathrm{Im} f = \dim W$

Dem

a) "$\Rightarrow$" Ip: f inj. Dem $\mathrm{Ker} f = \{0_V\}$

OBS $f: (V,+) \to (W,+)$ morfism de grupuri $\Rightarrow f(0_V) = 0_W$

Fie $x \in \mathrm{Ker} f \Rightarrow f(x) = 0_W$, dar $f(0_V) = 0_W$ $\Rightarrow f(x) = f(0_V)$, dar f inj $\Rightarrow x = 0_V$

$\Rightarrow \mathrm{Ker} f = \{0_V\}$.

"$\Leftarrow$" Ip: $\mathrm{Ker} f = \{0_V\}$. Dem că f e inj

Fie $x_1, x_2 \in V$ ai $f(x_1) = f(x_2) \Rightarrow f(x_1) - f(x_2) = 0_W \Rightarrow f(x_1 - x_2) = 0_W$

$x_1 - x_2 \in \mathrm{Ker}\, f = \{0_V\} \Rightarrow x_1 - x_2 = 0_V \Rightarrow x_1 = x_2 \Rightarrow f$ inj

b) "$\Rightarrow$" Ip: f surj $\Rightarrow \mathrm{Im} f = W \Rightarrow \dim \mathrm{Im} f = \dim W$

"$\Leftarrow$" Ip: $\left.\begin{array}{l}\dim \mathrm{Im} f = \dim W \\ \mathrm{Im} f \subseteq W \text{ subsp vect}\end{array}\right\} \Rightarrow \mathrm{Im} f = W$

<u>Teorema dimensiunii</u>

$f: V \to W$ apl. liniară

$\Rightarrow \dim V = \dim \mathrm{Ker} f + \dim \mathrm{Im} f$

<u>Dem</u>

$\mathrm{Ker} f \subseteq V$ subsp. vect

$\dim_{\mathbb{K}} \mathrm{Ker} f = p$, $\dim_{\mathbb{K}} V = n$, $p \leq n$.

Fie $R_0 = \{e_1, .., e_p\}$ reper în $\mathrm{Ker} f$.

Extindem R_0 la un reper în V

$R_0 \cup \{e_{p+1}, .., e_n\}$ reper în V.

Dem că $R = \{f(e_{p+1}), .., f(e_n)\}$ reper în $\mathrm{Im} f$

1 R este SLI

$$\sum_{j=p+1}^{n} a_j f(e_j) = 0_W \overset{f \text{ lin}}{\Rightarrow} f\Big(\sum_{j=p+1}^{n} a_j e_j\Big) = 0_W \Rightarrow$$

$$\sum_{j=p+1}^{n} a_j e_j \in \mathrm{Ker} f = \langle R_0 \rangle \Rightarrow \sum_{j=p+1}^{n} a_j e_j = \sum_{i=1}^{p} a_i e_i$$

$$\sum_{i=1}^{p} a_i e_i - \sum_{j=p+1}^{n} a_j e_j = 0_V \xrightarrow[\text{SLI}]{\text{reperul } V} \begin{array}{l} a_i = 0,\ i = \overline{1,p} \\ \boxed{a_j = 0,\ j = \overline{p+1,n}} \end{array}$$

2 R SG pt $\mathrm{Im} f \Leftrightarrow \mathrm{Im} f = \langle R \rangle$

$\forall y \in \mathrm{Im} f \Rightarrow \exists x \in V$ aî $f(x) = y$

$V = \langle \{e_1, .., e_p, e_{p+1}, .., e_n\} \rangle$

$$f(x) = f\Big(\underbrace{\sum_{i=1}^{p} a_i e_i}_{\substack{\| \\ x \\ \cap \\ \mathrm{Ker}(f)}} + \sum_{j=p+1}^{n} a_j e_j\Big) = \underbrace{f\Big(\sum_{i=1}^{p} a_i e_i\Big)}_{\substack{\| \\ 0_W}} + \underbrace{f\Big(\sum_{j=p+1}^{n} a_j e_j\Big)}_{\substack{\| \\ \sum_{j=p+1}^{n} a_j f(e_j)}}$$

$\Rightarrow \mathcal{R}$ e SG.

Deci $\mathcal{R}$ e reper în $\mathrm{Im}\, f$

$\dim V = n = p + n - p = \dim \mathrm{Ker}\, f + \dim \mathrm{Im}\, f$

Aplicatie $f: \mathbb{R}^3 \longrightarrow \mathbb{R}^3$, $f(\underset{(x_1,x_2,x_3)}{x}) = (x_1+x_2-x_3,\ x_1+x_2,\ x_1+x_2+x_3)$

a) f liniară

b) $\mathrm{Ker}\, f$, $\mathrm{Im}\, f = ?$ Precizați câte un reper în fiecare subspațiu.

Sol.

a) f liniară $\Leftrightarrow f(ax+by) = af(x)+bf(y), \forall x,y \in \mathbb{R}^3, \forall a,b \in \mathbb{R}$

$f(ax+by) = f(ax_1+by_1, ax_2+by_2, ax_3+by_3)$

$= (ax_1+by_1+ax_2+by_2-(ax_3+by_3), ax_1+by_1+ax_2+by_2, ax_1+by_1+ax_2+by_2+ax_3+by_3)$

$= a(x_1+x_2-x_3, x_1+x_2, x_1+x_2+x_3) + b(y_1+y_2-y_3, y_1+y_2, y_1+y_2+y_3)$

$= af(x) + bf(y)$

b) $\mathrm{Ker}\, f = \{x \in \mathbb{R}^3 \mid f(x) = 0_{\mathbb{R}^3}\} = S(A) = \{(x_1, -x_1, 0) \mid x_1 \in \mathbb{R}\}$

$$\begin{cases} x_1+x_2-x_3 = 0 \\ x_1+x_2 = 0 \\ x_1+x_2+x_3 = 0 \end{cases} \quad A = \left(\begin{array}{ccc|c} 1 & 1 & -1 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \end{array}\right) \quad \begin{cases} x_2 - x_3 = -x_1 \Rightarrow x_3 = 0 \\ x_2 = -x_1 \end{cases}$$

$\dim \mathrm{Ker}\, f = 3 - \mathrm{rg}\, A = 3-2 = 1$ $\quad \mathrm{Ker}\, f = \langle \underbrace{\{(1,-1,0)\}}_{\mathcal{R}_0} \rangle$

Extindem $\mathcal{R}_0$ la un reper în $\mathbb{R}^3$

$\mathrm{rg}\begin{pmatrix} 1 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} = 3$ $\quad \mathcal{R}_0 \cup \{e_1, e_3\}$ reper în $\mathbb{R}^3$

$\mathcal{R} = \{f(e_1), f(e_3)\}$ reper în $\mathrm{Im}\, f$

$f(e_1) = (1,1,1)$
$f(e_3) = (-1,0,1)$

Met 2 (pt reper în Im f).

$Im f = \{ y \in \mathbb{R}^3 \mid \exists x \in \mathbb{R}^3 \text{ a.î. } f(x) = y \}$

$$(*) \begin{cases} x_1 + x_2 - x_3 = y_1 \\ x_1 + x_2 = y_2 \\ x_1 + x_2 + x_3 = y_3 \end{cases}$$

(*) este sist compatibil $\Leftrightarrow rg A = rg \bar{A}$

$$A = \left(\begin{array}{ccc|c} 1 & \boxed{1} & \boxed{-1} & y_1 \\ 1 & \boxed{1} & \boxed{0} & y_2 \\ 1 & 1 & 1 & y_3 \end{array} \right)$$

$\Delta_p = \begin{vmatrix} 1 & -1 \\ 1 & 0 \end{vmatrix} \neq 0$, $\Delta_c = \begin{vmatrix} 1 & -1 & y_1 \\ 1 & 0 & y_2 \\ 1 & 1 & y_3 \end{vmatrix} = 0$

$$\begin{vmatrix} 1 & -1 & y_1 \\ 1 & 0 & y_2 \\ 2 & 0 & y_1 + y_3 \end{vmatrix} = 0 \Rightarrow \begin{vmatrix} 1 & y_2 \\ 2 & y_1 + y_3 \end{vmatrix} = 0$$

$l_3' = l_1 + l_3$

$y_1 - 2y_2 + y_3 = 0$

$Im f = \{ y \in \mathbb{R}^3 \mid y_1 - 2y_2 + y_3 = 0 \} = S(A')$ $\dim Im f = 3 - rg A' = 3 - 1 = 2$

$A' = (1 \ -2 \ 1) | 0$

$y_1 = 2y_2 - y_3$

$Im f = \{ (2y_2 - y_3, y_2, y_3) \mid y_2, y_3 \in \mathbb{R} \}$

$= y_2(2,1,0) + y_3(-1,0,1)$

$Im f = \langle \underbrace{\{ (2,1,0), (-1,0,1) \}}_{\text{reper în Im f}} \rangle$

<u>Prop</u> $f : V \to W$ apl. liniară

a) f inj $\Leftrightarrow \dim V = \dim Im f$

a) $Ker f = \{0_V\} \Rightarrow \dim Ker f = 0$

b) f surj $\Leftrightarrow \dim W = \dim Im f$

b) $\dim Im f = \dim W$

$\dim V = \dim Ker f + \dim W$

b) f bijectivă $\Leftrightarrow \dim V = \dim W$.

T. dim: $\dim V = \dim Ker f + \dim Im f$.

Teoremă V și W sunt sp. vect. izomorfe ($V \simeq W$)

$\Leftrightarrow \dim V = \dim W$

Dem

"$\Rightarrow$" Ip: $\exists f : V \to W$ izomorfism de sp vect $\begin{cases} f \text{ liniară} \\ f \text{ bijectivă} \end{cases}$

cf. prop. preced $\dim V = \dim W$

"$\Leftarrow$" $\dim V = \dim W = n$

$R = \{e_1, .., e_n\}$, $R' = \{e'_1, .., e'_n\}$ repere în V, resp W.

$f : V \to W$, $f(e_i) = e'_i$, $\forall i = \overline{1,n}$

Extindem f prin liniaritate

$f(x) = f(x_1 e_1 + ... + x_n e_n) = x_1 \underbrace{f(e_1)}_{e'_1} + ... + x_n \underbrace{f(e_n)}_{e'_n} = x'$

f bij $\Leftrightarrow$ $\forall \underbrace{x'}_{x_1 e'_1 + ... + x_n e'_n} \in W$, $\exists!\, x = x_1 e_1 + ... + x_n e_n$ aî $f(x) = x'$

$\Rightarrow f$ izomorfism de sp. vect i.e $V \simeq W$.

Prop $f : V \to W$ apl. liniară

a) f inj $\Leftrightarrow$ f transformă $\forall$ SLI din V într-un SLI din W
i.e. S e SLI în V $\Rightarrow$ $f(S)$ e SLI în W.

b) f surj $\Leftrightarrow$ f transformă $\forall$ SG din V într-un SG din W
i.e $V = \langle S \rangle \Rightarrow W = \langle f(S) \rangle$

c) f bij $\Leftrightarrow$ f transformă orice reper din V într-un reper în W.

Dem

a) "$\Rightarrow$" Ip: f inj $\Leftrightarrow \ker f = \{0_V\}$.

Fie $S = \{v_1, .., v_n\}$ SLI în V $\overset{?}{\Rightarrow}$ $f(S) = \{f(v_1), .., f(v_n)\}$ SLI în W.

$a_1 f(v_1)+\dots+a_n f(v_n) = 0_W \xrightarrow{f \text{ lin}} f(\underbrace{a_1v_1+\dots+a_nv_n}_{\in \operatorname{Ker} f = \{0_V\}}) = 0_W$

$\Rightarrow a_1 v_1+\dots+a_n v_n = 0_V \xRightarrow[SLI]{S=\{v_1,\dots,v_n\}} a_1 = \dots = a_n = 0_{\mathbb{K}}$.

"$\Leftarrow$" Ip $\forall S$ SLI în $V \Rightarrow f(S)$ SLI în W.

Dem că f inj i.e. $\operatorname{Ker} f = \{0_V\}$

Ip. abs. $\exists \underset{\neq 0_V}{x} \in \operatorname{Ker} f \Rightarrow \{x\}$ SLI în $V \xRightarrow{ip} \{f(x)\}$ SLI

$\Rightarrow f(x) \neq 0_W$

dar $f(x) = 0_W$ $(x \in \operatorname{Ker} f)$ ↯. Ip. este falsă

$\operatorname{Ker} f = \{0_V\}$

b) "$\Rightarrow$" Ip f surj $\Leftrightarrow \dim \operatorname{Im} f = \dim W$

$S = \{v_1,\dots,v_n\}$ $\quad \langle S \rangle = V \overset{?}{\Rightarrow} \langle f(S) \rangle = W$

$f(S) = \{f(v_1),\dots,f(v_n)\}$.

$W = \operatorname{Im} f$.

$\forall y \in W, \exists x \in V = \langle S \rangle$ aî $y = f(x) = f(a_1v_1+\dots+a_nv_n)$

$= a_1 f(v_1)+\dots+a_n f(v_n)$

$\Rightarrow W = \langle f(S) \rangle$.

"$\Leftarrow$" Ip: $V = \langle S \rangle$ și $W = \langle f(S) \rangle$.

C: f surj.

$\left.\begin{array}{l} W = \langle f(S) \rangle \\ V = \langle S \rangle \end{array}\right| \Rightarrow f(V) = W \Rightarrow f$ surj

c) Din a)+b)

Ru

02

Matricea asociată unei aplicații liniare

$f: V \to W$ aplicație liniară $\qquad \dim V = n,\ \dim W = m.$

$R_1 = \{e_1, .., e_n\} \xrightarrow{A} R_2 = \{\bar{e}_1, .., \bar{e}_m\}$ repere în V, resp W

$A \in \mathcal{M}_{m,n}(\mathbb{K})$

$$f(e_i) = \sum_{j=1}^{m} a_{ji}\bar{e}_j,\ \forall i = \overline{1,n}$$

prima col

$$f(e_1) = a_{11}\bar{e}_1 + a_{21}\bar{e}_2 + \dots + a_{m1}\bar{e}_m$$

$\vdots$

ultima col

$$f(e_n) = a_{1n}\bar{e}_1 + a_{2n}\bar{e}_2 + \dots + a_{mn}\bar{e}_m$$

$$\left.\begin{aligned} f(x) = y \iff f\Big(\sum_{i=1}^{n} x_i e_i\Big) &= \sum_{j=1}^{m} y_j \bar{e}_j \\ \sum_{i=1}^{n} x_i f(e_i) \\ = \sum_{i=1}^{n} x_i\Big(\sum_{j=1}^{m} a_{ji}\bar{e}_j\Big) &= \sum_{j=1}^{m}\Big(\sum_{i=1}^{n} a_{ji}x_i\Big)\bar{e}_j \end{aligned}\right\} \Rightarrow$$

$$y_j = \sum_{i=1}^{n} a_{ji}x_i \qquad \boxed{Y = AX},\quad A = [f]_{R_1,R_2}$$

Teorema de caracterizare a apl. liniare

$f: V \to W$ e liniară $\iff \exists A \in \mathcal{M}_{m,n}(\mathbb{K})$ a.î. coord lui $x \in V$ în rap cu reperul $R_1 = \{e_1, .., e_n\}$ din V și coord lui $y = f(x) \in W$ în rap cu reperul $R_2 = \{\bar{e}_1, .., \bar{e}_m\}$ din W, verifică $Y = AX$

OBS

$$\begin{array}{ccc} R_1 = \{e_1, .., e_n\} & \xrightarrow{A = [f]_{R_1,R_2}} & R_2 = \{\bar{e}_1, .., \bar{e}_m\} \\ C \downarrow & & \downarrow D \\ R_1' = \{e_1', .., e_n'\} & \xrightarrow{A' = [f]_{R_1',R_2'}} & R_2' = \{\bar{e}_1', .., \bar{e}_m'\} \end{array}$$

$$\boxed{A' = D^{-1}AC}$$

$$\operatorname{rg} A' = \operatorname{rg} A$$

Obs $f \in End(V) = \{f: V \to V \mid f \text{ lin}\}$

$$\begin{array}{ccc} R_1 = \{e_1, .., e_n\} & \xrightarrow{A = [f]_{R_1, R_1}} & R_1 = \{e_1, .., e_n\} \\ C \downarrow & & \downarrow C \\ R_1' = \{e_1', .., e_n'\} & \xrightarrow{A' = [f]_{R_1', R_1'}} & R_1' = \{e_1', .., e_n'\} \end{array}$$

$\boxed{A' = C^{-1} A C}$

Exemplu $f: \mathbb{R}^2 \to \mathbb{R}^2$, $f(x) = (x_1 + x_2, 2x_2)$

$R_0 = \{e_1 = (1,0), e_2 = (0,1)\}$ reperul canonic în $\mathbb{R}^2$

$R' = \{e_1' = e_1 - 2e_2, e_2' = e_1 + e_2\}$ reper în $\mathbb{R}^2$

a) $[f]_{R_0, R_0}$; b) $[f]_{R', R'}$

SOL

a) $f(e_1) = f(1,0) = (1,0) = \boxed{1} e_1 + \boxed{0} e_2$ $\quad A = \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix} = [f]_{R_0, R_0}$

$f(e_2) = f(0,1) = (1,2) = (1,0) + 2(0,1) = \boxed{1} e_1 + \boxed{2} e_2$

b) $f(e_1') = f(1,-2) = (-1,-4) = a e_1' + b e_2'$ $\quad A' = \begin{pmatrix} a & c \\ b & d \end{pmatrix}$

$(-1,-4) = a(1,-2) + b(1,1) = (a+b, -2a+b)$

$\begin{cases} a + b = -1 \\ -2a + b = -4 \end{cases}$ $\ominus$ $\quad \begin{cases} a = 1 \\ b = -2 \end{cases}$ $\quad A' = \begin{pmatrix} 1 & 0 \\ -2 & 2 \end{pmatrix}$

$3a = 3$

$f(e_2') = f(1,1) = (2,2) = c e_1' + d e_2' = (c+d, -2c+d)$

$\begin{cases} c + d = 2 \\ -2c + d = 2 \end{cases}$ $\quad \begin{matrix} c = 0 \\ d = 2 \end{matrix}$

$3c \;/ = 0$

OBS $R_0 \xrightarrow{C} R'$ $\quad C$ = matricea de trecere dintre cele 2 repere din $\mathbb{R}^2$

$R_0 = \{e_1, e_2\}$, $R' = \{e_1' = e_1 - 2e_2, e_2' = e_1 + e_2\}$

$C = \begin{pmatrix} 1 & 1 \\ -2 & 1 \end{pmatrix}$; $A' = C^{-1} A C$

$\det C = 3$, $C^T = \begin{pmatrix} 1 & -2 \\ 1 & 1 \end{pmatrix}$, $C^* = \begin{pmatrix} 1 & -1 \\ 2 & 1 \end{pmatrix}$

$$C^{-1} = \frac{1}{\det C} C^* = \begin{pmatrix} \frac{1}{3} & -\frac{1}{3} \\ \frac{2}{3} & \frac{1}{3} \end{pmatrix}$$

$$C^{-1}AC = \begin{pmatrix} \frac{1}{3} & -\frac{1}{3} \\ \frac{2}{3} & \frac{1}{3} \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ -2 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 \\ -2 & 2 \end{pmatrix} = A'$$

<u>OBS</u> $f : \mathbb{R}^2 \to \mathbb{R}^2$, $f(x_1, x_2) = (x_1 + x_2, 2x_2)$

f liniară $\Leftrightarrow f(x) = y \Leftrightarrow Y = AX$.

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} x_1 + x_2 \\ 2x_2 \end{pmatrix}$$

<u>Prop</u> $f : V \to W$ liniară

a) f inj $\Leftrightarrow \dim V = rg\, A$

b) f surj $\Leftrightarrow \dim W = rg\, A$

c) f bij $\Leftrightarrow A \in GL(n, \mathbb{K})$

Dem

a) f inj $\Leftrightarrow \operatorname{Ker} f = \{0_V\}$

$\operatorname{Ker} f = \{x \in V \mid f(x) = 0_W\}$

$= \{x \in V \mid AX = 0\} = S(A)$

$\dim \operatorname{Ker} f = \dim V - rg\, A = 0$

$\dim V = rg\, A$

b) f surj $\Leftrightarrow \dim \operatorname{Im} f = \dim W$.

T. dim: $\dim V = \underbrace{\dim \operatorname{Ker} f}_{\dim V - rg\, A} + \dim \operatorname{Im} f \Rightarrow \underbrace{\dim \operatorname{Im} f}_{\dim W} = rg\, A$

c) Consecință a subpunctelor a), b)

OBS

a)
$$\begin{tikzcd} V \arrow[r, "f"] \arrow[rr, bend right, "h"'] & W \arrow[r, "g"] & U \end{tikzcd} \qquad h = g \circ f$$

f, g liniare $\Rightarrow$ h liniară.

$x \longrightarrow f(x) = y \longrightarrow g(y) = z$

$$\left.\begin{array}{l} Z = A_g \cdot Y \\ Y = A_f \cdot X \end{array} \Rightarrow Z = A_g \cdot A_f \cdot X \atop Z = A_h X \right\} \Rightarrow \boxed{A_{g \circ f} = A_g \cdot A_f}$$

$x \xrightarrow{h} h(x) = z \qquad Z = A_h X$

b) $f : V \to W$ izomorfism de sp vect

$$\begin{tikzcd} V \arrow[r, "f"] \arrow[rr, bend right, "id_V"'] & W \arrow[r, "f^{-1}"] & V \end{tikzcd} \quad ; \quad \begin{tikzcd} W \arrow[r, "f^{-1}"] \arrow[rr, bend right, "id_W"'] & V \arrow[r, "f"] & W \end{tikzcd}$$

$\dim V = n = \dim W$

$I_n = A_{f^{-1}} \cdot A_f \qquad I_n = A_f \cdot A_{f^{-1}}$

$A_{f^{-1}} = (A_f)^{-1}$.

c) $Aut(V) = GL(V) = \{ \underbrace{f \in End(V)}_{(f : V \to V \text{ lin})} \mid f \text{ bij} \}$.

$\varphi : (GL(V), \circ) \longrightarrow (GL(n, \mathbb{K}), \cdot)$ izomorfism de grupuri

$\varphi(f) = A_f$

$\varphi(f \circ g) = \varphi(f) \cdot \varphi(g)$

φ bijecție.

<u>Def</u> (spațiul dual)

$(V,+,\cdot)_{/\mathbb{K}}$ sp vect

$(V^* = \{ f : V \longrightarrow \mathbb{K} \mid f \text{ liniară} \}, +, \cdot)_{/\mathbb{K}}$ sp. vectorial dual.

<u>Teoremă</u>

$V \simeq V^*$ (sp. vect. izomorfe).

<u>Dem</u>

$\mathcal{R} = \{e_1, .., e_n\}$ reper în V

Construim $\mathcal{R}^* = \{e_1^*, .., e_n^*\} \subset V^*$ aî

$e_i^* : V \longrightarrow \mathbb{K}$ $\quad e_i^*(e_j) = \delta_{ij} = \begin{cases} 1, & i=j \\ 0, & i \neq j \end{cases}$

$\forall i,j = \overline{1,n}$

$e_1^*(e_1) = 1, \ e_1^*(e_2) = 0, \ldots, \ e_1^*(e_n) = 0$

$\vdots$

$e_n^*(e_1) = 0, \ e_n^*(e_2) = 0, \ldots, \ e_n^*(e_n) = 1.$

Prelungim e_i^* prin liniaritate, $\forall i = \overline{1,n}$

$e_i^*(x) = e_i^*(x_1 e_1 + \ldots + x_n e_n) = x_1 e_i^*(e_1) + \ldots x_i \underbrace{e_i^*(e_i)}_{1} + \ldots$

$+ x_n e_i^*(e_n) = x_i, \ \forall i = \overline{1,n}$

$\mathcal{R}^*$ este reper pt V^*.

1) SLI $\quad a_1 e_1^* + \ldots + a_n e_n^* = 0$

$a_1 \underbrace{e_1^*(e_1)}_{1} + \ldots + a_n \underbrace{e_n^*(e_1)}_{0} = 0 \Rightarrow a_1 = 0$

$\vdots$

$a_1 \underbrace{e_1^*(e_n)}_{0} + \ldots + a_n \underbrace{e_n^*(e_n)}_{1} = 0 \Rightarrow a_n = 0$

2) SG. $\quad V^* = \langle \mathcal{R}^* \rangle$.

$\forall f \in V^*$ $\quad (f : V \longrightarrow \mathbb{K} \text{ lin})$ $\quad$ $\exists a_1, .., a_n \in \mathbb{K}$ aî $f = a_1 e_1^* + \ldots + a_n e_n^*$

$f(e_1) = a_1, .., f(e_n) = a_n$

$$f(x) = f(x_1 e_1 + \ldots + x_n e_n) = \underset{\substack{\parallel \\ e_1^*(x)}}{x_1} f(e_1) + \ldots + \underset{\substack{\parallel \\ e_n^*(x)}}{x_n} f(e_n) =$$

$$= \left(f(e_1) e_1^* + \ldots + f(e_n) e_n^*\right)(x), \ \forall x \in V$$

$$f = f(e_1) e_1^* + \ldots + f(e_n) e_n^*$$

Deci $\dim V = \dim V^* = n \Rightarrow V \simeq V^*$