Seminar 7

~~Conv~~

Studiați convergența simplă și uniformă pt următ. șiruri de fct.

a) $f_n : [0,\infty) \to \mathbb{R} \quad f_n(x) = \frac{x}{x+n} \quad \forall n \in \mathbb{N}^*$

Sol

Conv. simplu:

Fie $x \in [0,\infty)$

$\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} \frac{x}{x+n} = 0 \Rightarrow$

$f_n \xrightarrow[n\to\infty]{s} f$ unde $f : [0,\infty) \to \mathbb{R} \quad f(x) = 0$

Conv. simplu

$\sup_{x\in[0,\infty)} |f_n(x) - f(x)| = \sup_{x\in[0,\infty)} \left|\frac{x}{x+n} - 0\right| = \sup_{x\in[0,\infty)} \left|\frac{x}{x+n}\right|$

$\sup_{x\in[0,\infty)} \frac{x}{x+n} \geq \frac{n}{n+n} = \frac{1}{2} > 0$

$\downarrow n\to\infty$

0

$\Rightarrow f_n \not\xrightarrow[n\to\infty]{u} f$

b) $f_n : [2;3] \to \mathbb{R} \quad f_n(x) = \frac{x}{x+n} \quad \forall n \in \mathbb{N}$

Conv simplu:

Fie $x \in [2;3]$

$\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} \frac{x}{x+n} = 0$

$\Rightarrow f_n \xrightarrow[n\to\infty]{s} f$, cu $f:[2;3]\to\infty$ $f(x)=0$

<u>Conv. unif</u>

$$\sup_{x\in[2;3]} |f_n(x)-f(x)| = \sup_{x\in[2;3]} \left|\frac{x}{x+n}-0\right| = \sup_{x\in[2;3]} \left|\frac{x}{x+n}\right|$$

$$= \sup_{x\in[2;3]} \frac{x}{x+n}$$

Fie $f_n:[2;3]\to\mathbb{R}$ $f_n(x)=\frac{x}{x+n}$, $\forall n\in\mathbb{N}$

$$f_n'(x) = \frac{(x+n)-x}{(x+n)^2} = \frac{n}{(x+n)^2} > 0$$

x	2				3
$f_n'(x)$	+	+	+	+	+
$f_n(x)$	$\nearrow$				

Deci, $\sup_{x\in[2;3]} f_n(x) = f_n(3) = \frac{3}{3+n}$

$$\lim_{n\to\infty} \sup_{x\in[2;3]} f_n(x) = \lim_{n\to\infty} \frac{3}{3+n} = 0$$

$\Rightarrow f_n \xrightarrow[n\to\infty]{u} f$

c) $f_n:[0;\infty)\to\mathbb{R}$ $f_n(x)=\sqrt{x^2+\frac{1}{n}}$ $\forall n\in\mathbb{N}^*$

<u>Sol</u>

<u>Conv. simplă</u>

Fie $x\in[0,\infty)\to\mathbb{R}$

$$\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} \sqrt{x^2+\frac{1}{n}} = |x| = x$$

$\Rightarrow f_n \xrightarrow[n\to\infty]{s} f$, cu $f:[0;\infty)\to\mathbb{R}$ $f(x)=x$

Conv. uniformă

$$\lim_{n\to\infty} \sup_{x\in[0,\infty)} |f_n(x) - f(x)| = \lim_{n\to\infty} \sup_{x\in[0,\infty)} \left|\sqrt{x^2+\frac{1}{n}} - x\right|$$

$$\sup_{x\in[0,\infty)} \left|\sqrt{x^2+\frac{1}{n}} - x\right| = \sup_{x\in[0,\infty)} \left|\frac{\cancel{x^2}+\frac{1}{n}-\cancel{x^2}}{\sqrt{x^2+\frac{1}{n}}+x}\right|$$

$$= \sup_{x\in[0,\infty)} \frac{\frac{1}{n}}{\sqrt{x^2+\frac{1}{n}}+x} \leq \frac{\frac{1}{n}}{\sqrt{\frac{1}{n}}} = \sqrt{\frac{1}{n}}$$

$$\lim_{n\to\infty} \sqrt{\frac{1}{n}} = 0 \Rightarrow f_n \xrightarrow[n\to\infty]{u} f \quad \square$$

d) $f_n : [0;\infty) \to \mathbb{R}$ $f_n(x) = \frac{n}{n+x}$ $\forall$ $n\in\mathbb{N}^*$

Conv simplă

Fie $x\in[0;\infty)$

$$\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} \frac{n}{n+x} = 1$$

$\Rightarrow f_n \xrightarrow[n\to\infty]{s} f$ cu $f : [0;\infty) \to \mathbb{R}$ $f(x) = 1$

Conv. uniformă

$$\lim_{n\to\infty} \sup_{x\in[0;\infty)} |f_n(x) - f(x)| = ?$$

$$\sup_{x\in[0;\infty)} |f_n(x) - f(x)| = \sup_{x\in[0;\infty)} \left|\frac{n}{n+x} - 1\right|$$

$$= \sup_{x\in[0;\infty)} \left|\frac{-x}{n+x}\right| = \sup_{x\in[0;\infty)} \frac{x}{n+x}$$

Fie $g_n : [0;\infty) \to \mathbb{R}$ $g_n(x) = \frac{x}{n+x}$

$$g_n'(x) = \frac{n+\cancel{x} - \cancel{x}}{(n+x)^2} = \frac{n}{(n+x)^2} \geq 0$$

x	0				∞
$g_n'(x)$	$+$	$+$	$+$	$+$	$+$
$g_n(x)$	$\nearrow$				1

Deci, $\sup_{x\in[0;\infty)} g_n(x) = \lim_{x\to\infty} g_n(x) = 1$

Așadar, $\lim_{n\to\infty} \sup_{x\in[0,\infty)} g_n(x) = 1 \neq 0$

Prin urmare $f_n \not\xrightarrow[n\to\infty]{u} f$ $\square$

e) $f_n : (0;1] \to \mathbb{R}$ $f_n(x) = x^n$, $\forall n \in \mathbb{N}$

<u>Conv simplă:</u>

Fie $x \in (0;1)$

$\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} x^n = 0 \Rightarrow$

~~$f_n \xrightarrow[n\to\infty]{s} f$, unde $f:[0;1]$~~

Pt $x = 1$, $\lim_{n\to\infty} f_n(1) = 1$

Deci, $\lim_{n\to\infty} f_n(x) = \begin{cases} 0, & x \in (0;1) \\ 1, & x = 1 \end{cases}$

Așadar $f_n \xrightarrow[n\to\infty]{s} f$, unde $f:(0;1] \to \mathbb{R}$

$f(x) = \begin{cases} 0, & x \in (0;1) \\ 1, & x = 1 \end{cases}$

<u>Conv. uniformă</u>

$\left.\begin{array}{l} f_n \text{ cont. } \forall n \in \mathbb{N} \\ f \text{ nu e cont (în } 1) \end{array}\right| \Rightarrow f_n \not\xrightarrow[n\to\infty]{u} f$

f) $f_n : [\frac{1}{2}; 1] \to \mathbb{R} \quad f_n(x) = \frac{(1+x)^n}{e^{2nx}}, \forall n \in \mathbb{N}$

<u>Convergența simplă</u>:

$$\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} \frac{(1+x)^n}{e^{2nx}} = \lim_{n\to\infty} \left(\frac{1+x}{e^{2x}}\right)^n$$

Vrem să arătăm Fie $g : [\frac{1}{2}; 1] \to \mathbb{R} \quad g(x) = \cancel{\frac{1+x}{e^{2x}}} - 1$

$0 \leq g(x) < 1$

evident $\left(\frac{1+x}{e^{2x}}\right)^n$

$$\cancel{g'(x) = \frac{1 \cdot e^{2x} - (1+x)e^{2x} \cdot 2}{e^{4x}}}$$

$$\cancel{= \frac{e^{2x} - 2e^{2x} - 2xe^{2x}}{e^{4x}}}$$

$$\cancel{= -\frac{e^{2x} - 2xe^{2x}}{e^{4x}}}$$

$$\cancel{0 = \frac{e^{2x}(1+2x)}{e^{4x}}}$$

$\frac{1+x}{e^{2x}} < \frac{1}{1} \Rightarrow g(x) = 1 + x - e^{2x} < 0$

$g'(x) = 1 - e^{2x} \cdot 2 < 0 \; \forall x \in [\frac{1}{2}; 1]$

$\Rightarrow g$ - s. descresc

x	$\frac{1}{2}$		1
$g'(x)$	$-$	$-$	$-$
$g(x)$	$\frac{3}{2} - e$	$\longrightarrow$	$2 - e^2$

Deci, $g(x) \leq \frac{3}{2} - e < 0 \; \forall x \in [\frac{1}{2}; 1]$

Așadar, $1 + x < e^{2x} \; \forall x \in [\frac{1}{2}; 1]$

Prin urmare, $f_1(x) < 1 \; \forall x \in [\frac{1}{2}; 1]$

Fie $x \in [\frac{1}{2}; 1]$

$\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} (f_1(x))^n = 0 \Rightarrow$

$f_n \xrightarrow[n\to\infty]{s} f, \; f(x) = 0$

Converg unif

Avem:

1) $[\frac{1}{2}, 1]$ - mult compt

2) $(f_n)_n$ descresc $((f_1(x))^n)$ $\in (0,1)$

3) f_n, f cont $\forall n \in \mathbb{N}^*$

4) $f_n \xrightarrow[n\to\infty]{s} f$

Cf. Crit lui Dini $f_n \xrightarrow[n\to\infty]{u} f$ [ ]

g) $f_n : [\frac{1}{2}; \frac{\pi}{2}] \to \mathbb{R}$ $f_n(x) = (\cos x)^n$, $\forall n \in \mathbb{N}^*$

Conv. simplă

$0 \le \cos x < 1$ $\forall$ $x \in [\frac{1}{2}; \frac{\pi}{2}] \Rightarrow$

$\lim\limits_{n\to\infty} f_n(x) = \lim\limits_{n\to\infty} (\cos x)^n = 0$ $\forall$ $x \in [\frac{1}{2}; \frac{\pi}{2}]$

$\Rightarrow$ $f_n \xrightarrow[n\to\infty]{s} f$ unde $f(x) = 0$

Conv uniformă

$\underset{[\frac{1}{2}; \frac{\pi}{2}]}{x} \mapsto \underset{[0;1)}{\cos x}$ descrescătoare $\Rightarrow f_n$ descresc $\forall n \in \mathbb{N}^*$

Avem

1) $f_n, f : [\frac{1}{2}; \frac{\pi}{2}] \to \mathbb{R}$ $\forall$ $n \in \mathbb{N}^*$

2) f_n descresc

3) f continuă

4) $f_n \xrightarrow[n\to\infty]{s} f$

$\Rightarrow$ Cf T. Polya avem că $f_n \xrightarrow[n\to\infty]{u} f$

2. Fie $f_n, f, g : [a,b] \to \mathbb{R}$ $\forall n \in \mathbb{N}$ a.î. $(a,b \in \mathbb{R}, a<b)$

$f_n \xrightarrow[n\to\infty]{u} f$ și g cont. Arătați că

$f_n \cdot g \xrightarrow[n\to\infty]{u} f \cdot g$.

$|f_n(x)\, g(x) - f(x)\, g(x)| < \varepsilon$

$|g(x)\,(f_n(x) - f(x))| < \varepsilon$

$\underbrace{|g(x)|}_{\le M}\, \underbrace{|f_n(x) - f(x)|}_{\le \frac{\varepsilon}{M}} < \varepsilon$

$\left.\begin{array}{l} g:[a,b]\to\mathbb{R} \text{ cont} \\ [a,b] \text{ - compactă} \end{array}\right| \Rightarrow \exists M \ge 0$ a.î. $|g(x)| \le M$ $\forall x \in [a,b]$

$f_n \xrightarrow[n\to\infty]{u} f \Rightarrow \forall \varepsilon > 0$ $\exists n_\varepsilon \in \mathbb{N}$ a.î. $\forall$ $n \ge n_\varepsilon$ ~~avem~~ și $\forall x \in [a,b]$ avem

(majorez înt. cât vreau) $|f_n(x) - f(x)| < \frac{\varepsilon}{M}$

Fie $\varepsilon > 0$ (avem $n_\varepsilon \in \mathbb{N}$ de mai sus)

Fie $n \ge n_\varepsilon$ și $x \in [a,b]$

$|f_n(x)\, g(x) - f(x)\, g(x)| = \underbrace{|g(x)|}_{\le M}\, \underbrace{|f_n(x) - f(x)|}_{< \frac{\varepsilon}{M}}$

$< M \cdot \frac{\varepsilon}{M} = \varepsilon$

$f_n g \xrightarrow[n\to\infty]{u} f \cdot g$

3. Studiați conv. simplă și uniformă pt $(f_n)_n$ și $(f_n')_n$ unde $f_n : \mathbb{R} \to \mathbb{R}$ $f_n(x) = \frac{\operatorname{arctg} nx}{n}$ $\forall n \in \mathbb{N}^*$

Sol
• pt $(f_n)_n$

Conv. simplă

Fie $x \in \mathbb{R}$

$\operatorname{arctg} nx \in [-\frac{\pi}{2}; \frac{\pi}{2}]$ $\forall$ $n \in \mathbb{N}^*$, $x \in \mathbb{R}$

$-\frac{\pi}{2} \le \operatorname{arctg} nx \le \frac{\pi}{2}$ $/ \cdot \frac{1}{n}$

$-\frac{\pi}{2n} \le \underbrace{\frac{\operatorname{arctg} nx}{n}}_{\xrightarrow{n\to\infty} 0} \le \frac{\pi}{2n}$ $/ \lim_{n\to\infty}()$

$\Rightarrow \lim_{n\to\infty} f_n(x) = 0$

Deci, $f_n \xrightarrow[n\to\infty]{s} f$, $f(x) = 0$

Conv. uniformă

$\sup_{x\in\mathbb{R}} |f_n(x) - f(x)| = \sup_{x\in\mathbb{R}} \left| \frac{\operatorname{arctg} nx}{n} - 0 \right|$

$= \sup_{x\in\mathbb{R}} \left| \frac{\operatorname{arctg} nx}{n} \right| = \sup_{x\in\mathbb{R}} \frac{|\operatorname{arctg} nx|}{n} \le \frac{\pi}{2n}$

$\xrightarrow[n\to\infty]{} 0$

$\Rightarrow f_n \xrightarrow[n\to\infty]{u} f$

• pt $(f_n')_n$

$f_n'(x) = \frac{(\operatorname{arctg} nx)' \cdot n + \operatorname{arctg} nx \cdot (n')}{n^2}$

$$= \frac{\frac{1}{1+n^2x^2}\cdot \cancel{n^2} + \operatorname{arctg} nx \cdot 0}{\cancel{n^2}} = \frac{1}{1+n^2x^2}$$

<u>Conv. simplă</u>

Fie $x \in \mathbb{R}$

$$\lim_{n\to\infty} f_n'(x) = \lim_{n\to\infty} \frac{1}{1+n^2x^2} = \begin{cases} 0, & x \in \mathbb{R}^* \\ 1, & x = 0 \end{cases}$$

$\Rightarrow f_n' \xrightarrow[n\to\infty]{s} g$, unde $g(x) = \begin{cases} 0, & x \in \mathbb{R}^* \\ 1, & x = 0 \end{cases}$

<u>Conv. uniformă</u>

~~$\sup_{x\in\mathbb{R}} |f_n(x) - f(x)|$~~

$\left.\begin{array}{l} f_n' \text{ continuă pe } \mathbb{R} \\ g \text{ nu e cont în } 0 \end{array}\right| \Rightarrow f_n' \cancel{\xrightarrow[n\to\infty]{u}} g$

<u>Exc</u>

$f_n : [0, \pi] \to \mathbb{R}$ $f_n(x) = \frac{\cos nx}{n}$ $\forall n \in \mathbb{N}^*$

$(f_n)_n$, $(f_n')_n$