1) $f: \mathbb{R}^n \longrightarrow \mathbb{R}$ diferentiabilă și omogenă de grad p

adică $f(\lambda x_1, \lambda x_2, \ldots, \lambda x_n) = \lambda^p f(x_1, x_2, \ldots, X_n), \forall x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$

Arătați

$$x_1 \frac{\partial f}{\partial x_1}(x) + x_2 \frac{\partial f}{\partial x_2}(x) + \cdots + X_n \frac{\partial f}{\partial x_n}(x) = p f(x), \quad \forall x \in \mathbb{R}^n$$

Soluție

$$g: \mathbb{R} \times \mathbb{R}^n \longrightarrow \mathbb{R}, \quad g(\lambda, x_1, x_2, \ldots, x_n) = f(\underbrace{\lambda x_1, \lambda x_2, \ldots, \lambda x_n}_{\lambda x})$$

$$\frac{\partial g}{\partial \lambda}(\lambda, x) = \frac{\partial f}{\partial x_1}(\lambda x) \cdot x_1 + \frac{\partial f}{\partial x}(\lambda x) \cdot x_2 + \cdots + \frac{\partial f}{\partial x_n}(\lambda x) \cdot x_n \quad (1)$$

$$g(\lambda, x) = f(\lambda x) = \underline{\lambda^p f(x)} \Longrightarrow \frac{\partial g}{\partial \lambda}(\lambda, x) = p \lambda^{p-1} f(x) \quad (2)$$

$(1),(2) \Rightarrow \quad x_1 \cdot \frac{\partial f}{\partial x_1}(\lambda x) + \cdots + x_n \frac{\partial f}{\partial x_n}(\lambda x) = p\lambda^{p-1} f(x)$

$\lambda = 1 \Rightarrow \quad x_1 \frac{\partial f}{\partial x_1}(x) + \cdots + x_n \frac{\partial f}{\partial x_n}(x) = p f(x).$

2) $f : \mathbb{R}^2 \to \mathbb{R}$, $f(x,y) = e^{x+2y}$. Scrieți polinomul Taylor de grad 2 și 3 asociat lui f în $(0,0)$.

<u>Soluție</u>

$T_2(x,y) = f(0,0) + df(0,0)(x,y) + \frac{1}{2} d^2 f(0,0)(x,y)$

$= f(0,0) + \left(\frac{\partial f}{\partial x}(0,0) \cdot x + \frac{\partial f}{\partial y}(0,0) \cdot y\right) + \frac{1}{2}\left(\frac{\partial^2 f}{\partial x^2}(0,0) x^2 + 2 \frac{\partial^2 f}{\partial x \partial y}(0,0) x y + \right.$

$\left. + \frac{\partial^2 f}{\partial y^2}(0,0) y^2\right)$

$$f(x,y) = e^{x+2y} \qquad f(0,0) = 1$$

$$\frac{\partial f}{\partial x} = e^{x+2y}, \quad \frac{\partial f}{\partial y} = 2e^{x+2y},$$

$$\frac{\partial^2 f}{\partial x^2} = e^{x+2y}, \quad \frac{\partial^2 f}{\partial x \partial y} = 2e^{x+2y}, \quad \frac{\partial^2 f}{\partial y^2} = 4e^{x+2y}$$

$$\frac{\partial f}{\partial x}(0,0) = 1, \quad \frac{\partial f}{\partial y}(0,0) = 2, \quad \frac{\partial^2 f}{\partial x^2}(0,0) = 1, \quad \frac{\partial^2 f}{\partial x \partial y}(0,0) = 2, \quad \frac{\partial^2 f}{\partial y^2}(0,0) = 4$$

$$T_2(x,y) = 1 + (x+2y) + \frac{1}{2}\left(x^2 + 4xy + 4y^2\right)$$

$$= 1 + (x+2y) + \frac{1}{2}(x+2y)^2$$

$$\frac{\partial^3 f}{\partial x^3} = e^{x+2y}, \quad \frac{\partial^3 f}{\partial x^2 \partial y} = 2e^{x+2y}, \quad \frac{\partial^3 f}{\partial x \partial y^2} = 4e^{x+2y}, \quad \frac{\partial^3 f}{\partial y^3} = 8e^{x+2y}$$

$$T_3(x,y) = T_2(x,y) + \frac{1}{3!} d^3 f(0,0)(x,y) =$$

$$= T_2(x,y) + \frac{1}{6}\left(\frac{\partial f}{\partial x}(0,0)\, x + \frac{\partial f}{\partial y}(0,0)\, y\right)^{(3)}$$

$$= T_2(x,y) + \frac{1}{6}\left(\frac{\partial^3 f}{\partial x^3}(0,0)\, x^3 + 3\frac{\partial^3 f}{\partial x^2 \partial y}(0,0)\, x^2 y + 3\frac{\partial^2 f}{\partial x \partial y^2}(0,0)\, x y^2 + \frac{\partial^3 f}{\partial y^3}(0,0)\, y^3\right)$$

$$= T_2(x,y) + \frac{1}{6}\left(x^3 + 3\cdot 2\, x^2 y + 3\cdot 4\, x y^3 + 8 y^3\right)$$

$$= 1 + (x+2y) + \frac{1}{2}(x+2y)^2 + \frac{1}{6}(x+2y)^3 .$$

$$d^2f(0,0) = \frac{\partial^2 f}{\partial x^2}(0,0)\,dx^2 + 2\frac{\partial^2 f}{\partial x \partial y}(0,0)\,dx\,dy + \frac{\partial^2 f}{\partial y^2}(0,0)\,dy^2$$

$$d^2f(0,0)(a,b) = \frac{\partial^2 f}{\partial x^2}(0,0)\,a^2 + 2\frac{\partial^2 f}{\partial x \partial y}(0,0)\,ab + \frac{\partial^2 f}{\partial y^2}(0,0)\,b^2$$

$$d^2f(0,0)(a,b) = a^2 + 4ab + 4b^2$$

$$d^2f(0,0) = dx^2 + 4\,dx\,dy + 4\,dy^2$$

$$H_f(0,0) = \begin{pmatrix} \frac{\partial^2 f}{\partial x^2}(0,0) & \frac{\partial^2 f}{\partial x \partial y}(0,0) \\ \frac{\partial^2 f}{\partial x \partial y}(0,0) & \frac{\partial^2 f}{\partial y^2}(0,0) \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}$$

$$(a\ b)\begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = (a+2b\ ,\ 2a+4b)\begin{pmatrix} a \\ b \end{pmatrix} = a^2+4ab+4b^2$$

3) $f : (0,\infty)\times(0,\infty)\to\mathbb{R}$, $f(x,y) = g(xy) + \sqrt{xy}\cdot h\left(\frac{y}{x}\right)$

unde g și h sunt funcții de clasă C^1 pe $\mathbb{R}$. Arătați că

$$x^2\frac{\partial^2 f}{\partial x^2} - y^2\frac{\partial^2 f}{\partial y^2} = 0.$$

Soluție. $$\frac{\partial f}{\partial x}(x,y) = g'(xy)\cdot\frac{\partial}{\partial x}(xy) + \sqrt{y}\cdot\frac{1}{2\sqrt{x}}\cdot h\left(\frac{y}{x}\right) + \sqrt{xy}\cdot h'\left(\frac{y}{x}\right)\cdot\frac{\partial}{\partial x}\left(\frac{y}{x}\right)$$

$$= y\,g'(xy) + \frac{1}{2}x^{-\frac{1}{2}}y^{\frac{1}{2}}h\left(\frac{y}{x}\right) - x^{-\frac{3}{2}}y^{\frac{3}{2}}\cdot h'\left(\frac{y}{x}\right)$$

$$\frac{\partial^2 f}{\partial x^2}(x,y) = y^2 g''(xy) - \frac{1}{4}x^{-\frac{3}{2}}y^{-\frac{1}{2}}h\left(\frac{y}{x}\right) - \frac{1}{2}x^{-\frac{1}{2}}y^{\frac{1}{2}}\cdot h'\left(\frac{y}{x}\right)\cdot\frac{y}{x^2}$$
$$+\frac{3}{2}x^{-\frac{5}{2}}y^{\frac{3}{2}}h'\left(\frac{y}{x}\right) + x^{-\frac{3}{2}}y^{\frac{3}{2}}\cdot h''\left(\frac{y}{x}\right)\cdot\left(-\frac{y}{x^2}\right)$$

4) Fie $f:\mathbb{R}^2 \to \mathbb{R}$, $f(x,y) = (y-x^2)(y-3x^2)$

1) Să se arate că $(0,0)$ nu este punct de extrem local al fct. f.

2) Să se arate că $(0,0)$ este pct de minim local al funcției f de-a lungul oricărei drepte care trece prin origine.

<u>Soluție</u>: Fie $a \neq 0$.

$$f(a, 2a^2) = (2a^2 - a^2)(2a^2 - 3a^2) = -a^4 < 0, \quad f(0,0) = 0$$

$$f(a, 4a^2) = (4a^2 - a^2)\cdot(4a^2 - 3a^2) = 3a^4 > 0$$

Deci $(0,0)$ nu este pct de extrem local.

2) Consideram o dreaptă care trece prin origine

$$\begin{cases} x = at \\ y = bt \end{cases}, \ a, b \in \mathbb{R}, \ a^2+b^2 \neq 0.$$

Aratam că O este punct de minim al funcției

$$g: \mathbb{R} \to \mathbb{R}, \quad g(t) = f(at, bt) = (bt - a^2t^2)(bt - 3a^2t^2)$$

$$g(t) = 3a^4t^4 + b^2t^2 - 4a^2bt^3$$

$$\left.\begin{aligned} g'(t) &= 12a^4t^3 + 2b^2t - 12a^2bt^2, \quad & g'(0) = 0. \\ g''(t) &= 36a^4t^2 + 2b^2 - 24a^2bt \quad & g''(0) = 2b^2 \end{aligned}\right\} \Longrightarrow$$

$\Longrightarrow$ daca $b \neq 0$ atunci O este pct de minim local dacă $b=0$, $\left.\begin{matrix} g(t)=3a^4t^4 \\ a\neq 0 \end{matrix}\right\} \Rightarrow 0$ este pct de minim local al lui g

5) Să se determine punctele de extrem local ale funcției $f:\mathbb{R}^2\to\mathbb{R}$, $f(x,y)=\ln(1+|x-y|)$

Rezolvare. f are derivate parțiale pe $\mathbb{R}^2 \setminus \{(a,a) \mid a\in\mathbb{R}\}$

$$f(x,y)=\begin{cases} \ln(1+x-y), & x\geq y \\ \ln(1+y-x), & x<y \end{cases}$$

$$\frac{\partial f}{\partial x}(x,y)=\begin{cases} \dfrac{1}{1+x-y} & x>y \\ \dfrac{-1}{1+y-x} & x<y \end{cases} ; \quad \frac{\partial f}{\partial y}(x,y)=\begin{cases} -\dfrac{1}{1+x-y}, & x>y \\ \dfrac{1}{1+y-x}, & x<y \end{cases}$$

Fie $a \in \mathbb{R}$.

$$\lim_{\substack{x \to a \\ x < a}} \frac{f(x,a) - f(a,a)}{x-a} = \lim_{\substack{x \to a \\ x < a}} \frac{\ln(1+a-x)}{x-a}$$

$$= \lim_{\substack{t \to 0 \\ t < 0}} \frac{\ln(1-t)}{t} = -1$$

$$\lim_{\substack{x \to a \\ x > a}} \frac{f(x,a) - f(a,a)}{x-a} = \lim_{\substack{x \to a \\ x > a}} \frac{\ln(1+x-a)}{x-a} = 1$$

Asadar nu exista $\lim\limits_{x \to a} \frac{f(x,a) - f(a,a)}{x-a}$.

deci nu există $\frac{\partial f}{\partial x}(a,a)$, $\forall a \in \mathbb{R}$.

Analog se arată că pt $\forall a \in \mathbb{R}$ nu există $\frac{\partial f}{\partial y}(a, a)$.

$$f(x,y) - f(a,a) = \ln(1+|x-y|) - 0 \geq \ln(1) = 0, \ \forall (x,y) \in \mathbb{R}^2$$

Deci (a,a) este punct de minim local (chiar global) pentru f.

Remarcă! f are o infinitate de puncte de minim local și niciun punct critic.

6) Determinați punctele de extrem local ale funcției
$f:\mathbb{R}^2 \longrightarrow \mathbb{R}$, $f(x,y) = x^3+y^3-3x^2+6y^2+3x+12y+1$

Soluție $\mathbb{R}^2$ este mulțime deschisă, f este de clasă.

I. $\begin{cases} \frac{\partial f}{\partial x} = 0 \\ \frac{\partial f}{\partial y} = 0. \end{cases}$ $\frac{\partial f}{\partial x} = 3x^2-6x+3$

$\frac{\partial f}{\partial y} = 3y^2+12y+12.$

$\begin{cases} 3x^2-6x+3=0 \\ 3y^2+12y+12=0 \end{cases} \Longleftrightarrow \begin{cases} x=1 \\ y=-2 \end{cases}$

Deci f are un singur punct critic: $(1,-2)$

II. $\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}(3x^2 - 6x + 3) = 6x - 6$

$\frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}(3y^2 + 12y + 12) = 6y + 12$

$\frac{\partial^2 f}{\partial x \partial y} = 0.$

$$H_f(x,y) = \begin{pmatrix} \frac{\partial^2 f}{\partial x^2}(x,y) & \frac{\partial^2 f}{\partial x \partial y}(x,y) \\ \frac{\partial^2 f}{\partial x \partial y}(x,y) & \frac{\partial^2 f}{\partial y^2}(x,y) \end{pmatrix} = \begin{pmatrix} 6x-6 & 0 \\ 0 & 6y+12 \end{pmatrix}$$

$H_f(1,-2) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ — nu putem trage nicio concluzie.

(căutăm alte metode)

$$f(x,y) = \underline{x^3} + y^3 - \underline{3x^2} + 6y^2 + \underline{3}x + 12y + \underline{1}$$

$$= (x-1)^3 + (y+2)^3 - 6$$

Fie $a \in \mathbb{R}$, $f(a+1,-2) = a^3 - 6$

$f(a+1,-2) < -6$ dacă $a < 0$.

$f(a+1,-2) > -6$ daca $a > 0$.

$a = \frac{1}{n}$

$f\left(1+\frac{1}{n}, -2\right)$

In nice vecinătate a lui $(1,-2)$ există puncte în care f ia valori strict mai mari ca $f(1,-2)$ și există puncte în care f ia valori strict mai mici ca $f(1,-2)$. Deci $(1,-2)$ nu este pct de extrem local.