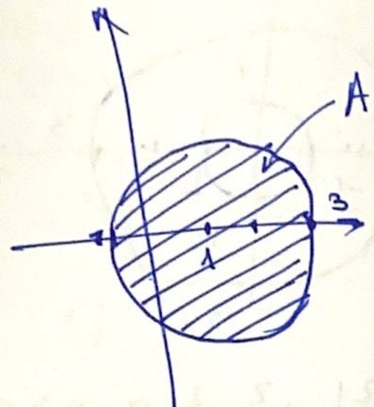
Seminar 14

a) $\iint_A y\,dx\,dy$, $A = \{ (x,y) \in \mathbb{R}^2 \mid (x-1)^2+y^2 \leq 4 \}$

Sol



A convexă şi mărg $\Rightarrow A \in \mathcal{J}(\mathbb{R}^2)$

A compactă (închisă + mărg)

Fie $f: A \to \mathbb{R}$ $f(x,y)=y$

f cont

S.V. $\begin{cases} x = 1 + r\cos\theta & r \in [0;\infty) \\ y = r\sin\theta & \theta \in [0;2\pi] \end{cases}$

$(x,y) \in A \Rightarrow (x-1)^2+y^2 \leq 4 \Rightarrow (\cancel{1}+r\cos\theta-\cancel{1})^2+(r\sin\theta)^2 \leq 4$

$r^2\cos^2\theta + r^2\sin^2\theta \leq 4$

$r^2 (\underbrace{\cos^2\theta + \sin^2\theta}_{=1}) \leq 4 \Rightarrow r^2 \leq 4 \Rightarrow \begin{cases} r \in [0;2] \\ \theta \in [0;2\pi] \end{cases}$

~~Fie~~ Fie $B = [0;2] \times [0;2\pi]$

$$\iint_A f(x,y)\,dx\,dy = \iint_B r\, f(1+r\cos\theta, r\sin\theta)\,dr\,d\theta$$

$$= \int_0^2 \left(\int_0^{2\pi} r \cdot r\sin\theta\, d\theta \right) dr$$

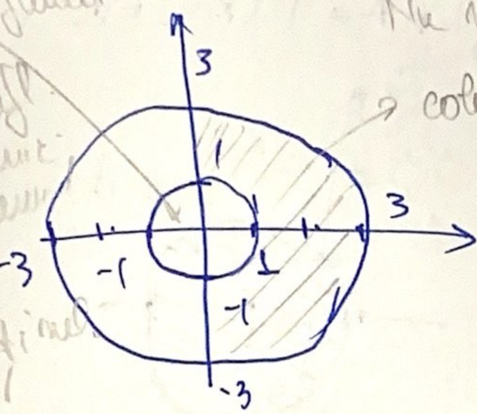
$$= \int_0^2 \left(r^2(-\cos\theta) \Big|_{\theta=0}^{\theta=2\pi} \right) dr$$

$$= \int_0^2 0\, dr = 0 \qquad \square$$

b) $\iint_A \sqrt{x^2+y^2}\,dx\,dy$, unde $A = \{(x,y)\in\mathbb{R}^2 \mid 1\le x^2+y^2\le 9, x\ge 0\}$

Sol   Nu mai e convexă

scot și jumăt aia, dar nu mă infl cu minus că ortem nu e în mulțime



$$A = \underbrace{\{(x,y)\in\mathbb{R}^2 \mid x^2+y^2\le 9, x\ge 0\}}_{\text{convexă + mărg} \to \text{măs. } J.} \setminus \underbrace{\{(x,y)\in\mathbb{R}^2 \mid x^2+y^2<1\}}_{\text{conv. + mărg} \to \text{măs } J.}$$

ex 14, pen. listă → $\boxed{\text{GREU}}$

Deci $A\in J(\mathbb{R}^2)$

A compactă

Fie $f: A\to\mathbb{R}$ $f(x,y)=\sqrt{x^2+y^2}$

f cont

S.V $\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$, $r\in[0,\infty)$, $\theta\in[0,2\pi)$

$(x,y)\in A \to \begin{cases} 1\le x^2+y^2\le 9 \\ x\ge 0 \end{cases} \Rightarrow \begin{cases} r\in[1,3] \\ \theta\in[0;\frac{\pi}{2}]\cup[\frac{3\pi}{2};2\pi] \end{cases}$

Fie $B=[1,3]\times\left([0;\frac{\pi}{2}]\cup[\frac{3\pi}{2};2\pi]\right)$

$$\iint_A f(x,y)\,dx\,dy = \iint_B r\, f(r\cos\theta, r\sin\theta)\,dr\,d\theta$$

$$= \int_1^3 \left(\int_0^{\frac{\pi}{2}} r\cdot\sqrt{r^2}\, d\theta + \int_{\frac{3\pi}{2}}^{2\pi} r\sqrt{r^2}\, d\theta \right) dr$$

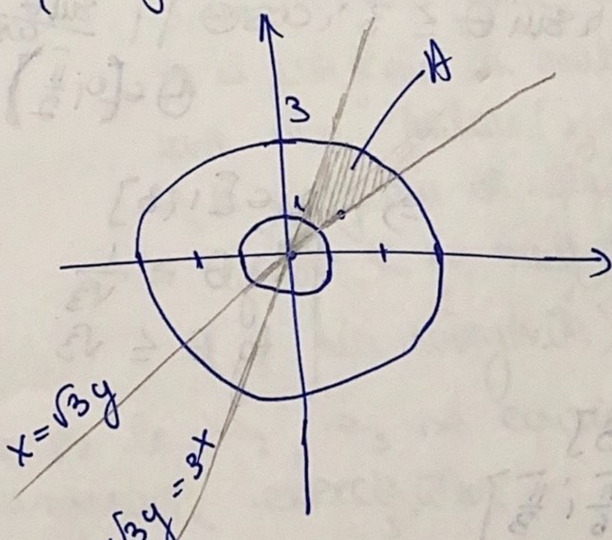
$$= \int_1^3 \left(r^2\cdot\theta \Big|_{\theta=0}^{\theta=\frac{\pi}{2}} + r^2\cdot\theta \Big|_{\theta=\frac{3\pi}{2}}^{\theta=2\pi} \right) dr$$

$$= \int_1^3 r^2\cdot\frac{\pi}{2} + r^2\cdot\frac{\pi}{2}\, dr = \pi \cdot \frac{r^3}{3}\Big|_{r=1}^{r=3}$$

$$= 26\frac{\pi}{3} \quad \square$$

c) $\iint_A \operatorname{arctg}\frac{y}{x}\, dxdy$, unde

$A = \{(x,y)\in\mathbb{R}^2 \mid 1 \le x^2+y^2 \le 9,\ x \le \sqrt{3}y \le 3x\}$



$x=\sqrt{3}y$

$$A = \underbrace{\{(x,y)\in\mathbb{R}^2 \mid x^2+y^2 \le 9,\ x \le \sqrt{3}y \le 3x\}}_{\text{conv. + mărg} \Rightarrow \text{măs J}} \setminus$$

$$\underbrace{\{(x,y)\in\mathbb{R}^2 \mid x^2+y^2 < 1\}}_{\text{convexă + mărg} \Rightarrow \text{măs J}}$$

Deci, $A \in \mathcal{J}(\mathbb{R}^2)$

Fie $f: A \to \mathbb{R}$ $f(x,y) = \text{arctg} \frac{y}{x}$

f cont

S.V $\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$ $r \in [0;\infty)$, $\theta \in [0;2\pi]$

$(x,y) \in A \Rightarrow \begin{cases} 1 \le x^2+y^2 \le 9 \\ x \le \sqrt{3}y \\ \sqrt{3}y \le 3x \end{cases}$

$\Rightarrow \begin{cases} 1 \le r^2 \le 9 \\ r\cos\theta \le \sqrt{3}\, r\sin\theta \\ \sqrt{3}\, r\sin\theta \le 3r\cos\theta \end{cases} \Rightarrow \begin{cases} r \in [1;3] \\ \frac{\sin\theta}{\cos\theta} \ge \frac{1}{\sqrt{3}} \\ \frac{\sin\theta}{\cos\theta} \le \sqrt{3} \end{cases}$

$\theta \in [0;\frac{\pi}{2})$

$\cos\theta \le \sqrt{3}\sin\theta$

$1 \le \sqrt{3}\frac{\sin\theta}{\cos\theta}$

$\Rightarrow \begin{cases} r \in [1;3] \\ tg\,\theta \ge \frac{1}{\sqrt{3}} \\ tg\,\theta \le \sqrt{3} \end{cases}$

$\Rightarrow r \in [1;3]$

$\theta \in [\frac{\pi}{6};\frac{\pi}{3}]$

Fie $B = [1;3] \times [\frac{\pi}{6};\frac{\pi}{3}]$

$\iint_A f(x,y)\,dx\,dy = \iint_B r\, f(r\cos\theta, r\sin\theta)\,dr\,d\theta$

$= \int_1^3 \left(\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} r\, \text{arctg}\left(\frac{\cancel{r}\sin\theta}{\cancel{r}\cos\theta}\right) d\theta \right) dr =$

$= \int_1^3 \left(\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} r\theta\, d\theta \right) dr = \int_1^3 r \cdot \frac{\theta^2}{2} \Big|_{\theta=\frac{\pi}{6}}^{\theta=\frac{\pi}{3}} dr$

$$= \int_1^3 \frac{r}{2}\left(\frac{\pi^2}{9} - \frac{\pi^2}{36}\right) dr = \int_1^3 \frac{r}{2}\cdot \frac{\cancel{3}\pi^2}{\cancel{36}\,12}\, dr$$

$$= \frac{\pi^2}{24}\cdot \frac{r^2}{2}\Bigg|_{r=1}^{r=3} = \frac{\pi^2}{\cancel{48}\,12}\cdot\frac{\cancel{(9-1)}\,4}{\cancel{2}} = \frac{\pi^2}{12}$$

Se știe iacobianul la examen, nu tb calc. de noi ;)

~~2. Def.~~

Observații:

$\boxed{1.}$ Atunci când calculăm integrale triple, nu mai este nevoie:

$\to_1$ să schițăm grafic mulțimile din $\mathbb{R}^3$

$\to_2$ să justificăm că mulțimile din $\mathbb{R}^3$ sunt măs. Jordan (și nici compacte)

$\to_3$ să justificăm că funcțiile sunt integr. R. pe mulț. din $\mathbb{R}^3$ (și nici mărginite)

Afirmațiile de la $\to_2$, $\to_3$ se consideră adev. din enunțurile exercițiilor

2. ~~Det~~:

a) $\iiint_A (xyz+y^2)\,dx\,dy\,dz$, unde $A = [-1;1]\times[2,3]\times[0,1]$

Sol: $\iiint_A (xyz+y^2)\,dx\,dy\,dz =$

$$\int_{-1}^{1}\left(\int_2^3\left(\int_0^1 (xyz+y^2)\,dz\right)dy\right)dx$$

$$\int_{-1}^{1}\left(\int_{2}^{3} xy\frac{z^2}{2}+\frac{y^2 \cdot z}{\cancel{z}}\Big|_{z=0}^{z=1} dy\right)dx$$

$$\int_{-1}^{1}\left(\int_{2}^{3}\frac{xy}{2}+y^2\, dy\right)dx$$

$$\int_{-1}^{1}\left(\frac{x}{2}\,\frac{y^2}{2}+\frac{y^3}{3}\Big|_{y=2}^{y=3}\right)dx$$

$$\int_{-1}^{1}\frac{x}{2}\cdot\frac{5}{2}+\frac{19}{3}\,dx$$

$$\frac{x^2}{2\cdot 2}\cdot\frac{5}{2}+\frac{19}{3}x\Big|_{-1}^{1} \Rightarrow \cancel{\frac{5}{8}+\frac{19}{3}+\frac{5}{8}+\frac{19}{3}}$$

$$\cancel{\frac{5}{4}+\frac{38}{3}} = \frac{\cancel{5}}{8}+\frac{19}{3}-\frac{\cancel{5}}{8}+\frac{19}{3}=\frac{38}{3}$$

b) $\iiint_A x\,dxdydz$, unde $A=[1,2]\times[0,1]\times[2,3]$

Sol: Rez. voi!!! $\square$

c) $\iiint_A (x^2+y^2)z\,dxdydz$, unde $A=\{(x,y,z)\in\mathbb{R}^3 \mid$

$(x,y)\in B,\ x^2+y^2\le z\le\sqrt{6-x^2-y^2}\}$

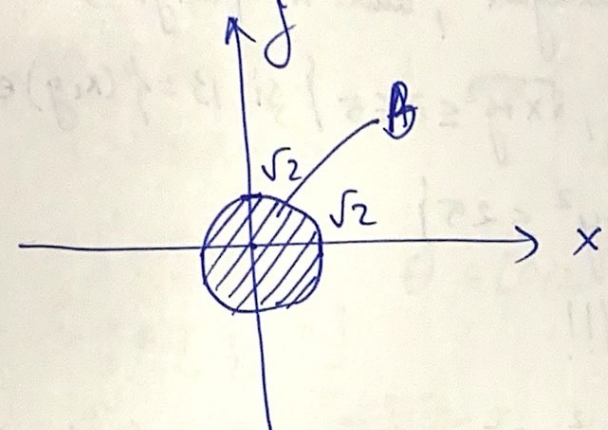
$B=\{(x,y)\in\mathbb{R}^2 \mid x^2+y^2\le 2\}$

$$\iiint_A (x^2+y^2)z\,dxdydz=\iint_B\left(\int_{x^2+y^2}^{\sqrt{6-x^2-y^2}}(x^2+y^2)z\,dz\right)dxdy$$

$$=\iint_B (x^2+y^2)\frac{z^2}{2}\Big|_{z=x^2+y^2}^{z=\sqrt{6-x^2-y^2}} dxdy$$

$$= \iint_B (x^2+y^2)\, \frac{6-x^2-y^2-(x^2+y^2)^2}{2}\, dx\, dy$$

$$= \iint_B \frac{(x^2+y^2)}{2} \left(6-x^2-y^2-(x^2+y^2)^2\right) dx\, dy$$



B convexă și mărg $\Rightarrow B \in J(\mathbb{R}^2)$

B compactă

Fie $f: B \to \mathbb{R}$ $\quad f(x,y) = \frac{(x^2+y^2)}{2}\left(6-x^2-y^2-(x^2+y^2)^2\right)$

S.V $\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$ $r \in [0;\infty)$, $\theta \in [0;2\pi]$

$(x,y) \in B \Rightarrow x^2+y^2 \le 2 \Rightarrow r^2 \le 2 \Rightarrow \begin{cases} r \in [0;\sqrt{2}] \\ \theta \in [0;2\pi] \end{cases}$

Fie $C = [0;\sqrt{2}] \times [0;2\pi]$

$$\iint_B f(x,y)\,dx\,dy = \iint_C r\, f(r\cos\theta, r\sin\theta)\, dr\, d\theta$$

$$= \int_0^{\sqrt{2}} \left(\int_0^{2\pi} r \cdot \frac{r^2}{2} (6-r^2-r^4)\, d\theta \right) dr$$

$$= \int_0^{\sqrt{2}} \cancel{2}\pi \cdot \frac{r^3}{\cancel{2}} (6-r^2-r^4)\, dr$$

$$= \pi \cdot \left(6 \cdot \frac{r^4}{4} \Big|_{r=0}^{r=\sqrt{2}} - \frac{r^6}{6} \Big|_{r=0}^{r=\sqrt{2}} - \frac{r^8}{8} \Big|_{r=0}^{r=\sqrt{2}} \right)$$

$$= \pi \left(6 - \frac{4}{3} - 2 \right) = \frac{8\pi}{3}$$

d) $\iiint_A xyz \, dx dy dz$, unde $A = \{ (x,y,z) \in \mathbb{R}^3 \mid (x,y) \in B, \sqrt{x^2+y^2} \le z \le 5 \}$ și $B = \{ (x,y) \in \mathbb{R}^2 \mid 9 \le x^2+y^2 \le 25 \}$

Sol Rez voi!!!

e) $\iiint_A \frac{x^2}{4} + \frac{y^2}{9} + \frac{z^2}{16} \, dx dy dz$, unde $A = \{ (x,y,z) \in \mathbb{R}^3 \mid \frac{x^2}{4} + \frac{y^2}{9} + \frac{z^2}{16} \le 1, z \le 0 \}$

θ pe podea

φ podea - tavan

Fie $f : A \to \mathbb{R}$ $f(x,y,z) = \frac{x^2}{4} + \frac{y^2}{9} + \frac{z^2}{16}$

S.V $\begin{cases} x = 2r \cos\theta \sin\varphi & r \in [0, \infty) \\ y = 3r \sin\theta \sin\varphi & \theta \in [0, 2\pi] \\ z = 4r \cos\varphi & \varphi \in [0, \pi] \end{cases}$

$(x,y,z) \in A \Rightarrow \begin{cases} \frac{x^2}{4} + \frac{y^2}{9} + \frac{z^2}{16} \le 1 \\ z \le 0 \end{cases}$

$$\begin{cases} \frac{\cancel{4} r^2 \cos^2\theta \sin^2\varphi}{\cancel{4}} + \frac{\cancel{9} r^2 \sin^2\theta \sin^2\varphi}{\cancel{9}} + \frac{\cancel{16} r^2 \cos^2\varphi}{\cancel{16}} \le 1 \\ \cancel{z \le 0} \quad 4r\cos\varphi \le 0 \end{cases}$$

$$\begin{cases} r^2 \sin^2\varphi\,(\underbrace{\cos^2\theta + \sin^2\theta}_{1}) + r^2\cos^2\varphi \le 1 \\ 4r\cos\varphi \le 0 \end{cases}$$

$$\begin{cases} r^2(\sin^2\varphi + \cos^2\varphi) \le 1 \\ 4r\cos\varphi \le 0 \end{cases} \Rightarrow \begin{cases} r^2 \le 1 \\ 4r\cos\varphi \le 0 \end{cases}$$

$$\Rightarrow \begin{cases} r \in [0;1] \\ \varphi \in [\frac{\pi}{2}; \pi] \end{cases} \Leftarrow \theta \in [0;2\pi]$$

Fie $B = [0;1] \times \cancel{[\frac{\pi}{2};\pi]} \times [0;2\pi] \times [\frac{\pi}{2};\pi]$

$$\iiint_A f(x,y,z)\,dx\,dy\,dz = \iiint_B 2\cdot 3\cdot 4\cdot r^2 \sin\varphi$$
$$f(2r\cos\theta\sin\varphi,\ 3r\sin\theta\sin\varphi,\ 4r\cos\varphi)\,dr\,d\theta\,d\varphi$$

$$= \int_0^1 \left(\int_0^{2\pi} \left(\int_{\frac{\pi}{2}}^{\pi} 24\cdot r^2 \sin\varphi \right) r^2 d\varphi \right) d\theta / dr \Big)$$

$$= 24\cdot \int_0^1 \left(\int_0^{2\pi} \left(r^4 \int_{\frac{\pi}{2}}^{\pi} \sin\varphi\, d\varphi \right) d\theta \right) dr$$

(se face calculul)

$$= \frac{48\pi}{5} \quad \square$$

f) $\iiint_A x\,dx\,dy\,dz$, unde $A = \{(x,y,z) \in \mathbb{R}^3 \mid x^2+y^2+z^2 \le 1,\ y \ge 0\}$