Calculati $\int_0^1 \frac{\ln(1+x)}{1+x^2}\,dx$ $\qquad \alpha(y)=0,\ \beta(y)=y$

calculand integrala $F(y)=\int_0^y \frac{\ln(1+xy)}{1+x^2}\,dx,\ y\geq 0$

Solutie.

$$F(y)=\int_{\alpha(y)}^{\beta(y)} f(x,y)\,dx,\quad F'(y)=\int_{\alpha(y)}^{\beta(y)} \frac{\partial f}{\partial y}(x,y)\,dx+f(\beta(y),y)\cdot\beta'(y)$$
$$-\ f(\alpha(y),y)\cdot\alpha'(y)$$

$$F'(y)=\int_0^y \frac{x}{(1+x^2)(1+xy)}\,dx+\frac{\ln(1+y^2)}{1+y^2}$$

$$\frac{x}{(1+x^2)(1+xy)} = \frac{A(y)}{1+yx} + \frac{x\cdot B(y) + C(y)}{1+x^2}$$

$$A(y)(1+x^2) + (1+yx)(B(y)x + C(y)) = x$$

$$\big(A(y) + y\cdot B(y)\big)x^2 + \big(C(y)\cdot y + B(y)\big)x + \big(A(y) + C(y)\big) = x$$

$$-A(y) = C(y), \quad A(y) = -yB(y)$$

$$B(y)\cdot y^2 + B(y) = 1 \Rightarrow B(y) = \frac{1}{1+y^2}, \quad A(y) = \frac{-y}{1+y^2}, \quad C(y) = \frac{y}{1+y^2}$$

$$F'(y) = \int_0^y \left(-\frac{y}{1+y^2}\cdot\frac{1}{1+yx} + \frac{1}{1+y^2}\cdot\frac{x+y}{1+x^2}\right)dx + \frac{\ln(1+y^2)}{1+y^2}$$

$$= \cancel{-\frac{1}{1+y^2}\cdot \ln(1+xy)}\Big|_{x=0}^{x=y} + \frac{1}{2(1+y^2)}\ln(1+x^2)\Big|_{x=0}^{x=y} +$$

$$+\frac{y}{1+y^2}\,\mathrm{arctg}\, x\Big|_{x=0}^{x=y} + \cancel{\frac{\ln(1+y^2)}{1+y^2}}$$

$$F'(y) = \frac{y}{1+y^2}\,\mathrm{arctg}\, y + \frac{\ln(1+y^2)}{2(1+y^2)}$$

$$F'(y) = \left(\frac{1}{2}\ln(y^2+1)\right)'\mathrm{arctg}\, y + \frac{1}{2}\ln(1+y^2)\left(\mathrm{arctg}\, y\right)'$$

$$= \left(\frac{1}{2}\ln(y^2+1)\cdot \mathrm{arctg}\, y\right)'$$

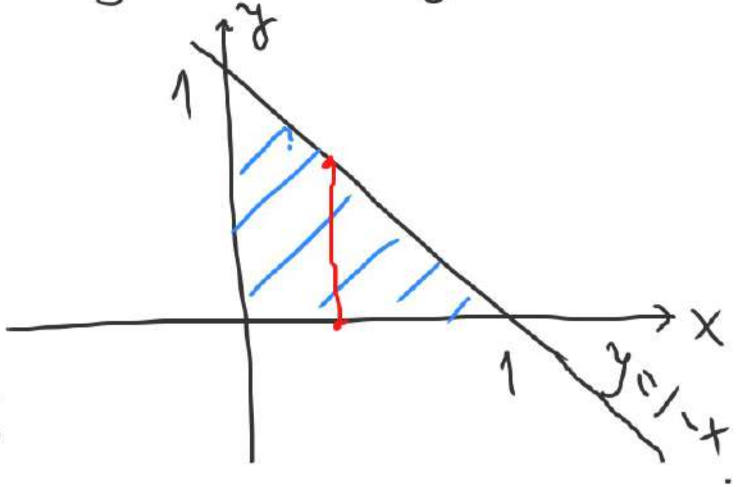
Există $C \in \mathbb{R}$ a.î.

$$F(y) = \frac{1}{2} \ln(1+y^2) \operatorname{arctg} y + C$$

$$F(0) = C = 0.$$

Deci $F(y) = \frac{1}{2} \ln(1+y^2) \operatorname{arctg} y$

$$\iint_D e^{(x+y)^2}\,dx\,dy \qquad D=\{(x,y)\in\mathbb{R}^2 \mid x+y\le 1,\ x,y\ge 0\}$$



$$\iint_D e^{(x+y)^2}\,dx\,dy = \int_0^1 \left(\underbrace{\int_0^{1-x} e^{(x+y)^2}\,dy}_{?}\right)dx$$

$D:\quad 0 \le x \le 1\ ;\ 0 \le y \le 1-x$

$$\mathring{D} = \{(x,y) \mid x+y<1,\ x>0,\ y>0\}$$

$$x = (x+y)\cdot\frac{x}{x+y} \qquad u = x+y\ ,\quad v = \frac{x}{x+y} \qquad x = uv$$

$$y = x+y-x \qquad\qquad y = u - uv = u(1-v)$$

Daca $u, v \in (0,1) \Rightarrow uv > 0,\ u - uv > 0,\ uv + (u - uv) = u < 1$

$\Rightarrow \forall (u,v) \in (0,1) \Rightarrow (uv,\ u-uv) \in \mathring{D}$

$\phi : (0,1) \times (0,1) \longrightarrow \mathring{D}$

$\phi(u,v) = (uv,\ u - uv)$

$\forall (x,y) \in \mathring{D},\ \exists!\ u,v \in (0,1)$ a.î. $\phi(u,v) = (x,y)$

Atunci

1) ϕ bijectivă ; 2) ϕ de clasă C^1

3) $\det J_\phi(u,v) = \begin{vmatrix} v & u \\ 1-v & -u \end{vmatrix} = -uv - u + uv = -u \neq 0$

Deci ϕ este difeomorfism de clasă C^1

$f(x,y) = e^{(x+y)^2}$ este cont și mărginită pe D

$\lambda(D \setminus \mathring{D}) = \lambda(Fr(D)) = 0$

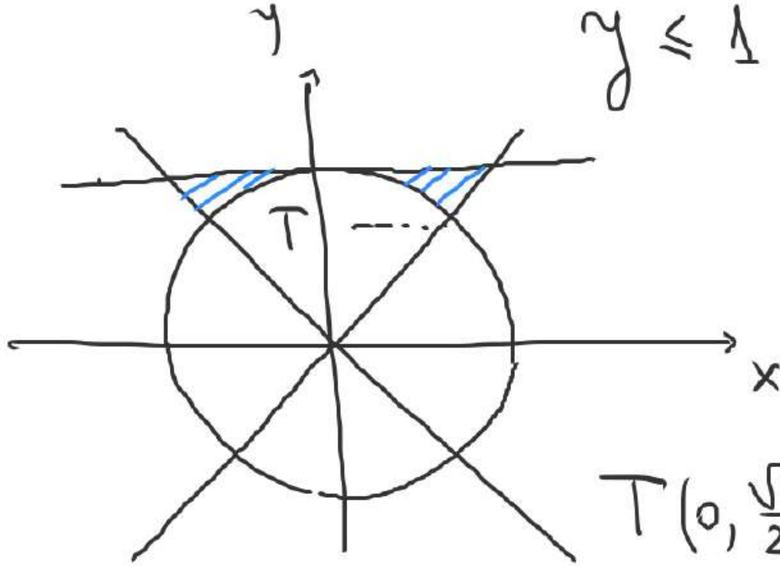
T. sch. var

$$\iint_D e^{(x+y)^2}\,dx\,dy = \iint_{\mathring{D}} e^{(x+y)^2}\,dx\,dy \xlongequal{\text{T. sch. var}} \iint_{\varphi^{-1}(\mathring{D})} e^{u^2}\cdot \left|\det J_\phi(u,v)\right| du\,dv$$

$$= \iint_{(0,1)\times(0,1)} e^{u^2} u\,du\,dv = \int_0^1 \left(\int_0^1 e^{u^2} u\,du\right) dv = \int_0^1 \left(\frac{e^{u^2}}{2}\Big|_0^1\right) dv$$

$$= \frac{e-1}{2}$$

Calculati $\iint\limits_D y\,dx\,dy$, $D = \{(x,y) \mid y \geq x,\ y \geq -x,\ x^2+y^2 \geq 1,\ y \leq 1\}$



$T\left(0, \frac{\sqrt{2}}{2}\right)$

$D' = D \cap \{(x,y) \mid x, y \geq 0\}$

$D' = \left\{(x,y) \mid \frac{\sqrt{2}}{2} \leq y \leq 1;\ \sqrt{1-y^2} \leq x \leq y\right\}$

$$\iint\limits_{D'} y\,dx\,dy = \int_{\frac{\sqrt{2}}{2}}^{1} \left(\int_{\sqrt{1-y^2}}^{y} y\,dx \right) dy$$

$$= \int_{\frac{\sqrt{2}}{2}}^{1} \left(y^2 - y\sqrt{1-y^2}\right) dy = \int_{\frac{\sqrt{2}}{2}}^{1} y^2\,dy - \int_{\frac{\sqrt{2}}{2}}^{1} y\sqrt{1-y^2}\,dy$$

$$= \frac{y^3}{3}\Big|_{\frac{\sqrt{2}}{2}}^{1} + \frac{1}{2}\int_{\frac{\sqrt{2}}{2}}^{1} \sqrt{1-y^2}\cdot(1-y^2)'\,dy$$

$$= \frac{1}{3}\left(1-\frac{1}{2\sqrt{2}}\right) + \frac{1}{2}\cdot\frac{(1-y^2)^{\frac{3}{2}}}{\frac{3}{2}}\Bigg|_{\frac{\sqrt{2}}{2}}^{1} = \frac{1}{3}\left(1-\frac{1}{2\sqrt{2}}\right) - \frac{1}{3}\left(\frac{1}{2}\right)^{\frac{3}{2}}$$

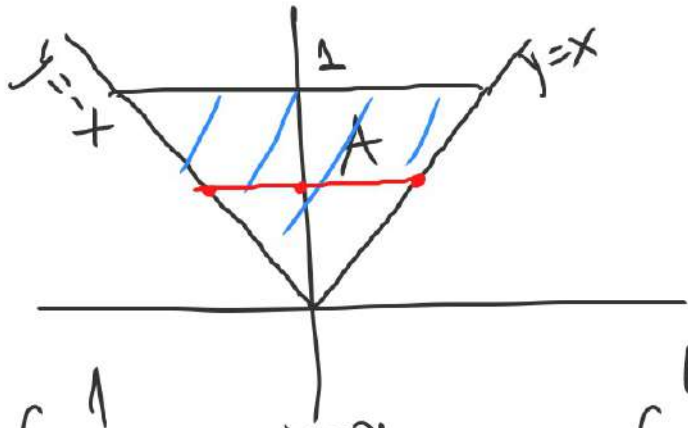
La fel pe $D'' = D \cap \{(x,y) \mid x \le 0,\ y \ge 0\}$

Obs: $D = A \setminus B$ $\quad A = \{(x,y) \mid y \ge x,\ y \ge -x,\ y \le 1\}$

$B = \{(x,y) \mid x^2+y^2 \le 1,\ y \ge x,\ y \ge -x\}$

$$\iint_D y\,dx\,dy = \iint_A y\,dx\,dy - \iint_B y\,dx\,dy$$

$A = \{(x,y) \mid y \geq x, y \geq -x, y \leq 1\}$

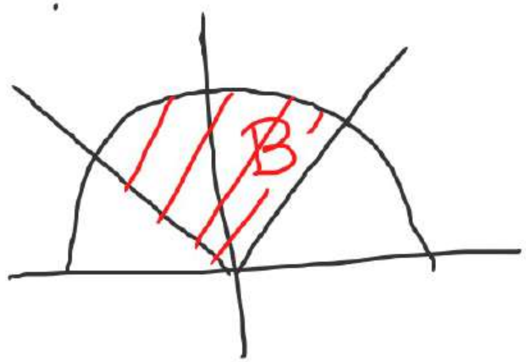


$A: \quad 0 \leq y \leq 1 \; , \; -y \leq x \leq y$

$$\iint_A y \, dx\,dy = \int_0^1 \left(\int_{-y}^{y} y \, dx \right) dy = \int_0^1 y \cdot x \Big|_{x=-y}^{x=y} dy = \int_0^1 2y^2 \, dy$$

$$= \frac{2}{3}$$

$B = \{(x,y) \mid x^2 + y^2 \leq 1, \; y \geq x, \; y \geq -x\}$



$$\begin{cases} x = r\cos\theta & r \in [0,1] \\ y = r\sin\theta & \theta \in \left[\frac{\pi}{4}, \frac{3\pi}{4}\right] \end{cases} \quad dx\,dy = r\,dr\,d\theta$$

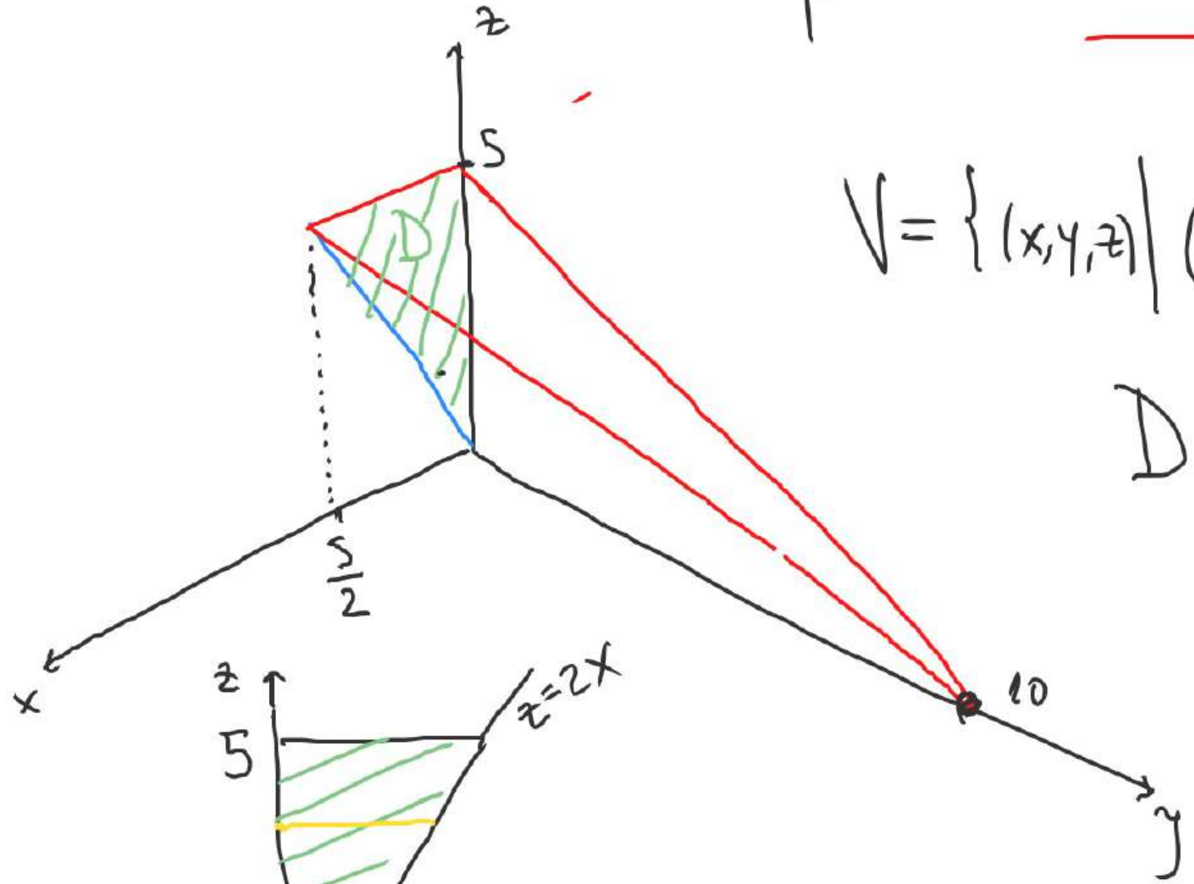
$$\iint_B y \, dx\,dy = \int_0^1 \left(\int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} r^2 \sin\theta \, d\theta \right) dr =$$

$$= \int_0^1 r^2\,dr \cdot \int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} \sin\theta\,d\theta = \frac{1}{3}\left(-\cos\theta\right)\Big|_{\frac{\pi}{4}}^{\frac{3\pi}{4}} = \frac{1}{3}\left(+\frac{\sqrt{2}}{2}+\frac{\sqrt{2}}{2}\right) = \frac{\sqrt{2}}{3}$$

$$\iint_D y\,dx\,dy = \frac{2}{3} - \frac{\sqrt{2}}{3}$$

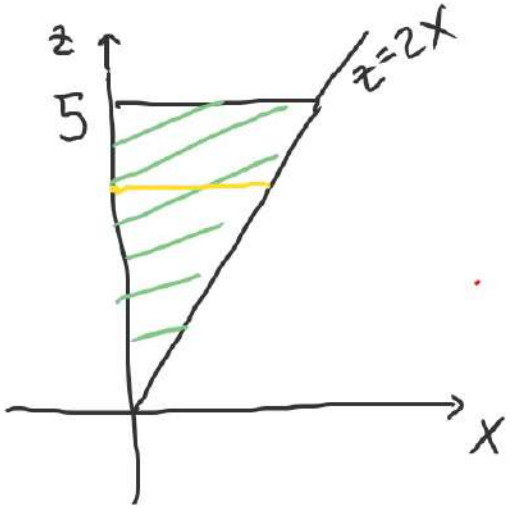
Calculati $\iiint_V x\,dx\,dy\,dz$ unde V este mărginită de

planele $x=0$, $y=0$, $y=10-2z$, $z=2x$



$$V=\{(x,y,z)\mid (x,z)\in D,\ 0\le y\le 10-2z\}$$

$$D=\{(x,z)\mid 0\le z\le 5,\ 0\le x\le \tfrac{z}{2}\}$$



$$\iiint\limits_V x\,dx\,dy\,dz = \iint\limits_D \left(\int_0^{10-2z} x\,dy \right) dx\,dz$$

$$= \iint\limits_D x(10-2z)\,dx\,dz = \int_0^5 \left(\int_0^{\frac{z}{2}} x(10-2z)\,dx \right) dz$$

$$= \int_0^5 (10-2z) \cdot \frac{x^2}{2} \Bigg|_{x=0}^{x=\frac{z}{2}} dz = \int_0^5 \frac{z^2}{8}(10-2z)\,dz = \dots$$