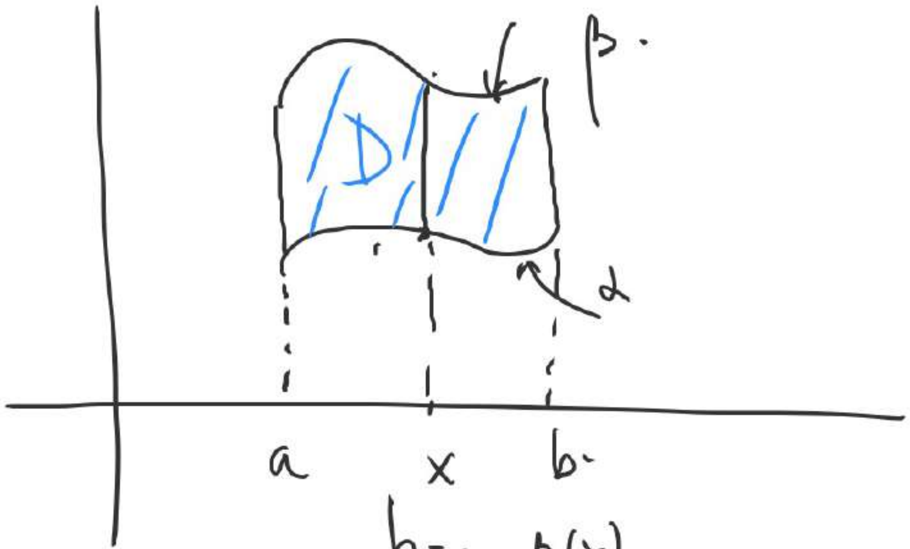
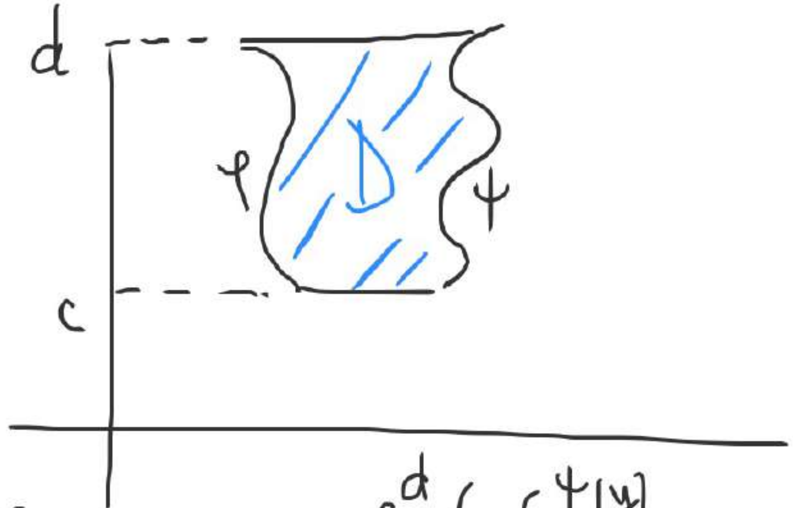


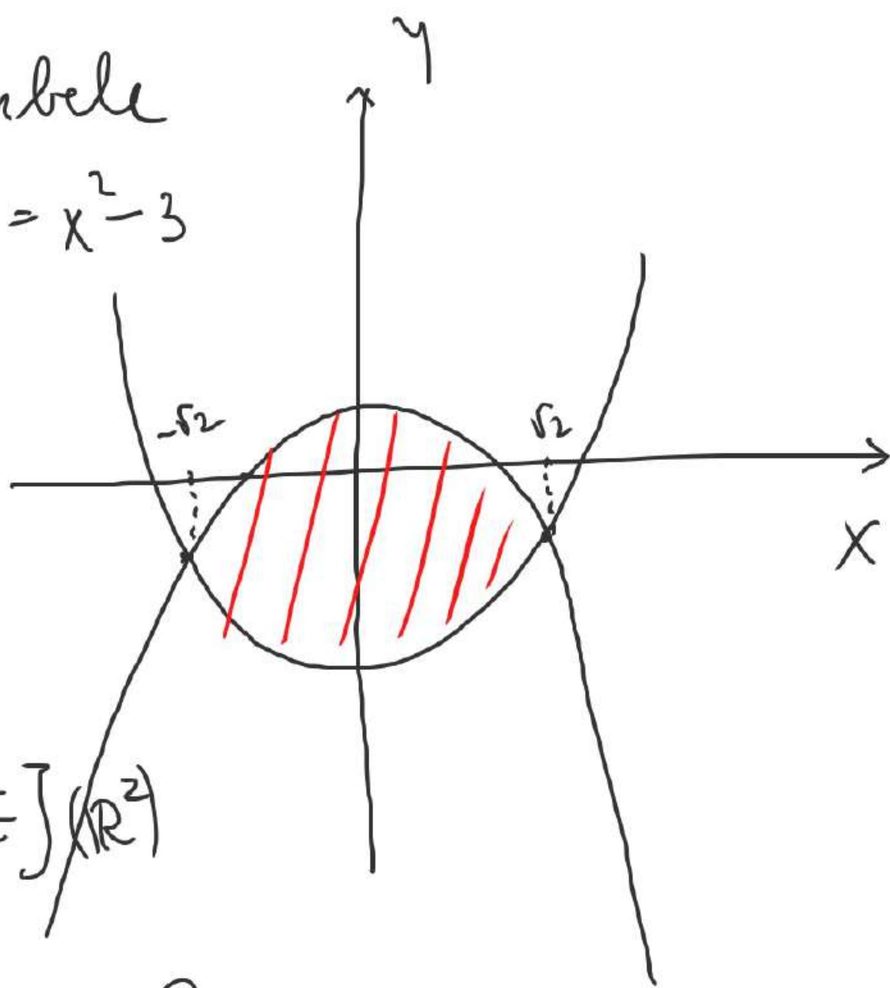
$$\iint_D f(x,y)\,dx = \int_a^b \left(\int_{\alpha(x)}^{\beta(x)} f(x,y)\,dy \right) dx$$



$$\iint_D f\,dx\,dy = \int_c^d \left(\int_{\varphi(y)}^{\psi(y)} f(x,y)\,dx \right) dy$$

$\iint_D x(y-1)\,dx\,dy$, D mărginită de curbele

$y = 1-x^2$ și $y = x^2-3$



$1-x^2 = x^2-3 \iff 2x^2 = 4 \iff x = \pm\sqrt{2}.$

$(-\sqrt{2}, -1),\ (\sqrt{2}, -1)$

$D = \{(x,y)\in\mathbb{R}^2 \mid -\sqrt{2}\le x\le\sqrt{2},\ x^2-3\le y\le 1-x^2\} \in J(\mathbb{R}^2)$

continue și deci integr. Riemann

$$\iint_D x(y-1)\,dx\,dy = \int_{-\sqrt{2}}^{\sqrt{2}}\left(\int_{x^2-3}^{1-x^2} x(y-1)\,dy\right)dx = \int_{-\sqrt{2}}^{\sqrt{2}} x\cdot\left(\frac{y^2}{2}-y\right)\Bigg|_{y=x^2-3}^{y=1-x^2} dx =$$

$$= \int_{-\sqrt{2}}^{\sqrt{2}} \left(\frac{x}{2}(1-x^2)^2 - \frac{x}{2}(x^2-3)^2 - x(1-x^2) + x(x^2-3) \right) = 0$$

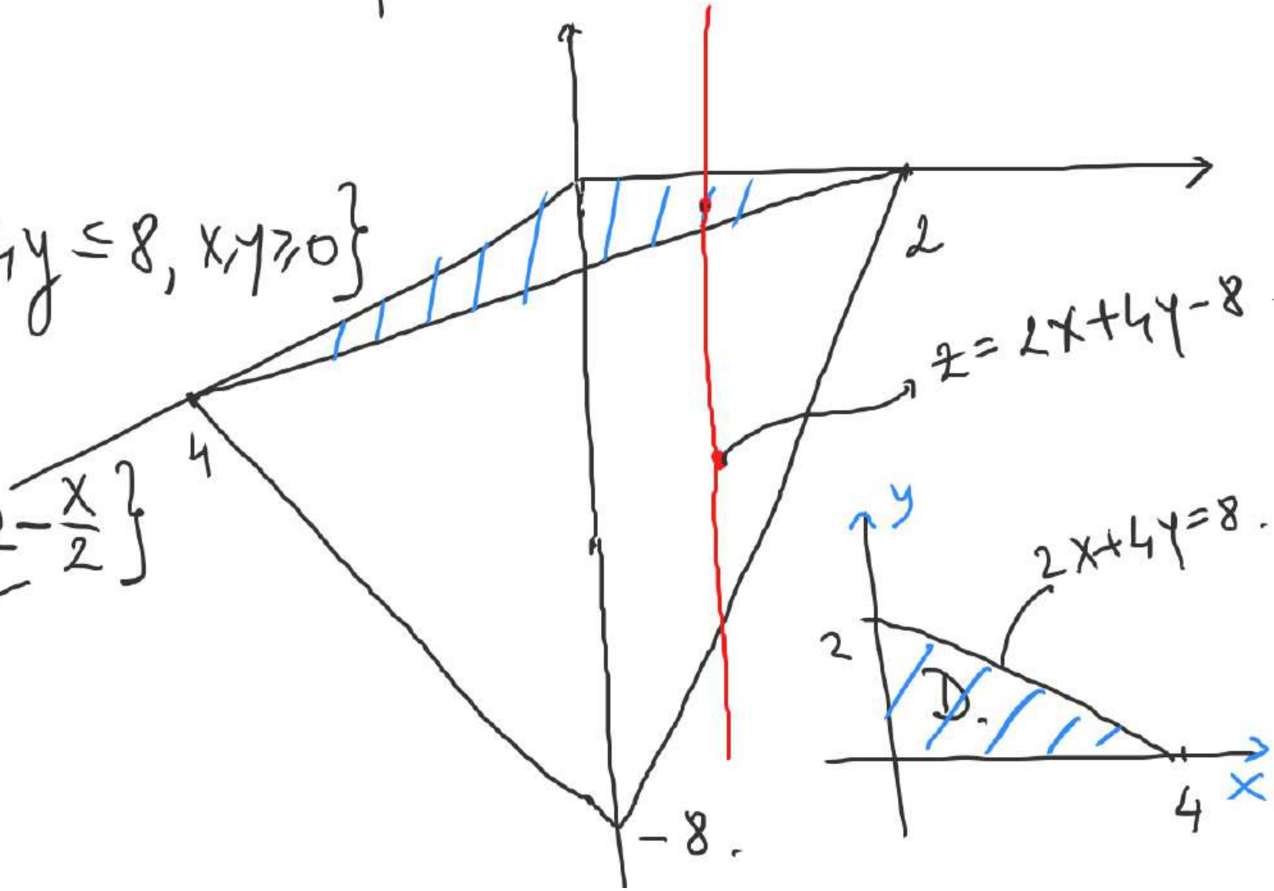
2) $\iiint\limits_V (x+y)\,dx\,dy\,dz$, $V = \{ (x,y,z) \in \mathbb{R}^3 \mid 2x+4y-z \le 8,\ x,y \ge 0,\ z \le 0 \}$

$pr_{xOy} V = D = \{ (x,y) \in \mathbb{R}^2 \mid 2x+4y \le 8,\ x,y \ge 0 \}$

$D = \{ (x,y) \in \mathbb{R}^2 \mid 0 \le x \le 4,\ \underbrace{0 \le y \le 2 - \frac{x}{2}}_{\text{cont}} \}$

$\Uparrow$

$J(\mathbb{R}^2)$.



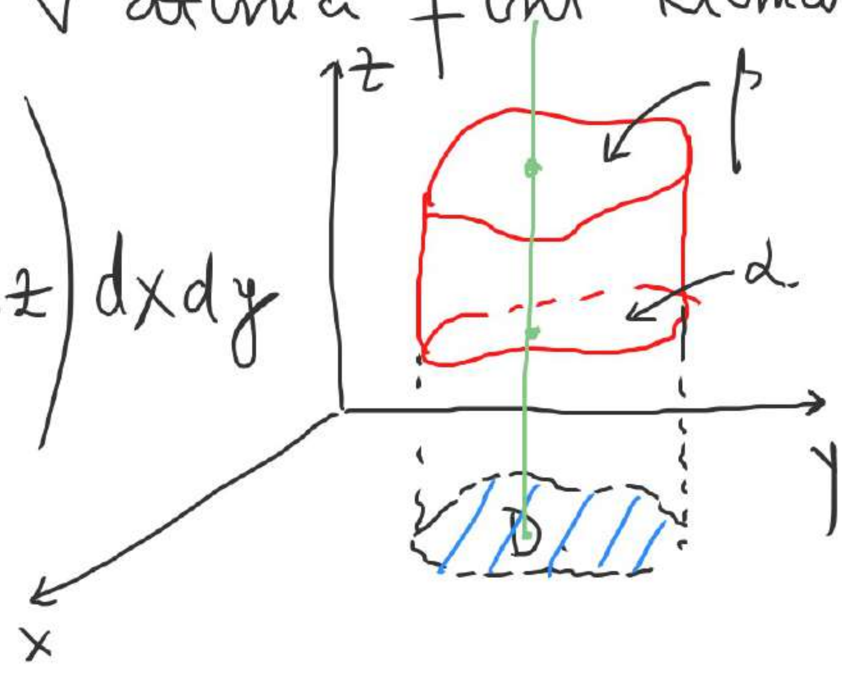
$$V = \{ (x,y,z) \in \mathbb{R}^3 \mid 2x+4y-8 \le z \le 0,\ (x,y) \in D \} \in J(\mathbb{R}^3)$$

<u>Propoziție</u>. Fie $D \in J(\mathbb{R}^2)$, $\alpha, \beta : D \to \mathbb{R}$ cont. și mărginite a.î. $\alpha(x,y) \le \beta(x,y)$, $\forall (x,y) \in D$. Atunci

$$V = \{ (x,y,z) \mid (x,y) \in D,\ \alpha(x,y) \le z \le \beta(x,y) \} \in J(\mathbb{R}^3)$$

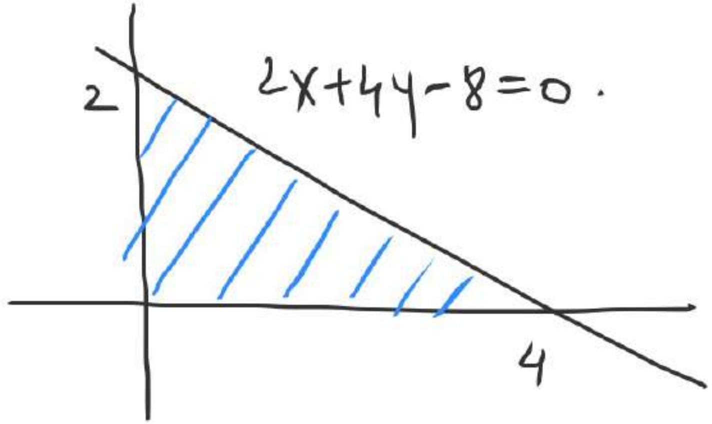
Dacă $f : V \to \mathbb{R}$ cont. și mărg pe V atunci f int Riemann

$$\iiint_V f(x,y,z)\,dx\,dy\,dz = \iint_D \left(\int_{\alpha(x,y)}^{\beta(x,y)} f(x,y,z)\,dz \right) dx\,dy$$



$$\iiint_V (x+y)\,dx\,dy\,dz = \iint_D \left(\int_{2x+4y-8}^{0} (x+y)\,dz \right) dx\,dy$$

$$= \iint_D (x+y)(-2x-4y+8)\,dx\,dy$$



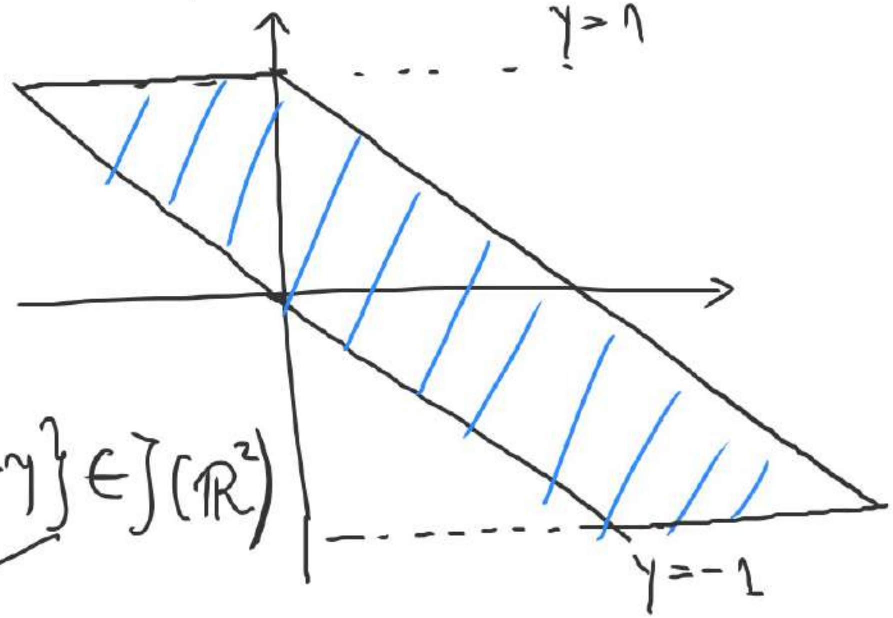
$$D = \left\{ (x,y) \;\middle|\; 0 \leq x \leq 4,\; 0 \leq y \leq 2-\frac{x}{2} \right\}$$

$$\|$$

$$\int_0^4 \left(\int_0^{2-\frac{x}{2}} (x+y)(-2x-4y+8)\,dy \right) dx = \cdots\cdots$$

3) $\iint\limits_D y\,dx\,dy$, D mărginit de dreptele

$x+y=0,\ x+y=1,\ y=1,\ y=-1$



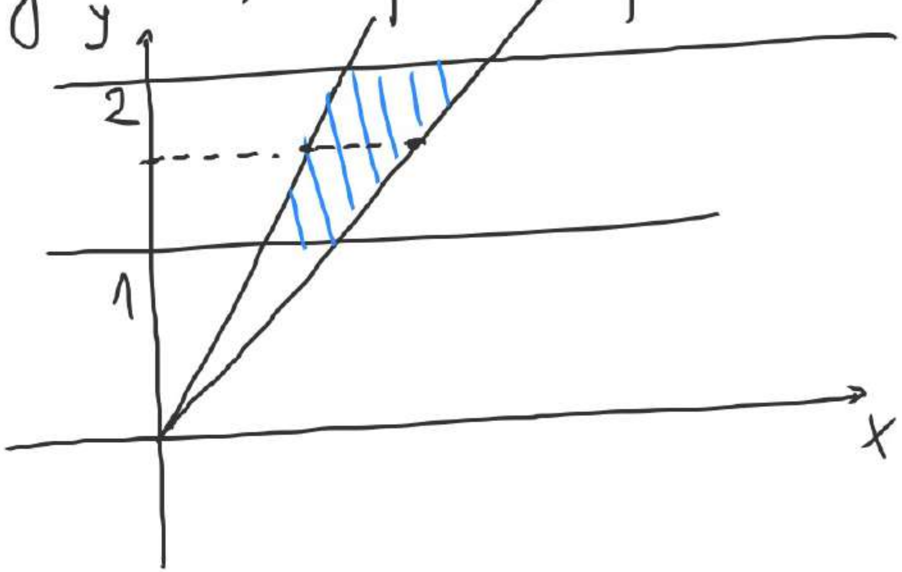
$D=\{(x,y)\mid -1\le y\le 1,\ \underbrace{-y\le x\le 1-y}_{\text{cont}}\}\in J(\mathbb{R}^2)$

$$\iint\limits_D y\,dx\,dy=\int_{-1}^{1}\left(\int_{-y}^{1-y} y\,dx\right)dy=\int_{-1}^{1} yx\Big|_{x=-y}^{x=1-y}dy=\int_{-1}^{1} y(1-\cancel{y}+\cancel{y})\,dy$$

$$=\frac{y^2}{2}\Big|_{-1}^{1}=0.$$

4) $\displaystyle\iint_D x\,dx\,dy$ D mărginit de dreptele

$y=2x,\quad y=4x,\quad y=1,\quad y=2$

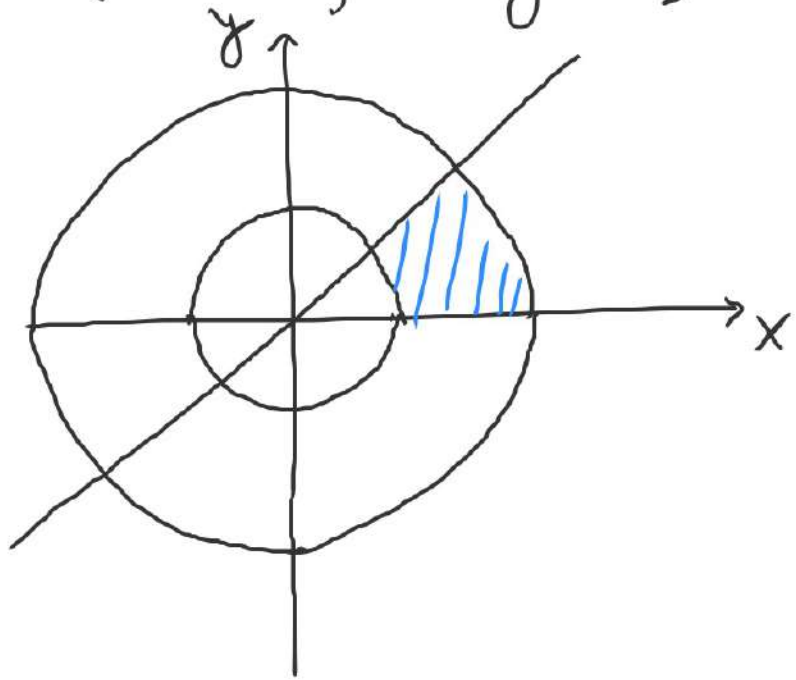


$$D=\left\{(x,y)\in\mathbb{R}^2 \;\middle|\; 1\le y\le 2,\ \frac{y}{4}\le x\le \frac{y}{2}\right\}$$

$$\iint_D x\,dx\,dy=\int_1^2\left(\int_{\frac{y}{4}}^{\frac{y}{2}} x\,dx\right)dy$$

$$=\int_1^2\left(\frac{x^2}{2}\Bigg|_{x=\frac{y}{4}}^{x=\frac{y}{2}}\right)dy=\int_1^2\frac{1}{2}\left(\frac{y^2}{4}-\frac{y^2}{16}\right)dy=\frac{3}{32}\cdot\frac{y^3}{3}\Bigg|_1^2=\frac{3}{32}\cdot\frac{7}{3}=\frac{7}{32}$$

5) $I = \iint_D y \, dx\,dy \quad , \quad D = \{(x,y) \in \mathbb{R}^2 \mid 1 \le x^2+y^2 \le 9, \; y \ge 0, \; y \le x\}$



$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$$

$$(x,y) \in D \iff \begin{cases} r \in [1,3] \\ \theta \in \left[0, \frac{\pi}{4}\right] \end{cases}$$

$$|\det J| = \left| \frac{D(x,y)}{D(r,\theta)} \right| = r \quad , \quad dx\,dy = r\,dr\,d\theta$$

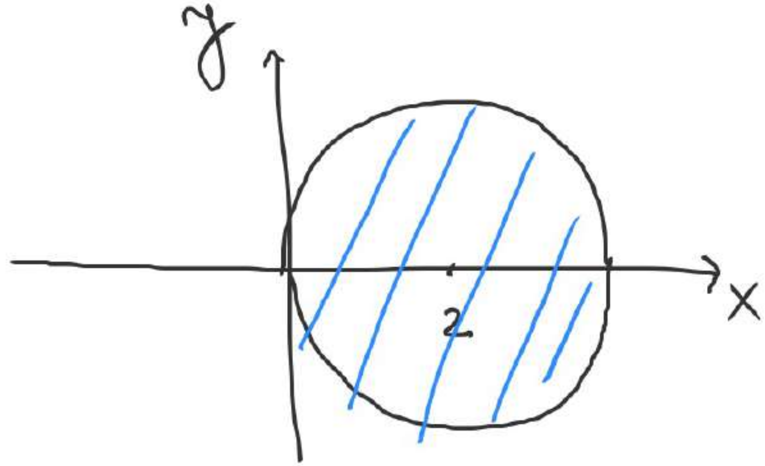
$$I = \iint_{[1,3]\times\left[0,\frac{\pi}{4}\right]} r\sin\theta \cdot r\,dr\,d\theta = \int_1^3 \left(\int_0^{\frac{\pi}{4}} r^2 \sin\theta \, d\theta \right) dr = \int_1^3 \left(-r^2\cos\theta \Big|_{\theta=0}^{\theta=\frac{\pi}{4}} \right) dr$$

$$= \int_1^3 r^2 \left(1 - \frac{\sqrt{2}}{2}\right) dr = \left(1 - \frac{\sqrt{2}}{2}\right) \cdot \frac{r^3}{3} \Big|_1^3 = \frac{8}{3}\left(1 - \frac{\sqrt{2}}{2}\right)$$

$$I = \iint_D xy \, dx\, dy \qquad D: \; x^2 + y^2 \leq 4x$$

$$\Updownarrow$$

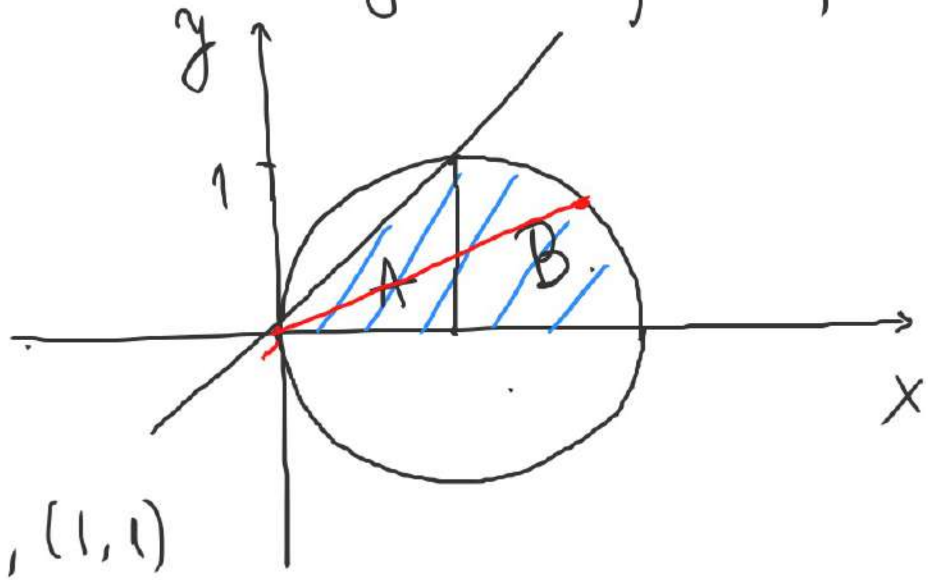
$$(x-2)^2 + y^2 \leq 4$$



$$\begin{cases} x - 2 = r\cos\theta \\ y = r\sin\theta \end{cases} \qquad |\det J| = r \qquad (x,y) \in D \Leftrightarrow \begin{cases} r \in [0,2] \\ \theta \in [0, 2\pi] \end{cases}$$

$$I = \int_0^2 \left(\int_0^{2\pi} (2 + r\cos\theta)\, r\sin\theta \cdot r \, d\theta \right) dr$$

6) $\iint\limits_D y\,dx\,dy$, D mărginit de curbele $x^2+y^2=2x$, $y=0$, $y=x$



$x^2+y^2=2x \iff x^2-2x+1+y^2=1$

$\underline{(x-1)^2+y^2=1}$

$\begin{cases} x^2+y^2=2x \\ y=x \end{cases} \iff \begin{cases} 2x^2=2x \\ y=x \end{cases}$ $(0,0)$, $(1,1)$

$D = A \cup B$, $A = \{(x,y)\in\mathbb{R}^2 \mid 0\le x\le 1,\ \underbrace{0\le y\le x}_{\text{cont}}\} \in J(\mathbb{R}^2)$

$B = \{(x,y)\in\mathbb{R}^2 \mid 1\le x\le 2,\ \underbrace{0\le y\le \sqrt{2x-x^2}}_{\text{cont.}}\} \in J(\mathbb{R}^2)$

$\Rightarrow D = A\cup B \in J(\mathbb{R}^2)$

$$\left.\begin{array}{l} D \text{ compactă} \\ f \text{ cont} \end{array}\right\} \Rightarrow \left.\begin{array}{l} f \text{ mărginită} \\ f \text{ continuă} \\ D \in \mathcal{J}(\mathbb{R}^2) \end{array}\right\} \Rightarrow f \text{ integrabilă.}$$

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$$

$D: \quad x^2+y^2 \leq 2x, \; y \leq x, \; y \geq 0. \qquad (x^2+y^2=r^2)$

$$(x,y)\in D \iff \begin{cases} r^2 \leq 2r\cos\theta \\ r\sin\theta \geq 0 \\ \sin\theta \leq \cos\theta \end{cases} \iff \begin{cases} 0 \leq r \leq 2\cos\theta \\ \theta \in \left[0, \frac{\pi}{4}\right]. \end{cases}$$

$|\det J| = r$

$dx\,dy = r\,dr\,d\theta$

$$D' = \left\{ (r,\theta) \;\middle|\; 0 \leq \theta \leq \frac{\pi}{4} \;;\; 0 \leq r \leq 2\cos\theta \right\} \qquad D \longleftrightarrow D'$$

$$\iint_D y\,dx\,dy = \iint_{D'} r\sin\theta \cdot \underbrace{|\det J|}_{\substack{\parallel \\ r}}\, dr\,d\theta = \iint_{D'} r^2 \sin\theta\, dr\,d\theta$$

$$= \int_0^{\frac{\pi}{4}} \left(\int_0^{2\cos\theta} r^2 \sin\theta\, dr \right) d\theta = \int_0^{\frac{\pi}{4}} \sin\theta \cdot \frac{r^3}{3}\Big|_{r=0}^{r=2\cos\theta} d\theta$$

$$= \int_0^{\frac{\pi}{4}} \frac{8}{3} \cdot \sin\theta \cdot \cos^3\theta\, d\theta = \int_0^{\frac{\pi}{4}} -\frac{8}{3} \cos^3\theta \cdot (\cos\theta)'\, d\theta$$

$$= -\frac{8}{3}\ \frac{\cos^4\theta}{4}\Big|_0^{\frac{\pi}{4}} = -\frac{2}{3}\left(\frac{1}{4} - 1\right) = \frac{1}{2}$$

Exercitii

1*) Fie $A \in \mathcal{J}(\mathbb{R}^n)$ cu $\lambda(A)=0$ si $f: A \to \mathbb{R}$ marginita.
Aratati ca f este integrabila Riemann si $\int_A f\,dx = 0$.

2*) Fie $f:[a,b]\times[a,b] \to \mathbb{R}$ continua. Aratati ca

$$\int_a^b \left(\int_a^y f(x,y)\,dx \right) dy = \int_a^b \left(\int_x^b f(x,y)\,dy \right) dx$$

3) Calculati $\iint_D \sqrt{x^2+y^2}\,dx\,dy$; $D = \{(x,y)\in\mathbb{R}^2 \mid 2x \le x^2+y^2 \le 4x,\ y \ge 0\}$

$\iiint_V y\,dx\,dy\,dz$; $V = \{(x,y,z)\in\mathbb{R}^3 \mid 2x+4y-z \le 8,\ x,y \ge 0,\ z \le -4\}$