2024

$\boxed{C_5}$

Operații cu subspații vectoriale.
Morfisme de spații vectoriale (Aplicații liniare)

Teorema Grassmann

$(V,+,\cdot)|_{\mathbb{K}}$ sp. vectorial finit generat și $V_1, V_2 \subset V$ subspații vectoriale.

$\Rightarrow \dim(V_1 + V_2) = \dim V_1 + \dim V_2 - \dim(V_1 \cap V_2)$,

unde $V_1 + V_2 = \langle V_1 \cup V_2 \rangle$.

Dem

$\dim(V) = n$, $\dim V_j = n_j$, $j = \overline{1,2}$, $\dim(V_1 \cap V_2) = p$

$n_j < n$, $p < n$, $j = \overline{1,2}$

Fie $R_0 = \{e_1, .., e_p\}$ reper în $V_1 \cap V_2$.

Extindem la $R_1 = \{e_1, .., e_p, f_{p+1}, .., f_{n_1}\}$ reper în V_1

—,,— $R_2 = \{e_1, .., e_p, g_{p+1}, .., g_{n_2}\}$ reper în V_2

Fie $R = \{e_1, .., e_p, f_{p+1}, .., f_{n_1}, g_{p+1}, .., g_{n_2}\}$.

Dem că R este reper în $V_1 + V_2$.

① R este SLI

$\forall a_1, .., a_p, b_{p+1}, .., b_{n_1}, c_{p+1}, .., c_{n_2} \in \mathbb{K}$ aî

$$\sum_{i=1}^{p} a_i e_i + \sum_{j=p+1}^{n_1} b_j f_j + \sum_{k=p+1}^{n_2} c_k g_k = 0_V$$

$$x = \underbrace{\sum_{i=1}^{p} a_i e_i + \sum_{j=p+1}^{n_1} b_j f_j}_{\in V_1} = \underbrace{-\sum_{k=p+1}^{n_2} c_k g_k}_{\in V_2} = \sum_{i=1}^{p} a_i' e_i \in V_1 \cap V_2$$

$$\underbrace{V_1 \qquad\qquad V_2}_{\in V_1 \cap V_2}$$

$x \in V_1 \cap V_2$

$$\sum_{i=1}^{p}(a_i-a'_i)\underline{e_i}+\sum_{j=p+1}^{n_1}b_j f_j=0 \overset{R_1 \text{ e SLI}}{\Longrightarrow} \begin{matrix} a_i-a'_i=0, \forall i=\overline{1,p} \\ b_j=0, \forall j=\overline{p+1,n_1} \end{matrix}$$

$$\sum_{i=1}^{p}a'_i e_i+\sum_{k=p+1}^{n_2}c_k g_k=0 \overset{R_2 \text{ SLI}}{\Longrightarrow} \begin{matrix} a'_i=0, \forall i=\overline{1,p} \\ c_k=0, \forall k=\overline{p+1,n_2} \end{matrix}$$

$$\Longrightarrow \begin{matrix} a_i=0, \forall i=\overline{1,p} \\ b_j=0, \forall j=\overline{p+1,n_1} \\ c_k=0, \forall k=\overline{p+1,n_2} \end{matrix} \rightarrow R \text{ este SLI}$$

② R este SG i.e $\langle R\rangle = V_1+V_2$.

„$\supseteq$" $\forall x\in V_1+V_2 \Rightarrow \exists x_1\in V_1, x_2\in V_2$ aî

$$x=x_1+x_2$$
$$=\Big(\underline{\sum_{i=1}^{p}a_i e_i}+\sum_{j=p+1}^{n_1}b_j f_j\Big)+\Big(\underline{\sum_{i=1}^{p}a'_i e_i}+\sum_{k=p+1}^{n_2}c_k g_k\Big)$$
$$=\sum_{i=1}^{p}(a_i+a'_i)e_i+\sum_{j=p+1}^{n_1}b_j f_j+\sum_{k=p+1}^{n_2}c_k g_k \in\langle R\rangle.$$

„$\subseteq$" Din construcție.

Deci $\langle R\rangle = V_1+V_2$

În concluzie $R=\{e_1,..,e_p, f_{p+1},..,f_{n_1}, g_{p+1},..,g_{n_2}\}$ reper în V_1+V_2

$\dim(V_1+V_2)=|R|=n_1+n_2-p=$
$=\dim V_1+\dim V_2-\dim(V_1\cap V_2)$

OBS În particular

$\dim(V_1\oplus V_2)=\dim V_1+\dim V_2$.

V_1+V_2 este sumă directă $\Leftrightarrow V_1\cap V_2=\{0_V\}$

OBS

a) $V = V_1 \oplus V_2$, R_k reper în V_k, $k=\overline{1,2} \Rightarrow$
$R = R_1 \cup R_2$ reper în V

b) Fie R reper în V și $R = R_1 \cup R_2$ (partiție), $V_k = \langle R_k \rangle$, $k=\overline{1,2} \Rightarrow V = V_1 \oplus V_2$.

Prop

$A \in \mathcal{M}_{m,n}(\mathbb{R})$

$S(A) = \{x \in \mathbb{K}^n \mid \overset{(m,n)(n,1)}{AX} = \overset{(m,1)}{0}\} \subset \mathbb{K}^n$

(mult. sol unui SLO) $x = \sum_{i=1}^{n} x_i e_i$

$R = \{e_1, .., e_n\}$ reper în $\mathbb{K}^n$, $A = (a_{ij})_{\substack{i=\overline{1,m} \\ j=\overline{1,n}}}$, $X = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$

a) $S(A) \subseteq \mathbb{K}^n$ subsp. vect

b) $\dim S(A) = n - \operatorname{rg} A$

Ex $(\mathbb{R}^3, +, \cdot)/\mathbb{R}$.

$V' = \left\{(x,y,z) \in \mathbb{R}^3 \mid \begin{cases} x - y + z = 0 \\ 2x + y - z = 0 \end{cases}\right\} = S(A)$

a) $\dim V'$; b) Precizați un reper în V'

SOL

a) $A = \begin{pmatrix} \boxed{\begin{matrix} 1 & -1 \\ 2 & 1 \end{matrix}} & \begin{matrix} 1 \\ -1 \end{matrix} \end{pmatrix}$ $\dim V' = 3 - \operatorname{rg}(A) = 3 - 2 = 1$

b) $\begin{cases} x - y = -z \\ 2x + y = z \end{cases}$
$\overline{3x \quad\; / \;\; = 0}$

$x = 0$
$y = z$
, $z = \alpha$

$V' = \{(0, \alpha, \alpha), \alpha \in \mathbb{R}\} = \langle \overbrace{\{(0,1,1)\}}^{R'} \rangle$

R' e SG, $\dim V' = 1 = |R'| \Rightarrow R'$ este reper în V'

Prop $(V,+,\cdot)/_{\mathbb{K}}$ sp. vect, $V' \subset V$ subsp. vect.
Coord. vectorilor din V', în raport cu $\forall$ reper, reprezintă soluțiile unui SLO i.e $\exists A \in \mathcal{M}_{m,n}(\mathbb{K})$ aî $V' = S(A)$

Ex $(\mathbb{R}^4,+,\cdot)/_{\mathbb{R}}$, $V' = \langle\{\overset{u}{\overset{\shortparallel}{(1,1,0,0)}}, \overset{v}{\overset{\shortparallel}{(1,0,1,-1)}}\}\rangle$.

a) Să se descrie V' printr-un sistem de ec. liniare (SLO)

b) $\mathbb{R}^4 = V' \oplus V''$; $V'' = ?$ subsp. complementar lui V'

c) Să se descompună $x = (1,1,2,1)$ în raport cu $\mathbb{R}^4 = V' \oplus V''$

SOL

a) $\{u,v\}$ SLI

$$\operatorname{rg}\begin{pmatrix}\boxed{\begin{matrix}1 & 1\\ 1 & 0\end{matrix}}\\ \begin{matrix}0 & 1\\ 0 & -1\end{matrix}\end{pmatrix} = 2 \xRightarrow[LI]{crit} \overset{R'}{\overset{\shortparallel}{\{u,v\}}} \text{ este SLI în } V'$$

R' este reper în V'

Fie $x \in V' \Rightarrow \exists\, a,b \in \mathbb{R}$ aî $x = au + bv$

$(x_1,x_2,x_3,x_4) = a(1,1,0,0) + b(1,0,1,-1) = (a+b, a, b, -b)$

$$\begin{cases} a+b = x_1\\ a = x_2\\ b = x_3\\ -b = x_4\end{cases} \qquad \left(\begin{array}{cc|c}\boxed{\begin{matrix}1 & 1\\ 1 & 0\end{matrix}} & & \begin{matrix}x_1\\ x_2\end{matrix}\\ \begin{matrix}0 & 1\\ 0 & -1\end{matrix} & & \begin{matrix}x_3\\ x_4\end{matrix}\end{array}\right)$$

$$\text{SC} \iff \begin{cases}\Delta_{c_1} = \begin{vmatrix}1 & 1 & x_1\\ 1 & 0 & x_2\\ 0 & 1 & x_3\end{vmatrix} = 0 \iff -x_3 + x_1 - x_2 = 0\\[2ex] \Delta_{c_2} = \begin{vmatrix}1 & 1 & x_1\\ 1 & 0 & x_2\\ 0 & -1 & x_4\end{vmatrix} = 0 \iff -x_4 + x_2 - x_1 = 0\end{cases}$$

$$V' = \left\{x \in \mathbb{R}^4 \;\middle|\; \begin{cases}x_1 - x_2 - x_3 = 0\\ -x_1 + x_2 - x_4 = 0\end{cases}\right\}$$

$A = \begin{pmatrix}1 & -1 & -1 & 0\\ -1 & 1 & 0 & -1\end{pmatrix}$; $V' = S(A)$

$d_{33} =$

b) $\mathbb{R}^4 = V' \oplus V''$

$$\det\begin{pmatrix} 1 & 1 & \boxed{\begin{matrix}0\\0\\0\\1\end{matrix}} & \boxed{\begin{matrix}1\\0\\0\\0\end{matrix}} \\ & & \| & \| \\ & & w & t \end{pmatrix} \neq 0$$

$$\det\begin{pmatrix} 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \end{pmatrix} \neq 0$$

$R'' = \{w, t\}$ (este SLI) reper în V''

$V'' = \langle R'' \rangle$

(nu este unic)

c) $x = (1,1,2,1) = x' + x'' \in V' \oplus V''$ (scriere unică)

$R = R' \cup R'' = \{u, v, w, t\}$ reper în $\mathbb{R}^4$.

$(1,1,2,1) = a(1,1,0,0) + b(1,0,1,-1) + c(0,0,0,1) + d(1,0,0,0)$

$= (a+b+d, a, b, -b+c)$

$a = 1$

$b = 2$

$-b+c = 1 \Rightarrow c = 3$

$a+b+d = 1 \Rightarrow d = \cancel{1} - \cancel{1} - 2 = -2$

$$(1,1,2,1) = \underbrace{1\cdot(1,1,0,0) + 2(1,0,1,-1)}_{x' = (3,1,2,-2)} + \underbrace{3(0,0,0,1) - 2(1,0,0,1}_{x'' = (-2,0,0\,1)}$$

<u>Ex</u> $(\mathbb{R}^4, +, \cdot)/\mathbb{R}$, $V' = \{(x,y,z,t) \in \mathbb{R}^4 \mid x+y-z-3t = 0\}$.

$V'' = \{(x,y,z,t) \in \mathbb{R}^4 \mid x+y+z+2t = 0\}$

Dem $\mathbb{R}^4 = V' + V''$, dar suma nu e directă

<u>SOL</u>

$\dim V' = 4 - \operatorname{rg} A' = 4 - 1 = 3$, $V' = S(A')$, $A' = (1\ 1\ -1\ -3)$

$\dim V'' = 4 - \operatorname{rg} A'' = 4 - 1 = 3$, $V'' = S(A'')$, $A'' = (1\ 1\ 1\ 2)$

$$V' \cap V'' = \left\{(x,y,z,t) \in \mathbb{R}^4 \;\middle|\; \begin{cases} x+y-z-3t = 0 \\ x+y+z+2t = 0 \end{cases}\right\} = S(A)$$

$\dim(V' \cap V'') = 4 - \operatorname{rg} A = 4 - 2 = 2$ $\quad A = \begin{pmatrix} 1 & 1 & -1 & -3 \\ 1 & 1 & 1 & 2 \end{pmatrix}$

$\dim(V' + V'') = 3 + 3 - 2 = 4$.

$$\left.\begin{array}{l} V' + V'' \subset \mathbb{R}^4 \\ \dim(V'+V'') = \dim \mathbb{R}^4 = 4 \end{array}\right\} \Rightarrow V'+V'' = \mathbb{R}^4.$$

nu e $\oplus$ deoarece $V' \cap V'' \neq \{0_V\}$.

OBS

$$\left.\begin{array}{l} V' \subset V \text{ subsp vect} \\ \text{dacă } \dim V' = \dim V = n \end{array}\right\} \Rightarrow V' = V.$$

Morfisme de spații vectoriale. (Aplicații liniare)

Def $(V_j, +, \cdot)|_{\mathbb{K}_j}$, $j = \overline{1,2}$ sp. vectoriale.

$f : V_1 \longrightarrow V_2$ s.n. aplicație semi-liniară $\Leftrightarrow$

1) $f(x+y) = f(x) + f(y)$

2) $\exists\ \theta : \mathbb{K}_1 \to \mathbb{K}_2$ izomorfism de corpuri aî

$f(\alpha x) = \theta(\alpha) f(x)$, $\forall x \in V_1$, $y \in V_1$, $\alpha \in \mathbb{K}_1$.

Dacă $\underset{\mathbb{K}}{\underset{\|}{\mathbb{K}_1 = \mathbb{K}_2}}$, $\theta : \mathbb{K} \to \mathbb{K}$, $\theta = id_{\mathbb{K}}$, at f s.n.

morfism de sp. vectoriale (aplicație liniară)

OBS a) $(V_j, +, \cdot)/_{\mathbb{R}}$, $j = \overline{1,2}$ sp. vect reale.

$\theta : \mathbb{R} \to \mathbb{R}$ automorfism de corpuri $\Rightarrow \theta = id_{\mathbb{R}}$.

$\forall f : V_1 \to V_2$ apl. semi-liniară $\Rightarrow$ f apl. liniară.

b) $(\mathbb{C}^n, +, \cdot)_{\mathbb{C}}$ sp. vect.

$\theta : \mathbb{C} \to \mathbb{C}$ autom. de corpuri, $\theta(z) = \overline{z}$

$f : \mathbb{C}^n \longrightarrow \mathbb{C}^n$, $f(z_1, .., z_n) = (\overline{z_1}, .., \overline{z_n})$

f este apl. semi-liniară (<u>nu</u> este liniară).

$(V_j, +, \cdot)/\mathbb{K}$, $f: V_1 \longrightarrow V_2$ apl. lin $\Longleftrightarrow$ 1) $f(x+y) = f(x) + f(y)$

2) $f(\alpha x) = \alpha f(x)$

$\forall x, y \in V_1, \alpha \in \mathbb{K}$

Dacă, în plus, f bijectivă, at f s.n. izomorfism de sp vect

Dacă $V_1 = V_2$, at f = automorfism de sp vect

$\mathrm{End}(V) = \{ f: V \rightarrow V \mid f \text{ liniară} \}$

$\mathrm{Aut}(V) = \{ f \in \mathrm{End}(V) \mid f \text{ bij} \}$.

OBS a)

$$\begin{tikzcd} V_1 \arrow[r, "f"] \arrow[rr, bend right, "h"'] & V_2 \arrow[r, "g"] & V_3 \end{tikzcd}$$

f, g apl lin $\Rightarrow h = g \circ f$ lin.

b) $f: V_1 \longrightarrow V_2$ apl. lin $\Rightarrow$

$f: (V_1, +) \longrightarrow (V_2, +)$ morf de grupuri și $f(0_{V_1}) = 0_{V_2}$

Exemple de apl. liniare

1) $f: V \longrightarrow V$, $f(x) = 0_V$, $f(x) = x$ apl lin.

2) $f: \mathbb{R}^n \longrightarrow \mathbb{R}^m$, $f(x) = y$, $Y = AX$ 71 —

3) $f: \mathcal{M}_m(\mathbb{R}) \longrightarrow \mathbb{R}$, $f(A) = Tr(A)$ apl lin.

$Tr(A+B) = Tr(A) + Tr(B)$

$Tr(\alpha A) = \alpha Tr(A)$

$f(A) = \det(A)$ NU este apl. liniară.

Prop de caract. a apl. liniare

$f: V_1 \longrightarrow V_2$ aplicație liniară $\Longleftrightarrow$

$f(ax + by) = a f(x) + b f(y), \forall x, y \in V_1, \forall a, b \in \mathbb{K}$.

$\Longleftrightarrow f\left(\sum_{i=1}^{n} a_i x_i\right) = \sum_{i=1}^{n} a_i f(x_i)$, $\forall x_1, \ldots, x_n \in V_1$

$\forall a_1, \ldots, a_n \in \mathbb{K}$.

Dem

"$\Rightarrow$" Ip: f liniară.

$\left.\begin{array}{l} x\in V_1 \\ a\in \mathbb{K} \end{array}\right\} \Rightarrow a\cdot x\in V_1$

$\left.\begin{array}{l} y\in V_1 \\ b\in \mathbb{K} \end{array}\right\} \Rightarrow by\in V_1$

$f(ax+by) \overset{1)}{=} f(ax)+f(by) \overset{2)}{=} a f(x) + b f(y)$

"$\Leftarrow$" Ip $f(ax+by) = af(x)+bf(y), \forall x,y\in V_1$, $\forall a,b\in\mathbb{K}$.

Fie $a=b=1_{\mathbb{K}}$ $\quad f(1_{\mathbb{K}}x+1_{\mathbb{K}}y) = f(x+y) = f(x)+f(y)$

Fie $b=0_{\mathbb{K}}$ $\quad f(a\cdot x+0_{\mathbb{K}}\cdot y) = f(a\cdot x) = af(x)$.

OBS $f: V_1\to V_2$ liniară

Dacă $V'\subset V_1$ subsp. vect $\Rightarrow f(V')\subset V_2$ subsp vect.

Dem Fie $\left.\begin{array}{l} y_1, y_2\in f(V') \\ a,b\in\mathbb{K} \end{array}\right. \Rightarrow ay_1+by_2\in f(V')$

$\exists x_1,x_2\in V'$ aî $y_1=f(x_1)$, $y_2=f(x_2)$

$ay_1+by_2 = af(x_1)+bf(x_2) = f(ax_1)+f(bx_2) = f(\underset{\overset{\parallel}{x\in V'}}{ax_1+bx_2})$

Def $f: V_1\to V_2$ apl. liniară

$Ker(f) = \{x\in V_1 \mid f(x) = 0_{V_2}\}$ nucleul lui f

$Im(f) = \{y\in V_2 \mid \exists x\in V_1 \text{ aî } f(x)=y\}$ imaginea lui f.

Prop $f: V_1\to V_2$ apl. lin.

a) $ker(f)\subset V_1$, $Im f\subset V_2$ subsp. vect

b) f inj $\Leftrightarrow$ $Ker f = \{0_{V_1}\}$

c) f surj $\Leftrightarrow$ $dim\, Im f = dim\, V_2$.

Dem

a) $\operatorname{Im} f = f(V_1) \subset V_2$ subsp (din obs)

$\operatorname{Ker} f \subset V_1$ subsp. vect

Fie $x_1, x_2 \in \operatorname{Ker} f \Rightarrow f(x_1) = f(x_2) = 0_{V_2}$.

Fie $a, b \in \mathbb{K}$

$$f(ax_1 + bx_2) = a \underset{0_{V_2}}{\underset{\parallel}{f(x_1)}} + b \underset{0_{V_2}}{\underset{\parallel}{f(x_2)}} = 0_{V_2} \Rightarrow ax_1 + bx_2 \in \operatorname{Ker} f$$

b) f inj $\overset{?}{\Rightarrow} \operatorname{Ker} f = \{0_{V_1}\}$.

Fie $x \in \operatorname{Ker} f \Rightarrow f(x) = 0_{V_2}$

dar $f(0_{V_1}) = 0_{V_2}$ $\Big| \Rightarrow f(x) = f(0_{V_1}) \overset{inj}{\Rightarrow} x = 0_{V_1}$

$\operatorname{Ker} f = \{0_{V_1}\} \overset{?}{\Rightarrow} f$ inj

Fie $x_1, x_2 \in V_1$ $f(x_1) = f(x_2) \Rightarrow f(x_1 - x_2) = 0_{V_2}$

$x_1 - x_2 \in \operatorname{Ker} f = \{0_{V_1}\} \Rightarrow x_1 = x_2$.

c) f surj $\Leftrightarrow \dim \operatorname{Im} f = \dim V_2$

"$\Rightarrow$" f surj $\Rightarrow \operatorname{Im} f = V_2 \Rightarrow \dim \operatorname{Im} f = \dim V_2$

"$\Leftarrow$" $\dim \operatorname{Im} f = \dim V_2$

dar $\operatorname{Im} f \subseteq V_2$ subsp vect $\Big| \overset{OBS}{\Rightarrow} \operatorname{Im} f = V_2 \Rightarrow f$ surj.