$$B(p,q) = \int_0^1 x^{p-1}(1-x)^{q-1}\,dx, \quad B(p,q) = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}$$

$$\Gamma(p) = \int_0^\infty x^{p-1}e^{-x}\,dx, \quad \Gamma(p+1) = p\,\Gamma(p); \; p,q>0.$$

$$\Gamma\left((n+1)!\right) = n!$$

1) Calculati $\displaystyle\int_0^1 \sqrt{x-x^2}\,dx$

$$\int_0^1 \sqrt{x(1-x)}\,dx = \int_0^1 x^{\frac{1}{2}}\cdot(1-x)^{\frac{1}{2}}\,dx = B\left(\frac{3}{2},\frac{3}{2}\right)$$

$$= \frac{\Gamma\left(\frac{3}{2}\right)\Gamma\left(\frac{3}{2}\right)}{\Gamma\left(\frac{3}{2}+\frac{3}{2}\right)} = \frac{\frac{1}{2}\Gamma\left(\frac{1}{2}\right)\cdot\frac{1}{2}\Gamma\left(\frac{1}{2}\right)}{\Gamma(3)} = \frac{1}{4}\cdot\frac{1}{2}\cdot\Gamma\left(\frac{1}{2}\right)^2 = \frac{\pi}{8}$$

$$B\left(\frac{1}{2},\frac{1}{2}\right)=\frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma(1)}=\Gamma\left(\frac{1}{2}\right)^2$$

$$B\left(\frac{1}{2},\frac{1}{2}\right)=\int_0^1\frac{1}{\sqrt{x(1-x)}}\,dx$$

$t=\sqrt{x}$, $x=t^2$ $\quad$ $x=0$, $t=0$; $x=1$, $t=1$ $\quad$ $dx=2t\,dt$

$$=\int_0^1\frac{1}{t\sqrt{1-t^2}}\cdot 2t\,dt=\int_0^1\frac{2}{\sqrt{1-t^2}}\,dt=2\arcsin t\Big|_0^1=\pi$$

$$\Gamma\left(\frac{1}{2}\right)=\sqrt{\pi}$$

2) $\int_0^\infty e^{-x^2}\,dx$

$$x^2 = t \qquad x=0 \quad t=0$$
$$x=\infty \quad t=\infty$$

$$x=\sqrt{t} \quad , \quad dx = \frac{1}{2\sqrt{t}}\,dt$$

$$\int_0^\infty e^{-x^2}\,dx = \int_0^\infty \frac{e^{-t}}{2\sqrt{t}}\,dt = \frac{1}{2}\int_0^\infty t^{-\frac{1}{2}} e^{-t}\,dt =$$

$$= \frac{1}{2}\Gamma\left(\frac{1}{2}\right) = \frac{\sqrt{\pi}}{2}.$$

$$y=-x \quad , \quad \int_0^\infty e^{-x^2}\,dx = \int_{-\infty}^0 e^{-y^2}\,dy \Rightarrow \int_{-\infty}^\infty e^{-x^2}\,dx = \sqrt{\pi}$$

3) Calculați $\int_0^\infty \sqrt{x}\, e^{-x^3} dx$

$x^3 = t \quad , \quad x = \sqrt[3]{t}$

$x = 0 \,;\, t = 0$

$x = \infty \,;\, t = \infty$

$dx = \frac{1}{3\sqrt[3]{t^2}} dt = \frac{1}{3} t^{-\frac{2}{3}} dt$

$$\int_0^\infty \sqrt{x}\, e^{-x^3} dx = \frac{1}{3} \int_0^\infty t^{\frac{1}{6}} \cdot e^{-t} \cdot t^{-\frac{2}{3}} dt =$$

$$= \frac{1}{3} \int_0^\infty t^{-\frac{1}{2}} e^{-t} dt = \frac{1}{3} \Gamma\left(\frac{1}{2}\right) = \frac{\sqrt{\pi}}{3}$$

5) $\displaystyle\int_0^1 \frac{1}{\sqrt{-\ln x}}\,dx$

$-\ln x = t \qquad x = e^{-t}$

$x = 0 \quad t = \infty$

$x = 1 \quad t = 0$

$dx = -e^{-t}\,dt$

$$\int_0^1 \frac{1}{\sqrt{-\ln x}}\,dx = -\int_\infty^0 \frac{1}{\sqrt{t}}\cdot e^{-t}\,dt = \int_0^\infty \frac{e^{-t}}{\sqrt{t}}\,dt$$

$$= \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$$

6) $\int_0^{\frac{\pi}{2}} \sin^4 x \cos^2 x \, dx$

$\sin^2 x = t \qquad \sin x = \sqrt{t} \quad , \quad x = \arcsin \sqrt{t}$

$x = 0 \; ; \; t = 0$

$x = \frac{\pi}{2} \; ; \; t = 1$

$dx = \frac{1}{\sqrt{1 - \sqrt{t}^2}} \cdot \frac{1}{2\sqrt{t}} \, dt$

$dx = \frac{1}{2\sqrt{1-t} \cdot \sqrt{t}} \, dt$

$\cos^2 x = 1 - \sin^2 x = 1 - t$

$$\int_0^{\frac{\pi}{2}} \sin^4 x \cos^2 x \, dx = \int_0^1 t^2 \cdot (1-t) \cdot \frac{1}{2\sqrt{t} \cdot \sqrt{1-t}} \, dt$$

$$= \frac{1}{2}\int_0^1 t^{\frac{3}{2}} \cdot (1-t)^{\frac{1}{2}}\, dt = \frac{1}{2} B\left(\frac{5}{2}, \frac{3}{2}\right)$$

$$= \frac{1}{2} \cdot \frac{\Gamma\left(\frac{5}{2}\right)\Gamma\left(\frac{3}{2}\right)}{\Gamma(4)} = \frac{1}{2} \cdot \frac{\frac{3\pi}{8}}{6} = \frac{\pi}{32}$$

$$\Gamma\left(\frac{5}{2}\right) = \frac{3}{2}\Gamma\left(\frac{3}{2}\right) = \frac{3}{2} \cdot \frac{1}{2}\Gamma\left(\frac{1}{2}\right) = \frac{3\sqrt{\pi}}{4}$$

$$\Gamma\left(\frac{3}{2}\right) = \frac{1}{2}\Gamma\left(\frac{1}{2}\right) = \frac{\sqrt{\pi}}{2}$$

$$\Gamma(4) = 3! = 6$$

8) $\int_0^{\frac{\pi}{2}} \sin^n x \, dx$

$\sin^2 x = t.$ $\quad x = \arcsin\sqrt{t}$, $\quad dx = \dfrac{dt}{2\sqrt{t(1-t)}}$

$x=0, \quad t=0$

$x=\frac{\pi}{2}, \quad t=1$

$$\int_0^{\frac{\pi}{2}} \sin^n x \, dx = \int_0^1 t^{\frac{n}{2}} \cdot \frac{1}{2\sqrt{t}\cdot\sqrt{1-t}} \, dt = \frac{1}{2}\int_0^1 t^{\frac{n-1}{2}} \cdot (1-t)^{-\frac{1}{2}} \, dt$$

$$= \frac{1}{2} B\left(\frac{n+1}{2}, \frac{1}{2}\right)$$

Pt $n=2k$

$$\int_0^{\frac{\pi}{2}} (\sin x)^{2k}\,dx = \frac{1}{2} B\left(\frac{2k+1}{2}, \frac{1}{2}\right) = \frac{1}{2} B\left(k+\frac{1}{2}, \frac{1}{2}\right)$$

$$= \frac{1}{2} \cdot \frac{\Gamma\left(k+\frac{1}{2}\right) \cdot \Gamma\left(\frac{1}{2}\right)}{\Gamma(k+1)}$$

$$\Gamma\left(k+\frac{1}{2}\right) = \left(k-\frac{1}{2}\right) \Gamma\left(k-\frac{1}{2}\right) = \left(k-\frac{1}{2}\right)\left(k-\frac{3}{2}\right) \Gamma\left(k-\frac{3}{2}\right)$$

$$= \left(k-\frac{1}{2}\right)\left(k-\frac{3}{2}\right) \cdots \frac{3}{2} \cdot \frac{1}{2}\, \Gamma\left(\frac{1}{2}\right)$$

$$= \frac{1}{2} \cdot \frac{(2k-1)(2k-3)\cdots 3\cdot 1}{2^k \cdot k!} \cdot \Gamma\left(\frac{1}{2}\right)\cdot\Gamma\left(\frac{1}{2}\right)$$

$$= \frac{\pi}{2} \cdot \frac{1\cdot 3\cdot 5\cdots(2k-1)}{2\cdot 4\cdot 6\cdots 2k}$$

Pt $n = 2k+1$. exercitiu.

Obs: $$\int_0^{\frac{\pi}{2}} \sin^n x \, dx = \int_0^{\frac{\pi}{2}} \cos^n x \, dx$$

$$x = \frac{\pi}{2} - u$$

Studiati convergenta integralei $\int_0^\infty \sin(x^2)\,dx$

$$x^2 = t \quad , \quad x = \sqrt{t} \qquad dx = \frac{1}{2\sqrt{t}}\,dt$$

$$\frac{1}{2}\int_0^\infty \frac{1}{\sqrt{t}} \sin t\, dt.$$

$$\int_0^1 \frac{1}{\sqrt{t}} \sin t\, dt$$

$$\lim_{\substack{t\to 0\\ t>0}} \frac{\sin t}{\sqrt{t}} = \lim_{\substack{t\to 0\\ t>0}} \sqrt{t}\cdot\frac{\sin t}{t} = 0.$$

Deoarece $t \to \dfrac{\sin t}{\sqrt{t}}$ este continuă pe $(0,1]$ și

$\lim\limits_{\substack{t \to 0 \\ t > 0}} \dfrac{\sin t}{\sqrt{t}} = 0$ rezultă că $\displaystyle\int_0^1 \frac{\sin t}{\sqrt{t}}\,dt$ este conv.

$$\int_1^\infty \frac{\sin t}{\sqrt{t}}\,dt$$

$$\left.\begin{array}{l} 1)\ \dfrac{1}{\sqrt{t}} \searrow 0 \\ 2)\ \left|\displaystyle\int_1^c \sin t\,dt\right| = \left|-\cos c + \cos 1\right| \le 2 \end{array}\right\} \xRightarrow{\text{Abel-Dirichlet}} \int_1^\infty \frac{\sin t}{\sqrt{t}}\,dt \text{ conv.}$$

$$\Longrightarrow \int_0^\infty \sin(x^2)\,dx \text{ conv.}$$

10) Să se determine mulțimea de convergență a seriei de funcții

$$\sum_{n=1}^{\infty} \underbrace{\left(1+\frac{1}{n}\right)^n \left(\frac{1-x}{1-2x}\right)^n}_{f_n(x)}, \quad x \in \mathbb{R} \setminus \left\{\frac{1}{2}\right\}$$

$$\lim_{n\to\infty} \sqrt[n]{|f(x)|} = \lim_{n\to\infty} \left(1+\frac{1}{n}\right) \cdot \left|\frac{1-x}{1-2x}\right|$$

dacă $\left|\frac{1-x}{1-2x}\right| < 1$ seria este convergentă dacă $\left|\frac{1-x}{1-2x}\right| > 1$ seria este divergentă.

dacă $\left|\frac{1-x}{1-2x}\right| = 1$ (adică $x=0$, $x=\frac{2}{3}$) seria

devine $\sum_{n=1}^{\infty}\left(1+\frac{1}{n}\right)^{n}$ − divergentă pt că $\left(1+\frac{1}{n}\right)^{n} \to e \neq 0$.

Mulțimea de conv. este

$$A = \left\{ x \in \mathbb{R} \setminus \left\{\frac{1}{2}\right\} \;\middle|\; \left|\frac{1-x}{1-2x}\right| < 1 \right\}$$

11) Să se determine mulțimea de convergență a seriei de funcții $\sum_{n=0}^{\infty} 2^n \sin \frac{x}{3^n}$, $x \in [-2,2]$

și să se stabilească dacă convergența este uniformă.

$$\sum_{n=0}^{\infty} f_n(x), \quad f_n : [-2,2] \to \mathbb{R}, \quad f_n(x) = 2^n \cdot \sin \frac{x}{3^n}$$

$$|f_n(x)| = \left| 2^n \cdot \sin \frac{x}{3^n} \right| \leq \left| 2^n \cdot \frac{x}{3^n} \right| = \left(\frac{2}{3}\right)^n \cdot |x| \leq 2 \cdot \left(\frac{2}{3}\right)^n$$

$\sum_{n=0}^{\infty} 2 \cdot \left(\frac{2}{3}\right)^n$ convergentă

$\xrightarrow{\text{Weierstrass}}$ seria converge unif și $A = [-2,2]$

12) Determinați mulțimea de convergență a seriei de puteri

$$\sum_{n=1}^{\infty} \frac{3^n \cdot n}{\sqrt{n^4+1}} x^n$$

$$R = \frac{1}{\rho} \left(\frac{1}{0} = \infty ; \frac{1}{\infty}, 0 \right), \quad a_n = \frac{3^n \cdot n}{\sqrt{n^4+1}}$$

$$\rho = \lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \to \infty} \frac{3^{n+1} (n+1)}{\sqrt{(n+1)^4+1}} \cdot \frac{\sqrt{n^4+1}}{3^n \cdot n} = 3 .$$

raza de conv $R = \frac{1}{3}$.

Pt $x = \frac{1}{3}$ seria devine $\sum_{n=1}^{\infty} \frac{n}{\sqrt{n^4+1}} \sim \sum_{n=1}^{\infty} \frac{n}{\sqrt{n^4}} = \sum_{n\geq 1} \frac{1}{n}$

divergentă.

Pt $x = -\frac{1}{3}$ seria devine $\sum_{n=1}^{\infty} \frac{(-1)^n \cdot n}{\sqrt{n^4+1}}$ conv (Leibniz).

Mulțimea de convergență este

$$A = \left[-\frac{1}{3}, \frac{1}{3}\right).$$

$$B(p,q) = \frac{\Gamma(p)\,\Gamma(q)}{\Gamma(p+q)}, \quad p, q > 0.$$

Demonstram relatia de mai sus pt $p, q \in \mathbb{N}^*$

$$p\,B(p, q+1) = q\,B(p+1, q)$$

$$B(p, q) = B(p, (q-1)+1) = \frac{q-1}{p} B(p+1, q-1)$$

$$= \frac{q-1}{p} \cdot B(p+1, (q-2)+1) = \frac{q-1}{p} \cdot \frac{q-2}{p+1} B(p+2, q-2) =$$

$$= \dots = \frac{(q-1)(q-2)\dots 2\cdot 1}{p\cdot(p+1)\dots(p+q-2)} B(p+q-1, 1)$$

$$B(p+q-1,1)=\int_0^1 x^{p+q-1-1}\cdot(1-x)^{1-1}\,dx=\int_0^1 x^{p+q-2}\,dx$$

$$=\frac{1}{p+q-1}$$

Deci

$$B(p,q)=\frac{(q-1)!}{p\cdot(p+1)\cdots(p+q-2)(p+q-1)}=\frac{(q-1)!\,(p-1)!}{(p+q-1)!}$$

$$=\frac{\Gamma(p)\,\Gamma(q)}{\Gamma(p+q)}.$$