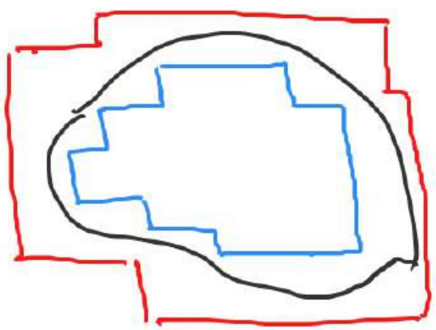
1) Fie $A = \{(x,y) \in \mathbb{R}^2 \mid 0 \le x \le 1, 0 \le y \le x\}$

1) Folosind definiția arătați că $A \in \mathcal{J}(\mathbb{R}^2)$

2) Folosind definiția integralei Riemann arătați că

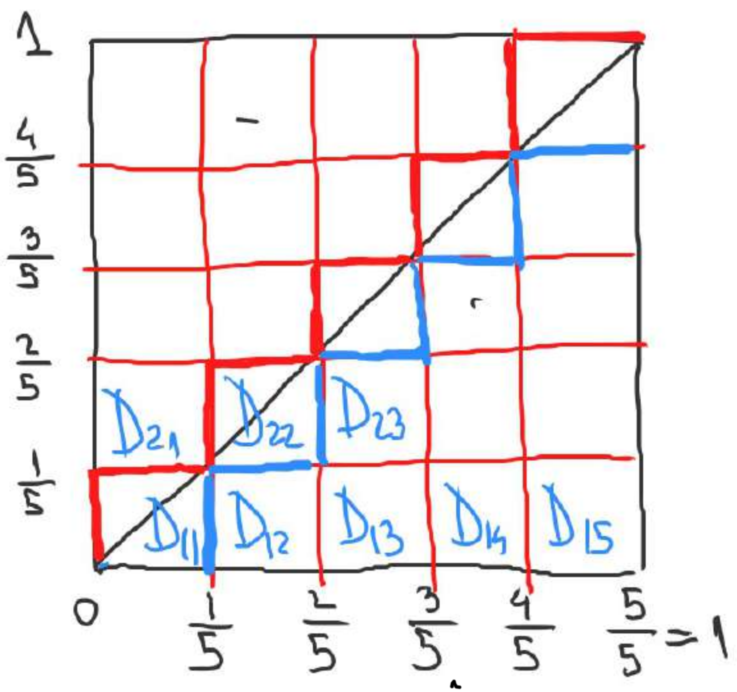
$$f : I = [0,1] \times [0,1] \to \mathbb{R}, \quad f(x,y) = \begin{cases} 1, & (x,y) \in A \\ 0, & (x,y) \in I \setminus A \end{cases}$$

este integrabilă Riemann și calc $\iint_I f(x,y)\,dx\,dy$



$$\lambda^*(A) = \inf\{\lambda(F) \mid F \in \mathcal{E}(\mathbb{R}^n), A \subset F\}$$

$$\lambda_*(A) = \sup\{\lambda(E) \mid E \in \mathcal{E}(\mathbb{R}^n), E \subset A\}$$



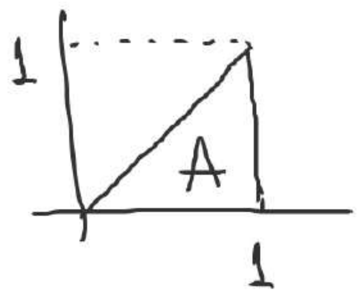
$I=[0,1]\times[0,1]$, $A=\{(x,y) \mid 0\le x\le 1, 0\le y\le x\}$

$F_n=\bigcup_{1\le i\le j\le n} D_{ij}$, $E_n=\bigcup_{1\le i<j\le n} D_{ij}$

$E_n, F_n\in \mathcal{E}(\mathbb{R}^n)$, $E_n\subset A\subset F_n$

$$\lambda(E_n)=\sum_{1\le i<j\le n} vol(D_{ij})=\frac{n(n-1)}{2}\cdot\frac{1}{n^2}=\frac{n^2-n}{2n^2}$$

$$\lambda(F_n)=\frac{n(n+1)}{2n^2}=\frac{n^2+n}{2n}$$



$n\ge 1$: $x_j=\frac{j}{n}$, $1\le j\le n$; $y_i=\frac{i}{n}$, $1\le i\le n$

$J_j=[x_{j-1}, x_j]$, $K_i=[y_{i-1}, y_i]$

$D_{ij}=J_j\times K_i$, $1\le i,j\le n$, $vol(D_{ij})=\frac{1}{n^2}$

$I=\bigcup_{1\le i,j\le n} D_{ij}$, $\Delta_n=\{D_{ij} \mid 1\le i,j\le n\}$ - diviz. a lui I

$$\frac{1}{2} = \sup_n \lambda(E_n) \leq \lambda_*(A) \leq \lambda^*(A) \leq \inf_n \lambda(F_n) = \frac{1}{2}$$

Deci $\lambda^*(A) = \lambda_*(A) = \frac{1}{2} \Rightarrow A \in \mathcal{J}(\mathbb{R}^2)$ și $\lambda(A) = \frac{1}{2}$.

2) $f(x,y) = \begin{cases} 1, & (x,y) \in A \\ 0, & (x,y) \in I \setminus A. \end{cases}$

$$s_{\Delta_n}(f) = \sum_{i,j=1}^{n} m_{ij} \cdot vol(D_{ij}) \qquad m_{ij} = \inf\{f(x,y) \mid (x,y) \in D_{ij}\}$$

$$S_{\Delta_n}(f) = \sum_{i,j=1}^{n} M_{ij}\, vol(D_{ij}) \qquad M_{ij} = \sup\{f(x,y) \mid (x,y) \in D_{ij}\}$$

$$m_{ij} = \begin{cases} 1; & i < j \\ 0; & i \geq j \end{cases} \qquad M_{ij} = \begin{cases} 1; & i \leq j \\ 0; & i > j \end{cases}$$

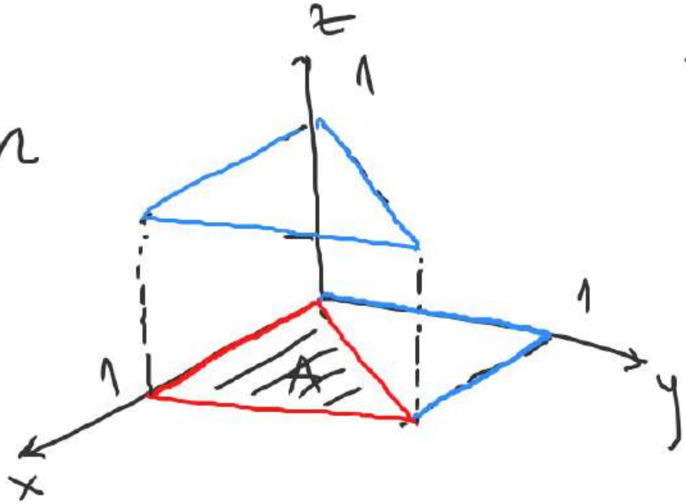
$$s_{\Delta_n}(f) = \sum_{1\le i<j\le n} 1\cdot \mathrm{vol}(D_{ij}) = \frac{(n-1)n}{2n^2} = \frac{n^2-n}{2n^2} = \frac{1}{2} - \frac{1}{2n}$$

$$S_{\Delta_n}(f) = \sum_{1\le i\le j\le n} 1\cdot \mathrm{vol}(D_{ij}) = \frac{n(n+1)}{2n^2} = \frac{1}{2} + \frac{1}{2n}$$

$$\frac{1}{2} = \sup_n s_{\Delta_n}(f) \le \sup_\Delta s_\Delta(f) = \underline{\int_I} f \le \overline{\int_I} f = \inf_\Delta S_\Delta(f) \le \inf S_{\Delta_n}(f) = \frac{1}{2}$$

$$\Rightarrow \overline{\int_I} f = \underline{\int_I} f \Rightarrow f \text{ int. Riemann}$$

și $\displaystyle\iint_I f(x,y)\,dx\,dy = \frac{1}{2}$



2) $A = \{(x,y) \in \mathbb{R}^2 \mid x + 2y = 0\}$ nu este măsurab. Jordan

Pt. că A este nemărginită

Teoremă $A \subset \mathbb{R}^n$ mărginită U.A.S.E.

1) $A \in \mathcal{J}(\mathbb{R}^n)$

2) $Fr(A) \in \mathcal{J}(\mathbb{R}^n)$ și $\lambda(Fr(A)) = 0 \iff \lambda^*(Fr(A)) = 0$

3) $\mathring{A}, \overline{A} \in \mathcal{J}(\mathbb{R}^n)$ și $\lambda(\mathring{A}) = \lambda(\overline{A})$.

---

3) $B = [0,1] \cap \mathbb{Q}$, $C = ([0,1] \times [0,1]) \cap \mathbb{Q}^2$

$\overline{B} = [0,1]$, $\mathring{B} = \emptyset$, $\overline{B}, \mathring{B} \in \mathcal{J}(\mathbb{R})$ dar $\underset{1}{\underset{\parallel}{\lambda(\overline{B})}} \neq \underset{0}{\underset{\parallel}{\lambda(\mathring{B})}}$.

Din Teoremă $\Rightarrow B \notin \mathcal{J}(\mathbb{R})$.

4) Fie $A \subset \mathbb{R}^n$ mărginită. Atunci

$$\left(A \in \mathcal{J}(\mathbb{R}^n) \text{ și } \lambda(A) = 0 \right) \Longleftrightarrow \lambda^*(A) = 0.$$

<u>Soluție.</u> "$\Longrightarrow$" evident

"$\Longleftarrow$" $0 \le \lambda_*(A) \le \lambda^*(A) = 0 \Longrightarrow \lambda^*(A) = \lambda_*(A) = 0$

$$\Downarrow$$

$$A \in \mathcal{J}(\mathbb{R}^n) \text{ și } \lambda(A) = 0.$$

5) Fie $A \subset \mathbb{R}^n$ mărg. UASE

1) $A \in \mathcal{J}(\mathbb{R}^n)$ și $\lambda(A) = 0$.

2) $\lambda^*(A) = 0$.

3) $\forall\, \varepsilon > 0, \exists\, F \in \mathcal{E}(\mathbb{R}^n)$ a.î. $A \subset F$ și $\lambda(F) < \varepsilon$.

Soluţie $1 \Leftrightarrow 2$ exerciţiul 4.

$3 \Rightarrow 2$. Fie $\varepsilon > 0$. Există $F \in \mathcal{E}(\mathbb{R}^n)$ a.î. $A \subset F$, $\lambda(F) < \varepsilon$

$$\lambda^*(A) = \inf\{\lambda(F) \mid F \in \mathcal{E}(\mathbb{R}^n), A \subset F\} < \varepsilon$$

$\Rightarrow \lambda^*(A) = 0$.

$2 \Rightarrow 3$ $\quad 0 = \lambda^*(A) = \inf\{\lambda(F) \mid F \in \mathcal{E}(\mathbb{R}^n), A \subset F\}$

Dacă $\varepsilon > 0$ există $F \in \mathcal{E}(\mathbb{R}^n)$, $A \subset F$ şi $\lambda(F) < \varepsilon$

Ex. $A = [0,1] \times \{1\}$, $B = ([0,1] \cap \mathbb{Q}) \times \{1\}$

Decideți dacă $A, B \in \mathcal{J}(\mathbb{R}^2)$ (vezi soluția la sfârșit).

6) Fie $A \subset \mathbb{R}^p$, $B \subset \mathbb{R}^q$ mărginite. Arătați că

$$\lambda^*(A \times B) \leq \lambda^*(A) \cdot \lambda^*(B), \quad \lambda_*(A \times B) \geq \lambda_*(A)\, \lambda_*(B).$$

Dacă $A \in \mathcal{J}(\mathbb{R}^p)$ și $B \in \mathcal{J}(\mathbb{R}^q)$ atunci $A \times B \in \mathcal{J}(\mathbb{R}^{p+q})$

și $\lambda(A \times B) = \lambda(A)\, \lambda(B)$.

Soluție. Fie $E \in \mathcal{E}(\mathbb{R}^p)$, $F \in \mathcal{E}(\mathbb{R}^q)$ a.i. $A \subset E$, $B \subset F$

Atunci $E \times F \in \mathcal{E}(\mathbb{R}^{p+q})$ și $A \times B \subset E \times F$

$$\lambda(E \times F) = \lambda(E) \cdot \lambda(F).$$

$$\lambda^*(A\times B) \le \inf\left\{ \underset{\lambda(E)\cdot\lambda(F)}{\overset{\|}{\lambda(E\times F)}} \;\middle|\; E\in\mathcal{E}(\mathbb{R}^p),\ F\in\mathcal{E}(\mathbb{R}^q),\ \begin{array}{l} A\subset E \\ B\subset F\end{array} \right\}$$

$$\|$$

$$\lambda^*(A)\cdot\lambda^*(B)$$

Analog $\lambda_*(A\times B) \ge \lambda_*(A)\cdot\lambda_*(B)$

2) A, B măs. Jordan.

$$\lambda^*(A\times B) \le \lambda^*(A)\cdot\lambda^*(B) \underset{\substack{A,B \text{ măs}\\ \text{Jordan}}}{=\!=\!=} \lambda_*(A)\cdot\lambda_*(B) \le \lambda_*(A\times B)$$

$\Longrightarrow$ $A\times B \in \mathcal{J}(\mathbb{R}^{p+q})$ și $\lambda(A\times B) = \lambda(A)\lambda(B)$.

7) $A = \{a\} \subset \mathbb{R}$ este măs. Jordan și $\lambda(A) = 0$

$E \subset \mathbb{R}$ este elementară dacă este o reuniune finită de intervale din $\mathbb{R}$.

$A \subset I = \left[a - \frac{\varepsilon}{2}, a + \frac{\varepsilon}{2}\right]$, $vol(I) = |I| = \varepsilon$, $I \in \mathcal{E}(\mathbb{R})$.

$\lambda^*(A) \leq \varepsilon$

$\Rightarrow \lambda^*(A) = 0$.

8) $A=[0,1]\times\{1\}$, $B=([0,1]\cap\mathbb{Q})\times\{1\}$.

$$\left.\begin{array}{l}[0,1]\in\mathcal{J}(\mathbb{R})\\ \{1\}\in\mathcal{J}(\mathbb{R}^2)\end{array}\right\}\xRightarrow{Ex\,6}\underbrace{[0,1]\times\{1\}}_{A}\in\mathcal{J}(\mathbb{R}^2) \text{ și } \lambda(A)=1\cdot 0=0.$$

Altfel. Fie $\varepsilon>0$.

$$\left.\begin{array}{l}A\subset\underbrace{[0,1]\times\left[1-\frac{\varepsilon}{2},1+\frac{\varepsilon}{2}\right]}_{F}\in\mathcal{E}(\mathbb{R}^2)\\ \lambda(F)=1\cdot\varepsilon=\varepsilon\end{array}\right\}\xRightarrow{Ex5}\begin{array}{l}A\in\mathcal{J}(\mathbb{R}^2)\\ \text{și } \lambda(A)=0.\end{array}$$

$B\subset A\Rightarrow 0\le\lambda^*(B)\le\lambda^*(A)=0.\Rightarrow\lambda^*(B)=0\Rightarrow B\in\mathcal{J}(\mathbb{R}^2)$ și $\lambda(B)=0$.

Altfel. $B\subset F$ unde F e ca mai sus. etc.