Seminar 11

1. Det pct. de extrem local ale fct f și precizați natura lor, unde:

a) $f: \mathbb{R}^2 \to \mathbb{R}$ $\quad f(x,y) = x^4 + y^4$

Sol $\quad \mathbb{R}^2$ deschisă, f continuă

$$\frac{\partial f}{\partial x} = 4x^3$$

$$\frac{\partial f}{\partial y} = 4y^3$$

$$\left.\begin{array}{l} \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \text{ cont pe } \mathbb{R}^2 \\ \mathbb{R}^2 \text{ deschisă} \end{array}\right| \xRightarrow[\text{de dif}]{\text{crit}} f \text{ dif pe } \mathbb{R}^2$$

$$\Rightarrow \begin{cases} \frac{\partial f}{\partial x} = 0 \\ \frac{\partial f}{\partial y} = 0 \end{cases} \Leftrightarrow \begin{cases} 4x^3 = 0 \\ 4y^3 = 0 \end{cases} \Leftrightarrow \begin{cases} x = 0 \\ y = 0 \end{cases}$$

Singurul pct critic al lui f este $(0,0)$

$$\frac{\partial^2 f}{\partial x^2} = 12x^2 \qquad \text{Obs că } f \text{ este de clasă } C^2$$

$$\frac{\partial^2 f}{\partial y^2} = 12y^2$$

$$\frac{\partial^2 f}{\partial x \partial y} = 0 = \frac{\partial^2 f}{\partial y \partial x}$$

$$Hf(x,y) = \begin{pmatrix} 12x^2 & 0 \\ 0 & 12y^2 \end{pmatrix} \quad \forall (x,y) \in \mathbb{R}^2$$

$$Hf(0,0) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$$\left.\begin{array}{l} \Delta_1 = 0 \\ \Delta_2 = \begin{vmatrix} 0 & 0 \\ 0 & 0 \end{vmatrix} = 0 \end{array}\right| \Rightarrow \text{crit nu decide}$$

$$\underset{x^4+y^4}{\overset{f(x,y)}{\parallel}} \quad \boxed{\geq} \quad \underset{0}{\overset{f(0,0)}{\parallel}} \quad \forall\ (x,y) \in \mathbb{R}^2$$

Deci, $(0,0)$ este pct de min. global al lui f, deci este şi pct de min. local

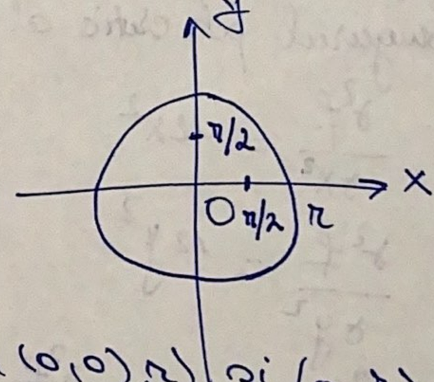
b) $f: \mathbb{R}^2 \to \mathbb{R}$ $\quad f(x,y) = -x^4 - y^4$

DE REZOLVAT $\qquad$ $(0,0)$ pct de max

c) $f: \mathbb{R}^2 \to \mathbb{R}$ $\quad f(x,y) = x^4 - y^4$

Ca la pct a), se arată că $(0,0)$ este sg pct critic al lui f şi că criteriul de stab. a pct de extrem local NU decide.

$$\underset{x^4-y^4}{\overset{f(x,y)}{\parallel}} \qquad \underset{0}{\overset{f(0,0)}{\parallel}}$$



$\forall\ r>0$ avem $\left(\frac{r}{2}, 0\right) \in B((0,0), r)$ şi $\left(0, \frac{r}{2}\right) \in B((0,0), r)$.

$\forall\ r>0$ avem $f\left(\frac{r}{2}, 0\right) = \frac{r^4}{16} > 0 = f(0,0)$ şi

$$f\left(0, \frac{r}{2}\right) = -\frac{r^4}{16} < 0 = f(0,0)$$

Deci $(0,0)$ nu e pct de extrem local al lui f

2. Ar. că ecuația $x\cos y + y\cos z + z\cos x = 1$ definește într-o vecinăt. a lui $(1,0,0)$ funcția implicită (unică) $z = z(x,y)$ și det $\frac{\partial z}{\partial x}(1,0)$, $\frac{\partial z}{\partial y}(1,0)$ și $dz(1,0)$.

Sol Fie $D = \mathbb{R}^3 \subset \mathbb{R}^{2+1}$

D deschisă

Fie $F: D \to \mathbb{R}$ $F(x,y,z) = x\cos y + y\cos z + z\cos x - 1$

1. $F(1,0,0) = 1\cdot\cos 0 + 0\cdot\cos 0 + 0\cdot\cos 1 - 1$

$= 1 - 1 = 0.$

2. $\frac{\partial F}{\partial x}(x,y,z) = \cos y - z\sin x$

$\frac{\partial F}{\partial y}(x,y,z) = -x\sin y + \cos z$

$\frac{\partial F}{\partial z}(x,y,z) = -y\sin z + \cos x$

$\frac{\partial F}{\partial x}, \frac{\partial F}{\partial y}, \frac{\partial F}{\partial z}$ continue $\Big| \Rightarrow F$ este de clasă C^1

3. $\frac{\partial F}{\partial z}(1,0,0) = -0\sin 0 + \cos 1 = \cos 1 \neq 0$

cf. TFi $\exists\, U = \mathring{U} \subset \mathcal{V}_{(1,0)}$

$\exists\, V = \mathring{V} \subset \mathcal{V}_0$

$\exists!\, z: U \to V$ aî:

a) $z(1,0) = 0$

b) $F(x,y,z(x,y)) = 0 \quad (\forall)\ (x,y) \in U$

c) z este de clasă C^1 și $\frac{\partial z}{\partial x}(x,y) = - \dfrac{\frac{\partial F}{\partial x}(x,y,z(x,y))}{\frac{\partial F}{\partial z}(x,y,z(x,y))} \quad \forall\ (x,y) \in U$

$$, \frac{\partial z}{\partial y}(x,y) = \frac{-\frac{\partial F}{\partial y}(x,y,z(x,y))}{\frac{\partial F}{\partial z}(x,y,z(x,y))} \quad \forall (x,y) \in U$$

Pt a det.

<u>V1</u> (Folosim a) și c))

$$\frac{\partial z}{\partial x}(x,y) = \frac{-(\cos y - z(x,y)\cdot \sin x)}{\cos x - y \sin z(x,y)}$$

$$\frac{\partial z}{\partial x}(1,0) = -\frac{(\cos 0 - z(1,0)\cdot \sin 0)}{\cos 1 - 0\cdot \sin z(1,0)}$$

$$\underset{\underset{z(1,0)=0}{\uparrow}}{=} -\frac{1}{\cos 1}$$

$$\frac{\partial z}{\partial y}(x,y) = \frac{-(\cos z(x,y) - x \sin y)}{\cos x - y \sin z(x,y)} \quad \forall (x,y) \in U$$

$$\Rightarrow \frac{\partial z}{\partial y}(1,0) = -\frac{(\cos z(1,0) - 1\cdot \sin 0)}{\cos 1 - 0\cdot \sin z(1,0)}$$

$$\underset{\underset{z(1,0)=0}{\uparrow}}{=} -\frac{1}{\cos 1}$$

z de clasă C^1 pe U $\Rightarrow$ z dif pe U

$\mathrm{dif}\ dz(1,0) : \mathbb{R}^2 \to \mathbb{R}$ $\quad dz(1,0)(u,v) = \left[\begin{pmatrix} \frac{\partial z}{\partial x}(1,0) & \frac{\partial z}{\partial y}(1,0) \end{pmatrix}^{t} \begin{pmatrix} u \\ v \end{pmatrix}\right]$

$= -\frac{1}{\cos 1}(u+v)$ i.e.

$dz(1,0) = \frac{-1}{\cos y}(dx+dy)$

$\underline{\underline{V_2}}$: Folosim a) şi b)

Cf. b) avem $F(x,y,z(x,y)) = 0 \ \forall (x,y) \in U$, deci

$x\cos y + y\cos(z(x,y)) + z(x,y)\cos x^{-1} = 0 \ \forall (x,y) \in U$ ⊛

Derivăm relația în raport cu x:

$\cos y + y \cdot -\sin(z(x,y)) \cdot \frac{\partial z}{\partial x} + \cancel{z(x,y) \cdot \sin x = 0}$

$\frac{\partial z}{\partial x}(x,y) \cdot \cos x + z(x,y) \cdot -\sin x = 0$

$\cos y - y\sin(z(x,y)) \cdot \frac{-1}{\cos 1} + \cancel{\frac{-1}{\cos 1}} \cdot \cos x + \cancel{\sin x} \cdot z(x,y) = 0$

$\Rightarrow \frac{\partial z}{\partial x}\left(-y\sin(z(x,y)) + \cos x\right) = -\cos y + z(x,y) \cdot \sin x$

$\frac{\partial z}{\partial x}(x,y) = \frac{-\cos y + z(x,y)\sin x}{-y\sin(z(x,y)) + \cos x} \ \forall (x,y) \in U$

$\frac{\partial z}{\partial x}(1,0) = \frac{-\cos 0 + 0}{-0 \cdot \sin 0 + \cos 1} = -\frac{1}{\cos 1}$

Derivăm rel ⊛ în raport cu y

$x \cdot -\sin y + \cos(z(x,y)) + y \cdot -\sin(z(x,y)) \cdot \frac{\partial z}{\partial y}(x,y)$

$+ \frac{\partial z}{\partial y}(x,y) \cdot \cos x = 0$

$\frac{\partial z}{\partial y}(x,y)\left(-y\sin(z(x,y)) + \cos x\right) = x\sin y \pm \cos(z(x,y))$

$$\frac{\partial z}{\partial y}(x,y) = \frac{x \sin y - \cos(z(x,y))}{-y \sin(z(x,y)) + \cos x} \quad \forall (x,y) \in U$$

$$\frac{\partial z}{\partial y}(1,0) = \frac{1 \cdot \sin 0 - \cos 0}{-0 \cdot \sin 0 + \cos 1} = \frac{-1}{\cos 1}$$

Pt a det $dz(1,0)$, procedăm ca la V_1

3. Fie $f: \mathbb{R}^3 \to \mathbb{R}$ $f(x,y,z) = xy + xz + yz$. Să se det pct de extrem local ale lui f cu legăturile

$-x + y + z = 1$ și $x - z = 0$

Sol

$\mathbb{R}^3$ deschisă

Fie $g_1, g_2 : \mathbb{R}^3 \to \mathbb{R}$ $\quad g_1(x,y,z) = -x + y + z - 1$

$g_2(x,y,z) = x - z$

Fie $A = \{(x,y,z) \in \mathbb{R}^3 \mid g_1(x,y,z) = 0,\ g_2(x,y,z) = 0\}$

$\frac{\partial f}{\partial x} = y + z \qquad \frac{\partial g_1}{\partial x} = -1 \qquad \frac{\partial g_2}{\partial x} = 1$

$\frac{\partial f}{\partial y} = x + z \qquad \frac{\partial g_1}{\partial y} = 1 \qquad \frac{\partial g_2}{\partial y} = 0$

$\frac{\partial f}{\partial z} = x + y \qquad \frac{\partial g_1}{\partial z} = 1 \qquad \frac{\partial g_2}{\partial z} = -1$

Fie $L: \mathbb{R}^3 \to \mathbb{R}$ $\quad L(x,y,z) = f(x,y,z) + \alpha g_1(x,y,z) + \beta g_2(x,y,z)$

$L(x,y,z) = \cancel{f(x,y,z)}$

$xy + xz + yz + \alpha \cdot (-x + y + z) + \beta(x - z)$

$$\frac{\partial L}{\partial x} = y+z + \alpha\cdot(-1) + \beta\cdot 1$$

$$\frac{\partial L}{\partial y} = x+z+\alpha\cdot 1 + \beta\cdot 0$$

$$\frac{\partial L}{\partial z} = x+y + \alpha\cdot 1 + \beta\cdot 1$$

$$\begin{cases} \frac{\partial L}{\partial x} = 0 \\ \frac{\partial L}{\partial y} = 0 \\ \frac{\partial L}{\partial z} = 0 \\ g_1(x,y,z) = 0 \\ g_2(x,y,z) = 0 \end{cases} \Rightarrow \begin{cases} 1)\ y+z-\alpha+\beta = 0 \\ 2)\ x+z+\alpha = 0 \Rightarrow \alpha = -2x \\ 3)\ x+y+\alpha+\beta = 0 \\ 4)\ -x+y+z = 1 \\ 5)\ x-z = 0 \Rightarrow x = z \end{cases} \Rightarrow \boxed{y=1}$$

Adunăm prima ec. cu a treia:

$2x + 2y = 0 \Rightarrow x+y = 0 \Rightarrow \boxed{x=-1}$, $\boxed{z=-1}$

$\alpha = 2$

$\beta = x+y+\alpha = -1+1+2 = 2$

$$\text{rang}\begin{pmatrix} \frac{\partial g_1}{\partial x}(-1,1,-1) & \frac{\partial g_1}{\partial y}(-1,1,-1) & \frac{\partial g_1}{\partial z}(-1,1,-1) \\ \frac{\partial g_2}{\partial x}(-1,1,-1) & \frac{\partial g_2}{\partial y}(-1,1,-1) & \frac{\partial g_2}{\partial z}(-1,1,-1) \end{pmatrix}$$

$$= \text{rang}\begin{pmatrix} -1 & 1 & 1 \\ 1 & 0 & -1 \end{pmatrix} = 2$$

Singurul punct staționar al lui f cu leg $g_1(x,y,z)=0$ și $g_2(x,y,z)=0$ este $(-1,1,-1)$.