C_{12} - GA

2024

Geometrie analitică euclidiană

$(\mathbb{R}^3, (\mathbb{R}^3, g_0), \varphi)$ sp. af. euclidian cu str. canonică

- Perpendiculara comună a 2 drepte necoplanare.

$$\mathcal{D}_1 : \frac{x_1 - a_1}{u_1} = \frac{x_2 - a_2}{u_2} = \frac{x_3 - a_3}{u_3} = t \iff \begin{cases} x_1 = a_1 + u_1 t \\ x_2 = a_2 + u_2 t \\ x_3 = a_3 + u_3 t, t \in \mathbb{R} \end{cases}$$

$$\mathcal{D}_2 : \frac{x_1 - b_1}{v_1} = \frac{x_2 - b_2}{v_2} = \frac{x_3 - b_3}{v_3} = s \iff \begin{cases} x_1 = b_1 + v_1 s \\ x_2 = b_2 + v_2 s \\ x_3 = b_3 + v_3 s, s \in \mathbb{R} \end{cases}$$

$\mathcal{R} = \{O; e_1, e_2, e_3\}$ reper cartezian ortonormat

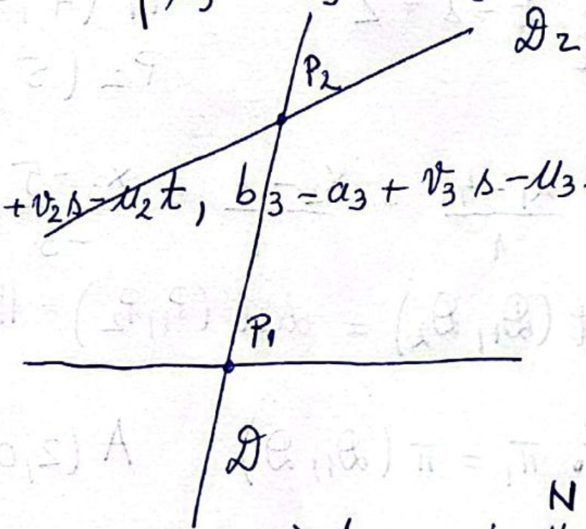
OBS
- $A(a_1, a_2, a_3) \in \mathcal{D}_1$, $V_{\mathcal{D}_1} = \langle \{u\} \rangle$, $u = (u_1, u_2, u_3)$

  $\overrightarrow{OA} = \sum_{i=1}^{3} a_i e_i$ $\qquad u = \sum_{i=1}^{3} u_i e_i$

- $B(b_1, b_2, b_3) \in \mathcal{D}_2$, $V_{\mathcal{D}_2} = \langle \{v\} \rangle$, $v = (v_1, v_2, v_3)$

  $\overrightarrow{AB} = (b_1 - a_1, b_2 - a_2, b_3 - a_3)$

$\mathcal{D}_1, \mathcal{D}_2$ necoplanare $\iff \begin{vmatrix} u_1 & v_1 & b_1 - a_1 \\ u_2 & v_2 & b_2 - a_2 \\ u_3 & v_3 & b_3 - a_3 \end{vmatrix} \neq 0$



$\mathcal{D} \perp \mathcal{D}_k$, $k = \overline{1,2}$

$\overrightarrow{P_1P_2} = (b_1 - a_1 + v_1 s - u_1 t, b_2 - a_2 + v_2 s - u_2 t, b_3 - a_3 + v_3 s - u_3 t)$

$$\begin{cases} \langle \overrightarrow{P_1P_2}, u \rangle = 0 \\ \langle \overrightarrow{P_1P_2}, v \rangle = 0 \end{cases} \Rightarrow t, s$$

OBS $\quad \pi_1 = \pi(\mathcal{D}_1, \mathcal{D})$, $A \in \pi_1$ $\qquad V_{\mathcal{D}} = \langle \{\overset{N}{\overset{\parallel}{u \times v}}\} \rangle$

$N_1 = N \times u$

$\pi_2 = \pi(\mathcal{D}_2, \mathcal{D})$, $B \in \pi_2$ $\qquad N_2 = N \times v$

$\pi_1 \cap \pi_2 = \mathcal{D}$

Aplicatie 2

$\mathcal{D}_1 : \frac{x_1-2}{1} = \frac{x_2}{2} = \frac{x_3-3}{1} = t \Rightarrow \begin{cases} x_1 = t+2 \\ x_2 = 2t \\ x_3 = t+3, t\in\mathbb{R} \end{cases}$

$\mathcal{D}_2 : \frac{x_1-1}{2} = \frac{x_2-3}{1} = \frac{x_3}{1} = s \Rightarrow \begin{cases} x_1 = 2s+1 \\ x_2 = s+3 \\ x_3 = s, s\in\mathbb{R} \end{cases}$

a) Să se scrie ec $\perp$ comune

b) $dist(\mathcal{D}_1, \mathcal{D}_2)$

SOL

a) $A(2,0,3)\in\mathcal{D}_1, u = (1,2,1)$ $\quad \overrightarrow{AB} = (-1,3,-3)$

$B(1,3,0)\in\mathcal{D}_2, v = (2,1,1)$

Cond. de necoplanaritate: $\begin{vmatrix} 1 & 2 & -1 \\ 2 & 1 & 3 \\ 1 & 1 & -3 \end{vmatrix} \neq 0$

$P_1(t+2, 2t, t+3)\in\mathcal{D}_1$;

$P_2(2s+1, s+3, s)\in\mathcal{D}_2$ $\quad \overrightarrow{P_1P_2} = (2s-t-1, s-2t+3, s-t-3)$

$\begin{cases} \langle\overrightarrow{P_1P_2}, u\rangle = 0 \\ \langle\overrightarrow{P_1P_2}, v\rangle = 0 \end{cases} \Rightarrow \begin{cases} 1(2s-t-1) + 2(s-2t+3) + 1(s-t-3) = 0 \\ 2(2s-t-1) + 1(s-2t+3) + 1(s-t-3) = 0 \end{cases}$

$\Rightarrow t = s = 2$ $\quad P_1(4,4,5)$ $\quad \overrightarrow{P_1P_2} = (1,1,-3)$

$P_2(5,5,2)$

$\mathcal{D} : \frac{x_1-4}{1} = \frac{x_2-4}{1} = \frac{x_3-5}{-3}$

$dist(\mathcal{D}_1, \mathcal{D}_2) = dist(P_1, P_2) = \|\overrightarrow{P_1P_2}\| = \sqrt{1+1+9} = \sqrt{11}$.

OBS

- $\pi_1 = \pi(\mathcal{D}_1, \mathcal{D})$ $\quad A(2,0,3)\in\pi_1$, $N = u\times v = \begin{vmatrix} e_1 & e_2 & e_3 \\ 1 & 2 & 1 \\ 2 & 1 & 1 \end{vmatrix} = (1,1,-3)$

$N_1 = N\times u = \begin{vmatrix} e_1 & e_2 & e_3 \\ 1 & 1 & -3 \\ 1 & 2 & 1 \end{vmatrix} = (7,-4,1)$

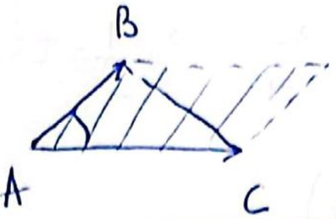
$\pi_1 : 7(x_1-2) - 4(x_2-0) + 1\cdot(x_3-3) = 0 \Rightarrow 7x_1 - 4x_2 + x_3 - 17 = 0$

- Analog $\pi_2 = \pi(\mathcal{D}_2, \mathcal{D})$

$\mathcal{D} : = \pi_1 \cap \pi_2$

Aria unui Δ în n = 3

$\{u, v\}$ SLI



$\|u \times v\| = \|u\| \cdot \|v\| \sin\alpha$

$A_{\Delta ABC} = \frac{1}{2}\|\overrightarrow{AB} \times \overrightarrow{AC}\| = \frac{1}{2}\sqrt{\Delta_1^2 + \Delta_2^2 + \Delta_3^2}$.

$$\overrightarrow{AB} \times \overrightarrow{AC} = \begin{vmatrix} e_1 & e_2 & e_3 \\ b_1 - a_1 & b_2 - a_2 & b_3 - a_3 \\ c_1 - a_1 & c_2 - a_2 & c_3 - a_3 \end{vmatrix} = (\Delta_1, \Delta_2, \Delta_3)$$

Exemplu

$A(1,0,1)$, $B(0,-1,0)$, $C(0,1,1) \Longrightarrow A_{\Delta ABC}$

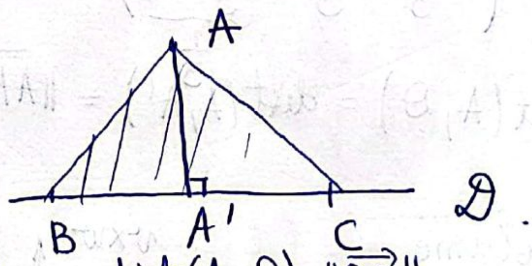
SOL $\overrightarrow{AB} = (-1,-1,-1)$, $\overrightarrow{BC} = (0,2,1)$

$$\overrightarrow{AB} \times \overrightarrow{AC} = \begin{vmatrix} e_1 & e_2 & e_3 \\ -1 & -1 & -1 \\ 0 & 2 & 1 \end{vmatrix} = (1, 1, -2)$$

$A_{\Delta ABC} = \frac{1}{2}\|\overrightarrow{AB} \times \overrightarrow{AC}\| = \frac{1}{2}\sqrt{1+1+4} = \frac{\sqrt{6}}{2}$.

- $Dist(A, \mathcal{D})$

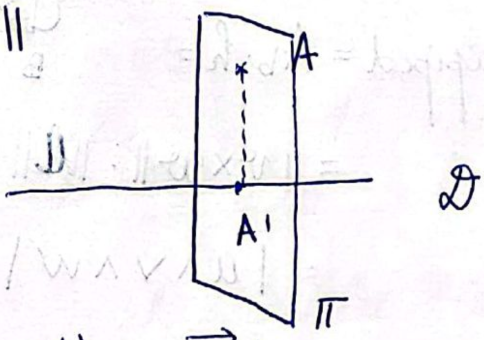
(M1) Fie $B, C \in \mathcal{D}$.



$A_{\Delta ABC} = \frac{1}{2}\|\overrightarrow{AB} \times \overrightarrow{AC}\| = \frac{dist(A,\mathcal{D}) \cdot \|\overrightarrow{BC}\|}{2}$

$Dist(A, \mathcal{D}) = \frac{\|\overrightarrow{AB} \times \overrightarrow{AC}\|}{\|\overrightarrow{BC}\|}$, $\forall B, C \in \mathcal{D}$

(M2) $\pi \ni A$, $\mathcal{D} \perp \pi$



$N_\pi = u_{\mathcal{D}}$

$\mathcal{D} \cap \pi = \{A'\}$

$dist(A, \mathcal{D}) = dist(A, A') = \|\overrightarrow{AA'}\|$.

Aplicatie

$\mathcal{D}: \frac{x_1}{1} = \frac{x_2}{-1} = \frac{x_3-1}{2} = t \Rightarrow \begin{cases} x_1 = t \\ x_2 = -t \\ x_3 = 2t+1, t \in \mathbb{R} \end{cases}$

$A(1,1,0)$

$dist(A, \mathcal{D})$

M1 $t=0 \quad B(0,0,1)$ $\qquad dist(A,\mathcal{D}) = \frac{\|\overrightarrow{AB} \times \overrightarrow{AC}\|}{\|\overrightarrow{BC}\|}$

$t=1 \quad C(1,-1,3)$

$\overrightarrow{AB} = (-1,-1,1)$; $\overrightarrow{AC} = (0,-2,3)$, $\overrightarrow{BC} = (1,-1,2)$

$$\overrightarrow{AB} \times \overrightarrow{AC} = \begin{vmatrix} e_1 & e_2 & e_3 \\ -1 & -1 & 1 \\ 0 & -2 & 3 \end{vmatrix} = (-1, 3, 2)$$

$$dist(A,\mathcal{D}) = \frac{\sqrt{1+9+4}}{\sqrt{1+1+4}} = \frac{\sqrt{14}}{\sqrt{6}} = \sqrt{\frac{7}{3}} = \frac{\sqrt{21}}{3}.$$

M2 $\pi \ni A(1,1,0)$ $\mathcal{D} \perp \pi \Rightarrow N_\pi = u_{\mathcal{D}} = (1,-1,2)$

$\pi: 1(x_1-1) + (-1)(x_2-1) + 2(x_3-0) = 0 \Rightarrow x_1 - x_2 + 2x_3 = 0$

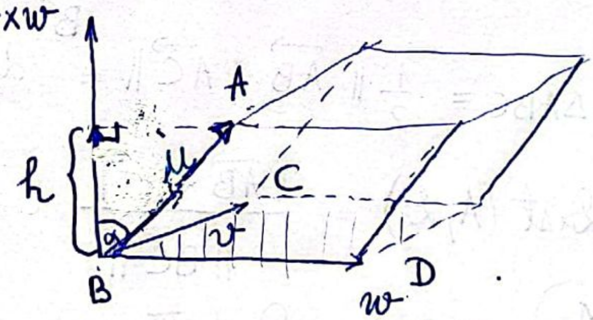
$\mathcal{D} \cap \pi = \{A'\}$ $\qquad t+t+4t+2 = 0 \Rightarrow 6t = -2 \Rightarrow t = -\frac{1}{3}$

$A'\left(-\frac{1}{3}, \frac{1}{3}, \underbrace{-\frac{2}{3}+1}_{\frac{1}{3}}\right)$, $\overrightarrow{AA'} = \left(-\frac{4}{3}, -\frac{2}{3}, \frac{1}{3}\right)$

$$dist(A,\mathcal{D}) = dist(A,A') = \|\overrightarrow{AA'}\| = \sqrt{\frac{16+4+1}{9}} = \frac{\sqrt{21}}{3}.$$

Volume

$\{u, v, w\}$ SLI



$V_{paralelipiped} = A_b \cdot h =$

$= \|v \times w\| \cdot \|u\| \cdot |\cos\alpha| = |\langle u, v \times w \rangle|$

$= |u \wedge v \wedge w| = |\Delta|$

$u = \overrightarrow{BA}$, $A(a_1, a_2, a_3)$

$v = \overrightarrow{BC}$ $\quad B(b_1, b_2, b_3)$

$w = \overrightarrow{BD}$ $\quad C(c_1, c_2, c_3)$

$D(d_1, d_2, d_3)$

$$\Delta = \begin{vmatrix} a_1-b_1 & a_2-b_2 & a_3-b_3 \\ c_1-b_1 & c_2-b_2 & c_3-b_3 \\ d_1-b_1 & d_2-b_2 & d_3-b_3 \end{vmatrix}$$

$$V_{paralelipiped} = \left| \det \begin{pmatrix} a_1 & a_2 & a_3 & 1 \\ b_1 & b_2 & b_3 & 1 \\ c_1 & c_2 & c_3 & 1 \\ d_1 & d_2 & d_3 & 1 \end{pmatrix} \right| = |\Delta|$$

Ec π
$B, C, D \in \pi$: $\begin{vmatrix} x_1 & x_2 & x_3 & 1 \\ b_1 & b_2 & b_3 & 1 \\ c_1 & c_2 & c_3 & 1 \\ d_1 & d_2 & d_3 & 1 \end{vmatrix} = 0 \Leftrightarrow ax_1 + bx_2 + cx_3 + d = 0$

$$V_{ABCD} = \frac{1}{6}|\Delta| = \frac{A_{BCD} \cdot dist(A, \pi)}{3}$$
$$= \frac{\|\vec{BC} \times \vec{BD}\|}{2} \cdot \frac{dist(A, \pi)}{3}$$

$$dist(A, \pi) = \frac{|\Delta|}{\|\vec{BC} \times \vec{BD}\|} = \frac{|aa_1 + ba_2 + ca_3 + d|}{\sqrt{a^2+b^2+c^2}}$$

Ex: $\pi : x_1 - 2x_2 + 3x_3 + 1 = 0$, $A(1,2,3)$ $\quad N = (1,-2,3)$
$dist(A, \pi)$.

SOL $dist(A, \pi) = \frac{|1 - 2\cdot 2 + 3\cdot 3 + 1|}{\sqrt{1+4+9}} = \frac{7}{\sqrt{14}} = \frac{\cancel{7}\sqrt{14}}{\cancel{14}_2} = \frac{\sqrt{14}}{2}$.

Conice în planul afin euclidian

Def O conică este LG. al punctelor $P(x_1, x_2)$ care în rap cu $R = \{O; e_1, e_2\}$ verifică:

$$\Gamma : f(x_1, x_2) = a_{11}x_1^2 + 2a_{12}x_1x_2 + a_{22}x_2^2 + 2b_1x_1 + 2b_2x_2 + c = 0$$

$\Gamma: \ f(x) = X^T A X + 2BX + c = 0.$

$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{12} & a_{22} \end{pmatrix} = A^T, \quad B = (b_1 \ \ b_2) \quad X = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$

$\tilde{A} = \begin{pmatrix} A & B^T \\ B & c \end{pmatrix} = \begin{pmatrix} \boxed{\begin{matrix} a_{11} & a_{12} \\ a_{12} & a_{22} \end{matrix}} & \begin{matrix} b_1 \\ b_2 \end{matrix} \\ \begin{matrix} b_1 & b_2 \end{matrix} & c \end{pmatrix}, \quad r = rg\,A, \ r' = rg\,\tilde{A}$

$\delta = \det A, \quad \Delta = \det \tilde{A}.$

- $\Delta \neq 0 \Leftrightarrow \Gamma$ nedegenerată
- $\Delta = 0 \Leftrightarrow \Gamma$ degenerată.

OBS $\Gamma_1 \equiv \Gamma_2$ congruente metric $\Leftrightarrow \exists\, \tau : \mathbb{R}^2 \to \mathbb{R}^2$ izometrie.

aî $\Gamma_2 = \tau(\Gamma_1)$ $\quad X' = CX + D$, $C \in O(2)$

Invarianții metrici: $\delta, \Delta, r, r', \frac{\Delta}{\delta}$

Conice def ca LG

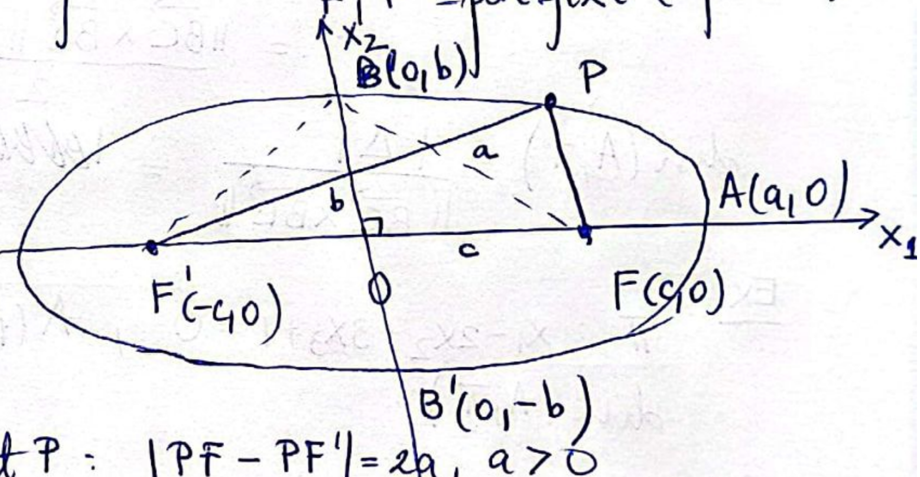
① Elipsa = LG al pct P: $PF + PF' = 2a, \ a > 0$

F, F' = pcte fixe (focare)

$\mathcal{E}: \frac{x_1^2}{a^2} + \frac{x_2^2}{b^2} = 1$

$a^2 = b^2 + c^2$

$a > b.$

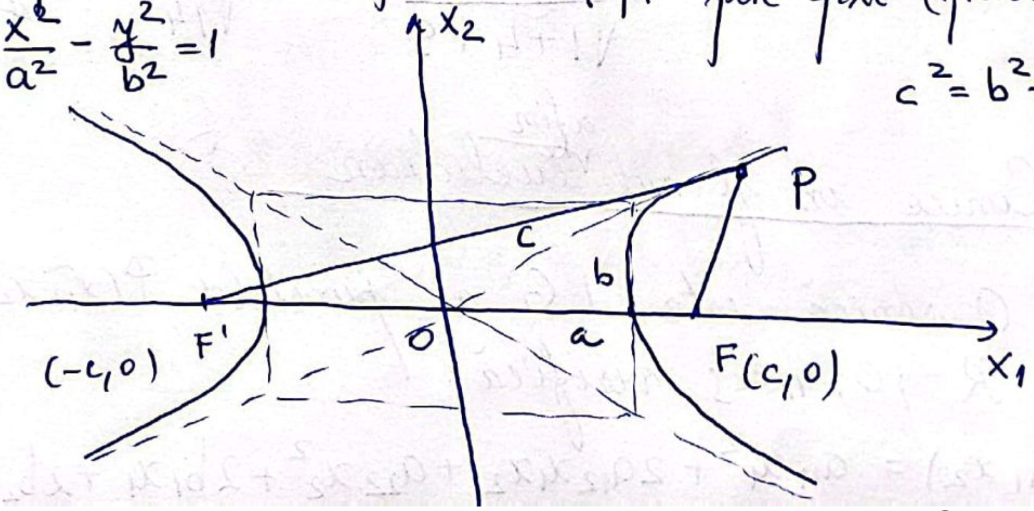


② Hiperbola = LG al pct P: $|PF - PF'| = 2a, \ a > 0$

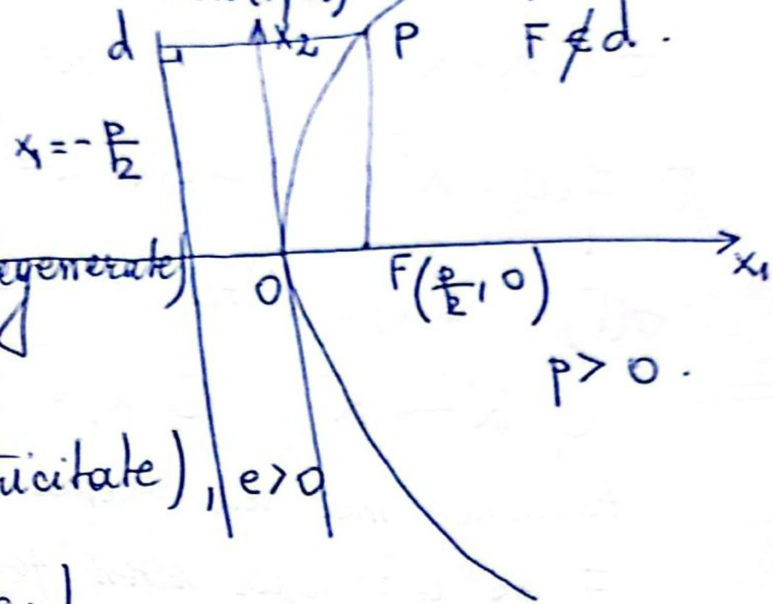
F, F' pcte fixe (focare)

$\mathcal{H}: \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

$c^2 = b^2 + a^2, \ c > a.$



) Parabola = LG al pct P : $\dfrac{dist(P,F)}{dist(P,d)} = 1$, F = focar
$\mathcal{P}: x_2^2 = 2p x_1$ d = dr. directoare
$F \notin d$.



$p > 0$.

Ⓣ (def. unitară conice nedegenerate)
LG al pct P:
$$\frac{dist(P,F)}{dist(P,d)} = e \text{ (excentricitate)}, \ e>0$$
reprez. o conică nedeg.)

$e = 1$ $\mathcal{P}$

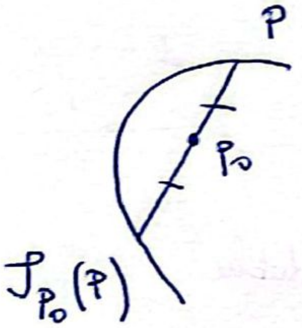
$e = \frac{c}{a} < 1$ $\mathcal{E}$

$e = \frac{c}{a} > 1$ $\mathcal{H}$.

<u>Aducerea la o formă canonică a unei conice cu centru unic</u> $(\delta \neq 0)$

<u>Def</u> Γ conică

P_0 s.n. <u>centru</u> $\Leftrightarrow [\forall P \in \Gamma, \ S_{P_0}(P) \in \Gamma]$



$$P_0 : \begin{cases} \frac{\partial f}{\partial x_1} = 0 \Rightarrow 2a_{11}x_1 + 2a_{12}x_2 + 2b_1 = 0 \\ \frac{\partial f}{\partial x_2} = 0 \quad 2a_{12}x_1 + 2a_{22}x_2 + 2b_2 = 0 \end{cases}$$

$$AX = -B^T$$

$\underset{\det A}{\overset{\shortparallel}{\delta}} \neq 0 \Rightarrow \exists!$ centrul. $P_0(x_1^0, x_2^0)$

<u>Prop</u> $\delta \neq 0 \Rightarrow f(x_1^0, x_2^0) = \dfrac{\Delta}{\delta}$

$\bullet\ \mathcal{R}=\{O; e_1, e_2\} \xrightarrow[\text{translație}]{\theta} \mathcal{R}'=\{P_0; e_1, e_2\} \xrightarrow[\text{rotație}]{\tau} \mathcal{R}''=\{P_0; e_1', e_2'\}$

$\theta: X = X' + X_0 \qquad X=\begin{pmatrix} x_1 \\ x_2 \end{pmatrix},\ X'=\begin{pmatrix} x_1' \\ x_2' \end{pmatrix},\ X_0=\begin{pmatrix} x_1^0 \\ x_2^0 \end{pmatrix}$

$\Gamma: f(x) = X^T A X + 2BX + c = 0$

$\theta(\Gamma): f(x) = X'^T A X' + \frac{\Delta}{\delta} = 0$

$Q: \mathbb{R}^2 \to \mathbb{R}^2,\ Q(x) = X'^T A X' = a_{11} x_1'^2 + 2a_{12} x_1' x_2' + a_{22} x_2'^2$

Aplicăm met. val. proprii

$\exists\ \{e_1', e_2'\}$ reper orton. format din versori proprii ai

$Q(x) = \lambda_1 x_1''^2 + \lambda_2 x_2''^2$

$e_k' = (l_k, m_k) \qquad R=\begin{pmatrix} l_1 & l_2 \\ m_1 & m_2 \end{pmatrix} \in SO(2)$

(o alegem)

$\tau: X' = RX''$ rotație.

$\tau(\theta(\Gamma)): \lambda_1 x_1''^2 + \lambda_2 x_2''^2 + \frac{\Delta}{\delta} = 0$

$\det(A-\lambda I_2)=0 \iff \lambda^2 - Tr(A)\lambda + \underbrace{\det(A)}_{\delta} = 0.$

$\lambda_1\lambda_2 = \delta \neq 0$

1) $\Delta \neq 0$

a) $\lambda_1\lambda_2 > 0$ — ϕ, Elipsă

b) $\lambda_1\lambda_2 < 0$ — Hiperbolă

2) $\Delta = 0 \qquad \lambda_1 x_1''^2 + \lambda_2 x_2''^2 = 0$

a) $\delta = \lambda_1\lambda_2 > 0 \qquad x_1 = x_2 = 0$ pct dublu

b) $\delta = \lambda_1\lambda_2 < 0$ — Drepte concurente.

$\Gamma: f(x) = \underline{7x_1^2 - 8x_1x_2 + x_2^2} - \underline{6x_1 - 12x_2} - 9 = 0.$

Să se aducă la o f. canonică, ef. izometrii în planul afin euclidian.

SOL

$$A = \begin{pmatrix} 7 & -4 \\ -4 & 1 \end{pmatrix} \quad B = \begin{pmatrix} -3 & -6 \end{pmatrix}, \quad \tilde{A} = \begin{pmatrix} 7 & -4 & -3 \\ -4 & 1 & -6 \\ -3 & -6 & -9 \end{pmatrix}$$

$\delta = \det A = 7 - 16 = -9 \neq 0$ (centru unic)

$\Delta = \det \tilde{A} = -9 \cdot 36 \neq 0$ (conică nedeg)

Centrul
$$\begin{cases} \frac{\partial f}{\partial x_1} = 14x_1 - 8x_2 - 6 = 0 \\ \frac{\partial f}{\partial x_2} = -8x_1 + 2x_2 - 12 = 0 \end{cases} \Rightarrow P_0(-3, -6)$$

$$\mathcal{R}_0 = \{O; e_1, e_2\} \xrightarrow{\theta} \mathcal{R}' = \{P_0; e_1, e_2\} \xrightarrow{\tau} \mathcal{R}'' = \{P_0; e_1', e_2'\}$$

$$\theta: X = X' + X_0 \iff \begin{cases} x_1 = x_1' + (-3) \\ x_2 = x_2' + (-6) \end{cases}$$

$$\theta(\Gamma): f(x) = \underline{7x_1'^2 - 8x_1'x_2' + x_2'^2} + \frac{\overset{36}{\Delta}}{\delta} = 0$$

$Q: \mathbb{R}^2 \to \mathbb{R}^2, \quad Q(x) = 7x_1'^2 - 8x_1'x_2' + x_2'^2$

$$P(\lambda) = 0 \Rightarrow \begin{vmatrix} 7-\lambda & -4 \\ -4 & 1-\lambda \end{vmatrix} = 0 \iff \lambda^2 - 8\lambda - 9 = 0$$
$$(\lambda + 1)(\lambda - 9) = 0$$

$\lambda_1 = -1 \qquad V_{\lambda_1} = \{x \in \mathbb{R}^2 \mid AX = -X\} = \langle \{e_1' = \frac{1}{\sqrt{5}}(1, 2)\} \rangle$

$\lambda_2 = 9 \qquad V_{\lambda_2} = \{x \in \mathbb{R}^2 \mid AX = 9X\} = \langle \{e_2' = \frac{1}{\sqrt{5}}(-2, 1)\} \rangle$

$$R = \frac{1}{\sqrt{5}} \begin{pmatrix} 1 & -2 \\ 2 & 1 \end{pmatrix} \in SO(2)$$

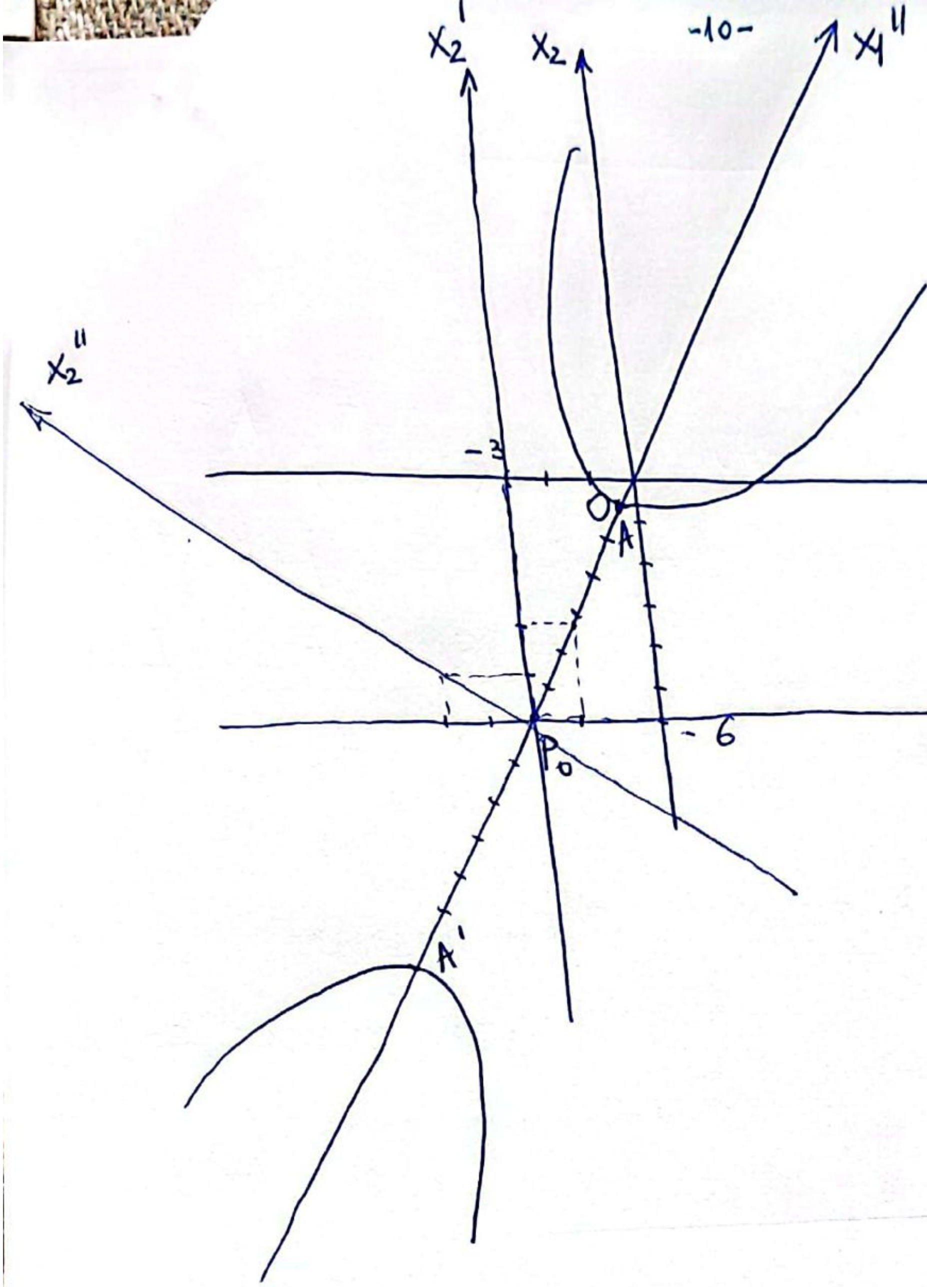
$\tau: X' = RX'' \qquad Q(x) = -x_1''^2 + 9x_2''^2$

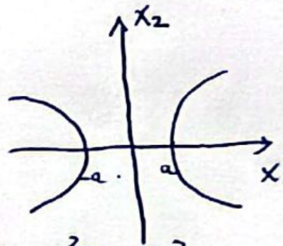
$$\tau(\theta(\Gamma)): -x_1''^2 + 9x_2''^2 + 36 = 0 \Rightarrow -x_1''^2 + 9x_2''^2 = -36$$

$$\frac{x_1''^2}{36} - \frac{x_2''^2}{4} = 1 \qquad \begin{matrix} a = 6 \\ b = 2 \end{matrix}$$

$e_1' = \frac{1}{\sqrt{5}}(1,2)$

$e_2' = \frac{1}{\sqrt{5}}(-2,1)$





$\mathcal{H}: \frac{x_1^2}{a^2} - \frac{x_2^2}{b^2} = 1$