7.03.2024 15/2024

C_2 - GA + C_3 - GA

Teorema Hamilton-Cayley. Polinom caracteristic

Fie $A \in \mathcal{M}_n(\mathbb{C})$

$$P_A(X) = \det(A - XI_n) = \begin{vmatrix} a_{11}-X & a_{12} \dots & a_{1n} \\ a_{21} & a_{22}-X \dots & a_{2n} \\ \vdots & & \\ a_{n1} & a_{n2} \dots & a_{nn}-X \end{vmatrix} =$$

$$= (-1)^n \left[X^n - \sigma_1 X^{n-1} + \sigma_2 X^{n-2} - \dots + (-1)^n \sigma_n \right]$$

polinomul caracteristic asociat matricei A,

unde σ_k = suma minorilor diagonali de ordin k, $k = \overline{1,n}$

i.e. $\sigma_1 = Tr(A)$ (urma matricei)

$$\sigma_2 = \sum_{1 \le i < j \le n} \begin{vmatrix} a_{ii} & a_{ij} \\ a_{ji} & a_{jj} \end{vmatrix} \quad (C_n^2 \text{ minori în sumă})$$

$$\sigma_3 = \sum_{1 \le i < j < k \le n} \begin{vmatrix} a_{ii} & a_{ij} & a_{ik} \\ a_{ji} & a_{jj} & a_{jk} \\ a_{ki} & a_{kj} & a_{kk} \end{vmatrix} \quad (C_n^3 \text{ minori în sumă})$$

$\vdots$

$\sigma_n = \det(A)$

Cazuri particulare

1) $n = 2$

$$P_A(X) = \begin{vmatrix} a_{11}-X & a_{12} \\ a_{21} & a_{22}-X \end{vmatrix} = X^2 - \sigma_1 X + \sigma_2 = X^2 - Tr(A)\,X + \det(A)$$

2) $n = 3$

$$P_A(X) = \begin{vmatrix} a_{11}-X & a_{12} & a_{13} \\ a_{21} & a_{22}-X & a_{23} \\ a_{31} & a_{32} & a_{33}-X \end{vmatrix} =$$

$$= (-1)^3 \left[X^3 - \sigma_1 X^2 + \sigma_2 X - \sigma_3 \right]$$

$$\sigma_2 = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} + \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix} + \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} = Tr(A^*)$$

$\sigma_3 = \det(A)$, $\sigma_1 = Tr(A)$

<u>Teorema Hamilton-Cayley</u>

$\forall A \in \mathcal{M}_n(\mathbb{K})$

(*) $P_A(A) = (-1)^n \left[A^n - \sigma_1 A^{n-1} + \sigma_2 A^{n-2} - \dots + (-1)^n \sigma_n I_n \right] = O_n$

<u>Cazuri particulare</u>

1) $n=2$ $\quad P_A(A) = A^2 - \underset{Tr(A)}{\sigma_1} A + \underset{\det A}{\sigma_2} I_2 = O_2$

2) $n=3$ $\quad P_A(A) = -\left[A^3 - \underset{Tr(A)}{\sigma_1} A^2 + \underset{Tr(A^*)}{\sigma_2} A - \underset{\det(A)}{\sigma_3} I_3 \right] = O_3$

<u>Aplicații THC</u>

(1) Calculul lui A^{-1} $(\det A \neq 0)$

(*) $\Big|\cdot A^{-1} \Rightarrow A^{-1}$

<u>Exemplu</u>

$A = \begin{pmatrix} 1 & 0 & 1 \\ 2 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \Rightarrow A^{-1}$

Th H-C: $\quad A^3 - \sigma_1 A^2 + \sigma_2 A - \sigma_3 I_3 = O_3$

$\sigma_1 = Tr(A) = 3$

$\sigma_2 = \begin{vmatrix} 1 & 0 \\ 2 & 1 \end{vmatrix} + \begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix} + \begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix} = 1 + 0 + 0 = 1.$

$\sigma_3 = \begin{vmatrix} 1 & 0 & 1 \\ 2 & 1 & 1 \\ 1 & 1 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 \\ 2 & 1 & -1 \\ 1 & 1 & 0 \end{vmatrix} = 1 \neq 0$

$C_3' = C_3 - C_1$

$A^3 - 3A^2 + A - I_3 = O_3 \; \Big| \cdot A^{-1} \Rightarrow A^2 - 3A + I_3 - A^{-1} = O_3$

$$\Rightarrow A^{-1} = A^2 - 3A + I_3 = \begin{pmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & -1 & 1 \end{pmatrix}$$

② Calcul pt A^n.

Exemplu

$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$ $\quad A^2 - Tr(A)A + det(A)I_2 = O_2$

$A^2 = Tr(A)A - det(A)I_2$

$A^n = x_n A + y_n I_2, \forall n \geq 1$ $\quad x_1 = 1, y_1 = 0$

$$\begin{cases} x_2 = Tr(A) = \sigma_1 \\ y_2 = -det(A) = -\sigma_2 \end{cases}$$

$A^{n+1} = A^n \cdot A$

$$\underline{x_{n+1}} A + \underline{\underline{y_{n+1}}} \cdot I_2 = x_n A^2 + y_n A$$

$$= x_n(\sigma_1 A - \sigma_2 I_2) + y_n A$$

$$= \underline{(\sigma_1 x_n + y_n)} A - \underline{\underline{x_n \sigma_2}} I_2$$

$$\begin{cases} x_{n+1} = \sigma_1 x_n + y_n \\ y_{n+1} = -x_n \sigma_2 \Rightarrow y_n = -x_{n-1}\sigma_2 \end{cases}$$

$$\boxed{x_{n+1} - \sigma_1 x_n + x_{n-1}\sigma_2 = 0}$$ recurența de ord al 2-lea.

Exemplu

Fie $A = \begin{pmatrix} 1 & 0 \\ 1 & 2 \end{pmatrix}$, a) $A^n = x_n A + y_n I_2$, $x_n, y_n = ?$

$\sigma_1 = TrA = 3$, $\sigma_2 = detA = 2$.

$$\begin{cases} c_1 + 2c_2 = 1 \\ c_1 + 4c_2 = 3 \end{cases} \ominus \quad \begin{matrix} c_2 = 1 \\ c_1 = -1 \end{matrix}$$

$2c_2 = 2$

$x_{n+1} - 3x_n + 2x_{n-1} = 0, \forall n \geq 2$, $x_1 = 1$, $x_2 = 3$

Asociem ec. caracteristică : $t^2 - 3t + 2 = 0 \Rightarrow t_1 = 1, t_2 = 2$

$x_n = c_1 t_1^n + c_2 t_2^n = c_1 + c_2 \cdot 2^n$, $\forall n \in \mathbb{N}^*$, $x_n = -1 + 2^n$

$$y_n = -\sqrt{2}\cdot x_{n-1} = -2\left(\overbrace{-1+2^{n-1}}^{-4}\right) = 2-2^n$$

$$A^n = (-1+2^n)A + (2-2^n)I_2, \ \forall n \in \mathbb{N}^*$$

b) $B = A^4 + A^3 + A^2 + A + I_2$

Det $a, b \in \mathbb{R}$ aî $B = aA + bI_2$.

OBS $P = X^4 + X^3 + X^2 + X + 1$, $Q = X^2 - 3X + 2$.

$$P = (X-1)(X-2)C + \underbrace{aX+b}_{R} \qquad \widetilde{P} = P$$

$$\begin{cases} P(1) = a+b = 5 \\ P(2) = 2a+b = 2^4+2^3+2^2+2+1 = \dfrac{2^5-1}{2-1} = 31 \end{cases}$$

$a = 31-5 = 26$ $\qquad R = 26X - 21$

$b = 5 - 26 = -21$

$B = P(A) = aA + bI_2 = 26A - 21I_2$.

③ Rezolvare de ec. matriceale binome din $\mathcal{M}_2(\mathbb{C})$

Exemplu $X^4 = A = \begin{pmatrix} -1 & -2 \\ 1 & 2 \end{pmatrix}$

$X = ?$ $X \in \mathcal{M}_2(\mathbb{C})$

$\det A = 0 = \det(X^4) = (\det X)^4 = 0 \Rightarrow \det X = 0$

Th H-C: $X^2 - Tr(X)X + \underbrace{\det X}_{0} \cdot I_2 = O_2$

$X^2 = Tr(X)X$

Prop $X^2 = \alpha X$, $\forall X \in \mathcal{M}_n(\mathbb{C})$, $\forall \alpha \in \mathbb{C}$ $\Rightarrow X^k = \alpha^{k-1}X$, $\forall k \geq 2$

$\underbrace{X^4}_{A} = (Tr(X))^3 X \ | Tr \Rightarrow \underbrace{Tr(A)}_{1} = Tr(X)^3\, Tr(X) = Tr(X)^4$

$z^4 - 1 = 0 \Rightarrow z \in \{\pm 1; \pm i\}$

$X = \frac{1}{(Tr(X))^3} A$ $\qquad X_{1,2} = \pm A$

$X_{3,4} = \frac{1}{\pm i} A = \mp i A$

Teorema Laplace

Fie $A \in \mathcal{M}_n(\mathbb{K})$

a) minor de ordin p, $p \le n$

$$\det(A_{I,J}) = \begin{vmatrix} a_{i_1 j_1} & \dots & a_{i_1 j_p} \\ \vdots & & \\ a_{i_p j_1} & \dots & a_{i_p j_p} \end{vmatrix}, \quad \begin{array}{l} I = \{i_1, \dots, i_p\} \\ J = \{j_1, \dots, j_p\} \\ 1 \le i_1 < \dots < i_p \le n \\ 1 \le j_1 < \dots < j_p \le n \end{array}$$

b) minor complementar lui $\det(A_{I,J})$

$\det(A_{\bar{I},\bar{J}})$ $\quad \bar{I} = \{1, \dots, n\} \setminus \{i_1, \dots, i_p\}$

$\bar{J} = \{1, \dots, n\} \setminus \{j_1, \dots, j_p\}$

(se obține din A suprimând liniile $l_{i_1}, \dots, l_{i_p}$
coloanele $c_{j_1}, \dots, c_{j_p}$)

c) complementul algebric al minorului $\det(A_{I,J})$

$$\kappa = (-1)^{i_1 + \dots + i_p + j_1 + \dots + j_p} \det(A_{\bar{I},\bar{J}})$$

Teorema Laplace

$\det(A) =$ suma produselor minorilor de ordinul p cu complemenții algebrici corespunzător, pentru p linii fixate $l_{i_1}, \dots, l_{i_p}$

(respectiv p coloane fixate $c_{j_1}, \dots, c_{j_p}$)

$$\det(A) = \sum_{1 \le j_1 < \dots < j_p \le n} (-1)^{i_1 + \dots + i_p + j_1 + \dots + j_p} \det(A_{I,J}) \det(A_{\bar{I},\bar{J}})$$

$$= \sum_{1 \le i_1 < \ldots < i_p \le n} (-1)^{i_1+\ldots+i_p+j_1+\ldots+j_p} \det(A_{I,J}) \det(A_{\bar{I},\bar{J}})$$

<u>OBS</u> pt $p=1 \Rightarrow$ dezvoltarea determinantului de pe o linie, resp. o coloană.

<u>Exemplu</u>

$$A = \begin{pmatrix} 1 & 1 & 2 & 3 \\ 1 & 1 & 3 & 4 \\ 2 & 5 & 1 & -1 \\ -1 & -2 & 2 & 4 \end{pmatrix} \qquad \det A = ?\ (\text{Th. L})$$

$p = 2$, l_1, l_2 fixate

$$\det A = (-1)^{1+2+1+2} \begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix} \cdot \begin{vmatrix} 1 & -1 \\ 2 & 4 \end{vmatrix} + (-1)^{1+2+1+3} \begin{vmatrix} 1 & 2 \\ 1 & 3 \end{vmatrix} \begin{vmatrix} 5 & -1 \\ -2 & 4 \end{vmatrix}$$

$$+ (-1)^{1+2+1+4} \begin{vmatrix} 1 & 3 \\ 1 & 4 \end{vmatrix} \cdot \begin{vmatrix} 5 & 1 \\ -2 & 2 \end{vmatrix} + (-1)^{1+2+2+3} \begin{vmatrix} 1 & 2 \\ 1 & 3 \end{vmatrix} \cdot \begin{vmatrix} 2 & -1 \\ -1 & 4 \end{vmatrix}$$

$$+ (-1)^{1+2+2+4} \begin{vmatrix} 1 & 3 \\ 1 & 4 \end{vmatrix} \cdot \begin{vmatrix} 2 & 1 \\ -1 & 2 \end{vmatrix} + (-1)^{1+2+3+4} \begin{vmatrix} 2 & 3 \\ 3 & 4 \end{vmatrix} \cdot \begin{vmatrix} 2 & 5 \\ -1 & -2 \end{vmatrix}.$$

$$= -5$$

<u>Sisteme de ec. algebrice de ordin 1 cu mai multe necunoscute.</u>

$(\mathbb{K}, +, \cdot)$ corp com.

$\circledast$ $AX = B$ (sistem de m ecuații cu n necunoscute)

$A \in \mathcal{M}_{m,n}(\mathbb{K})$, $X \in \mathcal{M}_{n,1}(\mathbb{K})$, $B \in \mathcal{M}_{m,1}(\mathbb{K})$

$A = (a_{ij})_{\substack{i=\overline{1,m} \\ j=\overline{1,n}}}$ $\quad X = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$, $B = \begin{pmatrix} b_1 \\ \vdots \\ b_m \end{pmatrix}$

$S(A) = \{ \underset{(x_1, \ldots, x_n)}{x \in \mathbb{K}^n} \mid AX = B \}$ mult. soluțiilor pt $\circledast$

OBS

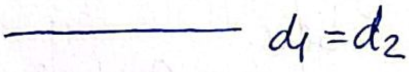
① $S(A) \neq \emptyset$ $\nearrow$ SCD (sol unică) ; $\searrow$ SCN (mai multe sol / o infinitate)

② $S(A) = \emptyset$ SI ($\nexists$ sol)

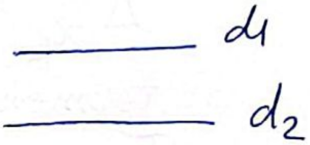
Cazuri particulare

1) $n = 2$ $\begin{cases} a_{11}x_1 + a_{12}x_2 = b_1 \\ a_{21}x_1 + a_{22}x_2 = b_2 \end{cases}$ $\cap$ 2 drepte în plan

SCD

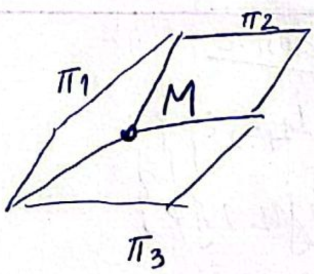


SCN

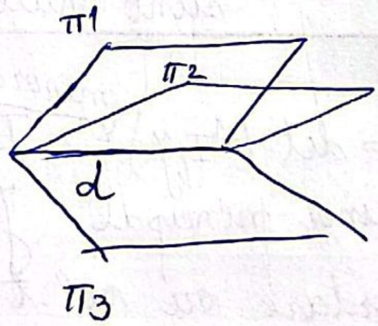


SI

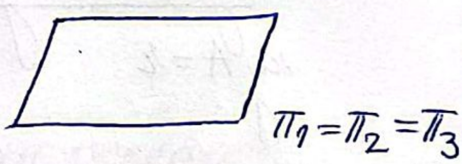
2) $n = 3$ $\begin{cases} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = b_1 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = b_2 \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = b_3 \end{cases}$ $\cap$ 3 plane în spațiu.



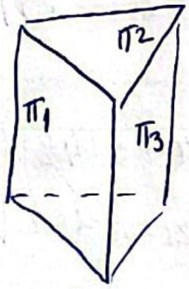
SCD



SC simplu N



SC dublu N.



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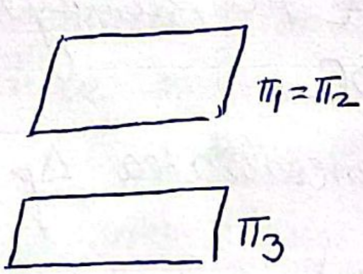
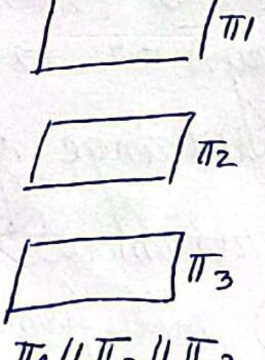


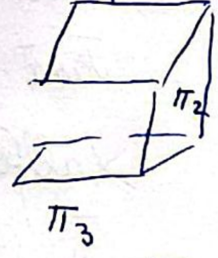
$\pi_1 \parallel \pi_3$

SI



$\pi_1 \parallel \pi_2 \parallel \pi_3$

SI



$\pi_1 \parallel \pi_3$

SI

Cazul general

$* \quad AX = B \qquad \bar{A} = (A|B)$ matricea extinsă.

- $n = m \quad A \in \mathcal{M}_n(\mathbb{K}) \quad , \; \Delta = \det A \neq 0 \Rightarrow$

$$\underbrace{A^{-1}AX}_{X} = A^{-1}B \qquad \Rightarrow (x_1, .., x_n) = \left(\frac{\Delta_{x_1}}{\Delta}, .., \frac{\Delta_{x_n}}{\Delta}\right)$$

(soluție unică – met Cra

Δ_{x_k} se obține înlocuind col. c_k cu col. termenilor liberi.

Teorema Kronecker - Capelli

$(*)$ este compatibil $\Leftrightarrow \operatorname{rg} A = \operatorname{rg} \bar{A}$

Teorema Rouché

$(*)$ este compatibil $\Leftrightarrow$ toți minorii caract (dc $\exists$)