Seminarul 4

imprimă

Repere. Coordonate. Operații. Subsp. vect

① $(\mathbb{R}^3, +, \cdot)/\mathbb{R}$, $R_0 = \{e_1, e_2, e_3\}$ reper canonic

$R' = \{e_1' = e_1 + 2e_2 + e_3,\ e_2' = e_1 + 7e_2 + e_3,\ e_3' = -e_1 + e_2 + e_3\}$

a) R' reper $\mathbb{R}^3$? $R_0 \xrightarrow{A} R'$, $A = ?$

b) $\text{coord}_{R'}(3, 2, 1) = ?$

sol | a) $A = \begin{pmatrix} 1 & 1 & -1 \\ 2 & 7 & 1 \\ 1 & 1 & 1 \end{pmatrix}$ $\det A = 7 - 2 + 1 + 7 - 1 - 2 = 10 \neq 0$

$\Rightarrow \text{rg}\, A = 3 = \text{nr. } \Rightarrow R'$ SLI $\Rightarrow$ (sist. liniar independent) R' SLI, card $R' = \dim$ $\Rightarrow$ SG

R_0, R' la fel or.; R_0 poz. or. (canon.)

b) $(3, 2, 1) = x = x_1' e_1' + x_2' e_2' + x_3' e_3' =$

$= x_1(1, 2, 1) + x_2(1, 7, 1) + x_3(-1, 1, 1) = \begin{cases} x_1 + x_2 - x_3 = 3 \\ 2x_1 + 7x_2 + x_3 = 2 \\ x_1 + x_2 + x_3 = 1 \end{cases}$

$\Delta = 10$, $\Delta_1 = \begin{vmatrix} 3 & 1 & -1 \\ 2 & 7 & 1 \\ 1 & 1 & 1 \end{vmatrix} = 21 - 2 + 1 + 7 - 3 - 2 = 22$

$\Delta_2 = \begin{vmatrix} 1 & 3 & -1 \\ 2 & 2 & 1 \\ 1 & 1 & 1 \end{vmatrix} = 2 - 2 + 3 + 2 - 1 - 6 = -2$

$\Delta_3 = \begin{vmatrix} 1 & 1 & 3 \\ 2 & 7 & 2 \\ 1 & 1 & 1 \end{vmatrix} = 7 + 6 + 2 - 21 - 2 - 2 = -10$

$\Rightarrow x_1 = \frac{\Delta_1}{\Delta} = \frac{22}{10}$, $x_2 = \frac{\Delta_2}{\Delta} = -\frac{2}{10}$, $x_3 = -1$

2. $(V, +, \cdot)/\mathbb{R}$ sp. vect. 3-dim

Fie $R = \{v_1, v_2, v_3\}$ reper în V

și $R' = \{v_1' = v_1,\ v_2' = v_1 + v_2,\ v_3' = v_1 + v_2 + v_3\}$

a) R' reper în V

$R \xrightarrow{A} R'$, $A = ?$

b) $v = (x_1, x_2, x_3)$ în rap cu R

v' în rap. cu $R' = ?$

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

$\left.\begin{array}{l} \text{rg}(A) = 3 \Rightarrow R' \text{ SLI} \\ |R'| = 3 = \dim_{\mathbb{R}} V \end{array}\right\} \Rightarrow R$ - reper

$$X = AX' \Rightarrow \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1' \\ x_2' \\ x_3' \end{pmatrix} = \begin{pmatrix} x_1' + x_2' + x_3' \\ x_2' + x_3' \\ x_3' \end{pmatrix}$$

$x_3' = x_3$

$x_2' = x_2 - x_3$ $\Rightarrow X' = \begin{pmatrix} x_1 - x_2 \\ x_2 - x_3 \\ x_3 \end{pmatrix}$

$x_1' = x_1 - x_2$

4. $(\mathbb{R}_3[X], +, \cdot)_{/\mathbb{R}}$

$V_1 = \{P \in \mathbb{R}_3[X] \mid P(0) = 0\}$

$V_2 = \{P \in \mathbb{R}_3[X] \mid P(1) = 0\}$

$V_3 = \{P \in \mathbb{R}_3[X] \mid P(0) = P(1) = 0\}$

a) V_i = subsp? $(i = \overline{1,3})$

$P = a_0 + a_1 X + a_2 X^2 + a_3 X^3$

$P = \tilde{P}$

$P(0) = 0 \Rightarrow a_0 = 0$

$\Rightarrow P = a_1 X + a_2 X^2 + a_3 X^3,\ Q = b_1 X + b_2 X^2 + b_3 X^3$

Fie $P, Q \in V_1 \Rightarrow P + Q = (a_1+b_1)X + (a_2+b_2)X^2 + (a_3+b_3)X^3 \in V_1$

$\alpha P = \alpha a_1 X + \alpha a_2 X^2 + \alpha a_3 X^3 \in V_1$

$V_1 = \{a_1 X + a_2 X^2 + a_3 X^3 \mid a_1, a_2, a_3 \in \mathbb{R}\} \subset \mathbb{R}_3[X]$

subsp. vect.

$P = a_1 X + a_2 X^2 + a_3 X^3 \in \langle \{X, X^2, X^3\} \rangle = V_1$ (SG) ①

$R_0 = \{1, X, X^2, X^3\}$ rep. canonic în $\mathbb{R}_3[X]$

$R_1 = \{X, X^2, X^3\} \subset R_0$ (SLI) $\Rightarrow R_1$ SLI ②

① + ② $\Rightarrow R_1$ rep. în V

$P \in V_2$

$\Rightarrow P(1) = 0 \Rightarrow a_0 + a_1 + a_2 + a_3 = 0$

$a_0 = -a_1 - a_2 - a_3$

$P = (-a_1 - a_2 - a_3) + a_1 X + a_2 X^2 + a_3 X^3$

$P = a_1(X-1) + a_2(X^2-1) + a_3(X^3-1) \in \langle \{X-1, X^2-1, X^3-1\} \rangle = R_2$

R_2 = sist. de generatori pt. V_2

Dem că R_2 = SLI

$R_0 = \{1, X, X^2, X^3\}$ rep. canonic pt. $\mathbb{R}_3[X]$

$R_2 = \{X-1, X^2-1, X^3-1\}$

$$\operatorname{rg}\begin{pmatrix} -1 & -1 & -1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = 3 = \text{maxim} \Rightarrow \left.\begin{matrix} R_2 = SLI \\ R_2 = SG \end{matrix}\right\} \Rightarrow R_2 = \text{reper în } V_2$$

$R_1 = \{X, X^2, X^3\}$

$R_2 = \{X-1, X^2-1, X^3-1\}$

$V_3 = \{P \in \mathbb{R}_3[X] \mid P(0) = P(1) = 0\}$

$P = a_0 + a_1 X + a_2 X^2 + a_3 X^2 = (-a_2 - a_3)X + a_2 X^2 + a_3 X^3 = a_2(X^2 - X) + a_3(X^3 - X) \in \langle \underbrace{\{X^2 - X, X^3 - X\}}_{R_3} \rangle$

$P(0) = 0 \Rightarrow a_0 = 0$

$P(1) = 0 \Rightarrow a_1 + a_2 + a_3 = 0 \Rightarrow a_1 = -a_2 - a_3$

$R_3 = SG$

$$\operatorname{rg}\begin{pmatrix} 0 & 0 \\ -1 & -1 \\ 1 & 0 \\ 0 & 1 \end{pmatrix} = 2 = \max. \Rightarrow R_3 = SLI$$

$\Rightarrow R_3$ = reper

c) $P_1 = X + 2X^2 + 3X^3 \in V_1$

$(1, 2, 3)$ - coord. în raport cu R_1

$P_2 = 1 + 2X^2 - 3X^3 \in V_2$

$= \underset{0}{\underset{\shortparallel}{a}}(X-1) + \underset{2}{\underset{\shortparallel}{b}}(X^2-1) + \underset{-3}{\underset{\shortparallel}{c}}(X^3-1)$

$(0, 2, -3)$ - coord. în raport cu R_2

$P_3 = X + 3X^2 - 4X^3 = 3(X^2 - X) - 4(X^3 - X)$

$(3, -4)$ - coord. în raport cu R_3

d) $R_1 = \{X, X^2, X^3\}$, $R_1 \cup \{1\}$ reper în $\mathbb{R}_3[X]$

$\mathbb{R}_3[X] = V_1 \oplus V_1'$, $V_1' = <\{1\}>$

$R_2 =$ reper în V_2

$R_2 = \{x-1, x^2-1, x^3-1\}$

$$\operatorname{rg}\left(\begin{array}{ccc|c} -1 & -1 & -1 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right) = 4 \Rightarrow V_2' = <\{1\}>$$

$R_3 = \{x^2 - x, x^3 - x\}$ - reper în V_3

$\mathbb{R}_3[X] = V_3 \oplus V_3'$

$$\left(\begin{array}{cc|cc} 0 & 0 & 0 & 1 \\ -1 & -1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{array}\right) = 4$$

$V_3' = <\{x^2, 1\}>$

e) $\mathbb{R}_3[X] = W_1 \oplus W_2 \oplus W_3$

Partiționăm reperul canonic în 3 submulțimi

$W_1 = <\{1, x\}>$

$\Rightarrow W_2 = <\{x^2\}>$

$W_3 = <\{x^3\}>$

$\mathbb{R}_3[X] = U_1 \oplus U_2 \oplus U_3 \oplus U_4$

$U_k = <\{x^{k-1}\}>$, $(\forall)\, k = \overline{1,4}$

Ex 6 $(\mathbb{R}^3,+,\cdot)|_{\mathbb{R}}$ $V' = \left\{ x \in \mathbb{R}^3 \;\middle|\; \begin{matrix} 2x_1 + x_2 = 0 \\ x_1 + 4x_3 = 0 \end{matrix} \right\}$

$= S(A)$.

a) reper în V'

b) subsp. compl. lui V' (i.e. $\mathbb{R}^3 = V' \oplus V''$)

c) desc. $x = (1,1,2)$ în rap. cu $\mathbb{R}^3 = V' \oplus V''$

a) $A = \begin{pmatrix} 2 & 1 \\ 1 & 0 \\ 0 & 4 \end{pmatrix}$ $\begin{cases} 2x_1 + x_2 = 0 \\ x = -4x_3 \end{cases} \Rightarrow x_2 = 8x_3$

$V' = \{(-4x_3, 8x_3, x_3) \mid x_3 \in \mathbb{R}\}$

$\parallel$

$R' = x_3(-4,8,1)$ SG pt. V' (1)

R' SLI (vector nenul) + (1) $\Rightarrow R'$ reper.

$\dim_{\mathbb{R}} V' = 1$

b) $\mathbb{R}^3 = V' \oplus V'' \Rightarrow \dim_{\mathbb{R}} V'' = 3-1 = 2$.

$rg \begin{pmatrix} -4 & 0 & 0 \\ 8 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} = 3 \Rightarrow V''$ sp. gen de $\{e_2, e_3\}$

$R' \cup \{e_2, e_3\}$ reper în $\mathbb{R}^3$

$(1,1,2) = a\underbrace{(-4,8,1)}_{v' \in V'} + \overbrace{b(0,1,0) + c(0,0,1)}^{v'' \in V''}$

$\Rightarrow \begin{cases} -4a = 1 \\ 8a + b = 1 \\ a + c = 2 \end{cases} \Rightarrow a = -\frac{1}{4},\ c = \frac{9}{4},\ b = 3$

$\Rightarrow v' = -\frac{1}{4}(-4,8,1) = (1,-2,-\frac{1}{4})$

$v'' = 3(0,1,0) + \frac{9}{4}(0,0,1) = (0,3,\frac{9}{4})$