TEMA 1, ANALIZA MATEMATICA I

1. Calculati (repetati cu cos inlocuit cu sin):
   a) $\bullet \cos(0)$ $\bullet \cos\left(\frac{\pi}{6}\right)$ $\bullet \cos\left(\frac{\pi}{4}\right)$ $\bullet \cos\left(\frac{\pi}{3}\right)$ $\bullet \cos\left(\frac{\pi}{2}\right)$ $\bullet \cos(\pi)$ $\bullet \cos\left(\frac{3\pi}{2}\right)$ $\bullet \cos(2\pi)$, $\cos(12\pi)$ $\bullet \cos(15\pi)$ $\bullet \cos\left(\frac{7\pi}{2}\right)$ $\bullet \cos\left(\frac{13\pi}{2}\right)$;
   b) $\bullet \cos\left(\frac{25\pi}{6}\right)$ $\bullet \cos\left(\frac{25\pi}{4}\right)$ $\bullet \cos\left(\frac{25\pi}{3}\right)$ $\bullet \cos\left(\frac{25\pi}{2}\right)$ $\bullet \cos\left(\frac{29\pi}{6}\right)$ $\bullet \cos\left(\frac{31\pi}{6}\right)$ $\bullet \cos\left(\frac{35\pi}{6}\right)$ $\bullet \cos\left(\frac{27\pi}{4}\right)$ $\bullet \cos\left(\frac{29\pi}{4}\right)$ $\bullet \cos\left(\frac{31\pi}{4}\right)$;
2. Calculati:
   a) $\bullet \int\limits_0^\pi \sin^2(9x)\,dx$ $\bullet \int\limits_0^\pi \cos^2(11x)\,dx$ $\bullet \int\limits_0^\pi \sin(6x)\sin(10x)\,dx$ $\bullet \int\limits_0^\pi \cos(4x)\cos(13x)\,dx$;
   b) $\bullet \int\limits_{-\pi}^{\pi} \cos(4x)\cos(11x)\,dx$ $\bullet \int\limits_{-\pi}^{\pi} \sin(5x)\sin(15x)\,dx$ $\bullet \int\limits_{-\pi}^{\pi} \cos^2(17x)\,dx$;
   c) $\bullet \int\limits_0^{\pi/2} \cos(8x)\cos(14x)\,dx$ $\bullet \int\limits_0^{\pi/2} \sin(15x)\sin(3x)\,dx$ $\bullet \int\limits_0^{\pi/2} \sin^2(22x)\,dx$;
3. Calculati:
   a) $\sum\limits_{n=0}^{\infty} \frac{1}{12^n}$, $\sum\limits_{n=0}^{\infty} \frac{(-1)^n}{6^n}$, $\sum\limits_{n=0}^{\infty} \left(-\frac{1}{16}\right)^n$, $\sum\limits_{n=0}^{\infty} \left(\frac{i}{7}\right)^n$, $\sum\limits_{n=0}^{\infty} \frac{1}{14^{2n}}$, $\sum\limits_{n=0}^{\infty} \left(-\frac{1}{19i}\right)^n$, $\sum\limits_{n=2}^{\infty} \frac{1}{8^n}$, $\sum\limits_{n=1}^{\infty} \frac{(-1)^n}{9^n}$, $\sum\limits_{n=3}^{\infty} \frac{4^{2n}}{5^{2n}}$;
   b) Calculati s_n si $\lim\limits_{n\to\infty} s_n$ pentru $s_n = \sum\limits_{k=1}^{n} \frac{1}{2^k}$, $s_n = \sum\limits_{k=2}^{n} \frac{(-1)^k}{3^k}$, $s_n = \sum\limits_{k=3}^{n+2} \frac{1}{5^{2k}}$, $s_n = \sum\limits_{k=4}^{n+1} \frac{(-1)^k}{9^{2k}}$.
4. Sa se studieze variatia functiilor pe domeniul indicat:

   a) $f(x) = \frac{\ln x}{x}$, $x \in (0,\infty)$;
   b) $f(x) = \cosh x = \frac{e^x+e^{-x}}{2}$, $x \in \mathbb{R}$;
   c) $f(x) = \sinh x = \frac{e^x-e^{-x}}{2}$, $x \in \mathbb{R}$;
   d) $f(x) = \begin{cases} \frac{\sin x}{x}, & x \in (0,\pi/2] \\ 1, & x = 0, \end{cases}$
   e) $f(x) = \begin{cases} \frac{\ln x}{x-1}, & x \in (0,\infty)\setminus\{1\} \\ 1, & x = 1, \end{cases}$
   f) $f(x) = 3x^4-8x^3-6x^2+24x-1$, $x \in \mathbb{R}$;
   g) $f(x) = x^3-\frac{9}{2}x^2+6x-1$, $x \in \mathbb{R}$;
   h) $f(x) = 2\sin x + \sin(2x)$, $x \in [0,2\pi)$;
   i) $f(x) = \begin{cases} 4-\frac{x^2}{2}, & x \in [0,2], \\ \frac{x}{2}+1, & x \in [2,\infty). \end{cases}$

5. Sa se studieze monotonia si marginirea urmatoarelor siruri:

   a) $x_n = \sum\limits_{k=0}^{n} \frac{1}{k!}$
   b) $x_n = \sum\limits_{k=1}^{n} \frac{1}{k(k+1)}$
   c) $x_n = \sum\limits_{k=1}^{n} \frac{1}{k^2}$
   d) $x_n = \sum\limits_{k=1}^{n} \frac{k+1}{k^3}$
   e) $x_n = \sum\limits_{k=1}^{n} \frac{(-1)^k}{k(k+1)}$
   f) $x_n = \sum\limits_{k=1}^{n} \frac{\sin k}{k}$

6. Criteriul Cesaro-Stolz:
   a) Calculati folosind criteriul Cesaro-Stolz $\lim\limits_{n\to\infty} \frac{1+\frac{1}{2}+\cdots+\frac{1}{n}}{\ln n}$.

b) $x_n = \frac{\sum_{k=1}^{n} k^4}{n^2(4n^2-1)}$;

c) $x_n = \frac{\sum\limits_{k=0}^{n}(-1)^k\sqrt{(n-k)^2+3}}{\sum\limits_{k=2}^{n}\ln k}$

d) $x_n = \frac{\sum\limits_{k=1}^{n}\frac{1}{\sqrt{k}}}{\sum\limits_{k=2}^{n}\ln k}$

e) $x_n = \frac{\sum\limits_{k=1}^{n}(2k)^4}{n^5}$

f) $x_n = \frac{n}{2^n}$

g) $x_n = \frac{\sum\limits_{k=1}^{n}\frac{1}{k}}{n}$

h) $x_n = \frac{\sum\limits_{k=1}^{n}\frac{1}{\sqrt{k}}}{\sqrt{n}}$

i) $x_n = \frac{1}{n}\left(\frac{a+b}{c+d} + \frac{a\sqrt{2}+b}{c\sqrt{2}+d} + \cdots + \frac{a\sqrt{n}+b}{c\sqrt{n}+d}\right)$, $a, b > 0$

j) $x_n = \frac{1}{n}\left(\frac{1}{\ln 2} + \frac{1}{\ln 3} + \cdots + \frac{1}{\ln n}\right)$

k) $x_n = \frac{1}{n}(\ln 2 + \ln 3 + \cdots + \ln n)$

l) $x_n = \frac{1^p+2^p+\cdots+n^p}{n^{p+1}}$, $p \in \mathbb{N}$

m) $x_n = \frac{n^3}{a^n}$

n) $x_n = \frac{1^2+4^2+7^2+\cdots+(3n+1)^2}{2n^3+5}$

o) $x_n = \frac{1+\frac{1}{\sqrt[3]{2}}+\frac{1}{\sqrt[3]{3}}+\cdots+\frac{1}{\sqrt[3]{n}}}{\sqrt[3]{n}}$

p) Aratati ca $\lim\limits_{n\to\infty} x_n = x$ implica $\lim\limits_{n\to\infty}\frac{x_1+x_2+\cdots+x_n}{n} = x$.

q) Fie $x_n > 0$ si $\lim\limits_{n\to\infty} x_n = x > 0$. Aratati ca $\lim\limits_{n\to\infty}\sqrt[n]{x_1x_2\cdots x_n} = x$.

r) Fie $c_n > 0$ cu $\lim\limits_{n\to\infty}\frac{c_{n+1}}{c_n} = c > 0$. Demonstrati ca $\lim\limits_{n\to\infty}\sqrt[n]{c_n} = c$. Construiti un contraexemplu care sa arate ca implicatia inversa nu este in general valabila. Folosind aceasta consecinta a criteriului Cesaro-Stolz, calculati $\lim\limits_{n\to\infty} c_n$ cu $c_n = \frac{\sqrt[n]{n!}}{n}$, $c_n = \sqrt[n]{n}$, $c_n = \sqrt[n]{n^8+2n^3+3}$, $c_n = \sqrt[n]{\frac{\arctan(1/n)}{(2n)!}}$, $c_n = \frac{\sqrt[n]{(n^2+1)(n^2+2)\cdots(2n^2)}}{n}$, $c_n = \sqrt[n]{(n+1)^{n+1}\sin 1\sin\frac{1}{2}\cdots\sin\frac{1}{n}}$.

s) Daca $y_n > 0$ este un sir cu $\lim\limits_{n\to\infty}(y_1+y_2+\cdots+y_n) = \infty$ si x_n e un sir de numere reale cu limita x, atunci $\lim\limits_{n\to\infty}\frac{\sum\limits_{k=1}^{n}x_ky_k}{\sum\limits_{k=1}^{n}y_k} = x$.

7. Puneti sub forma trigonometrica z si calculati z la puterea indicata:

a) $z = -1+i$, $z^{24} =?$, $z^{17} =?$

b) $z = \frac{1}{\sqrt{2}}(1-i)$, $z^{20} =?$

c) $z = -3+3i$, $z^{20} =?$

d) $z = -\sqrt{3}+i$, $z^{22} =?$, $z^{25} =?$

e) $z = -\frac{1}{\sqrt{2}} - \frac{i}{\sqrt{2}}$, $z^{28} =?$

f) $z = -1-i\sqrt{3}$, $z^{15} =?$

g) $z = -1+i\sqrt{3}$, $z^{11} =?$

h) $z = \frac{1}{2} - i\frac{\sqrt{3}}{2}$, $z^{26} =?$

i) $z = -\frac{\sqrt{3}}{2} - i\frac{1}{2}$, $z^{19} =?$

j) $z = \frac{\sqrt{3}}{2} - \frac{1}{2}i$, $z^{34} =?$

k) $z = -\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i$, $z^{16} =?$

l) $z = \sqrt{3}-i$, $z^{17} =?$

m) $z = 1-i$, $z^{18} =?$

n) $z = 20$, $z^{15} =?$

o) $z = -13$, $z^{17} =?$

p) $z = 10i$, $z^{101} =?$

q) $z = -9i$, $z^{13} =?$

r) $z = 1+\cos(\alpha)+i\sin(\alpha)$, $z^{53} =?$.

8. Siruri de numere complexe. Calculati:

a) Limita sirului $x_n = \frac{n^2}{2-n^2}$, $x_n = \sqrt{n+2}-\sqrt{n}$, $x_n = \frac{\sqrt{n^2+4}-\sqrt{n^2+1}}{\sqrt{n^2+9}-\sqrt{n^2+2}}$, $x_n = \sqrt[3]{n^3+an^2+1} - \sqrt[3]{n^3+bn^2+1}$;

b) $\lim\limits_{n\to\infty}\left(\frac{n^3-5n+3}{9n^3-2n+4} + i\cos\left(\frac{6n}{3n^2+2n+5}\right)\right)$, $\lim\limits_{n\to\infty}\left[\left(1+\frac{7}{n}\right)^n + i3^n\sin\left(\frac{1}{3^n}\right)\right]$, $\lim\limits_{n\to\infty}\left[\left(1-\frac{2}{n}\right)^n + i\cos\left(\frac{5}{n}\right)\right]$, $\lim\limits_{n\to\infty}\left[\sqrt{n+3}-\sqrt{n}+i\cos\left(\frac{\pi}{6^n}\right)\right]$,

$\lim\limits_{n\to\infty}\left[\sqrt[n]{n}+i9^n\sin\left(\frac{1}{9^n}\right)\right]$, $\lim\limits_{n\to\infty}\left(\frac{n^3+2}{15n^3-3}+i\cos\left(\frac{\pi n^2}{4n^2+2n+1}\right)\right)$,
$\lim\limits_{n\to\infty}\left[\sqrt{n+4}-\sqrt{n}+in\sin\left(\frac{12}{n}\right)\right]$, $\lim\limits_{n\to\infty}\left[\sqrt[n]{n+2}+in\sin\left(\frac{2}{n}\right)\right]$
$\lim\limits_{n\to\infty}\left(\frac{2n}{1+n^2}+i\frac{1-n^2}{1+n^2}\right)$, $\lim\limits_{n\to\infty}\frac{n}{n-i}$, $\lim\limits_{n\to\infty}\frac{2^n-3^ni}{3^n+4^ni}$,
$\lim\limits_{n\to\infty}\frac{2^n-3^ni}{3^n+4i}$, $\lim\limits_{n\to\infty}\left(\cos\left(\frac{1}{n}\right)+in\sin\left(\frac{1}{n}\right)\right)$, $\lim\limits_{n\to\infty}\left(1+\frac{z}{3n^2+2}\right)^{n^2}$, $z\in\mathbb{C}$,
$\lim\limits_{n\to\infty}\left(\frac{1+\sin\alpha+i\cos\alpha}{2}\right)^n$, $\alpha\in[0,2\pi]$.

9. Studiati convergenta seriei folosind CRITERIUL INTEGRAL:

   !a) $S=\sum\limits_{n=2}^{\infty}\frac{1}{n(\ln n)^p}$, $p>0$;
   b) $S=\sum\limits_{n=2}^{\infty}\frac{\ln n}{n^2}$;
   !c) $S=\sum\limits_{n=1}^{\infty}\frac{1}{n^p}$, $p>0$;

10. Studiati convergenta SERIILOR ARMONICE GENERALIZATE: !$S=\sum\limits_{n=1}^{\infty}\frac{1}{n}$; !$S=\sum\limits_{n=1}^{\infty}\frac{1}{\sqrt{n}}$; !$S=\sum\limits_{n=1}^{\infty}\frac{1}{n^6}$; $S=\sum\limits_{n=1}^{\infty}a^{\ln n}$, $a>0$;
11. Studiati convergenta seriei folosind CRITERIUL DE COMPARATIE:

   !a) $S=\sum\limits_{n=1}^{\infty}\frac{1}{7n^3-1}$;
   !b) $S=\sum\limits_{n=1}^{\infty}\frac{1}{\sqrt[3]{8n^5-1}}$;
   !c) $S=\sum\limits_{n=0}^{\infty}\frac{\sin(2n^2)}{4^n}$;
   d) $S=\sum\limits_{n=1}^{\infty}\ln\left(1+\frac{1}{n^p}\right)$, $p>0$;
   e) $S=\sum\limits_{n=1}^{\infty}\frac{1}{n}\sin\left(\frac{1}{n^p}\right)$, $p>0$;
   !f) $S=\sum\limits_{n=1}^{\infty}\frac{1}{\sqrt[3]{4n^3-1}}$;
   g) $S=\sum\limits_{n=1}^{\infty}\frac{\sqrt{n}}{\sqrt{n^5+n+1}}$;
   h) $S=\sum\limits_{n=1}^{\infty}\frac{n}{(\sqrt{n+1}+\sqrt{n})^p}$, $p>0$;
   !i) $\sum\limits_{n=1}^{\infty}\frac{1}{n^{p+\frac{1}{n}}}$, $p>0$;
   j) $S=\sum\limits_{n=1}^{\infty}\sin\left(\frac{1}{n^3}\right)$;
   k) $S=\sum\limits_{n=1}^{\infty}\frac{1}{n}(\sqrt{n+1}-\sqrt{n})^p$, $p>0$;
   l) $S=\sum\limits_{n=1}^{\infty}\frac{1}{n^p}\tan\left(\frac{1}{n}\right)$, $p>0$;

12. Studiati convergenta seriei folosind CRITERIUL RAPORTULUI:

   !a) $S=\sum\limits_{n=1}^{\infty}z^n\frac{n!}{n^n}$;
   !b) $S=\sum\limits_{n=0}^{\infty}\frac{z^n}{n!}$, $z\in\mathbb{C}$;
   !c) $S=\sum\limits_{n=0}^{\infty}\frac{1}{n!}$;
   !d) $S=\sum\limits_{n=1}^{\infty}\frac{6^n}{n!}$;
   !e) $S=\sum\limits_{n=1}^{\infty}2^n\frac{n!}{n^n}$;
   !f) $S=\sum\limits_{n=0}^{\infty}\frac{7^n}{(n+6)!}$;
   g) $S=\sum\limits_{n=1}^{\infty}(-1)^n\frac{2^n\sin n}{n!}$;
   h) $S=\sum\limits_{n=1}^{\infty}\frac{n+1}{n!}$;
   i) $S=\sum\limits_{n=1}^{\infty}\frac{n^2+n+1}{n!}$;
   j) $S=\sum\limits_{n=1}^{\infty}\frac{2^n(n+1)}{n!}$;
   k) $S=\sum\limits_{n=1}^{\infty}\frac{(2n-1)!!}{2^{2n}(n!)^2}$;
   l) $S=\sum\limits_{n=1}^{\infty}\frac{(n!)^2}{(2n)!}$;

m) $S = \sum_{n=1}^{\infty} n^2 \sin\left(\frac{\pi}{2^n}\right)$;

n) $S = \sum_{n=1}^{\infty} a^n \tan\left(\frac{\pi}{2^{n+1}}\right)$, $a > 0$;

o) $S = \sum_{n=1}^{\infty} \frac{z^n}{n^2+1}$, $z \in \mathbb{C}$;

13. Studiati convergenta seriei folosind CRITERIUL RADICALULUI:

!a) $S = \sum_{n=1}^{\infty} \frac{z^n}{n^p}$, $p > 0$, $z \in \mathbb{C}$;

!b) $S = \sum_{n=1}^{\infty} a^n \left(\frac{n}{n+1}\right)^{n^2}$, $a \in \mathbb{R}$;

c) $S = \sum_{n=1}^{\infty} \frac{n}{3^n}$;

d) $S = \sum_{n=1}^{\infty} \left(\frac{3n^2+1}{2n^2+5}\right)^n$;

e) $S = \sum_{n=1}^{\infty} (\arctan n)^{-n}$;

14 Studiati convergenta/convergenta absoluta a seriei folosind CRITERIUL LUI LEIBNIZ:

!a) $S = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{\sqrt{n}}$;

!b) $S = \sum_{n=2}^{\infty} (-1)^n \frac{\ln n}{n}$;

!c) $S = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{\sqrt[6]{n}}$;

!d) $S = \sum_{n=2}^{\infty} \frac{(-1)^n}{\ln n}$. Estimati numarul de termeni necesari calcularii sumei cu doua zecimale exacte;

e) $S = \sum_{n=1}^{\infty} (-1)^n \tan\left(\frac{\pi}{3n}\right)$;

f) $S = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{\arctan n}$;

15. Studiati convergenta seriei folosind CRITERIUL LUI DIRICHLET:

!a) $S = \sum_{n=1}^{\infty} \frac{\cos(n\alpha)}{n}$, $\alpha \in \mathbb{R}$;

!b) $S = \sum_{n=1}^{\infty} \frac{\sin(n\alpha)}{n}$, $\alpha \in \mathbb{R}$;

c) $S = \sum_{n=1}^{\infty} \frac{n \sin n}{\sqrt{n^3+4}}$;

d) $S = \sum_{n=1}^{\infty} \frac{\sin n \sin(1/n)}{\sqrt{n}}$;

e) $S = \sum_{n=1}^{\infty} \frac{\sin^2(3n)}{\sqrt{n}}$;

f) $S = \sum_{n=1}^{\infty} \frac{\sin^3 n}{n^p}$, p¿0;

16. Sa se determine $\overline{A}$, A', $\mathring{A}$ si $\mathrm{Fr}(A)$ pentru:

!a) $A = \mathbb{N}$;
!b) $A = [1, 9) \cup \{10, 12\}$;
!c) $A = [-3, 2) \cup (5, 7)$;
d) $A = \{1/n, n \in \mathbb{N}^*\}$;
e) $A = \mathbb{Q}$;
f) $A = \mathbb{Z}$;
g) $A = (-1, 2] \cup \{5\}$;
h) $A = [0, 2] \times (1, 4]$;
i) $A = \{x^2 + y^2 < 9\}$;
j) $A = \{x^2 + y^2 \le 25\}$;
k) $A = \{|x| > 1 \text{ sau } |y| > 2\}$;
l) $A = (\mathbb{Q} \cap [0, 1]) \times (\mathbb{Q} \cap [-1, 1])$;
m) $A = \{x^2 + y^2 < 4\} \cap \{y \ge x^2\}$;
n) $A = \{x + y = 1, x < 5\}$;
o) $A = \{x^2/4 + y^2 \le 1\} \cup \{x^2 + y^2 \le 9/4\}$.

17. Calculati normele infinit: $\|\cos x\|_\infty$ in $C([0, \pi/4])$; $\|x^2 - x^3\|_\infty$ in $C([0, 1])$; $\|\sin x\|_\infty$ in $C([0, \pi/3])$; $\|xe^{-x}\|_\infty$ in $C([0, 2])$; $\|\sin(2x) - \sin(x)\|_\infty$ in $C([0, \pi/2])$; $\|x - \ln x\|_\infty$ in $C([1/e, e])$; $\|x^4 - x^2\|_\infty$ in $C([0, 1])$; $\|\sin(nx)/n\|_\infty$ in $C([0, \pi])$.

18. Sa se studieze convergenta uniforma a sirului:
   !a) $f_n : [0,1] \to \mathbb{R},\ f_n(x) = x(1-x)^n$;
   !b) $f_n : [0,1] \to \mathbb{R},\ f_n(x) = x^2(1-x)^n$;
   !c) $f_n : [0,1] \to \mathbb{R},\ f_n(x) = \frac{n+x+1}{nx+n+1}$;
   !d) $f_n : [0,1] \to \mathbb{R},\ f_n(x) = \frac{2nx}{1+n^2x^2}$;
   !e) $f_n : [0,\pi/2] \to \mathbb{R},\ f_n(x) = \cos\left(\frac{x}{n}\right)$;
   !f) $f_n : [0,1] \to \mathbb{R},\ f_n(x) = \frac{n^2x}{n^2+x}$;
   g) $f_n : [0,1] \to \mathbb{R},\ f_n(x) = \frac{x^n}{1+x^{2n}}$;
   h) $f_n : [0,1] \to \mathbb{R},\ f_n(x) = (1-x)^n \sin(nx)$;
   i) $f_n : [0,1] \to \mathbb{R},\ f_n(x) = x^n + x + 1$;
   j) $f_n : [-1,1] \to \mathbb{R},\ f_n(x) = \frac{n\sin(x/n)+x+1}{4\sin(x^2/n)+n+1}$;
   k) $f_n : [-1,1] \to \mathbb{R},\ f_n(x) = \frac{nx^n+1}{x^n+n}$;
   l) $f_n : \mathbb{R} \to \mathbb{R},\ f_n(x) = \frac{x+1}{nx^2+1}$;
   m) $f_n : [1,2] \to \mathbb{R},\ f_n(x) = \frac{n}{n+x}$;
   n) $f_n : [1,\infty] \to \mathbb{R},\ f_n(x) = \frac{n}{n+x}$;
19. Sa se studieze convergenta uniforma pentru seriile de functii:
   a) $S(x) = \sum_{n=1}^{\infty} \frac{1}{n^2+x^2},\ x \in \mathbb{R}$;
   b) $S(x) = \sum_{n=1}^{\infty} \frac{1}{(x+n)(x+n+1)},\ x \in [0,\infty)$;
   c) $S(x) = \sum_{n=1}^{\infty} \arctan\left(\frac{2x}{x^2+n^4}\right),\ x \in \mathbb{R}$;
   d) $S(x) = \sum_{n=1}^{\infty} (1-x)x^n,\ x \in \mathbb{R}$;
   e) $S(x) = \sum_{n=1}^{\infty} \frac{\sin(nx)}{n^2},\ x \in \mathbb{R}$.

1. Cateva elemente de teorie:

- enuntati axioma Cantor-Dedekind; enuntati axioma lui Arhimede;
- enuntati un criteriu ce foloseste comparatia cu o serie geometrica;
- enuntati criteriul Cesaro-Stolz si consecintele lui;
- definiti convergenta si convergenta absoluta pentru serii;
- enuntati teorema lui Riemann relativa la convergenta neabsoluta si schimbarea ordinii de sumare;
- definiti notiunea de sir Cauchy in $\mathbb{R}$; definiti seria produs;
- enuntati testul/criteriul radical (criteriul Cauchy); enuntati testul raportului (criteriul lui d'Alembert) pentru serii; enuntati criteriul de comparatie; enuntati criteriul lui Dirichlet; enuntati criteriul lui Leibniz;
- definiti numarul e si demonstrati convergenta sirurilor care il definesc;
- aratati ca daca $\sum\limits_{n=1}^{\infty} x_n$ este convergenta, atunci $\lim\limits_{n\to\infty} x_n = 0$;
- Enunt criteriul integral. Aplicati-l la studiul seriilor armonice generalizate.

AXIOMA LUI ARHIMEDE: Multimea $\mathbb{N}$ nu e marginita superior in $\mathbb{R}$, adica oricare ar fi $x \in \mathbb{R}$, exista $n \in \mathbb{N}$ astfel incat $n > x$.

AXIOMA DE COMPLETITUDINE CANTOR-DEDEKIND: Daca $I_n = [a_n, b_n]$ este o familie numarabila de intervale INCHISE din $\mathbb{R}$ si $I_{n+1} \subset I_n$, atunci $\bigcap\limits_{n=1}^{\infty} I_n \neq \emptyset$.

CRITERIU CE FOLOSESTE COMPARATIA CU O SERIE GEOMETRICA: In demonstratia criteriilor raportului sau al radicalului se foloseste majorarea sau minorarea cu serii geometrice legate de limita L, deci la acest subiect enuntati fie criteriul raportului, fie pe cel al radicalului.

CRITERIUL CESARO-STOLZ SI CONSECINTELE LUI. Enuntul criteriului: Fie $(a_n)_n$ si $(b_n)_n$ doua siruri de numere reale cu proprietatile:

a) Sirul $(b_n)_n$ este strict monoton si nemarginit.
b) Exista $L = \lim\limits_{n\to\infty} \frac{a_{n+1}-a_n}{b_{n+1}-b_n}$ (L poate fi inclusiv $\pm\infty$).

Atunci exista si limita sirului raport a_n/b_n si, in plus, $\lim\limits_{n\to\infty} \frac{a_n}{b_n} = L$.

Consecintele criteriului C-S:

1. Daca $\lim\limits_{n\to\infty} x_n = x$, atunci $\lim\limits_{n\to\infty} \frac{x_1+x_2+\cdots+x_n}{n} = x$.
2. Daca $x_n > 0$ si $\lim\limits_{n\to\infty} x_n = x > 0$, atunci $\lim\limits_{n\to\infty} \sqrt[n]{x_1 x_2 \cdots x_n} = x$.
3. Daca $c_n > 0$ cu $\lim\limits_{n\to\infty} \frac{c_{n+1}}{c_n} = c > 0$, atunci $\lim\limits_{n\to\infty} \sqrt[n]{c_n} = c$.
4. Daca $y_n > 0$ este un sir cu $\lim\limits_{n\to\infty} (y_1 + y_2 + \cdots + y_n) = \infty$ si x_n e un sir de numere reale cu limita x, atunci $\lim\limits_{n\to\infty} \frac{\sum\limits_{k=1}^{n} x_k y_k}{\sum\limits_{k=1}^{n} y_k} = x$.

CONVERGENTA SI CONVERGENTA ABSOLUTA PENTRU SERII: O serie $S = \sum\limits_{n=1}^{\infty} x_n$ se numeste convergenta daca sirul sumelor partiale asociate seriei $S_N =$

$\sum_{n=1}^{N} x_n$ este convergent catre o limita finita cand $N \to \infty$. Seria $S = \sum_{n=1}^{\infty} x_n$ se numeste absolut convergenta daca seria modulelor $S' = \sum_{n=1}^{\infty} |x_n|$ este convergenta.

TEOREMA LUI RIEMANN relativa la convergenta neabsoluta si schimbarea ordinii de sumare: Fie $S = \sum_{n=0}^{\infty} x_n$ o serie de numere reale, CONVERGENTA NEABSOLUT (adica S este convergenta, dar nu e absolut). Atunci:

a) Exista o bijectie $\sigma : \mathbb{N} \to \mathbb{N}$ astfel incat $\sum_{n=0}^{\infty} x_{\sigma(n)}$ este divergenta;

b) Oricare ar fi $a \in \mathbb{R}$, exista o bijectie $\sigma : \mathbb{N} \to \mathbb{N}$ astfel incat $\sum_{n=0}^{\infty} x_{\sigma(n)} = a$.

SIR CAUCHY IN $\mathbb{R}$: Sirul de numere reale $(x_n)_n$ se numeste sir Cauchy (fundamental) daca oricare ar fi $\epsilon > 0$, exista un rang $N(\epsilon) \in \mathbb{N}$ astfel incat oricare ar fi $n > N(\epsilon)$ si $p \geq 1$ sa avem $|x_{n+p} - x_n| < \epsilon$ (adica sirul diferenta $|x_{n+p} - x_n|$ converge la 0 cand $n \to \infty$ pentru fiecare $p \geq 1$).

SERIE PRODUS: Seria $P = \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} a_k b_{n-k} \right)$ se numeste serie produs a seriilor $A = \sum_{n=0}^{\infty} a_n$ si $B = \sum_{n=0}^{\infty} b_n$. Daca seriile A si B sunt absolut convergente, atunci seria produs P este absolut convergenta si $P = AB$.

CRITERIUL/TESTUL RADICALULUI (CAUCHY): Fie $S = \sum_{n=1}^{\infty} z_n$ o serie de numere complexe pentru care exista $L = \lim_{n\to\infty} \sqrt[n]{|z_n|}$. Atunci:

- Daca $L < 1$, seria S este absolut convergenta;
- Daca $L > 1$, seria S este divergenta;
- Daca $L = 1$, testul radicalului nu da natura seriei S si trebuie facuta analiza suplimentara folosind alte rezultate.

CRITERIUL/TESTUL RAPORTULUI (D'ALEMBERT): Fie $S = \sum_{n=1}^{\infty} z_n$ o serie de numere complexe pentru care exista $L = \lim_{n\to\infty} \frac{|z_{n+1}|}{|z_n|}$. Atunci:

- Daca $L < 1$, seria S este absolut convergenta;
- Daca $L > 1$, seria S este divergenta;
- Daca $L = 1$, testul raportului nu da natura seriei S si trebuie facuta analiza suplimentara folosind alte rezultate.

CRITERIUL DE COMPARATIE LA LIMITA: Fie $S = \sum_{n=1}^{\infty} |x_n|$ si $T = \sum_{n=1}^{\infty} y_n$ doua serii cu termeni pozitivi astfel incat sa existe $L = \lim_{n\to\infty} \frac{|x_n|}{y_n}$. Atunci:

- Daca $L \in (0, \infty)$, seriile S si T au aceeasi natura.
- Daca $L = 0$ si T convergenta, atunci S este convergenta.
- Daca $L = \infty$ si T divergenta, atunci si S este divergenta.

CRITERIUL LUI LEIBNIZ: Fie seria alternanta $S = \sum\limits_{n=1}^{\infty} (-1)^{n+1} x_n$, cu $x_n > 0$, $(x_n)_n$ sir descrescator si tinzand la 0 cand $n \to \infty$. Atunci S este serie convergenta.

CRITERIUL LUI DIRICHLET: Fie seria de produse $S = \sum\limits_{n=1}^{\infty} y_n x_n$, cu

i) $(x_n)_n$ sir descrescator si tinzand la 0 cand $n \to \infty$;
ii) Exista $M > 0$ astfel incat $|y_1 + y_2 + \cdots + y_n| \leq M$, pentru orice $n \in \mathbb{N}^*$.

Atunci S este convergenta.

NUMARUL e. Definim e ca limita a sirurilor de numere rationale $x_n = \left(1 + \frac{1}{n}\right)^n$ si $y_n = \sum\limits_{k=0}^{n} \frac{1}{k!}$, adica $e = \lim\limits_{n\to\infty} x_n = \lim\limits_{n\to\infty} y_n$.

Demonstram ca sirul x_n este crescator si marginit.

Folosind binomul lui Newton,

$$\begin{aligned} x_n &= \left(1 + \frac{1}{n}\right)^n = \sum_{k=0}^{n} C_n^k \left(\frac{1}{n}\right)^k = 2 + \sum_{k=2}^{n} C_n^k \left(\frac{1}{n}\right)^k \\ &= 2 + \sum_{k=2}^{n} \frac{n(n-1)(n-2)\cdots(n-k+1)}{n^k k!} \\ &= 2 + \sum_{k=2}^{n} \left(1 - \frac{1}{n}\right)\left(1 - \frac{2}{n}\right)\cdots\left(1 - \frac{k-1}{n}\right)\frac{1}{k!}. \end{aligned} \tag{1}$$

Similar,

$$\begin{aligned} x_{n+1} &= 2 + \sum_{k=2}^{n+1} \left(1 - \frac{1}{n+1}\right)\left(1 - \frac{2}{n+1}\right)\cdots\left(1 - \frac{k-1}{n+1}\right)\frac{1}{k!} \\ &= 2 + \sum_{k=2}^{n} \left(1 - \frac{1}{n}\right)\left(1 - \frac{2}{n}\right)\cdots\left(1 - \frac{k-1}{n}\right)\frac{1}{k!} + \frac{1}{(n+1)^n}. \end{aligned}$$

Obtinem $x_{n+1} > x_n$ deoarece pentru fiecare $m = \overline{1, k-1}$ avem $1 - \frac{m}{n+1} > 1 - \frac{m}{n}$, adica x_n sir strict crescator.

Prin urmare, $x_1 = 2 < x_2 < \cdots < x_n$. Cum fractiile $1 - m/n$ sunt subunitare pentru fircare $m = \overline{1, n-1}$, putem majora coeficientii lui $1/k!$ in (1) cu 1, prin urmare,

$$x_n < 2 + \sum_{k=2}^{n} \frac{1}{k!} = y_n. \tag{2}$$

Se poate arata usor ca $y_{n+1} - y_n = 1/(n+1)! > 0$, deci y_n este strict crescator, deci $y_1 = 2 < y_2 < \cdots < y_n$.

De asemenea, $y_n = 2 + \sum\limits_{k=2}^{n} \frac{1}{k!} < z_n = 2 + \sum\limits_{k=2}^{n} \frac{1}{k(k-1)} = 2 + 1 - 1/n < 3$ (z_n se poate scrie ca o suma telescopica deoarece $\frac{1}{k(k-1)} = \frac{1}{k-1} - \frac{1}{k}$). Am aratat ca y_n monoton si marginit, deci convergent. Din marginirea lui y_n si (2), rezulta ca si x_n este monoton si marginit, deci convergent.

ARATATI CA DACA $\sum_{n=1}^{\infty} x_n$ ESTE CONVERGENTA, ATUNCI $\lim_{n\to\infty} x_n = 0$.

Cum seria $\sum_{n=1}^{\infty} x_n$ este convergenta, sirul sumelor partiale $S_N = \sum_{n=1}^{N} x_n$ este convergent la o limita finita S cand N tinde la infinit. Observam ca $x_N = S_N - S_{N-1}$, prin urmare, $\lim_{N\to\infty} x_N = \lim_{N\to\infty}(S_N - S_{N-1}) = S - S = 0$.

CRITERIUL INTEGRAL. Enunt: Fie $f : [n_0, \infty) \to \mathbb{R}_+$ o functie descrescatoare si $x_n = f(n)$. Atunci seria $S = \sum_{n=n_0}^{\infty} x_n$ este serie convergenta daca si numai daca sirul integralelor $I_n = \int_{n_0}^{n} f(x)\,dx$ este marginit.

Aplicatie pentru studiul convergentei seriei armonice generalizate de exponent $p > 0$, $S = \sum_{n=1}^{\infty} \frac{1}{n^p}$.

In acest caz, consideram $f(x) = \frac{1}{x^p} : [1, \infty) \to \mathbb{R}_+$.

Calculam $f'(x) = \frac{-p}{x^{p+1}}$ si observam ca $f' < 0$ pe $[1, \infty)$, deci f descrescatoare.

Calculam

$$I_n = \begin{cases} \int_1^n \frac{1}{x^p}\,dx = \int_1^n x^{-p}\,dx = \left.\frac{x^{-p+1}}{-p+1}\right|_1^n = \frac{n^{-p+1}}{-p+1} - \frac{1}{-p+1}, & p \neq 1 \\ \ln x\Big|_1^n = \ln n, & p = 1. \end{cases} \tag{3}$$

• Daca $p = 1$, $I_n = \ln n$ tinde la infinit cand $n \to \infty$, deci I_n nemarginit si seria $S = \sum_{n=1}^{\infty} \frac{1}{n}$ este divergenta.

• Daca $p < 1$, atunci n^{-p+1} din (3) inseamna n la o putere pozitiva, deci I_n tinde la infinit, I_n nemarginit si seria $S = \sum_{n=1}^{\infty} \frac{1}{n^p}$ este divergenta.

• Daca $p > 1$, atunci n^{-p+1} din (3) inseamna n la o putere negativa care tinde la 0 cand $n \to \infty$, deci I_n tinde la $1/(p-1)$, I_n marginit si seria $S = \sum_{n=1}^{\infty} \frac{1}{n^p}$ este convergenta.