Universitatea din Bucureşti
Facultatea de Matematică şi Informatică

Elemente de Calcul Ştiinţific
13–17/05/2024

CURS#10

12. Norme matriciale: definiţii; proprietăţi; norma matricială $\|\cdot\|_2$.
13. Descompunerea valorilor singulare (DVS): existenţa DVS; interpretare geometrică.

PROBLEME

1) Arătaţi că pentru orice $p \in \mathbb{N}^* \cup \{+\infty\}$, are loc inegalitatea

$$\|\mathbf{AB}\|_p \le \|\mathbf{A}\|_p \|\mathbf{B}\|_p, \quad \forall\ \mathbf{A} \in \mathscr{M}_{m,q}(\mathbb{R}), \quad \forall\ \mathbf{B} \in \mathscr{M}_{q,n}(\mathbb{R}).$$

2) Arătaţi că pentru orice $p \in \mathbb{N}^* \cup \{+\infty\}$ şi orice submatrice $\mathbf{B} \in \mathscr{M}_{q,r}(\mathbb{R})$ a matricei $\mathbf{A} \in \mathscr{M}_{m,n}(\mathbb{R})$, unde $1 \le q \le m$ şi $1 \le r \le n$, are loc inegalitatea

$$\|\mathbf{B}\|_p \le \|\mathbf{A}\|_p.$$

3) Fie $\mathbf{D} = \operatorname{diag}(\mu_1, \mu_2, \ldots, \mu_k) \in \mathscr{M}_{m,n}(\mathbb{R})$, unde $k \coloneqq \min\{m, n\}$. Arătaţi că pentru orice $p \in \mathbb{N}^* \cup \{+\infty\}$, $\|\mathbf{D}\|_p = \max\limits_{i=\overline{1,k}} |\mu_k|$.

4) Arătaţi că au loc inegalităţile

$$\|\mathbf{A}\|_2 \le \|\mathbf{A}\|_F \le \sqrt{n}\|\mathbf{A}\|_2, \quad \forall\ \mathbf{A} \in \mathscr{M}_{m,n}(\mathbb{R}),$$
$$\|\mathbf{A}\|_\triangle \le \|\mathbf{A}\|_2 \le \sqrt{mn}\|\mathbf{A}\|_\triangle, \quad \forall\ \mathbf{A} \in \mathscr{M}_{m,n}(\mathbb{R}),$$

unde

$$\|\mathbf{A}\|_\triangle \coloneqq \max_{\substack{i=\overline{1,m}\\ j=\overline{1,n}}} |a_{ij}|, \quad \forall\ \mathbf{A} = \big(a_{ij}\big)_{\substack{i=\overline{1,m}\\ j=\overline{1,n}}} \in \mathscr{M}_{m,n}(\mathbb{R}).$$

5) Arătaţi că au loc identităţile

$$\|\mathbf{A}\|_1 = \max_{j=\overline{1,n}} \sum_{i=1}^{m} |a_{ij}|, \quad \forall\ \mathbf{A} \in \mathscr{M}_{m,n}(\mathbb{R}),$$
$$\|\mathbf{A}\|_\infty = \max_{i=\overline{1,m}} \sum_{j=1}^{n} |a_{ij}|, \quad \forall\ \mathbf{A} \in \mathscr{M}_{m,n}(\mathbb{R}).$$

6) Arătaţi că au loc inegalităţile

$$\frac{1}{\sqrt{n}}\|\mathbf{A}\|_2 \le \|\mathbf{A}\|_1 \le \sqrt{m}\|\mathbf{A}\|_2, \quad \forall\ \mathbf{A} \in \mathscr{M}_{m,n}(\mathbb{R}),$$
$$\frac{1}{\sqrt{n}}\|\mathbf{A}\|_\infty \le \|\mathbf{A}\|_2 \le \sqrt{m}\|\mathbf{A}\|_\infty, \quad \forall\ \mathbf{A} \in \mathscr{M}_{m,n}(\mathbb{R}),$$
$$\frac{1}{\sqrt{mn}}\|\mathbf{A}\|_1 \le \|\mathbf{A}\|_\infty \le \sqrt{mn}\|\mathbf{A}\|_1, \quad \forall\ \mathbf{A} \in \mathscr{M}_{m,n}(\mathbb{R}).$$

7) Arătaţi că are loc identitatea

$$\left\|\mathbf{E} - \left(\mathbf{I}_n - \frac{\mathbf{v}\mathbf{v}^\mathsf{T}}{\mathbf{v}^\mathsf{T}\mathbf{v}}\right)\right\|_F^2 = \|\mathbf{E}\|_F^2 - \frac{\|\mathbf{E}\mathbf{v}\|_2^2}{\|\mathbf{v}\|_2^2}, \quad \forall\ \mathbf{E} \in \mathscr{M}_n(\mathbb{R}), \quad \forall\ \mathbf{v} \in \mathbb{R}^n \setminus \{\mathbf{0}_n\}.$$

8) Pentru orice $\mathbf{u} \in \mathbb{R}^m$ şi orice $\mathbf{v} \in \mathbb{R}^n$, definim $\mathbf{E} \coloneqq \mathbf{u}\mathbf{v}^\mathsf{T} \in \mathscr{M}_{m,n}(\mathbb{R})$.

Arătaţi că au loc identităţile

$$\|\mathbf{E}\|_F = \|\mathbf{E}\|_2 = \|\mathbf{u}\|_2\|\mathbf{v}\|_2,$$
$$\|\mathbf{E}\|_\infty \le \|\mathbf{u}\|_\infty\|\mathbf{v}\|_1.$$

9) Fie $\mathbf{A} \in \mathscr{M}_{m,n}(\mathbb{R})$, $\mathbf{y} \in \mathbb{R}^m$ şi $\mathbf{x} \in \mathbb{R}^n \setminus \{\mathbf{0}_n\}$. Definim

$$\mathbf{E} \coloneqq (\mathbf{y} - \mathbf{A}\mathbf{x})\mathbf{x}^\mathsf{T} / (\mathbf{x}^\mathsf{T}\mathbf{x}) \in \mathscr{M}_{m,n}(\mathbb{R}).$$

Demonstraţi că

$$\mathbf{E} = \arg\min\left\{\|\mathbf{B}\|_2 \;\middle|\; \mathbf{B} \in \mathscr{M}_{m,n}(\mathbb{R}),\ (\mathbf{A} + \mathbf{B})\mathbf{x} = \mathbf{y}\right\}.$$

10) Pentru orice $\mathbf{A} \in \mathscr{M}_{m,n}(\mathbb{R})$, au loc identităţile

$$\|\mathbf{QAZ}\|_F = \|\mathbf{A}\|_F, \quad \forall\ \mathbf{Q} \in \mathscr{M}_m(\mathbb{R}):\ \mathbf{Q}^\mathsf{T}\mathbf{Q} = \mathbf{I}_m, \quad \forall\ \mathbf{Z} \in \mathscr{M}_n(\mathbb{R}):\ \mathbf{Z}^\mathsf{T}\mathbf{Z} = \mathbf{I}_n,$$
$$\|\mathbf{QAZ}\|_2 = \|\mathbf{A}\|_2, \quad \forall\ \mathbf{Q} \in \mathscr{M}_m(\mathbb{R}):\ \mathbf{Q}^\mathsf{T}\mathbf{Q} = \mathbf{I}_m, \quad \forall\ \mathbf{Z} \in \mathscr{M}_n(\mathbb{R}):\ \mathbf{Z}^\mathsf{T}\mathbf{Z} = \mathbf{I}_n.$$

NORME MATRICIALE

__Motivație__: Anumite metode numerice de rezolvare a unui sistem liniar pot fi inacurate pt. matrice aproape singulare. Pt a cuantifica această noțiune este nevoie de introducerea distanței pe spațiul matricelor, i.e. norme matriciale.

1. DEFINIȚII

Cum $M_{m,n}(\mathbb{R}) \simeq \mathbb{R}^{mn}$, definiția normei matriciale pe $M_{m,n}(\mathbb{R})$ trebuie să fie echivalentă cu cea a normei vectoriale.

__DEFINIȚIE:__

$f : M_{m,n}(\mathbb{R}) \longrightarrow \mathbb{R}$ s.n. normă matricială dacă satisface următoarele proprietăți:

(i) $f(A) \geq 0$, $\forall A \in M_{m,n}(\mathbb{R})$ și

$f(A) = 0 \iff A = 0_{m,n}$;

(ii) $f(A+B) \le f(A) + f(B)$, $\forall A, B \in \mathcal{M}_{m,n}(\mathbb{R})$;

(iii) $f(\alpha A) = |\alpha| f(A)$, $\forall A \in \mathcal{M}_{m,n}(\mathbb{R})$, $\forall \alpha \in \mathbb{R}$.

Similar cu cazul normei vectoriale, și în cazul normei matriciale notăm

$$\|A\| := f(A)$$

OBS:

În algebra computațională, cele mai folosite norme matriciale sunt:

- norma Frobenius

$$\boxed{\|A\|_F := \left(\sum_{i=1}^{m} \sum_{j=1}^{n} |a_{ij}|^2\right)^{1/2}, \ \forall A \in \mathcal{M}_{m,n}(\mathbb{R})}$$

- norma p, $p \in \mathbb{N}^* \cup \{\infty\}$

$$\boxed{\|A\|_p := \sup_{\substack{x \in \mathbb{R}^n \\ x \neq 0_n}} \frac{\|Ax\|_p}{\|x\|_p}, \ \forall A \in \mathcal{M}_{m,n}(\mathbb{R})}$$

unde

$\underline{p \in \mathbb{N}^*}$: $\quad \|\underline{x}\|_p := \left(\sum_{j=1}^{n} |x_j|^p\right)^{1/p}$ $\quad , \forall \underline{x} \in \mathbb{R}^n$

$\underline{p = \infty}$: $\quad \|\underline{x}\|_p := \max_{j=\overline{1,n}} |x_j|$

Norma p sau <u>norma matricială</u> $\|\cdot\|_p$ pe $\mathcal{M}_{m,n}(\mathbb{R})$ <u>indusă de normele vectoriale</u> $\|\cdot\|_p$ pe $\mathbb{R}^m$ și $\mathbb{R}^n$ (subordonată normelor vectoriale pe $\mathbb{R}^m$ și $\mathbb{R}^n$).

<u>OBS:</u>

$$\boxed{\begin{aligned} \|A\|_p &:= \sup_{\substack{\underline{x} \in \mathbb{R}^n \\ \underline{x} \neq \underline{0}_n}} \left\|A\left(\frac{\underline{x}}{\|\underline{x}\|_p}\right)\right\|_p \\ &= \sup_{\substack{\underline{x} \in \mathbb{R}^n: \\ \|\underline{x}\|_p = 1}} \|A\underline{x}\|_p = \max_{\substack{\underline{x} \in \mathbb{R}^n: \\ \|\underline{x}\|_p = 1}} \|A\underline{x}\|_p \end{aligned}}$$

$\{\underline{x} \in \mathbb{R}^n \mid \|\underline{x}\|_p = 1\} \subset \mathbb{R}^n$ <u>compact</u>

$\|\cdot\|_p : \mathbb{R}^m \longrightarrow [0, \infty)$ <u>continuă</u>

DEFINIȚIE:

Normele matriciale $f_1 : \mathcal{M}_{m,n}(\mathbb{R}) \to \mathbb{R}$, $f_2 : \mathcal{M}_{m,q}(\mathbb{R}) \to \mathbb{R}$ și $f_3 : \mathcal{M}_{q,n}(\mathbb{R}) \to \mathbb{R}$ sn <u>consistente</u> dacă

(iv) $$\boxed{\begin{array}{l} f_1(AB) \le f_2(A)\, f_3(B), \\ \forall A \in \mathcal{M}_{m,q}(\mathbb{R}),\ \forall B \in \mathcal{M}_{q,n}(\mathbb{R}) \end{array}}$$

Proprietatea (iv) sn <u>submultiplicativitate</u>.

OBS:

Nu toate normele matriciale satisfac proprietatea de submultiplicativitate!

<u>EX</u>: $\|A\|_\Delta := \max\limits_{\substack{i=\overline{1,m} \\ j=\overline{1,n}}} |a_{ij}|$ normă pe $\mathcal{M}_{m,n}(\mathbb{R})$

$$A = B = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \Rightarrow AB = \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix}$$

$$\|A\|_\Delta = \|B\|_\Delta = 1, \quad \|AB\|_\Delta = 2.$$

OBS: Normele matriciale $\|\cdot\|_p$, unde $p \in \mathbb{N}^* \cup \{+\infty\}$, satisfac proprietatea

$$\boxed{\|A\underline{x}\|_p \leq \|A\|_p \, \|\underline{x}\|_p, \quad \forall A \in \mathcal{M}_{m,n}(\mathbb{R}), \; \forall \underline{x} \in \mathbb{R}^n}$$

2. PROPRIETĂȚI ALE NORMELOR MATRICIALE

$\forall A \in \mathcal{M}_{m,n}(\mathbb{R})$,

(i) $\|A\|_2 \leq \|A\|_F \leq \sqrt{n}\,\|A\|_2$

(ii) $\|A\|_\Delta \leq \|A\|_2 \leq \sqrt{mn}\,\|A\|_\Delta$

(iii) $\frac{1}{\sqrt{n}}\|A\|_2 \leq \|A\|_1 \leq \sqrt{m}\,\|A\|_2$

(iv) $\frac{1}{\sqrt{n}}\|A\|_\infty \leq \|A\|_2 \leq \sqrt{m}\,\|A\|_\infty$

(v) $\frac{1}{\sqrt{mn}}\|A\|_1 \leq \|A\|_\infty \leq \sqrt{mn}\,\|A\|_1$

(vi) $\boxed{\|A\|_1 = \max_{j=\overline{1,n}} \sum_{i=1}^{m} |a_{ij}|}$

(vii) $\boxed{\|A\|_\infty = \max_{i=\overline{1,m}} \sum_{j=1}^{n} |a_{ij}|}$

3. NORMA MATRICIALĂ $\|\cdot\|_2$

<u>**TEOREMA #1**</u> (norma matricială $\|\cdot\|_2$):

$\forall A \in \mathcal{M}_{m,n}(\mathbb{R})$, $\boxed{\|A\|_2^2 = \max\limits_{\substack{\lambda \text{ valoare proprie} \\ \text{a lui } A^TA}} \lambda}$

<u>Dem:</u>

Fie $\underline{z} \in \mathbb{R}^n$, $\|\underline{z}\|_2 = 1$, a.î. $\|A\underline{z}\|_2 = \|A\|_2$.

Atunci

$$\underline{z} := \arg \max_{\substack{\underline{x} \in \mathbb{R}^n: \\ \underline{x} \neq \underline{0}_n}} \frac{1}{2} \frac{\|A\underline{x}\|_2^2}{\|\underline{x}\|_2^2}$$

$$= \arg \max_{\substack{\underline{x} \in \mathbb{R}^n: \\ \underline{x} \neq \underline{0}_n}} \frac{1}{2} \frac{(A\underline{x})^T A\underline{x}}{\underline{x}^T \underline{x}}$$

Definim

$$g: \mathbb{R}^n \longrightarrow \mathbb{R}, \quad g(\underline{x}) := \frac{1}{2} \frac{\|A\underline{x}\|_2^2}{\|\underline{x}\|_2^2} \Longrightarrow$$

$$\Longrightarrow \nabla g(\underline{z}) = \underline{0}_n \in \mathbb{R}^n$$

$$\frac{\partial}{\partial x_i} g(\underline{x}) = \frac{1}{2(\underline{x}^T\underline{x})^2}\left[\frac{\partial}{\partial x_i}(\underline{x}^T A^T A \underline{x})(\underline{x}^T\underline{x}) - \right.$$

$$\left. - (\underline{x}^T A^T A \underline{x}) \frac{\partial}{\partial x_i}(\underline{x}^T\underline{x})\right]$$

- $$\frac{\partial}{\partial x_i}(\underline{x}^T A^T A \underline{x}) = \frac{\partial}{\partial x_i}\left[\sum_{j=1}^{n} x_j \left(\sum_{k=1}^{n} (A^T A)_{jk} x_k\right)\right]$$

$$= \sum_{j=1}^{n} \frac{\partial x_j}{\partial x_i}\left(\sum_{k=1}^{n} (A^T A)_{jk} x_k\right) +$$

$$+ \sum_{j=1}^{n} x_j \left(\sum_{k=1}^{n} (A^T A)_{jk} \frac{\partial x_k}{\partial x_i}\right)$$

$$= \sum_{j=1}^{n} \delta_{ij}\left(\sum_{k=1}^{n} (A^T A)_{jk} x_k\right)$$

$$+ \sum_{j=1}^{n} x_j \left(\sum_{k=1}^{n} (A^T A)_{jk} \delta_{ik}\right)$$

$$= \sum_{k=1}^{n} (A^T A)_{ik} x_k + \sum_{j=1}^{n} x_j (A^T A)_{ji}$$

$$= 2 \sum_{k=1}^{n} (A^T A)_{ik} x_k$$

$\bullet\ \dfrac{\partial}{\partial x_i}(\underline{x}^T\underline{x}) = \dfrac{\partial}{\partial x_i}\left(\sum_{k=1}^{n} x_k^2\right) =$

$= \sum_{k=1}^{n} \dfrac{\partial x_k}{\partial x_i} x_k + \sum_{k=1}^{n} x_k \dfrac{\partial x_k}{\partial x_i} = 2\sum_{k=1}^{n} x_k \delta_{ki}$

$= 2x_i$

Am obținut:

$$\frac{\partial g}{\partial x_i} = \frac{1}{2(\underline{x}^T\underline{x})^2}\Big[\Big(2\sum_{k=1}^{n}(A^TA)_{ik}x_k\Big)(\underline{x}^T\underline{x})$$

$$- (\underline{x}^T A^T A \underline{x})\, 2x_i\Big]$$

$$= \frac{1}{\underline{x}^T\underline{x}}\left[\sum_{k=1}^{n}(A^TA)_{ik}x_k - \left(\frac{\underline{x}^TA^TA\underline{x}}{\underline{x}^T\underline{x}}\right)x_i\right]$$

$$= \frac{1}{\underline{x}^T\underline{x}}\left[A^TA\underline{x} - \left(\frac{\underline{x}^TA^TA\underline{x}}{\underline{x}^T\underline{x}}\right)\underline{x}\right]_i$$

$$= \frac{1}{\|\underline{x}\|_2^2}\left(A^TA\underline{x} - \frac{\|A\underline{x}\|_2^2}{\|\underline{x}\|_2^2}\,\underline{x}\right)_i, \quad i=\overline{1,n} \Rightarrow$$

$$\nabla g(\underline{x}) = \underline{0}_n \iff \frac{\partial g}{\partial x_i}(\underline{x}) = 0,\ i=\overline{1,n}$$

$$\Longleftrightarrow \frac{1}{\|\underline{z}\|_2^2}\left(A^T A\,\underline{z} - \frac{\|A\underline{z}\|_2^2}{\|\underline{z}\|_2^2}\,\underline{z}\right)_i = 0,\ i=\overline{1,n}$$

$$\Longleftrightarrow \left((A^T A)\,\underline{z} - \underbrace{\|A\underline{z}\|_2^2}_{=\|A\|_2^2 =: \lambda}\,\underline{z}\right)_i = 0,\ i=\overline{1,n}$$

$$\Longleftrightarrow (A^T A)\,\underline{z} = \lambda\,\underline{z}$$

$$\underline{z}\in\mathbb{R}^n : \|\underline{z}\|_2 = 1$$

$$\|A\underline{z}\|_2 = \|A\|_2 =: \sqrt{\lambda}$$

Prin urmare, $\lambda = \|A\underline{z}\|_2^2 = \|A\|_2^2$ este valoare proprie a lui $A^T A$ corespunzătoare vectorului propriu unitar $\underline{z}\in\mathbb{R}^n$. Cf definiției $\|A\|_2$, rezultă afirmația.

$\square$

<u>COROLAR #1:</u>

$$\boxed{\|A\|_2 \le \sqrt{\|A\|_1\,\|A\|_\infty},\ \forall A\in\mathcal{M}_{m,n}(\mathbb{R})}$$

<u>Dem</u>: Fie $\underline{z}\in\mathbb{R}^n$, $\|\underline{z}\|_2=1$, a.î.

$$A^T A\,\underline{z} = \lambda\,\underline{z} \quad \text{și} \quad \lambda := \|A\|_2^2 \Longrightarrow$$

$$\| \lambda \underline{z} \|_1 = \| A^T A \underline{z} \|_1 \implies$$

$$\lambda \cancel{\| \underline{z} \|_1} \leq \| A^T \|_1 \, \| A \|_1 \, \cancel{\| \underline{z} \|_1} \implies$$

$$\| A \|_2^2 \leq \| A^T \|_1 \, \| A \|_1 = \| A \|_\infty \, \| A \|_1 \implies$$

$$\| A \|_2 \leq \sqrt{\| A \|_1 \, \| A \|_\infty} \qquad \square$$

3.6. DESCOMPUNEREA VALORILOR SINGULARE (DVS)

TEOREMA #1 (DVS):

$\forall A \in \mathcal{M}_{m,n}(\mathbb{R})$, $r := \min\{m,n\}$,

$\exists U \in \mathcal{M}_m(\mathbb{R}) : U^T U = I_m$

$\exists V \in \mathcal{M}_n(\mathbb{R}) : V^T V = I_n$

$\exists \Sigma := \operatorname{diag}(\sigma_1, \sigma_2, \dots, \sigma_r) \in \mathcal{M}_{m,n}(\mathbb{R})$,

cu $\sigma_1 \geq \sigma_2 \geq \dots \geq \sigma_r \geq 0$

$$\boxed{A = U \Sigma V^T} \qquad (1)$$

DEFINIȚIE:

Numerele $\sigma_1 \geq \sigma_2 \geq \dots \geq \sigma_r \geq 0$ sn <u>valorile singulare</u> ale matricei $A \in \mathcal{M}_{m,n}(\mathbb{R})$.

Descompunerea (1) sn <u>descompunerea valorilor singulare</u> (DVS) pentru matricea $A \in \mathcal{M}_{m,n}(\mathbb{R})$.

<u>Dem</u>: Folosim <u>exclusiv</u> teoria normei matriciale $\|\cdot\|_2$!

- $A \in \mathcal{M}_{m,n}(\mathbb{R}) \setminus \{0_{m,n}\} \Longrightarrow$

$$\|A\|_2 := \max_{\substack{\underline{x} \in \mathbb{R}^n: \\ \|\underline{x}\|_2 = 1}} \|A\underline{x}\|_2 =: \sigma_1 > 0 \Longrightarrow$$

Fie $\underline{x} \in \mathbb{R}^n$, $\|\underline{x}\|_2 = 1$, şi $\underline{y} \in \mathbb{R}^m$, $\|\underline{y}\|_2 = 1$, a.î.

$$\boxed{A\underline{x} = \sigma_1 \underline{y}, \quad \sigma_1 = \|A\|_2}$$

- Definim matricele (folosind metoda Gram-Schmidt):

$$\begin{cases} U := [\underline{y} \ \ U_1] \in \mathcal{M}_m(\mathbb{R}), \ U_1 \in \mathcal{M}_{m,m-1}(\mathbb{R}): \\ \qquad U^T U = I_m \\ V := [\underline{x} \ \ V_1] \in \mathcal{M}_n(\mathbb{R}), \ V_1 \in \mathcal{M}_{n,n-1}(\mathbb{R}): \\ \qquad V^T V = I_n \end{cases} \Longrightarrow$$

$$\begin{bmatrix} 1 & \underline{0}_{m-1}^T \\ \underline{0}_{m-1} & I_{m-1} \end{bmatrix} = I_m = U^T U = \begin{bmatrix} \underline{y}^T \\ U_1^T \end{bmatrix} [\underline{y} \ \ U_1]$$

$$= \begin{bmatrix} y^T y & y^T U_1 \\ U_1^T y & U_1^T U_1 \end{bmatrix} = \begin{bmatrix} 1 & \underline{0}_{m-1} \\ \underline{0}_{m-1} & I_{m-1} \end{bmatrix} \Rightarrow$$

$$\begin{cases} U_1^T U_1 = I_{m-1}, \text{ i.e. } U_1 \in \mathcal{M}_{m,m-1}(\mathbb{R}) \\ U_1^T y = \underline{0}_{m-1} \qquad \underline{\text{ortogonală}} \end{cases}$$

Analog, obținem:

$$\begin{cases} V_1^T V_1 = I_{n-1}, \text{ i.e. } V_1 \in \mathcal{M}_{n,n-1}(\mathbb{R}) \\ V_1^T \underline{x} = \underline{0}_{n-1} \qquad \underline{\text{ortogonală}} \end{cases}$$

Prin urmare, am obținut:

$$\begin{cases} \text{span}(\text{col } U_1) = (\text{span } y)^\perp \\ \text{span}(\text{col } V_1) = (\text{span } \underline{x})^\perp \end{cases}$$

- Astfel, rezultă că:

$$U^T A V = \begin{bmatrix} y^T \\ U_1^T \end{bmatrix} A \,[\underline{x} \; V_1] = \begin{bmatrix} y^T \\ U_1^T \end{bmatrix} [A\underline{x} \; AV_1]$$

$$= \begin{bmatrix} y^T(A\underline{x}) & y^T(AV_1) \\ U_1^T(A\underline{x}) & U_1^T(AV_1) \end{bmatrix}$$

$$= \begin{bmatrix} \underline{y}^T(\sigma_1 \underline{y}) & \underline{y}^T A V_1 \\ U_1^T(\sigma_1 \underline{y}) & U_1^T A V_1 \end{bmatrix}$$

$$= \begin{bmatrix} \sigma_1 (\underline{y}^T \underline{y}) & (V_1^T A^T \underline{y})^T \\ \sigma_1 (U_1^T \underline{y}) & U_1^T A V_1 \end{bmatrix} = \begin{bmatrix} \sigma_1 & \underline{w}^T \\ \underline{0}_{m-1} & B \end{bmatrix} =: A_1$$

unde

$$\begin{cases} \underline{w} := V_1^T A^T \underline{y} \in \mathbb{R}^{n-1} \\ B := U_1^T A V_1 \in \mathcal{M}_{m-1,n-1}(\mathbb{R}) \end{cases}$$

Atunci, obținem:

$$A_1 \begin{bmatrix} \sigma_1 \\ \underline{w} \end{bmatrix} = \begin{bmatrix} \sigma_1 & \underline{w}^T \\ \underline{0}_{m-1} & B \end{bmatrix} \begin{bmatrix} \sigma_1 \\ \underline{w} \end{bmatrix} = \begin{bmatrix} \sigma_1^2 + \underline{w}^T \underline{w} \\ B\underline{w} \end{bmatrix} \in \mathbb{R}^m \Rightarrow$$

$$\left\| A_1 \begin{bmatrix} \sigma_1 \\ \underline{w} \end{bmatrix} \right\|_2^2 = (\sigma_1^2 + \|\underline{w}\|_2^2)^2 + \|B\underline{w}\|_2^2 \geq$$

$$\geq (\sigma_1^2 + \|\underline{w}\|_2^2)^2$$

$$= \left\| \begin{bmatrix} \sigma_1 \\ \underline{w} \end{bmatrix} \right\|_2^2 (\sigma_1^2 + \|\underline{w}\|_2^2) \geq \left\| \begin{bmatrix} \sigma_1 \\ \underline{w} \end{bmatrix} \right\|_2^2 \sigma_1^2$$

cu egalitate $\Longleftrightarrow$ $\underline{w} = \underline{0}_{n-1}$ $\Longrightarrow$

Obținem astfel

$$\boxed{\|A_1\|_2 \geq \frac{\left\| A_1 \begin{pmatrix} \sigma_1 \\ \underline{w} \end{pmatrix} \right\|_2}{\left\| \begin{pmatrix} \sigma_1 \\ \underline{w} \end{pmatrix} \right\|_2} = \sqrt{\sigma_1^2 + \|\underline{w}\|_2^2} \geq \sigma_1 = \|A_1\|_2}$$

cu egalitate $\iff$ $\underline{w} = \underline{0}_{n-1}$.

Prin urmare, rezultă $\underline{w} = \underline{0}_{n-1}$ și deci

$$U^T A V = \begin{bmatrix} \sigma_1 & \underline{0}_{n-1}^T \\ \underline{0}_{m-1} & U_1^T A V_1 \end{bmatrix}$$

unde

$$\begin{cases} U_1 \in \mathcal{M}_{m,m-1}(\mathbb{R}): & U_1^T U_1 = I_{m-1} \\ V_1 \in \mathcal{M}_{n,n-1}(\mathbb{R}): & V_1^T V_1 = I_{n-1} \end{cases}$$

- Astfel, se continuă procedura de mai sus, în mod iterativ, de $(r-1)$ ori, mai întâi pentru $U_1^T A V_1 \in \mathcal{M}_{m-1,n-1}(\mathbb{R})$, până se obține DVS dată de (1).

$\square$