Seminarul 9

Spații vectoriale euclidiene. Reper ortonormal.

(1) $g : \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}$, formă biliniară. $G = \begin{pmatrix} 3 & 2 & 0 \\ 2 & 2 & 2 \\ 0 & 2 & 1 \end{pmatrix}$ mat. asoc. în rap. cu R_0

este $(\mathbb{R}^3, g)$ s. v. euclidian real?
$\iff g$ prod. scalar?

$G = G^T$
g formă bilin. $\Big\} \Rightarrow$ simetrică $\Rightarrow G \in L^s(\mathbb{R}^3, \mathbb{R}^3, \mathbb{R})$

$Q : \mathbb{R}^3 \to \mathbb{R}$, formă pătratică asociată

$Q(x) = \sum_{i=1}^{3} g_{ii} x_i^2 + 2 \sum_{i<j} g_{ij} x_i x_j = 3x_1^2 + 2x_2^2 + x_3^2 + 4x_1x_2 + 4x_2x_3$

Este Q poz. def.? (are sgn. $(3,0)$?)

Met. Jacobi:

$\Delta_1 = 3$

$\Delta_2 = \begin{vmatrix} 3 & 2 \\ 2 & 2 \end{vmatrix} = 2$

$\Delta_3 = \begin{vmatrix} 3 & 2 & 0 \\ 2 & 2 & 2 \\ 0 & 2 & 1 \end{vmatrix} = 6 - 12 - 4 = -10$

$\exists$ un reper a.î. $Q(x) = \frac{1}{\Delta_1} x_1^2 + \frac{\Delta_1}{\Delta_2} x_2^2 + \frac{\Delta_2}{\Delta_3} x_3^2 =$

$= \frac{1}{3} x_1^2 + \frac{3}{2} x_2^2 - \frac{1}{5} x_3^2 \Rightarrow$ sgn. $(2,1)$
$\Rightarrow$ nu e poz. def
$\Rightarrow$ nu e prod. scalar

(2) $(\mathbb{R}^3, g_0)$ $g_0 : \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}$, $g_0(x, y) = x_1 y_1 + x_2 y_2 + x_3 y_3$

$U = \left\{ x \in \mathbb{R}^3 \;\middle|\; \begin{cases} x_1 - x_3 = 0 \\ 2x_2 - x_3 = 0 \end{cases} \right\}$

a) $U^\perp = ?$

b) $R = R_1 \cup R_2$ reper ortonormat în $\mathbb{R}^3$ a.î.
R_1 rep. ortonormat în U și R_2 — " — $U^\perp$

a) $A = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 2 & -1 \end{pmatrix}$ $\quad \dim U = 3 - \text{rg} A$

$$\begin{cases} x_1 = x_3 \\ 2x_2 = x_3 \end{cases} \Rightarrow x_2 = \frac{1}{2} x_3 \Rightarrow U = \left\{ \left(x_3, \frac{1}{2} x_3, x_3 \right) \middle| x_3 \in \mathbb{R} \right\}$$

$$= \left\{ \frac{x_3}{2} (2, 1, 2) \middle| x_3 \in \mathbb{R} \right\} \Rightarrow$$

$$\Rightarrow U = \langle \underbrace{(2, 1, 2)}_{= f_1} \rangle \qquad \frac{f_1}{\|f_1\|} = \frac{1}{3} (2, 1, 2) = e_1'$$

$$\mathbb{R}^3 = U \oplus U^\perp \qquad x \in U^\perp \Rightarrow g_0(x, f_1) = 0 \Rightarrow$$

$$\Rightarrow 2x_1 + x_2 + 2x_3 = 0$$

$$U^\perp = \{ x \in \mathbb{R}^3 \mid x_2 = -2x_1 - 2x_3 \} =$$

$$= \{ (x_1, -2x_1 - 2x_3, x_3), \; x_1, x_3 \in \mathbb{R} \}$$

$$= x_1 \underbrace{(1, -2, 0)}_{= f_2} + x_3 \underbrace{(0, -2, 1)}_{= f_3}$$

$\dim U^\perp = 2$

$\{ f_2, f_3 \}$ reper arbitrar în $U^\perp$

$$\boxed{\langle a, b \rangle = g_0(a, b)}$$

Aplicăm Gram Schmidt:

$$e_2 = f_2$$

$$e_3 = f_3 - \frac{\langle f_3, e_2 \rangle}{\langle e_2, e_2 \rangle} e_2 = (0, -2, 1) - \frac{4}{5} \cdot (1, -2, 0) =$$

$$= \left(-\frac{4}{5}, -\frac{2}{5}, 1 \right) = \frac{1}{5} (-4, -2, 5)$$

$$\{ f_2, f_3 \} \longrightarrow \{ e_2, e_3 \} = \left\{ e_2' = \frac{1}{\sqrt{5}} (1, -2, 0), \; e_3' = \frac{1}{3\sqrt{5}} (-4, -2, 5) \right\}$$

$R_2 = \{ e_2', e_3' \}$

$R = R_1 \cup R_2$

④ $(\mathbb{C}, +, \cdot)|_{\mathbb{R}}$, $g : \mathbb{C}\times\mathbb{C} \to \mathbb{R}$ formă biliniară

$G = \begin{pmatrix} 1 & 2 \\ 2 & 5 \end{pmatrix}$ mat. asoc. lui G în rap. cu $R_0 = \{1, i\}$

a) $(\mathbb{C}, G)$ spv euclid. real?

$z = x_1 + i x_2 \equiv (x_1, x_2)$

$z' = y_1 + i y_2 \equiv (y_1, y_2)$

$g(z, z') = \sum_{i,j=1}^{2} g_{ij} x_i y_j = x_1 y_1 + 2 x_1 y_2 + 2 x_2 y_1 + 5 x_2 y_2$

Met. Jacobi

$\left.\begin{array}{l} \Delta_1 = 1 \\ \Delta_2 = 1 \end{array}\right\} \Rightarrow Q : \mathbb{R}^2 \equiv \mathbb{C} \to \mathbb{R}$ sgn $= (2, 0)$

$\Rightarrow Q$ poz. def $\Rightarrow$

$\Rightarrow g$ prod. scalar

$G = G^T$

b) $u = 2 - i \equiv (2, -1)$ versor în rap. cu $g \Rightarrow$

$\Rightarrow g(u, u) = Q(u) = 1$

$g(z, z) = Q(z) = x_1^2 + 4 x_1 x_2 + 5 x_2^2$

$\|u\|^2 = Q(u) = 2^2 + 4\cdot 2\cdot(-1) + 5(-1)^2 = 1$

$\|z\| = \sqrt{g(z,z)}$ $\quad g_0(u, u) = 4 + 1 = 5$

c) Să se det. $u^{\perp}$, în rap. cu g

$u^{\perp} = \{ z \in \mathbb{C} \mid g(\underset{x_1 + i x_2}{z}, \underset{2-i}{u}) = 0 \}$

$g(z, u) = 2x_1 - 2x_1 + 4x_2 - 5x_2 = -x_2$

$u^{\perp} = \{ z = x_1 + i x_2 \mid x_2 = 0,\ x_1 \in \mathbb{R} \} = \mathbb{R}$

$\mathbb{C} \simeq \mathbb{R}^2 = \langle u \rangle \oplus \underbrace{u^{\perp}}_{= \mathbb{R}}$

d) Să se ortonormeze R_0 în raport cu g

$R_0 = \{1, i\} \equiv \{\underbrace{(1,0)}_{f_1} \; \underbrace{(0,1)}_{f_2}\}$

G-S: $e_1 = f_1 = 1$

$e_2 = f_2 - \dfrac{g_0(f_2, e_1)}{g(e_1, e_1)} e_1 = i - \dfrac{2}{1} \cdot 1 = -2 + i = -u$

e) Să se afle int. dintre cercul unitate în $(\mathbb{C}, g_0)$ și $(\mathbb{C}, g)$

$\underset{\mathbb{R}^2}{(\mathbb{C}, g_0)}$ $A = \{ z \in \mathbb{C} \mid g_0(z, z) = Q_0(z) = x_1^2 + x_2^2 = 1 \}$

$= \{ z \overset{x_1 + i x_2}{=} \in \mathbb{C} \mid |z| = 1 \}$

$B = \{ z \in \mathbb{C} \mid g(z, z) = Q(z) = x_1^2 + 4 x_1 x_2 + 5 x_2^2 = 1 \}$

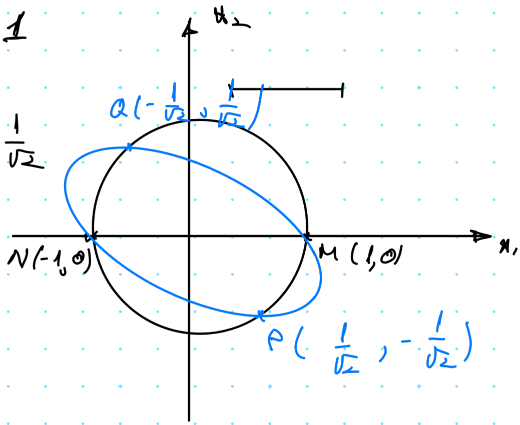
$A \cap B$
$$\begin{cases} x_1^2 + x_2^2 = 1 \\ x_1^2 + 4 x_1 x_2 + 5 x_2^2 = 1 \end{cases} \Rightarrow 4 x_1 x_2 + 4 x_2^2 = 0$$
$$\Rightarrow 4 x_2 (x_1 + x_2) = 0$$

$x_2 = 0 \Rightarrow x_1 = \pm 1$

sau

$x_1 + x_2 = 0 \Rightarrow x_1 = -x_2$

$\Rightarrow 2 x_1^2 = 1 \Rightarrow x_1 = \pm \dfrac{1}{\sqrt{2}}$



$A \cap B = \{M, N, P, Q\}$

⑤ $(\mathbb{R}^3, g_0)$ $R = \{ f_1 = (1, 2, 3), f_2 = (0, 1, 1), f_3 = \{1, 2, 5\} \}$

a) R reper în $\mathbb{R}^3$ (să se ortonormeze)

b) $f_1 \times f_2$

c) $f_1 \wedge f_2 \wedge f_3$

a) $\det\begin{pmatrix} 1 & 0 & 1 \\ 2 & 1 & 2 \\ 3 & 1 & 5 \end{pmatrix} = 2 \Rightarrow rg = 3$ (numărul de vectori SLI)

$\Rightarrow R = SLi$

$card\ R = \dim R^3 = 3 \Rightarrow R$ reper

Aplicăm G - S :

$e_1 = f_1$

$e_2 = f_2 - \dfrac{\langle f_2, e_1 \rangle}{\langle e_1, e_1 \rangle} f_1 = (0,1,1) - \dfrac{5}{14}(1,2,3) =$

$= \left(-\dfrac{5}{14}; \dfrac{4}{14}; -\dfrac{1}{14}\right) =$

$= \dfrac{1}{14}(-5; 4; -1)$

b) $f_1 \times f_2 = \begin{vmatrix} i & j & k \\ 1 & 2 & 3 \\ 0 & 1 & 1 \end{vmatrix} = (-1, -1, 1) = -i - j + k$

$R_0 = \{i, j, k\}$

$e_3 = e_1 \times e_2 = f_1 \times f_2 \quad R' = \{e_1, e_2, e_3\}$ reper ortogonal

c) $e_1' = \dfrac{e_1}{\|e_1\|} = \dfrac{1}{\sqrt{14}}(1, 2, 3) \qquad e_2' = \dfrac{e_2}{\|e_2\|} = \dfrac{1}{\sqrt{42}}(-5, 4, -1)$

$e_3' = \dfrac{1}{\sqrt{3}}(-1, -1, 1)$

$R'' = \{e_1', e_2', e_3'\}$ R. ortonormat

$f_1 \wedge f_2 \wedge f_3 = \begin{vmatrix} 1 & 2 & 3 \\ 0 & 1 & 1 \\ 1 & 2 & 5 \end{vmatrix} = 2$

(10) (V, g) spv. euc. real. UAE :

1) $x \perp y$

2) $\|x - y\|^2 = \|x\|^2 + \|y\|^2$

3) $\|x - y\| = \|x + y\|$, $\forall x, y \in V$

$$\|x-y\|^2 = g(x-y,\, x-y) = \underset{\underset{\|x^2\|}{\parallel}}{g(x,x)} - 2g(x,y) + \underset{\underset{\|y^2\|}{\parallel}}{g(y,y)}$$

$$\|x+y\|^2 = \|x\|^2 + 2\,g(x,y) + \|y^2\|$$

1) $x \perp y \Leftrightarrow g(x,y) = 0 \Rightarrow \|x-y\|^2 = \|x\|^2 + \|y\|^2 \Leftrightarrow 2)$

1) $x \perp y \Leftrightarrow \|x-y\|^2 = \|x+y\|^2 \Leftrightarrow 3)$