Exerciții

1. Determinați $\underline{\lim}\, x_n$, $\overline{\lim}\, x_n$ și precizați dacă există $\lim\limits_{n\to\infty} x_n$, unde:

a) $x_n = \dfrac{n}{2n+1}\left(\cos\dfrac{n\pi}{3}\right)^n \quad \forall n\in\mathbb{N}^*$.

b) $x_n = \dfrac{2+(-1)^n}{1+n^{(-1)^n}} + \sin\dfrac{n\pi}{2} \quad \forall n\in\mathbb{N}$.

c) $x_n = \sqrt[n]{4^{(-1)^n}+2} \quad \forall n\in\mathbb{N}^*$.

d) $x_n = \dfrac{1}{2}\left(n-2-3\left[\dfrac{n-1}{3}\right]\right)\left(n-3-3\left[\dfrac{n-1}{3}\right]\right)$

$\forall n\in\mathbb{N}^*$, unde $[a]$ reprezintă partea întreagă a lui a.

e) $x_n = \dfrac{\left(1-(-1)^n\right)\cdot 2^n+1}{2^n+3} \quad \forall n\in\mathbb{N}$.

f) $x_n = \dfrac{(1+\cos n\pi)\ln 3n+\ln n}{\ln 2n} \quad \forall n\in\mathbb{N}^*$.

2. Studiați convergența seriilor:

a) $\sum_{n=1}^{\infty} e^{-n^2}$.

b) $\sum_{n=1}^{\infty} \frac{(an)^n}{n!}$, $a>0$, $a \neq \frac{1}{e}$.

c) $\sum_{n=1}^{\infty} \frac{\sin(nx)}{2^n}$, $x \in \mathbb{R}$.

d) $\sum_{n=1}^{\infty} a^n \left(1+\frac{1}{n}\right)^n$, $a>0$.

e) $\sum_{n=1}^{\infty} \frac{\ln n}{n^3}$.

f) $\sum_{n=1}^{\infty} \ln\left(1+\frac{1}{2n}\right)$.

g) $\sum_{n=1}^{\infty} \left(a\,\frac{n^2+n+1}{n^2}\right)^n$, $a>0$.

h) $\sum_{n=1}^{\infty} \frac{a(a+1)\cdot\ldots\cdot(a+n-1)}{b\cdot(b+1)\cdot\ldots\cdot(b+n-1)}(c-2)^n$, $a>0$, $b>0$, $c>2$.

i) $\sum_{n=1}^{\infty} \frac{1}{n}\sqrt[n]{(n+1)(n+2)\cdot\ldots\cdot(n+n)}$.

j) $\sum_{n=2}^{\infty} \frac{1}{\sqrt[n]{\ln n}}$.

k) $\sum_{n=1}^{\infty} \left(e^{\sin\frac{1}{n}} - 1\right) x^n, \quad x > 0.$

l) $\sum_{n=2}^{\infty} \frac{1}{(\ln n)^3} \cdot x^n, \quad x > 0.$

m) $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}\sqrt[4]{n+2}} x^n, \quad x > 0.$

n) $\sum_{n=1}^{\infty} \frac{a^n + n}{a^{2n} + n^3}, \quad a > 0.$

o) $\sum_{n=1}^{\infty} \left(\sqrt{(n+1)(n+x)} - n\right)^n.$

p) $\sum_{n=1}^{\infty} \frac{(n!)^3}{(3n)!} x^n, \quad x > 0.$

q) $\sum_{n=1}^{\infty} \frac{a^n}{2^n + 5^n}, \quad a > 0.$

r) $\sum_{n=2}^{\infty} (-1)^n \frac{1}{\ln n}$.

s) $\sum_{n=1}^{\infty} \sin\left(\pi\sqrt{n^2+1}\right)$.

t) $\sum_{n=1}^{\infty} \frac{10 \cdot 18 \cdot \ldots \cdot (8n+2)}{10 \cdot 19 \cdot \ldots \cdot (9n+1)} x^n$, $x > 0$.

u) $\sum_{n=1}^{\infty} \frac{1 \cdot 3 \cdot 5 \cdot \ldots \cdot (2n-1)}{2 \cdot 4 \cdot 6 \cdot \ldots \cdot (2n)} \cdot \frac{1}{n}$.

v) $\sum_{n=1}^{\infty} \left(\operatorname{arctg} \frac{1}{n(n+1)}\right) x^n$, $x > 0$.

w) $\sum_{n=1}^{\infty} \frac{\sqrt{n-1}}{n(n+1)} x^n$, $x > 0$.

x) $\sum_{n=1}^{\infty} \frac{\sin(nx)}{n}$, $x \in \mathbb{R}$.

y) $\sum_{n=1}^{\infty} \frac{\sin n \sin \frac{1}{n}}{n}$.

z) $\sum_{n=1}^{\infty} \frac{x^n}{\sqrt[3]{n+1}\sqrt[5]{n+2}}$, $x > 0$.

3. Faceți analiza topologică a mulțimii $A \subset \mathbb{R}$, unde:

a) $A = \mathbb{R} \setminus \mathbb{Q}$.

b) $A = \mathbb{Q} \cap (0,1)$

c) $A = [0,4] \setminus \{2\}$.

d) $A = \left\{ \frac{n}{3n+1} \mid n \in \mathbb{N} \right\}$.

4. Fie $X = C([0,1]) = \{ f : [0,1] \to \mathbb{R} \mid f \text{ continuă} \}$,

$d_1 : X \times X \to \mathbb{R}$, $d_1(f,g) = \int_0^1 |f(x) - g(x)| \, dx$ și

$d_\infty : X \times X \to \mathbb{R}$, $d_\infty(f,g) = \max \{ |f(x) - g(x)| \mid x \in [0,1] \}$

a) Arătați că d_1 și d_∞ sunt distanțe pe X.

b) Fie $(f_n)_n \subset X$ și $f \in X$ a.î. $\lim\limits_{n\to\infty} f_n \overset{d_\infty}{=} f$.

Arătați că $\lim\limits_{n\to\infty} f_n \overset{d_1}{=} f$.

c) Fie $(f_n)_n \subset X$, $f_n : [0,1] \to \mathbb{R}$, $f_n(x) =$

$= \frac{1}{1+nx}$ $\forall n \in \mathbb{N}$. Arătați că $\lim\limits_{n\to\infty} f_n \overset{d_1}{=} f$ și

$\lim\limits_{n\to\infty} f_n \overset{d_\infty}{\neq} f$, unde $f : [0,1] \to \mathbb{R}$, $f(x) = 0$.