SEMINAR IX - 7.12.2023

Forme pătratice. Formă canonică

Metoda Gauss. Metoda Jacobi

$$Q : \mathbb{R}^3 \to \mathbb{R}, \quad Q(x) = x_1^2 + x_2^2 + x_3^2 + x_1x_2 + x_1x_3 + x_2x_3$$

a) $G = ?$

b) $g : \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}$ forma polară asociată

c) să se aducă Q la o formă canonică (Gauss / Jacobi)

Este Q pozitiv definită? Generalizare

a) $Q(x) = x_1^2 + x_2^2 + x_3^2 + x_1x_2 + x_1x_3 + x_2x_3 = \sum_{i=1}^{n} g_{ii} x_i^2 + 2 \sum_{i<j} g_{ij} x_i x_j$

$$G = \begin{pmatrix} 1 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & 1 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & 1 \end{pmatrix}$$

b) $g(x,y) = \frac{1}{2}\left(Q(x+y) - Q(x) - Q(y) \right)$

$$g(x,y) = \sum_{i,j=1}^{3} g_{ij} x_i y_j = x_1y_1 + \frac{1}{2}x_1y_2 + \frac{1}{2}x_1y_3 + \frac{1}{2}x_2y_1 + x_2y_2 + \frac{1}{2}x_2y_3 + \frac{1}{2}x_3y_1 + \frac{1}{2}x_3y_2 + x_3y_3$$

c) Metoda Jacobi

$\Delta_1 = 1$

$$\Delta_2 = \begin{vmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & 1 \end{vmatrix} = 1 - \frac{1}{4} = \frac{3}{4} \neq 0$$

$$\Delta_3 = \det G = \begin{vmatrix} 1 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & 1 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & 1 \end{vmatrix} = \frac{1}{2} \neq 0$$

$\exists R'$ în $\mathbb{R}^3$ a.î.

$$Q(x) = \frac{1}{\Delta_1} x_1'^2 + \frac{\Delta_1}{\Delta_2} x_2'^2 + \frac{\Delta_2}{\Delta_3} x_3'^2 =$$

$$= x_1'^2 + \frac{4}{3} x_2'^2 + \frac{3}{2} x_3'^2$$

Signatura $(3,0) \Rightarrow Q$ pozitiv definită

Metoda Gauss

$$Q(x) = \left(x_1 + \frac{1}{2}x_2 + \frac{1}{2}x_3\right)^2 - \frac{x_2^2}{4} - \frac{x_3^2}{4} - \frac{1}{2}x_2x_3 + x_2^2 + x_3^2 + x_2x_3$$

$$= \dots + \frac{3}{4}x_2^2 + \frac{3}{4}x_3^2 + \frac{1}{2}x_2x_3$$

$$= \dots + \frac{3}{4}\left(x_2^2 + \frac{2}{3}x_2x_3\right) + \frac{3}{4}x_3^2$$

$$= \dots + \frac{3}{4}\left(x_2 + \frac{x_3}{3}\right)^2 - \frac{x_3^2}{12} + \frac{3}{4}x_3^2$$

$$= \left(x_1 + \frac{1}{2}x_2 + \frac{1}{2}x_3\right)^2 + \frac{3}{4}\left(x_2 + \frac{x_3}{3}\right)^2 + \frac{2}{3}x_3^2$$

Schimbarea de reper $\begin{cases} x_1' = x_1 + \frac{1}{2}x_2 + \frac{1}{2}x_3 \\ x_2' = x_2 + \frac{1}{3}x_3 \\ x_3' = x_3 \end{cases}$ $\quad Q(x) = x_1'^2 + \frac{3}{4}x_2'^2 + \frac{2}{3}x_3'^2$

2) $Q: \mathbb{R}^3 \to \mathbb{R}$

$Q(x) = 2x_1x_2 - 6x_1x_3 - 6x_2x_3$

? forma canonică (Gauss / Jacobi), signatura

$$G = \begin{pmatrix} 0 & 1 & -3 \\ 1 & 0 & -3 \\ -3 & -3 & 0 \end{pmatrix}$$

Metoda Jacobi $\Delta_1 = 0$ nu convine

Metoda Gauss

$$Q(x) = 2x_1x_2 - \overbrace{6x_1x_3 - 6x_2x_3}^{6x_3(x_1+x_2)}$$

$$\begin{cases} x_1' = x_1 + x_2 \\ x_2' = x_1 - x_2 \end{cases}$$

$x_3' = x_3$

$$\begin{cases} x_1 = \frac{1}{2}(x_1' + x_2') \\ x_2 = \frac{1}{2}(x_1' - x_2') \\ x_3 = x_3' \end{cases}$$

$$Q(x) = \frac{1}{2}(x_1'^2 - x_2'^2) - 6x_1'x_3' = \frac{1}{2}(x_1'^2 - 3x_1'x_3') - \frac{1}{2}x_2'^2$$

$$= \frac{1}{2}\left(x_1' - \frac{3}{2}x_3'\right)^2 - \frac{9}{8}x_3'^2 - \frac{1}{2}x_2'^2$$

$x_1'' = x_1' - \frac{3}{2}x_3'$

$x_2' = x_2'$

$x_3'' = x_3'$

$$Q = \frac{1}{2}x_1''^2 - \frac{1}{2}x_2''^2 - \frac{9}{8}x_3''^2$$

signatura $(1, 2)$

g nu e produs scalar