SEMINAR 1:

ȘIRURI DE NUMERE REALE

Ex1 Calculați următoarele lim. de șiruri

a) $\lim\limits_{n\to+\infty} \frac{1\cdot 6\cdot 11\cdot\ldots\cdot(5n+1)}{1\cdot 7\cdot 13\cdot\ldots\cdot(6n+1)} =$

b) $\lim\limits_{n\to+\infty} \frac{\sqrt[n]{n!}}{n}$

c) $\lim\limits_{n\to+\infty} \frac{7n+5}{9n^2-2}\cdot\sin(2n)$

d) $\lim\limits_{n\to+\infty} \frac{1^{100}+2^{100}+\ldots+n^{100}}{n^{101}}$

e) $\lim\limits_{n\to+\infty} (\sqrt{n^2+n+1} - \sqrt[3]{n^3+1}$

a) $\lim\limits_{n\to+\infty} \frac{1\cdot 6\cdot 11\cdot\ldots\cdot(5n+1)}{1\cdot 7\cdot 13\cdot\ldots\cdot(6n+1)}$

Aplicăm criteriul raportului pentru șirul cu termeni pozitivi

Pas 1: $a_n = \frac{1\cdot 6\cdot 11\cdot\ldots\cdot(5n+1)}{1\cdot 7\cdot 13\cdot\ldots\cdot(6n+1)}$, $a_n>0, \forall n\in\mathbb{N}$

Pas 2: Se calculează $\lim \frac{a_{n+1}}{a_n} = l \in \overline{\mathbb{R}}$

$l<1 \Rightarrow \lim a_n = 0$

$l>1 \Rightarrow \lim a_n = \infty$

$l=1 \Rightarrow$ nu știm să aflăm lim cu crit. rap.

$$\lim_{n\to+\infty} \frac{a_{n+1}}{a_n} = \frac{\dfrac{\cancel{1\cdot 6\cdot 11\cdot\ldots\cdot(5n+1)}\cdot(5n+6)}{\cancel{1\cdot 7\cdot 13\cdot\ldots\cdot(6n+1)}\cdot(6n+7)}\begin{matrix}a\\ b\end{matrix}}{\dfrac{\cancel{1\cdot 6\cdot 11\cdot\ldots\cdot(5n+1)}}{\cancel{1\cdot 7\cdot 13\cdot\ldots\cdot(6n+1)}}\begin{matrix}c\\ d\end{matrix}} = \frac{a}{b}\cdot\frac{d}{c} = \frac{5n+6}{6n+7} = \frac{5}{6} < 1$$

$\Rightarrow \lim\limits_{n\to+\infty} a_n = 0$

b) $b_n = \frac{\sqrt[n]{n!}}{n}$, $n \in \mathbb{N}^*$

Se aplică criteriul rădăcinii / radicalului pt. șiruri cu termeni poz.

↳ $\sqrt[n]{a_n}$, $\frac{a_{n+1}}{a_n} = \ell \Rightarrow$ $= \ell$

Pas 1: Introducem totul sub rad:

$$b_n = \sqrt[n]{\frac{n!}{n^n}}$$

Notăm șirul de sub radical cu $c_n = \frac{n!}{n^n}$

Pas 2: Se calc. $\lim\limits_{n\to+\infty} \frac{c_{n+1}}{c_n} = \lim\limits_{n\to+\infty} \frac{\frac{(n+1)!}{(n+1)^{n+1}}}{\frac{n!}{n^n}} = \lim\limits_{n\to+\infty} \frac{\cancel{(n+1)!}}{(n+1)^{n+1}} \cdot \frac{n^n}{\cancel{n!}}$

$$= \lim_{n\to+\infty} \frac{(n+1)\cdot n^n}{(n+1)^{n+1}} = \lim_{n\to+\infty} \frac{n^n}{(n+1)^n} = \lim_{n\to+\infty} \left(\frac{n}{n+1}\right)^n \leftarrow \lim_{n\to+\infty} (1+x_n)^{\frac{1}{x_n}} = e$$

$$= \lim_{n\to+\infty} \left(1 + \frac{-1}{n+1}\right)^n = \lim_{n\to+\infty} \left(1+\frac{-1}{n+1}\right)^{n \cdot \frac{1}{\frac{-1}{n+1}} \cdot \frac{-1}{n+1}}$$

$$= e^{\lim\limits_{n\to+\infty} \frac{-n}{n+1}} = e^{-1} = \frac{1}{e}$$

$$\Rightarrow \lim_{n\to+\infty} b_n = \frac{1}{e}$$

c) $\lim\limits_{n\to+\infty} \underbrace{\frac{7n+5}{9n^2-2} \cdot \sin(2n)}_{c_n}$

$-1 \le \sin(2n) \le 1 \quad / \cdot \frac{7n+5}{9n^2-2} \neq 0 \quad (\Leftrightarrow)$

$$-\frac{7n+5}{9n^2-2} \le c_n \le \frac{7n+5}{9n^2-2} \Rightarrow \lim_{n\to+\infty} c_n = 0$$

$\searrow 0 \quad \downarrow \quad \swarrow 0 \qquad n \in \mathbb{N}^*$

d) $\lim\limits_{n\to+\infty} \frac{1^{100} + 2^{100} + \ldots + n^{100}}{n^{101}} = d_n$

LEMA STOLTZ-CESARO

$$x_n = \frac{1^{100}+2^{100}+\ldots+n^{100}}{n^{101}}$$
$y_n =$

$$\lim_{n\to+\infty} \frac{x_{n+1}-x_n}{y_{n+1}-y_n} = \lim_{n\to+\infty} \frac{(n+1)^{100}}{(n+1)^{101}-n^{101}} = \lim_{n\to+\infty} d_n$$

Folosim binomul lui Newton

$$\lim_{n\to+\infty} \frac{C_{100}^{0}\cdot n^{100} + C_{100}^{1}\cdot n^{99} + \ldots + C_{100}^{100}\cdot n^{0}}{\cancel{C_{101}^{0}\cdot n^{101}} + C_{101}^{1}\cdot n^{100} + \ldots + C_{101}^{101}\cdot n^{0} - \cancel{n^{101}}} = \frac{C_{100}^{0}\cdot n^{100} + C_{100}^{1}\cdot n^{99} + \ldots + C_{100}^{100}}{C_{101}^{1}\cdot n^{100} + \ldots + C_{101}^{101}\cdot n^{0}} = \frac{1}{101}$$

↳ aici avem 2 polinoame

deci luăm gradele $\left(\frac{n^{100} + 100\cdot n^{99} + \ldots + 1}{101\cdot n^{100} + \ldots + 1} \right)$

$$\Rightarrow \lim_{n\to+\infty} d_x = \frac{1}{101}$$

e) $\lim_{n\to+\infty} \left(\sqrt{n^2+n+1} - \sqrt[3]{n^3+1} \right) \overset{,,\infty-\infty"}{=}$

$$\lim_{n\to+\infty} n\sqrt{1+\frac{1}{n}+\frac{1}{n^2}} - n\sqrt[3]{1+\frac{1}{n^3}} = \lim_{n\to+\infty} n\left(\sqrt{1+\frac{1}{n}+\frac{1}{n^2}} - \sqrt{1+\frac{1}{n^3}} \right)$$

$$= \lim_{n\to+\infty} \frac{\sqrt{1+\frac{1}{n}+\frac{1}{n^2}} - \sqrt{1+\frac{1}{n^3}}}{\frac{1}{n}}$$

Facem cu schimbare de variabilă $y = \frac{1}{n}$

$$\lim_{y\to 0} \frac{\sqrt{1+y+y^2} - \sqrt{1+y^3}}{y} \overset{\frac{0}{0}"}{\underset{L'H}{=}} \frac{\frac{1+2y}{2\sqrt{1+y+y^2}} - \frac{3y^2}{2\sqrt{1+y^3}}}{1} = \frac{\frac{1+0}{2\cdot 1} - 0}{1} = \frac{1}{2}$$