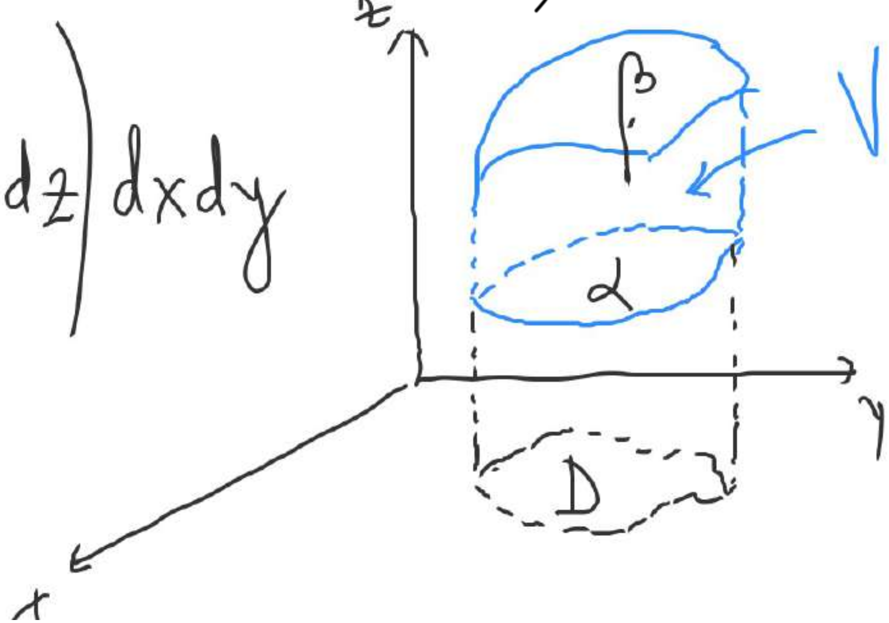
Propoziție 1. Fie $D \in \mathcal{J}(\mathbb{R}^2)$, $\alpha, \beta : D \to \mathbb{R}$ continue și mărg pe D cu $\alpha(x,y) \le \beta(x,y)$, $\forall (x,y) \in D$. Atunci

$$V = \{(x,y,z) \in \mathbb{R}^3 \mid (x,y) \in D,\ \alpha(x,y) \le z \le \beta(x,y)\}$$

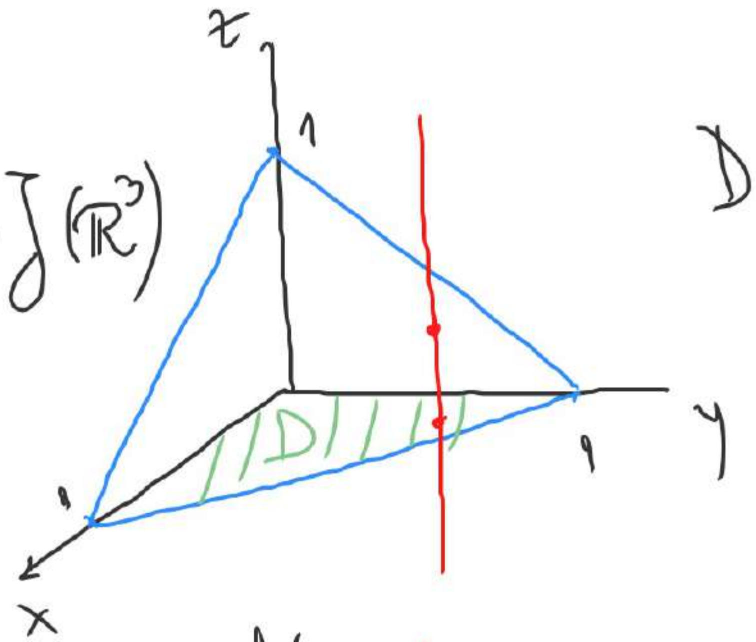
este măsurabilă Jordan. Dacă $f : V \to \mathbb{R}$ este continuă și mărginită atunci f este integrabilă Riemann și

$$\iiint_V f(x,y,z)\,dx\,dy\,dz = \iint_D \left(\int_{\alpha(x,y)}^{\beta(x,y)} f(x,y,z)\,dz \right) dx\,dy$$



$(D = pr_{xOy} V)$

Calculati $\iiint\limits_V x\,dx\,dy\,dz$, $V=\{(x,y,z) \mid x+y+z\le 1,\ x,y,z\ge 0\}$



$V=\{(x,y,z)\mid (x,y)\in D,\ \underbrace{0\le z\le 1-x-y}_{\text{cont + mărg}}\}\in J(\mathbb{R}^3)$

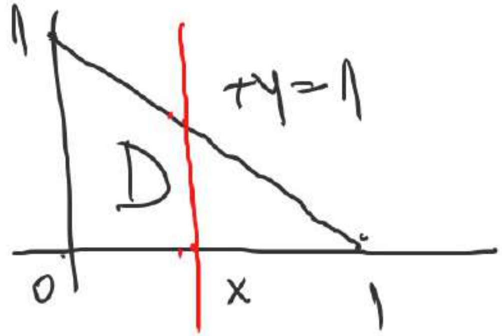
$D=pr_{xOy}V$

$D\in J(\mathbb{R}^2)$

$D=\{(x,y)\in\mathbb{R}^2 \mid x+y\le 1,\ x,y\ge 0\}$

$\shortparallel$

$\{(x,y)\in\mathbb{R}^2\mid 0\le x\le 1,\ \underbrace{0\le y\le 1-x}_{\text{cont}}\}\in J(\mathbb{R}^2)$.

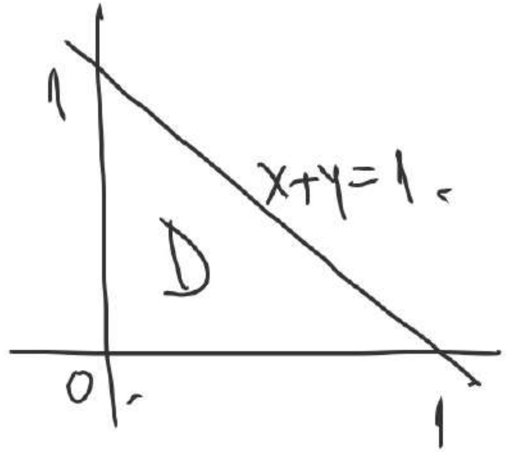


$f: V\to\mathbb{R}$, $f(x,y,z)=x$

f. int. R (cont și mărg pe V)

$$\iiint_V x\,dx\,dy\,dz = \iint_D \left(\int_0^{1-x-y} x\,dz \right) dx\,dy = \iint_D xz \Big|_{z=0}^{z=1-x-y} dx\,dy$$

$$= \iint_D x(1-x-y)\,dx\,dy = \iint_D \left(x(1-x) - yx \right) dx\,dy$$



$$= \int_0^1 \left(\int_0^{1-x} \left(x(1-x) - yx \right) dy \right) dx = \int_0^1 \left(x(1-x)\cdot y \Big|_{y=0}^{y=1-x} - x\cdot \frac{y^2}{2} \Big|_{y=0}^{y=1-x} \right) dx$$

$$= \int_0^1 \left(x(1-x)^2 - \frac{x(1-x)^2}{2} \right) dx = \frac{1}{2} \int_0^1 x(1-x)^2\,dx = \frac{1}{2} \int_0^1 (1-x)^2 \left(1-(1-x)\right) dx$$

$$= \frac{1}{2}\int_0^1 \left[(1-x)^2 - (1-x)^3\right] dx = \frac{1}{2}\int_0^1 \left[(x-1)^2 + (x-1)^3\right] dx$$

$$= \frac{1}{2}\cdot\frac{(x-1)^3}{3}\Bigg|_0^1 + \frac{1}{2}\cdot\frac{(x-1)^4}{4}\Bigg|_0^1 = \frac{1}{2}\left(\frac{1}{3} - \frac{1}{4}\right) = \frac{1}{24}.$$

Propoziție. Fie $A \in \mathcal{J}(\mathbb{R}^n)$ cu $\lambda(A)=0$ și $f: A \to \mathbb{R}$ mărginită.
Atunci f este integrabilă Riemann și $\int_A f(x)\,dx = 0$.

Dem (ex!)

Propoziție. Fie $A, B \in \mathcal{J}(\mathbb{R}^n)$ și $f: A\cup B \to \mathbb{R}$ mărginită.
Dacă f este integrabilă Riemann pe A și pe B atunci f este integrabilă Riemann pe $A\cup B$ și pe $A\cap B$ și

$$\int_{A\cup B} f(x)\,dx + \int_{A\cap B} f(x)\,dx = \int_A f(x)\,dx + \int_B f(x)\,dx$$

Corolar. Fie $A, B \in \mathcal{J}(\mathbb{R}^n)$ cu $\lambda(A \cap B) = 0$ și $f : A \cup B \to \mathbb{R}$ mărginită. Dacă f este int. R. pe A și pe B atunci f este integrabilă Riemann pe $A \cup B$ și

$$\int_{A \cup B} f(x)\,dx = \int_A f(x)\,dx + \int_B f(x)\,dx.$$

Propoziție. Fie $A, B \in \mathcal{J}(\mathbb{R}^n)$, $\lambda(B) = 0$ și $f : A \cup B \to \mathbb{R}$ mărg. Atunci f este integrabilă Riemann pe $A \cup B$ dacă și numai dacă f este integrabilă Riemann pe A.

În acest caz:
$$\int_{A \cup B} f(x)\,dx = \int_A f(x)\,dx$$

Schimbarea de variabilă pt. integrala multiplă

Teoremă. Fie $U, V \subset \mathbb{R}^n$ mulțimi deschise, $\phi : U \to V$ difeomorfism de clasă C^1 și $A \subset V$, $A \in \mathcal{J}(\mathbb{R}^n)$

Atunci:

1) $\phi^{-1}(A) \in \mathcal{J}(\mathbb{R}^n)$

2) Pt orice funcție $f : A \to \mathbb{R}$ integrabilă Riemann funcția $f \circ \phi \cdot |\det J_\phi| : \phi^{-1}(A) \to \mathbb{R}$ este int. Riemann și

$$\int_A f(x)\,dx = \int_{\phi^{-1}(A)} f \circ \phi(u) \cdot |\det J_\phi(u)|\,du$$

unde $\det J_\phi(u)$ este Jacobianul lui ϕ în u.

Obs: $f:[c,d]\to\mathbb{R}$ int. $\phi:[a,b]\to[c,d]$ de clasă C^1 bijectivă, $\phi(a)=c$, $\phi(b)=d$.

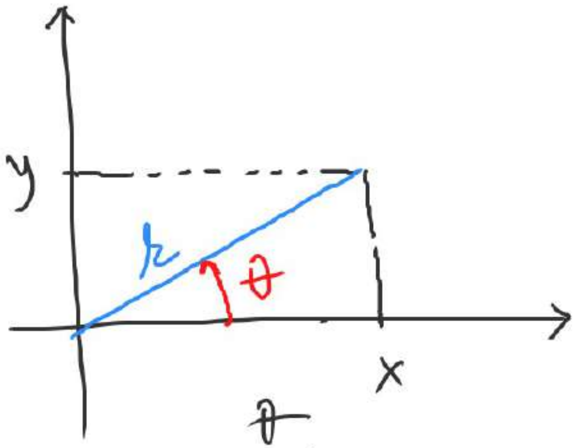
$$\int_c^d f(x)\,dx = \int_{\phi(a)}^{\phi(b)} f(x)\,dx = \int_a^b f(\phi(u))\cdot\phi'(u)\,du.$$

$$dx = \phi'(u)\,du$$

$$\overset{\parallel}{\int_{\phi^{-1}(c)}^{\phi^{-1}(d)}} f\circ\phi(u)\cdot\phi'(u)\,du.$$

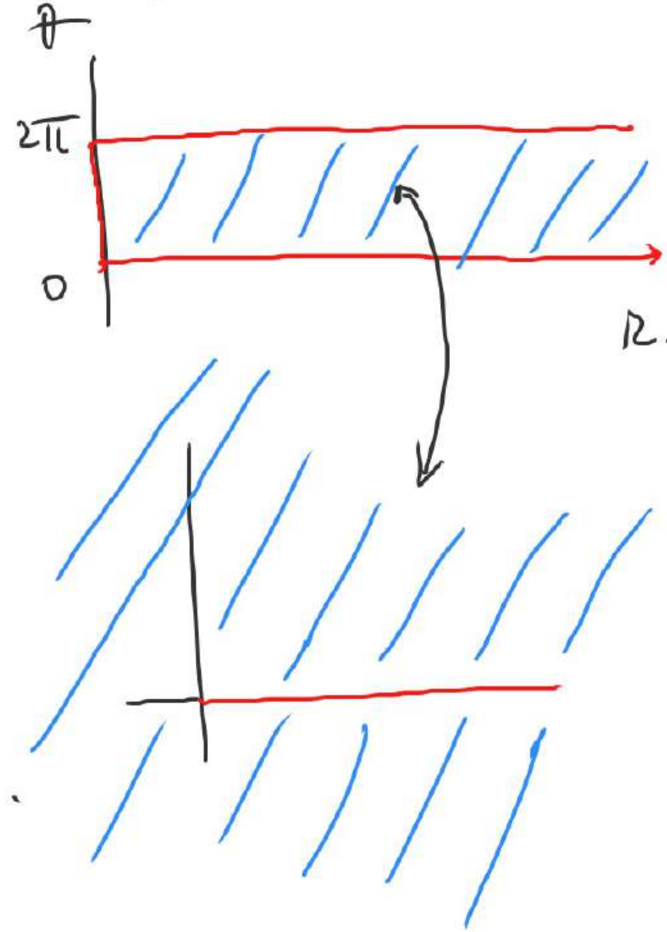
Trecerea la coordonate polare

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta . \end{cases}$$



$\phi : [0,\infty) \times [0, 2\pi] \longrightarrow \mathbb{R}^2$

$\phi(r,\theta) = (r\cos\theta, r\sin\theta)$, ϕ surjectivă.



Restricția lui ϕ la $(0,\infty) \times (0, 2\pi)$ este

un difeom. de clasă C^1 între

$$(0,\infty) \times (0, 2\pi) \xleftrightarrow[\phi]{\quad} \mathbb{R}^2 \setminus \{(x,0) \mid x \geq 0\}.$$

$$J_\phi(r,\theta) = \begin{pmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{pmatrix} = \begin{pmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{pmatrix}$$

$$\det J_\phi(r,\theta) = r\cos^2\theta + r\sin^2\theta = r \neq 0, \ \forall (r,\theta) \in (0,\infty) \times (0, 2\pi).$$

$$dx\,dy = |\det J_\phi(r,\theta)|\, dr\, d\theta = r\, dr\, d\theta$$

$A \in \mathcal{J}(\mathbb{R}^2)$, $f : A \to \mathbb{R}$ int. Riemann

$A' = A \setminus Ox$, $A' \in \mathcal{J}(\mathbb{R}^2)$, $A' \subset A$, $\lambda(A \setminus A') = \lambda(A \cap Ox) = 0$.

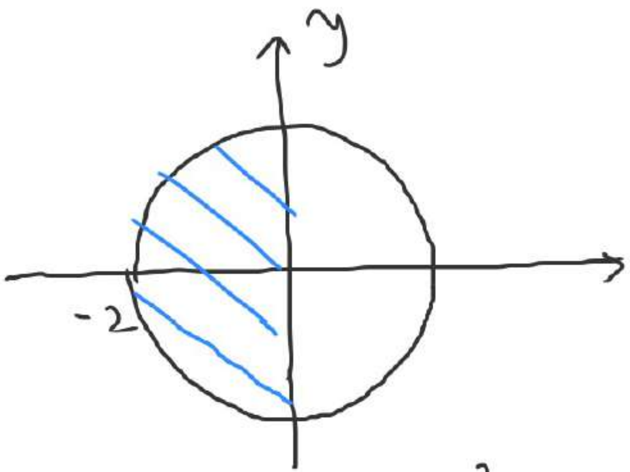
$\phi^{-1}(A'), \phi^{-1}(A) \in \mathcal{J}(\mathbb{R}^2)$; $\quad \lambda(\phi^{-1}(A) \setminus \phi^{-1}(A')) = 0$.

$$\iint_{A'} f(x,y)\,dx\,dy = \iint_{\phi^{-1}(A')} f(r\cos\theta, r\sin\theta)\cdot \underbrace{\left|\det J_\phi(r,\theta)\right|}_{\parallel \atop r.}\,dr\,d\theta$$

$$\parallel \qquad\qquad\qquad\qquad \parallel$$

$$\iint_{A} f(x,y)\,dx\,dy = \iint_{\phi^{-1}(A)} f(r\cos\theta, r\sin\theta)\cdot r\,dr\,d\theta$$

Calculați $\displaystyle\iint_D y^2\,dx\,dy$, $D = \{(x,y)\in\mathbb{R}^2 \mid x^2+y^2\le 4,\ x\le 0\}$



$y^2 \le 4-x^2$

$D = \{(x,y)\in\mathbb{R}^2 \mid -2\le x\le 0,\ -\sqrt{4-x^2}\le y\le\sqrt{4-x^2}\}$

$D \in J(\mathbb{R}^2) \Longleftarrow$ cont.

$f: D\to\mathbb{R},\ f(x,y)=y^2$

f cont + mărginită $\Longrightarrow$ f int. Riemann

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases} \qquad (x,y)\in D \Longleftrightarrow \begin{cases} r\in[0,2] \\ \theta\in\left[\frac{\pi}{2},\frac{3\pi}{2}\right] \end{cases}$$

$x^2+y^2\le 4 \Longleftrightarrow r^2\le 4$

$r\cos\theta\le 0 \Longleftrightarrow \cos\theta\le 0$

$\Updownarrow$

$\theta\in\left[\frac{\pi}{2},\frac{3\pi}{2}\right]$

$$D \longleftrightarrow [0,2] \times \left[\frac{\pi}{2}, \frac{3\pi}{2}\right]$$

$$dx\,dy = r\,dr\,d\theta$$

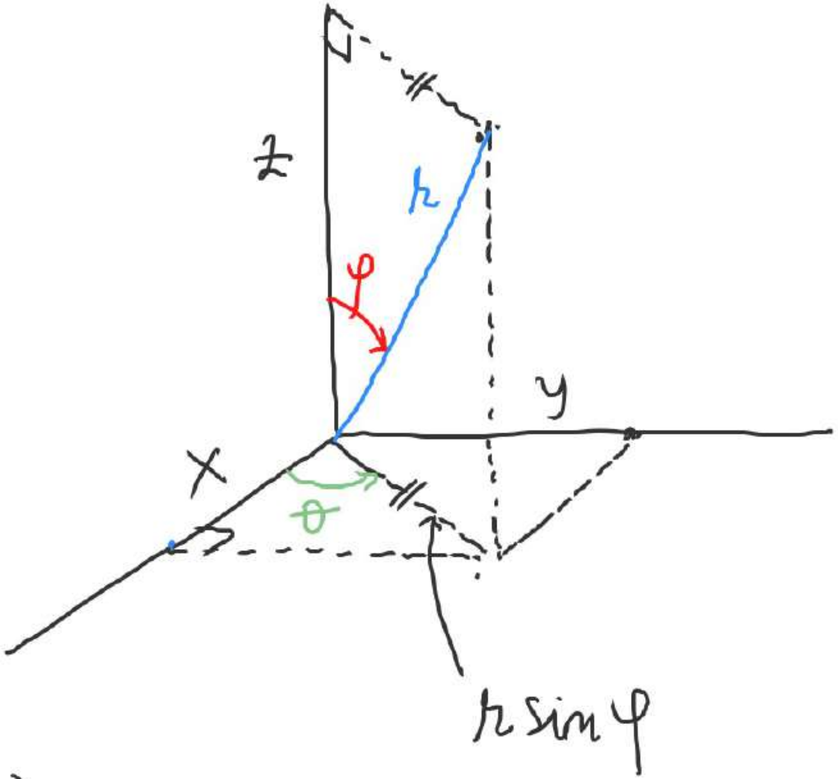
$$\iint_D y^2\,dx\,dy = \iint_{[0,2]\times\left[\frac{\pi}{2},\frac{3\pi}{2}\right]} r^2 \sin^2\theta \cdot r\,dr\,d\theta = \int_0^2 \left(\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} r^3 \sin^2\theta\,d\theta \right) dr =$$

$$\cos 2\theta = \cos^2\theta - \sin^2\theta = 2\cos^2\theta - 1 = 1 - 2\sin^2\theta \Rightarrow \sin^2\theta = \frac{1-\cos 2\theta}{2}$$

$$= \int_0^2 \left(r^3 \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \frac{1-\cos 2\theta}{2}\,d\theta \right) dr = \int_0^2 \frac{r^3}{2} \cdot \left(\theta - \frac{\sin 2\theta}{2}\right)\Bigg|_{\theta=\frac{\pi}{2}}^{\theta=\frac{3\pi}{2}} dr = \int_0^2 \frac{\pi r^3}{2}$$

$$= \frac{\pi r^4}{8}\Bigg|_0^2 = 2\pi.$$

Trecerea la coordonate sferice.



$$\begin{cases} x = r\sin\varphi\cos\theta \\ y = r\sin\varphi\sin\theta \\ z = r\cos\varphi \end{cases}$$

$$\phi : \underset{r}{[0,\infty)} \times \underset{\theta}{[0,2\pi]} \times \underset{\varphi}{[0,\pi]} \longrightarrow \mathbb{R}^3$$

$$\phi(r,\theta,\varphi) = (r\sin\varphi\cos\theta,\ r\sin\varphi\sin\theta,\ r\cos\varphi)$$

$$(0,\infty)\times(0,2\pi)\times(0,\pi) \xleftrightarrow{\phi \text{ difeom. de cls } C^1} \mathbb{R}^3 \setminus S.$$

$$S = \{(x,0,z) \in \mathbb{R}^3 \mid x \geq 0,\ z \in \mathbb{R}\}.$$

$$\det J_f(r,\theta,\varphi) = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \varphi} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \varphi} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial \varphi} \end{vmatrix}$$

$$= \begin{vmatrix} \cos\theta \sin\varphi & -r\sin\theta\sin\varphi & r\cos\theta\cos\varphi \\ \sin\theta\sin\varphi & r\cos\theta\sin\varphi & r\sin\theta\cos\varphi \\ \cos\varphi & 0 & -r\sin\varphi \end{vmatrix} =$$

$$= -\underline{r^2\cos^2\theta\sin}^3\varphi - r^2\sin^2\theta\sin\varphi\cos^2\varphi - r^2\cos^2\theta\sin\varphi\cos^2\varphi -$$

$$- \underline{r^2\sin^2\theta\sin}^3\varphi = -r^2\sin^3\varphi - r^2\sin\varphi\cos^2\varphi = -r^2\sin\varphi$$

$$\left|\det J_\phi(r,\theta,\varphi)\right| = r^2 \sin\varphi \qquad \left(dx\,dy\,dz = r^2 \sin\varphi\, dr\,d\theta\,d\varphi\right).$$

$V \in \mathcal{J}(\mathbb{R}^3)$, $f : V \to \mathbb{R}$ integrabilă Riemann

$V' = V - S$, $V' \in \mathcal{J}(\mathbb{R}^3)$, $\lambda(V \setminus V') = 0$.

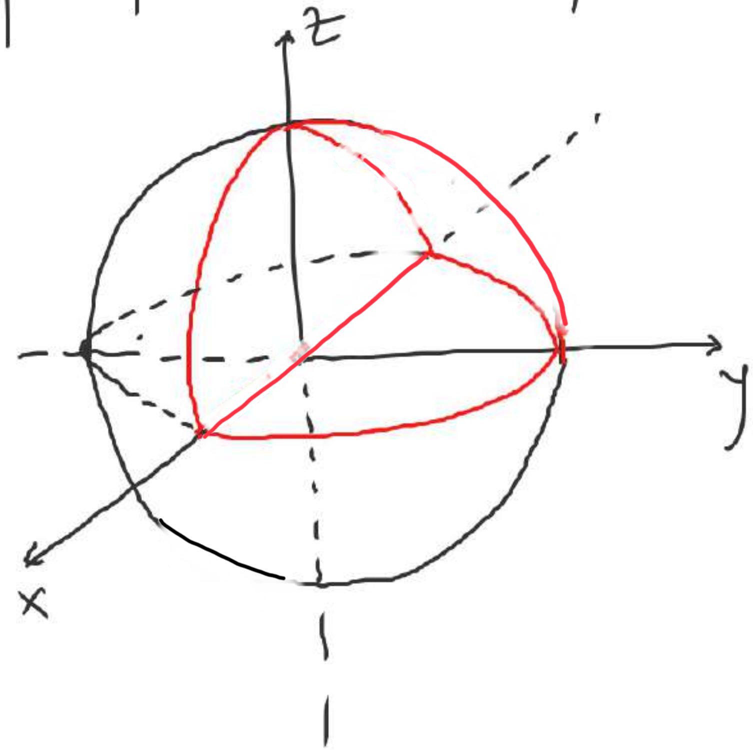
$\phi^{-1}(V'), \phi^{-1}(V) \in \mathcal{J}(\mathbb{R}^3)$, $\lambda(\phi^{-1}(V) \setminus \phi^{-1}(V')) = 0$.

$$\iiint_{V'} f(x,y,z)\,dx\,dy\,dz = \iiint_{\phi^{-1}(V')} f(r\cos\theta\sin\varphi, r\sin\theta\sin\varphi, r\cos\varphi)\cdot r^2\sin\varphi\,dr\,d\theta\,d\varphi$$

$$\|$$

$$\iiint_{V} f(x,y,z)\,dx\,dy\,dz = \iiint_{\phi^{-1}(V)} f(r\cos\theta\sin\varphi, r\sin\theta\sin\varphi, r\cos\varphi)\cdot r^2\sin\varphi\,dr\,d\theta\,d\varphi$$

Calculati $\iiint\limits_V z\,dx\,dy\,dz$, $V = \{(x,y,z) \mid x^2+y^2+z^2 \le 4,\ z \ge 0,\ y \ge 0\}$.



$$\begin{cases} x = r\cos\theta\sin\varphi \\ y = r\sin\theta\sin\varphi \\ z = r\cos\varphi \end{cases}$$

$$(x,y,z) \in V \iff \begin{cases} r \in [0,2] \\ \theta \in [0,\pi] \\ \varphi \in [0,\frac{\pi}{2}] \end{cases}.$$

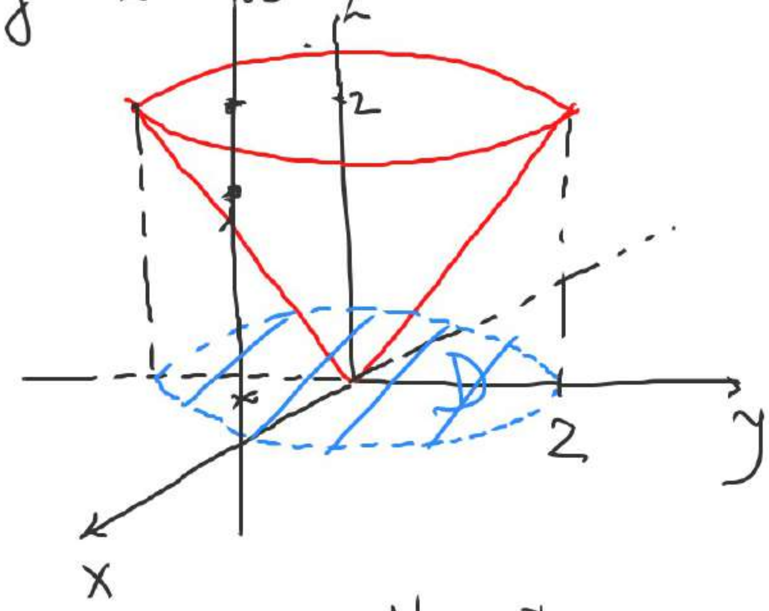
$$V \longleftrightarrow V' = [0,2] \times [0,\pi] \times [0,\tfrac{\pi}{2}] \qquad dx\,dy\,dz = r^2\sin\varphi\,dr\,d\theta\,d\varphi.$$

$$\iiint\limits_V z\,dx\,dy\,dz = \iiint\limits_{V'} r\cos\varphi \cdot r^2\sin\varphi\,dr\,d\theta\,d\varphi$$

$$= \int_0^2 \left(\int_0^{\pi} \left(\int_0^{\frac{\pi}{2}} r^3 \sin\varphi \cos\varphi \, d\varphi \right) d\theta \right) dr$$

$$= \int_0^2 r^3 dr \cdot \int_0^{\pi} d\theta \cdot \int_0^{\frac{\pi}{2}} \sin\varphi \cos\varphi \, d\varphi = \frac{r^4}{4}\Big|_0^2 \cdot \theta\Big|_0^{\pi} \cdot \frac{\sin^2\varphi}{2}\Big|_0^{\frac{\pi}{2}} = 2\pi$$

$$\iiint_V z(x^2+y^2)\,dx\,dy\,dz$$ unde V este o mulțime închisă mărginită de suprafețele $x^2+y^2=z^2$ și $z=2$.



$$V=\left\{(x,y,z)\ \middle|\ (x,y)\in D,\ \sqrt{x^2+y^2}\le z\le 2\right\}$$

$$D=\left\{(x,y)\in\mathbb{R}^2\ \middle|\ x^2+y^2\le 4\right\}.$$

$D\in \mathcal{J}(\mathbb{R}^2)$, $V\in \mathcal{J}(\mathbb{R}^3)$ (ex).

$pr_{xOy}V=D$.

$$\iiint_V z(x^2+y^2)\,dx\,dy\,dz=\iint_D\left(\int_{\sqrt{x^2+y^2}}^{2} z(x^2+y^2)\,dz\right)dx\,dy =$$

$$= \iint_D (x^2+y^2)\cdot \frac{z^2}{2}\Big|_{\sqrt{x^2+y^2}}^{2} dx\,dy = \frac{1}{2}\iint_D (x^2+y^2)\left(4-(x^2+y^2)\right)dx\,dy =$$

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases} \qquad (x,y)\in D \iff \begin{cases} r\in[0,2] \\ \theta\in[0,2\pi] \end{cases} \qquad dx\,dy = r\,dr\,d\theta$$

$$D: x^2+y^2 \le 4$$

$$= \frac{1}{2}\int_0^2 \left(\int_0^{2\pi} r^2(4-r^2)\cdot r\,d\theta\right) dr = \frac{1}{2}\int_0^2 2\pi r^3(1-r^2)\,dr = \pi\cdot\left(\frac{r^4}{4}-\frac{r^6}{6}\right)\Big|_0^2 = \cdots$$

Exerciții

1*) Fie $A \in \mathcal{J}(\mathbb{R}^n)$ cu $\lambda(A) = 0$ și $f : A \to \mathbb{R}$ mărginită.
Arătați că f este integrabilă Riemann și $\int_A f\,dx = 0$.

2*) Fie $f : [a,b] \times [a,b] \to \mathbb{R}$ continuă. Arătați că

$$\int_a^b \left(\int_a^y f(x,y)\,dx \right) dy = \int_a^b \left(\int_x^b f(x,y)\,dy \right) dx$$

3) Calculați $\iint_D \sqrt{x^2+y^2}\,dx\,dy$; $D = \{(x,y) \in \mathbb{R}^2 \mid 2x \le x^2+y^2 \le 4x,\ y \ge 0\}$

$\iiint_V y\,dx\,dy\,dz$; $V = \{(x,y,z) \in \mathbb{R}^3 \mid 2x+4y-z \le 8,\ x,y \ge 0,\ z \le -4\}$

Calculați

4) $\iint\limits_D (1+x)\,dx\,dy$, $D=\{(x,y)\in\mathbb{R}^2 \mid x^2+y^2\le 2y,\ y\le 2-x\}$

5) $\iint\limits_D (x+xy)\,dx\,dy$, D este trapezul determinat de $A(1,0)$, $B(5,0)$, $C(3,4)$, $D(1,4)$

6) $\iint\limits_D (x^2+y^2+xy)\,dx\,dy$, $D=\{(x,y)\in\mathbb{R}^2 \mid 4\le x^2+y^2\le 9,\ 0\le x\le y\}$

7) $\iint\limits_D (x+y)xy\,dx\,dy$, D este limitat de dreptele $x+y=-3$, $x+y=3$, $x-y=1$, $x-y=-1$

8) Aflați aria (adică masura Jordan) următoarelor mulțimi

i) $D = \{(x,y) \in \mathbb{R}^2 \mid x^2+y^2 \leq 4,\ y \leq x\}$

ii) $D = \{(x,y) \in \mathbb{R}^2 \mid 3 \leq 2x+2y \leq 4,\ y^2 \leq 2x\}$

iii) $D = \{(x,y) \in \mathbb{R}^2 \mid (x^2+y^2)^2 \leq a^2(x^2-y^2)\}$, $a>0$.

9) $\iint\limits_D \ln(x^2+y^2)\,dx\,dy$, $D = \{(x,y) \in \mathbb{R}^2 \mid 1 \leq x^2+y^2 \leq 4,\ y \leq \sqrt{3}x\}$

10) $\iint\limits_D \sqrt{xy}\,dx\,dy$, D este marginit de curbele

$y^2 = x,\ y^2 = 8x,\ xy = 1,\ xy = 8$.

11) $\iiint\limits_V x^2\,dx\,dy\,dz$, $V=\left\{(x,y,z)\in\mathbb{R}^3 \;\middle|\; z^2\le \frac{x^2}{4}+\frac{y^2}{9}\le 2z\right\}$

12) $\iiint\limits_V z\,dx\,dy\,dz$, $V=\left\{(x,y,z)\in\mathbb{R}^3 \;\middle|\; x^2+y^2\le 4z,\ 0\le z\le 1\right\}$

13) $\iiint\limits_V xyz\,dx\,dy\,dz$, $V=\left\{(x,y,z)\in\mathbb{R}^3 \;\middle|\; 9\le x^2+y^2\le z^2,\ 0\le z\le 5\right\}$

14) Calc. volumul (adică masura Jordan) următoarelor mulțimi

i) V este mărginit de suprafețele: $x^2+y^2+z^2=4$ și $x^2+y^2=3z$

ii) V este mărginit de suprafețele: $x^2+y^2=4z$, $x^2+y^2=4x$, $z=0$