C_9 - GA

Spații vectoriale euclidiene

Def $(V,+,\cdot)_{/\mathbb{R}}$, $g: V\times V \to \mathbb{R}$ s.n. produs scalar $\Leftrightarrow$

1) $g \in L^s(V,V;\mathbb{R})$ (formă biliniară simetrică)

2) g pozitiv definită i.e.

$$\begin{cases} g(x,x) = Q(x) > 0, \forall x \in V\setminus\{0_V\} \\ g(x,x) = Q(x) = 0 \Leftrightarrow x = 0_V \end{cases}, \quad Q: V\to\mathbb{R}$$

forma pătratică asociată

Not (V,g) $((E,g), (E,\langle\cdot,\cdot\rangle), (E,(\cdot,\cdot)))$ s.n. spațiu vectorial euclidian real

Def (V,g) s.v.e.r, $R=\{e_1,..,e_n\}$ reper

1) R s.n. reper ortogonal $\Leftrightarrow$ $g(e_i,e_j)=0$, $\forall i\neq j$, $i,j=\overline{1,n}$

2) R s.n. reper ortonormat $\Leftrightarrow$ $g(e_i,e_j)=\delta_{ij}=\begin{cases}0, & i\neq j\\ 1, & i=j\end{cases}$

(ortogonali și versori)

Prop (V,g) s.v.e.r.

$R=\{e_1,..,e_n\} \xrightarrow{C} R'=\{e'_1,..,e'_n\}$ repere ortonormate

$$\Rightarrow \left[C\in O(n) \Leftrightarrow C\cdot C^T = I_n\right]$$

Dem

$$e'_k = \sum_{i=1}^{n} c_{ik} e_i, \quad \forall k=\overline{1,n}$$

$$\delta_{kp} = g(e'_k,e'_p) = g\Big(\sum_{i=1}^{n} c_{ik} e_i, \sum_{j=1}^{n} c_{jp} e_j\Big) = \sum_{i,j=1}^{n} c_{ik} c_{jp} \underbrace{g(e_i,e_j)}_{\delta_{ij}}$$

$$= \sum_{i=1}^{n} c_{ik} c_{ip} \Rightarrow I_n = C^T C \Rightarrow C\in O(n)$$

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OBS

Dacă $\mathcal{R}, \mathcal{R}'$ sunt repere ortonormate, la fel orienta

$\Rightarrow C \in SO(n)$ (i.e $C \in O(n)$, $\det C = 1 > 0$)

OBS

A da un produs scalar $\Leftrightarrow$ a declara un reper ortonormat

„$\Rightarrow$" $g: V \times V \longrightarrow \mathbb{R}$ produs scalar dat

$\mathcal{R} = \{e_1, ..., e_n\}$ reper ortonormat, $g(e_i, e_j) = \delta_{ij}$, $\forall i,j = \overline{1,n}$

„$\Leftarrow$" $\mathcal{R}$ reper ortonormat $g(e_i, e_j) = \delta_{ij}$

$g: V \times V \longrightarrow \mathbb{R}$ produs scalar

$$g(x,y) = g\left(\sum_{i=1}^{n} x_i e_i, \sum_{j=1}^{n} y_j e_j\right) = \sum_{i,j=1}^{n} x_i y_j \overbrace{g(e_i, e_j)}^{\delta_{ij}}$$

$$= \sum_{i=1}^{n} x_i y_i = x_1 y_1 + \ldots + x_n y_n.$$

Exemplu

$g_0: \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R}$, $g_0(x,y) = x_1 y_1 + \ldots + x_n y_n$.

$\mathcal{R}_0 = \{e_1, ..., e_n\}$ reper canonic, $g_0(e_i, e_j) = \delta_{ij}$, $\forall i,j = \overline{1,n}$

g_0 = produs scalar canonic ; $\|x\| = \sqrt{g_0(x,x)}$ norma vectorului

$\mathcal{R}_0$ reper ortonormat

Def $(\mathbb{R}^3, g_0)$ s.v.e.r (cu str. canonică)

$x, y \in \mathbb{R}^2$

Definim produsul vectorial $x \times y$ astfel

I $S = \{x, y\}$ SLD $\Rightarrow x \times y = 0$

II $S = \{x, y\}$ SLI avem:

1) $\|x \times y\|^2 = \begin{vmatrix} g_0(x,x) & g_0(x,y) \\ g_0(y,x) & g_0(y,y) \end{vmatrix}$

2)

$g_0(x \times y, y) = 0$, $g_0(x \times y, x) = 0$

3) $R = \{x, y, x \times y\}$ este reper pozitiv orientat (la fel orientat cu R_0 = reperul canonic)

<u>OBS</u> $x \times y$ este un "determinant formal"

$R_0 = \{e_1, e_2, e_3\}$ reperul canonic în $\mathbb{R}^3$

$x = \sum_1^3 x_i e_i$, $y = \sum_1^3 y_j e_j$

$$x \times y = \begin{vmatrix} e_1 & e_2 & e_3 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{vmatrix}$$

$$= e_1 \begin{vmatrix} x_2 & x_3 \\ y_2 & y_3 \end{vmatrix} - e_2 \begin{vmatrix} x_1 & x_3 \\ y_1 & y_3 \end{vmatrix} + e_3 \begin{vmatrix} x_1 & x_2 \\ y_1 & y_2 \end{vmatrix}$$

$$= (x_2 y_3 - x_3 y_2,\ x_3 y_1 - x_1 y_3,\ x_1 y_2 - x_2 y_1)$$

<u>OBS</u> a) $x \times y = -y \times x$

b) $(x \times y) \times z = g_0(x, z) y - g_0(y, z) x$

c) <u>Identitatea Jacobi</u>

$$\sum_{x,y,z}^{c} (x \times y) \times z = (x \times y) \times z + (y \times z) \times x + (z \times x) \times y = 0$$

<u>Def</u> Produsul mixt

$(\mathbb{R}^3, g_0)$ s.v.e.r cu str. canonică, $\{x, y, z\} \subset \mathbb{R}^3$

$$z \wedge x \wedge y = g_0(z, x \times y) = \begin{vmatrix} z_1 & z_2 & z_3 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{vmatrix} = \begin{vmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ z_1 & z_2 & z_3 \end{vmatrix}$$

<u>OBS</u> a) $x \times y = \begin{vmatrix} x_1 & y_1 & e_1 \\ x_2 & y_2 & e_2 \\ x_3 & y_3 & e_3 \end{vmatrix}$; b) $z \wedge x \wedge y = x \wedge y \wedge z$

Exemplu $(\mathbb{R}^3, g_0)$, $g_0(x,y) = x_1y_1 + x_2y_2 + x_3y_3$.

$u = (1,-1,2)$, $v = (0,1,3)$, $w = (1,1,0)$

a) $u \times v$; b) $w \wedge u \wedge v$.

SOL

a) $$u \times v = \begin{vmatrix} e_1 & e_2 & e_3 \\ 1 & -1 & 2 \\ 0 & 1 & 3 \end{vmatrix} = e_1 \begin{vmatrix} -1 & 2 \\ 1 & 3 \end{vmatrix} - e_2 \begin{vmatrix} 1 & 2 \\ 0 & 3 \end{vmatrix} + e_3 \begin{vmatrix} 1 & -1 \\ 0 & 1 \end{vmatrix}$$
$$= (-5, -3, 1)$$

b) $w \wedge u \wedge v = g_0(w, u \times v) = 1(-5) + 1\cdot(-3) + 0\cdot 1 = -8$

sau $$w \wedge u \wedge v = \begin{vmatrix} 1 & 1 & 0 \\ 1 & -1 & 2 \\ 0 & 1 & 2 \end{vmatrix} = -8$$

OBS

a) $g_0(u \times v, u) = 0$, $g_0(u \times v, v) = 0$

b) $R = \{u, v, u \times v\}$ reper la fel orientat cu R_0.

c) $\|u\| = \sqrt{g_0(u,u)} = \sqrt{1+1+4} = \sqrt{6}$

$\|v\| = \sqrt{g_0(v,v)} = \sqrt{0+1+9} = \sqrt{10}$.

$g_0(u,v) = 0 - 1 + 6 = 5$

Problemă (V, g) s.v.e.r.

$$R \longrightarrow R' \longrightarrow R''$$

reper arbitrar — reper ortogonal — reper ortonormat.

Procedeul Gram-Schmidt

(V, g) s.v.e.r, $\dim V = n$

Fie $R = \{f_1, .., f_n\}$ reper arbitrar $\Rightarrow$

$\exists\, R' = \{e_1, .., e_n\}$ reper ortogonal aî $Sp\{f_1, .., f_i\} = Sp\{e_1, .., e_i\}$ $\forall i = \overline{1,n}$

Dem   Dem este inductivă

$f_1 \neq 0_V \quad e_1 = f_1$

Construim $e_2 = f_2 + \alpha_{21} e_1$ a.î. $\langle e_1, e_2 \rangle = 0$

$$\langle f_1, f_2 + \alpha_{21} e_1 \rangle = 0 \Rightarrow \langle f_1, f_2 \rangle + \alpha_{21} \langle f_1, e_1 \rangle = 0$$

$$\alpha_{21} = -\frac{\langle f_2, f_1 \rangle}{\langle f_1, f_1 \rangle} \Rightarrow e_2 = f_2 - \frac{\langle f_2, f_1 \rangle}{\langle f_1, f_1 \rangle} \cdot e_1$$

$$\begin{cases} f_1 = e_1 \\ f_2 = \frac{\langle f_2, e_1 \rangle}{\langle e_1, e_1 \rangle} \cdot e_1 + e_2 \end{cases} \Rightarrow Sp\{f_1, f_2\} = Sp\{e_1, e_2\}.$$

P_{k-1} adev : $\{e_1, .., e_{k-1}\}$ ortogonali și

$Sp\{e_1, .., e_i\} = Sp\{f_1, .., f_i\}, \forall i = \overline{1, k-1}$

$$e_k = f_k + \sum_{j=1}^{k-1} \alpha_{kj} e_j$$

$$\langle e_k, e_p \rangle = 0 \quad \forall p = \overline{1, k-1}$$

$$\langle f_k, e_p \rangle + \sum_{j=1}^{k-1} \alpha_{kj} \langle e_j, e_p \rangle = 0$$

$$\langle f_k, e_p \rangle + \alpha_{kp} \langle e_p, e_p \rangle = 0 \Rightarrow \alpha_{kp} = -\frac{\langle f_k, e_p \rangle}{\langle e_p, e_p \rangle}$$

$$\forall p = \overline{1, k-1}$$

$$e_k = f_k - \sum_{j=1}^{p-1} \frac{\langle f_k, e_j \rangle}{\langle e_j, e_j \rangle} e_j$$

$$\begin{cases} f_1 = e_1 \\ f_2 = \frac{\langle f_2, e_1 \rangle}{\langle e_1, e_1 \rangle} e_1 + e_2 \\ \vdots \\ f_k = \frac{\langle f_k, e_1 \rangle}{\langle e_1, e_1 \rangle} e_1 + \frac{\langle f_k, e_2 \rangle}{\langle e_2, e_2 \rangle} \cdot e_2 + \dots + \frac{\langle f_k, e_{k-1} \rangle}{\langle e_{k-1}, e_{k-1} \rangle} e_{k-1} + e_k \end{cases}$$

$\Rightarrow Sp\{f_1, .., f_i\} = Sp\{e_1, .., e_i\}, \forall i = \overline{1, k}$

$R' = \{e_1, \dots, e_n\}$ sistem de vectori ortogonali $\Rightarrow R$ e SLI

Fie $a_1, \dots, a_n \in \mathbb{R}$ a.î. $v = a_1 e_1 + \dots + a_n e_n = 0_{\mathbb{R}}$

$\langle a_1 e_1 + \dots + a_n e_n, e_1 \rangle = 0 \Rightarrow a_1 \langle e_1, e_1 \rangle + a_2 \langle e_2, e_1 \rangle + \dots + a_n \langle e_n, e_1 \rangle = 0$

$a_1 \|e_1\|^2 = 0 \Rightarrow a_1 = 0$

Analog $\langle v, e_k \rangle = 0, \forall k = \overline{2,n} \Rightarrow a_k = 0, \forall k = \overline{2,n}$

$\dim V = n = |R'| \Rightarrow R'$ reper ortogonal în V

$$\begin{cases} f_1 = e_1 \\ f_2 = \frac{\langle f_2, e_1 \rangle}{\langle e_1, e_1 \rangle} e_1 + e_2 \\ \vdots \\ e_n = \frac{\langle f_n, e_1 \rangle}{\langle e_1, e_1 \rangle} e_1 + \dots \frac{\langle f_n, e_{n-1} \rangle}{\langle e_{n-1}, e_{n-1} \rangle} e_{n-1} + e_n \end{cases}$$

$Sp\{f_1, \dots, f_i\} = Sp\{e_1, \dots, e_i\}, \forall i = \overline{1,n}$

$$\underbrace{R = \{f_1, \dots, f_n\}}_{\text{reper } \forall} \xrightarrow{A} \underbrace{R' = \{e_1, \dots, e_n\}}_{\text{reper ortogonal}} \xrightarrow{B} \underbrace{R'' = \left\{\frac{e_1}{\|e_1\|}, \dots, \frac{e_n}{\|e_n\|}\right\}}_{\text{reper ortonormat}}$$

OBS $v \in V, v \neq 0_V$ $\quad w = \frac{v}{\|v\|}, \|w\| = 1$

w este versor.

$\|w\|^2 = g(w, w) = g\left(\frac{1}{\|v\|} v, \frac{1}{\|v\|} v\right) = \frac{1}{\|v\|^2} g(v, v) = 1$

$\Rightarrow \|w\| = 1$

$$A^{-1} = \begin{pmatrix} 1 & \frac{\langle f_2, e_1 \rangle}{\langle e_1, e_1 \rangle} & \dots & \frac{\langle f_n, e_1 \rangle}{\langle e_1, e_1 \rangle} \\ 0 & 1 & & \vdots \\ 0 & 0 & & \frac{\langle f_n, e_{n-1} \rangle}{\langle e_{n-1}, e_{n-1} \rangle} \\ \vdots & \vdots & & \\ 0 & 0 & & 1 \end{pmatrix}, \quad B = \begin{pmatrix} \frac{1}{\|e_1\|} & 0 & 0 \\ 0 & \frac{1}{\|e_2\|} & 0 \\ \vdots & \vdots & \vdots \\ 0 & 0 & \frac{1}{\|e_n\|} \end{pmatrix}$$

$\det(A^{-1}) = \frac{1}{\det A} = 1, \det A = 1 > 0$ $\qquad \det B = \frac{1}{\|e_1\| \dots \|e_n\|} > 0$

Def (V, g) s.v.e.r.

a) $x \in V$, $x^{\perp} = \langle \{x\} \rangle^{\perp} = \{ y \in V \mid g(x, y) = 0 \} \subset V$ subsp. vectorial.

b) $U \subset V$ subsp vect, $U^{\perp} = \{ y \in V \mid g(x, y) = 0, \forall x \in U \}$

$U^{\perp} \subset V$ ssp vect.

Ex $(\mathbb{R}^3, g_0)$, $u = (1, 2, -1)$

a) $u^{\perp} = ?$. b) Det. un reper ortonormat în $u^{\perp}$

Sol

a) $u^{\perp} = \{ x \in \mathbb{R}^3 \mid \underbrace{g_0(x, u)}_{x_1 + 2x_2 - x_3} = 0 \}$, $\dim u^{\perp} = 2$.

a) Det $\mathcal{R}$ reper în $u^{\perp}$

$x_3 = x_1 + 2x_2$

$u^{\perp} = \{ (x_1, x_2, x_1 + 2x_2) = x_1 \underbrace{(1, 0, 1)}_{f_1} + x_2 \underbrace{(0, 1, 2)}_{f_2}, x_1, x_2 \in \mathbb{R} \}$

$= \langle \{ f_1, f_2 \} \rangle$

$\mathcal{R} = \{ f_1, f_2 \}$

Aplicăm Gram - Schmidt

$e_1 = f_1 = (1, 0, 1)$

$e_2 = f_2 - \frac{\langle f_2, e_1 \rangle}{\langle e_1, e_1 \rangle} \cdot e_1 = (0, 1, 2) - \frac{2}{2} (1, 0, 1)$

$= (0, 1, 2) - (1, 0, 1) = (-1, 1, 1)$

$\mathcal{R}' = \{ e_1, e_2 \}$ reper ortogonal.

$\mathcal{R}'' = \{ \frac{1}{\sqrt{2}} e_1, \frac{1}{\sqrt{3}} e_2 \}$ reper ortonormat.

$\| e_1 \| = \sqrt{\langle e_1, e_1 \rangle} = \sqrt{1 + 0 + 1} = \sqrt{2}$, $\| e_2 \| = \sqrt{3}$

Prop (V, g) s.v.e.r, $U \subseteq V$ subsp.v.

$\Rightarrow \exists! \, \underset{\text{ssp}}{U^\perp} \subseteq V$ a.î. $E = U \oplus U^\perp$ ↳ complement ortogonal.

Dem $U + U^\perp \underset{\text{ssp } V}{\subseteq} V$ și $U \oplus U^\perp \subseteq V$

Dem că $\oplus$ (suma este directă)

Fie $x \in U \cap U^\perp$

$g(x, y) = 0, \forall y \in U$ $\xrightarrow{g \text{ poz.def.}} x = 0_V$

Fie $y = x \Rightarrow g(x, x) = 0$

Dem că $V \subseteq U \oplus U^\perp$

Fie $\{e_1, .., e_k\}$ reper ortonormat în U

Fie $u \in V$, $u' = u - \underbrace{\sum_{i=1}^{k} \langle u, e_i \rangle e_i}_{u''}$, $u'' \in U$

$u = \underset{U}{u''} + \underset{U^\perp}{u'}$

Dem că $u' \in U^\perp$ $\{e_1, .., e_k\}$ reper în U ortonormat

Arătăm $\langle u', e_j \rangle = 0, \forall j = \overline{1, k}$

$\langle u - \sum_{i=1}^{k} \langle u, e_i \rangle e_i, e_j \rangle =$

$= \langle u, e_j \rangle - \underbrace{\sum_{i=1}^{k} \langle u, e_i \rangle \overbrace{\langle e_i, e_j \rangle}^{\delta_{ij}}}_{\langle u, e_j \rangle} = 0, \forall j = \overline{1, k}$

$\Rightarrow \langle u', x \rangle = 0, \forall x \in U$

$u = \underset{U}{u''} + \underset{U^\perp}{u'}$ Deci $V = U \oplus U^\perp$

Ex $(\mathbb{R}^3, g_0)$ $\quad U = \{x \in \mathbb{R}^3 \mid \underbrace{x_1 - x_2 + x_3}_{g((x_1,x_2,x_3),(1,-1,1))} = 0\}$, $\dim U = 2$

a) $U^\perp$, $\mathbb{R}^3 = U \oplus U^\perp$

b) $R = R_1 \cup R_2$ reper ortonormat în $\mathbb{R}^3$ cu

R_1 — " — U

R_2 — " — $U^\perp$

a) $U^\perp = \langle (1,-1,1) \rangle$ $\qquad R_2 = \{ \frac{1}{\sqrt{3}}(1,-1,1) \}$

$U = \{ (x_1, x_1 + x_3, x_3) \} = \{ x_1 \underbrace{(1,1,0)}_{f_1} + x_3 \underbrace{(0,1,1)}_{f_2} \mid x_1, x_3 \in \mathbb{R} \}$

$e_1 = f_1$

$e_2 = f_2 - \frac{\langle f_2, e_1 \rangle}{\langle e_1, e_1 \rangle} \cdot e_1 = (0,1,1) - \frac{1}{2}(1,1,0)$

$= \left(-\frac{1}{2}, \frac{1}{2}, 1\right) = \frac{1}{2}(-1,1,2)$

$R_1 = \{ \frac{1}{\sqrt{2}}(1,1,0), \frac{1}{\sqrt{6}}(-1,1,2) \}$ reper ortonormat în U

$u = \alpha v$, $\alpha > 0$, $\frac{u}{\|u\|} = \frac{v}{\|v\|}$