1) Să se găsească punctele de extrem local ale funcției $f : \mathbb{R}^2 \to \mathbb{R}$, $f(x,y) = xy\, e^{-(x^2+y^2)}$

<u>Soluție</u> Dom de def $\mathbb{R}^2$ al funcției f este o mulțime deschisă și f este de clasă C^2. Punctele de extrem ale lui f se găsesc printre punctele critice.

$$\frac{\partial f}{\partial x} = y e^{-(x^2+y^2)} - xy e^{-(x^2+y^2)} \cdot 2x = e^{-(x^2+y^2)} (y - 2x^2 y)$$

$$\frac{\partial f}{\partial y} = e^{-(x^2+y^2)} \left(x - 2xy^2 \right)$$

$$\begin{cases} \dfrac{\partial f}{\partial x}(x,y) = 0 \\ \dfrac{\partial f}{\partial y}(x,y) = 0 \end{cases} \iff \begin{cases} e^{-(x^2+y^2)}(y-2x^2y) = 0 \\ e^{-(x^2+y^2)}(x-2xy^2) = 0 \end{cases}$$

$$\begin{cases} y(1-2x^2) = 0 \\ x(1-2y^2) = 0 \end{cases}$$

$y(1-2x^2) = 0 \iff y = 0$ sau $x = \pm \frac{1}{\sqrt{2}}$

$$\left.\begin{array}{l} y = 0 \\ x(1-2y^2) = 0 \end{array}\right\} \Rightarrow x = 0.$$

$x = -\frac{1}{\sqrt{2}} \Rightarrow 1-2y^2 = 0 \Rightarrow y = -\frac{1}{\sqrt{2}}$ sau $y = \frac{1}{\sqrt{2}}$

$x = \frac{1}{\sqrt{2}} \Rightarrow 1-2y^2 = 0 \Rightarrow y = \frac{1}{\sqrt{2}}$ sau $y = -\frac{1}{\sqrt{2}}$

Punctele critice sunt

$$(0,0);\ \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right), \left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right), \left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right), \left(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$$

$$\frac{\partial f}{\partial x} = e^{-(x^2+y^2)}\left(y - 2x^2 y\right)$$

$$\frac{\partial^2 f}{\partial x^2} = -2x e^{-(x^2+y^2)}\left(y - 2x^2 y\right) - 4xy e^{-(x^2+y^2)} = e^{-(x^2+y^2)}\left(4x^3 y - 6xy\right)$$

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = -2y\left(y - 2x^2 y\right) e^{-(x^2+y^2)} +$$

$$+ \left(1 - 2x^2\right) e^{-(x^2+y^2)} = e^{-(x^2+y^2)}\left(1 - 2y^2 - 2x^2 + 4x^2 y^2\right).$$

$$\frac{\partial^2 f}{\partial y^2} = e^{-(x^2+y^2)}\left(4xy^3 - 6xy\right)$$

$$H_f(x,y) = \begin{pmatrix} \frac{\partial^2 f}{\partial x^2}(x,y) & \frac{\partial^2 f}{\partial x \partial y}(x,y) \\ \frac{\partial^2 f}{\partial y \partial x}(x,y) & \frac{\partial^2 f}{\partial y^2}(x,y) \end{pmatrix}$$

$\begin{cases} \Delta_1 > 0, \Delta_2 > 0 \to \text{min local} \\ \Delta_1 < 0, \Delta_2 > 0 \to \text{max local} \\ \Delta_2 < 0 \to \text{nu este pct de extrem local} \end{cases}$

$$= e^{-(x^2+y^2)} \cdot \begin{pmatrix} 4x^3y - 6xy & 1-2x^2-2y^2+4x^2y^2 \\ 1-2x^2-2y^2+4x^2y^2 & 4xy^3-6xy \end{pmatrix}$$

$$H_f(0,0) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \left.\begin{matrix} \Delta_1 = 0 \\ \Delta_2 = -1 \end{matrix}\right| \Longrightarrow (0,0) \text{ nu este pct de extrem local}$$

$$H_f\left(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right) = H_f\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) = \frac{1}{e}\begin{pmatrix} -2 & 0 \\ 0 & -2 \end{pmatrix}$$

$$\left.\begin{array}{l}\Delta_1 = -\frac{2}{e} < 0 \\ \Delta_2 = \frac{4}{e^2} > 0\end{array}\right\} \Rightarrow \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right), \left(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right) \text{ puncte de max local}$$

$$H_f\left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right) = H_f\left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) = \frac{1}{e}\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$$

$$\left.\begin{array}{l}\Delta_1 = \frac{2}{e} > 0 \\ \Delta_2 = \left|\frac{1}{e}\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}\right| = \frac{4}{e^2} > 0\end{array}\right\} \Rightarrow \left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right) \text{ și } \left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) \text{ pct. de minim local.}$$

<u>Exercițiu</u>. Determinați $\sup_{\mathbb{R}^2} f$ și $\inf_{\mathbb{R}^2} f$ unde

$$f(x,y) = e^{-(x^2+y^2)} \cdot xy$$

Exercițiu $f : K = \{(x,y) \in \mathbb{R}^2 \mid x^2+y^2 \leq 1\} \to \mathbb{R}$

$f(x,y) = x^2 + 2y^2$, $\sup_K f = ?$ $\inf_K f = ?$

Propoziție. $f : D = \overset{\circ}{D} \subset \mathbb{R}^2 \to \mathbb{R}$, de clasă C^2, $a = (x_0, y_0, z_0)$ pct critic

$$H_f(a) = \begin{pmatrix} \frac{\partial^2 f}{\partial x^2}(a) & \frac{\partial^2 f}{\partial x \partial y}(a) & \frac{\partial^2 f}{\partial x \partial z}(a) \\ \frac{\partial^2 f}{\partial y \partial x}(a) & \frac{\partial^2 f}{\partial y^2}(a) & \frac{\partial^2 f}{\partial y \partial z}(a) \\ \frac{\partial^2 f}{\partial z \partial x}(a) & \frac{\partial^2 f}{\partial z \partial y}(a) & \frac{\partial^2 f}{\partial z^2}(a) \end{pmatrix}$$

$\Delta_1 = \dfrac{\partial^2 f}{\partial x^2}(a)$,

$$\Delta_2 = \begin{vmatrix} \dfrac{\partial^2 f}{\partial x^2}(a) & \dfrac{\partial^2 f}{\partial x \partial y}(a) \\ \dfrac{\partial^2 f}{\partial y \partial x}(a) & \dfrac{\partial^2 f}{\partial y^2}(a) \end{vmatrix}, \quad \Delta_3 = \det H_f(a).$$

1) dacă $\Delta_1 > 0$, $\Delta_2 > 0$, $\Delta_3 > 0 \Rightarrow$ a este pct de min local

2) dacă $\Delta_1 < 0$, $\Delta_2 > 0$, $\Delta_3 < 0 \Rightarrow$ a este pct de max. local

3) dacă $(\Delta_1 \geq 0, \Delta_2 \geq 0, \Delta_3 \geq 0)$ sau $(\Delta_1 \leq 0, \Delta_2 \geq 0, \Delta_3 \leq 0)$ dar există j a.î. $\Delta_j = 0$ nu ne putem pronunța

4) în orice altă situație a nu este pct de extrem local.

Obs. Dacă $\Delta_1 \neq 0$, $\Delta_2 \neq 0$ și $\Delta_3 \neq 0$ dar nu suntem nici în cazul 1) și nici în cazul 2) atunci a nu este punct de extrem local.

<u>Propoziție</u>. Fie $f : D = \overset{\circ}{D} \subset \mathbb{R}^n \longrightarrow \mathbb{R}$, f de clasă C^2 și $a \in D$ punct critic.

1) dacă toate val. proprii ale lui $H_f(a)$ sunt strict pozitive atunci a este minim local

2) dacă toate val. proprii ale lui $H_f(a)$ sunt strict negative at. a este pct de maxim local

3) dacă $H_f(a)$ are o valoare proprie strict pozitivă și o val. proprie strict negativă atunci a nu

este punct de extrem local

4) în orice alt caz nu ne putem pronunța

Reamintim (Vezi cursul 9)

1) dacă $d^2f(a)$ este poz definită $\Rightarrow$ a pct de minim loc

2) dacă $d^2f(a)$ este neg definită $\Rightarrow$ a pct de max. loc

3) dacă $\exists u, v \in \mathbb{R}^n$ a.î. $d^2f(a)(u) > 0$ și $d^2f(a)(v) < 0$ atunci a nu este pct de extrem local

$f, g : \mathbb{R}^2 \to \mathbb{R}$

$f(x,y) = x^6 + y^2 \qquad g(x,y) = -x^6 + y^2$

$(0,0)$ pct critic pt f și g.

$H_f(x,y) = \begin{pmatrix} 30x^4 & 0 \\ 0 & 2 \end{pmatrix} \quad H_g(x,y) = \begin{pmatrix} -30x^4 & 0 \\ 0 & 2 \end{pmatrix}$

$H_f(0,0) = H_g(0,0) = \begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix}$

$d^2f(0,0)(u,v) = d^2g(0,0)(u,v) = 2v^2$ – pozitiv semidef.

$(0,0)$ minim loc pt f ; $(0,0)$ nu este pct de ext. local pt g

3) Determinați punctele de extrem local ale funcției $f : \mathbb{R}^3 \to \mathbb{R}$,

$$f(x,y,z) = x^2 + y^2 + z^2 + xy + xz + yz - x - y - 2z + 1$$

<u>Soluție</u>. $\mathbb{R}^3$ - deschisă, $f \in C^2(\mathbb{R})$.

$$\frac{\partial f}{\partial x} = 2x + y + z - 1$$

$$\frac{\partial f}{\partial y} = 2y + x + z - 1, \quad \frac{\partial f}{\partial z} = 2z + x + y - 2$$

$$\begin{cases} \frac{\partial f}{\partial x} = 0 \\ \frac{\partial f}{\partial y} = 0 \\ \frac{\partial f}{\partial z} = 0 \end{cases} \Longleftrightarrow \begin{cases} x = 0 \\ y = 0 \\ z = 1 \end{cases}$$

Puncte critice: $(0, 0, 1)$

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial^2 f}{\partial y^2} = \frac{\partial^2 f}{\partial z^2} = 2$$

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial x \partial z} = \frac{\partial^2 f}{\partial y \partial z} = 1.$$

<u>Metoda 1.</u>

$$H_f(x,y,z) = H_f(0,0,1) = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix}$$

$\Delta_1 = 2 > 0$

$\Delta_2 = \begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} = 3 > 0$ $\Rightarrow (0,0,1)$ pct de minim local.

$\Delta_3 = \begin{vmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{vmatrix} = 4 > 0$

Metoda II : Calculăm val proprii pt $H_f(0,0,1)$

$$H_f(0,0,1) = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix}$$

$$\lambda_1 \lambda_2 \lambda_3 = \det H_f(0,0,1)$$

$$\lambda_1 + \lambda_2 + \lambda_3 = Tr(H_f(0,0,1))$$

$$\det(H_f(0,0,1) - \lambda I_3) = 0.$$

$$\begin{vmatrix} 2-\lambda & 1 & 1 \\ 1 & 2-\lambda & 1 \\ 1 & 1 & 2-\lambda \end{vmatrix} = (2-\lambda)^3 + 2 - 3(2-\lambda) = 0 \Rightarrow \begin{cases} \lambda_1 = \lambda_2 = 1 \\ \lambda_3 = 4. \end{cases}$$

$\lambda_i > 0, \ \forall i = 1,2,3 \Rightarrow (0,0,1)$ punct de minim local

Metoda III. Folosim Metoda lui Gauss pt $d^2f(0,0,1)$

$$d^2f(0,0,1)(a,b,c) = \frac{\partial^2 f}{\partial x^2}(0,0,1)\,a^2 + \frac{\partial^2 f}{\partial y^2}(0,0,1)\,b^2 + \frac{\partial^2 f}{\partial z^2}(0,0,1)\,c^2$$

$$+ 2\frac{\partial^2 f}{\partial x \partial y}(0,0,1)\,ab + 2\frac{\partial^2 f}{\partial x \partial z}(0,0,1)\,ac + 2\frac{\partial^2 f}{\partial y \partial z}(0,0,1)\,bc$$

$$= 2a^2 + 2b^2 + 2c^2 + 2ab + 2ac + 2bc.$$

$$= \left(2a^2 + 2ab + 2ac + \frac{b^2}{2} + \frac{c^2}{2}\right) + \left(\frac{3b^2}{2} + 2bc + \frac{2}{3}c^2\right) + \frac{7}{6}c^2$$

$$= 2\left(a + \frac{b}{2} + \frac{c}{2}\right)^2 + \left(\sqrt{\frac{3}{2}}\,b + \sqrt{\frac{2}{3}}\cdot c\right)^2 + \frac{7}{6}c^2$$

$d^2f(0,0,1)(a,b,c) > 0, \ \forall (a,b,c) \in \mathbb{R}^3 \setminus \{(0,0,0)\}$

$\Rightarrow d^2 f(0,0,1)$ poz def $\Rightarrow (0,0,1)$ pct de min local.

4) Determinați punctele de extrem local ale funcției

$f: \mathbb{R}^3 \to \mathbb{R}$, $f(x,y,z) = x^4 + 4xy + 2y^2 + z^2 + 2z$

<u>Soluție</u> $Dom(f) = \mathbb{R}^3$ - deschisă, $f \in C^2(\mathbb{R})$.

$$\frac{\partial f}{\partial x} = 4x^3 + 4y, \quad \frac{\partial f}{\partial y} = 4x + 4y, \quad \frac{\partial f}{\partial z} = 2z + 2$$

$$\begin{cases} \frac{\partial f}{\partial x} = 0 \\ \frac{\partial f}{\partial y} = 0 \\ \frac{\partial f}{\partial z} = 0 \end{cases} \Longleftrightarrow \begin{cases} 4x^3 + 4y = 0 \\ 4x + 4y = 0 \Rightarrow y = -x \Rightarrow 4x^3 - 4x = 0 \\ 2z + 2 = 0 \Rightarrow z = -1 \end{cases}$$

$x_1 = 0$, $y_1 = 0$

$x_2 = 1$, $y_2 = -1$

$x_3 = -1$, $y_3 = 1$

Puncte critice: $(0,0,-1), (1,-1,-1), (-1,1,-1)$

$$\frac{\partial^2 f}{\partial x^2} = 12x^2, \quad \frac{\partial^2 f}{\partial y^2} = 4, \quad \frac{\partial^2 f}{\partial z^2} = 2$$

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x} = 4, \quad \frac{\partial^2 f}{\partial x \partial z} = \frac{\partial^2 f}{\partial z \partial x} = 0, \quad \frac{\partial^2 f}{\partial y \partial z} = \frac{\partial^2 f}{\partial z \partial y} = 0.$$

$$H_f(x,y,z) = \begin{pmatrix} 12x^2 & 4 & 0 \\ 4 & 4 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

$$H_f(1,-1,-1) = H_f(-1,1,-1) = \begin{pmatrix} 12 & 4 & 0 \\ 4 & 4 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

$\Delta_1 > 0$, $\Delta_2 = \begin{vmatrix} 12 & 4 \\ 4 & 4 \end{vmatrix} = 32 > 0$

$\Delta_3 = 64 > 0 \Rightarrow (1,-1,-1)$ și $(-1,1,-1)$ sunt pct de min local.

$$H_f(0,0,-1) = \begin{pmatrix} 0 & 4 & 0 \\ 4 & 4 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

$$\left.\begin{aligned} &\Delta_1 = 0 \\ &\Delta_2 = \begin{vmatrix} 0 & 4 \\ 4 & 4 \end{vmatrix} = -16 < 0 \\ &\Delta_3 = -32 < 0 \end{aligned}\right\} \Rightarrow (0,0,-1) \text{ nu este pct de extrem local}$$