Consultație

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1. Studiați convergența seriilor:

a) $\sum_{n=1}^{\infty} \frac{\sin(nx)}{2^n}$, $x \in \mathbb{R}$.

Sol.: $x_n = \frac{\sin(nx)}{2^n}$ $\forall n \in \mathbb{N}^*$.

$|x_n| = \frac{|\sin(nx)|}{2^n} \leq \frac{1}{2^n}$ $\forall n \in \mathbb{N}^*$.

Fie $y_n = \frac{1}{2^n}$ $\forall n \in \mathbb{N}^*$.

$\sum_{n=1}^{\infty} y_n = \sum_{n=1}^{\infty} \frac{1}{2^n} = \sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n$ conv. (serie geom., $q = \frac{1}{2}$).

Conform crit. de comp. cu ineg. avem că $\sum_{n=1}^{\infty} |x_n|$ este conv. Deci $\sum_{n=1}^{\infty} x_n$ este absolut conv.

Prin urmare $\sum_{n=1}^{\infty} x_n$ este conv. ($\forall x \in \mathbb{R}$). $\square$

b) $\sum_{n=1}^{\infty} \frac{\ln n}{n^3}$.

Sol.: $x_n = \frac{\ln n}{n^3}$ $\forall n \in \mathbb{N}^*$.

Avem $\ln n < \ln(n+1) < n$ $\forall n \in \mathbb{N}^*$.

Deci $\underbrace{\frac{\ln n}{n^3}}_{= x_n} < \frac{\cancel{n}}{n^{\cancel{3}2}} = \frac{1}{n^2} \quad \forall n \in \mathbb{N}^*$.

Fie $y_n = \frac{1}{n^2} \quad \forall n \in \mathbb{N}^*$.

$\sum_{n=1}^{\infty} y_n = \sum_{n=1}^{\infty} \frac{1}{n^2}$ conv. (serie arm. gen., $\alpha = 2$).

Conform crit. de comp. cu ineg. avem că $\sum_{n=1}^{\infty} x_n$ este conv. $\square$

c) $\sum_{n=1}^{\infty} \sin\left(\pi\sqrt{n^2+1}\right)$.

Sol.: $x_n = \sin\left(\pi\sqrt{n^2+1}\right) \quad \forall n \in \mathbb{N}^*$.

$$x_n = \sin\left(\overbrace{\pi\sqrt{n^2+1} - n\pi + n\pi}^{\pi\sqrt{n^2+1} + n\pi}\right) = \sin\left(\pi\sqrt{n^2+1} - n\pi\right)\cos(n\pi) +$$

$$+ \underbrace{\sin(n\pi)}_{= 0} \cos\left(\pi\sqrt{n^2+1} - n\pi\right) = (-1)^n \cdot \sin\left(\frac{\pi^2(n^2+1) - n^2\pi^2}{\pi\sqrt{n^2+1} + n\pi}\right) =$$

$$= (-1)^n \cdot \sin\left(\frac{\cancel{n^2\pi^2} + \pi^2 - \cancel{n^2\pi^2}}{\pi\sqrt{n^2+1} + n\pi}\right) = (-1)^n \sin\left(\frac{\pi^{\cancel{2}}}{\cancel{\pi}\sqrt{n^2+1} + n\cancel{\pi}}\right) =$$

$$= (-1)^n \sin\left(\frac{\pi}{\sqrt{n^2+1} + n}\right) \quad \forall n \in \mathbb{N}^*.$$

Fie $a_n = \sin\left(\frac{\pi}{\sqrt{n^2+1}+n}\right)$ $\forall n \in \mathbb{N}^*$.

$\left(\frac{\pi}{\sqrt{n^2+1}+n}\right)_{n\geq 1} \subset \left(0, \frac{\pi}{2}\right)$ (strict) descrescător.

$$\begin{array}{ccc} x & \longmapsto & \sin x \\ \cap & & \cap \\ \left(0,\frac{\pi}{2}\right) & & \mathbb{R} \end{array}$$ (strict) crescătoare.

Deci $(a_n)_n$ este (strict) descrescător.

$\lim_{n\to\infty} a_n = \sin 0 = 0$.

Conform Crit. lui Leibniz avem că $\sum_{n=1}^{\infty} x_n$ este conv. $\square$

d) $\sum_{n=1}^{\infty}\left(\operatorname{arctg}\frac{1}{n(n+1)}\right)x^n$, $x>0$.

Sol.: $x_n = \left(\operatorname{arctg}\frac{1}{n(n+1)}\right)x^n$ $\forall n \in \mathbb{N}^*$.

$$\begin{array}{ccc} x & \longmapsto & \operatorname{arctg} x \\ \cap & & \cap \\ \mathbb{R} & & \mathbb{R} \end{array}$$ (strict) crescătoare.

Deci $\operatorname{arctg}\frac{1}{n(n+1)} > \operatorname{arctg} 0 = 0 \quad \forall n \in \mathbb{N}^*$.

Așadar $x_n > 0 \quad \forall n \in \mathbb{N}^*$.

$$\lim_{x\to 0}\frac{\operatorname{arctg} x}{x} = 1 \Rightarrow \lim_{n\to\infty}\frac{\operatorname{arctg}\frac{1}{n(n+1)}}{\frac{1}{n(n+1)}} = 1.$$

Fie $y_n = \frac{1}{n(n+1)}x^n \quad \forall n \in \mathbb{N}^*$.

$$\lim_{n\to\infty}\frac{x_n}{y_n} = \lim_{n\to\infty}\frac{\left(\operatorname{arctg}\frac{1}{n(n+1)}\right)\cancel{x^n}}{\frac{1}{n(n+1)}\cdot\cancel{x^n}} = 1 \in (0,\infty).$$

$\xrightarrow[n\to\infty]{} 1$

Conform Crit. de comp. cu limită avem că $\sum_{n=1}^{\infty} x_n \sim \sum_{n=1}^{\infty} y_n$.

$$\sum_{n=1}^{\infty} y_n = \sum_{n=1}^{\infty}\frac{1}{n(n+1)}\cdot x^n,\quad x>0.$$

$$\lim_{n\to\infty}\frac{y_{n+1}}{y_n} = \lim_{n\to\infty}\frac{\frac{1}{\cancel{(n+1)}(n+2)}\cdot \cancel{x^{n+1}}^{\,x}}{\frac{1}{n\cancel{(n+1)}}\cdot\cancel{x^n}} = x.$$

Cf. Crit. rap. avem:

1) Dacă $x<1$ (i.e. $x\in(0,1)$), atunci $\sum_{n=1}^{\infty} y_n$ e conv.

2) Dacă $x>1$ (i.e. $x\in(1,\infty)$), atunci $\sum_{n=1}^{\infty} y_n$ e div.

3) Dacă $x=1$, atunci crit. nu decide.

Fie $x=1$.

$\sum_{n=1}^{\infty} y_n = \sum_{n=1}^{\infty} \frac{1}{n(n+1)}$ conv. (vezi Curs 2). $\square$

2. Studiați convergența simplă și uniformă pentru următoarele șiruri de funcții:

a) $f_n : [0,1] \to \mathbb{R}$, $f_n(x) = x^n(1-x^n)$ $\forall n\in\mathbb{N}^*$.

<u>Sol.</u>: <u>C. S.</u>

Fie $x\in[0,1]$.

$\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} x^n(1-x^n) = 0 \Rightarrow f_n \xrightarrow[n\to\infty]{s} f$, unde

$f:[0,1]\to\mathbb{R}$, $f(x)=0$.

<u>C. U.</u>

$\sup_{x\in[0,1]} |f_n(x)-f(x)| = \sup_{x\in[0,1]} |x^n(1-x^n)-0| =$

$$= \sup_{x\in[0,1]} x^n(1-x^n) \underset{x=\frac{1}{\sqrt[n]{2}}}{\geq} \left(\frac{1}{\sqrt[n]{2}}\right)^n\left(1-\left(\frac{1}{\sqrt[n]{2}}\right)^n\right) = \frac{1}{2}\cdot\left(1-\frac{1}{2}\right) =$$

$$= \underset{\substack{\vee \\ 0}}{\frac{1}{4}} \not\xrightarrow[n\to\infty]{} 0.$$

Deci $f_n \not\xrightarrow[n\to\infty]{u} f$. $\square$

b) $f_n : [0,1] \to \mathbb{R}$, $f_n(x) = x^n(1-x)^n \ \forall\, n \in \mathbb{N}^*$.

<u>Sol.:</u> <u>C. S.</u>

Fie $x \in [0,1]$.

$$\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty}\left[x(1-x)\right]^n = 0 \Rightarrow f_n \xrightarrow[n\to\infty]{s} f, \text{ unde}$$

$f : [0,1] \to \mathbb{R}$, $f(x) = 0$.

<u>C. U.</u>

$$\sup_{x\in[0,1]} |f_n(x) - f(x)| = \sup_{x\in[0,1]} \left|\left[x(1-x)\right]^n - 0\right| = \sup_{x\in[0,1]} \left[x(1-x)\right]^n.$$

Fie $g : [0,1] \to \mathbb{R}$, $g(x) = x(1-x) = x - x^2$.

$g'(x) = 1 - 2x \quad \forall x \in [0,1]$.

$g'(x) = 0 \iff x = \frac{1}{2}$.

x	0		$\frac{1}{2}$		1
$g'(x)$	$+$	$+\,+\,+\,+\,+$	0	$-\,-\,-\,-\,-$	
$g(x)$	0	$\nearrow$	$\frac{1}{4}$	$\searrow$	0

Deci $\sup\limits_{x \in [0,1]} [x(1-x)]^n = \left(\frac{1}{4}\right)^n \xrightarrow[n \to \infty]{} 0$.

Prin urmare $f_n \xrightarrow[n \to \infty]{u} f$. $\square$

3. Fie $A \subset \mathbb{R}$ o mulțime mărginită și $f : A \to \mathbb{R}$ o funcție a.î., $\forall (x_n)_n \subset A$, $(x_n)_n$ șir Cauchy, avem că $(f(x_n))_n$ șir Cauchy. Arătați că f este funcție uniform continuă.

<u>Sol.</u>: Pp. prin absurd că f nu este uniform continuă.

f funcție uniform continuă $\Leftrightarrow \forall \varepsilon>0, \exists \delta_\varepsilon>0$ a.î.

$\forall x, y \in A$ cu proprietatea că $|x-y|<\delta_\varepsilon$, avem

$|f(x)-f(y)|<\varepsilon$.

f nu este funcție uniform continuă $\Leftrightarrow \exists \varepsilon_0>0$ a.î. $\forall \delta>0$

$\exists x_\delta, y_\delta \in A$ cu proprietatea că $|x_\delta - y_\delta|<\delta$ și

$|f(x_\delta)-f(y_\delta)| \geq \varepsilon_0$.

Deci $\exists \varepsilon_0>0$ a.î. $\forall n \in \mathbb{N}^*$ $\left(\delta=\frac{1}{n}\right)$, $\exists x_{\frac{1}{n}}, y_{\frac{1}{n}} \in A$

cu proprietatea că $|x_{\frac{1}{n}} - y_{\frac{1}{n}}|<\frac{1}{n}$ și $|f(x_{\frac{1}{n}})-f(y_{\frac{1}{n}})| \geq \varepsilon_0$.

Notăm $x_{\frac{1}{n}} = x_n$ și $y_{\frac{1}{n}} = y_n$.

Așadar, $\exists \varepsilon_0>0$ a.î. $\forall n \in \mathbb{N}^*$, $\exists x_n, y_n \in A$ cu

proprietatea că $|x_n - y_n|<\frac{1}{n}$ și $|f(x_n)-f(y_n)| \geq \varepsilon_0$.

Avem $\lim\limits_{n\to\infty}(x_n-y_n)=0$.

$\left.\begin{array}{l}(x_n)_n \subset A \\ A \text{ mărginită}\end{array}\right\} \Rightarrow (x_n)_n$ mărginit.

Conform Lemei lui Cesaro $\exists (x_{n_k})_k \subset (x_n)_n$ a.î.

$(x_{n_k})_k$ convergent.

Avem $\lim\limits_{k\to\infty} (x_{n_k} - y_{n_k}) = 0$ și $(x_{n_k})_k$ conv.

Deci $(y_{n_k})_k$ e conv. și $\lim\limits_{k\to\infty} y_{n_k} = \lim\limits_{k\to\infty} x_{n_k}$.

Considerăm șirul $(z_p)_{p\geq 1}$ definit astfel:

$z_1 = x_{n_1}$, $z_2 = y_{n_1}$, $z_3 = x_{n_2}$, $z_4 = y_{n_2}$, ...

Avem că $\lim\limits_{p\to\infty} z_p = \lim\limits_{k\to\infty} x_{n_k} = \lim\limits_{k\to\infty} y_{n_k} \in \mathbb{R}$, deci

$(z_p)_p$ este șir convergent.

Prin urmare $(z_p)_p$ este șir Cauchy.

Conform ipotezei avem că $(f(z_p))_p$ este șir Cauchy.

Dar $\forall p \in \mathbb{N}^*$, p impar, avem $|f(z_p) - f(z_{p+1})| \geq \varepsilon_0$, contradicție.

Prin urmare f este uniform continuă. $\square$

4. Studiați uniform continuitatea funcției $f:(0,1]\to\mathbb{R}$, $f(x)=e^{\frac{1}{x}}$.

Sol.: Avem că f este uniform continuă dacă și numai dacă, $\forall (x_n)_n \subset (0,1]$, $\forall (y_n)_n \subset (0,1]$ cu proprietatea că $\lim\limits_{n\to\infty}(x_n-y_n)=0$, avem că $\lim\limits_{n\to\infty}(f(x_n)-f(y_n))=0$.

f nu este uniform continuă $\Leftrightarrow \exists (x_n)_n\subset(0,1]$, $\exists (y_n)_n\subset(0,1]$ a.î. $\lim\limits_{n\to\infty}(x_n-y_n)=0$ și $\lim\limits_{n\to\infty}(f(x_n)-f(y_n))\neq 0$.

Alegem $x_n=\frac{1}{\ln(n+1)}$ și $y_n=\frac{1}{\ln n}$ $\forall n\in\mathbb{N}, n\geq 3$.

Avem $\lim\limits_{n\to\infty}(x_n-y_n)=0$ și $\lim\limits_{n\to\infty}(f(x_n)-f(y_n))=$

$=\lim\limits_{n\to\infty}\left(e^{\ln(n+1)}-e^{\ln n}\right)=\lim\limits_{n\to\infty}(\cancel{n}+1-\cancel{n})=1\neq 0.$

Deci f nu este uniform continuă. $\square$

5. Studiați uniform continuitatea funcției $f:(0,1]\to\mathbb{R}$,

$f(x)=e^{-\frac{1}{x}}$.

Sol.: Avem $\lim\limits_{\substack{x\to 0\\ x>0}} f(x)=\lim\limits_{\substack{x\to 0\\ x>0}} e^{-\frac{1}{x}}=0$.

Funcția $\tilde{f}:[0,1]\to\mathbb{R}$, $\tilde{f}(x)=\begin{cases} e^{-\frac{1}{x}} ; & x\in(0,1] \\ 0 ; & x=0 \end{cases}$ este continuă

și $\tilde{f}\big|_{(0,1]}=f$.

Deci f este uniform continuă. $\square$

6. Studiați uniform continuitatea funcției $f:[0,\infty)\to\mathbb{R}$,

$f(x)=e^{x^3}$.

Sol.: Alegem $x_n=\sqrt[3]{\ln(n+1)}$ și $y_n=\sqrt[3]{\ln n}$ $\forall n\in\mathbb{N}^*$.

Avem $\lim\limits_{n\to\infty}(x_n-y_n)=\lim\limits_{n\to\infty}\left(\sqrt[3]{\ln(n+1)}-\sqrt[3]{\ln n}\right)=$

$$=\lim_{n\to\infty}\left(\frac{\ln(n+1)-\ln n}{\sqrt[3]{(\ln(n+1))^2}+\sqrt[3]{\ln(n+1)\ln n}+\sqrt[3]{(\ln n)^2}}\right)=$$

$a^3-b^3=(a-b)(a^2+ab+b^2)$

$$= \lim_{n\to\infty} \frac{\ln\left(\frac{n+1}{n}\right)}{\sqrt[3]{(\ln(n+1))^2} + \sqrt[3]{\ln(n+1)\ln n} + \sqrt[3]{(\ln n)^2}} = 0 \text{ și}$$

$$\lim_{n\to\infty} (f(x_n) - f(y_n)) = \lim_{n\to\infty} \left(e^{\ln(n+1)} - e^{\ln n} \right) =$$

$$= \lim_{n\to\infty} (\cancel{n} + 1 - \cancel{n}) = 1 \neq 0.$$

Deci f nu este uniform continuă. $\square$

7. Determinați mulțimea de convergență pentru următoarele serii de puteri:

a) $\sum_{n=0}^{\infty} \frac{(-1)^n \cdot 2^n}{(n+1)^2 \sqrt{3^n}} \cdot (x+2)^n.$

Sol.: Notăm $x+2=y$.

Seria de puteri devine $\sum_{n=0}^{\infty} \frac{(-1)^n \cdot 2^n}{(n+1)^2 \sqrt{3^n}} y^n.$

$$a_n = \frac{(-1)^n \cdot 2^n}{(n+1)^2 \sqrt{3^n}} \quad \forall n \in \mathbb{N}.$$

$$\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \lim_{n\to\infty} \left(\frac{|(-1)^{n+1} \cdot 2^{n+1}|}{|(n+2)^2 \sqrt{3^{n+1}}|} \cdot \frac{|(n+1)^2 \sqrt{3^n}|}{|(-1)^n \cdot 2^n|} \right) =$$

$$= \lim_{n\to\infty}\left(\frac{\overset{2}{\cancel{2^{n+1}}}}{(n+2)^2\underset{\sqrt{3}}{\cancel{\sqrt{3^{n+1}}}}}\cdot\frac{(n+1)^2\cdot\cancel{\sqrt{3^n}}}{\cancel{2^n}}\right) =$$

$$= \lim_{n\to\infty}\frac{2(n+1)^2}{\sqrt{3}\,(n+2)^2} = \frac{2}{\sqrt{3}}.$$

$$R = \frac{1}{\frac{2}{\sqrt{3}}} = \frac{\sqrt{3}}{2}.$$

Fie B mulțimea de convergență a seriei de puteri $\sum\limits_{n=0}^{\infty}\frac{(-1)^n 2^n}{(n+1)^2\sqrt{3^n}}y^n$.

Avem $\left(-\frac{\sqrt{3}}{2},\frac{\sqrt{3}}{2}\right)\subset B\subset\left[-\frac{\sqrt{3}}{2},\frac{\sqrt{3}}{2}\right]$.

Studiem dacă $-\frac{\sqrt{3}}{2}\in B$ și $\frac{\sqrt{3}}{2}\in B$.

Dacă $y=\frac{\sqrt{3}}{2}$, seria de puteri devine

$$\sum_{n=0}^{\infty}\frac{(-1)^n\cdot 2^n}{(n+1)^2\sqrt{3^n}}\cdot\left(\frac{\sqrt{3}}{2}\right)^n = \sum_{n=0}^{\infty}\frac{(-1)^n\cdot\cancel{2^n}}{(n+1)^2\cdot\cancel{\sqrt{3^n}}}\cdot\frac{\cancel{\sqrt{3^n}}}{\cancel{2^n}} =$$

$$= \sum_{n=0}^{\infty}(-1)^n\frac{1}{(n+1)^2}$$ convergentă (Crit. lui Leibniz).

Deci $\frac{\sqrt{3}}{2}\in B$.

Dacă $y=-\frac{\sqrt{3}}{2}$, seria de puteri devine

$$\sum_{n=0}^{\infty} \frac{(-1)^n \cdot 2^n}{(n+1)^2 \sqrt{3^n}} \cdot \left(-\frac{\sqrt{3}}{2}\right)^n = \sum_{n=0}^{\infty} \frac{\cancel{(-1)^n \cdot 2^n}}{(n+1)^2 \cancel{\sqrt{3^n}}} \cdot \frac{\cancel{\sqrt{3^n}}}{\cancel{(-1)^n \cdot 2^n}} =$$

$$= \sum_{n=0}^{\infty} \frac{1}{(n+1)^2}.$$

Fie $x_n = \frac{1}{(n+1)^2}$ $\forall n \in \mathbb{N}$ și $y_n = \frac{1}{n^2}$ $\forall n \in \mathbb{N}^*$.

Avem $x_n < y_n$ $\forall n \in \mathbb{N}^*$.

$\sum_{n=1}^{\infty} y_n = \sum_{n=1}^{\infty} \frac{1}{n^2}$ conv. (serie arm. gen., $\alpha = 2$).

Cf. crit. de comp. cu ineg. avem că $\sum_{n=1}^{\infty} x_n$ e conv. Deci $\sum_{n=0}^{\infty} x_n$ e conv.

Așadar $-\frac{\sqrt{3}}{2} \in B$.

Prin urmare $B = \left[-\frac{\sqrt{3}}{2}, \frac{\sqrt{3}}{2}\right]$.

Fie A mulțimea de conv. a seriei de puteri

$$\sum_{n=0}^{\infty} \frac{(-1)^n \cdot 2^n}{(n+1)^2 \sqrt{3^n}} \cdot (x+2)^n.$$

$$-\frac{\sqrt{3}}{2} \le \underbrace{y}_{x+2} \le \frac{\sqrt{3}}{2} \;\Big|-2 \iff -\frac{\sqrt{3}}{2} - 2 \le x \le \frac{\sqrt{3}}{2} - 2 \iff$$

$\iff \dfrac{-\sqrt{3}-4}{2} \le x \le \dfrac{\sqrt{3}-4}{2}$.

Deci $A = \left[\dfrac{-\sqrt{3}-4}{2}, \dfrac{\sqrt{3}-4}{2}\right]$. $\square$

b) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt[3]{n}\sqrt[4]{n+2}} \cdot (x-2)^n$.

Sol.: Notăm $x-2=y$.

Seria de puteri devine $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt[3]{n}\sqrt[4]{n+2}} \cdot y^n$.

$$a_n = \frac{1}{\sqrt[3]{n}\sqrt[4]{n+2}} \quad \forall\, n \in \mathbb{N}^*.$$

$$\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \lim_{n\to\infty}\left(\frac{1}{\sqrt[3]{n+1}\sqrt[4]{n+3}} \cdot \frac{\sqrt[3]{n}\sqrt[4]{n+2}}{1}\right) =$$

$$= \lim_{n\to\infty}\left(\sqrt[3]{\frac{n}{n+1}}\sqrt[4]{\frac{n+2}{n+3}}\right) = 1.$$

$$R = \frac{1}{1} = 1.$$

Fie B mulțimea de conv. a seriei de puteri

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt[3]{n}\sqrt[4]{n+2}} y^n.$$

Avem $(-1,1) \subset B \subset [-1,1]$.

Studiem dacă $-1 \in B$ și $1 \in B$.

Dacă $y=-1$, seria de puteri devine $\sum_{n=1}^{\infty} \frac{1}{\sqrt[3]{n}\sqrt[4]{n+2}} \cdot (-1)^n =$

$$= \sum_{n=1}^{\infty} (-1)^n \cdot \frac{1}{\sqrt[3]{n}\sqrt[4]{n+2}}$$ conv. (Crit. lui Leibniz).

Deci $-1 \in B$.

Dacă $y=1$, seria de puteri devine $\sum_{n=1}^{\infty} \frac{1}{\sqrt[3]{n}\sqrt[4]{n+2}} \cdot 1^n =$

$$= \sum_{n=1}^{\infty} \frac{1}{\sqrt[3]{n}\sqrt[4]{n+2}}.$$

Fie $x_n = \frac{1}{\sqrt[3]{n}\sqrt[4]{n+2}}$ $\forall n \in \mathbb{N}^*$.

Fie $y_n = \frac{1}{\sqrt[3]{n}\sqrt[4]{n}}$ $\forall n \in \mathbb{N}^*$.

Avem $\lim_{n\to\infty} \frac{x_n}{y_n} = \lim_{n\to\infty} \frac{\frac{1}{\sqrt[3]{n}\sqrt[4]{n+2}}}{\frac{1}{\sqrt[3]{n}\sqrt[4]{n}}} =$

$$= \lim_{n\to\infty} \left(\frac{1}{\cancel{\sqrt[3]{n}}\sqrt[4]{n+2}} \cdot \frac{\cancel{\sqrt[3]{n}}\sqrt[4]{n}}{1} \right) = \lim_{n\to\infty} \sqrt[4]{\frac{n}{n+2}} = 1 \in (0,\infty).$$

Conform Crit. de comp. cu limită avem că $\sum_{n=1}^{\infty} x_n \sim \sum_{n=1}^{\infty} y_n$.

$$\sum_{n=1}^{\infty} y_n = \sum_{n=1}^{\infty} \frac{1}{\sqrt[3]{n}\,\sqrt[4]{n}} = \sum_{n=1}^{\infty} \frac{1}{n^{\frac{1}{3}} \cdot n^{\frac{1}{4}}} =$$

$$= \sum_{n=1}^{\infty} \frac{1}{n^{\frac{7}{12}}}$$ divergentă (serie arm. gen., $\alpha = \frac{7}{12}$).

Deci $\sum_{n=1}^{\infty} x_n$ div.

Așadar $1 \notin B$.

Prin urmare $B = [-1, 1)$.

Fie A mulțimea de conv. a seriei de puteri din enunț.

$$-1 \leq \underset{\substack{\parallel \\ x-2}}{y} < 1 \,\Big|\, +2 \iff -1+2 \leq x < 1+2 \iff 1 \leq x < 3.$$

Deci $A = [1, 3)$. $\square$

8. Fie $f: \mathbb{R}^2 \to \mathbb{R}$, $f(x,y) = \begin{cases} \dfrac{x^5 y^2}{\sqrt{x^{16}+y^8}} & ; (x,y) \neq (0,0) \\ 0 & ; (x,y) = (0,0). \end{cases}$

a) Studiați continuitatea lui f.

b) Determinați $\frac{\partial f}{\partial x}$, $\frac{\partial f}{\partial y}$.

c) Studiați diferențiabilitatea lui f.

Sol.: a) f continuă pe $\mathbb{R}^2 \setminus \{(0,0)\}$ (operații cu funcții elementare).

Studiem continuitatea lui f în $(0,0)$.

Fie $(x,y) \in \mathbb{R}^2 \setminus \{(0,0)\}$.

$$|f(x,y) - f(0,0)| = \left| \frac{x^5 y^2}{\sqrt{x^{16}+y^8}} - 0 \right| =$$

$$= \frac{|x^5 y^2|}{\sqrt{x^{16}+y^8}} = |x| \frac{x^4 y^2}{\sqrt{x^{16}+y^8}} \leq \frac{1}{\sqrt{2}} |x| \xrightarrow[(x,y)\to(0,0)]{} 0.$$

$\leq \frac{1}{\sqrt{2}}$ (Explicație: $\frac{x^{16}+y^8}{2} \geq$

$\geq \sqrt{x^{16} y^8} = |x^8 y^4| = x^8 y^4 \Rightarrow$

$\Rightarrow \frac{\sqrt{x^{16}+y^8}}{\sqrt{2}} \geq \sqrt{x^8 y^4} = |x^4 y^2| =$

$$= x^4 y^2 \,\Big|\, : \sqrt{x^{16}+y^8} \Rightarrow$$

$$\Rightarrow \frac{1}{\sqrt{2}} \geq \frac{x^4 y^2}{\sqrt{x^{16}+y^8}}\Big)$$

Deci f este continuă în $(0,0)$.

b) Fie $(x,y) \in \mathbb{R}^2 \setminus \{(0,0)\}$.

$$\frac{\partial f}{\partial x}(x,y) = \frac{5x^4y^2\sqrt{x^{16}+y^8} - x^5y^2 \cdot \frac{1}{2\sqrt{x^{16}+y^8}} \cdot 16x^{15}}{x^{16}+y^8}.$$

$$\frac{\partial f}{\partial y}(x,y) = \frac{2x^5y\sqrt{x^{16}+y^8} - x^5y^2 \cdot \frac{1}{2\sqrt{x^{16}+y^8}} \cdot 8y^7}{x^{16}+y^8}.$$

$$\frac{\partial f}{\partial x}(0,0) = \lim_{t\to 0} \frac{f((0,0)+te_1) - f(0,0)}{t} =$$

$$= \lim_{t\to 0} \frac{f(t,0)-f(0,0)}{t} = \lim_{t\to 0} \frac{0-0}{t} = 0.$$

$$\frac{\partial f}{\partial y}(0,0) = \lim_{t\to 0} \frac{f((0,0)+te_2) - f(0,0)}{t} =$$

$$= \lim_{t\to 0} \frac{f(0,t)-f(0,0)}{t} = \lim_{t\to 0} \frac{0-0}{t} = 0.$$

c) $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}$ continue pe $\mathbb{R}^2 \setminus \{(0,0)\}$

$\mathbb{R}^2 \setminus \{(0,0)\}$ deschisă $\Bigg\} \Rightarrow f$ dif. pe $\mathbb{R}^2 \setminus \{(0,0)\}$.

Studiem diferențiabilitatea lui f în $(0,0)$.

Dacă f ar fi dif. în $(0,0)$, atunci $df(0,0) : \mathbb{R}^2 \to \mathbb{R}$,

$$df(0,0)(u,v) = \left[\begin{pmatrix} \underbrace{\frac{\partial f}{\partial x}(0,0)}_{=0} & \underbrace{\frac{\partial f}{\partial y}(0,0)}_{=0} \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} \right]^t = 0 \cdot u + 0 \cdot v = 0.$$

$$\lim_{(x,y)\to(0,0)} \frac{f(x,y) - f(0,0) - df(0,0)\big((x,y)-(0,0)\big)}{\|(x,y)-(0,0)\|} =$$

$$= \lim_{(x,y)\to(0,0)} \frac{\frac{x^5 y^2}{\sqrt{x^{16}+y^8}} - 0 - 0}{\sqrt{x^2+y^2}} = \lim_{(x,y)\to(0,0)} \frac{x^5 y^2}{\sqrt{x^{16}+y^8}\,\sqrt{x^2+y^2}}.$$

Alegem $x_n = \frac{1}{n}\ \forall n \in \mathbb{N}^*$ și $y_n = \frac{1}{n^2}\ \forall n \in \mathbb{N}^*$.

Avem $\lim_{n\to\infty} (x_n, y_n) = (0,0)$ și $\lim_{n\to\infty} \frac{x_n^5 y_n^2}{\sqrt{x_n^{16}+y_n^8}\,\sqrt{x_n^2+y_n^2}} =$

$$= \lim_{n\to\infty} \frac{\frac{1}{n^5} \cdot \frac{1}{n^4}}{\sqrt{\frac{1}{n^{16}}+\frac{1}{n^{16}}}\,\sqrt{\frac{1}{n^2}+\frac{1}{n^4}}} = \lim_{n\to\infty} \frac{\frac{1}{n^9}}{\frac{\sqrt{2}}{n^8} \cdot \frac{\sqrt{n^2+1}}{n^2}} =$$

$$= \lim_{n\to\infty} \frac{n^{10}}{\sqrt{2}\, n^{9}\sqrt{n^2+1}} = \frac{1}{\sqrt{2}} \neq 0.$$

$$\frac{n^{10}}{n^{9}\sqrt{n^2+1}} \xrightarrow{n\to\infty} 1$$

Deci $\lim\limits_{(x,y)\to(0,0)} \dfrac{x^5y^2}{\sqrt{x^{16}+y^8}\sqrt{x^2+y^2}} \neq 0.$

Așadar f nu e dif. în $(0,0)$. $\square$

10. Fie $f:(0,\infty)^3 \to \mathbb{R}$, $f(x,y,z)=xy^2z^3$. Determinați punctele de extrem local ale lui f cu legătura $x+2y+3z=1$.

Sol.: $E=(0,\infty)^3$ mulțime deschisă.

Fie $g: E\to\mathbb{R}$, $g(x,y,z)=x+2y+3z-1$ și

$A=\{(x,y,z)\in(0,\infty)^3 \mid g(x,y,z)=0\}$.

$$\frac{\partial f}{\partial x} = y^2z^3$$

$$\frac{\partial f}{\partial y} = 2xyz^3 \qquad \forall (x,y,z)\in(0,\infty)^3.$$

$$\frac{\partial f}{\partial z} = 3xy^2z^2$$

$$\frac{\partial g}{\partial x} = 1$$

$$\frac{\partial g}{\partial y} = 2 \qquad \forall (x,y,z) \in (0,\infty)^3.$$

$$\frac{\partial g}{\partial z} = 3$$

Observăm că f și g sunt de clasă C^1.

$$\operatorname{rang}\left(\frac{\partial g}{\partial x}(x,y,z)\ \ \frac{\partial g}{\partial y}(x,y,z)\ \ \frac{\partial g}{\partial z}(x,y,z)\right) =$$

$$= \operatorname{rang}\left(1 \ \ 2 \ \ 3\right) = 1 \quad \forall (x,y,z) \in (0,\infty)^3.$$

Fie $L : (0,\infty)^3 \to \mathbb{R}$, $L(x,y,z) = f(x,y,z) + \lambda g(x,y,z) =$

$= xy^2z^3 + \lambda(x + 2y + 3z - 1)$.

$$\begin{cases} \dfrac{\partial L}{\partial x} = 0 \\ \dfrac{\partial L}{\partial y} = 0 \\ \dfrac{\partial L}{\partial z} = 0 \\ g(x,y,z) = 0 \end{cases} \iff \begin{cases} y^2z^3 + \lambda = 0 \\ 2xyz^3 + 2\lambda = 0 \,|\, :2 \\ 3xy^2z^2 + 3\lambda = 0 \,|\, :3 \\ x + 2y + 3z = 1 \end{cases} \iff \begin{cases} y^2z^3 = -\lambda \\ xyz^3 = -\lambda \\ xy^2z^2 = -\lambda \\ x + 2y + 3z = 1 \end{cases}$$

$$\cancel{y^2}\cancel{z^3}=-\lambda=x\cancel{y}\cancel{z^3} \Rightarrow y=x.$$

$$\cancel{x}\cancel{y}z^3=-\lambda=\cancel{x}\cancel{y^2}\cancel{z^2} \Rightarrow z=y.$$

$$\Rightarrow x=y=z.$$

$$x+2y+3z=1 \underset{\substack{\uparrow \\ x=y=z}}{\Rightarrow} x+2x+3x=1 \Rightarrow 6x=1 \Rightarrow x=\frac{1}{6}.$$

Deci $x=y=z=\frac{1}{6}$ și $\lambda=-\frac{1}{6^5}$.

Singurul punct staționar al lui f cu legătura $g(x,y,z)=0$ este $\left(\frac{1}{6},\frac{1}{6},\frac{1}{6}\right)$.

Avem $L:(0,\infty)^3\to\mathbb{R}$, $L(x,y,z)=xy^2z^3-\frac{1}{6^5}(x+2y+3z-1)$.

$$\frac{\partial^2 L}{\partial x^2}=0\ ;\quad \frac{\partial^2 L}{\partial y^2}=2xz^3;\quad \frac{\partial^2 L}{\partial z^2}=6xy^2z;$$

$$\frac{\partial^2 L}{\partial x\partial y}=2yz^3=\frac{\partial^2 L}{\partial y\partial x};\quad \frac{\partial^2 L}{\partial x\partial z}=3y^2z^2=\frac{\partial^2 L}{\partial z\partial x};$$

$$\frac{\partial^2 L}{\partial y\partial z}=6xyz^2=\frac{\partial^2 L}{\partial z\partial y}\qquad \forall (x,y,z)\in(0,\infty)^3.$$

Observăm că L este de clasă C^2.

Fie $F\left(\frac{1}{6},\frac{1}{6},\frac{1}{6}\right):\mathbb{R}^3\to\mathbb{R}$, $F\left(\frac{1}{6},\frac{1}{6},\frac{1}{6}\right)(u)=$
$= d^2L\left(\frac{1}{6},\frac{1}{6},\frac{1}{6}\right)(u)^2$.

Avem $F\left(\frac{1}{6},\frac{1}{6},\frac{1}{6}\right)=\frac{\partial^2 L}{\partial x^2}\left(\frac{1}{6},\frac{1}{6},\frac{1}{6}\right)dx^2+$

$+\frac{\partial^2 L}{\partial y^2}\left(\frac{1}{6},\frac{1}{6},\frac{1}{6}\right)dy^2+\frac{\partial^2 L}{\partial z^2}\left(\frac{1}{6},\frac{1}{6},\frac{1}{6}\right)dz^2+$

$+2\frac{\partial^2 L}{\partial x\partial y}\left(\frac{1}{6},\frac{1}{6},\frac{1}{6}\right)dxdy+2\frac{\partial^2 L}{\partial x\partial z}\left(\frac{1}{6},\frac{1}{6},\frac{1}{6}\right)dxdz+$

$+2\frac{\partial^2 L}{\partial y\partial z}\left(\frac{1}{6},\frac{1}{6},\frac{1}{6}\right)dydz=\frac{2}{6^4}dy^2+\frac{6}{6^4}dz^2+$

$+\frac{4}{6^4}dxdy+\frac{6}{6^4}dxdz+\frac{12}{6^4}dydz.$

Diferențiem legătura $g(x,y,z)=0$ (i.e. $x+2y+3z-1=0$) în (x,y,z) și obținem $1\cdot dx+2dy+3dz=0$.

În punctul $\left(\frac{1}{6},\frac{1}{6},\frac{1}{6}\right)$ ultima relație devine

$dx+2dy+3dz=0.$

Deci $dx = -2dy - 3dz$.

Fie $F\left(\frac{1}{6}, \frac{1}{6}, \frac{1}{6}\right)_{leg} : \mathbb{R}^{3-1} \to \mathbb{R}$,

$$F\left(\frac{1}{6}, \frac{1}{6}, \frac{1}{6}\right)_{leg} = \frac{2}{6^4} dy^2 + \frac{6}{6^4} dz^2 +$$

$$+ \frac{4}{6^4}(-2dy - 3dz)dy + \frac{6}{6^4}(-2dy - 3dz)dz +$$

$$+ \frac{12}{6^4} dydz = \frac{2}{6^4} dy^2 + \frac{6}{6^4} dz^2 - \frac{8}{6^4} dy^2 - \cancel{\frac{12}{6^4} dzdy} -$$

$$- \frac{12}{6^4} dydz - \frac{18}{6^4} dz^2 + \cancel{\frac{12}{6^4} dydz} = - \frac{6}{6^4} dy^2 -$$

$$- \frac{12}{6^4} dz^2 - \frac{12}{6^4} dydz = - \frac{6}{6^4}\left(dy^2 + 2dydz + dz^2\right) -$$

$$- \frac{6}{6^4} dz^2 = - \frac{6}{6^4}\left[(dy + dz)^2 + dz^2\right].$$

Avem $F(1,1,1)_{leg}(\underset{\substack{\| \\ (u_2,u_3) \\ \cap \\ \mathbb{R}^2}}{u}) = -\frac{6}{6^4}\left[(u_2+u_3)^2 + u_3^2\right] \; \forall \underset{\substack{\| \\ (u_2,u_3)}}{u} \in \mathbb{R}^2$.

Observăm că $F(1,1,1)_{leg}(u) \leq 0 \quad \forall u \in \mathbb{R}^2$ și

$F(1,1,1)_{leg}(u) = 0 \Leftrightarrow u = (0,0)$.

Deci $\left(\frac{1}{6}, \frac{1}{6}, \frac{1}{6}\right)$ este punct de maxim local al lui f cu legătura $x+2y+3z=1$. $\square$

11. Det. $\int_0^{\frac{\pi}{2}} (\sin x)^{\frac{3}{2}} (\cos x)^{\frac{1}{2}} dx$.

Sol.: $B(x,y) = 2\int_0^{\frac{\pi}{2}} (\sin t)^{2x-1} (\cos t)^{2y-1} dt \quad \forall x,y \in (0,\infty)$.

$$\begin{cases} 2x-1 = \frac{3}{2} \\ 2y-1 = \frac{1}{2} \end{cases} \Leftrightarrow \begin{cases} x = \frac{5}{4} \\ y = \frac{3}{4} \end{cases}.$$

$$\int_0^{\frac{\pi}{2}} (\sin x)^{\frac{3}{2}} (\cos x)^{\frac{1}{2}} dx = \frac{1}{2} \cdot \underbrace{2\int_0^{\frac{\pi}{2}} (\sin t)^{2\cdot\frac{5}{4}-1} (\cos t)^{2\cdot\frac{3}{4}-1} dt}_{= B\left(\frac{5}{4}, \frac{3}{4}\right)} =$$

$$= \frac{1}{2} \cdot B\left(\frac{5}{4}, \frac{3}{4}\right).$$

$$B\left(\frac{5}{4}, \frac{3}{4}\right) = \frac{\Gamma\left(\frac{5}{4}\right)\Gamma\left(\frac{3}{4}\right)}{\Gamma\left(\frac{5}{4}+\frac{3}{4}\right)} = \frac{\Gamma\left(\frac{5}{4}\right)\Gamma\left(\frac{3}{4}\right)}{\Gamma(2)}.$$

$$\Gamma\left(\frac{5}{4}\right) = \Gamma\left(1+\frac{1}{4}\right) = \frac{1}{4}\Gamma\left(\frac{1}{4}\right).$$

$$\Gamma\left(\frac{5}{4}\right)\Gamma\left(\frac{3}{4}\right) = \frac{1}{4}\Gamma\left(\frac{1}{4}\right)\Gamma\left(1-\frac{1}{4}\right) = \frac{1}{4} \cdot \frac{\pi}{\sin\frac{\pi}{4}} = \frac{1}{4} \cdot \frac{\pi}{\frac{\sqrt{2}}{2}} =$$

$$= \frac{1}{\cancel{4}_{2}} \cdot \frac{\cancel{2}\pi}{\sqrt{2}} = \frac{\pi}{2\sqrt{2}}.$$

$$B\left(\frac{5}{4}, \frac{3}{4}\right) = \frac{\frac{\pi}{2\sqrt{2}}}{1!} = \frac{\pi}{2\sqrt{2}}.$$

$$\int_0^{\frac{\pi}{2}} (\sin x)^{\frac{3}{2}} (\cos x)^{\frac{1}{2}} dx = \frac{1}{2} \cdot \frac{\pi}{2\sqrt{2}} = \frac{\pi}{4\sqrt{2}} = \frac{\pi\sqrt{2}}{8}. \square$$