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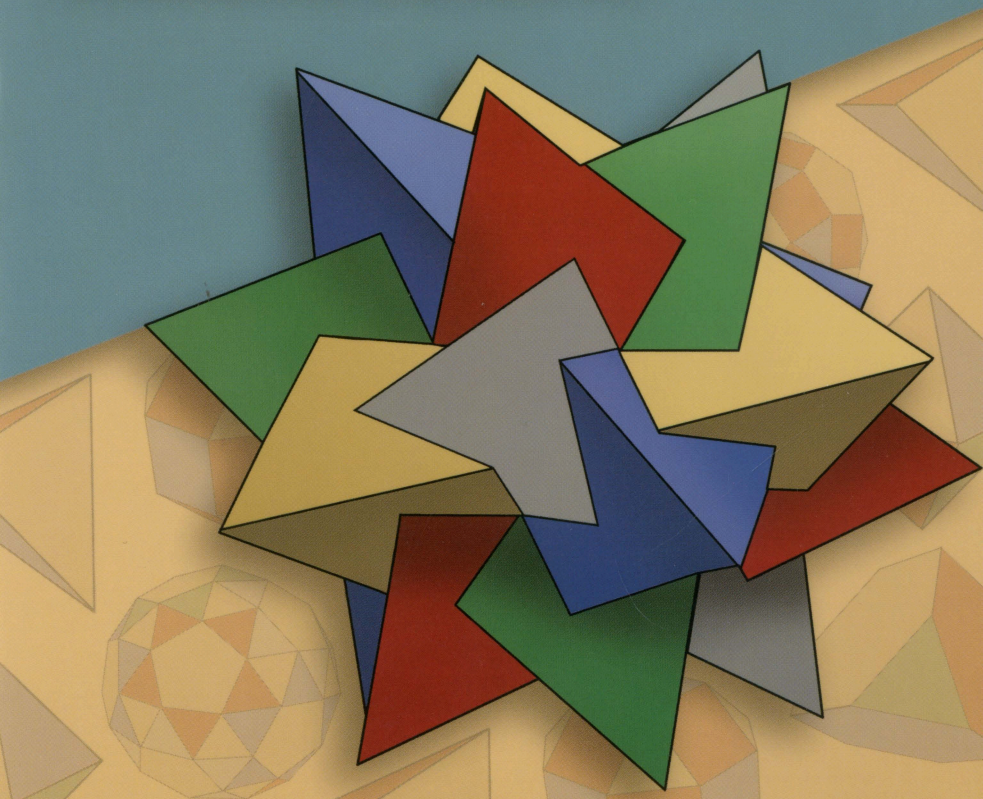
Elementary Geometry

Ilka Agricola
Thomas Friedrich



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Elementary Geometry

Ilka Agricola
Thomas Friedrich

Translated by
Philip G. Spain

AMS
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Providence, Rhode Island

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Contents

Preface to the English Edition v

Preface to the German Edition vii

Chapter 1. Introduction: Euclidean space 1

Exercises 6

Chapter 2. Elementary geometrical figures and their properties 9

§2.1. The line 9

§2.2. The triangle 19

§2.3. The circle 45

§2.4. The conic sections 63

§2.5. Surfaces and bodies 77

Exercises 89

Chapter 3. Symmetries of the plane and of space 99

§3.1. Affine mappings and centroids 99

§3.2. Projections and their properties 105

§3.3. Central dilations and translations 108

§3.4. Plane isometries and similarity transforms 114

§3.5. Complex description of plane transformations 127

§3.6. Elementary transformations of the space $\mathcal{E}^3$ 131

§3.7. Discrete subgroups of the plane transformation group 139
§3.8. Finite subgroups of the spatial transformation group 151
Exercises 156

Chapter 4. Hyperbolic geometry 167
§4.1. The axiomatic development of elementary geometry 167
§4.2. The Poincaré model 174
§4.3. The disc model 183
§4.4. Selected properties of the hyperbolic plane 185
§4.5. Three types of hyperbolic isometries 189
§4.6. Fuchsian groups 194
Exercises 204

Chapter 5. Spherical geometry 209
§5.1. The space $\mathbb{S}^2$ 209
§5.2. Great circles in $\mathbb{S}^2$ 211
§5.3. The isometry group of $\mathbb{S}^2$ 215
§5.4. The Möbius group of $\mathbb{S}^2$ 216
§5.5. Selected topics in spherical geometry 218
Exercises 226

Bibliography 229

List of Symbols 235

Index 237

Preface to the English Edition

We thank the American Mathematical Society, and Edward G. Dunne in particular, for commissioning a translation of our original book and for his willingness to print the whole book in four-color. We seized the opportunity to make infinitely many infinitesimal corrections that have been observed since the first German edition, all of which are not worth listing here. We thank all our colleagues and students who helped to identify them. The book is intended for both undergraduate mathematics students, to introduce them to an advanced point of view on geometry, and for mathematics teachers, as a reference and source book.

This edition, like the original, has its own homepage,

http://www.ams.org/bookpages/stml-43

and any further corrections to errors, mathematical and typographical, will be posted there as they come to our attention. We also intend to present there additional material and a collection of related Web links that we hope the reader may find useful. We will be happy to receive and respond to any comments. In particular, any student who encounters difficulties in solving the Exercises is invited to outline the problem to us by e-mail.

For teachers in schools and universities we have prepared a small volume with hints for solutions, available from us on request.

The Bibliography has been chosen to signpost other material which may be helpful to our readers, and to whet their appetites for geometry and its ramifications. It is not a list of prerequisites.

We thank our translator, Dr. Philip G. Spain, for aiding us in making our book available to a wider public. In its present form the book owes a lot to his expertise, and we are very much indebted to him for an exceptionally pleasant and intensive collaboration.

Berlin, August 2007

Ilka Agricola
Thomas Friedrich

Acknowledgment

The impressions of the ornaments on pages 145 and 165 are printed from Owen Jones, *Grammatik der Ornamente* (unchanged reprint from the First Edition of 1856), 1987, by gracious permission of Greno Verlag, Nördlingen, Germany. The original English edition appeared under the title *The Grammar of Ornament* and has often been reprinted.

Preface to the German Edition

for Julius

Geometry occupies an extensive part of the mathematical syllabus in German schools. In middle school, one starts with properties of the elementary plane figures (line, triangle, circle), elementary transformations of the plane, and surfaces and bodies in space. In high school one comes to analytic geometry, trigonometry, advanced transformations, special curves and the conic sections. Elements of non-Euclidean geometry can be covered in further optional courses. Altogether we have a broad spectrum of geometrical themes that the mathematics teacher can present to his pupils. During the study for the teacher's diploma at university the syllabus starts with lectures on linear algebra and analytic geometry in the first year, followed by lectures on elementary geometry in the second year. This is to present these geometrical themes to the prospective teacher in a mathematically systematic form. If one considers the university education in Germany over a longer time span, it is easy to recognize that in the lectures on linear algebra the geometric themes are reduced step by step, often almost completely masked out.

In all, one gains the impression that the course on elementary geometry, with its clearly defined contents, forms the main part of the geometric education for teachers in training.

This book arose from a one-semester lecture course on "elementary geometry" for future teachers in their second year of study at the Humboldt-Universität in Berlin. The students had already attended the first-year courses on linear algebra and calculus; in the first chapter we present a summary of some aspects of these lectures. Our treatment of elementary geometry assumes this fundamental knowledge, although in a large part of the text they will hardly be needed. Accordingly, this text is intended as a companion book to such a course and seminars. Further, we hope that the book will be used by working teachers as a compendium of the curriculum for elementary geometry. Selected parts of the text are also suitable for good high school pupils and ideally might be used as a foundation for independent study or projects.

Chapter 2 is devoted to the elementary geometric figures and their properties. We begin with the incidence theorems for lines and then turn our attention to the triangle. After the congruence and similarity theorems, we apply in particular the theorems of Menelaus and Ceva in order to treat the intersection points of the special lines in a triangle. Further, we discuss the incircle, circumcircle and excircle of the triangle, its area, and its relation to the radii of the circle. We treat the circle similarly, and discuss in particular the Feuerbach circle, and the Simson and Steiner lines. With a view to the underlying hyperbolic geometry in Chapter 4 we already introduce a section on inversion in the circle here. The conic sections follow, with the derivation of their general equation, their eccentricity and parameters, as well as the determination of the focus and directrix. Some striking properties of the conic sections are proved directly in the text; the reader will find some other properties in the Exercises at the end of Chapter 2. Then we turn to surfaces and bodies in space. We derive the formulae for the surface area of a surface of revolution and also the formula for the volume of a body of revolution: we prove Euler's polyhedron theorem (for convex polyhedra) and finish Chapter 2 with the classification of the Platonic bodies.

Chapter 3 deals with the symmetries of Euclidean space. We describe affine mappings briefly, also the linear mappings corresponding to them, and the centroid of a finite weighted point system. Parallel projections onto a plane along a line and onto a line along a plane are the first examples of affine mappings. Then we treat central dilations and translations exhaustively. First we characterize them through a common geometric property and deduce that together they form a nonabelian group of transformations of space to itself. Then, for the plane, we determine in detail their compositions and discuss as an application the dilation centers of two circles, with whose help one can construct the common tangents to two circles. Next follows the study of isometries of the plane. First come examples of axis reflections, translations and rotations, and again we study their compositions. Fixed points are important: An isometry of the plane with three noncollinear fixed points is the identity. Analogously we characterize all isometries with exactly two fixed points, with one fixed point, and also the fixed point free isometries. The group generated by all isometries and central dilations consists of the similarity transforms of the plane. In a similar way we treat the transformations of three-dimensional space. First we study the composition of distinct such mappings and then turn again to the description of the fixed point sets of spatial isometries. These fixed point sets yield a classification of the isometries of $\mathcal{E}^3$. The last two sections of this chapter are devoted to the study of the discrete isometry groups of Euclidean space. In the case of the plane we treat the cyclic rotation groups, the dihedral group and lattice. We deduce a necessary condition for the point group of a discrete isometry group of the plane and finally obtain a classification of all the groups in question. In the case of space we restrict ourselves to classifying the finite isometry groups. These are the invariance groups of the Platonic bodies and of the symmetry group of a pyramid or of a cylinder with regular polygonal base. The tetrahedron group, the cube group, and also the dodecahedron group are described completely.

We begin Chapter 4 with the axiomatics of elementary geometry and the significance of the parallel axiom. We construct hyperbolic geometry in the upper half plane, in which lines are Euclidean circular arcs or half lines. We treat various expressions for the hyperbolic distance

of two points. This can as well be represented by a cross ratio as by a direct formula. In particular, the triangle inequality holds, and the hyperbolic plane is a metric space. We determine its isometry group and derive the formulae for the hyperbolic length of a curve and for the hyperbolic area of a region. By means of the Cayley transform we pass to the disc model of hyperbolic geometry. We then treat selected properties of geometric figures in the hyperbolic plane. We compute the perimeter of a circle, its hyperbolic area, and derive the hyperbolic Pythagoras theorem as well as other formulae from trigonometry. The formula for the area of a triangle and its angle defect is proved completely. In the Exercises the reader will find a number of results in hyperbolic elementary geometry that are analogous to those of Euclidean geometry. These concern pairs of hyperbolic lines, triangles and their notable points, the incircle and circumcircle of a triangle, and also the horocycle. In a further section we present the classification of the isometries into elliptic, parabolic and hyperbolic transformations both by means of the Jordan normal form and also through their fixed point sets. We study in detail the question of the type of the commutator of two isometries. The last section of this chapter is devoted to Fuchsian groups. Here we are dealing with discrete subgroups of the isometry group of the hyperbolic plane. As well as a series of examples of such groups we introduce their limit sets and prove that these sets have either $0, 1, 2$ or infinitely many points. Fuchsian groups with no more than two limit points are called elementary. We classify all elementary Fuchsian groups.

Spherical geometry is treated in the last chapter in imitation of hyperbolic geometry. We consider the set of all points of the two-dimensional sphere $\mathbb{S}^2$. The great circles play the rôle of spherical lines and realize the shortest distance between two points in spherical space. We determine the isometry group and also the group of all conformal mappings completely. To conclude we prove the most important formulae of spherical trigonometry and study the polar triangle associated to each spherical triangle. From this we obtain the formulae for the areas of spherical lunes and triangles and various inequalities between the side lengths and angles.

At the end of each chapter the reader will find a selection of Exercises that have regularly been assigned to our auditors as homework. Any student who encounters difficulties in solving these Exercises is warmly invited to outline his problem to us by e-mail. We will endeavor to help. For teachers in schools and universities we have prepared a small volume with hints for solutions, which is available from us on request. Moreover, the German edition of the book has its own Internet page,

http://www-irm.mathematik.hu-berlin.de/~agricola/elemgeo.html

One will find there a list of all known typographical errors, and pdf-files of all the pages on which pictures appear that are multicolored in the original but are printed here in black-and-white on grounds of cost. There is also a collection of www-links on elementary geometry, though it makes no claim to completeness.

We thank the participants in our seminars for numerous suggestions that have led to extending and improving the text. Dr. sc. Hubert Gollek and Dr. Christof Puhle have read through the whole manuscript and have indicated necessary corrections in many chapters. Not last we thank Frau Schmickler-Hirzebruch of Vieweg Verlag for her willingness to print some pages of this book in two-tone. We are aware that this is a rare (if also much desired) privilege. We hope that this will not remain an isolated case in the mathematical literature and that the reader will appreciate and enjoy this enrichment of the text, which was not to be taken for granted.

Berlin, December 2004

Ilka Agricola
Thomas Friedrich

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List of Symbols

$\mathcal{A}_4$ – tetrahedron group, 151

$\mathcal{A}_5$ – dodecahedron group, 153

$|AB|$ – length of a segment, 1

$AB : BC$ – division ratio, 10

a, b, c – triangle sides, 19

α, β, γ – triangle angles, 19

$\vec{a} \times \vec{b}$ – vector product, 5

$\mathcal{B}$ – group of transformations, 108

$\mathcal{CS}(r_1, r_2, h)$ – frustum of a cone, 79

$\mathcal{C}(Z, r)$ – circle, 45

$\hat{\mathbb{C}}$ – extended complex numbers, 59

C_n – cyclic group, 140

$\mathrm{Conf}_0(\mathbb{S}^2)$ – Möbius group, 217

$d(A, B)$ – distance between points, 1

$\mathbb{D}^2$, 183

D_n – dihedral group, 140

$\Delta(A, B, C)$ – triangle, 19

$\mathcal{E}^n$ – affine space, 1

F_1, F_2 – foci, 67

$\mathrm{F}(\mathcal{C})$ – surface of revolution, 77

$\mathrm{Fix}(\gamma)$ – fixed point set, 190

G – centroid, 103

$[\gamma, \gamma_1]$ – commutator, 191

H – half line, 113

$\mathbb{HS}^2$ – hemisphere, 224

$\mathbb{H}^2$ – hyperbolic plane, 174

$\hat{\mathbb{H}}^2$ – extended hyperbolic plane, 189

$h_{(\Omega,k)}$ – central dilation, 108

$\mathcal{I}$ – group of isometries, 114

$\mathcal{I}^+$ – positive isometries, 123

$\mathcal{I}^-$ – negative isometries, 123

$\mathcal{I}_O$ – isometries with fixed point, 116

$\mathcal{I}_O^+$ – positive isometries with fixed point, 120

$\mathcal{I}_O^-$ – negative isometries with fixed point, 120

$\mathrm{K}(\mathcal{C})$ – body of revolution, 77

$\mathrm{L}(A, B)$ – line through A and B, 6

$L(\mathcal{C})$ – curve length, 3

$L(f)$ – vectorial mapping, 101

$\Lambda(\Gamma)$ – limit set, 196

$\mathrm{m}(A)$ – area, 4

m_A – point masses, 101

$\mathcal{N}_\pi$ – pole of great circle, 212

$\mathrm{O}(V(\mathcal{E}^n))$ – orthogonal group, 115

Ω – center of dilation, 108

P_c – centroid, 38

P_cc – circumcenter, 38

P_ic – incenter, 35

P_oc – orthocenter, 38

π, Π – plane, 63, 105

$\Pi(\Gamma)$ – parabolic points, 196

$\mathrm{PSL}(2, \mathbb{C})$, 60

$\mathrm{PSL}(2, \mathbb{R})$, 177

$\mathrm{PSL}(2, \mathbb{Z})$, 194

$\hat{\mathbb{R}}$ – extended real line, 176

$\mathbb{R}^n$ – Euclidean space, 1

$r_{(\Omega,k)}$ – rotation, 118

$\mathcal{S}$ – similarity transformation group, 124

$\mathcal{S}^+$ – positive similarities, 124

$\mathcal{S}^-$ – negative similarities, 124

$\mathcal{S}_4$ – octahedron group, 152

s_L – reflection in line, 117

s_π – reflection in plane, 132

$\mathrm{S}_\mathcal{C}$ – circle reflection, 59

$\mathrm{SL}(2, \mathbb{C})$, 60

$\mathrm{SL}(2, \mathbb{R})$, 177

$\mathbb{S}^2$ – spherical space, 209

$\mathcal{T}$ – translation group, 115

$t_{\vec{v}}$ – translation, 110

$\mathrm{vol}(A)$ – volume, 4

$V(\mathcal{E}^n)$ – Euclidean vector space, 1

ψ_A – exterior angle in triangle, 19

$(z_1 : z_2 ; z_3 : z_4)$ – cross ratio, 62

Index

affine mapping, 99
alternate angle, 22
altitude, 19, 33
  common, 205
  in hyperbolic triangle, 206
  theorem, 25
angle
  central, 51
  inscribed, 51
  peripheral, 51
angle bisection theorem for ellipse, 71
angle bisectors
  in Euclidean triangle, 35
  in hyperbolic triangle, 206
angle cosine theorem, 220
angle sum
  in Euclidean triangle, 21
  in hyperbolic triangle, 188
  in spherical triangle, 224
antipodal points, 211
antiprism, 87
Archimedean bodies, 87
area
  Euclidean triangle, 40
  hyperbolic, 182
  hyperbolic triangle, 188
ASA theorem, 26
asymptotes of the hyperbola, 72
asymptotic triangle, 187
autopolar triangle, 226
axis reflection, 117

barycenter, 103
Beltrami, E. (1835–1899), 173
Berger, Marcel (1927–), 86
betweenness relation, 168
body of revolution, 77
  volume of, 78
Bolyai, J. (1802–1860), 172
boundary circle, 187
boundary parallel lines, 204

Caspar, Donald L. D. (1927–), 164
Cassini curve, 75
cathetus, 24
  theorem, 24
Cavalieri's principle, 4, 79, 81, 82
Cayley transform, 183
central angle, 51
central dilation, 108
centroid, 34, 38, 91, 101, 104
  in hyperbolic triangle, 206
  weighted, 102, 115
Ceva's theorem, 30, 32, 92
chord of a circle, 46
chord theorem, 48
chordal quadrilateral, 56
circle, 45
  chord theorem, 48

Feuerbach, 40, 53
generalized, 57, 175
hyperbolic, 185
inversion in, 58
reflection in, 58
secant theorem, 47
spherical, 218
tangent theorem, 46
circular cone, 78
circumcenter
of Euclidean triangle, 36
circumcircle
of Euclidean triangle, 36, 43
of hyperbolic triangle, 206
collinear points, 6
commutator, 191
complex numbers, 5
cone, 81
volume, 81
conformal
group, 217
mapping, 60, 114
congruence theorems, 26, 226
congruent sets, 3
conic section, 63
directrix, 67
eccentricity, 65
focus, 67
parameter, 65
polar equation, 65
conjugate matrices, 189
coordinates
polar, 63
spherical, 210
cosine law
Euclidean, 20
hyperbolic, 205
spherical, 220
cotangent formula, 221
cross ratio, 61, 176
cube (hexahedron), 86, 152
cube group, 156
cuboctahedron, 88, 97
large, 88
cyclic group, 140, 156
cylinder, 82
volume of, 82

Delambre's equations, 226
Desargues' theorem, 18
diametrically opposite points, 211
dihedral group, 140, 156
dilation center (of two circles), 112
dilation factor, 108
direction vector, 6
directrix, 67
disc model, 183
distance
Euclidean, 1
hyperbolic, 176
in geometric plane, 170
spherical, 214
divergent lines, 205
division ratio, 10
dodecahedron, 86, 153
group, 153, 156
truncated, 87

eccentricity, 65
Einstein, Albert (1879–1955), 173
ellipse, 66
angle bisection theorem, 71
length, 71
surface, 71
ellipsoid, 78
elliptic
isometry group, 215
line, 211
space, 209
transformation, 190
Euclidean
cosine law, 20
distance, 1
isometry group, 2
length, 3
line, 6
plane, 5
sine law, 40
space, 1
triangle, 19
angle bisectors, 35
area, 40
circumcenter, 36
circumcircle, 43
excircle, 44
incenter, 36

incircle, 36, 43
Mollweide's equations, 91
Napier's equations, 91
orthocenter, 33
perpendicular bisectors, 36
side bisectors, 34
Euler line, 38
Euler's polyhedron formula, 82
exact sequence, 116
excircle of Euclidean triangle, 44
extended
complex plane, 176
complex plane, 59, 216
real line, 176
exterior angle, 20

Fermat problem, 24
Feuerbach circle, 40, 53
first Steiner line, 55
five element formula, 220
fixed point
of hyperbolic isometry, 190
focus, 67
focus-directrix pair, 67
football, 87
fractional linear transformation, 60
frieze, 144
group, 146, 164
Fuchsian group, 194
elementary, 201
first kind, 204
nonelementary, 201
second kind, 204
Fuller, Buckminster (1895–1983), 95
function (fractional linear), 60

Gauss, C. F. (1777–1855), 172
generalized circle, 57, 175
geometric plane, 168
glide reflection, 118
golden
ratio, 50
rectangle, 89, 154
section, 49, 89, 154
great circle, 211
pole, 212
group
cyclic, 140, 156
dihedral, 140, 156
dodecahedron, 153, 156
Fuchsian, 194
elementary, 201
first kind, 204
nonelementary, 201
second kind, 204
octahedron, 152, 156
orthogonal, 115
tetrahedron, 151, 156

Haeckel, Ernst (1834–1919), 86
half space, 82
hemisphere, 212
Heron's formula, 41
hexagonal lattice, 146
hexahedron (cube), 86, 152
snub, 88
truncated, 87
Hippocrates, lunes of, 90
homothety, 108
horocycle, 206
hyperbola, 66
asymptotes, 72
hyperbolic
area, 182
boundary circle, 187
circle, 185
cosine law, 205
distance, 176
isometry group, 180
length, 181
line, 175
lines
boundary parallel, 204
divergent, 205
plane, 174
disc model, 183
Poincaré model, 174
sine law, 206
transformation, 190
triangle
altitude, 206
angle bisectors, 206
area, 188
asymptotic, 187
circumcircle, 206

incircle, 206
orthocenter, 206
perpendicular bisector, 206
side bisector, 206
hyperboloid
one-sheeted, 78
two-sheeted, 78

icosadeltahedron, 95, 163
icosahedron, 86, 154
truncated, 87
icosidodecahedron, 88
large, 88
incenter, 35, 36
incidence theorem, 9
converse, 13
in space, 15, 107
oriented, 11, 31
incircle
of Euclidean triangle, 36, 43
of hyperbolic triangle, 206
inscribed angle, 51
inversion in circle, 58
isobarycenter, 103
isoceles triangle, 20
isometry, 1, 114, 171
group
of elliptic space, 215
of Euclidean space, 2
of hyperbolic plane, 180
negatively/positively oriented, 123
isoperimetric problem, 42

Jordan measure, 4
Jordan normal form, 190

Klein, Felix (1849–1925), 144, 173
Klein four group, 144
Klug, Aaron (1926–), 164

Lambert projection, 210
lattice, 146
leg, 24
lemniscate, 76
length
Euclidean, 3
hyperbolic, 181
spherical, 219
lever law, 101
l'Huilier's equation, 227
limit circle, 206
limit set, 196
line
elliptic, 211
Euclidean, 6
Euler, 38
first Steiner, 55
hyperbolic, 175
second Steiner, 55, 126, 159
Simson, 54, 126, 159
lines
hyperbolic
boundary parallel, 204
divergent, 205
parallel, 6, 169
skew, 6
Lobachevsky, N. I. (1793–1856), 172
lune, 212
of Hippocrates, 90

Möbius group, 217
mapping
affine, 99
angle-preserving (= conformal), 60, 114
orientation-preserving/reversing, 123
vectorial, 101
mass, 102
median theorem, 34, 104
Menelaus' theorem, 30
midtriangle, 38
modular group, 194
Mollweide's equations, 91
Morley's theorem, 91

Nagel point, 92
Napier's equations
for Euclidean triangle, 91
for spherical triangle, 227
Napoleon's theorem, 130
nine point circle, 40

octahedron, 86, 152
group, 152, 156
truncated, 87
octant, 219

one-sheeted hyperboloid, 78
order axioms, 168
orientation-preserving/reversing
  mapping, 123
oriented
  angle, 117
  incidence theorem, 11, 31
ornament group, 146
orthocenter, 91
  in Euclidean triangle, 33
  in hyperbolic triangle, 206
orthogonal group, 115
orthogonal projection, 105

Pappus' theorem, 13
parabola, 66
parabolic
  mirror, 94
  point, 196
  transformation, 190
parallel axiom, 172
parallelepiped, 5
parameter of conic section, 65
Pasch's Axiom, 169
pentagon, 50, 161
perfect set, 204
peripheral angle, 51
perpendicular bisector
  in Euclidean triangle, 36
  in hyperbolic triangle, 206
  planes, 132
picornavirus, 163
plane
  Euclidean, 5
  geometric, 168
  hyperbolic, 174
    disc model, 183
    Poincaré model, 174
  reflection, 132
Platonic body, 85
Poincaré, Henri (1854–1912), 173
Poincaré model, 174
point
  group, 144
  projection, 105
  reflection, 108, 136
  space, 1
  symmetry group, 144
polar, 212
  coordinates, 63, 185
  equation of conic section, 65
  triangle, 222
pole, 212
polyhedron, 82
  dual, 96
polytope, 82
  regular, quasiregular, 85
primitive pythagorean triple, 23
principal congruence group, 194
prism, 87
problem
  Fermat, 24
  isoperimetric, 42
projection, 105
  Lambert, 210
  stereographic, 210
Ptolemy's theorem, 62
Pythagoras' theorem
  Euclidean, 22
  hyperbolic, 186
  spherical, 221
Pythagoras triple, 23

quadratic lattice, 146

Radiolaria, 86
rectangular face centered lattice,
    146
rectangular lattice, 146
reflection
  glide, 118
  in circle, 58
  in great circle, 216
  in hyperbolic line, 180
  in plane, 132
  in point, 108, 136, 216
relation (betweenness), 168
revolution
  ellipsoid of, 78
  hyperboloid of, 78
rhombic lattice, 146
Riemann, Bernhard (1826–1866),
    173
rotation, 135
  half, 133
rotation-dilation, 124

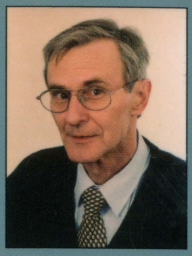
SAS theorem, 26
secant of circle, 45
secant theorem, 47
second Steiner line, 55, 126, 159
segment (in absolute geometry), 169
semiperimeter, 41
semiregular polyhedron, 87
sets
  congruent, 3, 171
  Jordan measurable, 4
  perfect, 204
  similar, 3
side bisector
  in Euclidean triangle, 34, 104
  in hyperbolic triangle, 206
side cosine law, 220
similar sets, 3
similarity transform, 124
Simson line, 54
sine law, 40
  hyperbolic, 206
  spherical, 220
skew lattice, 146
skew reflection, 114
snub
  hexahedron, 88
space, elliptic, 209
spherical
  angle cosine theorem, 220
  circle, 218
  coordinates, 210
  cosine law, 220
  cotangent formula, 221
  distance, 214
  excess, 226
  five element formula, 220
  lune, 212
  octant, 219
  polar triangle, 222
  sine law, 220
  triangle, 220
    area, 224
    autopolar, 226
    Delambre's equations, 226
    l'Huilier's equation, 227
    Napier's equations, 227
  trigonometry, 220
star polygon, 96
Steiner, Jacob (1796–1863), 126
Steiner line
  first, 55
  second, 55, 126, 159
stereographic projection, 210
supplementary angle, 213
surface of revolution, 77
  surface area of, 80
Sylvester and Gallai theorem, 90
symmetry point, 205

tangent
  quadrilateral, 56
  theorem, 46
  to circle, 45
  to two circles, 113
tetrahedron, 86, 104, 151, 157
  group, 151, 156
  truncated, 87
Thales' theorem, 15, 50, 107
theorem
  alternate angle, 22
  angle bisectors, 35
  angle cosine, 220
  central angle, 51
  Ceva, 30, 32, 92
  congruence, 226
  Desargues, 18
  ellipse
    angle bisection, 71
  exterior angle, 20
  incidence, 9
    in space, 15, 107
  median, 34, 104
  Menelaus, 30
  Morley, 91
  Napoleon, 130
  oriented incidence, 11
  Pappus, 13
  Ptolemy, 62
  Pythagoras
    Euclidean, 22
    hyperbolic, 186
    spherical, 221
  side cosine, 220
  spherical
    angle cosine, 220

side cosine, 220
Sylvester and Gallai, 90
Thales, 15, 50
tiling, 144
torus, 78
transformation, 108
complex, 128
elliptic, 190
hyperbolic, 190
parabolic, 190
translation, 1, 108
domain, 144
group, 115
subgroup, 144
transversal, 30
vertex, 32
triangle
asymptotic, 187
congruence theorems, 26
Euclidean, 19
angle bisectors, 35
angle sum, 21
area, 40
centroid, 38
circumcenter, 36
circumcircle, 43
excircle, 44
incenter, 35, 36
incircle, 36, 43
Mollweide's equations, 91
Napier's equations, 91
orthocenter, 33
perpendicular bisectors, 36
side bisectors, 34
hyperbolic
angle bisectors, 206
area, 188
circumcircle, 206
incircle, 206
orthocenter, 206
perpendicular bisector, 206
side bisector, 206
isoceles, 20
leg, 20
right-angled
altitude theorem, 25
cathetus theorem, 24
Pythagoras' theorem, 22
semiperimeter, 41
similarity theorem, 30
spherical, 220
area, 224
autopolar, 226
Delambre's equations, 226
excess, 226
l'Huilier's equation, 227
Napier's equations, 227
polar, 222
surface, 220
transversal, 30
trigonometry (spherical), 220
truncated
dodecahedron, 87
hexahedron, 87
icosahedron, 87
octahedron, 87
tetrahedron, 87
two-sheeted hyperboloid, 78

vector
mapping, 101
product, 5
projection, 105
triple product, 5
vectorial
isometry, 115
mapping, 101
vertex
angle, 213
transversal, 32

weighted centroid, 102, 115
Weyl, Hermann (1885–1955), 86
Wiles, Andrew (1953–), 24



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