ⓒ3 – GA Sisteme liniare

Algoritmul general

$rg A = r$ $\exists \Delta_p = \det(A_{I,J})$ minor de ordin r $I = \{i_1, .., i_p\}$

minor principal $J = \{j_1, .., j_p\}$

Δ_c se obtin prin bordare cu col. t. liberi și prin adăugarea unei linii l_i, $i \in \{1, .., n\} \setminus I$.

1) Dacă $\exists$ un minor $\Delta_c \neq 0$, at $rg \bar{A} = r+1$ SI

2) Dacă $rg \bar{A} = r \Rightarrow$ SC

Fără a restrânge generalitatea $\Delta_p = \begin{vmatrix} a_{11} .. & a_{1r} \\ \dot{a}_{r1} ... & a_{rr} \end{vmatrix}$

(altfel renumerotăm)

Fie ⊛ sist. format din primele r ec (ec. principale) (celelalte ec. sunt combinații liniare ale primelor r ec)

a) Dacă $m > n$ (nr ec > nr. necunoscutelor)

a1). $rg A = rg \bar{A} = n$ SCD ($x_1, .., x_n$ var pr, ∄ var secundare

a2) $rg A = rg \bar{A} = r < n$ $x_1, .., x_r$ var principale.

$x_{r+1}, .., x_n$ var secundare

Det $x_1, .., x_r$ în funcție de var. secundare.

b) Dacă $m < n$ (nr ec < nr. necunoscutelor)

$rg A = rg \bar{A} = r \leq m$ SCN.

OBS SLO $AX = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}$ $A \in \mathcal{M}_{m,n}(\mathbb{K})$

Un SLO este întotdeauna compatibil.

a) $m = n$, $A \in \mathcal{M}_n(\mathbb{K})$ → $\Delta \neq 0$ $\exists! (0, .., 0)$ SCD

→ $\Delta = 0$ SCN
∃ și sol nenule.

b) $m \neq n$ b1) $m > n$ → $rg A = r = n$ SCD

→ $rg A = r < n$ SCN

b1) $m < n$ SCN

Def 2 sisteme s.n. echivalente ⇔ au aceeași mulțime de soluții.

Teoremă Aplicarea transformărilor elementare asupra liniilor (resp a coloanelor) matricei extinse $\bar{A} = (A|B)$ conduce la matrice extinse ale unor sistem echivalente cu sist ⊛ $AX = B$

Metoda eliminării Gauss-Jordan

Exemplu
$$\begin{cases} -x + 2y - 3z = -2 \\ 2x - 6y + 9z = 3 \\ -3x + 2y + 2z = -3 \end{cases}$$
$\bar{A} = (A|B)$

$$\bar{A}=\left(\begin{array}{ccc|c}-1 & 2 & -3 & -2\\ 2 & -6 & 9 & 3\\ -3 & 2 & 2 & -3\end{array}\right)\sim\left(\begin{array}{ccc|c}1 & -2 & 3 & 2\\ 0 & -2 & 3 & -1\\ 0 & \enclose{circle}{-4} & 11 & 3\end{array}\right)$$

$$\sim\left(\begin{array}{ccc|c}1 & -2 & 3 & 2\\ 0 & -2 & 3 & -1\\ 0 & 0 & 5 & 5\end{array}\right)\sim\left(\begin{array}{ccc|c}1 & -2 & 3 & 2\\ 0 & 1 & -\frac{3}{2} & \frac{1}{2}\\ 0 & 0 & 1 & 1\end{array}\right)$$

$z=1$

$y-\frac{3}{2}z=\frac{1}{2}\Rightarrow y=2$

$x-2y+3z=2\Rightarrow x=4-3+2=3$

$(x,y,z)=(3,2,1)$

Spații vectoriale

Fie $(\mathbb{K},+,\cdot)$ corp com. și V m. nevidă

Spunem că V are o structură de spațiu vectorial dacă $\exists$ "$+$" $: V\times V\longrightarrow V$ (lege internă)

"$\cdot$" $:\mathbb{K}\times V\longrightarrow V$ (lege externă)

aî 1) $(V,+)$ grup abelian

2) $a\cdot(b\cdot x)=(a\cdot b)\cdot x$

3) $(a+b)\cdot x=a\cdot x+b\cdot x$

4) $a\cdot(x+y)=a\cdot x+a\cdot y$

5) $1_{\mathbb{K}}\cdot x=x$, $\forall x,y\in V$ (vectori), $\forall a,b\in\mathbb{K}$ (scalari)

Not $(V,+,\cdot)|_{\mathbb{K}}$

Exemple

a) $(\mathbb{K},+,\cdot)$ corp $\Rightarrow(\mathbb{K},+,\cdot)|_{\mathbb{K}}$ sp v $\begin{cases}(\mathbb{R},+,\cdot)/\mathbb{R}\\ (\mathbb{C},+,\cdot)/\mathbb{C}\\ (\mathbb{Z}_p,+,\cdot)/\mathbb{Z}_p\end{cases}$

b) $\mathbb{K}'\subseteq\mathbb{K}$ subcorp $\Rightarrow(\mathbb{K},+,\cdot)/_{\mathbb{K}'}$ sp.v.

$(\mathbb{R},+,\cdot)/\mathbb{Q}$, $(\mathbb{C},+,\cdot)/_{\mathbb{R}}$,

c) $(V_1,\oplus,\odot)/_{\mathbb{K}}$, $(V_2,\boxplus,\boxdot)/_{\mathbb{K}}$ sp vect $\Rightarrow (V_1\times V_2,+,\cdot)/_{\mathbb{K}}$ spV

$+ : (V_1\times V_2)\times(V_1\times V_2)\longrightarrow V_1\times V_2$

$(x_1,x_2)+(y_1,y_2)\overset{def}{=}(x_1\oplus y_1, x_2\boxplus y_2)$

$\cdot : \mathbb{K}\times(V_1\times V_2)\longrightarrow V_1\times V_2$

$a\cdot(x_1,x_2)=(a\odot x_1, a\boxdot x_2)$,

$\forall (x_1,x_2),(y_1,y_2)\in V_1\times V_2, \forall a\in\mathbb{K}$.

C. particular $(V_1=V_2=\mathbb{R},+,\cdot)/_{\mathbb{R}} \Rightarrow (\mathbb{R}^2,+,\cdot)/\mathbb{R}$ spV.

$(x_1,x_2)+(y_1,y_2)=(x_1+y_1, x_2+y_2)$

$a(x_1,x_2)=(ax_1, ax_2)$

Analog $(\mathbb{R}^n,+,\cdot)/\mathbb{R}$ spV

d) $(\mathcal{M}_{m,n}(\mathbb{K}),+,\cdot)/\mathbb{K}$. spV

e) $(\mathbb{K}[X],+,\cdot)/\mathbb{K}$ spV.

$(\mathbb{K}_n[X]=\{P\in\mathbb{K}[X] \mid \text{grad}\, P\le n\},+,\cdot)$ spV.

f) $(\mathcal{C}(I)=\{f:I\to\mathbb{R} \mid f \text{ cont}\},+,\cdot)/\mathbb{R}$ sp vect al fct continue

$(\mathcal{D}(I)=\{f:I\to\mathbb{R}\mid f \text{ derivabilă}\},+,\cdot)/\mathbb{R}$ sp vect al fct derivabile

$(\mathcal{I}(I)=\{f:I\to\mathbb{R} \mid f \text{ integrabilă Riemann}\},+,\cdot)/\mathbb{R}$ sp vect al fct. integrabile.

Def (subspațiu vectorial)

$(V,+,\cdot)/\mathbb{K}$ sp vect, $V'\subseteq V$ subm. nevidă

V' s.n. subspațiu vectorial $\Leftrightarrow$ 1) este inclus la adunarea vectorilor
2) și la înmulțirea cu scalari

i.e. 1) $\forall x,y\in V' \Rightarrow x+y\in V'$
2) $\forall a\in\mathbb{K}, x\in V' \Rightarrow ax\in V'$

Scanat cu CamScanner

OBS $V' \subseteq V$ ssp vect $\Rightarrow (V', +, \cdot)/_{\mathbb{K}}$ sp vect (cu operațiile induse).

Prop (caracterizare a ssp. vect)

$(V, +, \cdot)|_{\mathbb{K}}$, și $V' \subseteq V$ subm. nevidă

V' ssp. vect $\iff \left[\forall x, y \in V', \forall a, b \in \mathbb{K} \Rightarrow ax + by \in V'\right]$

$\iff \left[\begin{array}{l} \forall x_1, .., x_n \in V' \\ \forall a_1, .., a_n \in \mathbb{K} \end{array} \Rightarrow a_1 x_1 + \ldots + a_n x_n \in V'\right]$

Dem

"$\Rightarrow$" V' ssp vect.

$$\left.\begin{array}{l} a \in \mathbb{K} \\ x \in V' \end{array} \Rightarrow a \cdot x \in V' \atop \begin{array}{l} b \in \mathbb{K} \\ y \in V' \end{array} \Rightarrow b \cdot y \in V' \right\} \Rightarrow ax + b \cdot y \in V'$$

"$\Leftarrow$" $ax + by \in V'$, $\forall x, y \in V'$, $\forall a, b \in \mathbb{K}$.

$a = b = 1_{\mathbb{K}} \Rightarrow \underbrace{1_{\mathbb{K}} \cdot x + 1_{\mathbb{K}} \cdot y}_{x+y} \in V'$

$b = 0_{\mathbb{K}} \Rightarrow \underbrace{a \cdot x + 0_{\mathbb{K}} \cdot y}_{a \cdot x} \in V'$

Exemple de ssp vect

1) $(V, +, \cdot)|_{\mathbb{K}} \Rightarrow \{0_V\}, V \subseteq V$ ssp. v.

2) $n < m$ $\quad \mathbb{R}^n \subset \mathbb{R}^m$ ssp vect.

3) $(\mathcal{M}_n(\mathbb{R}), +, \cdot)$

$V' = \{A \in \mathcal{M}_n(\mathbb{R}) \,/\, A = \mathrm{diag}(a_1, .., a_n)\}$ ssp v

$V'' = \left\{A \in \mathcal{M}_n(\mathbb{R}) \,/\, A = \begin{pmatrix} a_1 & \ldots & a_n \\ 0 & \ldots & 0 \\ \vdots & & \vdots \\ 0 & \ldots & 0 \end{pmatrix}\right\}$ ssp v

$V''' = \{A \in \mathcal{M}_n(\mathbb{R}) \,/\, Tr(A) = 0\}$ ssp v

$\mathcal{M}_n^s(\mathbb{R}) = W = \{A \in \mathcal{M}_n(\mathbb{R}) \,/\, A = A^T\}$ ssp vect al matricelor simetrice.

$\mathcal{M}_n^a(\mathbb{R}) = W' = \{A \in \mathcal{M}_n(\mathbb{R}) \,/\, A = -A^T\}$ ssp vect al matricelor antisimetrice.

Scanat cu CamScanner

$GL(n,\mathbb{R})$
$O(n)$
$SO(n)$
$SL(n,\mathbb{R})$
$\subset \mathcal{M}_n(\mathbb{R})$ **NU** sunt ssp vect.

4) $(\mathbb{R}^2,+,\cdot)/\mathbb{R}$, $\underset{\text{ssp}V}{V} = \{(x,y)\in\mathbb{R}^2 \mid ax+by=0,\ a^2+b^2>0\}$
dreapta care trece prin origine.

$(\mathbb{R}^3,+,\cdot)/\mathbb{R}$, $\underset{\text{ssp}V}{V'} = \{(x,y,z)\in\mathbb{R}^3 \mid ax+by+cz=0,\ a^2+b^2+c^2>0\}$
plan care trece prin origine

$(\mathbb{R}^n,+,\cdot)/\mathbb{R}$. $V'' = \{(x_1,..,x_n)\in\mathbb{R}^n \mid a_1x_1+...+a_nx_n=0,\ a_1^2+...+a_n^2>0\}$
hiperplan care trece prin orig.

5) $(\mathbb{R}^n,+,\cdot)|\mathbb{R}$
$S(A) = \{(x_1,..,x_n)\in\mathbb{R}^n \mid \underset{(m,n)}{A}\,\underset{(n,1)}{X} = \underset{(m,1)}{0}\} \subset \mathbb{R}^n$ ssp v.

($\cap$ de m hiperplane) care trec prin origine.

Subspațiul vectorial generat de o mulțime nevidă

$(V,+,\cdot)|_{\mathbb{K}}$ sp vect, $S\subset V$ subm. nevidă

$<S> = \{x\in V \mid x = a_1x_1+...+a_nx_n$, unde $x_1,..,x_n\in S$, $a_1,..,a_n\in\mathbb{K}\}$

- Dacă $V=<S>$, at S s.n. sistem de generatori (SG)

V este un spațiu vectorial finit generat $\Leftrightarrow$
$\exists$ S mulțime finită a.î. $V = \underset{\text{sp. vect. generat de } S}{\underline{<S>}}$.

OBS a) $S \subset <S>$
b) $<S>$ = cel mai mic subsp-vect, care conține S.
c) $<\phi> = \{0_V\}$ Convenție
2

Def $(V,+,\cdot)|_{\mathbb{K}}$, $S\subseteq V$ subm. nevidă
1) S s.n. sistem liniar independent (SLI) $\Leftrightarrow$
$\forall x_1,..,x_n\in S$ a.î. $a_1x_1+...+a_nx_n = 0_V \Rightarrow a_1=...=a_n=0_{\mathbb{K}}$
$\forall a_1,..,a_n\in\mathbb{K}$

Prop $\{\underset{\neq 0_V}{x}\}$ este SLI

Dem Fie $a \in \mathbb{K}$ aî $a \cdot x = 0_V$

Pp. prin absurd că $a \neq 0_{\mathbb{K}}$, $(\mathbb{K},+,\cdot)$ corp $\Big| \Rightarrow \exists a^{-1} \in \mathbb{K}$

$\underbrace{a^{-1} \cdot a}_{1_{\mathbb{K}}} \cdot x = a^{-1} \cdot 0_V \Rightarrow \underset{\underset{x}{\parallel}}{1_{\mathbb{K}} \cdot x} = 0_V$ Contrad $(x \neq 0_V)$

Pp. este falsă $\Rightarrow a = 0_{\mathbb{K}} \Rightarrow \{\underset{\underset{0_V}{\nparallel}}{x}\}$ este SLI

Def S s.n. sistem liniar dependent (SLD)

$\Leftrightarrow \exists\, x_1, \dots, x_n \in S$ aî $a_1 x_1 + \dots + a_n x_n = 0_V$.

$\exists\, a_1, \dots, a_n \in \mathbb{K}$, nu toți nuli

Def $(V,+,\cdot)|_{\mathbb{K}}$ sp vect, $S \subset V$ subm. nevidă

S s.n. bază $\Leftrightarrow \begin{cases} 1) \ S \text{ este SLI} \\ 2) \ S \text{ este SG.} \end{cases}$

Exemple

1) $(\mathbb{R},+,\cdot)|_{\mathbb{R}}$ $B_0 = \{1\}$ baza canonică.

$\{\underset{\underset{0_{\mathbb{R}}}{\nparallel}}{1}\} \Rightarrow$ SLI, $\forall x \in \mathbb{R} \Rightarrow x = x \cdot 1 \Rightarrow \{1\}$ SG.

$\forall \underset{\underset{0_{\mathbb{R}}}{\nparallel}}{a} \in \mathbb{R} \Rightarrow B = \{a\}$ bază

2) $(\mathbb{R}^2,+,\cdot)|_{\mathbb{R}}$, $B_0 = \{(1,0), (0,1)\}$ baza canonică

SLI: $\forall a,b \in \mathbb{R}$ aî $\underset{\underset{(a,b)}{\parallel}}{a(1,0) + b(0,1)} = (0,0) \Rightarrow \begin{cases} a = 0 \\ b = 0 \end{cases} \Rightarrow$ SLI

SG: $\forall (x,y) = (x,0) + (0,y) = x(1,0) + y(0,1) \Rightarrow$ SG.

3) $(\mathcal{M}_{m,n}(\mathbb{R}),+,\cdot)|_{\mathbb{R}}$ $B_0 = \{E_{ij}\}$ $i = \overline{1,m}$, $j = \overline{1,n}$ $m \cdot n$ matrice

$$E_{ij} = \begin{array}{c} \\ i \end{array} \left(\begin{array}{ccc} 0 & \overset{j}{\vdots} & 0 \\ \cdots & 1 & \\ 0 & | & 0 \end{array} \right)$$

4) $(\mathbb{K}[X],+,\cdot)|_{\mathbb{R}}$ nu e sp v. finit generat.

$B_0 = \{1, X, X^2, \dots\}$.

$(\mathbb{K}_n[X],+,\cdot)|_{\mathbb{R}}$ $\quad$ $P \in \mathbb{K}_n[X]$ $\qquad \mathbb{R}^{n+1}$

$$\overset{\|}{a_0+a_1X+\dots+a_nX^n} \equiv \overset{\cup}{(a_0,a_1,\dots,a_n)}$$

$B_0=\{1,X,X^2,\dots,X^n\}$

Prop a) $\forall$ subm. nevidă a unui SLI este SLI

$S=\{x_1,\dots,x_n\}$ SLI $\Rightarrow S'=\{x_1,\dots,x_{n-1}\}$ SLI

$$a_1x_1+\dots+a_{n-1}x_{n-1}=0_V$$
$$\|$$
$$a_1x_1+\dots+a_{n-1}x_{n-1}+0_{\mathbb{K}}\cdot x_n=0_V$$
$$\xRightarrow[S\ SLI]{} a_1=\dots=a_{n-1}=0_{\mathbb{K}} \Rightarrow S' \text{ e SLI}$$

b) $\forall$ supramulțime a unui SLD este SLD.

$S=\{x_1,\dots,x_n\}$ SLD $\Rightarrow S'=S\cup\{x_{n+1}\}$ SLD.

$\exists\, a_1,\dots,a_n \in \mathbb{K}$, nu toți nuli, a.î. $a_1x_1+\dots+a_nx_n=0_V$

$$\| $$
$$a_1x_1+\dots+a_nx_n+0_{\mathbb{K}}\cdot x_{n+1}$$

c) $\forall$ supramulțime a unui SG este SG.

$V=\langle S\rangle \Rightarrow V=\langle S\cup\{x_{n+1}\}\rangle$, $S=\{x_1,\dots,x_n\}$.

$\forall x\in V$, $\exists\, x_1,\dots,x_n \in S$, $a_1,\dots,a_n\in\mathbb{K}$ a.î. $x=a_1x_1+\dots+a_nx_n$

$$\|$$
$$a_1x_1+\dots+a_nx_n+0_K\cdot x_{n+1}.$$

Teoremă $(V,+,\cdot)/\mathbb{K}$ sp. vect. f. generat

$\forall B_1,B_2$ baze în $V \Rightarrow |B_1|=|B_2|=n=\dim_{\mathbb{K}} V$ (dimensiunea lui V)

OBS $\dim_K V=n$

a) n = nr. maxim de vectori care formează SLI

b) n = nr. minim de vect. care formează SG.

Prop UAE $(V,+,\cdot)|_{\mathbb{K}}$, $\dim_{\mathbb{K}} V = n$, $S = \{x_1, .., x_n\}$

1) S bază
2) S e SLI
3) S e SG.