2024. $\mathbb{C}_7$ - GA

Def $(V,+,\cdot)|_{\mathbb{K}}$ sp. vect

$(V^* = \{f : V \to \mathbb{K} \mid f \text{ aplicație liniară}\}, +, \cdot)|_{\mathbb{K}}$ spațiul vectorial dual lui V

$(f+g)(x) := f(x) + g(x)$

$(af)(x) := af(x)$, $\forall a \in \mathbb{K}, \forall x \in V$

Teoremă

$$V \simeq V^*$$

Dem

Fie $\mathcal{R} = \{e_1, \dots, e_n\}$ reper în V

$\mathcal{R}^* = \{e_1^*, \dots, e_n^*\}$ — " — V^*, unde

$e_i^* : V \to \mathbb{K}$ apl. lin, $e_i^*(e_j) = \delta_{ij} = \begin{cases} 1, & i=j \\ 0, & i \neq j \end{cases}$

$e_i^*(x) = e_i^*\left(\sum_{j=1}^{n} x_j e_j\right) = \sum_{j=1}^{n} x_j \underbrace{e_i^*(e_j)}_{\delta_{ij}} = x_i$, $\forall i = \overline{1,n}$

$\varphi : V \to V^*$, $\varphi(e_i) = e_i^*$, $\forall i = \overline{1,n}$

φ izom. de sp. vect.

$[\varphi]_{\mathcal{R},\mathcal{R}^*} = I_n$.

Vectori proprii. Valori proprii. Diagonalizare

Problemă $f \in \mathrm{End}(V)$

$\exists$ un reper $\mathcal{R} = \{e_1, \dots, e_n\}$ în V a.î. $[f]_{\mathcal{R},\mathcal{R}} = A = \mathrm{diag}(\lambda_1 \dots \lambda_n)$? $= \begin{pmatrix} \lambda_1 & 0 & \dots & 0 \\ \vdots & & \ddots & \\ 0 & & & \lambda_n \end{pmatrix}$

$f(e_i) = \lambda_i e_i$, $\forall i = \overline{1,n}$

$\Leftrightarrow$

Def $\underset{\neq 0_V}{x} \in V$ s.n. vector propriu $\Leftrightarrow \exists \lambda \in \mathbb{K}$ a.î. $f(x) = \lambda x$.
(valoare proprie)

OBS $f(0_V) = f(0_{\mathbb{K}} \cdot x) = 0_{\mathbb{K}} \cdot f(x) = 0_V$

Not $V_\lambda = \{x \in V \mid f(x) = \lambda x\}$ subspațiul propriu corespunzător valorii proprii λ

OBS
a) $V_\lambda \subseteq V$ subsp. vectorial
b) V_λ subsp. invariant al lui f i.e. $f(V_\lambda) \subseteq V_\lambda$

$$f(\underset{\underset{V_\lambda}{\cap}}{x}) = \lambda x \in V_\lambda$$

Polinomul caracteristic

$f \in End(V)$, $R = \{e_1, .., e_n\}$ reper în V, $[f]_{R,R} = A$

$\underset{\neq 0_V}{x}$ vector propriu coresp. valorii proprii λ

$$f(x) = \lambda x \Rightarrow f\left(\sum_{i=1}^{n} x_i e_i\right) = \lambda \sum_{j=1}^{n} x_j e_j$$

$$\sum_{i=1}^{n} x_i \boxed{f(e_i)} = \lambda \sum_{j=1}^{n} x_j e_j \Rightarrow$$

$$\sum_{i=1}^{n} x_i \left(\sum_{j=1}^{n} a_{ji} e_j\right) = \lambda \sum_{j=1}^{n} x_j e_j$$

$$\sum_{j=1}^{n} \left(\sum_{i=1}^{n} a_{ji} x_i\right) e_j = \lambda \sum_{j=1}^{n} x_j e_j \xRightarrow[\text{reper}]{R}$$

$$\sum_{i=1}^{n} a_{ji} x_i = \lambda x_j = \lambda \sum_{i=1}^{n} \delta_{ij} x_i \Rightarrow \quad \forall j = \overline{1,n}$$

- $$\sum_{i=1}^{n} (a_{ji} - \lambda \delta_{ji}) x_i = 0, \ \forall j = \overline{1,n}$$

SLO • are sol. nenulă $\Leftrightarrow \det(a_{ij} - \lambda \delta_{ij}) = 0$

$P_A(\lambda) = \det(A - \lambda I_n) = 0$ (polinomul caracteristic)

$P_A(\lambda) = (-1)^n \left[\lambda^n - \sigma_1 \lambda^{n-1} + \ldots + (-1)^n \sigma_n\right]$

σ_k = suma minorilor diagonali de ordinul k, $\forall k = \overline{1,n}$

<u>Prop</u> Polinomul caracteristic este un invariant la schimbarea de reper

<u>Dem</u> $R = \{e_1, \ldots, e_n\} \xrightarrow{C} R' = \{e_1', \ldots, e_n'\}$, $f \in End(V)$

$A = [f]_{R,R}$ $\quad A' = [f]_{R',R'}$, $A' = C^{-1}AC$.

$$\det(A' - \lambda I_n) = \det(C^{-1}AC - \lambda C^{-1} I_n C) = \det\left[C^{-1}(A - \lambda I_n)C\right] = \det(C^{-1})\det(A - \lambda I_n) \cdot \det C = \det(A - \lambda I_n)$$

$\boxed{OBS}$ Valorile proprii = rădăcinile <u>din $\mathbb{K}$</u> ale polinomului caracteristic.

<u>Exemplu</u> $(\mathbb{R}^2, +, \cdot)/\mathbb{R}$, $J : \mathbb{R}^2 \to \mathbb{R}^2$, $J(x_1, x_2) = (-x_2, x_1)$

$R_0 = \{e_1, e_2\}$

$A = [J]_{R_0,R_0} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$

$J(x) = y \Leftrightarrow AX = Y \Leftrightarrow \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -x_2 \\ x_1 \end{pmatrix}$

SAU $\begin{cases} J(e_1) = J(1,0) = (0,1) = 0 \cdot e_1 + 1 \cdot e_2 \\ J(e_2) = J(0,1) = (-1,0) = -e_1 + 0 \cdot e_2 \end{cases}$ $\quad A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$

$P(\lambda) = \det(A - \lambda I_2) = \begin{vmatrix} -\lambda & -1 \\ 1 & -\lambda \end{vmatrix} = \lambda^2 + 1$

$P(\lambda) = 0 \Rightarrow \lambda_{1,2} = \pm i$ $\quad J$ nu are valori proprii.

<u>OBS</u> $P(\lambda) = 0 \Rightarrow (\lambda - \lambda_1)^{m_1} \cdot \ldots \cdot (\lambda - \lambda_k)^{m_k} = 0$

$\lambda_1, \ldots, \lambda_k$ răd. distincte și $m_1, \ldots, m_k$ multiplicități.

Not $\sigma(f) = \{\lambda_1, \ldots, \lambda_k\}$ spectrul lui f

$$Spec(f) = \{\underbrace{\lambda_1 = \ldots = \lambda_1}_{m_1 \text{ ori}} < \underbrace{\lambda_2 = \ldots = \lambda_2}_{m_2 \text{ ori}} < \ldots < \underbrace{\lambda_k = \ldots = \lambda_k}_{m_k \text{ ori}}\}$$

Prop Vectorii proprii coresp. la valori proprii distinte formează un SLI.

Dem Dem prin ind după nr. de vectori proprii.

x vect. proprii $\Rightarrow \{x\}$ SLI
$x \neq 0_V$

Pp. adev. P_{n-1} $\{v_1, .., v_{n-1}\}$ vectori proprii coresp. la valorile proprii $\lambda_1, .., \lambda_{n-1}$ dist $\Rightarrow$ formează SLI

dem $P_{n-1} \Rightarrow P_n$.

Fie $\{v_1, .., v_n\}$ vect. proprii coresp. la val pr. dist $\lambda_1, .., \lambda_n$.

Fie $a_1, .., a_n \in \mathbb{K}$ ai $\circledast$ $a_1 v_1 + ... + a_n v_n = 0_V \overset{dem}{\Longrightarrow} a_1 = .. = a_n = 0_{\mathbb{K}}$

$\circledast \,|\, f \Rightarrow f(a_1 v_1 + ... + a_n v_n) = f(0_V)$

$a_1 \underset{\lambda_1 v_1}{f(v_1)} + ... + a_{n-1} \underset{\lambda_{n-1} \cdot v_{n-1}}{f(v_{n-1})} + a_n \underset{\lambda_n v_n}{f(v_n)} = 0_V$ (1)

$\lambda_1, .., \lambda_n$ dist $\Rightarrow \lambda_n \neq 0$ (eventual renumerotăm indicii)

$\circledast \,|\cdot \lambda_n \Rightarrow a_1 v_1 \lambda_n + ... + a_{n-1} v_{n-1} \lambda_n + a_n v_n \lambda_n = 0_V$ (2)

(1) − (2) $\quad a_1 \underbrace{(\lambda_1 - \lambda_n)}_{\neq 0} v_1 + ... + a_{n-1} \underbrace{(\lambda_{n-1} - \lambda_n)}_{\neq 0} v_{n-1} = 0_V$

$\overset{P_{n-1}}{\Longrightarrow} \quad a_1 = ... = a_{n-1} = 0_{\mathbb{K}} \overset{\circledast}{\Longrightarrow} a_n v_n = 0_V \Rightarrow a_n = 0$

$\{v_1, .., v_{n-1}\}$ SLI

$\Rightarrow \{v_1, .., v_n\}$ SLI

Prop $f \in End(V)$, λ = valoare proprie $\Rightarrow \dim V_\lambda \leq m_\lambda$

V_λ = subsp. propriu al lui λ, m_λ = multiplicitatea.

Dem $V_\lambda \subseteq V$ subsp vect, $\dim V_\lambda = m_\lambda$

$R_0 = \{e_1, .., e_{m_\lambda}\}$ reper în V_λ. Îl extindem la

$R = \{e_1, .., e_{m_\lambda}, e_{m_\lambda + 1}, .., e_n\}$ reper în V.

$$A=[f]_{R,R}=\begin{pmatrix} \boxed{\begin{matrix}\lambda & 0 & 0\\ 0 & \lambda & 0\\ \vdots & & \vdots\\ 0 & 0 & \lambda\end{matrix}} & \\ \vdots & \vdots & \vdots & \\ 0 & 0 & 0 & \end{pmatrix}$$

$f(e_1)=\lambda e_1$

$\vdots$

$f(e_{n_\lambda})=\lambda e_{n_\lambda}$

$f(e_j)=\sum_{k=1}^{n} a_{kj}e_k$

$\forall j=\overline{n_\lambda+1,n}$

$$P(x)=\det(A-xI_n)=\begin{vmatrix}\lambda-x & 0 \dots & 0 & \\ 0 & \lambda-x \dots & \vdots & \\ \vdots & & \lambda-x & \\ 0 & \dots & 0 & \end{vmatrix}=(\lambda-x)^{n_\lambda}Q(x)$$

$\dim V_\lambda = n_\lambda \le m_\lambda$

Teorema (de diagonalizare)

$f\in End(V)$

$\exists$ un reper $R=\{e_1,\dots,e_n\}$ în V a.î. $[f]_{R,R}=A=$ diagonală $\Leftrightarrow$

1) toate răd. pol. caracteristic $\in \mathbb{K}$
(i.e. $\lambda_1,\dots,\lambda_k\in\mathbb{K}$, $\lambda_1,\dots,\lambda_k$ răd dist)

2) ~~dim~~ dim. subsp. proprii coincid cu multiplicitățile
(i.e. $\dim V_{\lambda_j}=m_j$, $\forall j=\overline{1,k}$, $m_1+\dots+m_k=n=\dim V$)

Dem

"$\Rightarrow$" Ip: $\exists R=\{e_1,\dots,e_n\}$ reper în V a.î.

$$A=[f]_{R,R}=\begin{pmatrix}\mu_1 & & O\\ & \ddots & \\ O & & \mu_n\end{pmatrix}\in\mathcal{M}_n(\mathbb{K})$$

Eventual renumerotăm indicii (sch. de reper) a.î.

$$A=\begin{pmatrix}\overbrace{\lambda_1\dots\lambda_1}^{m_1\text{ ori}} & & O\\ & \ddots & \\ O & & \overbrace{\lambda_k\dots\lambda_k}^{m_k\text{ ori}}\end{pmatrix}$$

$m_1+\dots+m_k=n$.

$P(\lambda)=\det(A-\lambda I_n)=(\lambda_1-\lambda)^{m_1}\cdot\ldots\cdot(\lambda_k-\lambda)^{m_k}$

$\lambda_1,\dots,\lambda_k$ sunt răd dist pt pol. caract, $\lambda_j\in\mathbb{K}$, $\forall j=\overline{1,k}$

Cf. prop. preced. $\dim V_{\lambda_j}\le m_j$, $\forall j=\overline{1,k}$ (1)

$$\left.\begin{array}{ll} R_1 = \{e_1, \dots, e_{m_1}\} \subset V_{\lambda_1} & \Rightarrow \dim V_{\lambda_1} \geq m_1 \\ \vdots & \\ R_k = \{e_{m_1+\dots+m_{k-1}+1}, \dots, e_n\} \subset V_{\lambda_k} & \Rightarrow \dim V_{\lambda_k} \geq m_k \end{array}\right\} (2)$$

Din (1), (2) $\Rightarrow \dim V_{\lambda_j} = m_j$, $\forall j = \overline{1,k}$

"$\Leftarrow$" Ip. 1) $\lambda_1, \dots, \lambda_k \in K$, $\lambda_1, \dots, \lambda_k$ răd. dist. pt pol. caract.

2) $\dim V_{\lambda_j} = m_j$, $\forall j = \overline{1,k}$ $\quad m_1 + \dots + m_k = n$.

Fie R_j reper în V_{λ_j}, $j = \overline{1,k}$

Considerăm $R = R_1 \cup \dots \cup R_k$. $\quad |R| = n = \dim V$.

Dem că R este reper în V

Este suficient să dem că R este SLI.

$$\underbrace{\sum_{j=1}^{m_1} a_j e_j}_{f_1 \in V_{\lambda_1}} + \dots + \underbrace{\sum_{j=m_1+\dots+m_{k-1}+1}^{n} a_j e_j}_{f_k \in V_{\lambda_k}} = 0_V$$

Pp. abs. $\exists f_{i_1}, \dots f_{i_p}$ nenule $i_1, \dots, i_p \in \{1, \dots, k\}$.

$f_{i_1} + \dots + f_{i_p} = 0_V$ $\quad \{f_{i_1}, \dots, f_{i_p}\}$ SLD Contrad.

(vect. pr. coresp. la val proprii dist. formează SLI)

În concluzie $\quad f_1 = \sum_{j=1}^{m_1} a_j e_j = 0 \xRightarrow{R_1 \text{ reper}} a_j = 0, \forall j = \overline{1,m_1}$

$\vdots$

$f_k = \sum_{j=m_1+\dots+m_{k-1}+1}^{n} a_j e_j = 0 \xRightarrow{R_k \text{ reper}} a_j = 0, \forall j = \overline{m_1+\dots+m_{k-1}+1, n}$

R este SLI $\Rightarrow R$ reper în V

$$A = [f]_{R,R} = \begin{pmatrix} \overbrace{\lambda_1 \dots \lambda_1}^{m_1} & & 0 \\ & \ddots & \\ 0 & & \underbrace{\lambda_k \dots \lambda_k}_{m_k} \end{pmatrix} \qquad m_1 + \dots + m_k = n$$

<u>OBS</u> $V = V_{\lambda_1} \oplus \dots \oplus V_{\lambda_k}$

Aplicatie 2 Fie $f \in End(\mathbb{R}^3)$, $f(x) = (x_1, x_2 + x_3, 2x_3)$

Det $\mathcal{R}$ reper in $\mathbb{R}^3$ aî $[f]_{\mathcal{R},\mathcal{R}} = A' =$ diagonala

sol

- $A = [f]_{\mathcal{R}_0,\mathcal{R}_0} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 2 \end{pmatrix}$

- $P_A(\lambda) = \det(A - \lambda I_3) = \begin{vmatrix} 1-\lambda & 0 & 0 \\ 0 & 1-\lambda & 1 \\ 0 & 0 & 2-\lambda \end{vmatrix} = (1-\lambda)^2(2-\lambda)$

$\begin{cases} \lambda_1 = 1, & m_1 = 2 \\ \lambda_2 = 2, & m_1 = 1 \end{cases}$

- $V_{\lambda_1} = \{ x \in \mathbb{R}^3 \mid f(x) = \lambda_1 x \}$

$AX = \lambda_1 X \Rightarrow (A - \lambda_1 I_3)X = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

$\lambda_1 = 1 \quad \underbrace{(A - I_3)}_{M} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

$V_{\lambda_1} = S(M) \quad \dim V_{\lambda_1} = 3 - rg\, M = 3 - 1 = 2 = m_1$ ✓

$x_3 = 0$

$V_{\lambda_1} = \{ (x_1, x_2, 0) \mid x_1, x_2 \in \mathbb{R} \} = < \underbrace{\{ e_1 = (1,0,0), e_2 = (0,1,0) \}}_{\mathcal{R}_1} >$

$V_{\lambda_2} = \{ X \in \mathbb{R}^3 \mid f(x) = 2X \}$

$AX = 2X \Rightarrow (A - 2I_3)X = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow \overbrace{\begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & 0 \end{pmatrix}}^{N} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

$\dim V_{\lambda_2} = 3 - rg(N) = 3 - 2 = 1 = m_2$ ✓

$\begin{cases} -x_1 = 0 \\ -x_2 + x_3 = 0 \end{cases} \quad \begin{matrix} x_1 = 0 \\ x_2 = x_3 \end{matrix}$

$V_{\lambda_2} = \{ (0, x_3, x_3) \mid x_3 \in \mathbb{R} \} = < \underbrace{\{ (0,1,1) \}}_{\mathcal{R}_2} >$

$\mathcal{R} = \mathcal{R}_1 \cup \mathcal{R}_2$ reper in $V = \mathbb{R}^3 = V_{\lambda_1} \oplus V_{\lambda_2}$.

$A' = [f]_{\mathcal{R},\mathcal{R}} = \begin{pmatrix} 1 & & 0 \\ & 1 & \\ 0 & & 2 \end{pmatrix}$

Proiecții și simetrii

Def $p : V_1 \oplus V_2 \to V_1 \oplus V_2$ apl. lin.

p s.n. proiecție pe V_1 $\Leftrightarrow$ $p(\underset{V_1}{\underset{\cap}{x_1}} + \underset{V_2}{\underset{\cap}{x_2}}) = x_1$.

Prop $p \in End(V)$

p = proiecție $\Leftrightarrow$ $p \circ p = p$.

Dem "$\Rightarrow$" $p : V_1 \oplus V_2 \to V_1 \oplus V_2$ $\quad p(\overbrace{x_1 + x_2}^{x}) = x_1$. pr. pe V_1

$\underline{p(p(\overbrace{x_1+x_2}^{x}))} = p(x_1) = p(\underset{V_1}{\underset{\cap}{x_1}} + \underset{V_2}{\underset{\cap}{0}}) = x_1 = \underline{p(x)}$, $\forall x \in V \Rightarrow p \circ p = p$

"$\Leftarrow$" $p \in End(V)$ a.î. $p \circ p = p$.

$V = \underset{Im\, p}{\underset{\shortparallel}{V_1}} \oplus \underset{Ker\, p}{\underset{\shortparallel}{V_2}}$ $\quad p(x_1 + x_2) = x_1$

$x \in Im\, p \cap Ker\, p$.

$\exists\, z \in V$ a.î. $x = p(z)$ $| p$

$p(x) = p(p(z))$

$0_V = p(z) = x \Rightarrow$ "$\oplus$"

$Im\, p \oplus Ker\, p \subseteq V$ (din construcție).

Dem $V \subseteq Im\, p \oplus Ker\, p$.

$x = \underset{Im\, p}{\underset{\cap}{\overbrace{p(x)}^{x_1}}} + \underset{Ker\, p}{\underset{\cap}{\overbrace{x - p(x)}^{x_2}}}$ (dem)

$p(x_2) = p(x) - p(p(x)) = 0_V$

Def $s \in End(V)$ s s.n. simetrie $\Leftrightarrow$ $s \circ s = id_V$ (involuție).
(caracteristica)

Prop $(V, +, \cdot) / \mathbb{K}$, $char\, \mathbb{K} \neq 2$ $(1 + 1 \neq 0_{\mathbb{K}})$

p proiecție pe V_1 $\Leftrightarrow$ $s = 2p - id_V$ este simetrie față de V_1

$s = 2p - id_V$

$s \circ s = id_V \iff (2p - id_V) \circ (2p - id_V) = id_V \iff$

$4 p \circ p - 2p - 2p + id_V = id_V \iff$

$p \circ p = p.$