Seminar 4

1. Studiați converg. (natura) seriilor

a) $\sum_{n=1}^{\infty} \frac{\cos\frac{1}{n} \cos n}{n}$

Soluție

$\sum_{n} \frac{\cos n}{n} \to$ converg de ora trecută

Aplicăm criteriul Abel-Dirichlet (II)

Fie $x_n = \cos\frac{1}{n}, \forall n \in \mathbb{N}^*$

$y_n = \frac{\cos n}{n}, \forall n \in \mathbb{N}^*$

$-1 \leq x_n \leq 1, \forall n \in \mathbb{N}^* \Rightarrow (x_n)_n$ mărginit ①

$\underset{(0, \frac{\pi}{2})}{x} \longmapsto \underset{(0,1)}{\cos x}$ descrescăt

$\left(\frac{1}{n}\right)_n$ descresc

$\frac{1}{n} \in (0; \frac{\pi}{2}) \; \forall n \in \mathbb{N}^*$

$\Rightarrow (x_n)_n$ ~~de~~cresc (comp de 2 descresc) ②

Deci, $(x_n)_n$ este monoton și mărginit I

$\sum_{n=1}^{\infty} y_n = \sum_{n=1}^{\infty} \frac{\cos n}{n}$ - conv (vezi ult. ex. din seminar 3) II

I + II $\Rightarrow \sum_{n} x_n y_n$ converg ☑

b) $\sum_{n=1}^{\infty} \frac{(-1)^n \sqrt{n} + 1}{n}$

$$x_n = \frac{(-1)^n \sqrt{n} + 1}{n} = \underbrace{\frac{(-1)^n \sqrt{n}}{n}}_{a_n} + \underbrace{\frac{1}{n}}_{b_n}$$

Fie $a_n = (-1)^n \cdot \frac{\sqrt{n}}{n}$

$b_n = \frac{1}{n}$

$\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \frac{1}{n}$ diverg (serie armonică gen. cu $\alpha = 1$)

$a_n = (-1)^n \cdot c_n \qquad c_n = \frac{\sqrt{n}}{n} = \sqrt{\frac{1}{n}}$ | crit $\Rightarrow$ Leibniz

$\lim_{n\to\infty} c_n = 0$

$(c_n)_n$ descresc

$\sum_{n=1}^{\infty} a_n$ converg

$\sum_{n=1}^{\infty} x_n$ divergentă $\square$

2. a) Det. $\lim_{x\to 0} \frac{1-\cos x}{x^2} = \frac{1}{2}$

b) Stud. converg. seriei $\sum_{n=1}^{\infty} \left(1-\cos\frac{1}{n}\right) x^n$, $x>0$

a) $\lim_{x\to 0} \frac{1-\cos x}{x^2} \overset{0/0}{\underset{l'H}{=}} \lim_{x\to 0} \underbrace{\frac{\sin x}{2x}}_{\to 1} = \frac{1}{2}$

b) $\sum_{n=1}^{\infty} \left(1-\cos\frac{1}{n}\right) x^n$

Fie $x_n = \left(1-\cos\frac{1}{n}\right) x^n$

$$\lim_{n\to\infty} \frac{x_{n+1}}{x_n} = \lim_{n\to\infty} \left(1-\cos\frac{1}{n+1}\right)\cdot x^{n+1} \cdot \frac{1}{\left(1-\cos\frac{1}{n}\right)x^{n}}$$

$$\lim_{n\to\infty} \frac{\frac{1-\cos\frac{1}{n+1}}{\left(\frac{1}{n+1}\right)^2}\cdot x\cdot\left(\frac{1}{n+1}\right)^2}{\frac{1-\cos\frac{1}{n}}{\frac{1}{n^2}}\cdot\frac{1}{n^2}}$$

$$= \lim_{n\to\infty} \underbrace{\frac{n^2}{(n+1)^2}}_{\to 1} \cdot \frac{\overbrace{\frac{1-\cos\frac{1}{n+1}}{\left(\frac{1}{n+1}\right)^2}}^{\to\frac{1}{2}}}{\underbrace{\frac{1-\cos\frac{1}{n}}{\left(\frac{1}{n}\right)^2}}_{\to\frac{1}{2}}} \cdot x$$

$= x$

Cf. crit raportului:

1) Dacă $x<1$ $(x\in(0,1))$, $\sum_{n=1}^{\infty} x_n$ converg

2) Dacă $x>1$ $(x\in(1,\infty))$, $\sum_{n=1}^{\infty} x_n$ diverg

3) Dacă $x=1$, $\sum_{n=1}^{\infty}$ criter nu decide

$x_n \overset{pt\ x=1}{=} \sum_{n=1}^{\infty}\left(1-\cos\frac{1}{n}\right)$, $\forall\, n\in\mathbb{N}^*$

Considerăm $y_n = \frac{1}{n^2}$, $\forall\, n\in\mathbb{N}^*$

$$\lim_{n\to\infty}\frac{x_n}{y_n} = \lim_{n\to\infty}\frac{1-\cos\frac{1}{n}}{\frac{1}{n^2}} = \frac{1}{2} \in (0,\infty)$$

Cf. crit de comp cu limită avem că

$$\sum_{n=1}^{\infty} x_n \sim \sum_{n=1}^{\infty} y_n$$

$\sum_n y_n$ converg (serie armonică cu $\alpha = 2 > 1$)

Deci, $\sum_{n} x_n$ converg.

③ Fie $n \in \mathbb{N}^*$ şi $d_1 : \mathbb{R}^n \times \mathbb{R}^n$

$d_1(x, y) = \sum_{i=1}^{n} |x_i - y_i| = |x_1 - y_1| + \dots + |x_n - y_n|$

$x = (x_1, x_2 \dots x_n)$, $y = (y_1, y_2 \dots y_n)$

a) Arătaţi că d_1 e metrică (sau distanţă) pe $\mathbb{R}^n$

Sol

• $d_1(x, y) \geq 0 \ \forall x, y \in \mathbb{R}^n$ (e sumă de module)

• $d_1(x, y) = 0 \Leftrightarrow \sum_{i=1}^{n} |x_i - y_i| = 0 \Leftrightarrow$

$|x_i - y_i| = 0 \ \forall i \in \overline{1,n} \Leftrightarrow x_i = y_i \ \forall i \in \overline{1,n}$

$\Leftrightarrow x = y, \forall x, y \in \mathbb{R}^n$

• $d_1(x, y) = \sum_{i=1}^{n} |x_i - y_i| = \sum_{i=1}^{n} |-(x_i - y_i)|$

$= \sum_{i=1}^{n} |y_i - x_i| = d_1(y, x), \ \forall x, y \in \mathbb{R}^n$

• Fie $x, y, z \in \mathbb{R}^n$

Arătam că $d_1(x, z) \leq d_1(x, y) + d_1(y, z)$

$d_1(x, z) = \sum_{i=1}^{n} |x_i - z_i| = \sum_{i=1}^{n} |\underbrace{x_i - y_i}_{a} + \underbrace{y_i - z_i}_{b}| \leq$

$\sum_{i=1}^{n} |x_i - y_i| + \sum_{i=1}^{n} |y_i - z_i| = d_1(x, y) + d_1(y, z)$

Deci, $d_1(x, y) \leq d_1(x, y) + d_1(y, z)$

$\Rightarrow$ Deci d_1 este metrică pe $\mathbb{R}^n$ $\square$

b) Fie $n \in \mathbb{N}^*$, $(x^k)_k \subset \mathbb{R}^n$, $x^k = (x_1^k \dots x_n^k)$, $x \in \mathbb{R}^n$, $x = (x_1 \dots x_n)$. Arătați că $\lim\limits_{k\to\infty} x^k \overset{d_1}{=} x$ dacă și numai dacă $\lim\limits_{k\to\infty} x_i^k = x_i \quad \forall\ i = \overline{1,n}$

<u>Sol</u>

"$\Rightarrow$"

Știm că $\lim\limits_{k\to\infty} x^k \overset{d_1}{=} x \Rightarrow \lim\limits_{k\to\infty} d_1(x^k, x) = 0$

$\forall\ \varepsilon > 0$, $\exists\ k_\varepsilon \in \mathbb{N}$ a.î. $\forall\ k \geq k_\varepsilon$ avem $d_1(x^k, x) < \varepsilon \Rightarrow \forall\ \varepsilon > 0\ \exists\ k_\varepsilon \in \mathbb{N}$ a.î. $\forall\ k \geq k_\varepsilon$ avem $\sum\limits_{i=1}^{n} |x_i^k - x_i| < \varepsilon\ \forall\ i = \overline{1,n}$

$\Rightarrow \forall\ \varepsilon > 0$, $\exists\ k_\varepsilon \in \mathbb{N}$ a.î. $\forall\ k \geq k_\varepsilon$ avem $|x_i^k - x_i| < \varepsilon \quad \forall\ i = \overline{1,n}$

Deci, $\lim\limits_{k\to\infty} x_i^k = x_i$, $\forall\ i = \overline{1,n}$

"$\Leftarrow$" Fie $i \in \{1, 2, \dots, n\}$

$\lim\limits_{k\to\infty} x_i^k = x_i \Rightarrow \forall\ \varepsilon > 0\ \exists\ k_\varepsilon \in \mathbb{N}$ a.î. $\forall\ k \geq k_\varepsilon$ avem $|x_i^k - x_i| < \frac{\varepsilon}{n}$

Fie $\varepsilon > 0$. Alegem $k_\varepsilon = \max\{k_\varepsilon^1, \dots, k_\varepsilon^n\}$

Atunci, $\forall\ k \geq k_\varepsilon$ avem $\sum\limits_{i=1}^{n} |x_k^i - x_i| < \underbrace{\frac{\varepsilon}{n} + \dots + \frac{\varepsilon}{n}}_{\text{de } n \text{ ori}} = \varepsilon$

, i.e. $\forall\ k \geq k_\varepsilon$ avem $d_1(x^k, x) < \varepsilon$, i.e.

$\lim\limits_{k\to\infty} \overset{d_1}{=} x$ $\square$

$$\left(\frac{1}{3}, \frac{2n}{5n+8}\right) \xrightarrow[n\to\infty]{d_1} \left(0, \frac{2}{5}\right)$$

4. Fie $n \in \mathbb{N}^*$ și $d_2 \overset{not}{=} d : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$

$$d(x,y) = \sqrt{\sum_{i=1}^{n} (x_i - y_i)^2}$$

Arătați că d e metrică pe $\mathbb{R}^n$

Sol

Reamintim ineg. Cauchy-Buniakowski-Schwartz (CBS)

" Fie $n \in \mathbb{N}^*$ și $a_1, \dots, a_n, b_1, \dots, b_n \in \mathbb{R}$

Atunci $\left(\sum_{i=1}^{n} a_i b_i\right)^2 \leq \left(\sum_{i=1}^{n} a_i^2\right)\left(\sum_{i=1}^{n} b_i^2\right)$

1) $d(x,y) \geq 0$

2) $d(x,y) = 0 \Leftrightarrow x = y$

3) $d(x,y) = d(y,x)$

se demonstr.

4) Fie $x, y, z \in \mathbb{R}^n$

$$d(x,z) = \sqrt{\sum_{i=1}^{n} (x_i - z_i)^2} = \sqrt{\sum_{i=1}^{n} (x_i - y_i + y_i - z_i)^2}$$

$$= \sqrt{\sum_{i=1}^{n} (x_i - y_i)^2 + 2(x_i - y_i)(y_i - z_i) + (y_i - z_i)^2}$$

$$= \sqrt{\sum_{i=1}^{n} (x_i - y_i)^2 + 2\sum_{i=1}^{n} (x_i - y_i)(y_i - z_i) + \sum_{i=1}^{n} (y_i - z_i)^2} \leq$$

CBS

$$\left|\sum_{i=1}^{n}(x_i-y_i)(y_i-z_i)\right| \le \sqrt{\sum_{i=1}^{n}(x_i-y_i)^2}\sqrt{\sum_{i=1}^{n}(y_i-z_i)^2}$$

$$\le \sqrt{\sum_{i=1}^{n}(x_i-y_i)^2+\sum_{i=1}^{n}(y_i-z_i)^2+2\sqrt{\sum_{i=1}^{n}(x_i-y_i)^2}\sqrt{\sum_{i=1}^{n}(y_i-z_i)^2}}$$

$$= \sqrt{\left(\sqrt{\sum_{i=1}^{n}(x_i-y_i)^2}+\sqrt{\sum_{i=1}^{n}(y_i-z_i)^2}\right)^2}$$

$$= \sqrt{\sum_{i=1}^{n}(x_i-y_i)^2}+\sqrt{\sum_{i=1}^{n}(y_i-z_i)^2}$$

$$= d(x,y)+d(y,z)$$

Deci, d este metrică pe $\mathbb{R}$

5. Fie $n\in\mathbb{N}^*$ şi $d_1, d_2 \overset{not}{=} d$ ca mai sus.
Arătaţi că $\exists\ a,b\in(0,\infty)$ a.î.
$a d_1(x,y) \le d(x,y) \le b d_1(x,y),\ \forall x,y\in\mathbb{R}^n$

Sol

$$d_1(x,y) = \sum_{i=1}^{n}|x_i-y_i|$$

$$= \sum_{i=1}^{n}|x_i-y_i|\cdot 1 \underset{CBS}{\le} \sqrt{\sum_{i=1}^{n}(x_i-y_i)^2}$$

$$\sqrt{\sum_{i=1}^{n}1^2} = \sqrt{n}\ d(x,y)\ \forall x,y\in\mathbb{R}^n$$

Alegem $a=\frac{1}{\sqrt{n}}$

$$d(x,y) = \sqrt{\sum_{i=1}^{n} (x_i - y_i)^2} = \sqrt{\sum_{i=1}^{n} |x_i - y_i|^2} \leq$$

$$\sqrt{\sum_{i=1}^{n} |x_i - y_i|^2} = \sum_{i=1}^{n} |x_i - y_i| = d_1(x,y) \; \forall x,y \in \mathbb{R}^n$$

Alegem $b = 1$

$\square$