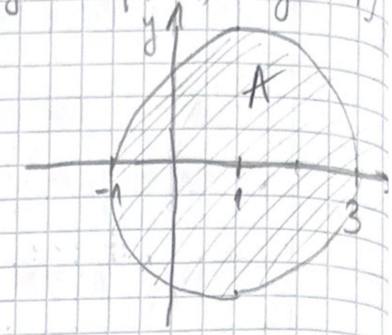
Seminar 14

18.01.2024

1) Det a) $\iint_A y\,dx\,dy$, unde $A=\{(x,y)\in\mathbb{R}^2 \mid (x-1)^2+y^2\le 4\}$



Sol:

A convexă și mărginită

$\Rightarrow A\in \mathcal{J}(\mathbb{R}^2)$

A compactă

Fie $f: A\to\mathbb{R}$, $f(x,y)=y$.

Gh ex14

f cont

SV: $\begin{cases} x=1+r\cos\theta \\ y=1+r\sin\theta \end{cases}$, $r\in[0,\infty)$, $\theta\in[0,2\pi]$

$(x,y)\in A \Rightarrow (x-1)^2+y^2\le 4 \Rightarrow (1+r\cos\theta-1)^2+(r\sin\theta)^2\le 4 \Rightarrow$

$\Rightarrow r^2\cos^2\theta+r^2\sin^2\theta\le 4 \Rightarrow r^2\le 4 \Rightarrow \begin{cases} r\in[0,2] \\ \theta\in[0,2\pi] \end{cases}$

Fie $B=[0,2]\times[0,2\pi]$.

$$\iint_A f(x,y)\,dx\,dy=\iint_B r f(1+\cos\theta; r\sin\theta)\,dr\,d\theta=$$

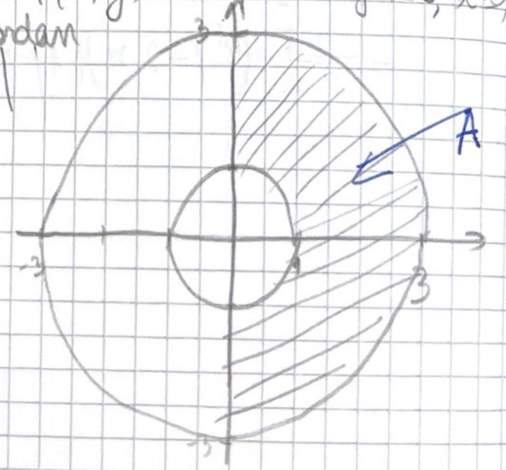
$$=\int_0^2\left(\int_0^{2\pi} r\cdot r\sin\theta\,d\theta\right)dr=\int_0^2\left(r^2(-\cos\theta)\Big|_{\theta=0}^{\theta=2\pi}\right)dr=$$

$$=\int_0^2 0\,dr=0.$$

b) $\iint_A \sqrt{x^2+y^2}\,dx\,dy$, unde $A=\{(x,y)\in\mathbb{R}^2 \mid 1\le x^2+y^2\le 9,\ x\ge 0\}$.

Sol: $A=\underbrace{\{(x,y)\in\mathbb{R}^2 \mid x^2+y^2\le 9, x\ge 0\}}_{\text{convexă}\Rightarrow\text{măs. Jordan}}$

$\setminus \underbrace{\{(x,y)\in\mathbb{R}^2 \mid x^2+y^2<1\}}_{\text{convexă}\Rightarrow\text{măs. Jordan}}$.



$\Rightarrow A\in\mathcal{J}(\mathbb{R}^2)$

Fie $f: A\to\mathbb{R}$, $f(x,y)=\sqrt{x^2+y^2}$.

f cont.

SV: $\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$, $r \in [0, \infty)$, $\theta \in [0, 2\pi]$

$(x,y) \in A \Rightarrow \begin{cases} 1 \le x^2+y^2 \le 9 \\ x \ge 0 \end{cases} \Rightarrow \begin{cases} 1 \le r^2 \le 9 \\ r\cos\theta \ge 0 \end{cases} \Rightarrow$

$\Rightarrow \begin{cases} r \in [1,3] \\ \theta \in [0, \frac{\pi}{2}] \cup [\frac{3\pi}{2}, 2\pi] \end{cases}$

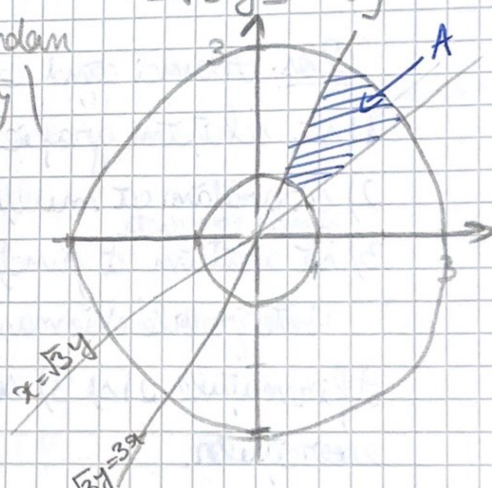
Fie $B = [1,3] \times \left([0, \frac{\pi}{2}] \cup [\frac{3\pi}{2}, 2\pi]\right)$

$\iint_A f(x,y)\,dx\,dy = \iint_B r f(r\cos\theta, r\sin\theta)\,dr\,d\theta =$

$= \int_1^3 \left(\int_0^{\frac{\pi}{2}} r\sqrt{r^2}\,d\theta + \int_{\frac{3\pi}{2}}^{2\pi} r\sqrt{r^2}\,d\theta \right) dr = \int_1^3 \left(r^2 \frac{\pi}{2} + r^2 \frac{\pi}{2} \right) dr =$

$= \pi \cdot \frac{r^3}{3}\Big|_{r=1}^{r=3} + \frac{\pi}{2} \cdot \frac{r^3}{3}\Big|_{r=1}^{r=3} = \frac{26\pi}{3}$.

c) $\iint_A \operatorname{arctg} \frac{y}{x}\,dx\,dy$, unde $A = \{(x,y) \in \mathbb{R}^2 \mid 1 \le x^2+y^2 \le 9,$
$x \le \sqrt{3}y \le 3x\}$



Sol: convexă și mărg. $\Rightarrow$ măs. Jordan

$A = \{(x,y) \in \mathbb{R}^2 \mid x^2+y^2 \le 9,\ x \le \sqrt{3}y \le 3x\} \setminus$
$\setminus \{(x,y) \in \mathbb{R}^2 \mid x^2+y^2 \le 1\}$

convexă și mărg $\Rightarrow$ măs Jordan

Deci $A \in \mathcal{J}(\mathbb{R}^2)$

A compactă

Fie $f: A \to \mathbb{R}$, $f(x,y) = \operatorname{arctg} \frac{y}{x}$

f cont

SV: $\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$, $r \in [0, \infty)$, $\theta \in [0, 2\pi]$

$(x,y) \in A \Rightarrow \begin{cases} 1 \le x^2+y^2 \le 9 \\ x \le \sqrt{3}y \\ \sqrt{3}y \le 3x \end{cases} \Rightarrow \begin{cases} 1 \le r^2 \le 9 \\ r\cos\theta \le \sqrt{3}r\sin\theta \\ r\sin\theta \le \sqrt{3}r\cos\theta \end{cases} \Rightarrow$

$\Rightarrow \begin{cases} r \in [1,3] \\ \frac{\sin\theta}{\cos\theta} \ge \frac{1}{\sqrt{3}} \\ \frac{\sin\theta}{\cos\theta} \le \sqrt{3} \end{cases} \Rightarrow \theta \in (0, \frac{\pi}{2})$, $r \in [1,3]$

$$\Rightarrow \begin{cases} r \in [1,3] \\ \operatorname{tg}\theta \geq \frac{1}{\sqrt{3}} \\ \operatorname{tg}\theta \leq \sqrt{3} \end{cases} \Rightarrow \begin{cases} r \in [1,3] \\ \theta \in \left[\frac{\pi}{6}, \frac{\pi}{3}\right] \end{cases}$$

Fie $B = [1,3] \times \left[\frac{\pi}{6}, \frac{\pi}{3}\right]$

$$\iint_A f(x,y)\,dx\,dy = \iint_B r\, f(r\cos\theta, r\sin\theta)\,dr\,d\theta =$$

$$= \int_1^3 \left(\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \left(r \operatorname{arctg}\left(\frac{r\sin\theta}{r\cos\theta}\right) d\theta \right) \right) dr = \int_1^3 \left(\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} r \cdot \theta \, d\theta \right) dr =$$

$$= \int_1^3 \left(r \cdot \frac{\theta^2}{2} \right) \Bigg|_{\theta=\frac{\pi}{6}}^{\theta=\frac{\pi}{3}} = \int_1^3 \frac{r}{2}\left(\frac{\pi^2}{9} - \frac{\pi^2}{36}\right) dr = \int_1^3 \frac{3\pi^2}{36} \cdot \frac{1}{2}\, r\, dr =$$

$$= \frac{\pi^2}{24} \cdot \frac{r^2}{2} \Bigg|_{r=1}^{r=3} = \frac{\pi^2}{48}(9-1) = \frac{\pi^2}{6}. \quad \square$$

Obs: Atunci când calculăm integr. triple, nu este nevoie:

1) să schițăm grafic mulțimile din $\mathbb{R}^3$

2) să arătăm că mulțimile din $\mathbb{R}^3$ sunt măsurabile Jordan (și nici compacte)

3) să arătăm că funcțiile definite pe mulțimi din $\mathbb{R}^3$ sunt integrabile Riemann (și nici mărginite)

Afirmațiile 2) și 3) se consideră adevărate din enunțurile exercițiilor.

③. Det:

a) $\iiint_A (xyz + y^2)\, dx\, dy\, dz$; unde $A = [-1,1] \times [2,3] \times [0,1]$

Sol: $\iiint_A (xyz + y^2)\, dx\,dy\,dz = \int_{-1}^{1} \left(\int_{2}^{3} \left(\int_{0}^{1} (xyz + y^2)\, dz \right) dy \right) dx$

$= \int_{-1}^{1} \left(\int_{2}^{3} \left(xy \frac{z^2}{2} + y^2 z \right) \Big|_{z=0}^{z=1} dy \right) dx = \int_{-1}^{1} \left(\int_{2}^{3} \left(\frac{xy}{2} + y^2 \right) dy \right) dx$

$= \int_{-1}^{1} \left(x \frac{y^2}{4} + \frac{y^3}{3} \right) \Big|_{y=2}^{y=3} dx = \int_{-1}^{1} \left(x \cdot \frac{9-4}{4} + \frac{27-8}{3} \right) dx =$

$= \frac{5}{4} \cdot \frac{x^2}{2} \Big|_{x=-1}^{x=1} + \frac{19}{3} \Big|_{x=-1}^{x=1} = 0 + \frac{38}{3} = \frac{38}{3}$

b) $\iiint_A x\, dx\, dy\, dz$, $A = [1,2] \times [0,1] \times [2,3]$

c) $\iiint_A (x^2+y^2) \cdot z\, dx\, dy\, dz$, unde

$A = \{ (x,y,z) \in \mathbb{R}^3 \mid (x,y) \in B,\ x^2+y^2 \le z \le \sqrt{6-x^2-y^2} \}$

și $B = \{ (x,y) \in \mathbb{R}^2 \mid x^2+y^2 \le 2 \}$

$\iiint_A (x^2+y^2) z\, dx\,dy\,dz = \iint_B \left(\int_{x^2+y^2}^{\sqrt{6-x^2-y^2}} (x^2-y^2) \cdot z\, dz \right) dx\,dy$

$= \iint_B \left(\frac{(x^2+y^2) z^2}{2} \Big|_{x^2+y^2}^{\sqrt{6-x^2-y^2}} \right) dx\, dy =$

$= \iint_B \frac{x^2+y^2}{2} \left(6 - x^2 - y^2 - (x^2+y^2)^2 \right) dx\, dy$

B convexă + mărg
B compactă $\Rightarrow B \in \mathcal{J}(\mathbb{R}^2)$

Fie $f: B \to \mathbb{R}$, $f(x,y) = \frac{(x^2-y^2)}{2} \cdot \left(6 - x^2 - y^2 - (x^2+y^2)^2\right)$

f cont

SV: $\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$, $r \in [0, \infty)$, $\theta \in [0, 2\pi]$

$(x,y) \in B \Rightarrow x^2+y^2 \le 2 \Rightarrow r^2 \le 2$

Fie $C = [0, \sqrt{2}] \times [0, 2\pi]$

$$\iint_B f(x,y)\,dx\,dy = \iint_C r f(r\cos\theta, r\sin\theta)\,dr\,d\theta =$$

$$= \iint_C \left(r \cdot \frac{r^2}{2}\left(6 - r^2 - (r^2)^2\right)\right) dr\,d\theta =$$

$$= \int_0^{\sqrt{2}} \left(\int_0^{2\pi} \frac{r^3}{2}\left(6 - r^2 - r^4\right) d\theta \right) dr = \int_0^{\sqrt{2}} \theta \left(\frac{r^3}{2} \cdot \left(6 - r^2 - r^4\right) \right)\Big|_{\theta=0}^{\theta=2\pi} dr$$

$$= 2\pi \int_0^{\sqrt{2}} \frac{r^3}{2}\left(6 - r^2 - r^4\right) dr = \left(6\pi \cdot \frac{r^4}{4} - \pi \cdot \frac{r^6}{6} - \pi \cdot \frac{r^8}{8}\right)\Big|_{r=0}^{r=\sqrt{2}} = \frac{8}{3}\pi$$

d) $\iiint_A xyz\,dx\,dy\,dz$, unde $A = \{(x,y,z) \in \mathbb{R}^3 \mid (x,y) \in B, \sqrt{x^2+y^2} \le z \le 5\}$ și $B = \{(x,y) \in \mathbb{R}^2 \mid 9 \le x^2 + y^2 \le 25\}$

e) $\iiint_A \left(\frac{x^2}{4} + \frac{y^2}{9} + \frac{z^2}{16}\right) dx\,dy\,dz$, unde

$A = \{(x,y,z) \in \mathbb{R}^3 \mid \frac{x^2}{4} + \frac{y^2}{9} + \frac{z^2}{16} \le 1, z \le 0\}$

Sol: Fie $f : A \to \mathbb{R}$, $f(x,y,z) = \frac{x^2}{4} + \frac{y^2}{9} + \frac{z^2}{16}$

SV: $\begin{cases} x = 2r\cos\theta\sin\varphi \\ y = 3r\sin\theta\sin\varphi \\ z = 4r\cos\varphi \end{cases}$, $r \in [0, \infty)$, $\theta \in [0, 2\pi]$, $\varphi \in [0, \pi]$

$$(x,y,z) \in A \Rightarrow \begin{cases} \frac{x^2}{4} + \frac{y^2}{9} + \frac{z^2}{16} \le 1 \\ z \le 0 \end{cases} \Rightarrow$$

$$\Rightarrow \begin{cases} \frac{4r^2\cos^2\theta\sin^2\varphi}{4} + \frac{9r^2\sin^2\theta\sin^2\varphi}{9} + \frac{16r^2\cos^2\varphi}{16} \le 1 \\ 4r\cos\varphi \le 0 \end{cases}$$

$$\Rightarrow \begin{cases} r^2 \underbrace{\left(\cos^2\theta \cdot \sin^2\varphi + \sin^2\theta \cdot \sin^2\varphi + \cos^2\varphi\right)}_{=1} \le 1 \\ r\cos\varphi \le 0 \quad | : r > 0 \end{cases} \Rightarrow \begin{cases} r^2 \le 1 \\ \cos\varphi \le 0 \end{cases}$$

$$\Rightarrow \begin{cases} r \in [0, 1] \\ \cos\varphi \le 0 \quad \Rightarrow \varphi \in \left[\frac{\pi}{2}, \pi\right] \end{cases}$$

Fie $B = [0,1] \times [0, 2\pi] \times [\frac{\pi}{2}, \pi]$

Deci $\iiint_A f(x,y,z)\, dx\, dy\, dz = \iiint_B (2 \cdot 3 \cdot 4 \cdot r^2 \cdot \sin\varphi) \cdot$

$\cdot f(2r\cos\theta\sin\varphi, 3r\sin\theta\sin\varphi, 4r\cos\varphi)\, dr\, d\theta\, d\varphi =$

$= \int_0^1 \left(\int_0^{2\pi} \left(\int_{\frac{\pi}{2}}^{\pi} (24r^2 \sin\varphi \cdot r^2)\, d\varphi \right) d\theta \right) dr =$

$= \int_0^1 \left(\int_0^{2\pi} \left(24r^4 \, (-\cos\varphi) \Big|_{\varphi=\frac{\pi}{2}}^{\varphi=\pi} \right) d\theta \right) dr =$

$= \int_0^1 \left(\int_0^{2\pi} 24r^4\, d\theta \right) dr = \int_0^1 \left(24r^4\theta \Big|_{\theta=0}^{\theta=2\pi} \right) dr =$

$= \int_0^1 48\pi r^2\, dr = 48\pi \frac{r^5}{5} \Big|_{r=0}^{r=1} = \frac{48}{5}\pi$

T f) $\iiint_A x\, dx\, dy\, dz$, unde $A = \{(x,y,z) \in \mathbb{R}^3 \mid x^2+y^2+z^2 \le 1, y \ge 0\}$