C1. Linear harmonic oscillator

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T.O. Cheche

1.1 Linear harmonic oscillator. Examples.
1.2 The solution of linear harmonic oscillator equation.
1.3 Energy conservation.
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1.1 Linear harmonic oscillator. Examples.

The vector **A** starts from the x axis at the initial moment $t_0 = 0$ a circular motion with constant angular velocity ω and its tip describes a circle of radius $A = |\mathbf{A}|$. At the moment t the angle swept by **A** measured counter-clockwise from the x axis is $\alpha(t)= \omega t$. The initial conditions are formally written as follows

$$t(0) = t_0 = 0\,,\quad x(0) = A\,,\; y(0) = 0 \tag{1}$$

where $x(0)$ and $y(0)$ are the *components* at the initial moment of vector $\mathbf{A}(t) = \mathbf{x}(t) + \mathbf{y}(t) = x(t)\mathbf{i} + y(t)\mathbf{j}$ on the two perpendicular axes at the moment $t(0)$, with **i** and **j** as unit vectors of x and y axes, respectively. The component of a vector on an axis is a scalar equal the magnitude of the vector projection on the axis if the projection and axis have the same direction and minus the magnitude of the vector projection on the axis if the projection and axis have opposite directions.

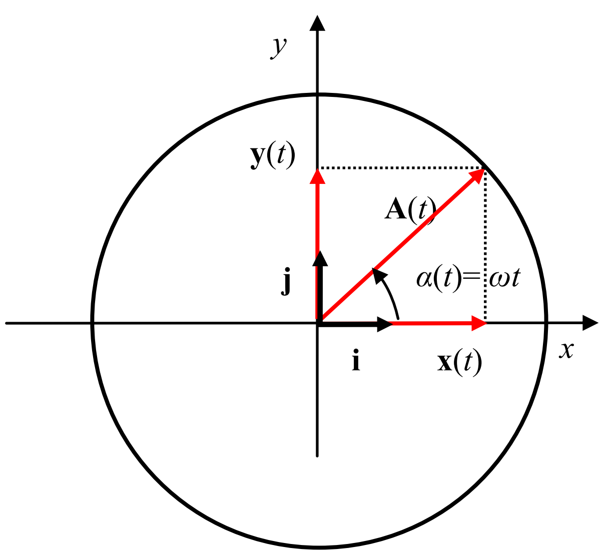


Fig. 1 Uniform rotation of vector **A** and its projections.

See
https://www.animations.physics.unsw.edu.au/jw/SHM.htm
https://en.wikipedia.org/wiki/Simple_harmonic_motion#/media/File:Simple_Harmonic_Motion_Orbit.gif

At the moment t we have projection of the motion on the two axes x and y with the components as follows

$$x(t) = A\cos\omega t\,,\ y(t) = A\sin\omega t\,. \tag{2}$$

The time derivatives emerges as the velocity with the components on the two axes

$$v_x = x'(t) \equiv \frac{dx(t)}{dt} \equiv \dot{x} = -A\omega\sin\omega t\,,\ v_y = y'(t) \equiv \frac{dy(t)}{dt} \equiv \dot{y} = A\omega\cos\omega t \tag{3}$$

The time derivatives of velocity generates the acceleration with the components on the two axes

$$a_x = x''(t) \equiv \frac{d^2x(t)}{dt^2} \equiv \ddot{x} = -A\omega^2\cos\omega t\,,\ a_y = y''(t) \equiv \frac{d^2y(t)}{dt^2} \equiv \ddot{y} = -A\omega^2\sin\omega t \tag{4}$$

By comparing (2) and (4) we obtain

$$\ddot{x} + \omega^2 x = 0\,,\ \ddot{y} + \omega^2 y = 0 \tag{5}$$

Any of the equations (5) represent the mathematical equation of the *linear harmonic oscillator* (LHO). Equation (5) is a *linear* (no power of $\ddot{x}, \dot{x}$ or x, for example) *second* order differential (time derivative of maximum second order) equation with *constant* coefficients and *homogenous* (at the right hand side of the equality one has zero). Intuitively, in Fig. 1 the projection of the tip of **A** generates an oscillatory motion on each of the two axes, x and y.

Examples.

1.1 *Spring one-dimensional oscillations on the horizontal plane.*

One considers an ideal spring (massless) tied to a body of point mass m, which oscillates on the horizontal plane along some direction under an elastic force $\mathbf{F}_e$ of the form $\mathbf{F}_e = -k\mathbf{x}$, where k is the so called *elastic constant* of the spring (positively defined) and $\mathbf{x}$ is the vector *displacement* along the oscillation direction with the tail in the equilibrium position and the tip in the actual position of the body during oscillation. The horizontal plane is located at the surface of some planet (Earth, e.g.). In addition, one considers an ideal frictionless motion. The forces acting on the body are represented in Fig.2.

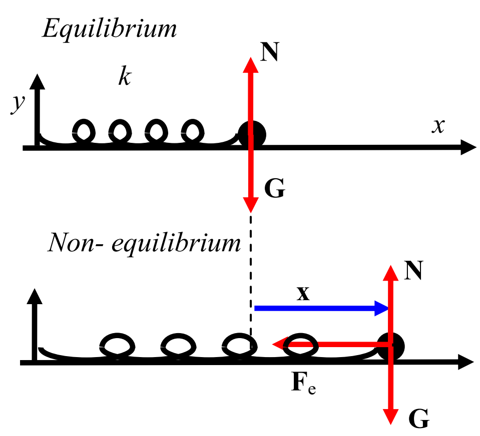


Fig. 2 One-dimensional oscillations in the horizontal plane.

By using the second Newton law we can write

$$\mathbf{N} + \mathbf{G} + \mathbf{F}_e = m\mathbf{a}\,, \tag{6}$$

where $\mathbf{G}$ is the gravity force acting on the body (force generated by the planet mass), $\mathbf{N}$ is the reaction force acting on the body from the horizontal plane, and $\mathbf{a} = \ddot{\mathbf{x}}$ is the acceleration of the body. As the body moves along the x axis (located in the horizontal plane) the forces $\mathbf{N}$ and $\mathbf{G}$ cancel each other and one has

$$\mathbf{F}_e = m\mathbf{a} \tag{7}$$

or by introducing the elastic force expression

$$-k\mathbf{x} = m\ddot{\mathbf{x}} \tag{8}$$

By writing the above vector equation for the component of vectors (reminder: scalar value, positive if the vector and axis has the same sense and negative if opposite) one obtains (just give up to the vector writing, a bold character according to our notations)

$$\ddot{x} + \frac{k}{m}x = 0 \text{ or } \ddot{x} + \omega^2 x = 0\,, \tag{9}$$

where $\omega^2 = k/m$, that is an LHO equation type.

1.2 *Spring one-dimensional oscillations in the vertical plane.*

We have the same problem as before but the motion is in the vertical direction now, along y axis.

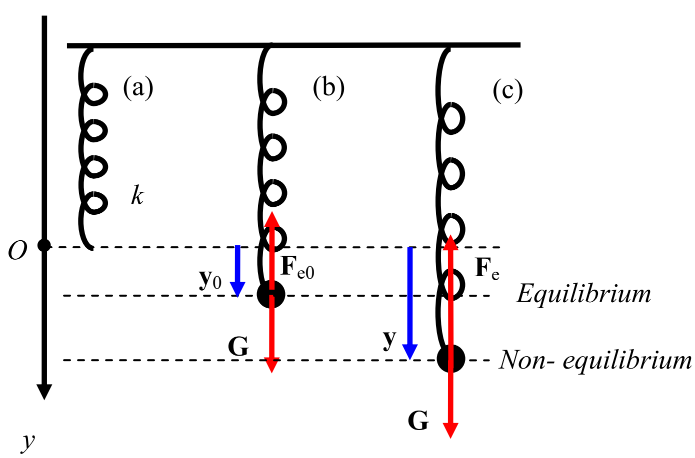


Fig. 3 One-dimensional oscillations in the vertical direction.

For the equilibrium position from Fig. 3b we can write

$$\mathbf{G} + \mathbf{F}_{e0} = 0 \tag{10}$$

or replacing $\mathbf{G} = m\mathbf{g}$ (where $\mathbf{g}$ is the gravity acceleration), $\mathbf{F}_{e0} = -k\mathbf{y}_0$, and by using the components of vectors on y axis

$$mg - ky_0 = 0 \tag{11}$$

where $g = |\mathbf{g}| > 0$ is the magnitude of vector $\mathbf{g}$ and $y_0 = |\mathbf{y_0}| > 0$.

For non-equilibrium, during the oscillation we can write

$$\mathbf{G} + \mathbf{F}_e = m\mathbf{a} \quad \text{or} \quad \mathbf{G} - k\mathbf{y} = m\ddot{\mathbf{y}} \tag{12}$$

and for the y component

$$mg - ky = m\ddot{y}\,. \tag{13}$$

By using (11), equation (13) is written

$$m\ddot{y} + k(y - y_0) = 0 \;\text{ or }\; \ddot{y} + \omega^2 y = g\,. \tag{14}$$

Equation (14) is also a LHO equation but *non-homogenous* (at the right hand side of the equation one has a non-zero function). In this case eq. (14) can be transformed to a proper LHO equation by a change of variable

$$y = u + g/\omega^2 = u + mg/k\,. \tag{15}$$

By replacing (15) in (14) one has

$$\ddot{u} + \omega^2 u = 0\,, \tag{16}$$

which indeed is a homogenous LHO.

1.3 *Mathematical pendulum*

In ideal conditions (massless and inextensible string, the body is a point mass, no friction, no motion of the planet Earth) the small angle oscillation of the gravitational pendulum suspended from a support is described by a LHO for the angle as a variable: after the pendulum is moved out of equilibrium position (with tensed string) it will swing back and forth at constant angular amplitude. This problem is known as the mathematical pendulum problem. In terms of the second Newton law the motion is described by

$$\mathbf{T} + \mathbf{G} = m\mathbf{a}\,, \tag{17}$$

where $\mathbf{T}$ is the tension force in the string. The LHO equation can be found starting with (17). Another way of finding the LHO for the mathematical pendulum is by using the energy conservation principle, which states that in the absence of energy dissipation the mechanical energy is conserved. By the convention related to the relative value (with respect to a reference) of the potential energy, it is convenient to shift the origin of y axis in O (see Fig. 4) and write

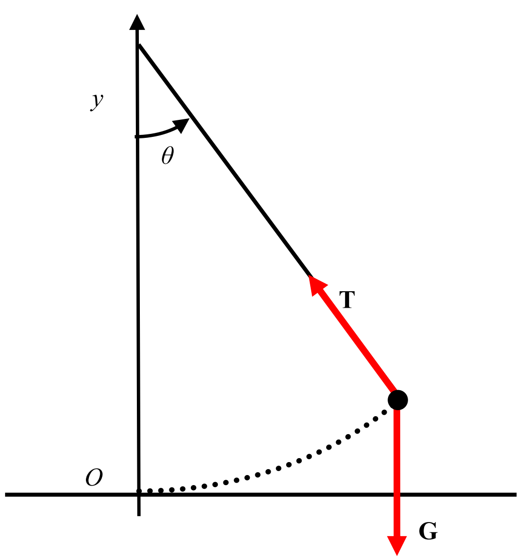


Fig.4 Idealized gravitational pendulum (massless string).

$$E = U + T = mgl(1-\cos\theta) + \frac{mv^2}{2}$$
$$= mgl(1-\cos\theta) + \frac{ml^2\dot{\theta}^2}{2} = constant \quad , \quad (18)$$

where U is the gravitational potential energy, T is the kinetic energy, l is the length of string, and θ is the angle with the vertical. By taking the time derivative and considering energy conservation ($\dot{E}=0$) one obtains (by simplifying $\dot{\theta}$)

$$\ddot{\theta} + \frac{g}{l}\sin\theta = 0 \quad (19)$$

In the limit of small angles ($\sin\theta < 5°$) $\sin\theta \cong \theta$ and

$$\ddot{\theta} + \omega^2\theta = 0 \quad (20)$$

where $\omega^2 = g/l$. Equation (20) is a homogenous LHO.

2. **Solution of LHO**

For the solution of the second order linear and homogenous differential LHO one needs its *characteristic equation*; it is obtained by searching for the solution of the form e^{qt}. Introducing this solution type in eq. (9)

$$\ddot{x} + \omega^2 x = 0$$

one obtains $q^2 + \omega^2 = 0$, that is, $q = \pm i\omega$. According to the theory of differential equations, the solution is a linear combination of solutions for $q = \pm i\omega$ of the form,

$$x = C_1\exp(i\omega t) + C_2\exp(-i\omega t), \quad (21)$$

Its time derivative is the velocity

$$\dot{x} = i\omega\left[C_1\exp(i\omega t) - C_2\exp(-i\omega t)\right]. \quad (22)$$

For the initial conditions, we choose, $x(0) = x_0$, and $\dot{x}(0) = v_0$. With these initial conditions, we have

$$x_0 = C_1 + C_2,$$
$$\frac{v_0}{i\omega} = C_1 - C_2$$

and obtain

$$C_1 = C_2^* = \left(x_0 - i\,v_0/\omega\right)/2. \quad (23)$$

By introducing (23) in (21) we have

$$x = \frac{x_0 - i\,v_0/\omega}{2}\exp(i\omega t) + \frac{x_0 + i\,v_0/\omega}{2}\exp(-i\omega t)$$
$$= \frac{x_0}{2}\left[\exp(i\omega t) + \exp(-i\omega t)\right] + \frac{v_0}{2i\omega}\left[\exp(i\omega t) - \exp(-i\omega t)\right] \quad (24)$$
$$= x_0\cos\omega t + \frac{v_0}{\omega}\sin\omega t$$

where one used the Euler's equations:

$$\cos\alpha = \frac{\exp(i\alpha) + \exp(-i\alpha)}{2}$$
$$\sin\alpha = \frac{\exp(i\alpha) - \exp(-i\alpha)}{2i}$$

Homework: Prove the above Euler's equations.
Hint: use the Taylor expansion about 0 for $\cos\alpha$ and $\sin\alpha$.

Equation (24) can *also* be written as

$$x(t) = A\cos(\omega t + \alpha) \tag{25}$$

where A is named *amplitude*, ω *angular frequency*, α *initial phase*. By comparing the last equality from (24) and (25) one has

$$x_0 = A\cos\alpha$$
$$v_0 = -\omega A\sin\alpha$$

that is, by using $\sin^2\alpha + \cos^2\alpha = 1$ we obtain

$$A = \sqrt{\frac{v_0^2}{\omega^2} + x_0^2} \text{ and } \tan\alpha = -\omega/(v_0 x_0) \tag{26}$$

In addition, for the velocity we have from (25) the expression

$$\begin{aligned} \dot{x} &= -A\omega\sin(\omega t + \alpha) \\ &= -A\omega\sin\omega t\cos\alpha - A\omega\cos\omega t\sin\alpha \\ &= -x_0\omega\sin\omega t + v_0\cos\omega t \end{aligned} \tag{27}$$

which is the same as that obtained from (24) by time derivative.
The *period* T of motion is *by definition* the necessary time for a complete oscillation:

$$T = \frac{2\pi}{\omega}. \tag{28}$$

Thus at time t and $t+T$, by using eq. (25), one obtains the displacement and velocity are the same:

$$\begin{aligned} x(t+T) &= A\cos(\omega(t+T) + \alpha) \\ &= A\cos(\omega t + 2\pi + \alpha) = A\cos(\omega t + \alpha) = x(t) \end{aligned}$$

and

$$\begin{aligned} \dot{x}(t+T) &= -\omega A\sin(\omega(t+T) + \alpha) = -\omega A\sin(\omega t + 2\pi + \alpha) \\ &= -\omega A\sin(\omega t + 2\pi + \alpha) = -\omega A\sin(\omega t + \alpha) = \dot{x}(t) \end{aligned}$$

1.3 Energy conservation.

The mechanical energy of the LHO is written as

$$E = U_e + T\,,$$

where U_e is the elastic potential energy, $U_e = kx^2/2$, and T is the kinetic energy $T = mv^2/2$. By replacing the elongation and velocity from (25) and (27), respectively, one obtains

$$E = \frac{kx^2}{2} + \frac{mv^2}{2} = \frac{kA^2}{2}\cos^2(\omega t + \alpha) + \frac{m\omega^2 A^2}{2}\sin^2(\omega t + \alpha)$$
$$= \frac{m\omega^2 A^2}{2}\left[\cos^2(\omega t + \alpha) + \sin^2(\omega t + \alpha)\right] = \frac{m\omega^2 A^2}{2}$$
$$= \frac{kA^2}{2} = constant$$

1.4 Application

Solution for **1.2** *Spring one-dimensional oscillations in the vertical plane.*

The *initial conditions* $y(0) = Y_0$, $\dot{y}(0) = 0$

Equation (16) has as solution

$$u = A\cos(\omega t + \alpha)$$

By using (15)

$$y = u + g/\omega^2 = u + mg/k$$

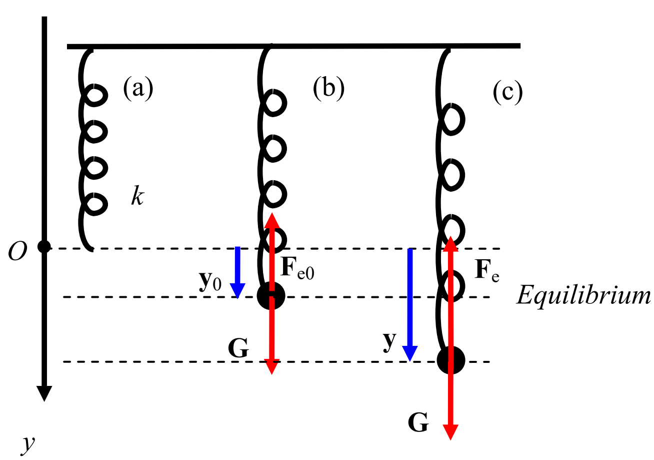
we have

$$y = A\cos(\omega t + \alpha) + g/\omega^2 = A\cos(\omega t + \alpha) + mg/k$$
$$\dot{y} = -\omega A\sin(\omega t + \alpha)$$

$$y(0) = Y_0 = A\cos\alpha + mg/k$$
$$\dot{y}(0) = 0 = -\omega A\sin\alpha$$

=>

$$\alpha = 0$$
$$A = Y_0 - mg/k$$
$$y = (Y_0 - mg/k)\cos\left(\sqrt{k/m}\,t\right) + mg/k$$



In[251]:= $\mathtt{Y0 = 1;\ \ m = 4.;\ \ k = 10;\ \ g = 1;}$

$$\mathtt{T} = 2 * \pi * \sqrt{\frac{\mathtt{m}}{\mathtt{k}}}$$

$$\mathtt{y[t\_]} := \left(\mathtt{Y0} - \frac{\mathtt{m} * \mathtt{g}}{\mathtt{k}}\right) * \mathtt{Cos}\left[\sqrt{\frac{\mathtt{k}}{\mathtt{m}}} * \mathtt{t}\right] + \frac{\mathtt{m} * \mathtt{g}}{\mathtt{k}};$$

$$\mathtt{Plot}\left[\left\{\mathtt{y[t]},\ \frac{\mathtt{m} * \mathtt{g}}{\mathtt{k}}\right\},\ \{\mathtt{t},\ 0,\ 1 * \mathtt{T}\}\right]$$

$$\mathtt{Print}\left[\text{"A="},\ \mathtt{Y0} - \frac{\mathtt{m} * \mathtt{g}}{\mathtt{k}}\right];$$

$$\mathtt{Print}\left[\text{"y0=}\frac{\mathtt{m} * \mathtt{g}}{\mathtt{k}}\text{="},\ \mathtt{y0} = \frac{\mathtt{m} * \mathtt{g}}{\mathtt{k}}\right];$$

$$\mathtt{Print[\text{"y[0]="},\ y[0]];}$$

$$\mathtt{Print}\left[\text{"y[}\frac{\mathtt{T}}{4}\text{]="},\ \mathtt{y}\left[\frac{\mathtt{T}}{4}\right]\right];$$

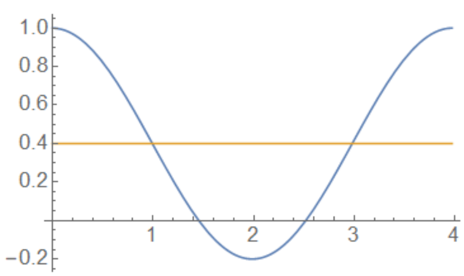
$$\mathtt{Print}\left[\text{"y[}\frac{\mathtt{T}}{2}\text{]="},\ \mathtt{y}\left[\frac{2 * \mathtt{T}}{4}\right]\right];$$

$$\mathtt{Print}\left[\text{"y[}\frac{3 * \mathtt{T}}{4}\text{]="},\ \mathtt{y}\left[\frac{3 * \mathtt{T}}{4}\right]\right];$$

$$\mathtt{Print}\left[\text{"y[T]="},\ \mathtt{y}\left[\frac{4 * \mathtt{T}}{4}\right]\right];$$

Out[252]= 3.97384

Out[254]=



A=0.6

$\text{y0=}\frac{\mathtt{m} * \mathtt{g}}{\mathtt{k}}\text{=0.4}$

y[0]=1.

$\text{y[}\frac{\mathtt{T}}{4}\text{]=0.4}$

$\text{y[}\frac{\mathtt{T}}{2}\text{]=-0.2}$

$\text{y[}\frac{3 * \mathtt{T}}{4}\text{]=0.4}$

y[T]=1.