2024.

C_{11} - GA

Endomorfisme simetrice

$(E, \langle \cdot, \cdot \rangle)$ s.v.e.r , $f \in End(E)$

$f \in Sim(E) \Leftrightarrow \langle f(x), y \rangle = \langle x, f(y) \rangle, \forall x, y \in E$

- $f \in Sim(E) \Rightarrow$ toate răd. pol. caract. sunt reale.
  $\Rightarrow$ vect. pr. coresp. la val proprii dist sunt ortog.

<u>Prop</u> $f \in Sim(E)$, $U \subseteq E$ subsp. invar al lui f

a) $U^\perp$ subsp invariant al lui f

b) $f/_{U^\perp} : U^\perp \to U^\perp$ endomorfism simetric

<u>Dem</u>

a) Fie $x \in U^\perp$. Dem că $f(x) \in U^\perp$

Fie $y \in U$. Arătăm că $\langle f(x), y \rangle = 0$

$\langle f(x), y \rangle = \langle \underset{U^\perp}{\underset{\cap}{x}}, \underset{U}{\underset{\cap}{f(y)}} \rangle = 0 \Rightarrow U^\perp$ invariant.

b) Din a).

<u>Teoremă</u> $f \in Sim(E)$

$\Rightarrow \exists R' =$ reper ortonormat format din versori proprii

aî $[f]_{R', R'} =$ diagonală.

<u>Dem</u> Fie R reper ortonormat arbitrar, $A = [f]_{R,R}$

$P(\lambda) = \det(A - \lambda I_n) = 0$ are toate răd. reale.

Fie $\lambda_1 =$ o răd și $e_1 =$ versorul propriu

$f(e_1) = \lambda_1 e_1 \Rightarrow \langle \{e_1\} \rangle \subset E$ subsp invar al lui f

$\langle \{e_1\} \rangle^\perp$ — " —

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-2

$f|_{<\{e_1\}>^\perp} : <\{e_1\}>^\perp \longrightarrow <\{e_1\}>^\perp$ endom. sim.

Fie λ_2 = val proprie pt $f|_{<\{e_1\}>^\perp}$ și e_2 = versor propriu i.e $f(e_2) = \lambda_2 e_2$

$f(e_1) = \lambda_1 e_1$

$f(e_2) = \lambda_2 e_2$

$<\{e_1, e_2\}>$ subsp. invar al lui f

$<\{e_1, e_2\}>^\perp$ —||—

$f|_{<\{e_1 e_2\}>^\perp} : <\{e_1, e_2\}>^\perp \longrightarrow <\{e_1, e_2\}>^\perp$ endom. sim

Fie λ_3 val pr, e_3 = versor propriu

După n pași construim $R' = \{e_1, \dots, e_n\}$

sistem de n versori ortogonali $\Rightarrow R'$ e SLI

dar $|R'| = \dim E = n$

$\Rightarrow R' = \{e_1, \dots, e_n\}$ reper ortonormat

$f(e_i) = \lambda_i e_i \ , \ i = \overline{1,n}$ $\qquad A' = [f]_{R', R'} = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}$

<u>OBS</u>

a) $f \in End(E)$ simetric $\Rightarrow \dim V_{\lambda_i} = m_i, \ i = \overline{1,k}$

$\lambda_1, \dots, \lambda_k$ = val pr. distincte

$m_1 + \dots + m_k = n$.

$E = V_{\lambda_1} \oplus \dots \oplus V_{\lambda_k}$.

$R' = R_1 \cup \dots \cup R_k$, R_i reper în $V_{\lambda_i}, i = \overline{1,k}$

b) $A = A^T \rightarrow f \in Sim(E)$

$\searrow Q : E \rightarrow \mathbb{R}$ formă pătratică

$Q(x) = \langle x, f(x) \rangle, \forall x \in E, \ Q(x) = \sum_{i,j=1}^{n} a_{ij} x_i x_j$

Q se poate aduce la o formă canonică utilizând metoda valorilor proprii sau prin schimbări de reper ortonormat sau printr-o transf. ortogonală.

Spații afine euclidiene. Geometrie analitică euclidiană

Def $(\mathcal{A}, V_{/\mathbb{K}}, \varphi)$ s.n. spațiu afin dacă

1) $\mathcal{A} \neq \phi$ (mulțime de puncte)

2) $V_{/\mathbb{K}}$ sp. vectorial director

3) $\varphi : \mathcal{A} \times \mathcal{A} \to V$ structură afină

a) $\varphi(A,B) + \varphi(B,C) = \varphi(A,C)$, $\forall A,B,C \in \mathcal{A}$

b) $\exists\, O \in \mathcal{A}$ a.î. $\varphi_O : \mathcal{A} \to V$, $\varphi_O(A) = \varphi(O,A)$ $\forall A \in \mathcal{A}$ bijecție.

$\varphi(A,B) = \overrightarrow{AB}$; $\dim \mathcal{A} = \dim V = n$.

OBS De fapt b) $\forall O$ este satisf. cond.

Ex $(\underset{\mathcal{A}}{\mathbb{R}^n}, \underset{V}{\mathbb{R}^n}{}_{/\mathbb{R}}, \varphi)$, $\varphi : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$, $\varphi(u,v) = v - u$ $\forall u, v \in \mathbb{R}^n$

str. afină canonică

Def $(\mathbb{R}^n, \mathbb{R}^n{}_{/\mathbb{R}}, \varphi)$ sp. afin cu str. af. canonică

$M \subset \mathbb{R}^n$ subm. de puncte

$Af(M) = \{ \sum_{i=1}^{k} a_i P_i \,,\ \sum_{i=1}^{k} a_i = 1,\ P_i \in M,\ a_i \in \mathbb{R},\ k \in \mathbb{N}^* \}$

sp. afin generat de M.

Def $\mathcal{A}' \subset \mathbb{R}^n$ varietate liniară sau subspațiu afin

$\Leftrightarrow \forall P_1, P_2 \in \mathcal{A}' \Rightarrow \mathcal{A}f\{P_1, P_2\} \subset \mathcal{A}'$

Ex $(\mathbb{R}^n, \mathbb{R}^n/_{\mathbb{R}}, \varphi)$

$\mathcal{A}' = \{x \in \mathbb{R}^n \mid AX = B\}$

$V' = \{x \in \mathbb{R}^n \mid AX = O_{m,1}\}$ subsp. director.

Def $\mathcal{A}', \mathcal{A}'' \subseteq \mathbb{R}^n$ subsp. afine.

$\mathcal{A}' \parallel \mathcal{A}'' \Leftrightarrow V' \subseteq V''$ sau $V'' \subseteq V'$

Ex $\mathcal{A}' = \{x \in \mathbb{R}^3 \mid x_1 + 2x_2 + 3x_3 = 1\}$

$\mathcal{A}'' = \{x \in \mathbb{R}^3 \mid x_1 + 2x_2 + 3x_3 = 2\}$

$V' = \{x \in \mathbb{R}^3 \mid x_1 + 2x_2 + 3x_3 = 0\}$

$V'' = \{x \in \mathbb{R}^3 \mid x_1 + 2x_2 + 3x_3 = 0\}$

$V' = V'' \Rightarrow \mathcal{A}' \parallel \mathcal{A}''$ (plane paralele)

Def $(\mathcal{E}, (E, \langle\cdot,\cdot\rangle), \varphi)$ s.n. spațiu afin euclidian

$\Leftrightarrow$ este un sp. afin și sp. director = sp. vect. euclidian.

Def $(\mathcal{E}, (E, \langle\cdot,\cdot\rangle), \varphi)$

$\mathcal{E}_1, \mathcal{E}_2 \subset \mathcal{E}$ subsp. afine.

a) $\mathcal{E}_1, \mathcal{E}_2$ sunt perpendiculare $\Leftrightarrow E_1 \perp E_2$

b) $\mathcal{E}_1, \mathcal{E}_2$ sunt normale $\Leftrightarrow E = E_1 \oplus E_1^{\perp}$

$= E_2 \oplus E_2^{\perp}$

$E_1 = E_2^{\perp}, \quad E_2 = E_1^{\perp}$

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Geometrie analitică euclidiană

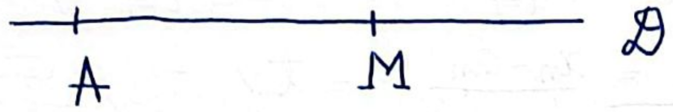
$(\mathbb{R}^n, (\mathbb{R}^n, g_0), \varphi)$, $\varphi : \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R}^n$, $\varphi(u, v) = v - u$.

$\mathcal{R} = \{O; e_1, .., e_n\}$ reper cartezian ortonormat

$O \in \mathcal{A} = \mathbb{R}^n$, $\{e_1, .., e_n\}$ reper ortonormat în $V = \mathbb{R}^n$

① Ec. unei drepte afine în $\mathbb{R}^n$

a)



$A \in \mathcal{D}$

$V_{\mathcal{D}} = \langle \{u\} \rangle$

$u \neq 0_{\mathbb{R}^n}$.

$\forall M \in \mathcal{D} \Rightarrow \overrightarrow{AM} \in V_{\mathcal{D}}$

$\exists t \in \mathbb{R}$ aî $\underset{\varphi(A,M)}{\overrightarrow{AM}} = v u$; $\overrightarrow{OA} = \sum_{i=1}^{n} a_i e_i$; $A(a_1, .., a_n)$

$\overrightarrow{OM} = \sum_{i=1}^{n} x_i e_i$

$v = \sum_{i=1}^{n} v_i e_i$

$\overrightarrow{AM} = \overrightarrow{OM} - \overrightarrow{OA} = \sum_{i=1}^{n} (x_i - a_i) e_i = t \sum_{i=1}^{n} v_i e_i$

$\mathcal{D} : x_i - a_i = t v_i$, $\forall i = \overline{1,n}$ ec. parametrice.
$t \in \mathbb{R}$

$\mathcal{D} : \dfrac{x_1 - a_1}{v_1} = .. = \dfrac{x_n - a_n}{v_n} = t$

<u>Convenție</u> Dc. $\exists\, i_0 \in \overline{1,n}$ aî $v_{i_0} = 0$, at $x_{i_0} - a_{i_0} = 0$.

<u>Ex</u> $(\mathbb{R}^3, (\mathbb{R}^3, g_0), \varphi)$ $A(1, 2, 3)$, $v = (3, 1, 1)$

$\mathcal{R} = \{O; e_1, e_2, e_3\}$, $O \in \mathbb{R}^3$, $\{e_1, e_2, e_3\}$ reper canonic în $\mathbb{R}^3$

Ec dreptei $\mathcal{D} \ni A$, $V_{\mathcal{D}} = \langle \{v\} \rangle$

$\mathcal{D} : \dfrac{x_1 - 1}{3} = \dfrac{x_2 - 2}{1} = \dfrac{x_3 - 3}{1} = t \Leftrightarrow \begin{cases} x_1 = 3t + 1 \\ x_2 = t + 2 \\ x_3 = t + 3 \end{cases}$ ec param.
$t \in \mathbb{R}$

b)  $\mathcal{D}$ $\quad A \neq B$, $A, B \in \mathcal{D}$

$\exists t \in \mathbb{R}$ a.î. $\quad V_{\mathcal{D}} = \langle \{\overrightarrow{AB}\} \rangle$

$\mathcal{D}: x_i - a_i = t(b_i - a_i), \forall i = \overline{1,n}$ $\quad V = \overrightarrow{AB} = \sum_{i=1}^{n} (b_i - a_i) e_i$

ec. param.

$\forall M \in \mathcal{D}, \exists t \in \mathbb{R}$ a.î. $\overrightarrow{AM} = t\overrightarrow{AB}$

$$\sum_{i=1}^{n} (x_i - a_i) e_i = t \sum_{i=1}^{n} (b_i - a_i) e_i$$

$$\mathcal{D}: \frac{x_1 - a_1}{b_1 - a_1} = \dots = \frac{x_n - a_n}{b_n - a_n} = t$$

<u>Convenție</u> Dc $\exists i_0 \in \overline{1,n}$ a.î. $b_{i_0} - a_{i_0} = 0$, at $x_{i_0} - a_{i_0} = 0$

<u>Ex</u> $n = 3$, $A(1,1,3)$, $B(2,5,6)$

Ec dreptei $\mathcal{D} \ni A, B$ $\quad \overrightarrow{AB} = (2-1, 5-1, 6-3) = (1,4,3)$

$$\mathcal{D}: \frac{x_1 - 1}{1} = \frac{x_2 - 1}{4} = \frac{x_3 - 3}{3} = t \Leftrightarrow \begin{cases} x_1 = t+1 \\ x_2 = 4t+1 \\ x_3 = 3t+3, \ t \in \mathbb{R} \end{cases}$$

Poz. relativă a 2 drepte afine $\mathcal{D}_1, \mathcal{D}_2$.

$\mathcal{D}_1: x_i - a_i = t u_i, \forall i = \overline{1,n}$ $\quad A(a_1, .., a_n) \in \mathcal{D}_1$,

$\mathcal{D}_2: x_i - b_i = s v_i \quad \forall i = \overline{1,n}$ $\quad B(b_1, .., b_n) \in \mathcal{D}_2$

$V_{\mathcal{D}_1} = \langle \{u\} \rangle$; $V_{\mathcal{D}_2} = \langle \{v\} \rangle$

$\mathcal{D}_1 \cap \mathcal{D}_2: \quad t u_i + a_i = s v_i + b_i, \ i = \overline{1,n}$

$$\begin{cases} t u_1 - s v_1 = b_1 - a_1 \\ \vdots \\ t u_n - s v_n = b_n - a_n \end{cases} \qquad \left(\begin{array}{cc|c} u_1 & -v_1 & b_1 - a_1 \\ \vdots & \vdots & \vdots \\ u_n & -v_n & b_n - a_n \end{array}\right)$$

<u>Caz particular</u>

$n = 3$

$$\begin{array}{cccc} & u & -V & \overrightarrow{AB} \\ C = & \left(\begin{array}{cc|c} u_1 & -V_1 & b_1 - a_1 \\ u_2 & -V_2 & b_2 - a_2 \\ u_3 & -V_3 & b_3 - a_3 \end{array}\right) \end{array}$$

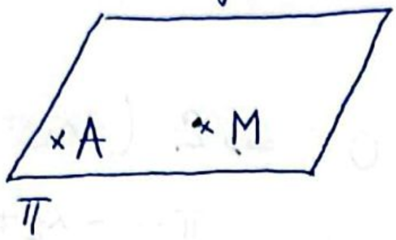
$\mathcal{D}_1, \mathcal{D}_2$ necoplanare $\Leftrightarrow$ $rg\, C = 2$, $rg\, \overline{C} = 3$

$$\Delta_c = -\begin{vmatrix} u_1 & v_1 & b_1-a_1 \\ u_2 & v_2 & b_2-a_2 \\ u_3 & v_3 & b_3-a_3 \end{vmatrix} \neq 0 \quad (\mathcal{D}_1, \mathcal{D}_2 \text{ necoplanare})$$

$$\begin{array}{ccc} u & v & \overline{AB} \end{array}$$

② Ec. unui plan afin (ssp 2-dim) în $\mathbb{R}^n$.

a)



$A \in \pi$

$V_\pi = \langle \{u, v\} \rangle$

$\{u, v\}$ SLI

$\forall M \in \pi$, $\exists\, t, s \in \mathbb{R}$ a.î. $\overrightarrow{AM} = tu + sv$

$$\sum_{i=1}^{n} (x_i - a_i) e_i = t \sum_{i=1}^{n} u_i e_i + s \sum_{i=1}^{n} v_i e_i$$

$\pi: \; x_i - a_i = t u_i + s v_i$, $\forall i = \overline{1,n}$, $t, s \in \mathbb{R}$.

Ex $n = 3$. $A(1,-1,2)$, $u = (2,3,1)$, $v = (4,1,3)$.

$\pi \ni A$, $V_\pi = \langle \{u, v\} \rangle$

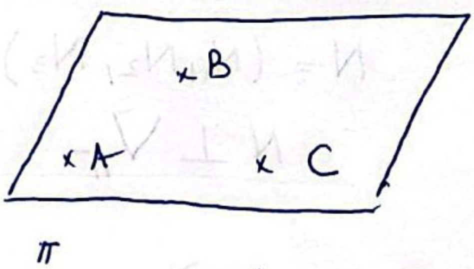
$$\pi: \begin{cases} x_1 - 1 = t\cdot 2 + s\cdot 4 \\ x_2 + 1 = t\cdot 3 + s\cdot 1 \\ x_3 - 2 = t\cdot 1 + s\cdot 3 \end{cases} \iff \begin{vmatrix} x_1-1 & 2 & 4 \\ x_2+1 & 3 & 1 \\ x_3-2 & 1 & 3 \end{vmatrix} = 0$$

$t, s \in \mathbb{R}$ ec param

$\pi: (x_1 - 1)\, 8 - (x_2 + 1)\, 2 + (x_3 - 2)(-10) = 0$

$N = (8, -2, -10) = u \times v$

b)



$A, B, C \in \pi$

$A \in \pi$, $V_\pi = \langle \{\overrightarrow{AB}, \overrightarrow{AC}\} \rangle$

$\{\overrightarrow{AB}, \overrightarrow{AC}\}$ SLI

(A, B, C nu sunt coliniare)

$\forall M \in \pi$, $\exists\, t, s \in \mathbb{R}$ a.î. $\overrightarrow{AM} = t\overrightarrow{AB} + s\overrightarrow{AC}$ $\quad \forall i = \overline{1,n}$

$x_i - a_i = t(b_i - a_i) + s(c_i - a_i)$

Ex $n=3$, $A(1,1,1)$, $B(-1,1,1)$, $C(2,0,0)\in\pi$

$\overrightarrow{AB}=(-2,0,0)$

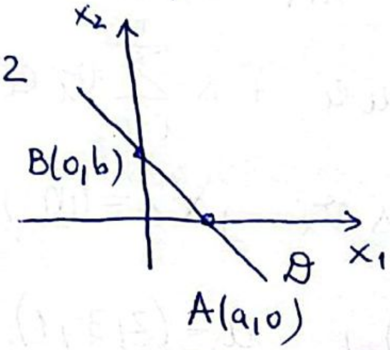
$\overrightarrow{AC}=(1,-1,-1)$

$$\pi:\begin{cases}x_1-1=-2t+s\\ x_2-1=0\cdot t+s(-1)\\ x_3-1=0t+s(-1), \quad t,s\in\mathbb{R}\end{cases}$$

$$\pi:\begin{vmatrix}\cancel{x_1-1} & \cancel{-2} & \cancel{1}\\ x_2-1 & 0 & -1\\ x_3-1 & 0 & -1\end{vmatrix}=0 \Rightarrow 2(-x_2+1+x_3-1)=0$$

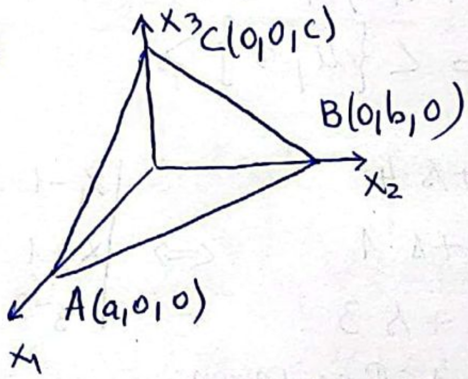
$\pi:\ -x_2+x_3=0$

OBS $n=2$



$\mathcal{D}:\ \frac{x_1}{a}+\frac{x_2}{b}=1$

$n=3$

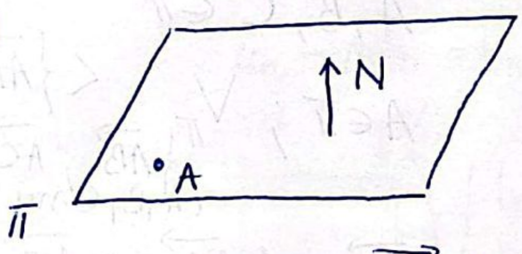


$\pi:\ \frac{x_1}{a}+\frac{x_2}{b}+\frac{x_3}{c}=1$

OBS $A,B,C\in\pi$:

$$\begin{vmatrix}x_1 & x_2 & x_3 & 1\\ a_1 & a_2 & a_3 & 1\\ b_1 & b_2 & b_3 & 1\\ c_1 & c_2 & c_3 & 1\end{vmatrix}=0$$

c)



$N=(N_1,N_2,N_3)$ normala la plan

$N\perp V_\pi$

$\forall M\in\pi \Rightarrow \overrightarrow{AM}\perp N \Rightarrow \langle\overrightarrow{AM},N\rangle=0$

$\pi:\ (x_1-a_1)N_1+(x_2-a_2)N_2+(x_3-a_3)N_3=0$

ex $n=3$ $\quad A(1,2,3)\in\Pi$, $\quad u=(0,1,3)$, $v=(4,5,0)$
vectori directori

a) N

b) $\Pi=?$

sol a) $N = u\times v = \begin{vmatrix} e_1 & e_2 & e_3 \\ 0 & 1 & 3 \\ 4 & 5 & 0 \end{vmatrix} = (-15, 12, -4)$

b) $\langle \overrightarrow{AM}, N\rangle = 0 \Rightarrow \langle (x_1-1, x_2-2, x_3-3), (-15,12,-4)\rangle = 0$

$\Pi \quad -15(x_1-1)+12(x_2-2)-4(x_3-3)=0$

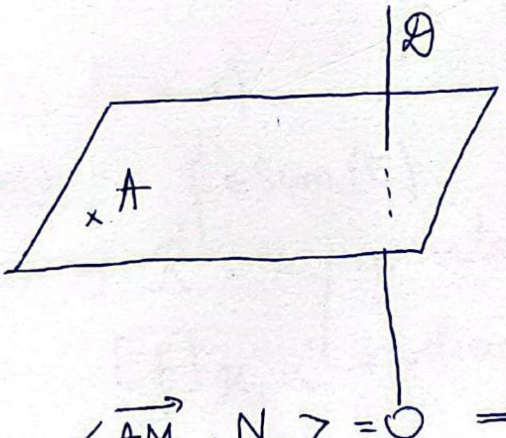
OBS $\quad \Pi: \ ax_1+bx_2+cx_3+d=0$, $N=(a,b,c)$
ec. generală a planului $\quad a^2+b^2+c^2>0$.

$N=(-15,12,-4)$, $A(1,2,3)\in\Pi$

$\Pi: \ -15x_1+12x_2-4x_3+d=0$

$A\in\Pi \Rightarrow -15\cdot 1+12\cdot 2-4\cdot 3+d=0 \Rightarrow d$.

d)



$\mathcal{D}: \ \dfrac{x_1-b_1}{\alpha_1} = \dots = \dfrac{x_n-b_n}{\alpha_n} = t$

$A(a_1,\dots,a_n)\in\Pi$

$u_{\mathcal{D}} = (\alpha_1,\dots,\alpha_n) = N_\Pi$

$\Pi: \ \langle \overrightarrow{AM}, N_\Pi\rangle = 0 \Rightarrow (x_1-a_1)\alpha_1+\dots+(x_n-a_n)\alpha_n=0$