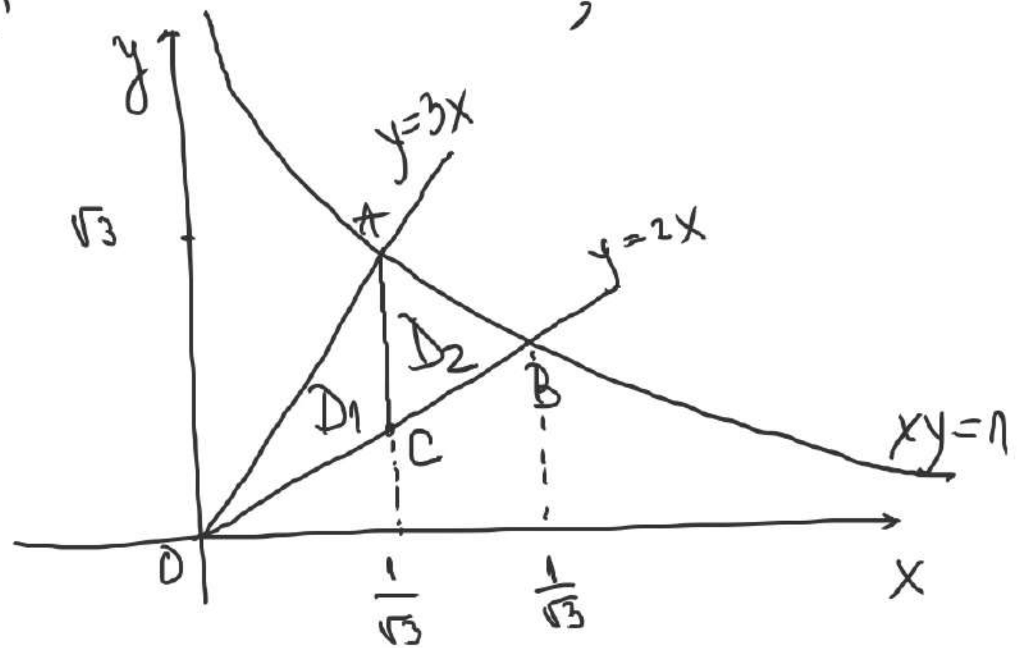
1) Fie $D \subset \mathbb{R}^2$ în primul cadran mărginită de curbele

$y=2x$, $y=3x$, $xy=1$

Arătați că $D \in \mathcal{J}(\mathbb{R}^2)$ și calculați $\lambda(D)$.



$$\begin{cases} xy=1 \\ y=3x \end{cases} \Rightarrow 3x^2=1 \Rightarrow x=\frac{1}{\sqrt{3}},\ y=\sqrt{3} \quad A\left(\frac{1}{\sqrt{3}}, \sqrt{3}\right).$$

$$\begin{cases} xy=1 \\ y=2x \end{cases} \Rightarrow x=\frac{1}{\sqrt{2}},\ y=\sqrt{2},\ B\left(\frac{1}{\sqrt{2}}, \sqrt{2}\right)$$

$$D_1 = \left\{ (x,y) \in \mathbb{R}^2 \;\middle|\; 0 \le x \le \frac{1}{\sqrt{3}},\ 2x \le y \le 3x \right\}$$

$$f, g : \left[0, \frac{1}{\sqrt{3}}\right] \to \mathbb{R}$$

$f(x) = 2x$, $g(x) = 3x$. integr. Riemann pt că sunt continue.

$$\Longrightarrow D_1 = \Gamma_{f,g} \in \mathcal{J}(\mathbb{R}^2)$$

$$D_2 = \left\{ (x,y) \in \mathbb{R}^2 \;\middle|\; \frac{1}{\sqrt{3}} \le x \le \frac{1}{\sqrt{2}},\ 2x \le y \le \frac{1}{x} \right\}$$

$$u, v : \left[\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{2}}\right] \to \mathbb{R}, \quad u(x) = 2x,\ v(x) = \frac{1}{x}$$

u și v sunt integrabile Riemann fiind continue

$$\Longrightarrow D_2 = \Gamma_{u,v} \in \mathcal{J}(\mathbb{R}^2)$$

$$D = D_1 \cup D_2 \ , \ D_1 \cap D_2 = [AC] \in \mathcal{J}(\mathbb{R}^2)$$

$$\lambda(D) = \lambda(D_1) + \lambda(D_2) - \lambda(D_1 \cap D_2) = \lambda(D_1) + \lambda(D_2) - \lambda([AC])$$

Avem

$$\lambda([AC]) = \lambda\left(\left\{\frac{1}{\sqrt{3}}\right\} \times [\sqrt{2}, \sqrt{3}]\right) = \lambda\left(\left\{\frac{1}{\sqrt{3}}\right\}\right) \cdot \lambda\left([\sqrt{2}, \sqrt{3}]\right) = 0.$$

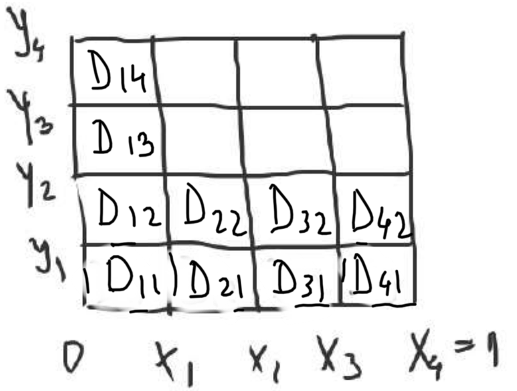
$$\lambda(D) = \lambda(D_1) + \lambda(D_2)$$

$$\lambda(D_1) = \int_0^{1/\sqrt{3}} (g(x) - f(x))\, dx = \int_0^{1/\sqrt{3}} x\, dx = \left.\frac{x^2}{2}\right|_0^{\frac{1}{\sqrt{3}}} = \frac{1}{6}$$

$$\lambda(D_2) = \int_{\frac{1}{\sqrt{3}}}^{\frac{1}{\sqrt{2}}} (v(x) - u(x))\, dx = \int_{\frac{1}{\sqrt{3}}}^{\frac{1}{\sqrt{2}}} \left(\frac{1}{x} - 2x\right) dx = \dots$$

2) $f: I=[0,1]\times[0,1]\to\mathbb{R}$, $f(x,y)=x+y$. Folosind definiția, arătați că f este integrabilă Riemann și calc $\iint\limits_{I} f$.

Soluție



$n\geq 1$,

$x_i=\frac{i}{n}$, $1\leq i\leq n$

$y_j=\frac{j}{n}$, $1\leq j\leq n$.

$D_{ij}=[x_{i-1},x_i]\times[y_{j-1},y_j]$. , $\Delta_n=\{D_{ij}\mid 1\leq i,j\leq n\}$, $\mathrm{vol}(D_{ij})=\frac{1}{n^2}$

$m_{ij}=\inf\{f(x,y)\mid (x,y)\in D_{ij}\}$, $M_{ij}=\sup\{f(x,y)\mid (x,y)\in D_{ij}\}$

$m_{ij}=\frac{i-1+j-1}{n}=\frac{i+j-2}{n}$ $\quad M_{ij}=\frac{i+j}{n}$

$$S_{\Delta_n}(f) = \sum_{1\le i,j\le n} M_{ij}\cdot \mathrm{vol}(D_{ij}) = \sum_{1\le i,j\le n} \frac{i+j}{n}\cdot\frac{1}{n^2} =$$

$$= \frac{1}{n^2}\sum_{1\le i,j\le n}(i+j) = \frac{1}{n^3}\cdot\sum_{j=1}^{n}\left(\sum_{i=1}^{n}(i+j)\right) = \frac{1}{n^3}\sum_{j=1}^{n}\left(nj+\sum_{i=1}^{n} i\right)$$

$$= \frac{1}{n^3}\sum_{j=1}^{n}\left(nj+\frac{n(n+1)}{2}\right) = \frac{1}{n^3}\left[n\cdot\frac{n(n+1)}{2}+n\cdot\frac{n(n+1)}{2}\right] = \frac{n^3+n^2}{n^3} = \frac{n+1}{n}$$

$$s_{\Delta_n}(f) = \sum_{1\le i,j\le n} m_{ij}\,\mathrm{vol}(D_{ij}) = \sum_{1\le i,j\le n} \frac{i+j-2}{n}\cdot\frac{1}{n^2}$$

$$= \sum_{1\le i,j\le n}\frac{i+j}{n^3} - \sum_{1\le i,j\le n}\frac{2}{n^3} = \frac{n+1}{n} - \frac{2n^2}{n^3} = \frac{n-1}{n}.$$

$$\sup_{n\geq 1} s_{\Delta_n}(f) \leq \sup_{\Delta} s_{\Delta}(f) = \underline{\int_I} f \leq \overline{\int_I} f = \inf_{\Delta} S_{\Delta}(f) \leq \inf_{n\geq 1} S_{\Delta_n}(f)$$

$$\sup_{n\geq 1} s_{\Delta_n}(f) = \sup_{n\geq 1} \frac{n-1}{n} = 1. \qquad \inf_{n\geq 1} S_{\Delta_n}(f) = \inf_{n\geq 1} \frac{n+1}{n} = 1.$$

$$\Longrightarrow \overline{\int_I} f = \underline{\int} f \Rightarrow f \text{ integr. Riemann și } \int_I f(x,y)\,dx\,dy = 1$$

<u>Exercițiu</u> Calc $\iint_I (x+y)\,dx\,dy$ cu Fubini. $I = [0,1]\times[0,1]$.

$$\iint_D (xy + y^2 + 1)\, dx\, dy \qquad D = [0,2] \times [0,1]$$

$f(x,y) = xy + y^2 + 1$ continuă pe D.

$$\iint_D (xy + y^2 + 1)\, dx\, dy = \int_0^2 \left(\int_0^1 (xy + y^2 + 1)\, dy \right) dx = (*)$$

$$\int_0^1 (xy + y^2 + 1)\, dy = \left(x \cdot \frac{y^2}{2} + \frac{y^3}{3} + y \right) \Bigg|_{y=0}^{y=1} = \frac{x}{2} + \frac{4}{3}.$$

$$(*) = \int_0^2 \left(\frac{x}{2} + \frac{4}{3} \right) dx = \left(\frac{x^2}{4} + \frac{4x}{3} \right) \Bigg|_0^2 = 1 + \frac{8}{3} = \frac{11}{3}$$

$$= \int_0^1 \left(\frac{6x}{x^2+1} + 24\right) dx = 3\int_0^1 \frac{2x}{x^2+1} dx + 24\int_0^1 dx$$

$$= 3\ln(x^2+1)\Big|_0^1 + 24x\Big|_0^1 = 3\ln 2 + 24.$$

$$\iiint_V \left(\frac{xy}{x^2+1}+y+z^2\right) dx\,dy\,dz \qquad V=[0,1]\times[0,2]\times[0,3].$$

$f(x,y,z)=\frac{xy}{x^2+1}+y+z^2$ continuă pe V

$$\iiint_V \left(\frac{xy}{x^2+1}+y+z^2\right) dx\,dy\,dz = \int_0^1\left(\int_0^2\left(\int_0^3\left(\frac{xy}{x^2+1}+y+z^2\right)dz\right)dy\right)dx$$

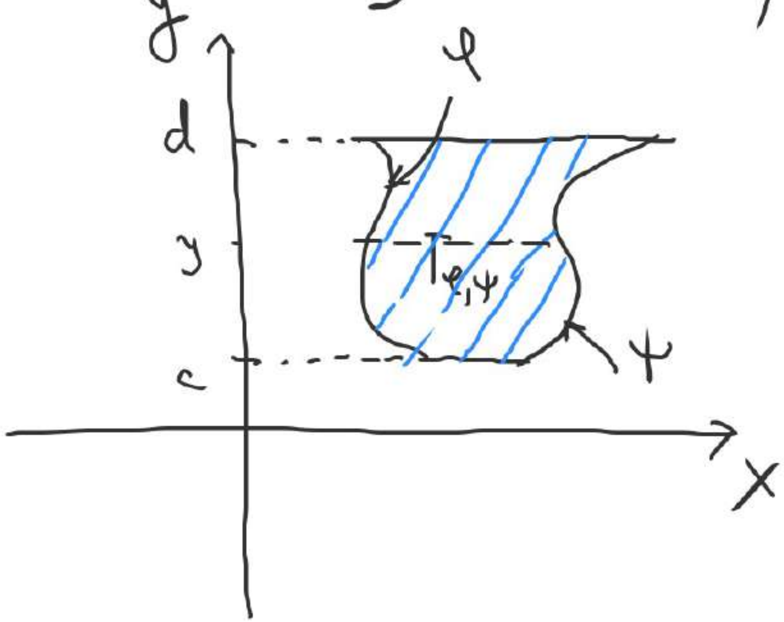
$$= \int_0^1\left(\int_0^2\left(\frac{xy}{x^2+1}\cdot z+y\cdot z+\frac{z^3}{3}\right)\Bigg|_{z=0}^{z=3} dy\right)dx = \int_0^1\left(\int_0^2\left(\frac{3xy}{x^2+1}+3y+9\right)dy\right)dx$$

$$= \int_0^1\left(\frac{3x}{x^2+1}\cdot\frac{y^2}{2}+3\cdot\frac{y^2}{2}+9y\right)\Bigg|_{y=0}^{y=2} dx$$

Propoziție. Fie $\varphi, \psi : [c,d] \to \mathbb{R}$ integrabile Riemann a.î. $\varphi(y) \leq \psi(y)$, $\forall y \in [c,d]$. Atunci,

$$\Gamma_{\varphi,\psi} = \left\{ (x,y) \in \mathbb{R}^2 \,\middle|\, y \in [c,d],\ \varphi(y) \leq x \leq \psi(y) \right\} \in \mathcal{J}(\mathbb{R}^2)$$

și $$\lambda(\Gamma_{\varphi,\psi}) = \int_c^d \left(\psi(y) - \varphi(y)\right) dy$$



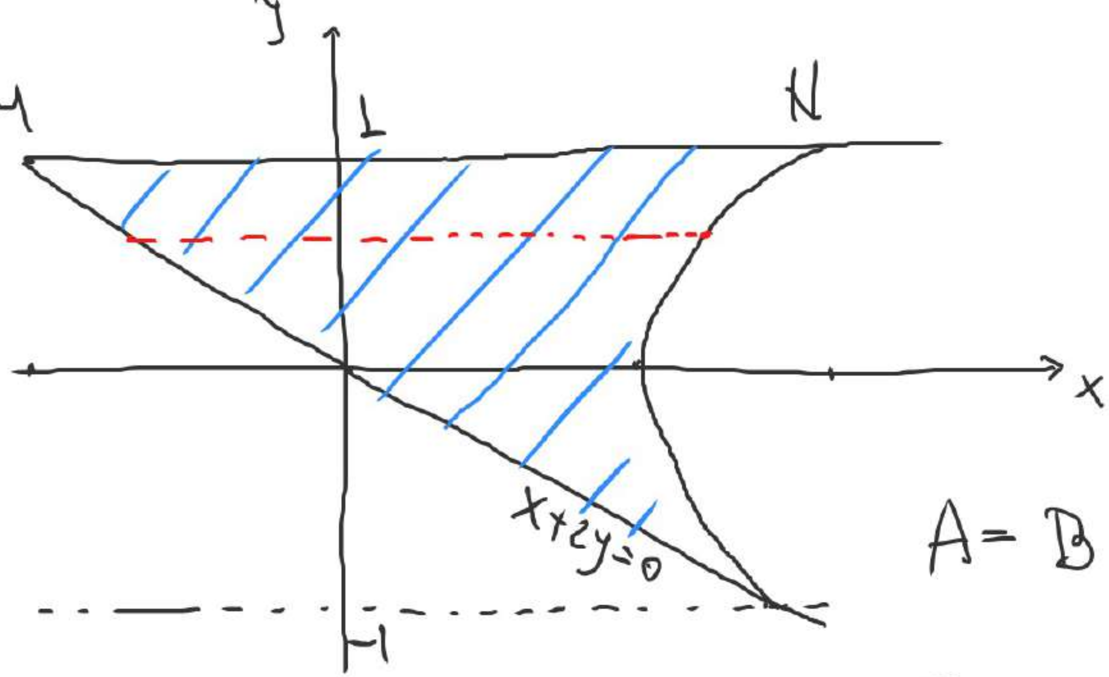
În plus, $G_\varphi, G_\psi \in \mathcal{J}(\mathbb{R}^2)$ și

$$\lambda(G_\varphi) = \lambda(G_\psi) = 0.$$

4) $A = \{(x,y) \in \mathbb{R}^2 \mid x \leq y^2+1 \ , \ |y| \leq 1 \ , \ x+2y \geq 0\}$

$B = \{(x,y) \mid x < y^2+1 \ , \ |y| < 1 \ , \ x+2y \geq 0\}$

Aratati ca $A, B \in \mathcal{J}(\mathbb{R}^2)$ si calc. $\lambda(A)$ si $\lambda(B)$.



$A = B \cup [MN] \cup G_\psi$

$B = A \setminus (G_\varphi \cup [MN])$.

$$A = \{(x,y) \mid -1 \leq y \leq 1, \ \underbrace{-2y}_{\varphi} \leq x \leq \underbrace{y^2+1}_{\psi}\}$$

$$\left.\begin{array}{l} \varphi, \psi : [-1,1] \to \mathbb{R} \\ \varphi(y) = -2y \,,\; \psi(y) = y^2+1 \\ \varphi, \psi \text{ cont și deci int. R.} \end{array}\right\} \Longrightarrow A = \Gamma_{\varphi,\psi} \in \mathcal{J}(\mathbb{R}^2)$$

$$\lambda(A) = \int_{-1}^{1} \big(\psi(y) - \varphi(y)\big)\, dy = \int_{-1}^{1} (y^2+2y+1)\, dy = \int_{-1}^{1} (y+1)^2\, dy$$

$$= \left.\frac{(y+1)^3}{3}\right|_{-1}^{1} = \frac{8}{3}.$$

$$B = A \setminus \big(G_\psi \cup [MN]\big)$$

$[MN] \in \mathcal{J}(\mathbb{R}^2)$, $\lambda([MN]) = 0$,

ψ int. R $\Longrightarrow G_\psi \in \mathcal{J}(\mathbb{R}^2)$

$\lambda(G_\psi) = 0$.

Deci $\lambda(B) = \lambda(A)$.

6) Fie $J \subset \mathbb{R}^n$ interval și $f : J \longrightarrow [0, \infty)$ integrabilă a.î. $\int_J f(x)\,dx = 0$. Arătați că $B = \{x \in J \mid f(x) > 0\}$ este neglijabilă Lebesgue.

Soluție.

Fie $x_0 \in J$, a.î. f cont. în x_0. Dacă $f(x_0) > 0$ atunci există un interval K a.i. $x_0 \in K$ și $f(x) > \frac{f(x_0)}{2}$, $\forall x \in K$

$$\int_K f(x)\,dx \geq \frac{f(x_0)}{2} \cdot \mathrm{vol}(K) > 0 \Rightarrow \int_J f(x)\,dx > 0 \text{ absurd.}$$

Așadar $f(x_0) = 0$.

$B \subset D_f = \{ x \in] \mid f$ nu e continuă în $x \}$

f int. Riemann $\xRightarrow{\text{Crit. Lebesgue}}$ $\left. \begin{array}{l} D_f \text{ neglijabilă Lebesgue} \\ B \subset D_f \end{array} \right\} \Rightarrow$

$\Rightarrow B$ neglijabilă Lebesgue.

<u>Exercițiu</u> Dați o soluție fără să folosiți Crit. Lebesgue.