SEMINAR $\overline{XI}$ - 15.12.2023

1. $(\mathbb{R}^3, g_0)$, $f \in End(\mathbb{R}^3)$ $\quad A = [f]_{R_0, R_0} = \frac{1}{9}\begin{pmatrix} 8 & 1 & -4 \\ 1 & 8 & 4 \\ -4 & 4 & 7 \end{pmatrix}$

a) $f \in O(\mathbb{R}^3)$ de speță 2

b) $\varphi$ și axa de rotație

c) $R = \{e_1, e_2, e_3\}$

a) $A \cdot A^T = \frac{1}{81} \cdot \begin{pmatrix} 8 & 1 & -4 \\ 1 & 8 & 4 \\ -4 & 4 & 7 \end{pmatrix}\begin{pmatrix} 8 & 1 & -4 \\ 1 & 8 & 4 \\ -4 & 4 & 7 \end{pmatrix} = \frac{1}{81}\begin{pmatrix} 81 & 0 & 0 \\ 0 & 81 & 0 \\ 0 & 0 & 81 \end{pmatrix} = I_3$

$\Rightarrow A \in O(3)$

$\det A = \frac{1}{9^3}\begin{vmatrix} 8 & 1 & -4 \\ 1 & 8 & 4 \\ -4 & 4 & 7 \end{vmatrix} = \frac{1}{9^3}\begin{vmatrix} 0 & 1 & 0 \\ -63 & 8 & 36 \\ -36 & 4 & 9 \end{vmatrix} = (-1)\frac{1}{9^3}\begin{vmatrix} -63 & 36 \\ -36 & 9 \end{vmatrix} =$

$= -1 \Rightarrow f$ este de speța 2

$\operatorname{Tr} A = \frac{1}{9} \cdot 9 = 1 = -1 + 2\cos\varphi$

$$\left.\begin{array}{l} 2\cos\varphi = 2 \Leftrightarrow \cos\varphi = 1 \\ \varphi \in [-\pi, \pi] \end{array}\right\} \Rightarrow \varphi = 0$$

$f(x) = -x$

$AX = -X$

$(A + I_3)X = O_3$

$$\frac{1}{9}\begin{pmatrix} 17 & 1 & -4 \\ 1 & 17 & 4 \\ -4 & 4 & 2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{vmatrix} 17 & 1 & -4 \\ 1 & 17 & 4 \\ -4 & 4 & 2 \end{vmatrix} = 2\begin{vmatrix} 17 & 1 & -2 \\ 1 & 17 & 2 \\ -4 & 4 & 1 \end{vmatrix} = 2\begin{vmatrix} 9 & 9 & -2 \\ 9 & 9 & 2 \\ 0 & 0 & 1 \end{vmatrix} = 0$$

$\operatorname{rg}(A + I_3) = 2$

$\det\begin{pmatrix} 17 & 1 \\ 1 & 17 \end{pmatrix} \neq 0$

$$\begin{cases} 17x_1 + x_2 = 4x_3 \\ x_1 + 17x_2 = -4x_3 \end{cases} \qquad 16(x_1 + x_2) = 0 \quad \Rightarrow x_2 = -x_1 \qquad 16x_1 = 4x_3 \Rightarrow x_1 = \frac{1}{4}x_3$$

$x_2 = -\frac{1}{4}x_3$

$(x_1, x_2, x_3) = \left(\frac{1}{4}x_3, -\frac{1}{4}x_3, x_3\right)$

$= \frac{1}{4}x_3(1, -1, 4)$

versorul axei

c) $e_1^{\perp} = \{x \in \mathbb{R}^3 \mid \langle x, e_1 \rangle = 0\}$

$x_1 - x_2 + 4x_3 = 0$

$x_1 + 4x_3 = x_2$

$e_1^{\perp} = \{(x_1, x_1 + 4, x_3) \mid x_1, x_3 \in \mathbb{R}\}$

$e_1^{\perp} = \langle\{(1, 1, 0), (0, 4, 1)\}\rangle$

$\{f_1, f_2\}$ reper arbitrar în $e_1^\perp$

Aplicăm Gauss - Schmidt

$e_2' = f_2$

$$e_3' = f_3 - \frac{\langle f_3, e_2' \rangle}{\langle e_2', e_2' \rangle} \cdot e_2' = (0,4,1) - \frac{4}{2}(1,1,0)$$

$$= (0,4,1) - (2,2,0) = (-2,2,1)$$

$\{e_2', e_3'\}$ reper ortogonal în $e_1^\perp$

$$e_2 = \frac{e_2'}{\|e_2'\|} = \frac{1}{\sqrt{2}}(1,1,0)$$

$$e_3 = \frac{e_3'}{\|e_3'\|} = \frac{1}{3}(-2,2,1)$$

$\{e_2, e_3\}$ reper ortonormat în $e_1^\perp$

$R = \{e_1, e_2, e_3\}$ reper ortonormat a.î.

$$[f]_{R,R} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

2) $(\mathbb{R}^3, g_0)$ $\quad u = (1, 1, 0)$

a) $< \{u\} >^{\perp} = ?$

Precizați un reper ortonormat

b) Să se determine transf. ortogonală de speță 1, care este rotație de ∢ orientat $\varphi = \frac{\pi}{2}$ și axă $< \{u\} >$

a) $u^{\perp} = \{x \in \mathbb{R}^3 \mid <u, x> = 0\} = \{(x_1, \underset{\shortparallel}{-x_1}, x_3) \mid x_1, x_3 \in \mathbb{R}\}$

$x_1 + x_2 = 0$ $\quad\quad x_1 \underbrace{(1, -1, 0)}_{f_2} + x_3 \underbrace{(0, 0, 1)}_{f_3}$

$x_2 = -x_1$

$\{f_2, f_3\}$ reper ortonormat în $u^{\perp}$

Aplicăm Gram-Schmidt

$e_2' = f_2 = (1, -1, 0)$

$e_3' = f_3 - \frac{<f_3, e_2'>}{<e_2', e_2'>} \cdot e_2' = (0, 0, 1) - 0 = f_3$

$e_2 = \frac{e_2'}{\|e_2'\|} = \frac{1}{\sqrt{2}} e_2' = \frac{1}{\sqrt{2}} (1, -1, 0)$

$e_3 = \frac{e_3'}{\|e_3'\|} = (0, 0, 1)$

$\Rightarrow \{e_2, e_3\}$ ortonormat în $u^{\perp}$

$e_1 = \frac{u}{\|u\|} = \frac{1}{\sqrt{2}} (1, 1, 0)$ versorul axei

$R = \{e_1, e_2, e_3\}$

$R_0 \xrightarrow{C} R$

$$C = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & \sqrt{2} \end{pmatrix}$$

$A = [f]_{R_0, R_0} = ?$

$$A' = [f]_{R,R} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}$$

$f: \mathbb{R}^3 \to \mathbb{R}^3$

$$f(x) = \frac{1}{\sqrt{2}} \left( x_1 + x_2 - \sqrt{2} x_3 ,\ x_1 + x_2 + \sqrt{2} x_3 ,\ \sqrt{2}(x_1 + x_2) \right)$$

$A' = C^{-1} A C$

$$A = C A' C^T = \frac{1}{2} \begin{pmatrix} 1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & \sqrt{2} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & \sqrt{2} \end{pmatrix} =$$

$$= \frac{1}{2} \begin{pmatrix} 1 & 0 & -1 \\ 1 & 0 & 1 \\ 0 & \sqrt{2} & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & \sqrt{2} \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 1 & 1 & -\sqrt{2} \\ 1 & 1 & \sqrt{2} \\ \sqrt{2} & \sqrt{2} & 0 \end{pmatrix}$$

4. $(\mathbb{R}^3, g_0)$, $f: \mathbb{R}^3 \to \mathbb{R}^3$, $f(x) = g_0(x, u)$

$u = (1, -1, 2)$

a) Să se arate că $f \in \mathrm{Sim}(\mathbb{R}^3)$; $f = ?$

b) Să se afle $Q: \mathbb{R}^3 \to \mathbb{R}$ formă pătratică asociată

Să se aducă $Q$ la o formă canonică, efectuând o transformare ortogonală $h$

a) $f(x) = (x_1 - x_2 + 2x_3)(1, -1, 2) = (x_1 - x_2 + 2x_3,\ -x_1 + x_2 - 2x_3,$
$2x_1 - 2x_2 + 4x_3)$

$$A = [f]_{R_0, R_0} = \begin{pmatrix} 1 & -1 & 2 \\ -1 & 1 & -2 \\ 2 & -2 & 4 \end{pmatrix} = A^T \Rightarrow f \in \mathrm{Sim}(\mathbb{R}^3)$$

$f(x) = y \Leftrightarrow Y = AX \Leftrightarrow f$ liniară (endomorfism)

b) $Q: \mathbb{R}^3 \to \mathbb{R}, \quad Q(x) = x_1^2 + x_2^2 + 4x_3^2 - 2x_1x_2 + 4x_1x_3 - 4x_2x_3$

Aplicăm metoda valorilor proprii

$$P(\lambda) = \det(A - \lambda I_3) = 0 \Rightarrow \begin{vmatrix} 1-\lambda & -1 & 2 \\ -1 & 1-\lambda & -2 \\ 2 & -2 & 4-\lambda \end{vmatrix} = 0$$

$$\begin{vmatrix} -\lambda & -\lambda & 0 \\ -1 & 1-\lambda & -2 \\ 2 & -2 & 4-\lambda \end{vmatrix} \Rightarrow -\lambda \begin{vmatrix} 1 & 1 & 0 \\ -1 & 1-\lambda & 2 \\ 2 & -2 & 4-\lambda \end{vmatrix} = 0 \Rightarrow$$

$$\Rightarrow -\lambda \begin{vmatrix} 1 & 0 & 0 \\ -1 & 2-\lambda & -2 \\ 2 & -4 & 4-\lambda \end{vmatrix} = 0 \Rightarrow -\lambda[(2-\lambda)(4-\lambda) - 8] = 0$$

$$\Rightarrow -\lambda^2(\lambda - 6) = 0$$

$\lambda_1 = 0 \quad m_1 = 2$

$\lambda_2 = 6 \quad m_2 = 1$

$V_{\lambda_1} = \{x \in \mathbb{R}^3 \mid f(x) = 0\} = \operatorname{Ker} f$

$$A \cdot X = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$x_1 = x_2 - x_3$

$$V_{\lambda_1} = \{(x_2 - 2x_3, x_2, x_3) \mid x_2, x_3 \in \mathbb{R}\} = \langle \{\underset{f_1}{\underset{\shortparallel}{(1,1,0)}}, \underset{f_2}{\underset{\shortparallel}{(-2,0,1)}}\} \rangle$$

$e_1' = f_1$

$$e_2' = f_2 - \frac{\langle f_2, e_1' \rangle}{\langle e_1', e_1' \rangle} \cdot e_1' = (-2,0,1) - \frac{-2}{2}(1,1,0) = (-1,1,1)$$

$$e_1 = \frac{1}{\sqrt{2}}(1,1,0) \qquad e_2 = \frac{1}{\sqrt{3}}(-1,1,1)$$

$V_{\lambda_2} = \{ x \in \mathbb{R}^3 \mid f(x) = 6x \}$

$AX = 6X \Rightarrow (A - 6i_3)X = O_{3,1}$

$$\Rightarrow \begin{pmatrix} -5 & -1 & 2 \\ -1 & -5 & -2 \\ 2 & -2 & -2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{vmatrix} 5 & -1 & 2 \\ 1 & -5 & -2 \\ 2 & -2 & -2 \end{vmatrix} = 2 \begin{vmatrix} -5 & -1 & 2 \\ -1 & -3 & -2 \\ 1 & -1 & -1 \end{vmatrix} = 2 \begin{vmatrix} -5 & -6 & -3 \\ -1 & -6 & -3 \\ 1 & 0 & 0 \end{vmatrix} = 0 \Rightarrow$$

$\Rightarrow rg = 2$

$$\begin{cases} -5x_1 - x_2 = -2x_3 \\ -x_1 - 5x_2 = 2x_3 \end{cases} \Rightarrow \begin{cases} -6(x_1 + x_2) = 0 \Rightarrow x_2 = -x_1 \\ -4x_1 = -2x_3 \Rightarrow x_1 = \frac{1}{2}x_3 \end{cases}$$

$V_{\lambda_2} = \{ (\frac{1}{2}x_3, -\frac{1}{2}x_3, x_3) \mid x_3 \in \mathbb{R} \}$

$= < \{ (1, -1, 2) \} >$ $e_3 = \frac{1}{\sqrt{6}}(1, -1, 2)$

$R = \{e_1, e_2, e_3\}$ reper ortonormat în $\mathbb{R}^3$

$$[f]_{R,R} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 6 \end{pmatrix}$$

$Q(x) = 6x_3'^2$ $(1, 0)$ signatura

$R_0 \xrightarrow{C} R$

$$C = \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{6}} \\ 0 & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} \end{pmatrix}$$

$h \in O(\mathbb{R}^3)$ $[h]_{R_0,R_0} = C$

$h(x) = \left( \frac{1}{\sqrt{2}}x_1 - \frac{1}{\sqrt{3}}x_2 + \frac{1}{\sqrt{6}}x_3, \frac{1}{\sqrt{2}}x_1 + \frac{1}{\sqrt{3}}x_2 - \frac{1}{\sqrt{6}}x_3, \frac{1}{\sqrt{3}}x_2 + \frac{1}{\sqrt{6}}x_3 \right)$

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