Teorema Hamilton-Cayley

$\forall A \in \mathcal{M}_n(K)$ își anulează polinomul caracteristic

i.e. $P_A(A) = O_n \Leftrightarrow A^n - \sigma_1 A^{n-1} + \ldots + (-1)^n I_n \sigma_n = O_n.$

Dem

Notăm $M = A - XI_n$

$P_A(X) = \det M = (-1)^n (X^n - \sigma_1 X^{n-1} + \ldots + \sigma_n (-1)^n)$

$M \cdot M^* = I_n \cdot \det M = (-1)^n (X^n - \sigma_1 X^{n-1} + \ldots + \sigma_n (-1)^n) I_n = (*)$

$M^* = X^{n-1} B_{n-1} + X^{n-2} B_{n-2} + \ldots + X B_1 + B_0$

$(A - XI_n)(X^{n-1} B_{n-1} + X^{n-2} B_{n-2} + \ldots + X B_1 + B_0) = (*)$

$$\left\{\begin{array}{rl|l}
A B_0 &= (-1)^{2n} \sigma_n I_n & \cdot \\
-B_0 + A B_1 &= (-1)^{2n-1} \sigma_{n-1} I_n & A \\
-B_1 + A B_2 &= (-1)^{2n-2} \sigma_{n-2} I_n & A^2 \\
 & & \vdots \\
-B_{n-2} + A B_{n-1} &= (-1)^{n+1} \sigma_1 I_n & \\
-B_{n-1} &= (-1)^n I_n & A^n
\end{array}\right. \quad \oplus$$

$$O_n = (-1)^n \left[A^n - \sigma_1 A^{n-1} + \ldots + (-1)^n \sigma_n I_n \right]$$

$\Rightarrow A^n - \sigma_1 A^{n-1} + \ldots + (-1)^n \sigma_n I_n = O_n$

În particular, pt $n = 2$: $A^2 - \sigma_1 A + \sigma_2 I_2 = O_2$

$\sigma_1 = \operatorname{Tr} A$

$\sigma_2 = \det A$

[redacted]

Prop. $A \in \mathcal{M}_{m,n}(\mathbb{R})$

$S(A) = \{x \in \mathbb{R}^n \mid AX = 0\} \subset \mathbb{R}^n$ subspațiu vectorial

și $\dim_{\mathbb{R}} S(A) = n - rg(A)$

Dem.

$$\left.\begin{array}{l} \forall\, x, y \in S(A) \\ \forall\, a, b \in \mathbb{R} \end{array}\right\} \overset{?}{\Longrightarrow} ax + by \in S(A)$$

$$\left.\begin{array}{l} AX = 0 \\ AY = 0 \end{array}\right\} \Rightarrow A(aX + bY) = 0 \Rightarrow S(A) \subset \mathbb{R}^n \text{ subsp vect}$$

Fie $rg A = r$. Fără a restrânge generalitatea,

$x_1, \dots, x_r$ = variabile principale

$x_{r+1} = \lambda_1, \dots, x_n = \lambda_p$ variabile secundare, $p = n - r$.

$$\begin{cases} x_1 = \alpha_{11}\lambda_1 + \dots + \alpha_{1p}\lambda_p \\ x_r = \alpha_{r1}\lambda_1 + \dots + \alpha_{rp}\lambda_p \end{cases}$$

Sol sist

$$(*)\begin{cases} (x_1, \dots, x_r, \lambda_1, \dots, \lambda_p) = \\ = (\alpha_{11}\lambda_1 + \dots + \alpha_{1p}\lambda_p, \dots, \alpha_{r1}\lambda_1 + \dots + \alpha_{rp}\lambda_p, \lambda_1, \dots, \lambda_p) = \\ = \lambda_1 \underbrace{(\alpha_{11}, \dots, \alpha_{r1}, 1, 0, \dots, 0)}_{y_1} + \dots + \lambda_p \underbrace{(\alpha_{1p}, \dots, \alpha_{rp}, 0, \dots, 0, 1)}_{y_p} \end{cases}$$

$S(A) = \langle \{y_1, \dots, y_p\} \rangle$

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$R = \{y_1, \dots, y_p\}$ SG pentru $S(A)$

Dem că R este SLI

Fie $\lambda_1, \dots, \lambda_p \in \mathbb{R}$ ai $\lambda_1 y_1 + \dots + \lambda_p y_p = 0_{\mathbb{R}^n}$

$\overset{(*)}{\Rightarrow} (x_1, \dots, x_r, \lambda_1, \dots, \lambda_p) = (0, \dots, 0) \Rightarrow \lambda_1 = \dots = \lambda_p = 0 \Rightarrow$ SLI