Seminar 6

② $f: \mathbb{R}^3 \to \mathbb{R}^3$, $f(x_1, x_2, x_3) = (x_1 + 2x_2 + x_3,\ 2x_1 + 5x_2 + 3x_3,\ -3x_1 - 7x_2 - 4x_3)$

a) f lin. ?
b) $\ker f = ?$ un reper în $\ker f$?
c) $\operatorname{Im} f = ?$ — " — $\operatorname{Im} f$?
d) $[f]_{R_0, R_0} = A = ?$, R_0 reper canonic în $\mathbb{R}^3$

a) $f(x) = y \iff \boxed{AX = Y}$ *

coef. scriși pe linie

$$A = \begin{pmatrix} 1 & 2 & 1 \\ 2 & 5 & 3 \\ -3 & -7 & -4 \end{pmatrix}; \quad X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \qquad A = [f]_{R_0, R_0} = \begin{pmatrix} 1 & 2 & 1 \\ 2 & 5 & 3 \\ -3 & -7 & -4 \end{pmatrix}$$

Obs: $R_0 = \{e_1, e_2, e_3\}$; $f(e_i) = \sum_{j=1}^{3} a_{ji} e_j$

$f(e_1) = f(1,0,0) = (1, 2, -3) = 1e_1 + 2e_2 - 3e_3$
$f(e_2) = f(0,1,0) = (2, 5, -7) = 2e_1 + 5e_2 - 7e_3$
$f(e_3) = f(0,0,1) = (1, 3, -4) = 1e_1 + 3e_2 - 4e_3$

$$A = \begin{pmatrix} 1 & 2 & 1 \\ 2 & 5 & 3 \\ -3 & -7 & -4 \end{pmatrix}$$

$\Rightarrow f$ lin. din *

b) $\ker f = \{x \in \mathbb{R}^3 \mid f(x) = 0_{\mathbb{R}^3}\} = \{x \in \mathbb{R}^3 \mid AX = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}\} = S(A)$

$\dim \ker f = \underset{\substack{\| \\ 3}}{\dim \mathbb{R}^3} - \operatorname{rg} A = 3 - 2 = 1$

$$\begin{vmatrix} 1 & 2 & 1 \\ 2 & 5 & 3 \\ -3 & -7 & -4 \end{vmatrix} \overset{L_1 + = L_2 + L_3}{=} \begin{vmatrix} 0 & 0 & 0 \\ 2 & 5 & 3 \\ -3 & -7 & -4 \end{vmatrix} = 0 \Rightarrow \operatorname{rg} A = 2$$

$$A = \begin{pmatrix} 1 & 2 & 1 \\ 2 & 5 & 3 \\ -3 & -7 & -4 \end{pmatrix} \qquad \begin{cases} x_1 + 2x_2 = -x_3 \quad | \cdot (-2) \\ 2x_1 + 5x_2 = -3x_3 \end{cases} \Rightarrow \begin{cases} -2x_1 - 4x_2 = 2x_3 \\ 2x_1 + 5x_2 = -3x_3 \end{cases}$$

$$x_2 = -x_3$$

$x_1 = -x_3 - 2x_2 = x_3$

$\mathrm{Ker}\, f = \{(x_3, -x_3, x_3) \mid x_3 \in \mathbb{R}^3\} = \{x_3(1,-1,1) \mid x_3 \in \mathbb{R}\}$

$R = \{(1,-1,1)\}$ reper în $\mathrm{Ker}\, f$

c) Metoda 1

extindem R la un reper în $\mathbb{R}^3$

$$\mathrm{rg}\begin{pmatrix} 1 & 0 & 1 \\ -1 & 0 & 0 \\ 1 & 1 & 0 \end{pmatrix} = 3 \text{ (maxim)}$$

$R \cup \{e_3, e_1\}$ rep. în $\mathbb{R}^3$

$\{f(e_3), f(e_1)\}$ reper în $\mathrm{Im}\, f$ (cf. T dim curs 6)

$f(e_3) = (1, 3, -4)$

$f(e_1) = (1, 2, -3)$

T. dim.: $\underbrace{\dim \mathbb{R}^3}_{3} = \dim \mathrm{Im}\, f + \underbrace{\dim \mathrm{Ker}\, f}_{1} \Rightarrow$

$\Rightarrow \dim \mathrm{Im}\, f = 2$

c) Metoda 2

$$A = \begin{pmatrix} 1 & 2 & 1 \\ 2 & 5 & 3 \\ -3 & -7 & -4 \end{pmatrix} \qquad (*) \begin{cases} x_1 + 2x_2 + x_3 = y_1 \\ 2x_1 + 5x_2 + 3x_3 = y_2 \\ -3x_1 - 7x_2 - 4x_3 = y_3 \end{cases}$$

cf. teoriei

$Y \in \mathrm{Im}\, f \Rightarrow \exists X \in \mathbb{R}^3$ a.î. $f(X) = Y$

$(*)$ SC $\Rightarrow \mathrm{rg}\, A = \mathrm{rg}\, \bar{A} = 2$

$\Rightarrow \Delta_c = 0 \Rightarrow$
$$\begin{vmatrix} 1 & 2 & y_1 \\ 2 & 5 & y_2 \\ -3 & -7 & y_3 \end{vmatrix} = \begin{vmatrix} 0 & 0 & y_1 + y_2 + y_3 \\ 2 & 5 & y_2 \\ -3 & -7 & y_3 \end{vmatrix} =$$

$$= (y_1 + y_2 + y_3) \cdot \begin{vmatrix} 2 & 5 \\ -3 & -7 \end{vmatrix} = y_1 + y_2 + y_3 = 0$$

$\mathrm{Im}\, f = \{y \in \mathbb{R}^3 \mid y_1 + y_2 + y_3 = 0\}$ $\qquad y_3 = -y_1 - y_2$

$\Rightarrow \operatorname{Im} f = \{ (y_1, y_2, -y_1-y_2) \mid y_1, y_2 \in \mathbb{R} \}$

$$\overset{\parallel}{y_1(1,0,-1) + y_2(0,1,-1)} \quad ; \quad \{(1,0,-1); (0,1,-1)\} \text{ reper în } \operatorname{Im} f$$

Obs.: $\left.\begin{array}{l} f: \mathbb{R}^2 \to \mathbb{R}^3 \\ \dim \ker f = 2 \end{array}\right) \Rightarrow f = \text{funcț. ct. nulă}$

④ $f: \mathbb{R}_3[X] \to \mathbb{R}_2[X]$; $f(P) = P'$

a) $A = [f]_{R_0, R_0'}$ unde R_0, R_0' sunt rep. canonice în $\mathbb{R}_3[X], \mathbb{R}_2[X]$

$R_0 = \{1, X, X^2, X^3\}$, $R_0' = \{1, X, X^2\}$

b) $\dim \ker f$, $\dim \operatorname{Im} f$

$P = a_1 + a_2 X + a_3 X^2 + a_4 \cdot X^3 \equiv (a_1, a_2, a_3, a_4) \in \mathbb{R}^4$

$f(P) = a_2 + 2a_3 X + 3a_4 X^2 \equiv (a_2, 2a_3, 3a_4) \in \mathbb{R}^3$

$f: \mathbb{R}^4 \to \mathbb{R}^3$, $f(a_1, a_2, a_3, a_4) = (a_2, 2a_3, 3a_4)$

$$A = \underbrace{\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix}}_{\in \mathcal{M}_{3,4}(\mathbb{R})} \quad ; \quad A \cdot \begin{pmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{pmatrix} = \begin{pmatrix} a_2 \\ 2a_3 \\ 3a_4 \end{pmatrix}$$

b) $\ker f = \{ a \in \mathbb{R}^4 \mid Aa = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \} = S(A)$

$\dim \ker f = \underset{\substack{\parallel \\ 4}}{\dim \mathbb{R}^4} - \operatorname{rg} A = 4 - 3 = 1$

T. dim. : $\dim \mathbb{R}^4 = \dim \ker f + \dim \operatorname{Im} f \Rightarrow \dim \operatorname{Im} f = 3$

Obs: $\langle\{1\}\rangle = \ker f$ $\qquad f(P) = P' = 0$

$\qquad \{a, a \in \mathbb{R}\}$

$f(1) = 0 = 0 \cdot 1 + 0X + 0X^2$

$f(X) = 1 = 1 \cdot 1 + 0X + 0X^2$ $\Big|$ $f(X^2) = 2X$, $f(X^3) = 3X^2$

$$A = \begin{pmatrix} 0 & & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix}$$

(9) $f : \mathbb{R}^3 \to \mathbb{R}^3$ lin.

$f(v_i) = u_i, \quad i = \overline{1,3}$ $\quad A' = \begin{pmatrix} 2 & 1 & 0 \\ 3 & 3 & 0 \\ -1 & 1 & 1 \end{pmatrix}$

a) $f = ?$

b) $[f]_{R_0, R_0}$

c) $\ker f = ?$ $\quad \operatorname{Im} f = ?$

$\ker f = \{0_{\mathbb{R}^3}\}$ $\quad \operatorname{Im} f = \mathbb{R}^3$

$v_1 = (-1, 1, 1) \qquad u_1 = 2u_1 + 3v_2 - v_3$

$v_2 = (1, 1, 1) \qquad u_2 = v_1 + 3u_2 + v_3$

$v_3 = (0, 2, 1) \qquad u_3 = v_3$

a) $\det \begin{pmatrix} -1 & 1 & 0 \\ 1 & 1 & 2 \\ 1 & 1 & 1 \end{pmatrix} = \begin{vmatrix} -1 & 1 & 0 \\ 1 & 1 & 2 \\ 1 & 1 & 1 \end{vmatrix} \neq 0$ $\qquad R_1 = \{v_1, v_2, v_3\}$ reper în $\mathbb{R}^3$

$u_1 = (-2+3, 2+3-2, 2+3-1) = (1, 3, 4)$

$u_2 = (-1+3, 1+3+2, 1+3-1) = (2, 6, 5)$

$u_3 = (0, 2, 1)$

$\det \begin{pmatrix} 1 & 2 & 0 \\ 3 & 6 & 2 \\ 4 & 5 & 1 \end{pmatrix} = \begin{vmatrix} 1 & 2 & 0 \\ 3 & 6 & 2 \\ 4 & 5 & 1 \end{vmatrix} \neq 0$ $\qquad R_2 = \{u_1, u_2, u_3\}$ reper în $\mathbb{R}^3$

c) f duce reper în reper $\Rightarrow$ f izomorf. $\Rightarrow \begin{cases} f \text{ inj.} \Rightarrow \ker f = \{0_{\mathbb{R}^3}\} \\ f \text{ surj.} \Rightarrow \operatorname{Im} f = \mathbb{R}^3 \end{cases}$

b)
$$\begin{tikzcd} R_0 \arrow[r, "A"] \arrow[d, "C"'] & R_0' \arrow[d, "D"] \\ R_1 \arrow[r, "A'"] & R_2 \end{tikzcd}$$

$A' = D^{-1} A C$

$A = D A' C^{-1}$

$C = \begin{pmatrix} -1 & 1 & 0 \\ 1 & 1 & 2 \\ 1 & 1 & 1 \end{pmatrix}$

$D = \begin{pmatrix} 1 & 2 & 0 \\ 3 & 6 & 2 \\ 4 & 5 & 1 \end{pmatrix}$

$f(v_1) = f(-e_1 + e_2 + e_3) = -f(e_1) + f(e_2) + f(e_3)$

$\;\; \| $

$u_1 = (1, 3, 4) = e_1 + 3e_2 + 4e_3$

$f(u_2) = f(e_1 + e_2 + 2e_3) = f(e_1) + f(e_2) + 2f(e_3)$

$\;\; \| $

$u_2 = (2, 6, 5) = 2e_1 + 6e_2 + 5e_3$

$f(u_3) = f(2e_2 + e_3) = 2f(e_2) + f(e_3)$

$u_3 = (0, 2, 1) = 2e_2 + e_3$

$$\begin{cases} -f(e_1) + f(e_2) + f(e_3) = (1,3,4) \\ f(e_1) + f(e_2) + f(e_3) = (2,6,5) \\ 2f(e_2) + f(e_3) = (0,2,1) \end{cases} \qquad A = \begin{pmatrix} \frac{1}{2} & -\frac{3}{2} & 3 \\ \frac{3}{2} & -\frac{5}{2} & 7 \\ \frac{1}{2} & -\frac{7}{2} & 8 \end{pmatrix}$$

ec 2 − ec 1 : $\Rightarrow 2f(e_1) = (1,3,1) \Rightarrow f(e_1) = \left(\frac{1}{2}, \frac{3}{2}, \frac{1}{2}\right)$

analog : $\Longrightarrow f(e_2) = \left(-\frac{3}{2}; -\frac{5}{2}; -\frac{7}{2}\right)$

$f(e_3) = (3, 7, 8)$

$f : \mathbb{R}^3 \to \mathbb{R}^3, \; f(x) = \left(\frac{1}{2}x_1 + \frac{3}{2}x_2 + \frac{1}{2}x_3, \; -\frac{3}{2}x_1 - \frac{5}{2}x_2 - \frac{7}{2}x_3, \; 3x_1 + 7x_2 + 8x_3\right)$

(14) $f : \mathbb{R}^3 \to \mathbb{R}^3, \; f(x) = (x_1 - 2x_2 + 5x_3, \; mx_1 + 3x_2 - x_3, \; x_2 - 3x_3)$

a) $m = ?$ a.î. f inj. ; b) Pt $m = 1$, $\operatorname{Im} f = ?$; c) $m = -\frac{8}{3}$, $\dim \operatorname{Im} f = ?$

a) $\begin{pmatrix} 1 & -2 & 5 \\ m & 3 & -1 \\ 0 & 1 & -3 \end{pmatrix} = A = [f]_{R_0, R_0}$ $\quad \det A = \begin{vmatrix} 1 & -2 & 5 \\ m & 3 & -1 \\ 0 & 1 & -3 \end{vmatrix} =$

$= -9 + 0 + 5m - 0 - 6m + 1$

$= -m - 8 \neq 0$

f inj $\Rightarrow m \in \mathbb{R} \setminus \{-8\}$

$\Longrightarrow f$ surj (la End!)

$\ker f = \left\{ x \in \mathbb{R}^3 \mid Ax = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \right\}$

f inj $\Longleftrightarrow \ker f = \{0_{\mathbb{R}^3}\}$

b) $m = 1 \Rightarrow \det \neq 0 \Rightarrow \ker f = \{0_{\mathbb{R}^3}\} \Rightarrow$ surj $\Rightarrow \operatorname{Im} f = \mathbb{R}^3$

c) $m = -\frac{8}{3} \Rightarrow \det \neq 0 \Rightarrow f$ inj $\Rightarrow f$ surj $\Rightarrow \operatorname{Im} f = \mathbb{R}^3 \Rightarrow \dim \operatorname{Im} f = 3$

(7) $f \in End(V)$

$f^2 = \Theta \Rightarrow g = id_V + f \in Aut(V)$

Sol: $A = [f]_{R,R}$

$A^2 = \Theta_n \Rightarrow I_n + A$ e inv.

$I_n^2 - A^2 = I_n$

$(I_n + A)(I_n - A) = I_n$; $(I_n - A)(I_n + A) = I_n \Rightarrow (I_n + A)^{-1} = (I_n - A)$

$g^{-1} = id_V - f$