2024

C4 - GA

Spații vectoriale. Repere. Schimbare de coordonate. Operații cu subspații

$(V,+,\cdot)|_{\mathbb{K}}$ sp. vect

$S = \{v_1, \dots, v_n\}$ sn

1) SLI $\iff$ $\forall a_1, \dots, a_n \in \mathbb{K}$, $\forall x_1, \dots, x_n \in S$: $a_1 x_1 + \dots + a_n x_n = 0_V \Rightarrow a_i = 0$ $\forall i = \overline{1,n}$

2) SLD $\iff$ $\exists x_1, \dots, x_n \in S$, $\exists a_1, \dots, a_n \in \mathbb{K}$, nu toți nuli aî $a_1 x_1 + \dots + a_n x_n = 0_V$

3) SG $\iff$ $\forall x \in V$, $\exists a_1, \dots, a_n \in \mathbb{K}$ aî $x = a_1 x_1 + \dots + a_n x_n$

$V = \langle S \rangle$

V s.n. finit generat $\iff$ $\exists$ S finit SG i.e $\langle S \rangle = V$

4) bază $\iff$ $\begin{cases} 1) \text{ SLI} \\ 2) \text{ SG} \end{cases}$

Teorema schimbului

$(V,+,\cdot)|_{\mathbb{K}}$ sp vect, f.gen.

$\{x_1, \dots, x_n\}$ SG
$\{y_1, \dots, y_n\}$ SLI $\Rightarrow \{y_1, \dots, y_n\}$ SG

Dem

$V = \langle \{x_1, \dots, x_n\} \rangle \Rightarrow \exists a_1, \dots, a_n \in \mathbb{K}$ aî $y_1 = a_1 x_1 + \dots + a_n x_n$

$\underset{y_1}{\overset{\cup\!\!\!\!\!\!\;\;}{V}}$

Ip. abs $a_1 = \dots = a_n = 0_{\mathbb{K}} \Rightarrow y_1 = 0_V$

$\{0_V, y_2, \dots, y_n\}$ SLI (din ip)

$a_1 \cdot 0_V + 0 \cdot y_2 + \dots + 0 y_n = 0_V \Rightarrow$ SLD Contrad.

$a_1 \neq 0_{\mathbb{K}}$

Ip. este falsă. Fie $a_1 \neq 0_{\mathbb{K}}$

$y_1 = a_1x_1 + \ldots + a_nx_n \mid \cdot a_1^{-1} \Rightarrow x_1 = a_1^{-1}y_1 - a_1^{-1}a_2x_2 - \ldots - a_1^{-1}a_nx_n$ 2)

$x_1 \in \langle\{y_1, x_2, \ldots, x_n\}\rangle \Rightarrow \langle\{x_1, \ldots, x_n\}\rangle = \langle\{y_1, x_2, \ldots, x_n\}\rangle = V$

$y_2 \in V \Rightarrow \exists\, b_1, a_2, \ldots, a_n \in \mathbb{K}$ a.î. $y_2 = b_1y_1 + a_2x_2 + \ldots + a_nx_n$

Pp. abs $a_2 = \ldots = a_n = 0_{\mathbb{K}} \Rightarrow y_2 = b_1y_1 \Rightarrow$

$b_1y_1 - \underset{\substack{\neq \\ 0_{\mathbb{K}}}}{1} \cdot y_2 + 0 \cdot y_3 + \ldots + 0 \cdot y_n = 0_V \Rightarrow \{y_1, \ldots, y_n\}$ SLD abs.

Pp. este falsă. Fie $a_2 \neq 0_{\mathbb{K}}$ (altfel renumerotăm indicii)

$x_2 = a_2^{-1}(y_2 - b_1y_1 - a_3x_3 - \ldots - a_nx_n)$

$x_2 \in \langle\{y_1, y_2, x_3, \ldots, x_n\}\rangle \Rightarrow \langle\{x_1, \ldots, x_n\}\rangle = \langle\{y_1, y_2, x_3, \ldots, x_n\}\rangle$

Analog, după un nr finit de pași $\Rightarrow$

$V = \langle\{x_1, \ldots, x_n\}\rangle = \langle\{y_1, \ldots, y_n\}\rangle \Rightarrow \{y_1, \ldots, y_n\}$ SG.

Prop

card $\forall$SG (finit) $\geq$ card $\forall$SLI

Dem

Fie $\{x_1, \ldots, x_n\}$ SG și $\{y_1, \ldots, y_n, y_{n+1}\}$. Dem că $\{y_1, \ldots, y_{n+1}\}$ este SLD.

1) $\{y_1, \ldots, y_n\}$ SLI $\xRightarrow{\text{Th. schimbului}}$ $\{y_1, \ldots, y_n\}$SG

$\underset{\substack{\cup \\ y_{n+1}}}{V} = \langle\{y_1, \ldots, y_n\}\rangle \Rightarrow \exists\, a_1, \ldots, a_n \in \mathbb{K}$ a.î.

$y_{n+1} = a_1y_1 + \ldots + a_ny_n$

$a_1y_1 + \ldots + a_ny_n - \underset{\substack{\neq \\ 0_{\mathbb{K}}}}{1_{\mathbb{K}}} \cdot y_{n+1} = 0_V \Rightarrow$

$\Rightarrow \{y_1, \ldots, y_n, y_{n+1}\}$ SLD.

2) $\{y_1, \dots, y_n\}$ SLD $\Rightarrow$ $\{y_1, \dots, y_n\} \cup \{y_{n+1}\}$ (o supramulțime) SLD.

__Teoremă__

$(V, +, \cdot)/\mathbb{K}$ sp.v, finit generat

$\Rightarrow \forall B_1, B_2$ baze : $\text{card } B_1 = \text{card } B_2 = n = \dim_{\mathbb{K}} V$

$\text{card } B_1 = |B_1|$, $\text{card } B_2 = |B_2|$ (dimensiunea)

__Dem__

① B_1 SG, B_2 SLI $\Rightarrow |B_1| \geq |B_2|$

② B_2 SG, B_1 SLI $\Rightarrow |B_2| \geq |B_1|$

$\Longrightarrow |B_1| = |B_2| = n$ (este un invariant)

__Def__ $(V, +, \cdot)/\mathbb{K}$ sp. vect, $B = \{e_1, \dots, e_n\}$ bază

B s.n. __reper__ dacă este o bază ordonată

__Prop__ $(V, +, \cdot)/\mathbb{K}$ sp. vect, $R = \{e_1, \dots, e_n\}$ reper

$\forall x \in V, \ \exists! (x_1, \dots, x_n) \in \mathbb{K}^n$

coordonatele lui x în raport cu reperul R (componentele)

__Dem__

$V = \langle R \rangle \Rightarrow \exists x_1, \dots, x_n \in \mathbb{K}$ aî $x = x_1 e_1 + \dots + x_n e_n$ (1)

Pp. abs $\exists x_1', \dots, x_n' \in \mathbb{K}$ aî

$x = x_1' e_1 + \dots + x_m' e_n$ (2)

Din (1),(2) $(x_1 - x_1') e_1 + \dots + (x_n - x_m') e_n = 0_V \xRightarrow{R \text{ este SLI}}$

$\Rightarrow x_1 - x_1' = 0$
$\vdots$
$x_n - x_m' = 0$
$\Rightarrow x_k = x_k', \ \forall k = \overline{1,n}$

Modificarea coordonatelor la schimbarea de repere

$(V,+,\cdot)/_{\mathbb{K}}$ sp. vect, finit generat

$\mathcal{R}=\{e_1,..,e_n\} \xrightarrow{A=(a_{ij})_{i,j=\overline{1,n}}} \mathcal{R}'=\{e_1',..,e_n'\}$ repere în V

$$e_i' = \sum_{j=1}^{n} a_{ji}\, e_j \ , \ \forall i=\overline{1,n}$$

$$\forall x = \underline{\sum_{j=1}^{n} x_j e_j}$$

$$= \sum_{i=1}^{n} x_i' e_i' = \sum_{i=1}^{n} x_i' \left(\sum_{j=1}^{n} a_{ji} e_j\right) = \sum_{j=1}^{n}\left(\sum_{i=1}^{n} a_{ji} x_i'\right) \cdot e_j$$

$$\Rightarrow x_j = \sum_{i=1}^{n} a_{ji} x_i' \ , \ \forall j=\overline{1,n}$$

$$X = AX' \qquad X=\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}, \ X'=\begin{pmatrix} x_1' \\ \vdots \\ x_n' \end{pmatrix}, \ A=(a_{ij})_{i,j=\overline{1,n}}$$

Prop $(V,+,\cdot)/_{\mathbb{K}}$ sp.v. finit generat

$$\mathcal{R}=\{e_1,..,e_n\} \xrightarrow{A} \mathcal{R}'=\{e_1',..,e_n'\} \xrightarrow{B} \mathcal{R}''=\{e_1'',..,e_n''\}$$

with $\mathcal{R} \xrightarrow{C} \mathcal{R}''$.

Atunci $C=AB$

Dem

$$e_i'' = \sum_{k=1}^{n} \underline{c_{ki}\, e_k} \ , \ \forall i=\overline{1,n}$$

$$e_i'' = \sum_{j=1}^{n} b_{ji}\, e_j' = \sum_{j=1}^{n} b_{ji}\left(\sum_{k=1}^{n} a_{kj}\, e_k\right)$$

$$= \sum_{k=1}^{n}\left(\underline{\sum_{j=1}^{n} a_{kj} b_{ji}}\right) e_k$$

$$c_{ki} = \sum_{j=1}^{n} a_{kj}\, b_{ji} \Rightarrow C=AB$$

Prop $(V,+,\cdot)|_{\mathbb{K}}$ sp.v, f.generat

$\mathcal{R}=\{e_1,\dots,e_n\} \xrightarrow{A} \mathcal{R}'=\{e_1',\dots,e_n'\}$ repere

$\Rightarrow A\in GL(n,\mathbb{K})$

(matricea de trecere de la $\mathcal{R}$ la $\mathcal{R}'$ este inversabilă)

Dem

$$\begin{tikzcd} \mathcal{R} \arrow[r, "A"] \arrow[rr, bend right, "I_n"'] & \mathcal{R}' \arrow[r, "B"] & \mathcal{R} \end{tikzcd} \qquad AB=I_n$$

$$\Rightarrow B=A^{-1}\Rightarrow A\in GL(n,\mathbb{K})$$

$$\begin{tikzcd} \mathcal{R}' \arrow[r, "B"] \arrow[rr, bend right, "I_n"'] & \mathcal{R} \arrow[r, "A"] & \mathcal{R}' \end{tikzcd} \qquad B\cdot A=I_n$$

Def $\mathcal{R}$ și $\mathcal{R}'$ s.n. repere la fel orientate $\Leftrightarrow \det A>0$

opus orientate $\Leftrightarrow \det A<0$.

OBS Relația „a fi orientate" este o relație de echivalență.

a) Reflexivă $\mathcal{R}\xrightarrow{I_n}\mathcal{R}$ $\det(I_n)>0$. $\checkmark_1$

b) Simetrică $\mathcal{R}\xrightarrow{A}\mathcal{R}'$ $\det(A)>0\Rightarrow \det(A^{-1})>0$ $\checkmark_2$

$\mathcal{R}'\xrightarrow{A^{-1}}\mathcal{R}$ $\qquad \det(A^{-1})=\frac{1}{\det A}$

c) Tranzitivă $\begin{tikzcd} \mathcal{R} \arrow[r, "A"] \arrow[rr, bend right, "C"'] & \mathcal{R}' \arrow[r, "B"] & \mathcal{R}'' \end{tikzcd}$ $\begin{array}{l}\det A>0\\ \det B>0\end{array} \Rightarrow \det C=\det(A\cdot B) =\det A\cdot\det B>0$

Pe mulțimea reperelor se consideră 2 clase de echivalență.

A alege o orientare = a preciza un reper pozitiv orientat

Criteriu de LI

Fie $(V,+,\cdot)|_{\mathbb{K}}$ sp.v, f.generat, $\dim_{\mathbb{K}}V=n$.

$S=\{v_1,\dots,v_m\}\subset V$ sistem de vectori $m\le n$

$\Rightarrow S$ este SLI $\Leftrightarrow$ matricea componentelor vectorilor din S, în raport cu $\forall$ reper, are rangul maxim $=m$ $\checkmark_2$

Dem Fie $\mathcal{R}=\{e_1,\dots,e_n\}$ reper în V

$$v_i=\sum_{j=1}^{n} v_{ji}e_j,\ \forall i=\overline{1,m}$$

Scanat cu CamScanner

S este SLI $\Leftrightarrow \left[\forall a_1,\dots,a_m \in \mathbb{K}: \sum_{i=1}^{m} a_i v_i \Rightarrow a_1 = \dots = a_m = 0_{\mathbb{K}}\right.$

$$\sum_{i=1}^{m}\left(\sum_{j=1}^{n} a_i v_{ji} e_j\right)$$

$$\sum_{j=1}^{n}\Big(\underline{\sum_{i=1}^{m} a_i v_{ji}}\Big) e_j = 0_V \xRightarrow[R\,e\,SLI]{} (*) \sum_{i=1}^{m} v_{ji} a_i = 0, \forall j = \overline{1,n}$$

(*) SLO de n ec cu m $(a_1,\dots,a_m)$ necunoscute

(*) are numai sol nulă $\Leftrightarrow$ $rg\, C = m = \text{maxim}$

$$C = (v_{ji})_{\substack{j=\overline{1,n}\\ i=\overline{1,m}}}$$

<u>OBS</u> $R = \{e_1,\dots,e_n\} \xrightarrow{A} R' = \{e_1',\dots,e_n'\}$

$$v_i = \sum_{k=1}^{n} v_{ki}' e_k' = \sum_{k=1}^{n} v_{ki}' \sum_{j=1}^{n} a_{jk} e_j =$$

$$\left.\begin{array}{l} v_i = \sum_{j=1}^{n}\Big(\underline{\sum_{k=1}^{n} a_{jk} v_{ki}'}\Big) e_j \\ \text{dar } v_i = \sum_{j=1}^{m} \underline{v_{ji}}\, e_j \end{array}\right\} \Rightarrow v_{ji} = \sum_{k=1}^{n} a_{jk} v_{ki}'$$

$C = AC'$ $\qquad C' = (v_{ki}')_{\substack{k=\overline{1,n}\\ i=\overline{1,m}}}$

$$\left.\begin{array}{l} rg\, C = rg(AC') \\ A \in GL(n,\mathbb{K}) \end{array}\right| \Rightarrow rg\, C = rg\, C' = m = \text{maxim}.$$

<u>Aplicație</u>

$(\mathbb{R}^2, +, \cdot)/\mathbb{R}$, $R_0 = \{e_1 = (1,0), e_2 = (0,1)\}$ reperul canonic (pozitiv orientat) convenție.

Fie $R' = \{e_1' = (2,1), e_2' = (3,0)\}$.

a) R' este reper în $\mathbb{R}^2$

b) $R_0 \xrightarrow{A} R'$, $R' \xrightarrow{B} R_0$, $A, B = ?$

R_0, R' sunt la fel orientate?

c) Fie $x = (1,2)$. Să se afle coordonatele în raport cu R'.

2a) $e_1' = (2,1) = 2e_1 + 1\cdot e_2$ $\quad A = \begin{pmatrix} \boxed{\begin{matrix}2\\1\end{matrix}} & \boxed{\begin{matrix}3\\0\end{matrix}} \end{pmatrix}$

$e_1' = a_{11}e_1 + a_{21}e_2$

$e_2' = (3,0) = 3e_1 + 0\cdot e_2$

$rg A = 2 = max \Rightarrow R'$ este SLI

dar $|R'| = 2 = \dim_{\mathbb{R}} \mathbb{R}^2$ $\Big\} \Rightarrow R'$ reper

b) $R_0 \xrightarrow{A} R'$, $R' \xrightarrow{A^{-1}} R_0$

$\det A = -3 < 0$ sunt opus orientate R', R_0

c) $x = (1,2) = 1\cdot e_1 + 2e_2$ $\quad$ $(1,2)$ coord în rap cu R_0.

$x = x_1'\cdot e_1' + x_2' e_2' = x_1'(2,1) + x_2'(3,0) = (2x_1' + 3x_2', x_1')$

$\overset{\shortparallel}{(1,2)}$

$\begin{cases} 2x_1' + 3x_2' = 1 \\ x_1' = 2 \end{cases} \Rightarrow x_2' = \frac{1}{3}(1-4) = -1$ $\quad$ $(2,-1)$ coord. lui x în rap. cu reperul R'

SAU $\quad X = AX'$ $\quad \begin{pmatrix}1\\2\end{pmatrix} = \begin{pmatrix}2 & 3\\1 & 0\end{pmatrix}\begin{pmatrix}2\\-1\end{pmatrix}$

$X' = A^{-1}X$

Operații cu subspații vectoriale

$(V,+,\cdot)/\mathbb{K}$ sp.v, f. generat, $V' \subset V$ subm. nevidă

$V' \subset V$ subspațiu vectorial $\Leftrightarrow \forall x_1,..,x_n \in V'$, $a_1,..,a_n \in \mathbb{K}$: $a_1x_1 + ... + a_nx_n \in V'$

Prop $V_1, V_2 \subset V$ subsp. vect $\Rightarrow V_1 \cap V_2$ subsp. vectorial.

Dem $\forall x, y \in V_1 \cap V_2$, $\forall a, b \in \mathbb{K}$ $\quad$ $x,y \in V_1$ și $x,y \in V_2$

$\Downarrow$ $\quad$ $\Downarrow$

$ax+by \in V_1$ și $ax+by \in V_2$

$\Rightarrow ax+by \in V_1 \cap V_2$.

OBS În general, $V_1 \cup V_2$ nu e subspațiu vect

Considerăm $\underset{\text{not } V_1+V_2}{\langle V_1 \cup V_2 \rangle} = \left\{ \sum_{i=1}^{n} a_i x_i \,,\, x_i \in V_1 \cup V_2,\, a_i \in \mathbb{K},\, i = \overline{1,n} \right\}$

$\mathcal{A}_{33}$

Prop $\underset{\langle V_1 \cup V_2 \rangle}{V_1 \overset{\shortparallel}{} + V_2} = \{ v_1 + v_2 \mid v_1 \in V_1, v_2 \in V_2 \}$

Dem

„$\subseteq$" Fie $v \in V_1 + V_2 = \langle V_1 \cup V_2 \rangle$

$v = \sum_{i=1}^{m} a_i v_i \quad , \quad v_i \in V_1 \cup V_2, i = \overline{1,m}$

Renumerotăm indicii ai $v_1, .., v_k \in V_1$

$v_{k+1}, .., v_m \in V_2$.

$$v = \underbrace{\sum_{i=1}^{k} a_i v_i}_{v_1 \in V_1} + \underbrace{\sum_{j=k+1}^{m} a_j v_j}_{v_2 \in V_2} = v_1 + v_2 .$$

„$\supseteq$" $\underset{v_1 \in V_1, v_2 \in V_2}{v_1 + v_2} \in \langle V_1 \cup V_2 \rangle$

(combinație liniară particulară).

Def. $(V, +, \cdot)/\mathbb{K}$, $V_1, V_2 \subseteq V$ subsp. vect.

Spunem că $V_1 + V_2$ este sumă directă și notăm $V_1 \oplus V_2$

dacă $V_1 \cap V_2 = \{0_V\}$

Prop

$V_1 + V_2$ este sumă directă $\Leftrightarrow \forall x \in V_1 + V_2 \Rightarrow \exists! \, v_1 \in V_1, v_2 \in V_2$ ai $x = v_1 + v_2$

Dem

„$\Rightarrow$" $V_1 \oplus V_2 \Leftrightarrow V_1 \cap V_2 = \{0_V\}$

Pp. abs $v_1, v_1' \in V_1$, $v_2, v_2' \in V_2$ ai $\underset{V_1+V_2}{\overset{\cap}{x}} = v_1 + v_2 = v_1' + v_2'$

$\underset{V_1}{v_1 - v_1'} = \underset{V_2}{v_2' - v_2} \in \underset{\{0_V\}}{V_1 \cap V_2}$

$v_1 - v_1' = 0_V \Rightarrow v_1' = v_1$

$v_2 - v_2' = 0_V \Rightarrow v_2' = v_2$

„$\Leftarrow$" Pp. abs $\exists \, \overset{0_V}{\underset{}{u}} \neq \in V_1 \cap V_2$

$x = v_1 + v_2 = \underset{V_1}{(v_1 + u)} + \underset{V_2}{(v_2 - u)}$ scrierea nu e unică

Contradicție.

Pp. este falsă $V_1 \cap V_2 = \{0_V\}$