Seminar 13

1. Det $\int_{-\infty}^{\infty} \frac{x}{1+x^4} dx$

Sol $\left(\int_{-\infty}^{\infty} \frac{x}{1+x^4} dx = \int_{-\infty}^{0} \frac{x}{1+x^4} dx + \int_{0}^{\infty} \frac{x}{1+x^4} dx \right)$

$$\int_{-\infty}^{0} \frac{x}{1+x^4} dx = \lim_{c \to -\infty} \int_{c}^{0} \frac{x}{1+x^4} dx =$$

$$\lim_{c \to -\infty} \int_{c}^{0} 2 \cdot \frac{1}{2} \cdot \frac{x}{1+x^4} dx = \lim_{c \to -\infty} \frac{1}{2} \int_{c^2}^{0} \frac{1}{1+t^2} dt$$

Notăm $x^2 = t$

$2x\,dx = dt$

$x = c \Rightarrow t = c^2$

$x = 0 \Rightarrow t = 0$

$$= \lim_{c \to -\infty} \frac{1}{2} \operatorname{arctg} t \Big|_{c^2}^{0}$$

$$= \lim_{c \to -\infty} \left(\frac{1}{2} \operatorname{arctg} 0 - \frac{1}{2} \operatorname{arctg} c^2 \right)$$

$$= 0 - \frac{1}{2} \cdot \frac{\pi}{2}$$

$$= -\frac{\pi}{4} \quad \text{(conv)}$$

$$\int_{0}^{\infty} \frac{x}{1+x^4} dx = \lim_{c \to \infty} \int_{0}^{c} \frac{x}{1+x^4} dx$$

Notăm: $x^2 = t$

$2x\,dx = dt$

$x = 0 \to t = 0$

$x = c \to t = c^2$

$$\lim_{c \to \infty} \frac{1}{2} \int_{0}^{c^2} \frac{1}{1+t^2} = \lim_{c \to \infty} \frac{1}{2} \operatorname{arctg} t \Big|_{0}^{c^2}$$

$$= \frac{1}{2} \left(\frac{\pi}{2} - 0 \right) = \frac{\pi}{4} \quad \text{(conv)}$$

$$\int_{-\infty}^{\infty} \frac{x}{1+x^4}\,dx = \int_{-\infty}^{0} \frac{x}{1+x^4}\,dx + \int_{0}^{\infty} \frac{x}{1+x^4}\,dx$$

$$= -\frac{\pi}{4} + \frac{\pi}{4} = 0$$

2. Studiați convergența/natura urm. integrale improprii

a) $\int_1^{\infty} \frac{1}{x^4+1}\,dx$

Sol: Fie $f, g : [1;\infty) \to [0;\infty)$ $f(x) = \frac{1}{x^4+1}$

$g(x) = \frac{1}{x^4}$

Avem $0 \le f(x) \le g(x)$ $\forall x \in [1;\infty)$

$$\int_1^{\infty} g(x)\,dx = \int_1^{\infty} \frac{1}{x^4}\,dx = \lim_{d\to\infty}\int_1^{d} x^{-4}\,dx = \lim_{d\to\infty} \frac{x^{-3}}{-3}\Big|_1^{d}$$

$$= \lim_{d\to\infty}\left(-\frac{1}{3x^3}\right)\Big|_1^{d} = \lim_{d\to\infty}\left(-\frac{1}{3d^3}+\frac{1}{3}\right) = \frac{1}{3} \in \mathbb{R}$$

Deci, $\int_1^{\infty} g(x)dx$ e conv.

Cf crit de comp cu ~~f~~ ineg avem că $\int_1^{\infty} f(x)\,dx$ e conv. $\square$

b) $\int_2^{\infty} \frac{1}{\sqrt{x}-1}\,dx$

Sol Fie $f, g : [2;\infty) \to (0;\infty)$ $f(x) = \frac{1}{\sqrt{x}-1}$

$g(x) = \frac{1}{\sqrt{x}}$

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = \lim_{x\to\infty} \frac{\sqrt{x}}{\sqrt{x}-1} = 1 \in (0;\infty)$$

Cf crit de comp cu limită avem că

$$\int_2^{\infty} f(x)\,dx \sim \int_2^{\infty} g(x)\,dx$$

$$\int_2^\infty g(x)\,dx = \lim_{d\to\infty}\int_2^d x^{-\frac{1}{2}}\,dx = \lim_{d\to\infty} 2\sqrt{x}\Big|_2^\infty$$

$$= \lim_{d\to\infty}\left(2\sqrt{d} - 2\sqrt{2}\right) = +\infty$$

Deci, $\int_2^\infty g(x)\,dx$ e div.

Cf crit de comp cu limită avem că $\int_2^\infty f(x)\,dx$ e div. $\square$

c) $\int_1^\infty \frac{1}{\sqrt{x}+1}\,dx$ comp cu limită

Sol: Rez. voi !!! $\square$

d) $\int_1^\infty \sin\frac{1}{x^{11}}\,dx$ $\quad \frac{1}{x^{11}} \le \sin\frac{1}{x^{11}}$

Sol

$\frac{1}{x^{11}} \in (0;1] \;\forall\, x\in[1;\infty) \Rightarrow \sin\frac{1}{x^{11}} > 0 \;\forall\, x\in[1;\infty)$

$(0;1] \subset (0;\frac{\pi}{2})$

Fie $f:[1;\infty)\to(0;\infty)$ $f(x) = \sin\frac{1}{x^4}$

$x \in (0;\frac{\pi}{2}) \longrightarrow \sin x \in \mathbb{R}$ e cresc

$x \in [1;\infty) \longrightarrow \frac{1}{x^{11}}$ e descresc

$\Rightarrow f$ descresc

Cf crit integral al lui Cauchy avem că

$$\int_1^\infty f(x)\,dx \sim \sum_{n=1}^\infty f(n)$$

$$\sum_{n=1}^\infty f(n) = \sum_{n=1}^\infty \sin\frac{1}{n^{11}}$$

Fie $x_n = \sin \frac{1}{n^{11}}, \forall n \in \mathbb{N}^*$

$y_n = \frac{1}{n^{11}}, \forall n \in \mathbb{N}^*$

$$\lim_{n\to\infty} \frac{x_n}{y_n} = \lim_{n\to\infty} \frac{\sin \frac{1}{n^{11}}}{\frac{1}{n^{11}}} = 1 \in (0;\infty)$$

Cf crit de comp cu limită (pt serii) avem că

$$\sum_{n=1}^{\infty} x_n \sim \sum_{n=1}^{\infty} y_n$$

$\sum_{n=1}^{\infty} y_n = \sum_{n=1}^{\infty} \left(\frac{1}{n^{11}}\right)$ conv (serie armonică gen cu $\alpha = 11$)

Deci $\sum_{n=1}^{\infty} x_n$ e conv. Așadar $\int_1^{\infty} f(x)\,dx$ e conv

3. Folosind, eventual, fct Γ și β det:

$$\Gamma(x) = \int_0^{\infty} t^{x-1} e^{-t}\,dt$$

$$\beta(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1}\,dt$$

a) $\int_0^{\infty} e^{-x^2}\,dx = \frac{1}{2}\int_0^{\infty} t^{-\frac{1}{2}} \cdot e^{-t}\,dt$

S.V $= x^2 = t$

$\Downarrow$

$x = \sqrt{t}$

$dx = \frac{1}{2\sqrt{t}}\,dt$

$x = 0 \to t = 0$

$x = \infty \to t = \infty$

$$= \frac{1}{2}\int_0^{\infty} t^{\frac{1}{2}-1} \cdot e^{-t}\,dt = \frac{1}{2}\Gamma\left(\frac{1}{2}\right) = \frac{\sqrt{\pi}}{2}$$

b) $\int_{-\infty}^{\infty} e^{-x^2} dx$

Sol : Rez voi ! $\square$

c) $\int_0^{\infty} x^6 e^{-2x} dx = \frac{1}{2}\int_0^{\infty} \frac{t^6}{2^6} \cdot e^{-t} dt$

notam $2x = t \Rightarrow x = \frac{t}{2}$

$dx = \frac{1}{2} dt$

$x = 0 \to t = 0$

$x = \infty \to t = \infty$

$= \frac{1}{2^7}\int_0^{\infty} t^{7-1} e^{-t} dt = \frac{1}{2^7}\Gamma(7) = \frac{6!}{2^7}$

d) $\int_0^{\infty} \sqrt{x} \cdot e^{-x^3} dx = \frac{1}{3}\int_0^{\infty} t^{\frac{1}{6}} \cdot e^{-t} \cdot t^{-\frac{2}{3}} dt$

notam $t = x^3 \Rightarrow x = t^{\frac{1}{3}}$

$dx = \frac{1}{3} t^{-\frac{2}{3}} dt$

$x = 0 \to t = 0$

$x = \infty \to t = \infty$

$= \frac{1}{3}\int_0^{\infty} t^{\frac{1}{6}-\frac{2}{3}} \cdot e^{-t} dt$

$= \frac{1}{3}\int_0^{\infty} t^{-\frac{1}{2}} e^{-t} dt = \frac{1}{3}\int_0^{\infty} t^{\frac{1}{2}-1} e^{-t} dt$

$= \frac{1}{3} \cdot \Gamma\left(\frac{1}{2}\right) = \frac{\sqrt{\pi}}{3}$ $\square$

e) $\int_0^2 \frac{x^2}{\sqrt{2-x}} dx = \int_0^2 \frac{x^2}{\sqrt{2\left(1-\frac{x}{2}\right)}} dx =$

notam $\frac{x}{2} = t \Rightarrow x = 2t$

$dx = 2dt$

$x = 0 \to t = 0$

$x \to 2 \to t \to 1$

$$= \int_0^1 \frac{4t^2}{\sqrt{2(1-t)}} \cdot 2 \cdot dt = \frac{8}{\sqrt{2}} \int_0^1 t^2 (1-t)^{-\frac{1}{2}}$$

$$= \frac{8}{\sqrt{2}} \int_0^1 t^{3-1} (1-t)^{\frac{1}{2}-1} dt \, dt = \frac{8}{\sqrt{2}} B(3, \tfrac{1}{2})$$

$$B(3, \tfrac{1}{2}) = \frac{\Gamma(3)\Gamma(\frac{1}{2})}{\Gamma(3+\frac{1}{2})} = \frac{2!\sqrt{\pi}}{\Gamma(3+\frac{1}{2})} =$$

$$\Gamma(3+\tfrac{1}{2}) = \Gamma(1+\underbrace{2+\tfrac{1}{2}}) = (2+\tfrac{1}{2})\Gamma(2+\tfrac{1}{2})$$

$$= \frac{5}{2}\Gamma(1+\underbrace{\tfrac{1}{2}+1}) = \frac{5}{2}(\tfrac{1}{2}+1)\Gamma(1+\tfrac{1}{2})$$

$$= \frac{5}{2} \cdot \frac{3}{2} \cdot \frac{1}{2}\Gamma(\tfrac{1}{2}) = \frac{15}{8} \cdot \sqrt{\pi}$$

$$B(3, \tfrac{1}{2}) = \frac{2!\cancel{\sqrt{\pi}}}{15 \cdot \cancel{\sqrt{\pi}}} \cdot 8 = \frac{16}{15}$$

$$\int_0^2 \frac{x^2}{\sqrt{2-x}} dx = \frac{8}{\sqrt{2}} \cdot \frac{16}{15} \overset{8\sqrt{2}}{=} \frac{64\sqrt{2}}{15}$$

f) $\int_0^{\frac{\pi}{2}} (\sin t)^{\frac{5}{2}} (\cos t)^{\frac{3}{2}} dt$

Sol: $\begin{cases} 2x-1 = \frac{5}{2} \Rightarrow 2x = \frac{7}{2} \Rightarrow x = \frac{7}{4} \\ 2y-1 = \frac{3}{2} \Rightarrow 2y = \frac{5}{2} \Rightarrow y = \frac{5}{4} \end{cases}$

$$\int_0^{\frac{\pi}{2}} (\sin t)^{\frac{5}{2}} (\cos t)^{\frac{3}{2}} dt = \frac{1}{2} \cdot \underbrace{2\int_0^{\frac{\pi}{2}} (\sin t)^{2\cdot\frac{7}{4}-1} (\cos t)^{2\cdot\frac{5}{4}-1} dt}_{\beta(\frac{7}{4}; \frac{5}{4})}$$

$$= \frac{1}{2}\beta(\tfrac{7}{4}; \tfrac{5}{4})$$

$$\beta\left(\frac{7}{4}, \frac{5}{4}\right) = \frac{\Gamma\left(\frac{7}{4}\right)\Gamma\left(\frac{5}{4}\right)}{\Gamma(3)} = \frac{\Gamma\left(\frac{7}{4}\right)\Gamma\left(\frac{5}{4}\right)}{2!}$$

$$\Gamma\left(\frac{7}{4}\right) = \Gamma\left(1+\frac{3}{4}\right) = \frac{3}{4}\Gamma\left(\frac{3}{4}\right)$$

$$\Gamma\left(\frac{5}{4}\right) = \Gamma\left(1+\frac{1}{4}\right) = \frac{1}{4}\Gamma\left(\frac{1}{4}\right)$$

$$\Gamma\left(\frac{7}{4}\right)\Gamma\left(\frac{5}{4}\right) = \frac{3}{4}\Gamma\left(\frac{3}{4}\right)\frac{1}{4}\Gamma\left(\frac{1}{4}\right) = \frac{3}{16}\Gamma\left(1-\frac{1}{4}\right)$$

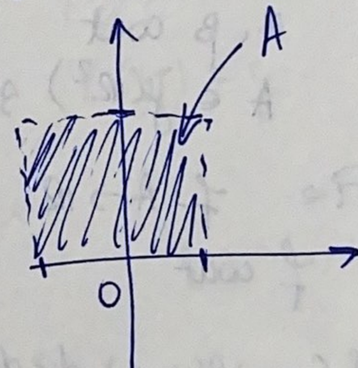
$$\Gamma\left(\frac{1}{4}\right) = \frac{3}{16}\cdot\frac{\pi}{\sin\frac{\pi}{4}} = \frac{3}{16}\cdot\frac{\pi}{\cancel{\sqrt{2}}}\times\sqrt{2} = \frac{3\pi\cdot\sqrt{2}}{16}$$

$$\beta\left(\frac{7}{4}; \frac{5}{4}\right) = \frac{3\pi\sqrt{\cancel{2}}}{16\cdot\cancel{2}} = \frac{3\pi\sqrt{2}}{32}$$

$$\int_0^{\frac{\pi}{2}} (\sin t)^{5/2}(\cos t)^{3/2}\,dt = \frac{1}{2}\beta\left(\frac{7}{4}; \frac{5}{4}\right) = \frac{1}{2}\,\frac{3\pi\sqrt{2}}{32} = \frac{3\sqrt{2}\,\pi}{64}$$

4. Determinați

a) $\iint_A y\,dx\,dy$, unde $A = [-1,1]\times[0,2]$



$A = [-1,1]\times[0,2]$

$\Rightarrow A \in \mathcal{J}(\mathbb{R}^2)$ şi A compactă

Fie $f: A \to \mathbb{R}$ $f(x,y) = y$

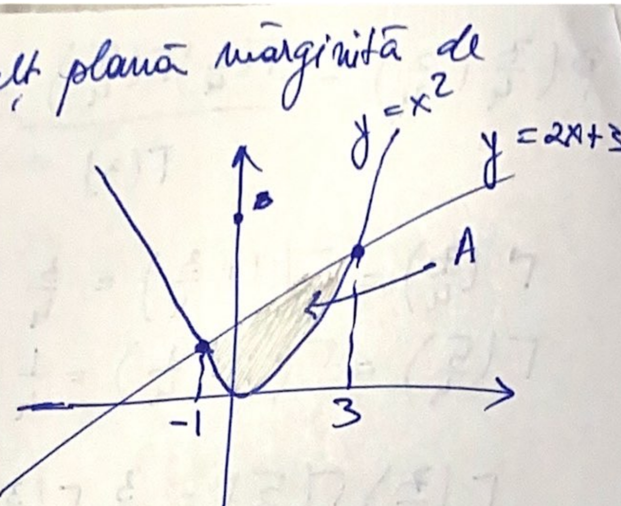
Avem $\iint_A f(x,y)\,dx\,dy = \int_{-1}^{1}\left(\int_0^2 y\,dy\right)dx$

$$= \int_{-1}^{1} \frac{y^2}{2}\Big|_{y=0}^{y=2} dx = \int_{-1}^{1} 2\,dx = 2x\Big|_{x=-1}^{x=1}$$

$$= 2-(-2) = 4$$

b) $\iint_A x\,dx\,dy$, unde A e mulț. plană mărginită de

$y = x^2$ și $y = 2x+3$



$x^2-2x-3=0$

$(x-3)(x+1)=0$

Facem proiectie pe OX

Det. pct de intersecție dintre parabolă și dreaptă

$$\begin{cases} y = x^2 \\ y = 2x+3 \end{cases} \Leftrightarrow x^2 = 2x+3 \Rightarrow x^2-2x-3=0$$

$(x-3)(x+1)=0$

$\Rightarrow x_1 = 3 \Rightarrow y_1 = 9$

$x_2 = -1 \Rightarrow y_2 = 1$

$A = \{(x,y) \in \mathbb{R}^2 \mid x \in [-1;3],\ x^2 \le y \le 2x+3\}$

Fie $\alpha, \beta\ [-1;3] \to \mathbb{R}$ $\alpha(x) = x^2$ $\beta(x) = 2x+3$

α, β cont

$A \in J(\mathbb{R}^2)$ și A compactă

Fie $f: A \to \mathbb{R}$ $f(x,y) = x$

f cont

$$\iint_A f(x,y)\,dx\,dy = \int_{-1}^{3}\left(\int_{x^2}^{2x+3} x\,dy\right)dx$$

$$= \int_{-1}^{3} (x\cdot y)\Big|_{y=x^2}^{y=2x+3} dx = \int_{-1}^{3} x\left(2x+3-x^2\right)dx$$

$$= \int_{-1}^{3} -x^3+2x^2+3x\,dx = -\frac{x^4}{4}\Big|_{x=-1}^{x=3} +$$

$$2\,\frac{x^3}{3}\Big|_{-1}^{3} + 3\,\frac{x^2}{2}\Big|_{-1}^{3} = -20+\frac{56}{3}+12 = \frac{32}{3} \quad \square$$

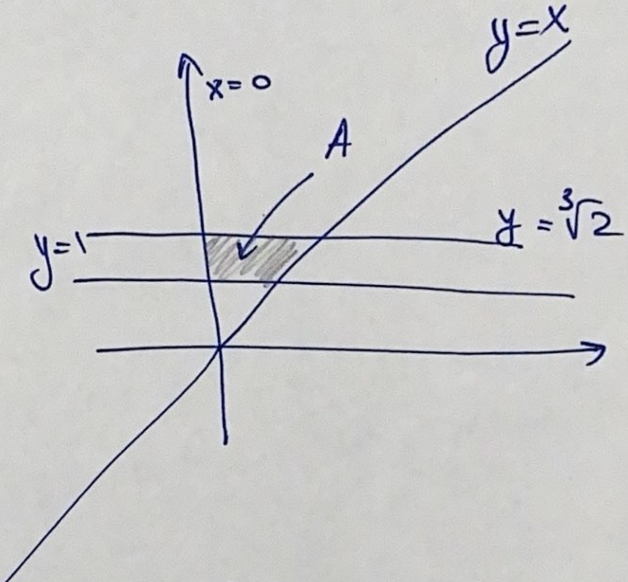
c) $\iint_A x \, dx dy$, unde A este mulț. plană limitată de

$y = -x^2 - x + 2$ și $y = x - 1$

Sol Rez. voi! $\square$

d) $\iint_A x \, dx dy$, unde A e mulț plană mărg de

$x = 0$, $y = 1$, $y = \sqrt[3]{2}$, $y = x$



Sol

Proiecție pe OX: pe y
îl fix a și tb spartă
integrala

$A = \{ (x,y) \in \mathbb{R}^2 \mid y \in [1, \sqrt[3]{2}],$
$0 \le x \le y \}$

Fie $\varphi, \psi : [1, \sqrt[3]{2}] \to \mathbb{R}$ $\varphi(y) = 0$
$\psi(y) = y$

$A \in \mathcal{J}(\mathbb{R}^2)$ și A compactă

Fie $f : A \to \mathbb{R}$ $f(x,y) = x$

f cont

$$\iint_A f(x,y) \, dx \, dy = \int_1^{\sqrt[3]{2}} \left(\int_0^y x \, dx \right) dy$$

$$= \int_1^{\sqrt[3]{2}} \left(\frac{x^2}{2} \Big|_{x=0}^{x=y} \right) dy = \int_1^{\sqrt[3]{2}} \frac{y^2}{2} \, dy$$

$$= \frac{1}{2} \frac{y^3}{3} \Big|_{y=1}^{y=\sqrt[3]{2}} = \frac{1}{2 \cdot 3} (2-1) = \frac{1}{6} \quad \square$$