(C13)

Conice - formă canonică

$(\mathbb{R}^2, (\mathbb{R}^2, g_0), \varphi)$ sp. afin euclidian cu str. afină canonică

$\Gamma: \quad f(x) = a_{11}x_1^2 + 2a_{12}x_1x_2 + a_{22}x_2^2 + 2b_1x_1 + 2b_2x_2 + c = 0.$

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{12} & a_{22} \end{pmatrix} = A^T \ (\operatorname{rg} A \geq 1), \quad \tilde{A} = \begin{pmatrix} a_{11} & a_{12} & b_1 \\ a_{12} & a_{22} & b_2 \\ b_1 & b_2 & c \end{pmatrix}$$

$\delta = \det A, \quad \Delta = \det \tilde{A}.$

I. $\delta \neq 0 \quad \exists!\ P_0$ centrul conicei

II. $\delta = 0 \quad$ (Γ nu are centru unic)

$$R = \{O; e_1, e_2\} \xrightarrow[\text{rotatie}]{\tau} R' = \{O; e_1', e_2'\} \xrightarrow[\text{translatie}]{\theta} R'' = \{P_0; e_1', e_2'\}.$$

$Q: \mathbb{R}^2 \to \mathbb{R}, \quad Q(x) = a_{11}x_1^2 + 2a_{12}x_1x_2 + a_{22}x_2^2.$

$\det(A - \lambda I_2) = 0 \Leftrightarrow \lambda^2 - \operatorname{Tr}(A)\lambda + \underset{0}{\underset{\parallel}{\det(A)}} = 0$

$\lambda(\lambda - \operatorname{Tr}(A)) = 0 \qquad \lambda_1 = \operatorname{Tr}(A) \neq 0, \ \lambda_2 = 0.$

$V_{\lambda_1} = \langle\{e_1'\}\rangle, \ V_{\lambda_2} = \langle\{e_2'\}\rangle \qquad e_1', e_2'$ versori proprii

$\tau: X = RX' \qquad R = \begin{pmatrix} l_1 & l_2 \\ m_1 & m_2 \end{pmatrix} \in O(2); \ e_k' = (l_k \ m_k), \ k = \overline{1,2}$

Alegem $R \in SO(2)$

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} l_1 & l_2 \\ m_1 & m_2 \end{pmatrix} \begin{pmatrix} x_1' \\ x_2' \end{pmatrix} \Rightarrow \begin{aligned} x_1 &= l_1x_1' + l_2x_2' \\ x_2 &= m_1x_1' + m_2x_2' \end{aligned}$$

$\tau(\Gamma): \quad \lambda_1 x_1'^2 + 2b_1(l_1x_1' + l_2x_2') + 2b_2(m_1x_1' + m_2x_2') + c = 0$

$$\lambda_1 x_1'^2 + 2\underbrace{(b_1l_1 + b_2m_1)}_{b_1'}x_1' + 2\underbrace{(b_1l_2 + b_2m_2)}_{b_2'}x_2' + c = 0$$

$\tau(\Gamma): \quad \underline{\lambda_1 x_1'^2} + 2b_1'x_1' + 2b_2'x_2' + c = 0 \qquad \tilde{A} = \begin{pmatrix} \boxed{\begin{matrix} \lambda_1 & 0 \\ 0 & 0 \end{matrix}} & \begin{matrix} b_1' \\ b_2' \end{matrix} \\ \begin{matrix} b_1' & b_2' \end{matrix} & c \end{pmatrix}$

$\Delta = -b_2'^2 \lambda_1 \qquad (\lambda_1 \neq 0)$

a) $\Delta \neq 0 \Rightarrow b_2' \neq 0$

$\theta(\Gamma): \lambda_1 x_1'^2 + 2b_1' x_1' + 2b_2' x_2' + c = 0$

$\lambda_1\left(x_1'^2 + 2\cdot\frac{b_1'}{\lambda_1}x_1' + \left(\frac{b_1'}{\lambda_1}\right)^2\right) + 2b_2' x_2' + c' = 0$

$\Rightarrow \lambda_1\left(x_1' + \frac{b_1'}{\lambda_1}\right)^2 + 2b_2'\left(x_2' + \frac{c'}{2b_2'}\right) = 0$

$$\begin{cases} x_1'' = x_1' + \frac{b_1'}{\lambda_1} \\ x_2'' = x_2' + \frac{c'}{2b_2'} \end{cases} \Rightarrow \tau : \begin{pmatrix} x_1' \\ x_2' \end{pmatrix} = \begin{pmatrix} x_1'' \\ x_2'' \end{pmatrix} + \begin{pmatrix} -\frac{b_1'}{\lambda_1} \\ -\frac{c'}{2b_2'} \end{pmatrix}$$

$X = RX'$ $\qquad X' = X'' + X_0$ (translația)

$\tau \circ \theta : X = RX'' + RX_0 \qquad RX_0 = \begin{pmatrix} \alpha \\ \beta \end{pmatrix},\ P_0(\alpha, \beta)$

$\tau(\theta(\Gamma)) : \lambda_1 x_1''^2 + 2b_1' x_2'' = 0$ parabolă.

b) $\Delta = 0$ (Γ degenerată) $\Rightarrow b_2' = 0$

$\theta(\Gamma): \lambda_1 x_1'^2 + 2b_1' x_1' + c = 0$

$\lambda_1\left(x_1' + \frac{b_1'}{\lambda_1}\right)^2 + c' = 0$

$$\begin{cases} x_1'' = x_1' + \frac{b_1'}{\lambda_1} \\ x_2'' = x_2' \end{cases} \Rightarrow \tau : \begin{pmatrix} x_1' \\ x_2' \end{pmatrix} = \begin{pmatrix} x_1'' \\ x_2'' \end{pmatrix} + \begin{pmatrix} -\frac{b_1'}{\lambda_1} \\ 0 \end{pmatrix}$$

$X' = X'' + X_0.$

$\tau(\theta(\Gamma)): \lambda_1 x_1''^2 + c' = 0$

b1) $c' = 0$ dreaptă dublă

b2) $c' \neq 0 \Rightarrow x_1''^2 = \frac{-c'}{\lambda_1}$ $\quad \emptyset$, drepte ∥.

<u>OBS</u> $\delta \neq 0.$ $\qquad \lambda_1 x_1''^2 + \lambda_2 x_2''^2 + \frac{\Delta}{\delta} = 0.$

a) $\Delta \neq 0$ $\quad \delta = \lambda_1\lambda_2 > 0$ Elipsă, $\emptyset$
$\qquad \delta = \lambda_1\lambda_2 < 0$ Hiperbolă

b) $\Delta = 0$ $\quad \lambda_1 x_1''^2 + \lambda_2 x_2''^2 = 0$ $\quad \delta > 0$ Pct dublu
$\qquad \delta < 0$ 2 drepte concurente

Δ (natura)	δ (genul)	Tipul conicei
$\Delta \neq 0$	$\delta > 0$	ϕ, Elipsă
	$\delta < 0$	Hiperbolă
	$\delta = 0$	Parabolă
$\Delta = 0$	$\delta > 0$	Pct dublu
	$\delta < 0$	Drepte concurente.
	$\delta = 0$	ϕ, drepte //

Exemplu ($\delta = 0$)

În planul euclidian considerăm conica:

$$\Gamma: f(x) = \underline{x_1^2 - 4x_1x_2 + 4x_2^2} - \underline{6x_1 + 2x_2} + 1 = 0.$$

Să se aducă la o formă canonică, utilizând izometrii. Reprezentare grafică.

SOL

$$A = \begin{pmatrix} 1 & -2 \\ -2 & 4 \end{pmatrix}, \delta = \det A = 0, \tilde{A} = \begin{pmatrix} 1 & -2 & -3 \\ -2 & 4 & 1 \\ -3 & 1 & 1 \end{pmatrix}$$

$$\Delta = \begin{vmatrix} -8 & 1 & -3 \\ 1 & 3 & 1 \\ 0 & 0 & 1 \end{vmatrix} = -24 - 1 = -25 \neq 0$$

$$R = \{O; e_1, e_2\} \xrightarrow[\text{rotatie}]{\tau} R' = \{O; e_1', e_2'\} \xrightarrow[\text{translatie}]{\theta} R'' = \{P_0; e_1', e_2'\}$$

$\tau: X = RX'$

Fie $Q: \mathbb{R}^2 \to \mathbb{R}$, $Q(x) = x_1^2 - 4x_1x_2 + 4x_2^2$

$\lambda^2 - 5\lambda = 0 \qquad \lambda_1 = 5, \lambda_2 = 0$

$V_{\lambda_1} = \{x \in \mathbb{R}^2 \mid AX = 5X\} = \{(x_1, -2x_1) / x_1 \in \mathbb{R}\} = <\{(1,-2)\}$

$$(A - 5I_2)X = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \begin{pmatrix} -4 & -2 \\ -2 & -1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \qquad \begin{matrix} -2x_1 - x_2 = 0 \\ x_2 = -2x_1 \end{matrix}$$

$$e_1' = \frac{1}{\sqrt{5}}(1, -2)$$

$V_{\lambda_2} = \{x \in \mathbb{R}^2 \mid AX = 0\} = \{(2x_2, x_2) \mid x_2 \in \mathbb{R}\} = <\{(2,1)\}>$

$x_1 - 2x_2 = 0 \Rightarrow x_1 = 2x_2$

$$e_2' = \frac{1}{\sqrt{5}}(2, 1)$$

$\tau : X = \frac{1}{\sqrt{5}}\begin{pmatrix} 1 & 2 \\ -2 & 1 \end{pmatrix} X' \Rightarrow \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \frac{1}{\sqrt{5}}\begin{pmatrix} 1 & 2 \\ -2 & 1 \end{pmatrix}\begin{pmatrix} x_1' \\ x_2' \end{pmatrix}$

$R = \frac{1}{\sqrt{5}}\begin{pmatrix} 1 & 2 \\ -2 & 1 \end{pmatrix} \in SO(2)$

$x_1 = \frac{1}{\sqrt{5}}(x_1' + 2x_2')$

$x_2 = \frac{1}{\sqrt{5}}(-2x_1' + x_2')$

$\Gamma : f(x_1, x_2) = \underbrace{x_1^2 - 4x_1x_2 + 4x_2^2} - 6x_1 + 2x_2 + 1 = 0$

$\tau(\Gamma) : \quad 5x_1'^2 - \frac{6}{\sqrt{5}}(x_1' + 2x_2') + \frac{2}{\sqrt{5}}(-2x_1' + x_2') + 1 = 0$

$5x_1'^2 - \frac{\cancel{10}^{2\sqrt{5}}}{\cancel{\sqrt{5}}}x_1' - \frac{\cancel{10}^{2\sqrt{5}}}{\cancel{\sqrt{5}}}x_2' + 1 = 0$

$x_1'^2 - \frac{2}{\sqrt{5}}x_1' - \frac{2}{\sqrt{5}}x_2' + \frac{1}{5} = 0.$

$\left(x_1' - \frac{1}{\sqrt{5}}\right)^2 - \frac{2}{\sqrt{5}}x_2' = 0.$

Fie $\begin{cases} x_1'' = x_1' - \frac{1}{\sqrt{5}} \\ x_2'' = x_2' \end{cases}$ $\quad \theta : \begin{pmatrix} x_1' \\ x_2' \end{pmatrix} = \begin{pmatrix} x_1'' \\ x_2'' \end{pmatrix} + \begin{pmatrix} \frac{1}{\sqrt{5}} \\ 0 \end{pmatrix}$

$X' = X'' + X_0.$

$\theta \circ \tau : X = RX'' + RX_0$

$RX_0 = \frac{1}{\sqrt{5}}\begin{pmatrix} 1 & 2 \\ -2 & 1 \end{pmatrix}\begin{pmatrix} \frac{1}{\sqrt{5}} \\ 0 \end{pmatrix} = \frac{1}{5}\begin{pmatrix} 1 & 2 \\ -2 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{1}{5}\begin{pmatrix} 1 \\ -2 \end{pmatrix}$

$P_0\left(\frac{1}{5}, -\frac{2}{5}\right).$

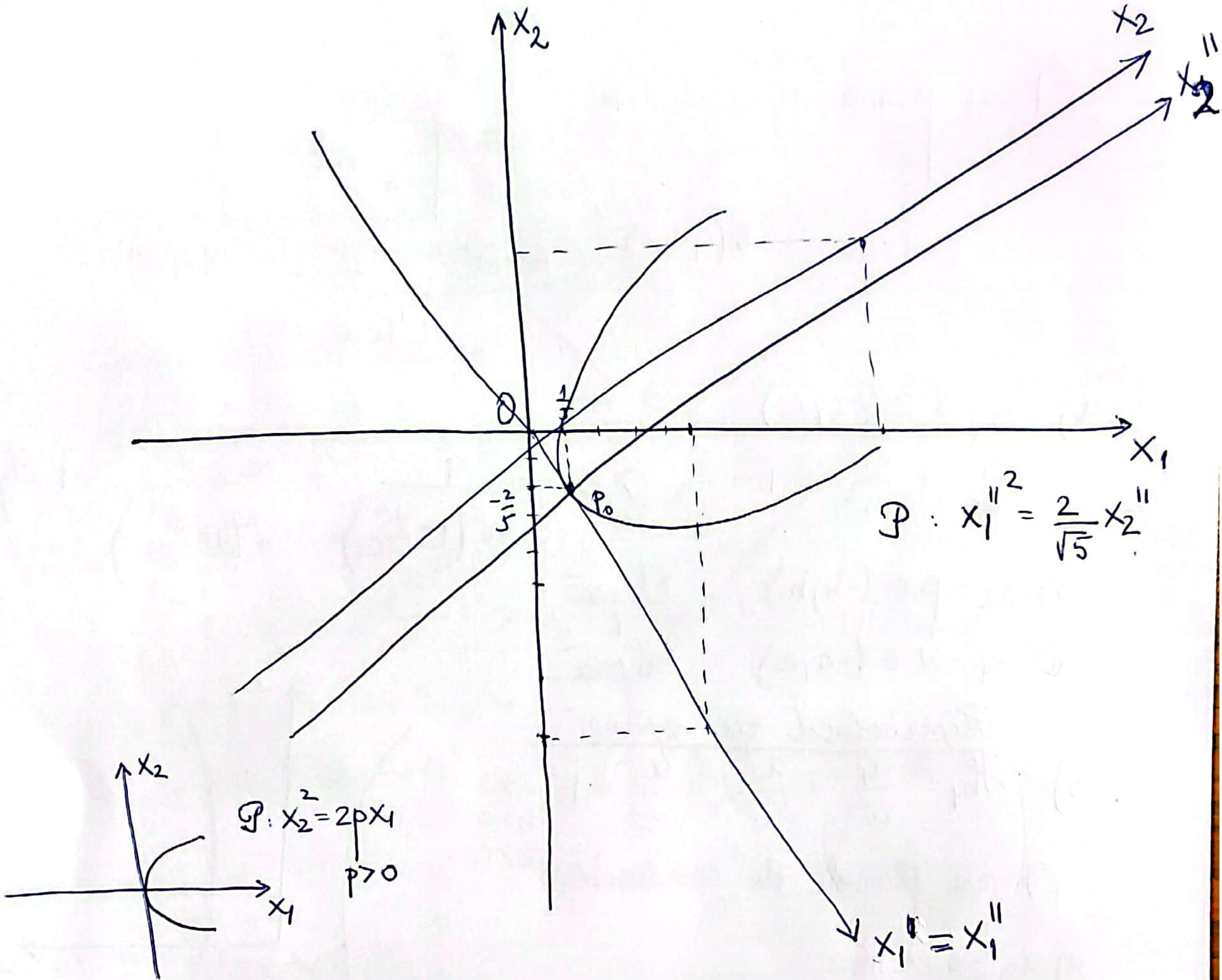
$\theta(\tau(\Gamma)) : \; x_1''^2 - \frac{2}{\sqrt{5}}x_2'' = 0$ Parabolă

OBS $R = \frac{1}{\sqrt{5}}\begin{pmatrix} 1 & 2 \\ -2 & 1 \end{pmatrix} = \begin{pmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{pmatrix}$ $\quad \sin\varphi = -\frac{2}{\sqrt{5}}$

$\cos\varphi = \frac{1}{\sqrt{5}}$

$\varphi = -\operatorname{arctg} 2, \quad \varphi \in [-\pi, \pi].$ $\quad \operatorname{tg}\varphi = -2$

$e_1' = \frac{1}{\sqrt{5}}(1,-2)$, $e_2' = \frac{1}{\sqrt{5}}(2,1)$, $P_0\left(\frac{1}{5}, -\frac{2}{5}\right)$



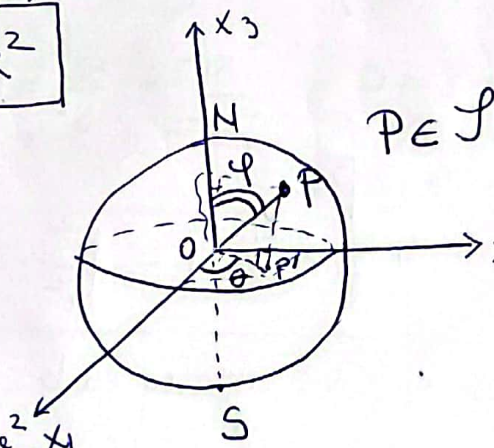
ec. gen.

Cuadrice studiate pe ecuații reduse

① **Sfera** $\mathcal{S}(A(a,b,c),R): (x_1-a)^2 + (x_2-b)^2 + (x_3-c)^2 = R^2$.

Dacă $A = O$: $\boxed{x_1^2 + x_2^2 + x_3^2 = R^2}$

$\theta \in [0,2\pi)$, $\varphi \in [0,\pi]$



$P \in \mathcal{S}(O;R)$

$P(R\sin\varphi\cos\theta, R\sin\varphi\sin\theta, R\cos\varphi)$

$\cap$ cu planele de coordonate:

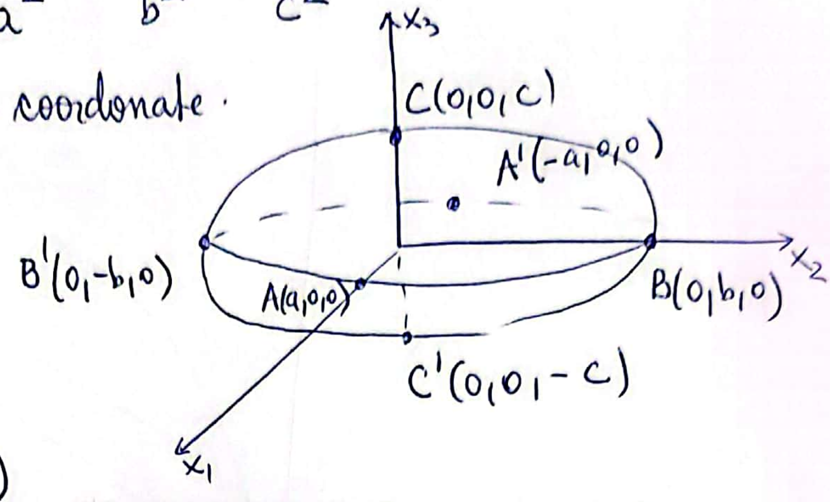
a) $x_3 = \gamma \in (-R,R) \Rightarrow x_1^2 + x_2^2 = R^2 - \gamma^2 > 0$

b) $x_2 = \beta$ Analog cercuri

c) $x_1 = \alpha$

2). <u>Elipsoidul</u> : $\frac{x_1^2}{a^2}+\frac{x_2^2}{b^2}+\frac{x_3^2}{c^2}=1$, $a>0, b>0, c>0$.

$\cap$ cu planele de coordonate.



a) $x_3=\gamma\in(-c,c)$

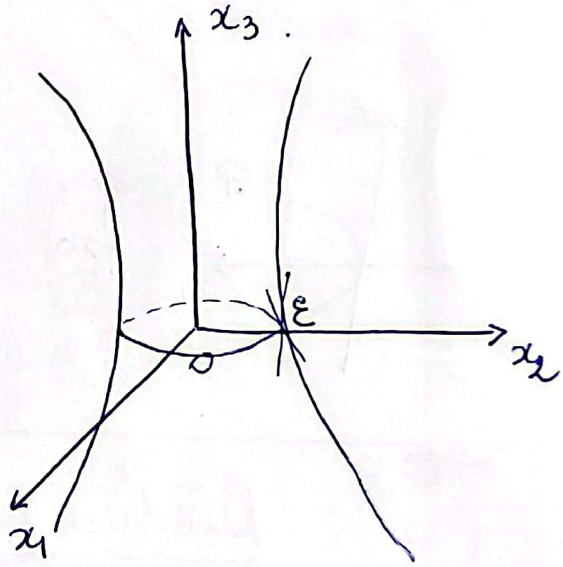
$$\frac{x_1^2}{a^2}+\frac{x_2^2}{b^2}=1-\frac{\gamma^2}{c^2}>0 \Rightarrow \frac{x_1^2}{a^2\left(1-\frac{\gamma^2}{c^2}\right)}+\frac{x_2^2}{b^2\left(1-\frac{\gamma^2}{c^2}\right)}=1 \text{ Elipsă}$$

b) $x_2=\beta\in(-b,b)$ Elipsă

c) $x_1=\alpha\in(-a,a)$ Elipsă.

<u>Hiperboloid cu o pânză</u>

3). $\mathcal{H}_1: \frac{x_1^2}{a^2}+\frac{x_2^2}{b^2}-\frac{x_3^2}{c^2}=1$, $a>0$, $b>0$, $c>0$



$\cap$ cu planele de coordonate.

a) $x_3=\gamma\in\mathbb{R}$

$$\frac{x_1^2}{a^2}+\frac{x_2^2}{b^2}=1+\frac{\gamma^2}{c^2}\Rightarrow$$

$$\frac{x_1^2}{a^2\left(1+\frac{\gamma^2}{c^2}\right)}+\frac{x_2^2}{b^2\left(1+\frac{\gamma^2}{c^2}\right)}=1. \text{ Elipsa}$$

($\gamma=0$ $\frac{x_1^2}{a^2}+\frac{x_2^2}{b^2}=1$ Elipsa colier)

b) $x_2=\beta$

$$\frac{x_1^2}{a^2}-\frac{x_3^2}{c^2}=1-\frac{\beta^2}{b^2},$$

$\beta\neq\pm b$ Hiperbolă.

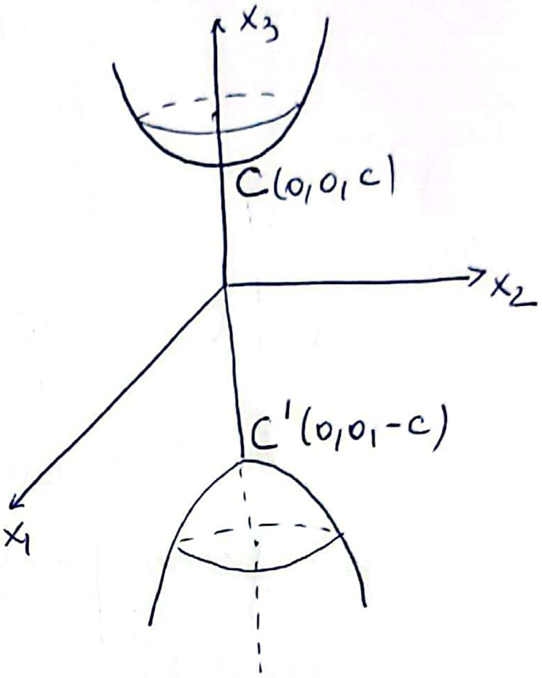
$\beta=\pm b$ $\frac{x_1^2}{a^2}-\frac{x_3^2}{c^2}=0$

$x_3=\pm\frac{c}{a}x_1$ plane.

$\cap$ cu $x_2=\beta\Rightarrow$ drepte.

c) $x_1=\alpha$ Analog cu b.

<u>Teoremă</u> $\mathcal{H}_1$ = cuadrică dublu riglată (i.e. $\exists$ 2 familii de generatoare G_1, G_2 și prin fiecare punct trece câte o dreaptă din fiecare familie)



4) $\mathcal{H}_2$ $\quad -\frac{x_1^2}{a^2}-\frac{x_2^2}{b^2}+\frac{x_3^2}{c^2}=1$

Hiperboloid cu 2 pânze.

$\cap$ cu planele de coordonate.

a) $x_3=\gamma\in(-\infty,-c)\cup(c,\infty);\ |\gamma|>c$

$\frac{x_1^2}{a^2}+\frac{x_2^2}{b^2}=\frac{\gamma^2}{c^2}-1>0$ Elipsa

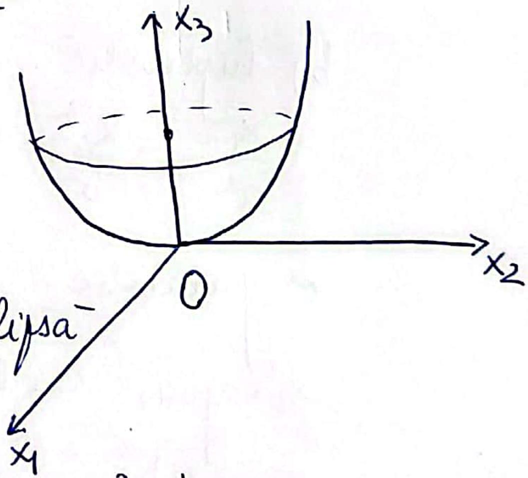
$(\gamma\in\{\pm c\}\Rightarrow C, C')$

b) $x_2=\beta \Rightarrow -\frac{x_1^2}{a^2}+\frac{x_3^2}{c^2}=1+\frac{\beta^2}{b^2}$ Hiperbola.

c) $x_1=\alpha \quad -\frac{x_2^2}{b^2}+\frac{x_3^2}{c^2}=1+\frac{\alpha^2}{a^2}$ — // —

5) Paraboloid eliptic.

$P_e:\ \frac{x_1^2}{a^2}+\frac{x_2^2}{b^2}=2x_3\geq 0$



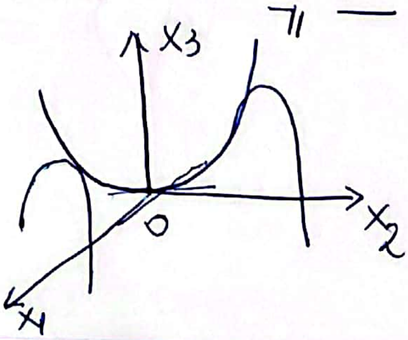
$\cap$ cu planele de coord.

a) $x_3=\gamma\geq 0 \Rightarrow \frac{x_1^2}{a^2}+\frac{x_2^2}{b^2}=2\gamma$ Elipsa

$\gamma=0 \Rightarrow O(0,0,0)$

b) $x_2=\beta \quad \frac{x_1^2}{a^2}=2x_3-\frac{\beta^2}{b^2}=2\left(x_3-\frac{\beta^2}{2b^2}\right)$ Parabola

c) $x_1=\alpha \quad \frac{x_2^2}{b^2}=2x_3-\frac{\alpha^2}{a^2}$ — // —



6) <u>Paraboloid hiperbolic</u>

$P_h:\ \frac{x_1^2}{a^2}-\frac{x_2^2}{b^2}=2x_3$

$\cap$ cu planele de coordonate $\quad \mathcal{P}_h : \dfrac{x_1^2}{a^2} - \dfrac{x_2^2}{b^2} = 2x_3$

$a > 0,\ b > 0$

a) $x_3 = \gamma \Rightarrow \dfrac{x_1^2}{a^2} - \dfrac{x_2^2}{b^2} = 2\gamma$

($\gamma \neq 0$ Hiperbolă)

($\gamma = 0 \Rightarrow x_2 = \pm \dfrac{b}{a} x_1 \cap$ cu $x_3 = 0$ drepte conc.)

b) $x_2 = \beta \Rightarrow \dfrac{x_1^2}{a^2} = 2x_3 + \dfrac{\beta^2}{b^2} = 2\left(x_3 + \dfrac{\beta^2}{2b^2}\right)$ Parabolă.

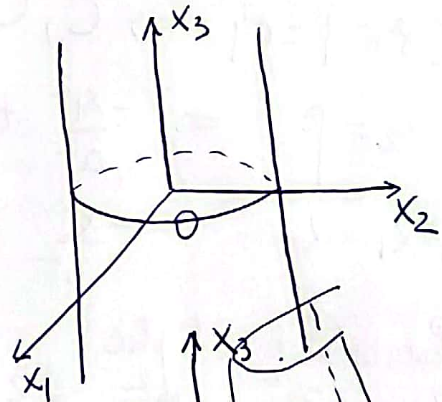
c) $x_1 = \alpha \Rightarrow \dfrac{x_2^2}{b^2} = -2x_3 + \dfrac{\alpha^2}{a^2} = -2\left(x_3 - \dfrac{\alpha^2}{2a^2}\right)$ Parabolă.

<u>Teoremă</u> $\mathcal{P}_h$ = cuadrică dublu riglată ($\exists$ 2 fam de generatoare și prin fiecare punct trece câte o dreaptă din fiecare familie).

7) Cilindrul

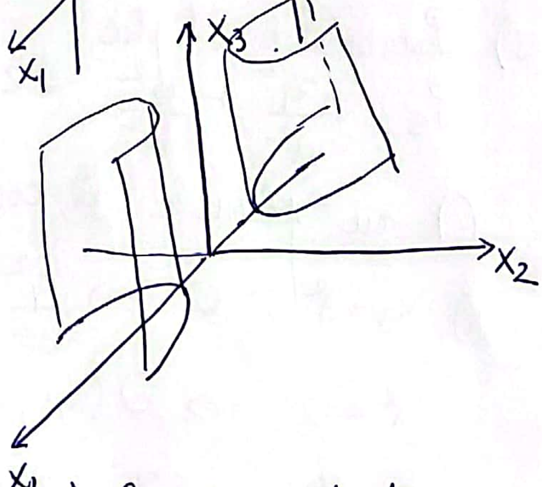
a) eliptic.

$\dfrac{x_1^2}{a^2} + \dfrac{x_2^2}{b^2} = 1,\ x_3 \in \mathbb{R}$



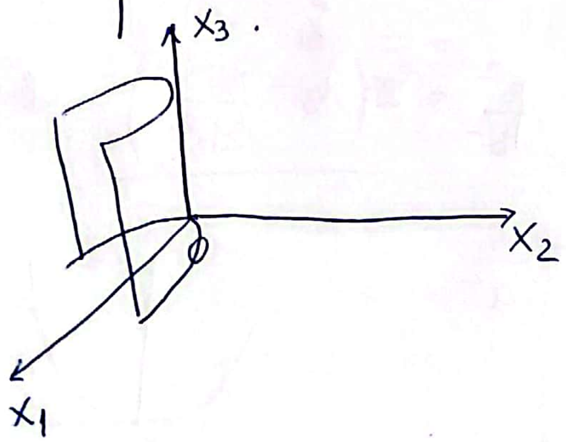
b) hiperbolic.

$\dfrac{x_1^2}{a^2} - \dfrac{x_2^2}{b^2} = 1,\ x_3 \in \mathbb{R}$



c) parabolic.

$x_2^2 = 2px_1,\ x_3 \in \mathbb{R}$.



8) Conul pătratic.

$\dfrac{x_1^2}{a^2} + \dfrac{x_2^2}{b^2} - \dfrac{x_3^2}{c^2} = 0$