22.11.2023.

SEMINAR 8:

SERII DE PUTERI

$$\sum_{n\geq 0} f_n(x) = \sum_{n\geq 0} a_n (x-x_0)^n$$

$f_n: \mathbb{R}\to\mathbb{R}$, $f_n(x) = a_n (x-x_0)^n$, $\forall x \in \mathbb{R}$

Serii de puteri 1) $\rightarrow$ raza de convergență $R = \dfrac{1}{\lim \sqrt[n]{|a_n|}} \in \overline{\mathbb{R}}$ sau $R = \dfrac{1}{\lim \left|\dfrac{a_{n+1}}{a_n}\right|}$

2) $\rightarrow$ interval de convergență $(x_0-R, x_0+R) \subseteq \mathbb{R}$

3) $\rightarrow$ mulțimea de convergență $A \subseteq \mathbb{R}$, $x_0 \in A$

$(x_0-R, x_0+R) \subseteq A \subseteq [x_0-R, x_0+R]$

4) $\rightarrow$ suma seriei de puteri (funct.) $f: A \to \mathbb{R}$

$f(x_0) = a_0$

f este cont pe A $\quad$ f este fct. de clasă C^∞ pe (x_0-R, x_0+R)

$f|_{(x_0-R, x_0+R)}$

Serii de puteri remarcabile:

Dacă $\exists \lim\limits_{n\to+\infty} \dfrac{|a_{n+1}|}{|a_n|} \in \overline{\mathbb{R}}_+$

atunci $R = \dfrac{1}{\lim\limits_{n\to+\infty} \dfrac{|a_{n+1}|}{|a_n|}}$

1) $\sum\limits_{n\geq 0} x^n = \dfrac{1}{1-x}$, $\forall x \in (-1,1)$

2) $\sum\limits_{n\geq 0} (-1)^n \cdot x^n = \dfrac{1}{1+x}$, $\forall x \in (-1,1)$

3) $\sum\limits_{n\geq 0} \dfrac{x^n}{n!} = e^x$, $\forall x \in \mathbb{R}$

4) $\sum\limits_{n\geq 0} \dfrac{(-1)^n \cdot x^{2n}}{(2n)!} = \cos x$, $\forall x \in \mathbb{R}$

5) $\sum\limits_{n\geq 0} \dfrac{(-1)^n x^{2n+1}}{(2n+1)!} = \sin x$, $\forall x \in \mathbb{R}$

Exerciții:

E1 Să se det. R, A și f pt. următoarele serii de puteri

a) $\sum (-1)^n (n+1) x^n$

b) $\sum_{n\geq 0} \frac{x^{2n+1}}{2n+1}$

$$R = \frac{1}{\lim \sqrt[n]{|a_n|}}$$

a) $x_0 = 0$

$a_n = (-1)^n (n+1)$, $\forall n \in \mathbb{N}$

$a_0 = 1$

Avem $\underbrace{(-1)^n(n+1)}_{a_n}x^n$ și $\sum f_n(x) = \sum a_n(x-x_0)^n$

Dacă alegem $x_0 = 0$ avem $a_n x^n \Rightarrow a_n = (-1)^n (n+1)$

1) $R = \frac{1}{\lim \left|\frac{a_{n+1}}{a_n}\right|}$, $\lim_{n\to+\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to+\infty} \left|\frac{(n+2)(-1)^{n+1}}{(n+1)(-1)^n}\right| = \lim_{n\to+\infty} \frac{n+2}{n+1}$

poz.

poz. ↳ modulul „anulează" $(-1)^{n+1}$ și $(-1)^n$

$$= \lim_{n\to+\infty} \frac{\cancel{n}(1+\frac{2}{n})}{\cancel{n}(1+\frac{1}{n})} = 1 \in \mathbb{R}_+^* \Rightarrow R = 1$$

2) 3) $\begin{cases} A \subseteq \mathbb{R} \\ 0 \in A \\ (-1,1) \subseteq A \subseteq [-1,1] \end{cases}$

Pt. ca $-1 \in A$, seria de nr. reale $\sum_{n\geq 0} (-1)^n (n+1)\cdot(-1)^n$ este conv.

Pt. ca $-1 \notin A$, seria —"— este div.

$$\sum_{n\geq 0} (-1)^n (n+1)(-1)^n = \sum_{n\geq 0} (n+1) \quad \left((-1)^{n+n} = (-1)^{2n} = 1\right)$$

$\lim_{n\to+\infty} n+1 = +\infty \neq 0 \Rightarrow$ seria este divergentă $\Rightarrow -1 \notin A$

$x = 1 \Rightarrow \sum (-1)^n (n+1) 1^n = \sum_{n\geq 0} (-1)^n (n+1)$

$\left.\begin{aligned} &\lim_{n\to+\infty} (-1)^{2n}(2n+1) = \infty \\ &\lim_{n\to+\infty} (-1)^{2n+1}(2n+2) = -\infty \end{aligned}\right| \Rightarrow \nexists \lim_{n\to+\infty} (-1)^n (n+1) \Rightarrow$

$\sum_{n\geq 0} (-1)^n (n+1) =$ div. $\Rightarrow 1 \notin A$

Deci $A = (-1,1)$

5) $f: A=(-1,1) \to \mathbb{R}$, $f(x) = \sum_{n\geq 0} (-1)^n (n+1) \cdot x^n$

$f(0)=1$

f cont pe $(-1,1)$

f funcție de clasa C^∞ pe $(-1,1)$

$$\int f(x)\,dx = \sum_{n\geq 0} \int \underbrace{(-1)^n (n+1)}_{\text{constantă}} \cdot \overset{\text{integrat } = \frac{x^{n+1}}{n+1}}{x^n}\,dx = \sum_{n\geq 0} (-1)^n \cancel{(n+1)} \cdot \frac{x^{n+1}}{\cancel{n+1}} + C =$$

$$= \sum_{n\geq 0} [(-1)^n \cdot x^{n+1}] + C$$

$$= \sum_{n\geq 0} (-1)^n \cdot x^n \cdot x + C = x \sum_{n\geq 0} (-1)^n \cdot x^n + C$$

scoatem x și obținem o serie de put. remarcabile la care știm suma

$$\int f(x)\,dx = x \cdot \underbrace{\frac{1}{1+x}}_{\text{suma seriei } \sum_{n\geq 0} (-1)^n \cdot x^n} + C \Rightarrow \int f(x)\,dx = \frac{x}{1+x} \Big|'$$

$\forall x \in (-1,1)$

$$f(x) = \left(\frac{x}{1+x} + C\right)' = \cancel{\frac{(1+x)' \cdot x - 1 \cdot x}{(1+x)^2}}$$

$$= \cancel{(x\,2)}$$

$$= \frac{x' \cdot (1+x) - (1+x)' \cdot x}{(1+x)^2}$$

$$= \frac{1+\cancel{x}-\cancel{x}}{(1+x)^2} = \frac{1}{(1+x)^2}$$

b) $$\sum_{n\geq 0} \frac{x^{2n+1}}{2n+1} = \frac{1}{\underbrace{2n+1}_{\alpha}} \cdot (x-0)^{\overset{\alpha}{2n+1}} = a_\alpha (x-x_0)^\alpha \Rightarrow a_{2n+1} = \frac{1}{2n+1}$$

$x_0 = 0$

$\cancel{a_n} = \frac{1}{2n+1}$, a_{2n+1}

$a_{2n} = 0$

$a_0 = 0$

$$R = \frac{1}{\lim \sqrt[n]{|a_n|}}$$

$$\lim_{n\to+\infty} \sqrt[2n]{|a_{2n}|} = 0 \quad (1)$$

$$\lim_{n\to+\infty} \sqrt[2n+1]{|a_{2n+1}|} = \lim_{n\to+\infty} \sqrt[2n+1]{\left|\frac{1}{2n+1}\right|}$$

$$\lim_{n\to+\infty} \sqrt[n]{x_n} = ? \quad , \quad \lim_{n\to+\infty} \frac{x_{n+1}}{x_n} = 1$$

$$\lim_{n\to+\infty} \frac{|a_{2(n+1)+1}|}{|a_{2n+1}|} = \lim_{n\to+\infty} \frac{|a_{2n+3}|}{|a_{2n+1}|}$$

$$= \lim_{n\to+\infty} \frac{\frac{1}{2n+3}}{\frac{1}{2n+1}} = \lim_{n\to+\infty} \frac{2n+1}{2n+3} = 1$$

$$\Rightarrow \lim_{n\to+\infty} \sqrt[2n+1]{|a_{2n+1}|} = 1 \quad (2)$$

(1)+(2) $\Rightarrow \lim_{n\to+\infty} \sqrt[n]{|a_n|} = \max\{0,1\} = 1 \Rightarrow R=1$

$A \subset \mathbb{R}$

$0 \in A$

$(-1,1) \subseteq A \subseteq [-1,1]$

$$\sum_{n\geq 0} \frac{1}{2n+1} \sim \sum_{n\geq 1} \frac{1}{n} = \sum \frac{1}{n^1} \text{ div.}$$

$$x=-1 \Rightarrow \sum_{n\geq 0} \frac{(-1)^{2n+1}}{2n+1} = \sum_{n\geq 0} \frac{-1}{2n+1} = -\sum_{n\geq 0} \frac{1}{2n+1} \text{ - div} \quad -1 \notin A$$

$$x=1 \Rightarrow \sum_{n\geq 0} \frac{1^{2n+1}}{2n+1} = \sum_{n\geq 1} \frac{1}{2n+1} \text{ = div.} \quad 1 \notin A$$

$$f: A=(-1,1) \to \mathbb{R}, \quad f(x) = \sum_{n\geq 0} \frac{x^{2n+1}}{2n+1}$$

$f(0)=0$

f cont. pe $(-1,1)$

f este de clasă C^∞ pe $(-1,1)$

(- . . ?)

$$\int f(x)\,dx = \sum \int \frac{x^{2n+1}}{2n+1}\,dx = \sum \frac{1}{2n+1} \cdot \frac{x^{2n+2}}{2n+2} + \mathcal{C}$$

$\forall x \in (-1,1)$

???

$$= \frac{1}{2n+1} \cdot \frac{1}{2n+2} \cdot \sum x^{2n+2}$$

$$\int f(x)\,dx = \frac{1}{(2n+1)(2n+2)} \cdot \frac{1}{1-x} \Rightarrow f(x) = \frac{1}{2n+1} \cdot \frac{1}{2n+2} \cdot \frac{-1}{(1-x)^2}$$

$$= -\frac{1}{(2n+1)(2n+2)\cdot(1-x)^2} \quad \text{si}$$