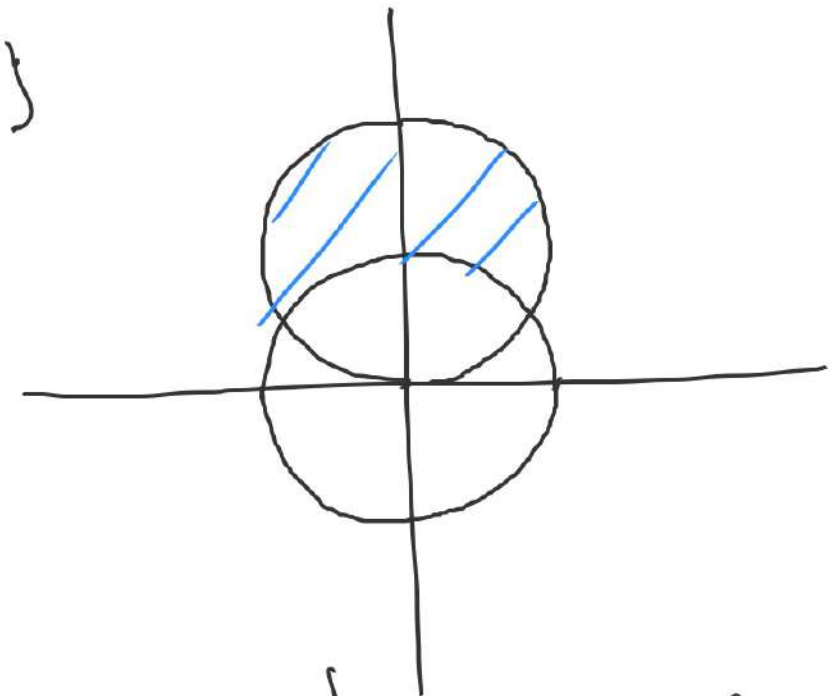
$$\iint_D x\,dx\,dy \qquad D = \{(x,y) \mid 1 \le x^2+y^2 \le 2y\}$$



$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$$

$$(x,y) \in D \iff \begin{cases} 1 \le r^2 \\ r^2 \le 2r\sin\theta \end{cases} \iff 1 \le r \le 2\sin\theta \iff \begin{cases} 1 \le r \le 2\sin\theta \\ 1 \le 2\sin\theta \end{cases}$$

$$\iff \begin{cases} \theta \in \left[\frac{\pi}{6}, \frac{5\pi}{6}\right] \\ 1 \le r \le 2\sin\theta \end{cases}$$

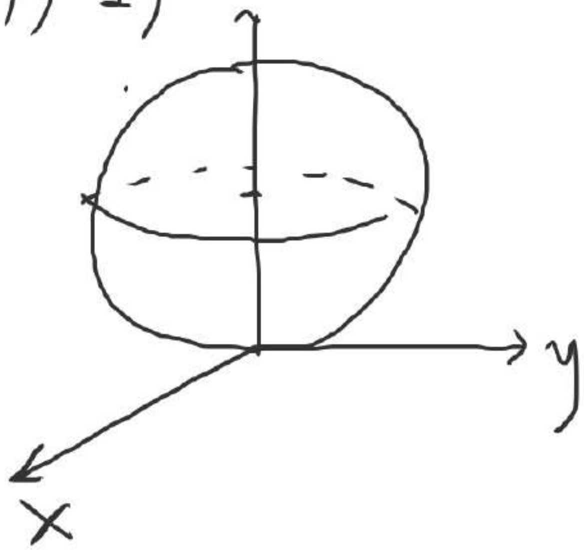
$$= \int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} \left(\int_{1}^{2\sin\theta} r\cos\theta \cdot r \, dr \right) d\theta$$

$$= \int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} \cos\theta \cdot \frac{r^3}{3} \Bigg|_{r=1}^{r=2\sin\theta} d\theta = \frac{1}{3} \int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} \left(8\sin^3\theta \, \overbrace{\cos\theta}^{(\sin\theta)' \, =} - \cos\theta \right) d\theta$$

$$= \frac{8}{3} \cdot \frac{\sin^4\theta}{4} \Bigg|_{\frac{\pi}{6}}^{\frac{5\pi}{6}} - \frac{\sin\theta}{3} \Bigg|_{\frac{\pi}{6}}^{\frac{5\pi}{6}} = 0$$

$$J = \iiint\limits_V z\sqrt{x^2+y^2+z^2}\, dx\,dy\,dz \;, \quad V = \{(x,y,z) \in \mathbb{R}^3 \mid x^2+y^2+z^2 \le 2z\}$$

$$V = B((0,0,1), 1).$$



$$\begin{cases} x = r\sin\varphi\cos\theta \\ y = r\sin\varphi\sin\theta \\ z = r\cos\varphi \end{cases}$$

$$(x,y,z) \in V \iff r^2 \le 2r\cos\varphi \iff r \le 2\cos\varphi$$

$$\iff \begin{cases} 0 \le \theta \le 2\pi \\ 0 \le \varphi \le \frac{\pi}{2} \\ 0 \le r \le 2\cos\varphi \end{cases} \qquad dx\,dy\,dz = r^2\sin\varphi\, dr\,d\theta\,d\varphi$$

$$J = \int_0^{2\pi}\left(\int_0^{\frac{\pi}{2}}\left(\int_0^{2\cos\varphi} r^2\cos\varphi \cdot r^2\sin\varphi \, dr\right)d\varphi\right)d\theta$$

$$= \int_0^{2\pi}\left(\int_0^{\frac{\pi}{2}}\cos\varphi\sin\varphi \cdot \frac{r^5}{5}\Bigg|_{r=0}^{r=2\cos\varphi} d\varphi\right)d\theta =$$

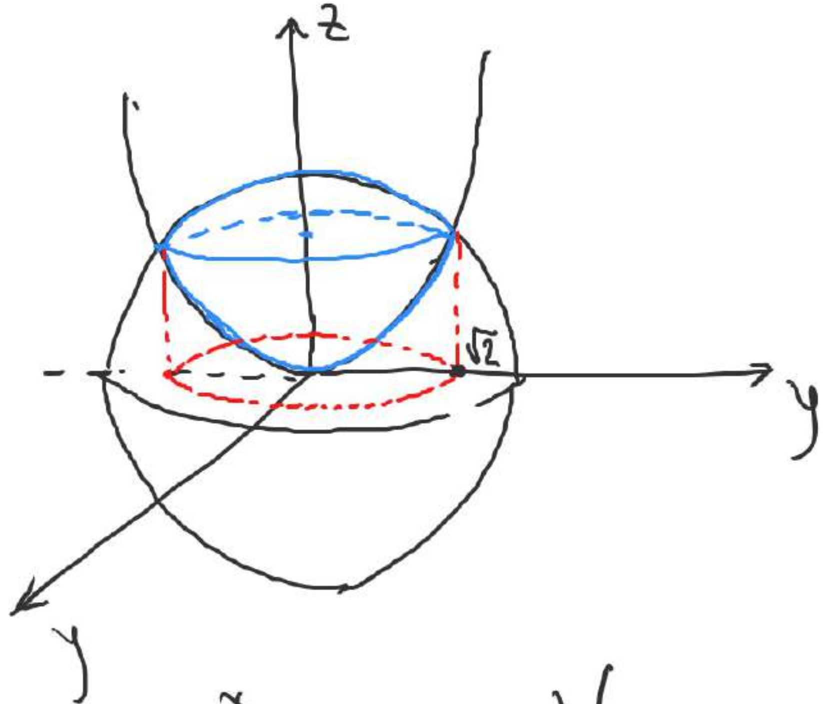
$$\int_0^{2\pi}\left(\int_0^{\frac{\pi}{2}}\frac{32}{5}\cos^6\varphi\sin\varphi \, d\varphi\right)d\theta = \int_0^{2\pi} - \frac{32}{5}\cdot\frac{\cos^7\varphi}{7}\Bigg|_0^{\frac{\pi}{2}} d\theta$$

$$= 2\pi \cdot \frac{32}{35}.$$

$$\iiint_V (x^2+y^2)\cdot z \, dx\,dy\,dz$$, V este limitata de sfera $x^2+y^2+z^2=6$ si paraboloidul $z=x^2+y^2$

$$\begin{cases} z = x^2+y^2 \\ x^2+y^2+z^2=6 \end{cases}$$

$$z^2+z=6 \Rightarrow z=2$$



Intersectia sferei cu paraboloidul este cercul $x^2+y^2=2$ din planul $z=2$.

$$D = pr_{xOy} V$$

$$D = \{ (x,y) \in \mathbb{R}^2 \mid x^2+y^2 \le 2 \}$$

$$V = \{ (x,y,z) \mid (x,y) \in D, \quad x^2+y^2 \le z \le \sqrt{6-x^2-y^2} \}$$

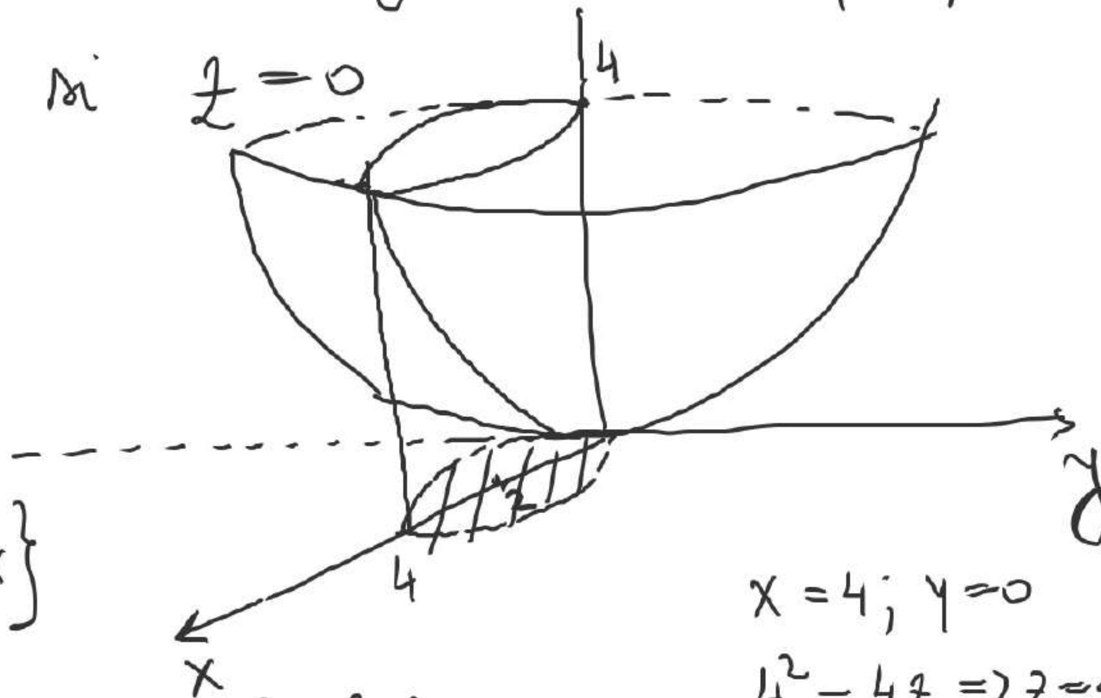
$$\iiint_V (x^2+y^2) z \, dx\,dy\,dz = \iint_D \left(\int_{x^2+y^2}^{\sqrt{6-x^2-y^2}} (x^2+y^2) z \, dz \right) dx\,dy$$

$$= \iint_D (x^2+y^2) \cdot \frac{z^2}{2} \Bigg|_{z=x^2+y^2}^{\sqrt{6-x^2-y^2}} dx\,dy = \frac{1}{2} \iint_D (x^2+y^2)\left(6-x^2-y^2-(x^2+y^2)^2\right) dx\,dy$$

se trece la coordonate polare.

Găsiți volumul mulțimii V mărginită de suprafețele

$x^2+y^2=4z$, $x^2+y^2=4x$ și $z=0$



$$x^2+y^2=4x \iff (x-2)^2+y^2=4$$

$$pr_{xOy} V = D = \{(x,y) \mid x^2+y^2 \le 4x\}$$

$x=4;\ y=0$

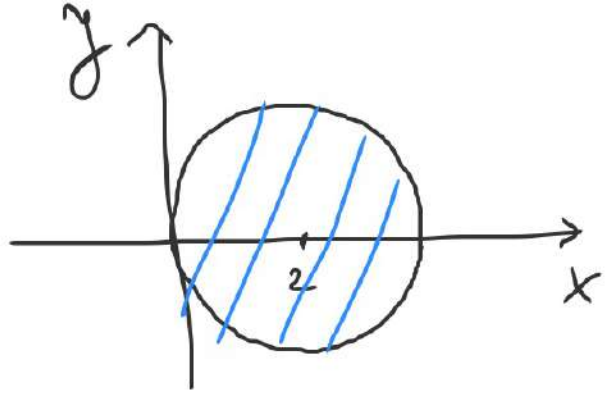
$4^2=4z \Rightarrow z=4.$

$$V=\left\{(x,y,z) \;\middle|\; (x,y)\in D,\ 0\le z\le \frac{x^2+y^2}{4}\right\}$$

$$\lambda(V)=\iiint_V dx\,dy\,dz=\iint_D\left(\int_0^{\frac{x^2+y^2}{4}} dz\right)dx\,dy=\iint_D \frac{x^2+y^2}{4}\,dx\,dy$$

$$\iint_D \frac{x^2+y^2}{4}\,dx\,dy$$

$$D:\ (x-2)^2+y^2 \le 4 \iff x^2+y^2 \le 4x$$



$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$$

$$(x,y)\in D \iff r^2 \le 4r\cos\theta$$

$$(x,y)\in D \iff \begin{cases} \theta \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \\ 0 \le r \le 4\cos\theta \end{cases}$$

$$dx\,dy = r\,dr\,d\theta$$

$$\iint_D \frac{x^2+y^2}{4}\,dx\,dy = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left(\int_0^{4\cos\theta} \frac{r^2}{4}\cdot r\,dr \right) d\theta = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{r^4}{16}\Bigg|_{r=0}^{r=4\cos\theta} d\theta =$$

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 16 \cos^4\theta \, d\theta = \int_0^{\frac{\pi}{2}} 32 \cos^4\theta \, d\theta$$

Exerciții

1) Calculați $\displaystyle\int_0^y \frac{\ln(1+yx)}{1+x^2}\,dx \ , \ y>0$

2) $\displaystyle\iint_D e^{(x+y)^2}\,dx\,dy \ , \quad D=\{(x,y)\in\mathbb{R}^2 \mid x+y\le 1,\ x,y\ge 0\}$