Teoremă.

Fie $f_n : [a,b] \to \mathbb{R}$ integrabile Riemann; $f : [a,b] \to \mathbb{R}$ a.î. $f_n \xrightarrow{u} f$. Atunci f este int. Riemann și

$$\lim_{n\to\infty} \int_a^b f_n(x)\,dx = \int_a^b f(x)\,dx$$

1) $$\lim_{n\to\infty} \int_0^1 (1+x^2)\,\frac{ne^x + xe^{-x}}{n+x}\,dx$$

$f_n : [0,1] \to \mathbb{R}$, $f_n(x) = (1+x^2)\cdot \dfrac{ne^x + xe^{-x}}{n+x}$

$\lim\limits_{n\to\infty} f_n(x) = e^x(1+x^2)$, $f(x) = e^x(1+x^2)$, $x \in [0,1]$

$$\left|f_n(x)-f(x)\right| = \left|(1+x^2)\frac{ne^x+xe^{-x}}{n+x}-(1+x^2)e^x\right|$$

$$= (1+x^2)\left|\frac{\cancel{ne^x}+xe^{-x}-\cancel{ne^x}-xe^x}{n+x}\right| = (1+x^2)\frac{xe^x-xe^{-x}}{n+x}$$

$$= (1+x^2)\frac{xe^x(1-e^{-2x})}{n+x} = \frac{1+x^2}{n+x}\cdot x\left(e^x-\frac{1}{e^x}\right) \leq \frac{1+1}{n+0}\cdot 1\cdot e = \frac{2e}{n}$$

pt $x\in[0,1]$.

$$\left|f_n(x)-f(x)\right| \leq \frac{2e}{n}, \ \forall x\in[0,1].$$

$$\sup_{x\in[0,1]}\left|f_n(x)-f(x)\right| \leq \frac{2e}{n} \xrightarrow{n\to\infty} 0. \text{ Deci } f_n \xrightarrow{u} f.$$

$$\Rightarrow \lim_{n\to\infty} \int_0^1 f_n(x)\,dx = \int_0^1 (1+x^2)e^x\,dx = \dots$$

2) Calculați $\lim_{n\to\infty} \int_0^1 \ln(1+x^n)\,dx$

$f_n : [0,1] \to \mathbb{R}$, $f_n(x) = \ln(1+x^n)$

$$\lim_{n\to\infty} f_n(x) = f(x) = \begin{cases} 0, & 0 \le x < 1 \\ \ln 2; & x = 1. \end{cases}$$

$f_n \xrightarrow{s} f$

$f_n \stackrel{u}{\not\to} f$

$\ln(1+x) \le x, \ \forall x > -1$

$$0 \le \ln(1+x^n) \le x^n, \forall x \in [0,1] \Rightarrow 0 \le \int_0^1 \ln(1+x^n)\,dx \le \int_0^1 x^n\,dx$$

$$\left.\begin{array}{l} 0 \le \int_0^1 \ln(1+x^n)\,dx \le \dfrac{1}{n+1} \\ \lim\limits_{n\to\infty} \dfrac{1}{n+1} = 0 \end{array}\right\} \Longrightarrow \lim_{n\to\infty} \int_0^1 \ln(1+x^n)\,dx = 0$$

Studiați convergența următoarelor integrale improprii și în caz de convergență calculați

3) $$\int_0^1 \frac{1}{\sqrt{x}}\,dx$$

$$\lim_{\substack{c\to 0\\ c>0}} \int_c^1 \frac{1}{\sqrt{x}}\,dx = \lim_{c\searrow 0}\left(2\sqrt{x}\Big|_c^1\right) = \lim_{c\searrow 0}\left(2-2\sqrt{c}\right) = 2$$

deci $\int_0^1 \frac{1}{\sqrt{x}}\,dx$ este conv și $\int_0^1 \frac{1}{\sqrt{x}}\,dx = 2$.

2) $\displaystyle\int_0^\infty \frac{x^2}{1+x^3}\,dx = \lim_{c\to\infty} \int_0^c \frac{x^2}{1+x^3}\,dx$

$1+x^3 = t \qquad 3x^2\,dx = dt$

$x=0\ ,\ t=1$

$x=c\ ,\ t=1+c^3$

$\displaystyle = \lim_{c\to\infty} \frac{1}{3}\int_1^{1+c^3} \frac{1}{t}\,dt = \lim_{c\to\infty} \frac{1}{3}\ln t\Big|_1^{1+c^3} = \lim_{c\to\infty} \frac{\ln(1+c^3)}{3}$

$= \infty$

$\Rightarrow$ int. este divergentă.

3) $\displaystyle\int_3^{\infty} \frac{1}{x^2-3x+2}\,dx$

$$\frac{1}{x^2-3x+2} = \frac{1}{x-2} - \frac{1}{x-1}$$

$$\lim_{c\to\infty} \int_3^{c} \left(\frac{1}{x-2} - \frac{1}{x-1}\right) dx = \lim_{c\to\infty} \left(\ln(c-2) - \ln(c-1)\right)\Big|_3^{c}$$

$$= \lim_{c\to\infty} \left(\ln(c-2) - \ln(c-1) - \ln 1 + \ln 2\right) = \lim_{c\to\infty} \ln \frac{(c-2)2}{c-1}$$

$$= \ln 2$$

$\Rightarrow$ int. este conv. și $\displaystyle\int_3^{\infty} \frac{1}{x^2-3x+2}\,dx = \ln 2.$

4) $\int_a^\infty \frac{1}{x^\alpha}\,dx,\ a>0$ este conv dacă $\alpha>1$

divergentă dacă $0<\alpha\leq 1$

Fie $\alpha \neq 1$

$$\lim_{c\to\infty}\int_a^c \frac{1}{x^\alpha}\,dx = \lim_{c\to\infty} \left.\frac{x^{-\alpha+1}}{-\alpha+1}\right|_a^c$$

$$= \lim_{c\to\infty} \frac{c^{-\alpha+1}}{-\alpha+1} - \frac{a^{-\alpha+1}}{-\alpha+1} = \begin{cases} \dfrac{a^{-\alpha+1}}{\alpha-1}; & \alpha>1 \\ +\infty; & \alpha<1 \end{cases}$$

Pt $\alpha=1$

$$\lim_{c\to\infty}\int_a^c \frac{1}{x}\,dx = \lim_{c\to\infty}\left(\ln x\Big|_a^c\right) = +\infty.$$

1) Exerciții

$$\cos x + \cos 2x + \cdots + \cos nx = \frac{\sin\frac{nx}{2}\cos\frac{n+1}{2}x}{\sin\frac{x}{2}}, \quad x \neq 0.$$

Indicație $z = \cos x + i\sin x$, $z + z^2 + \cdots + z^n$

$$\frac{1}{2} + \cos x + \cos 2x + \cdots + \cos nx = \frac{\sin\left(n+\frac{1}{2}\right)x}{2\sin\frac{x}{2}}, \quad x \neq 0.$$

$$\Big\Downarrow \; x \to 2x$$

$$1 + 2\cos 2x + 2\cos 4x + \cdots 2\cos 2nx = \frac{\sin(2n+1)x}{\sin x}, \quad x \neq 0.$$

2) Lema lui Riemann. Fie $f : [a,b] \to \mathbb{R}$ integr. R.
Atunci $\lim\limits_{n\to\infty} \int_a^b f(x)\sin nx\,dx = \lim\limits_{n\to\infty}\int_a^b f(x)\cos nx\,dx = 0.$

5) Calculați $\displaystyle\int_0^{\infty} \frac{\sin x}{x}\, dx$.

Int. $\displaystyle\int_0^{\infty} \frac{\sin x}{x}\, dx$ este conv. (Vezi cursul)

$$\int_0^{\infty} \frac{\sin x}{x}\, dx = \lim_{c \to \infty} \int_0^{c} \frac{\sin x}{x}\, dx = \lim_{n \to \infty} \int_0^{(2n+1)\frac{\pi}{2}} \frac{\sin x}{x}\, dx$$

$x = (2n+1)y \qquad x = 0 \;,\; y = 0 \qquad dx = (2n+1)dy$

$\qquad\qquad\qquad\quad\; x = (2n+1)\frac{\pi}{2} \;,\; y = \frac{\pi}{2}$

$$= \lim_{n \to \infty} \int_0^{\frac{\pi}{2}} \frac{\sin (2n+1)y}{y}\, dy$$

$$1+2\cos 2x+2\cos 4x+\dots+2\cos 2nx=\frac{\sin(2n+1)x}{\sin x},\ x\neq 0$$

$x \longrightarrow \dfrac{\sin(2n+1)x}{x}$ poate fi prel. prin continuitate în 0.

și integrând egalitatea de mai sus obținem.

$$\frac{\pi}{2}=\int_0^{\frac{\pi}{2}}\frac{\sin(2n+1)x}{\sin x}\,dx,\ \forall n\in\mathbb{N}\quad(2)$$

$$\left(\text{pt că } \int_0^{\frac{\pi}{2}}\cos 2kx\,dx=\frac{1}{2k}\sin 2kx\Big|_0^{\frac{\pi}{2}}=0\right)$$

$$f(x)=\begin{cases} \dfrac{1}{\sin x}-\dfrac{1}{x}=\dfrac{x-\sin x}{x\sin x}, & x\in\left(0,\frac{\pi}{2}\right] \\ 0, & x=0 \end{cases}$$

$$\lim_{x\to 0}\frac{x-\sin x}{x\cdot\sin x}=0 \Rightarrow f \text{ continuă pe } \left[0,\frac{\pi}{2}\right] \text{ și deci int R}$$

$$\xrightarrow{\text{Lema Riemann}} \lim_{n\to\infty}\int_0^{\frac{\pi}{2}} f(x)\sin(2n+1)x\,dx=0.$$

$$\Rightarrow \int_0^{\frac{\pi}{2}}\left[\frac{\sin(2n+1)x}{\sin x}-\frac{\sin(2n+1)x}{x}\right]dx=$$

$$\overset{(2)}{\Longrightarrow} \lim_{n\to\infty}\int_0^{\frac{\pi}{2}}\frac{\sin(2n+1)x}{x}\,dx=\frac{\pi}{2} \overset{(1)}{\Longrightarrow} \int_0^{\infty}\frac{\sin x}{x}\,dx=\frac{\pi}{2}$$

$$\lim_{x\to 0} \frac{x-\sin x}{x\sin x} = \lim_{x\to 0.} \frac{x-\sin x}{x^2} \cdot \frac{x}{\sin x} = 0.$$

$$\lim_{x\to 0} \frac{x-\sin x}{x^2} = \lim_{x\to 0} \frac{1-\cos 2x}{2x} = \lim_{x\to 0} \frac{2\sin 2x}{2} = 0$$

$$\lim_{x\to 0} \frac{x}{\sin x} = 1$$

6) $$\int_0^\infty x e^{-x}\,dx = \lim_{c\to\infty} \int_0^c x e^{-x}\,dx$$

$$\int_0^c x e^{-x}\,dx = \int_0^c x\,(-e^{-x})'\,dx = -xe^{-x}\Big|_0^c + \int_0^c e^{-x}\,dx$$

$$= -ce^{-c} - e^{-x}\Big|_0^c = -ce^{-c} - e^{-c} + 1$$

$$\lim_{c \to \infty} \int_0^c x e^{-x}\, dx = 1$$

$$\int_0^\infty x e^{-x}\, dx = 1.$$

Exercitiu: Arătați că integrala improprie

$$\int_0^1 \frac{1}{x^\alpha}\, dx$$ este converg. dacă $0 < \alpha < 1$

și diverg. dacă $\alpha \geq 1$.

Exerciții:

1) Calculați $\int_0^{\frac{\pi}{2}} (\sin x)^{2n} dx$ și $\int_0^{2\pi} (\sin x)^{2n+1} dx$ cu ajutorul funcției beta a lui Euler.

2) Arătați că integrala improprie $\int_0^{\infty} \frac{\ln x}{x^2+a^2} dx$, $a>0$ este convergentă și determinați valoarea ei

3) Fie $f:[0,\infty) \longrightarrow [0,\infty)$ o funcție continuă a.î. $\int_0^{\infty} f(x)\,dx < \infty$. Arătați că

$$\lim_{n \to \infty} \frac{1}{n} \int_0^{n} x f(x)\,dx = 0.$$