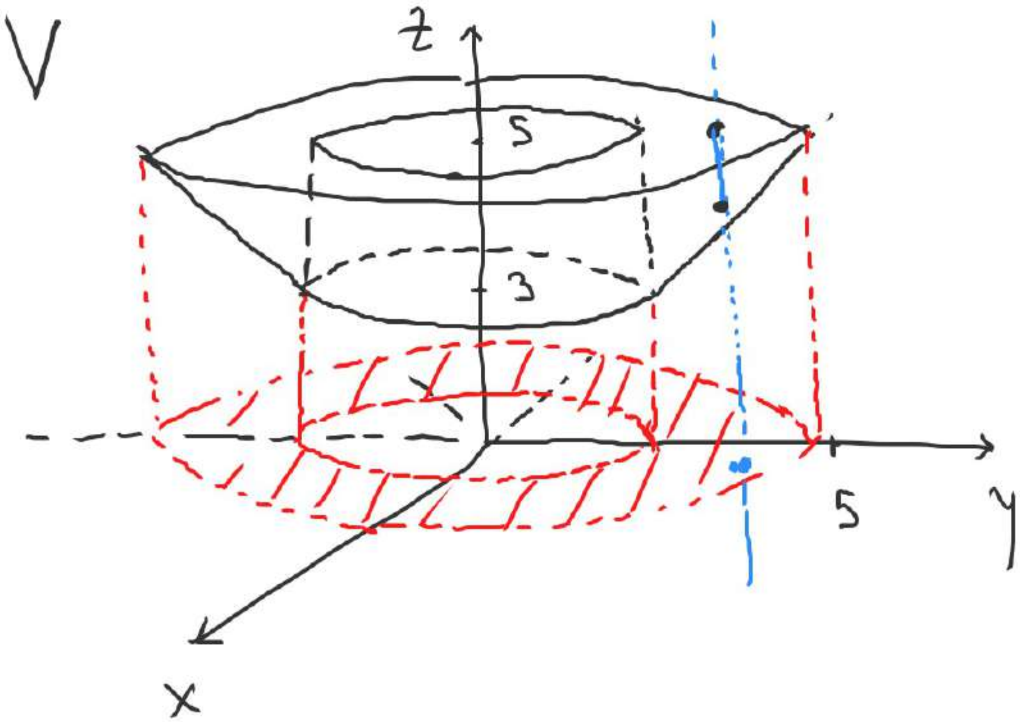
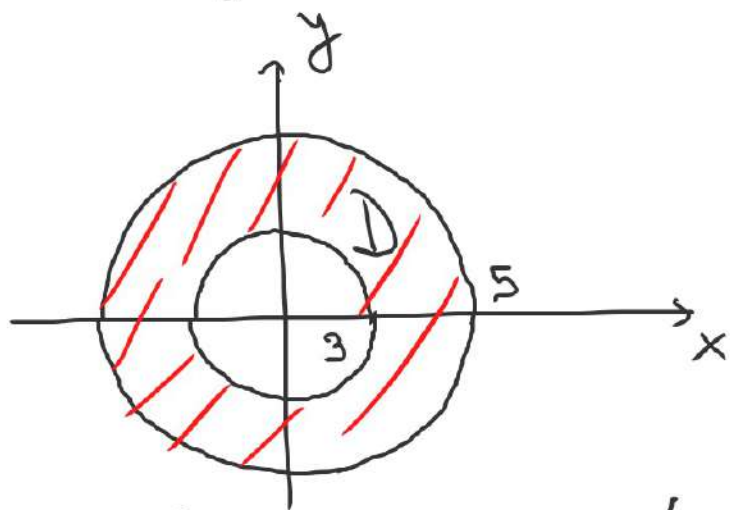
$$I = \iiint_V xyz \, dx\,dy\,dz \qquad V = \{(x,y,z) \mid 9 \le x^2+y^2 \le z^2,\ 0 \le z \le 5\}$$



$x^2+y^2=z^2$ - con

$x^2+y^2=9$ - cilindru.



$$V = \{(x,y,z) \in \mathbb{R}^3 \mid (x,y) \in D,\ \sqrt{x^2+y^2} \le z \le 5.\} \qquad D = pr_{xoy} V$$

$$D = \{(x,y) \in \mathbb{R}^2 \mid 9 \le x^2+y^2 \le 25\}$$

$$I = \iint_D \left(\int_{\sqrt{x^2+y^2}}^{5} xyz \, dz \right) dx\,dy = \iint_D \frac{xyz^2}{2} \Big|_{\sqrt{x^2+y^2}}^{5} dx\,dy$$

$$= \frac{1}{2} \iint_D xy(25 - x^2 - y^2)\, dx\,dy$$

$D: \ 9 \le x^2+y^2 \le 25.$

$$\begin{cases} x = r\cos\theta, & r \in [3,5] \\ y = r\sin\theta & \theta \in [0, 2\pi]. \end{cases}$$

$dx\,dy = r\,dr\,d\theta$

$\parallel$

$$\frac{1}{2} \iint_{[3,5]\times[0,2\pi]} (25 - r^2) \cdot r^2 \cos\theta \sin\theta \, r\, dr\, d\theta = \frac{1}{2} \int_3^5 \left(\int_0^{2\pi} r^3 (25 - r^2) \sin\theta \cos\theta \, d\theta \right) dr$$

$$= \frac{1}{2}\int_3^5 (25-r^2)\,r^3\,dr \cdot \int_0^{2\pi} \sin\theta\cos\theta\,d\theta$$

$$= \frac{1}{2}\left(25\,\frac{r^4}{4} - \frac{r^6}{6}\right)\Bigg|_3^5 \cdot \ \frac{\sin^2\theta}{2}\Bigg|_0^{2\pi} = 0.$$

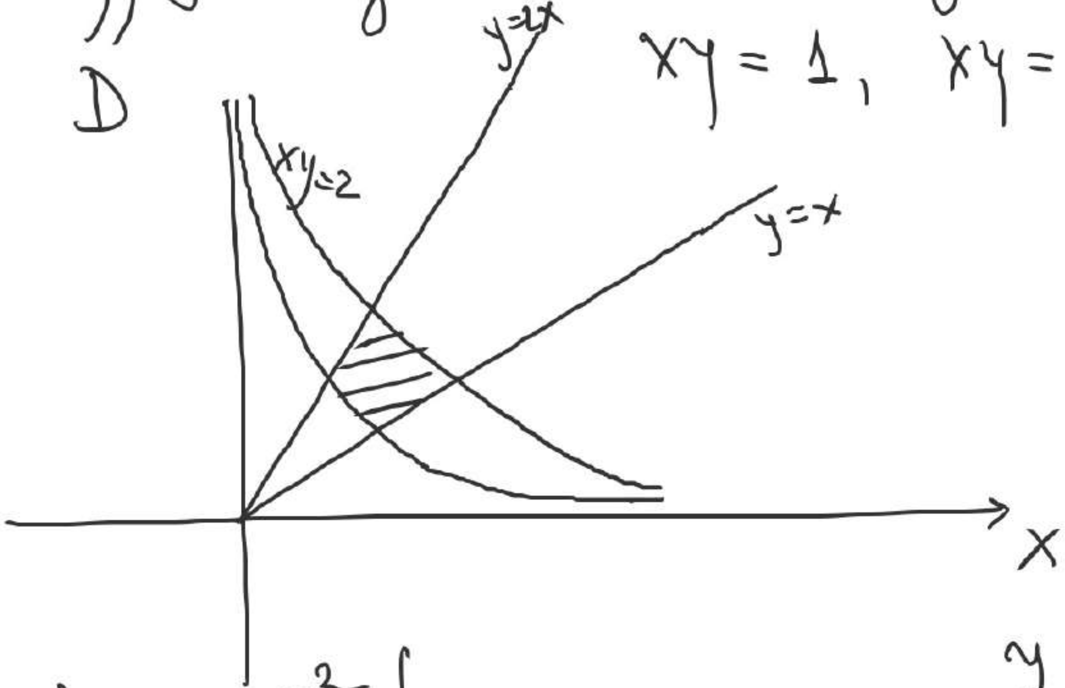
Sau.

$$V_z = \{(x,y) \mid 9 \le x^2+y^2 \le z^2\}$$

Prop 3, curs 19 $\Longrightarrow$ $I = \int_3^5 \left(\iint_{V_z} xyz\,dx\,dy\right) dz = \dots$

2) $\iint\limits_D y\,dx\,dy$ $\quad D$ este mărg. de curbele

$xy = 1,\ xy = 2,\ y = x$ și $y = 2x$.



$$D = \left\{ (x,y) \in \mathbb{R}^2 \ \middle|\ 1 \le xy \le 2;\ 1 \le \frac{y}{x} \le 2 \right\}$$

$$U = (0,\infty) \times (0,\infty),\quad T : U \longrightarrow U,\quad T(x,y) = \left(xy, \frac{y}{x} \right)$$

Fie $u, v > 0$.

$$\begin{cases} u = xy \\ v = \dfrac{y}{x} \end{cases} \Longleftrightarrow \begin{cases} y = vx \\ u = yx = vx^2 \end{cases} \Longleftrightarrow \begin{cases} x^2 = \dfrac{u}{v} \\ y = vx \end{cases} \Longleftrightarrow \begin{cases} x = \sqrt{\dfrac{u}{v}} \\ y = v \cdot \sqrt{\dfrac{u}{v}} = \sqrt{uv} \end{cases}$$

Deci $\forall (u,v) \in U$, $\exists ! (x,y)$ a.i. $T(x,y) = (u,v)$

T bijectivă $\quad T^{-1}(u,v) = (x,y) = \left(\sqrt{\dfrac{u}{v}}, \sqrt{uv} \right)$

$$\det J_{T^{-1}}(u,v) = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} \dfrac{1}{v} \cdot \dfrac{1}{2\sqrt{\frac{u}{v}}} & -\dfrac{u}{v^2} \dfrac{1}{2\sqrt{\frac{u}{v}}} \\ \dfrac{\sqrt{v}}{2\sqrt{u}} & \dfrac{\sqrt{u}}{2\sqrt{v}} \end{vmatrix} =$$

$$= \frac{1}{2v} \sqrt{\frac{v}{u}} \cdot \frac{1}{2} \sqrt{\frac{u}{v}} + \frac{1}{4} \cdot \frac{u}{v^2} \cdot \frac{v}{u} = \frac{1}{2v} \neq 0.$$

T^{-1} bijectivă, de clasă C^1 și $\det J_{T^{-1}}(u,v) \neq 0$, $\forall (u,v) \in U$

$\Longrightarrow T^{-1}$ difeomorfism de clasă C^1.

$$T^{-1}(u,v) = (x,y) = \left(\sqrt{\frac{u}{v}}, \sqrt{uv} \right)$$

$$(x,y) \in D \Longleftrightarrow \begin{cases} 1 \leq xy \leq 2 \\ 1 \leq \frac{y}{x} \leq 2 \end{cases} \Longleftrightarrow T(x,y) \in [1,2] \times [1,2]$$

$$\Longleftrightarrow (x,y) \in T^{-1}\left([1,2] \times [1,2] \right) = T^{-1}(A), \quad A = [1,2] \times [1,2]$$

$A \leftrightarrow D$.

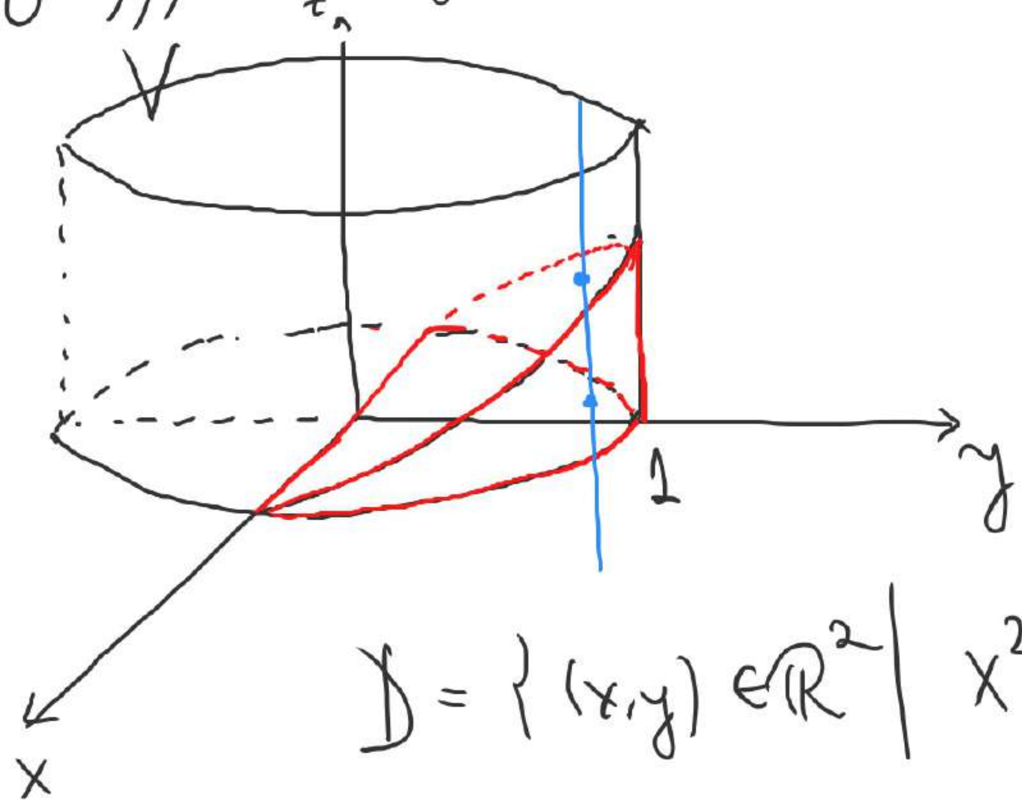
$$\iint_D y \, dx\,dy = \iint_{T^{-1}(A)} y \, dx\,dy = \iint_A \sqrt{uv} \cdot \left| \det J_{T^{-1}}(u,v) \right| du\,dv$$

$$= \iint_A \sqrt{uv} \cdot \frac{1}{2v} \, du \, dv = \frac{1}{2} \int_1^2 \left(\int_1^2 \sqrt{u} \cdot \frac{1}{\sqrt{v}} \, dv \right) du =$$

$$= \frac{1}{2} \int_1^2 \sqrt{u} \cdot 2\sqrt{v} \Big|_{v=1}^{v=2} du = \frac{1}{2} \left(2\sqrt{2} - 2 \right) \cdot \frac{u^{\frac{3}{2}}}{\frac{3}{2}} \Bigg|_1^2$$

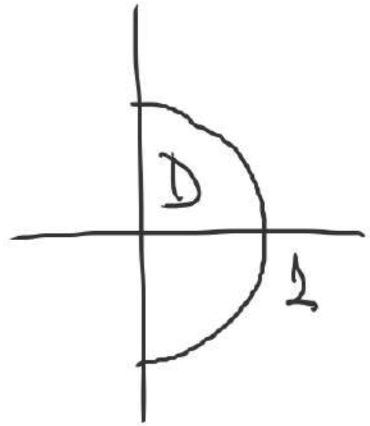
$$= \left(\sqrt{2} - 1 \right) \cdot \frac{2\sqrt{2} - 1}{\frac{3}{2}} = \frac{2}{3} \left(\sqrt{2} - 1 \right) \left(2\sqrt{2} - 1 \right).$$

3) $\mathcal{J} = \iiint\limits_V z\,dx\,dy\,dz \quad , \quad V = \{(x,y,z)\in\mathbb{R}^3 \mid x^2+y^2\le 1,\ 0\le z\le y\}$



$x^2+y^2=R^2$ – cilindru

$D = pr_{xOy} V$



$$D = \{(x,y)\in\mathbb{R}^2 \mid x^2+y^2\le 1,\ y\ge 0\}$$

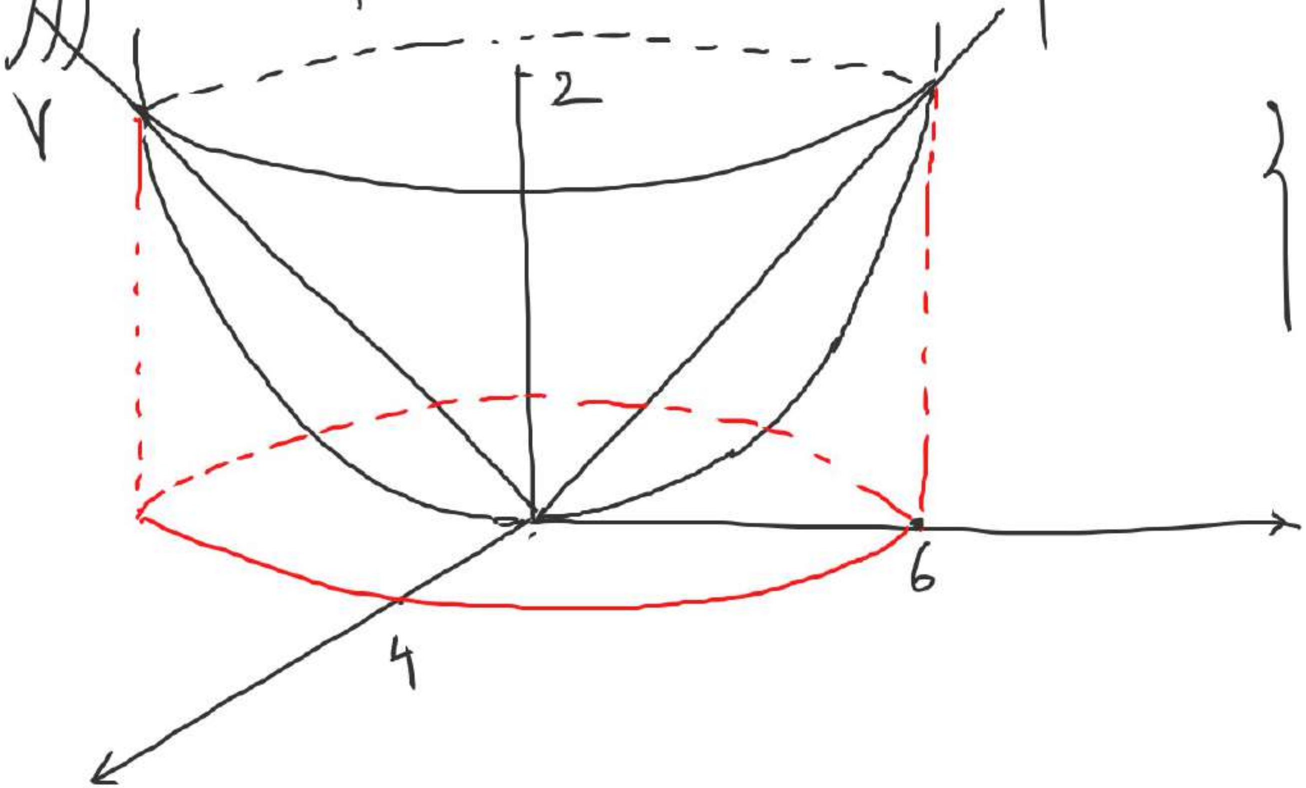
$$V = \{(x,y,z) \mid (x,y)\in D,\ 0\le z\le y\}$$

$$J = \iint_D \left(\int_0^y z\,dz \right) dx\,dy = \iint_D \frac{z^2}{2} \Big|_{z=0}^{z=y} dx\,dy = \iint_D \frac{y^2}{2}\,dx\,dy$$

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases} \quad r \in [0,1], \quad \theta \in [0,\pi] \quad dx\,dy = r\,dr\,d\theta$$

$$= \int_0^1 \left(\int_0^\pi \frac{r^3 \sin^2\theta}{2}\,d\theta \right) dr = \int_0^1 \frac{r^3}{2}\,dr \cdot \int_0^\pi \sin^2\theta\,d\theta = \dots$$

4) $\iiint\limits_V x^2\,dx\,dy\,dz = \quad V = \left\{ (x,y,z) \;\middle|\; z^2 \le \frac{x^2}{4} + \frac{y^2}{9} \le 2z \right\}$



$$\begin{cases} \frac{x^2}{4} + \frac{y^2}{9} = 2z \\ \frac{x^2}{4} + \frac{y^2}{9} = z^2 \end{cases}$$

$z^2 = 2z \Rightarrow z \in \{0, 2\}$

Intersectia dintre con si paraboloid este

elipsa $\quad \frac{x^2}{4} + \frac{y^2}{9} = 4$ din planul $z = 2$

$$\Updownarrow$$

$$\frac{x^2}{16} + \frac{y^2}{36} = 1$$

$$D = \mathrm{pr}_{xOy} V = \left\{ (x,y) \in \mathbb{R}^2 \,\middle|\, \frac{x^2}{16} + \frac{y^2}{36} \leq 1 \right\}$$

$$V = \left\{ (x,y,z) \in \mathbb{R}^3 \,\middle|\, (x,y) \in D,\ \frac{x^2}{8} + \frac{y^2}{18} \leq z \leq \sqrt{\frac{x^2}{4} + \frac{y^2}{9}} \right\}$$

$$\iiint_V x^2 \, dx\,dy\,dz = \iint_D \left(\int_{\frac{x^2}{8} + \frac{y^2}{16}}^{\sqrt{\frac{x^2}{4} + \frac{y^2}{9}}} x^2 \, dz \right) dx\,dy$$

$$= \iint_D x^2 \left(\sqrt{\frac{x^2}{4} + \frac{y^2}{9}} - \frac{x^2}{8} - \frac{y^2}{16} \right) dx\,dy = (*)$$

$$\begin{cases} x = 4r\cos\theta & r \in [0,1] \\ y = 6r\sin\theta & \theta \in [0, 2\pi] \end{cases}, \quad dx\,dy = 24 r \, dr \, d\theta$$

$$\frac{x^2}{4} + \frac{y^2}{16} = 4r^2$$

$$(*) = \int_0^1 \left(\int_0^{2\pi} 16 r^2 \cos^2\theta \cdot (2r - 2r^2) \cdot 24 r \, d\theta \right) dr$$

$$= \int_0^1 \left(\int_0^{2\pi} 32 \cdot 24 (r^4 - r^5) \cdot \cos^2\theta \, d\theta \right) dr = 32 \cdot 24 \pi \left(\frac{1}{5} - \frac{1}{6} \right)$$

$$\int_0^{2\pi} \cos^2\theta \, d\theta = \int_0^{2\pi} \frac{\cos 2\theta + 1}{2} \, d\theta = \pi$$