Seminar 1

① $A = \begin{pmatrix} a & b & c \\ b & c & a \\ c & a & b \end{pmatrix}$ a) $\det A = \Delta = ?$

b) $\Delta = 0 \Leftrightarrow a+b+c=0$ sau $a=b=c$

Δ matrice circulară

$$\Delta = (a+b+c)(-(b-c)^2 \pm (a-b)(a-c))$$

$$= -(a+b+c)(c^2-2bc+b^2+a^2+bc) = -(a+b+c)(a^2+b^2+c^2-ac-bc-ab) = -\frac{1}{2}(a+b+c)(a^2-2ab+b^2+a^2-2ac+c^2+b^2-2bc+c^2) = -\frac{1}{2}(a+b+c)((a-b)^2+(a-c)^2+(b-c)^2)$$

2. Fie $A = \begin{pmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{pmatrix} \in \mathcal{M}_3(\mathbb{R})$

$$\det(A) = V(a,b,c) = \begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 \\ a & b-a & c-a \\ a^2 & (b-a)(b+a) & (c-a)(c+a) \end{vmatrix} = (b-a)(c-a)\underbrace{\begin{vmatrix} 1 & 1 \\ b+a & c+a \end{vmatrix}}_{c-b} = (c-b)(c-a)(b-a)$$

3. Fie $A = \begin{pmatrix} 1+a^2 & ba & ca \\ ba & 1+b^2 & cb \\ ca & bc & 1+c^2 \end{pmatrix} \in \mathcal{M}_3(\mathbb{R})$

$\det(A^*) = ?$

$$\begin{vmatrix} 1+a^2 & 0+ba & 0+ca \\ 0+ab & 1+b^2 & 0+cb \\ 0+ac & 0+bc & 1+c^2 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} + \begin{vmatrix} a^2 & 0 & 0 \\ ab & 1 & 0 \\ ac & 0 & 1 \end{vmatrix} + \begin{vmatrix} 1 & ab & 0 \\ 0 & b^2 & 0 \\ 0 & bc & 1 \end{vmatrix} + \begin{vmatrix} 1 & 0 & ca \\ 0 & 1 & cb \\ 0 & 0 & c^2 \end{vmatrix} = 1+a^2+b^2+c^2$$

$$\det(A^*) = (\det(A))^2 = (1+a^2+b^2+c^2)^2$$

4. $A = \begin{pmatrix} 2 & -1 & 3m+4 \\ 1 & m & 1 \\ -1 & -1 & 0 \end{pmatrix} \in \mathcal{M}_3(\mathbb{Z})$

a) $m = ?$ a.î. $A^{-1} \in \mathcal{M}_3(\mathbb{Z})$

$$A^{-1} \in \mathcal{M}_3(\mathbb{Z}) \Leftrightarrow \det(A^{-1}) \in \mathbb{Z} \Leftrightarrow \frac{1}{\det(A)} \in \mathbb{Z} \Rightarrow \det(A) \in \{\pm 1\}$$

$$\begin{vmatrix} 2 & -1 & 3m+4 \\ 1 & m & 1 \\ -1 & -1 & 0 \end{vmatrix} = \begin{vmatrix} 2 & -3 & 3m+4 \\ 1 & m-1 & 1 \\ -1 & 0 & 0 \end{vmatrix} = (-1)^5 \begin{vmatrix} -3 & 3m+4 \\ m-1 & 1 \end{vmatrix} \cdot (-1) = -3-(m-1)(3m+4) =$$

$$= -3-(3m^2+4m-3m-4) = 3m^2+m-1$$

$3m^2+m-1 = 1 \Leftrightarrow 3m^2+m-2=0$

$\Delta = 1+24 = 25$

$m_{1,2} = \frac{-1\pm 5}{6} \begin{cases} -1 \\ \frac{2}{3} \notin \mathbb{Z} \end{cases}$

$3m^2+m-1=-1$

$\Rightarrow m(3m+1)=0 \Rightarrow m=0$ sau $m=\frac{1}{3} \notin \mathbb{Z}$

$\Rightarrow m \in \{-1, 0\}$

b) $m=0 \Rightarrow A=\begin{pmatrix} 2 & -1 & 4 \\ 1 & 0 & 1 \\ -1 & -1 & 0 \end{pmatrix}$

$A^{-1}=?$

I. $\Delta_T=-1$

$A^T=\begin{pmatrix} 2 & 1 & -1 \\ -1 & 0 & -1 \\ 4 & 1 & 0 \end{pmatrix}$ $\quad A^*=\begin{pmatrix} 1 & -4 & -1 \\ -1 & 4 & 2 \\ -1 & 3 & 1 \end{pmatrix}$

$A^{-1}=\begin{pmatrix} -1 & 4 & 1 \\ 1 & -4 & -2 \\ 1 & -3 & -1 \end{pmatrix}$

II. Gauss-Jordan

$$(A|I_3)=\left(\begin{array}{ccc|ccc} 2 & -1 & 4 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ -1 & -1 & 0 & 0 & 0 & 1 \end{array}\right) \sim \begin{pmatrix} 1&0&1&0&1&0 \\ 2&-1&4&1&0&0 \\ 1&1&0&0&0&1 \end{pmatrix} \sim \begin{pmatrix} 1&0&1&0&1&0 \\ 0&-1&2&1&-2&0 \\ 0&-1&1&0&1&1 \end{pmatrix} \sim \begin{pmatrix} 1&0&1&0&1&0 \\ 0&-1&2&1&-2&0 \\ 0&0&-1&-1&3&1 \end{pmatrix} \sim \begin{pmatrix} 1&0&1&0&1&0 \\ 0&1&-2&-1&2&0 \\ 0&0&1&1&-3&-1 \end{pmatrix} \sim \left(\begin{array}{ccc} 1&0&0 \\ 0&1&0 \\ 0&0&1 \end{array} \begin{array}{ccc} -1&4&1 \\ 1&-4&-2 \\ 1&-3&-1 \end{array}\right)$$

$L_2'=L_2-2L_1$

$L_3'=L_3+L_1$

$L_3'=L_3-L_2$

$C=I_3 \quad B=A^{-1}$

5. $A\in \mathcal{M}_2(\mathbb{C})$, $(\exists)\, k\geq 2$ a.î. $A^k=O_2 \Rightarrow A^2=O_2$

Din C-H: $A^2-\operatorname{tr}(A)A+\det(A)I_2=O_2$

$A^k=O_2 \Rightarrow \det(A)=0 \Rightarrow A^2=\operatorname{tr}(A)\cdot A \Rightarrow A^k=(\operatorname{tr}(A))^{k-1}\cdot A=O_2 \;|\operatorname{Tr} \Rightarrow (\operatorname{tr}(A))^{k-1}\cdot\operatorname{tr}(A)=0$

$\Rightarrow \operatorname{tr}(A)=0$

$\Rightarrow A^2=O_2$

8. $A\in\mathcal{M}_n(\mathbb{R})$

a) $A^2=O_n$, at. I_n-A, I_n+A inv.

$I_n=I_n^2-A^2$

$I_n=(I_n-A)(I_n+A)=(I_n+A)(I_n-A)$

$\Rightarrow (I_n-A)^{-1}=I_n+A$

$(I_n+A)^{-1}=I_n-A$

Obs

a) $A^2=\alpha A \Rightarrow A^k=\alpha^{k-1}A$

$(\forall)k\geq 2$

b) $\operatorname{Tr}(\alpha A)=\alpha\operatorname{Tr}(A)$

Obs

$n=2$, $A^2=O_2 \Rightarrow \det(A)=0$

$\underbrace{A^2}_{O_2}-\operatorname{Tr}(A)A+\underbrace{\det(A)}_{0}I_2=O_2 \Rightarrow \operatorname{Tr}(A)A=O_2 \;|\operatorname{Tr}$

$\Rightarrow (\operatorname{Tr}(A))^2=0 \Rightarrow \operatorname{Tr}(A)=0$

$A=\begin{pmatrix} a & b \\ c & d\end{pmatrix}^2$ $\quad ad-bc=0$, $a+d=0$ $\quad I_2+A=\begin{pmatrix} a+1 & b \\ c & d+1\end{pmatrix}$

$\det(I_2+A)=\begin{vmatrix} a+1 & b \\ c & d+1\end{vmatrix}=(a+1)(d+1)-bc=$

$=\underbrace{ad-bc}_{0}+\underbrace{a+d}_{0}+1=1$

b) $A^3 = O_3 \Rightarrow I_n + A,\ I_n - A$ inv.

$I_n = I_n^3 - A^3 = (I_n - A)(I_n^2 + A + A^2) = (I_n^2 + A + A^3)(I_n - A)$

$(I_n - A)^{-1} = I_n + A + A^2$

Analog $I_n = I_n^3 + A^3$, $(I_n + A)$ inv.

$(I_n + A)^{-1} = I_n - A + A^2$

11. $A = \begin{pmatrix} 1 & 2 & 3 & 1 \\ 2 & 0 & a & 1 \\ 0 & 1 & 3 & b \end{pmatrix} \in \mathcal{M}_{3,4}(\mathbb{R})$

$a, b = ?$ a.î. $rg(A) = 2$

$\Delta_1 = \begin{vmatrix} 1 & 2 & 3 \\ 2 & 0 & a \\ 0 & 1 & 3 \end{vmatrix} = \begin{vmatrix} 1 & 2 & 3 \\ 0 & -4 & a-6 \\ 0 & 1 & 3 \end{vmatrix} = \begin{vmatrix} -4 & a-6 \\ 1 & 3 \end{vmatrix} = -12 - a + 6 = -a - 6$

$\Rightarrow -a - 6 = 0,\ a = -6$

$\Delta_2 = \begin{vmatrix} 1 & 2 & 1 \\ 2 & 0 & 1 \\ 0 & 1 & b \end{vmatrix} = 2 - 1 - 4b = 1 - 4b$

$1 - 4b = 0 \Rightarrow b = \frac{1}{4}$

Ex

c) $A = \begin{pmatrix} \hat{2} & \hat{2} \\ \hat{1} & \hat{3} \end{pmatrix} \in \mathcal{M}_2(\mathbb{Z}_6)$

A - inv ?

$(\mathbb{Z}_6, +, \cdot)$ - inel

$U(\mathbb{Z}_6) = \{\hat{1}, \hat{5}\}$ (prime cu 6)

Prop.

$A \in \mathcal{M}_2(\mathbb{Z}_6)$ inv. $\Leftrightarrow \det(A) \in U(\mathbb{Z}_6)$

$\det(A) = \hat{4} \notin U(\mathbb{Z}_6) \Rightarrow A$ nu este inv.

d) $A = \begin{pmatrix} \hat{1} & \hat{2} \\ \hat{3} & \hat{4} \end{pmatrix} \in \mathcal{M}_2(\mathbb{Z}_5)$

$\det(A) = \hat{4} - \hat{1} = \hat{3} \in U(\mathbb{Z}_5)$ (5 - prim)

$\Rightarrow A$ inv.

$\left(\begin{array}{cc|cc} \hat{1} & \hat{2} & \hat{1} & \hat{0} \\ \hat{3} & \hat{4} & \hat{0} & \hat{1} \end{array}\right) \sim \left(\begin{array}{cc|cc} \hat{1} & \hat{2} & \hat{1} & \hat{0} \\ \hat{0} & \hat{3} & \hat{2} & \hat{1} \end{array}\right) \sim \left(\begin{array}{cc|cc} \hat{1} & \hat{2} & \hat{1} & \hat{0} \\ \hat{0} & \hat{1} & \hat{4} & \hat{2} \end{array}\right) \sim \left(\begin{array}{cc|cc} \hat{1} & \hat{0} & \hat{3} & \hat{1} \\ \hat{0} & \hat{1} & \hat{4} & \hat{2} \end{array}\right)$

$L_2 = L_2 + 2L_1$

(8) 13) $A = \begin{pmatrix} a^3 & 3a^2 & 3a & 1 \\ a^2 & a^2+2a & 2a+1 & 1 \\ a & 2a+1 & a+2 & 1 \\ 1 & 3 & 3 & 1 \end{pmatrix}$ $\det A =$

$= \begin{vmatrix} a^3-1 & 3(a^2-1) & 3(a-1) & 0 \\ a^2-1 & a^2+2a-3 & 2(a-1) & 0 \\ a-1 & 2(a-1) & a-1 & 0 \\ 1 & 3 & 3 & 1 \end{vmatrix} = (a-1)^3 \begin{vmatrix} a^2+a+1 & 3(a+1) & 3 \\ a+1 & a+3 & 2 \\ 1 & 2 & 1 \end{vmatrix}$

$= (a-1)^3 \begin{vmatrix} a^2+a-2 & 3(a-1) & 3 \\ a-1 & a-1 & 2 \\ 0 & 0 & 1 \end{vmatrix} = (a-1)^5 \begin{vmatrix} a+2 & 3 \\ 1 & 1 \end{vmatrix} = (a-1)^6$