https://www.youtube.com/watch?v=t-_VPRCtiUg&t=51s

https://www.youtube.com/watch?v=5JbpcsH80us

https://www.youtube.com/watch?v=XggxeuFDaDU

https://www.youtube.com/watch?v=cBkdZ71SpPs

Driven oscillator

Oscilatii fortate

1. Driven oscillator - solution with phasors

$$\ddot{x} + \frac{r}{m}\dot{x} + \frac{k}{m}x = \frac{F_0}{m}\cos\Omega t$$

$$\ddot{x} + 2b\dot{x} + \omega^2 x = \frac{F_0}{m}\cos\Omega t$$

Solution of the form: $x(t)=x_0(t)+x_p(t)$ with $x_0(t)=e^{-bt}(C_1e^{-i\omega' t}+C_2e^{-i\omega' t})$ for *underdamped* oscillations.

The particular solution $x_p(t)$ is chosen at time infinity, when the *stationary* regime (steady state) is assumed to be obtained.

$$x_p = B\cos(\Omega t+\beta)$$

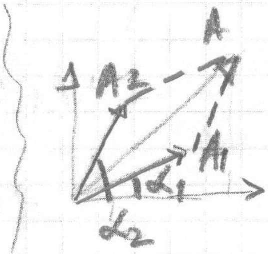
$$(-\Omega^2+\omega^2)B\cos(\Omega t+\beta) - 2b\Omega B\sin(\Omega t+\beta) = \frac{F_0}{m}\cos\Omega t$$

$$\underbrace{B(-\Omega^2+\omega^2)}_{A_1}\cos(\Omega t+\underbrace{\beta}_{\alpha_1}) + \underbrace{2b\Omega B}_{A_2}\cos(\Omega t+\underbrace{\beta+\tfrac{\pi}{2}}_{\alpha_2}) = \underbrace{\boxed{\frac{F_0}{m}}}_{A}\cos\Omega t \qquad A\cos(\Omega t+\alpha)$$

Assume: $\omega > \Omega$

$$A_1\cos(\Omega t+\alpha_1) + A_2\cos(\Omega t+\alpha_2) = A\cos(\Omega t+0)$$

$$A = \frac{F_0}{m} = \sqrt{A_1^2+A_2^2+2A_1A_2\cos(\underbrace{\alpha_2-\alpha_1}_{\frac{\pi}{2}})} = \sqrt{A_1^2+A_2^2}$$



$$\begin{cases}\cos\alpha = \cos 0 = 1 = \dfrac{A_1\cos\beta + A_2\cos(\beta+\frac{\pi}{2})}{\sqrt{A_1^2+A_2^2}}\\[2ex] \sin\alpha = 0 = A_1\sin\beta + A_2\sin(\beta+\frac{\pi}{2})\end{cases}$$

$$\Rightarrow \begin{cases}A_1\cos\beta - A_2\sin\beta = \sqrt{A_1^2+A_2^2} & \| \times A_1\\ A_2\cos\beta + A_1\sin\beta = 0 & | \times A_2\end{cases}$$

$$\begin{cases}\cos\beta = \dfrac{\sqrt{A_1^2+A_2^2}\cdot A_1}{A_1^2+A_2^2} = \dfrac{A_1}{\sqrt{A_1^2+A_2^2}} = \dfrac{(-\Omega^2+\omega^2)B}{\sqrt{A_1^2+A_2^2}} \\[2ex] \sin\beta = -\dfrac{A_2}{A_1}\cos\beta = -\dfrac{A_2}{\sqrt{A_1^2+A_2^2}} = \dfrac{-2b\Omega B}{\sqrt{A_1^2+A_2^2}} < 0 \Rightarrow \boxed{\beta\in(\pi, 2\pi)}\end{cases}$$

$>0 \Leftrightarrow \boxed{\beta\in(0, \pi/2) \;\&\; (3\pi/2, 2\pi)}$, $\omega>\Omega$

$<0 \Leftrightarrow \boxed{\beta\in(\pi/2, 3\pi/2)}$

$$\operatorname{tg}\beta = -\frac{A_2}{A_1} = \frac{2b\Omega}{\Omega^2-\omega^2}$$

$\Downarrow$ $\boxed{\beta\in(3\pi/2, 2\pi/2)}$

Case a): $\beta \in (3\pi/2, 2\pi/2) \wedge \omega>\Omega$

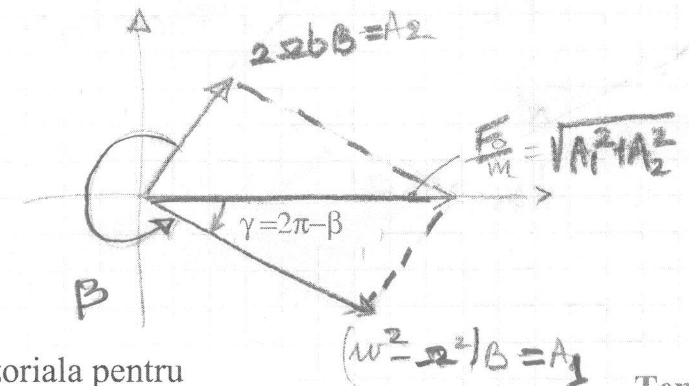


Diagrama fazoriala pentru oscilatii fortate, caz ω>Ω

$$\sin\beta = \frac{-2\Omega bB}{F_0/m}$$

$$\sin\beta = -\sin(2\pi-\beta)$$

$$= -\sin\gamma = -\frac{2\Omega bB}{F_0/m}$$

Tema: Obtineti diagrama fazoriala pentru oscilatii fortate in cazul ω<Ω.

1. Driven oscillator – solution 2 (complex numbers)

$$\ddot{x} + 2b\dot{x} + \omega^2 x = \frac{F_0}{m}\cos\Omega t \qquad (1)$$

$$x(t) = Re[z(t)] \qquad (2)$$

$$\Rightarrow \ddot{z} + 2b\dot{z} + \omega^2 z = \frac{F_0}{m} e^{i\Omega t} \qquad (3)$$

because $Re[\ddot{z}] = Re\left[\frac{d^2 z}{dt^2}\right] = \frac{d^2}{dt^2} Re(z) = \ddot{x}$

Search for particular solution, $x_p(t)$ as $Re[z_p(t)]$ and guess $z_p(t) = C e^{i\Omega t}$. One replaces

$z_p(t)$ in (3) $\Rightarrow$

$$\left(-\Omega^2 + 2ib\Omega + \omega^2\right) z_p = \frac{F_0}{m} e^{i\Omega t} \qquad (4)$$

$$z_p(t) = \frac{\frac{F_0}{m} e^{i\Omega t}}{\omega^2 - \Omega^2 + 2ib\Omega} \qquad (5)$$

$$x_p(t) = Re[z_p(t)] \qquad (6)$$

$$\Rightarrow x_p(t) = \frac{F_0}{m} Re\left[\frac{\cos\Omega t + i\sin\Omega t}{(\omega^2-\Omega^2)^2 + 4b^2\Omega^2}\left(\omega^2 - \Omega^2 - 2ib\Omega\right)\right]$$

$$= \frac{F_0}{m}\left[\frac{\cos\Omega t\,(\omega^2-\Omega^2) + \sin\Omega t\; 2b\Omega}{(\omega^2-\Omega^2)^2 + 4b^2\Omega^2}\right]$$

$$\equiv B(\Omega)\,\frac{\left[(\omega^2-\Omega^2)\cos\Omega t + 2b\Omega\sin\Omega t\right]}{\sqrt{(\omega^2-\Omega^2)^2 + 4b^2\Omega^2}}$$

2. Resonance curve
Curba de rezonanta

$$B(\Omega) = \frac{F_0}{m\sqrt{(\omega^2-\Omega^2)^2+4b^2\Omega^2}} = \frac{F_0}{m\sqrt{f(\Omega)}}$$

$$f(\Omega) = \Omega^4 + (4b^2 - 2\omega^2)\Omega^2 + \omega^4$$

$$B = B_{max} \Leftrightarrow f(\Omega) = \min \Rightarrow$$

$$\begin{cases} \Omega^2_{min} = \left(\frac{-b}{2a}\right) = -2b^2 + \omega^2; & (\omega^2 > 2b^2 \text{ cond. de rezonanță}) \\ \Omega_{rez} \equiv \Omega_{min} \end{cases}$$

$$B_{max} = B(\Omega_{min}) = \frac{F_0}{m\sqrt{(\omega^2-\omega^2+2b^2)^2+4b^2(\omega^2-2b^2)}} =$$

$$= \frac{F_0}{2mb\sqrt{\omega^2-b^2}}$$

Close of resonance ($\Omega \simeq \Omega_{rez}$) and *low* damping ($\omega >> b$) aproximation:

$$\begin{cases} \omega^2 - \Omega^2 = (\omega-\Omega)(\omega+\Omega) \simeq 2\omega(\omega-\Omega) \\ 4b^2\Omega^2 \simeq 4b^2\omega^2; \quad B_{max} \simeq \frac{F_0}{2mb\omega} \end{cases} \Rightarrow$$

$$B(\Omega) \simeq \frac{F_0}{m\sqrt{4\omega^2(\omega-\Omega)^2+4b^2\omega^2}} \simeq \frac{F_0}{2m\omega\sqrt{(\omega-\Omega)^2+b^2}}$$

$$= \frac{F_0}{2mb\omega\sqrt{1+\frac{(\omega-\Omega)^2}{b^2}}} = \frac{B_{max}}{\sqrt{1+\frac{(\omega-\Omega)^2}{b^2}}}$$

If $B(\Omega) = \frac{B_{max}}{\sqrt{2}} \Rightarrow (\omega-\Omega)^2 = b^2 \Leftrightarrow |\omega-\Omega| = b \Rightarrow$

$$\Rightarrow \Omega = \omega \pm b.$$