C10

Transformări ortogonale. Endomorfisme simetrice.

Def $(E_i, \langle \cdot,\cdot \rangle_i)$, $i=\overline{1,2}$ s.v.r.e.

$f: E_1 \to E_2$ liniară

f s.n. aplicație ortogonală $\Leftrightarrow \langle f(x), f(y)\rangle_2 = \langle x, y\rangle_1$, $\forall x, y \in E_1$.

Def $(E, \langle \cdot,\cdot \rangle)$ s.v.e.r.

$f \in End(E)$

f transformare ortogonală $\Leftrightarrow \langle f(x), f(y)\rangle = \langle x, y\rangle$, $\forall x, y \in E$.

Prop $f: E_1 \to E_2$ aplicație ortogonală ; $\|x\|_1 = \sqrt{\langle x, x\rangle_1}$

1) $\|f(x)\|_2 = \|x\|_1$, $\forall x \in E_1$

2) f inj

Dem

1) f apl. ortog.

$\langle f(x), f(y)\rangle_2 = \langle x, y\rangle_1$, $\forall x, y \in E_1$

Fie $y = x \Rightarrow \underbrace{\langle f(x), f(x)\rangle_2}_{\|f(x)\|_2^2} = \underbrace{\langle x, x\rangle_1}_{\|x\|_1^2} \Rightarrow \|f(x)\|_2 = \|x\|_1$, $\forall x \in E_1$

2) f inj $\Leftrightarrow Ker f \overset{def}{=} \{x \in E_1 \mid f(x) = 0_{E_2}\} = \{0_{E_1}\}$.

Fie $x \in Ker f$

$\underbrace{\|f(x)\|_2}_{0} = \|x\|_1 \Rightarrow x = 0_{E_1} \Rightarrow Ker f = \{0_{E_1}\} \Rightarrow f$ inj

Not $O(E) = \{f: E \to E$ liniară $\mid f$ transf. ortogonală$\}$.

Teoremă $f \in End(E)$

$f \in O(E) \Leftrightarrow \|f(x)\| = \|x\|$, $\forall x \in E$, unde $(E, \langle \cdot,\cdot \rangle)$ s.v.e.r

Dem

"$\Rightarrow$" cf prop preced

"$\Leftarrow$"

$f \in End(E)$

$\|f(x+y)\|^2 = \|x+y\|^2 \Rightarrow \langle f(x+y), f(x+y)\rangle = \langle x+y, x+y\rangle, \forall x,y \in$

$\underbrace{\langle f(x), f(x)\rangle}_{\substack{\| \\ \|f(x)\|^2 \\ \| \, ip. \\ \|x\|^2}} + 2\langle f(x), f(y)\rangle + \underbrace{\langle f(y), f(y)\rangle}_{\substack{\| \\ \|f(y)\|^2 \\ \| \, ip. \\ \|y\|^2}} = \underbrace{\langle x, x\rangle}_{\substack{\| \\ \|x\|^2}} + 2\langle x, y\rangle + \underbrace{\langle y, y\rangle}_{\substack{\| \\ \|y\|}}$

$\Rightarrow \langle f(x), f(y)\rangle = \langle x, y\rangle, \ \forall x,y \in E \Rightarrow f \in O(E)$.

<u>OBS</u> $(E, \langle \cdot,\cdot \rangle)$ $\quad R = \{e_1, .., e_n\}$ reper ortonormat

$f \in O(E)$ $\quad (\langle e_i, e_j\rangle = \delta_{ij}, \forall i,j = \overline{1,n})$

$A = [f]_{R,R}$, $\quad f(e_i) = \sum_{k=1}^{n} a_{ki} e_k, \ \forall i = \overline{1,n}$

$\langle f(e_i), f(e_j)\rangle = \langle e_i, e_j\rangle = \delta_{ij}, \ \forall i,j = \overline{1,n}$

$\|$

$\langle \sum_{k=1}^{n} a_{ki} e_k, \sum_{r=1}^{n} a_{rj} e_r \rangle$

$\sum_{k,r=1}^{n} a_{ki} a_{rj} \underbrace{\langle e_k, e_r\rangle}_{\substack{\| \\ \delta_{rk}}} = \delta_{ij} \Rightarrow \sum_{k=1}^{n} a_{ki} a_{kj} = \delta_{ij}$

$A^T A = I_n$

$A \in O(n)$

$R = \{e_1, .., e_n\} \xrightarrow{C} R' = \{e'_1, .., e'_n\}$ repere orton.

$A = [f]_{R,R}$ $\quad A' = [f]_{R',R'}$ $\quad A' = C^T A C$

$A'^T A' = \underbrace{(C^T A C)^T}_{\substack{\| \\ C^T A^T (C^T)^T}} (C^T A C) = C^T A^T \underbrace{C C^T}_{I_n} A C = C^T A^T A C = I_n.$

<u>Prop</u> $f \in O(E) \Leftrightarrow A = [f]_{R,R} \in O(n), \ \forall R = $ reper ortonormat

OBS $f \in O(E) \Leftrightarrow$ schimbare de reper ortonormat

$R = \{e_1, \dots, e_n\} \xrightarrow{A} R' = \{e_1', \dots, e_n'\}$ repere orton.

$A = [f]_{R,R'}$ $\quad f(e_i) = e_i'$, $\forall i = \overline{1,n}$

Prop $f \in O(E) \Rightarrow$ valorile proprii $\in \{-1, 1\}$

SOL Fie λ valoare proprie. $\exists \underset{\underset{0_E}{\#}}{x} \in E$ aî $f(x) = \lambda x$.

$\langle f(x), f(x) \rangle = \langle x, x \rangle$, $\forall x \in E$

$\lambda^2 \|x\|^2 = \|x\|^2 \Rightarrow \|x\|^2 (\lambda^2 - 1) = 0 \Rightarrow \lambda^2 = 1 \Rightarrow \boxed{\lambda = \pm 1}$

Prop $f \in O(E)$, $U \subseteq E$ subsp. vect invariant al lui f (i.e. $f(U) \subseteq U$)

a) $f(U) = U$

b) $U^\perp \subseteq E$ subsp. inv.

Mai mult. $f|_{U^\perp} : U^\perp \to U^\perp$ transf. ortog. (din a), b))

Dem

a) f inj, $f : U \to f(U)$ bij + liniara

$\left.\begin{array}{l} \dim U = \dim f(U) \\ \text{dar } f(U) \subseteq U \end{array}\right\} \Rightarrow f(U) = U$

b) $\forall x \in U^\perp \Rightarrow f(x) \in U^\perp$

Fie $z \in U$. Dem $\langle f(x), z \rangle = 0$

$\exists y \in U$ aî $\overset{\overset{f(U)}{\shortparallel}}{z} = f(y)$; $\langle f(x), f(y) \rangle = \langle \underset{U^\perp}{\underset{\cap}{x}}, \underset{U}{\underset{\cap}{y}} \rangle = 0$

Clasificare transf. ortogonale

① $n = 1$ $\quad O(E) = \{id_E, -id_E\}$.

② $n = 2$. $f \in O(E)$, $R = \{e_1, e_2\}$ reper ortonormat în E

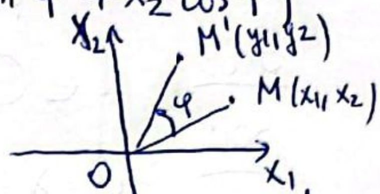
$A = [f]_{R,R} \in O(2)$

a) $\det A = 1$ (f de speță 1) $\quad A = \begin{pmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{pmatrix} = A_\varphi$, $\varphi \in (-\pi, \pi]$.

$f = R_\varphi$ rotație de $\sphericalangle$ orientat φ

$f : \mathbb{R}^2 \to \mathbb{R}^2$, $f(x_1, x_2) = (\overbrace{x_1 \cos\varphi - x_2 \sin\varphi}^{y_1}, \overbrace{x_1 \sin\varphi + x_2 \cos\varphi}^{y_2})$

$\underset{(2,2)}{A} \underset{(2,1)}{\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}} = \underset{(2,1)}{\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}}$



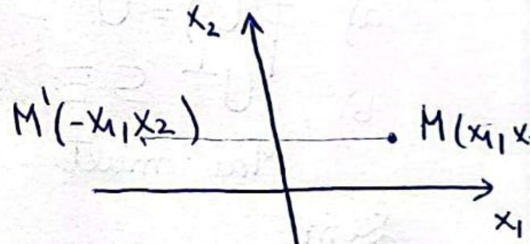
$Tr(A) = 2\cos\varphi$ = invariant la sch. de repere ortonormat.

b) $\det A = -1$ (f de speță 2) $\quad A = \begin{pmatrix} \cos\varphi & \sin\varphi \\ \sin\varphi & -\cos\varphi \end{pmatrix}$

$A = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$

f = simetrie față de $e_1^\perp$

$f : \mathbb{R}^2 \to \mathbb{R}^2$, $f(x_1, x_2) = (-x_1, x_2)$



<u>Teoremă</u> $n = 2$, $f \in O(E)$, $f \neq id_E$

$\Rightarrow f$ se poate scrie ca o „$\circ$" de cel mult 2 simetrii ortogonale față de drepte.

<u>Dem</u> 1) f de speță 1

$\det A = 1$. $\left(A_f = [f]_{R,R}\right.$, R = reper orton$\left.\right)$

Fie s' = simetrie ortog, $\det(A_{s'}) = -1$

$s = s' \circ f$ simetrie ortog

$s' \circ s = \underbrace{s' \circ s'}_{id_E} \circ f \Rightarrow \boxed{f = s' \circ s}$

2) f de speță 2

$f = s$ sim. ortog $\quad A_f = A_s$ are $\det = -1$.

3) $n = 3$, $f \in O(E)$, $A = [f]_{R,R}$ reper orton.

$P(\lambda) = \det(A - \lambda I_3) = 0$ (pol. de grd al 3-lea cu coef reali)

$\exists$ cel puțin o răd reală $\lambda \in \{-1, 1\}$.

Fie e_1 = versorul propriu coresp. valorii proprii λ $f(e_1) = \pm e_1$.

$U = \langle\{e_1\}\rangle \subset E$ subsp. invariant al lui $f \xRightarrow{prop} U^\perp \subset E$ sp. invar. al lui f

$E = U \oplus U^\perp$

$f|_{U^\perp} : U^\perp \to U^\perp$ transf ortog.

I. f este de speță 1 $(\det A = 1)$

Ia) $\lambda = 1$, $f(e_1) = e_1$. $A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \multicolumn{2}{c}{\tilde{A}} \\ 0 & & \end{pmatrix}$

$\det \tilde{A} = 1$, $A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\varphi & -\sin\varphi \\ 0 & \sin\varphi & \cos\varphi \end{pmatrix}$

$f = R_\varphi$ rotație de unghi orientat φ în planul $\langle\{e_1\}\rangle^\perp$, cu axa de rotație $\langle\{e_1\}\rangle$

$$\boxed{\begin{array}{l} \mathrm{Tr}A = 1 + 2\cos\varphi \text{ invariant} \\ \text{Axa de rotație: } f(x) = x. \end{array}}$$

Ib) $\lambda = -1$, $f(e_1) = -e_1$. $A = \begin{pmatrix} -1 & 0 & 0 \\ 0 & \multicolumn{2}{c}{\tilde{A}} \\ 0 & & \end{pmatrix}$

$\det \tilde{A} = -1$ $A = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

$\{e_1, e_2, e_3\}$

$\{e_3, e_1, e_2\}$ $\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\pi & -\sin\pi \\ 0 & \sin\pi & \cos\pi \end{pmatrix}$

$f : \mathbb{R}^3 \to \mathbb{R}^3$, $f(x) = (\underset{x_1'}{\underbrace{x_1}}, \underbrace{x_2\cos\varphi - x_3\sin\varphi}_{x_2'}, \underbrace{x_2\sin\varphi + x_3\cos\varphi}_{x_3'})$

$A_\varphi \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} x_1' \\ x_2' \\ x_3' \end{pmatrix}$ $f = R_\varphi$

II. f este de speța a-2-a $\qquad \det A = -1$

II a) $\lambda = 1 \qquad f(e_1) = e_1 \qquad A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \tilde{A} & \\ 0 & & \end{pmatrix}$

$\det \tilde{A} = -1 \qquad A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

$\{e_1, e_2, e_3\}$

$\{e_2, e_1, e_3\} \qquad \begin{pmatrix} -1 & 0 & 0 \\ 0 & \boxed{1} & \boxed{0} \\ 0 & \boxed{0} & \boxed{1} \end{pmatrix} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & \cos 0 & -\sin 0 \\ 0 & \sin 0 & \cos 0 \end{pmatrix}$

II b) $\lambda = -1 \qquad f(e_1) = -e_1 \qquad A = \begin{pmatrix} -1 & 0 & 0 \\ 0 & \tilde{A} & \\ 0 & & \end{pmatrix}$

$\det \tilde{A} = 1 \qquad A = \begin{pmatrix} -1 & 0 & 0 \\ 0 & \cos\varphi & -\sin\varphi \\ 0 & \sin\varphi & \cos\varphi \end{pmatrix}$

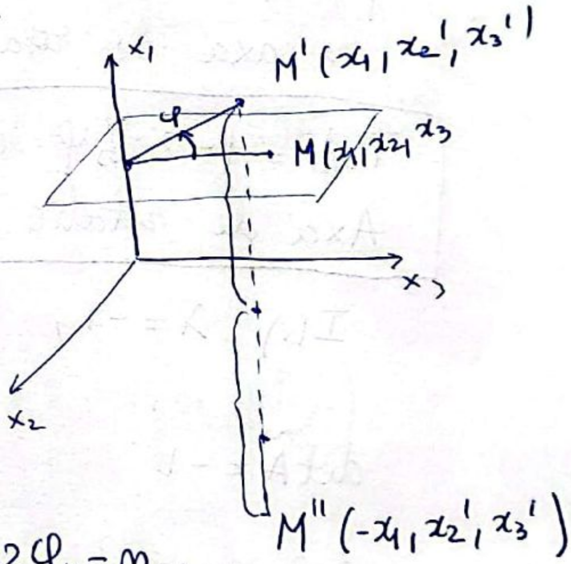
R_φ = rotație de ∢ or. φ în planul $\langle e_1 \rangle^\perp$

$f = s \circ R_\varphi \qquad$, s = simetrie ortog. față de $\langle e_1 \rangle^\perp$

$f: \mathbb{R}^3 \to \mathbb{R}^3$, $f(x) = (\underbrace{-x_1}_{x_1'}, \underbrace{x_2 \cos\varphi - x_3 \sin\varphi}_{x_2'}, \underbrace{x_2 \sin\varphi + x_3 \cos\varphi}_{x_3'})$

$\operatorname{Tr} A = -1 + 2\cos\varphi$ invariant.

Axa: $f(x) = -x$



4. $n \geq 4$.

$\exists\, \mathcal{R}$ reper. ortonormat a.î.

$A = \begin{pmatrix} 1 & & & & & \\ & \ddots & & & & \\ & & 1 & & & \\ & & & -1 & & \\ & & & & \ddots & \\ & & & & & -1 \\ & & & & & & \boxed{A_{\varphi_1}} \\ & & & & & & & \ddots \\ & & & & & & & & \boxed{A_{\varphi_k}} \end{pmatrix} = [f]_{\mathcal{R},\mathcal{R}}$. $\qquad$ (1 de r ori, -1 de s ori)

$r + s + 2\varphi_k = n$.

Teoremă (Cartan) $n \geq 2$, $f \in O(E)$, $f \neq id_E$

f se poate scrie ca o compunere de cel mult n simetrii ortogonale față de hiperplane.

Def $(E, \langle \cdot, \cdot \rangle)$ s.v.e.r, $f \in End(E)$.

f s.n. endomorfism simetric $\iff \langle f(x), y \rangle = \langle x, f(y) \rangle, \forall x, y \in E$

Not $Sim(E) = \{ f \in End(E) \mid f \text{ simetric} \}$

OBS $R = \{e_1, \dots, e_n\} \xrightarrow{C} R' = \{e_1', \dots, e'\}$ repere orton., $C \in O(n)$

$A = [f]_{R,R}$ $\qquad A' = [f]_{R', R'}$

$\langle f(e_i), e_j \rangle = \langle e_i, f(e_j) \rangle$

$\langle \sum_{k=1}^{n} a_{ki} e_k, e_j \rangle = \langle e_i, \sum_{r=1}^{n} a_{rj} e_r \rangle$

$\sum_{k=1}^{n} a_{ki} \underbrace{\langle e_k, e_j \rangle}_{\delta_{kj}} = \sum_{r=1}^{n} a_{rj} \underbrace{\langle e_i, e_r \rangle}_{\delta_{ir}} \Rightarrow a_{ji} = a_{ij}$

$\forall i, j = \overline{1,n} \Rightarrow A = A^T$

$A' = C^T A C$

$(A')^T = (C^T A C)^T = C^T \underbrace{A^T}_{A} C = C^T A C = A'$

Prop $f \in Sim(E) \iff A = [f]_{R,R}$ = simetrică, $\forall R$ = reper orton.

Prop $f \in Sim(E) \Rightarrow$ vectorii proprii coresp. la valori proprii dist. sunt $\perp$

Dem

Fie $\lambda \neq \mu$ valori proprii $\Rightarrow \exists x, y \in E$ ($x, y \neq 0$) a.î. $f(x) = \lambda x$, $f(y) = \mu y$

$\langle \underbrace{f(x)}_{\lambda x}, y \rangle = \langle x, \underbrace{f(y)}_{\mu y} \rangle \Rightarrow \lambda \langle x, y \rangle = \mu \langle x, y \rangle$

$\langle x, y \rangle (\lambda - \mu) = 0 \Rightarrow \langle x, y \rangle = 0$

Teoremă

$f \in Sim(E) \Rightarrow$ toate răd. pol. caract. sunt reale.

Prop $f \in Sim(E)$, $U \subseteq E$ subsp. invariant al lui f

a) $U^\perp$ subsp. invariant al lui f.

b) $f|_{U^\perp} : U^\perp \to U^\perp$ endom. simetric

$A = A^T$ $\longrightarrow$ $Q : E \to \mathbb{R}$

$Q(x) = \sum_{i,j=1}^{n} a_{ij} x_i x_j$

$\searrow$ $f \in Sim(E)$

$[f]_{R,R} = A$

$\langle x, f(x) \rangle = Q(x)$