SEMINAR 6

ȘIRURI DE FUNCȚII:

Șiruri de funcții → convergența simplă | $f_n \xrightarrow{S} f$ |
$\Updownarrow$
→ convergența uniformă | $f_n \xrightarrow{u} f$ |

Ex1: Să se studieze convergența simplă și uniformă a urm. șirului de fct.:

a) $f_n : [0,1] \to \mathbb{R}$, $f_n(x) = \dfrac{x^n}{\sqrt{1+x^n}}$ $\forall x \in [0,1]$, $\forall n \geq 1$

b) $f_n : \mathbb{R} \to \mathbb{R}$, $f_n(x) = \dfrac{nx}{1+n^2x^2}$ $\forall x \in \mathbb{R}$, $\forall n \in \mathbb{N}$

c) $f_n : (0,+\infty) \to \mathbb{R}$, $f_n(x) = \dfrac{nx}{nx^2+2}$, $\forall x \in (0,+\infty)$, $\forall n \in \mathbb{N}$

d) $f_n : \mathbb{R} \to \mathbb{R}$, $f_n(x) = \dfrac{\sin(nx)}{\sqrt{n^2+x^2}}$, $\forall x \in \mathbb{R}$, $\forall n \geq 1$

CONV. SIMP.

a) $\lim\limits_{n\to+\infty} f_n(x) = \lim\limits_{n\to+\infty} \dfrac{x^n}{\sqrt{1+x^n}} = \begin{cases} 0, & x \in [0,1) \\ \dfrac{1}{\sqrt{2}}, & x = 1 \end{cases}$

Cum $x \in [0,1]$

$A = \{x \in D \mid \lim\limits_{n\to+\infty} f_n(x) \in \mathbb{R}\} = [0,1]$

↳ mulțimea pe care va avea loc conv. simplă

$f : A \to \mathbb{R}$, $f(x) = \lim\limits_{x\to+\infty} f_n(x) = \begin{cases} 0, & x \in [0,1) \\ \dfrac{1}{\sqrt{2}}, & x = 1 \end{cases}$

$f_n \xrightarrow{S} f$

CONV. UNIF. ← teorema zice că dacă a) f conv. uniformă ⇒ f_n cont. + f cont.

$\left.\begin{array}{l} f_n \text{ funcție cont. pe } [0,1] \quad \forall n \geq 1 \\ f \text{ nu este funcție cont. pe } [0,1] \text{ (dif. în 1)} \end{array}\right\} \Rightarrow f_n$ nu este conv. uniformă

$f_n \not\xrightarrow{u} f$

b) CONV. SIMPLĂ

$f_n : \mathbb{R} \to \mathbb{R}$, $f_n(x) = \frac{nx}{1+n^2x^2}$, $\forall x \in \mathbb{R}$, $\forall n \in \mathbb{N}$

Fie $x \in \mathbb{R}$

$$\lim_{n \to +\infty} f_n(x) = \lim_{n \to +\infty} \frac{nx}{1+n^2x^2} = 0$$

$A = \mathbb{R}$

$f : A \to \mathbb{R}$, $f(x) = 0$

$f_n \xrightarrow{s} f$

CONV. UNIF. !

$$f_n \xrightarrow{u} f \iff \lim_{n \to +\infty} \left(\sup_{x \in A} |f_n(x) - f(x)| \right) = 0$$

$g : A \subseteq \mathbb{R} \to \mathbb{R}$

$\exists \sup_{x \in A} g(x) \in \overline{\mathbb{R}}$ = cel mai mic majorat al funcției g

$\sup_{x \in A} g(x) \geq g(y)$ $\forall y \in A$

$g(y) \leq c$ $\forall y \in A \Rightarrow \sup_{x \in A} g(x) \leq c$

Fie $n \in \mathbb{N}$

$$\sup_{x \in A = \mathbb{R}} |f_n(x) - f(x)| = \sup_{x \in \mathbb{R}} \left| \frac{nx}{1+n^2x^2} - 0 \right| = \sup_{x \in \mathbb{R}} \underbrace{\frac{n \cdot |x|}{1+n^2x^2}}_{g(x)}$$

$$g(7) = \frac{7n}{1+49n^2}$$

$$g\left(\frac{1}{n}\right) = \frac{\cancel{n} \cdot \frac{1}{\cancel{n}}}{1 + \cancel{n^2} \cdot \frac{1}{\cancel{n^2}}} = \frac{1}{2} \Rightarrow \sup_{x \in \mathbb{R}} g(x) \geq \frac{1}{2}$$

$$\Rightarrow \sup_{x \in \mathbb{R}} |f_n(x) - f(x)| \geq \frac{1}{2} \overset{n \to +\infty}{\Rightarrow} \lim_{n \to +\infty} \left(\sup_{x \in \mathbb{R}} |f_n(x) - f(x)| \right) \geq \frac{1}{2} \Rightarrow \neq 0$$

$\Rightarrow f_n \overset{u}{\not\to} f$

c) Fie $x \in (0,+\infty)$

$$\lim_{n\to+\infty} f_n(x) = \lim_{n\to+\infty} \frac{nx}{nx^2+2} = \frac{x}{x^2} = \frac{1}{x}$$

$A = (0,+\infty)$

$f : (0,+\infty) \to \mathbb{R}, \ f(x) = \frac{1}{x}$

$f_n \xrightarrow{S} f$

Fie $n \in \mathbb{N}$

$$\sup_{x\in(0,+\infty)} |f_n(x) - f(x)| = \sup_{x\in(0,+\infty)} \left| \frac{\overset{x}{nx}}{nx^2+2} - \frac{\overset{nx^2+2}{1}}{x} \right| = \sup_{x\in(0,+\infty)} \left| \frac{nx^2 - nx^2 - 2}{x(nx^2+2)} \right|$$

$$= \sup_{x\in(0,+\infty)} \left| \frac{-2}{x(nx^2+2)} \right| = \sup_{x\in(0,+\infty)} \underbrace{\frac{+2}{x(nx^2+2)}}_{g(x)}$$ nu prea pot găsi o val pt. x ca g(x) să nu mai conțină n

$$g(x)' = \left(\frac{2}{x(nx^2+2)} \right)' = 2 \cdot \left(\frac{1}{nx^3+2x} \right)' = \frac{-2 \cdot (3nx^2+2)}{(nx^3+2x)^2} \quad \forall x \in (0,+\infty)$$

$g'(x) = 0 \Rightarrow 3nx^2+2 = 0 \Rightarrow x^2 = \frac{-2}{3n}$ (nu are sol. reale)

x	0	$+\infty$
$g'(x)$	$- - - - - - -$	
$g(x)$	$+\infty \searrow$	0

$$\lim_{\substack{x\to 0\\ x>0}} \frac{2}{nx^3+2x} = \frac{2}{0+} = +\infty \qquad \lim_{x\to+\infty} \frac{2}{nx^3+2x} = 0$$

Din tabel am aflat sup

$$\sup_{x\in(0,+\infty)} g(x) = +\infty \Rightarrow \sup_{x\in(0,+\infty)} |f_n(x) - f(x)| = +\infty$$

$$\Rightarrow \lim_{n\to+\infty} \sup_{x\in(0,+\infty)} |f_n(x) - f(x)| = +\infty \neq 0 \Rightarrow f_n \not\xrightarrow{u} f$$

d) $f_n(x) = \frac{\sin(nx)}{\sqrt{n^2+x^2}}, \quad \forall x \in \mathbb{R}, \ \forall n \geq 1$

Fie $x \in \mathbb{R}$

$$\lim_{n\to+\infty} f_n(x) = \lim_{n\to+\infty} \frac{\sin(nx)}{\sqrt{n^2+x^2}} = \underbrace{\sin(nx)}_{\text{mărginit}} \cdot \underbrace{\frac{1}{\sqrt{n^2+x^2}}}_{\to 0} = 0$$

$A=\mathbb{R}$, $f:\mathbb{R}\to\mathbb{R}$, $f(x)=0$

$f_n \xrightarrow{s} f$

fie $n\in\mathbb{N}^*$

$$\sup_{x\in A} |f_n(x)-f(x)| = \sup_{x\in\mathbb{R}} \left| \frac{\sin(nx)}{\sqrt{n^2+x^2}} - 0 \right| = \sup_{x\in\mathbb{R}} \underbrace{\frac{|\sin(nx)|}{\sqrt{n^2+x^2}}}_{g(x)}$$

$$g\left(\frac{1}{n}\right) = \frac{\sin\left(n\cdot\frac{1}{n}\right)}{\sqrt{n^2+\frac{1}{n^2}}} = \frac{|\sin 1|}{\sqrt{n^2+\frac{1}{n^2}}} \to 0$$

$$g(x) \le \frac{1}{\sqrt{n^2+x^2}} \le \frac{1}{\sqrt{n^2}} \le \frac{1}{n}$$

$$\underset{\downarrow\, 0}{0} \le \underset{\downarrow\, 0}{\sup_{x\in\mathbb{R}} |f_n(x)-f(x)|} \le \underset{\downarrow\, 0}{\frac{1}{n}} \Rightarrow f_n \xrightarrow{u} f \quad \text{pt. că } \sup_{x\in\mathbb{R}} |\ldots| = 0$$