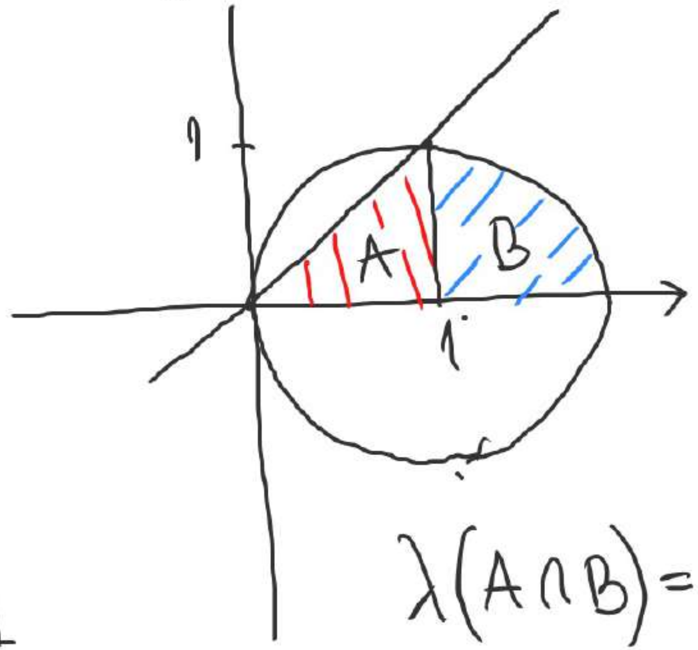
1) $\iint_D y\,dx\,dy$ D mărg. de $x^2+y^2=2x$, $y=0$ și $y=x$

Metoda II



$A=\{(x,y) \mid 0\le x\le 1,\ 0\le y\le x\}$

$B=\{(x,y) \mid 1\le x\le 2,\ y\le \sqrt{2x-x^2}\}$

$A, B \in \mathcal{J}(\mathbb{R}^2)$.

$x^2+y^2=2x$
$\Updownarrow$
$(x-1)^2+y^2=1$

$\lambda(A\cap B)=\lambda(\{1\}\times[0,1])=0$, $D=A\cup B$

$$\int_D y\,dx\,dy=\int_A y\,dx\,dy+\int_B y\,dx\,dy$$

$$\int_A y\,dx\,dy=\int_0^1\left(\int_0^x y\,dy\right)dx=\int_0^1\left(\frac{y^2}{2}\Big|_{y=0}^{y=x}\right)dx=\int_0^1\frac{x^2}{2}\,dx=\frac{x^3}{6}\Big|_0^1=\frac{1}{6}$$

$$B = \{(x,y) \mid 1 \le x \le 2,\ 0 \le y \le \sqrt{2x-x^2}\} = \{(x,y) \mid (x-1)^2 + y^2 \le 1,\ y \ge 0,\ x-1 \ge 0\}$$

$$\begin{cases} x = 1 + r\cos\theta \\ y = r\sin\theta \end{cases},\quad r \in [0,1],\ \theta \in [0, \tfrac{\pi}{2}]$$

$$(x-a)^2 + (y-b)^2 \le c^2$$

$$\begin{cases} x = a + r\cos\theta \\ y = b + r\sin\theta \end{cases},\quad r \in [0,c],\ \theta \in [0, 2\pi].$$

$$|\det J| = r,\quad dx\,dy = r\,dr\,d\theta$$

$$\iint_B y\,dx\,dy = \iint_{[0,1]\times[0,\pi/2]} r\sin\theta \cdot r\,dr\,d\theta = \int_0^1 \left(\int_0^{\pi/2} r^2 \sin\theta\,d\theta \right) dr$$

$$= \int_0^1 \left(-r^2 \cos\theta \Big|_0^{\pi/2} \right) dr = \int_0^1 r^2\,dr = \frac{1}{3}$$

$$\left(\iint_B y\,dx\,dy = \int_1^2 \left(\int_0^{\sqrt{2x-x^2}} y\,dy \right) dx = \int_1^2 \frac{2x-x^2}{2}\,dx = \frac{1}{3} \right)$$

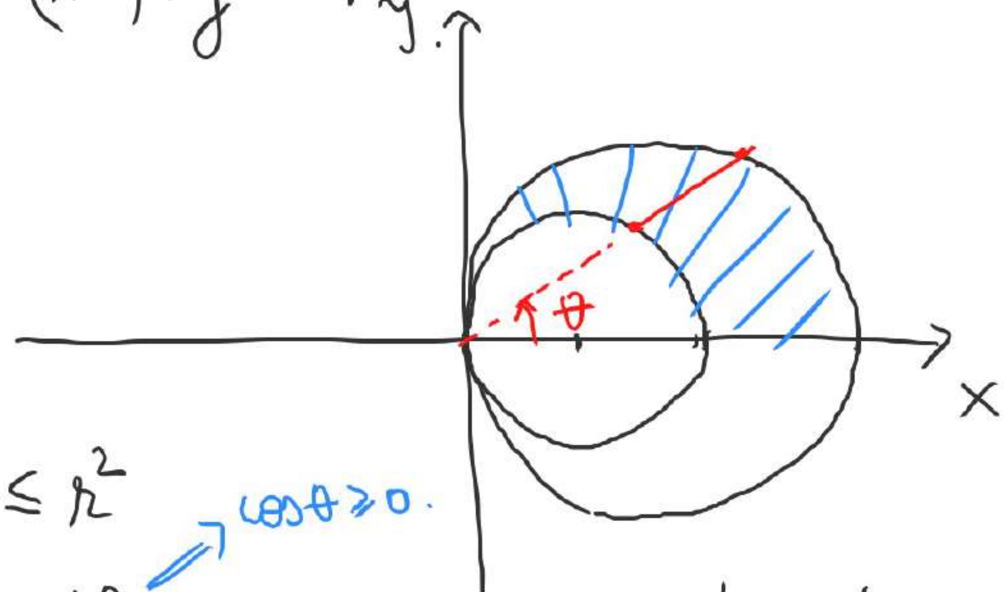
$$\iint_D y\,dx\,dy = \frac{1}{6} + \frac{1}{3} = \frac{1}{2}$$

2) $\displaystyle\iint_D \sqrt{x^2+y^2}\, dx\, dy$, $D = \{(x,y) \in \mathbb{R}^2 \mid 2x \leq x^2+y^2 \leq 4x,\ y \geq 0\}$

$$2x \leq x^2+y^2 \iff y^2+x^2-2x+1 \geq 1 \iff (x-1)^2+y^2 \geq 1.$$

$$x^2+y^2 \leq 4x \iff x^2-4x+4+y^2 \leq 4 \iff (x-2)^2+y^2 \leq 4$$

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$$



$$(x,y) \in D \iff \begin{cases} 2x \leq x^2+y^2 \\ x^2+y^2 \leq 4x \\ y \geq 0 \end{cases} \iff \begin{cases} 2r\cos\theta \leq r^2 \\ r^2 \leq 4r\cos\theta \\ r\sin\theta \geq 0 \end{cases}$$

$\Rightarrow \cos\theta \geq 0.$

$$\iff \begin{cases} 2\cos\theta \leq r \leq 4\cos\theta \\ \theta \in [0, \pi/2] \end{cases}$$

$$A=\left\{(r,\theta)\in\mathbb{R}^2 \;\middle|\; \theta\in\left[0,\tfrac{\pi}{2}\right],\ 2\cos\theta\le r\le 4\cos\theta\right\},\quad dx\,dy = r\,dr\,d\theta$$

$$\iint_D \sqrt{x^2+y^2}\,dx\,dy = \iint_A r\cdot r\,dr\,d\theta = \int_0^{\frac{\pi}{2}}\left(\int_{2\cos\theta}^{4\cos\theta} r^2\,dr\right)d\theta$$

$$= \int_0^{\frac{\pi}{2}} \frac{r^3}{3}\Bigg|_{2\cos\theta}^{4\cos\theta} dr = \int_0^{\frac{\pi}{2}} \frac{56}{3}\cos^3\theta\,d\theta$$

$$= \frac{56}{3}\int_0^{\frac{\pi}{2}} \cos\theta\,\big(\overbrace{1-\sin^2\theta}^{\cos^2\theta}\big)\,d\theta = \frac{56}{3}\int_0^{\frac{\pi}{2}}\cos\theta\,d\theta - \frac{56}{3}\int_0^{\frac{\pi}{2}}\sin^2\theta\cos\theta\,d\theta$$

$$= \frac{56}{3}\sin\theta\Big|_0^{\frac{\pi}{2}} - \frac{56}{3}\int_0^{\frac{\pi}{2}}\sin^2\theta\cdot(\sin\theta)'\,d\theta = \frac{56}{3} - \frac{56}{3}\cdot\frac{\sin^3\theta}{3}\Bigg|_0^{\frac{\pi}{2}} = \frac{56}{3}\cdot\frac{2}{3}$$

3) Coordonate polare generalizate.

$$\begin{cases} x = a r\cos\theta \\ y = b r\sin\theta \end{cases}, \quad r \in [0, \infty), \quad \theta \in [0, 2\pi], \quad a, b > 0.$$

Se folosesc dacă integrăm pe o mulțime mărginită de elipsa $\frac{x^2}{a^2} + \frac{y^2}{b^2} = c^2$

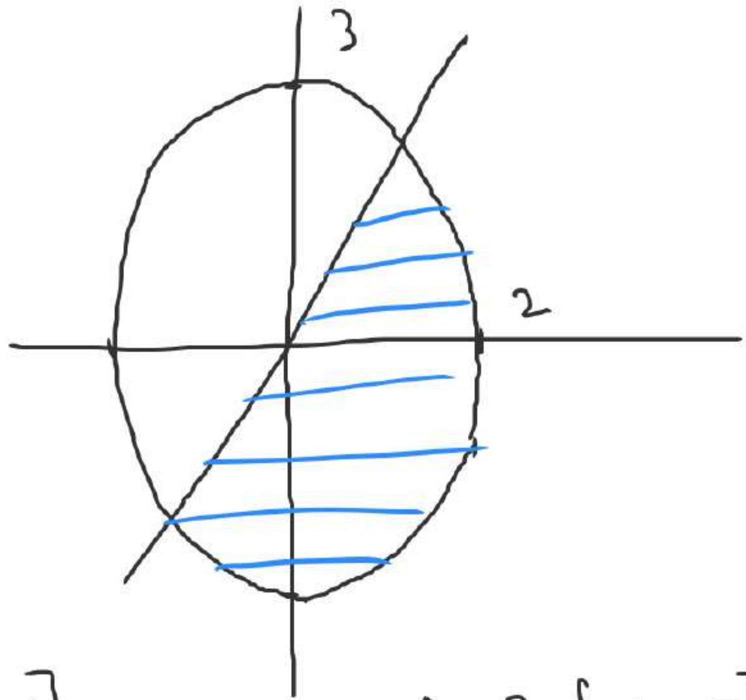
$\phi : [0, \infty) \times [0, 2\pi] \to \mathbb{R}^2$, $\phi(r, \theta) = (a r\cos\theta, b r\sin\theta)$

$r > 0$, $\theta \in (0, 2\pi)$, $\det J_\phi(r, \theta) = \frac{D(x, y)}{D(r, \theta)} = \begin{vmatrix} a\cos\theta & -a r\sin\theta \\ b\sin\theta & b r\cos\theta \end{vmatrix} = abr.$

$\left|\det J_\phi(r, \theta)\right| = abr$, $\underline{dx\,dy = abr\,dr\,d\theta}$

$$3\iint_D \frac{1}{\sqrt{4-\frac{x^2}{4}-\frac{y^2}{9}}}\,dx\,dy\ ,\quad D=\left\{(x,y)\in\mathbb{R}^2 \,\middle|\, \frac{x^2}{4}+\frac{y^2}{9}\le 1\ ,\ 2y\le 3x\right\}$$

$$\begin{cases} x = 2r\cos\theta \\ y = 3r\sin\theta \end{cases}$$



$$(x,y)\in D \iff \begin{cases} \dfrac{4r^2\cos^2\theta}{4}+\dfrac{9r^2\sin^2\theta}{9}\le 1 \\ 2\cdot 3r\sin\theta \le 3\cdot 2r\cos\theta \end{cases}$$

$$\iff \begin{cases} r^2\le 1 \\ \sin\theta\le\cos\theta \end{cases} \iff \begin{cases} r\in[0,1] \\ \theta\in\left[-\frac{3\pi}{4},\frac{\pi}{4}\right] \end{cases}.$$

$$A=[0,1]\times\left[-\frac{3\pi}{4},\frac{\pi}{4}\right]$$

$$dx\,dy = 2\cdot 3\,r\,dr\,d\theta = 6\,r\,dr\,d\theta$$

$$\iint_D \frac{1}{\sqrt{4-\frac{x^2}{4}-\frac{y^2}{9}}}\,dx\,dy = \iint_A \frac{1}{\sqrt{4-r^2}}\cdot 6r\,dr\,d\theta$$

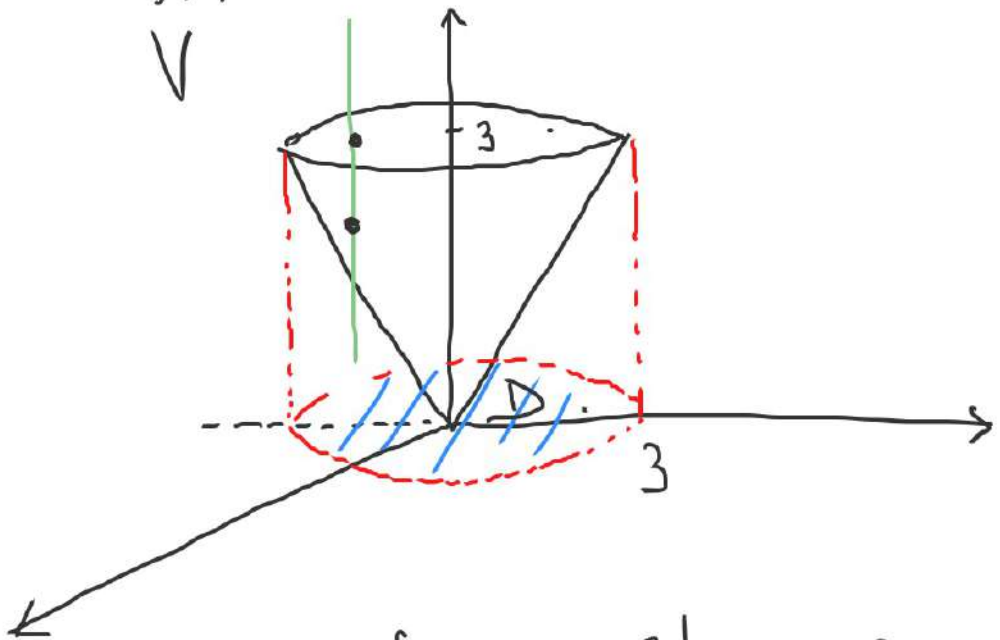
$$= \int_0^1 \left(\int_{-\frac{3\pi}{4}}^{\frac{\pi}{4}} \frac{6r}{\sqrt{4-r^2}}\,d\theta \right) dr = \int_0^1 \frac{6\pi r}{\sqrt{4-r^2}}\,dr$$

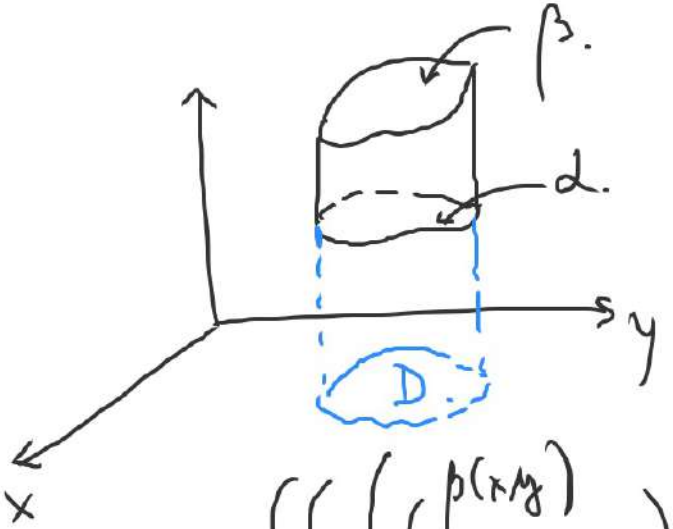
$$4-r^2=u \qquad -2r\,dr = du, \quad \begin{array}{ll} r=0, & u=4 \\ r=1 & u=3 \end{array}$$

$$= -\int_3^4 \frac{-3\pi}{\sqrt{u}}\,du \quad \int_3^4 \frac{3\pi}{\sqrt{u}}\,du = 6\pi\sqrt{u}\,\Big|_3^4 = 6\pi\left(2-\sqrt{3}\right)$$

4) $\iiint\limits_V (x^2+y^2)\,dx\,dy\,dz$, $V = \{(x,y,z) \in \mathbb{R}^3 \mid x^2+y^2 \le z^2,\ 0 \le z \le 3\}$

$\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{z^2}{c^2}$ con





$D = pr_{xOy} V = \{(x,y) \in \mathbb{R}^2 \mid x^2+y^2 \le 9\}$

$V = \{(x,y,z) \mid (x,y) \in D,\ \underset{\alpha}{\sqrt{x^2+y^2}} \le z \le \underset{\beta}{3}\}$

$D \in J(\mathbb{R}^2)$, $V \in J(\mathbb{R}^3)$

$$\iiint\limits_V f\,dx\,dy\,dz = \iint\limits_D \left(\int_{\alpha(x,y)}^{\beta(x,y)} f(x,y,z)\,dz \right) dx\,dy$$

$V = \{(x,y,z) \mid (x,y) \in D,\ \alpha(x,y) \le z \le \beta(x,y)\}$

$D = pr_{xOy} V$ $\qquad \alpha, \beta : D \to \mathbb{R}$

f cont și mărginită. Putem aplica Prop 3 / Curs 17

$$\iiint_V (x^2+y^2)\,dx\,dy\,dz = \iint_D \left(\int_{\sqrt{x^2+y^2}}^{3} (x^2+y^2)\,dz \right) dx\,dy$$

$$= \iint_D (x^2+y^2)\cdot z \Big|_{\sqrt{x^2+y^2}}^{3} dx\,dy = \iint_D (x^2+y^2)\left(3-\sqrt{x^2+y^2}\right) dx\,dy$$

$D: x^2+y^2 \leq 9$

$$\begin{cases} x = r\cos\theta, & r\in[0,3] \\ y = r\sin\theta & \theta\in[0,2\pi] \end{cases}$$

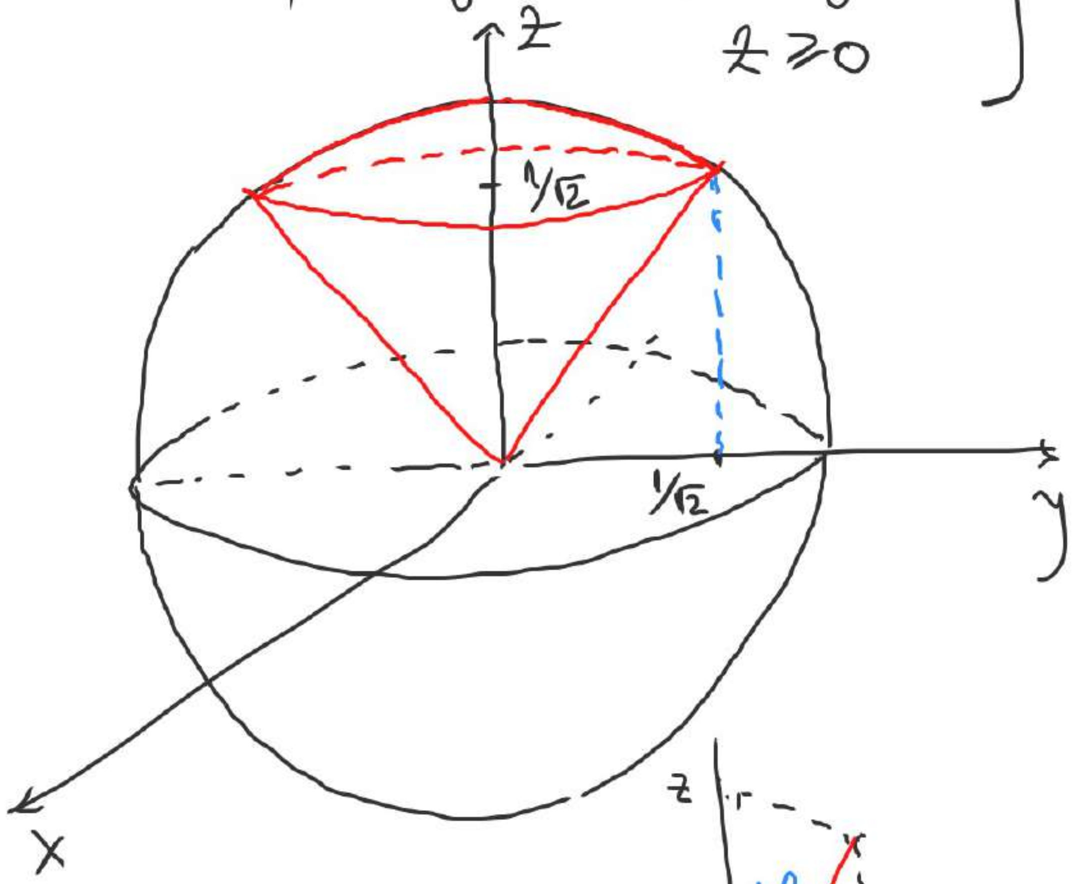
$dx\,dy = r\,dr\,d\theta$

$$= \iint_{[0,3]\times[0,2\pi]} r^2(3-r)\cdot r\,dr\,d\theta = \int_0^3 \left(\int_0^{2\pi} r^3(3-r)\,d\theta \right) dr$$

$$= \int_0^3 2\pi\, r^3(3-r)\,dr = \dots$$

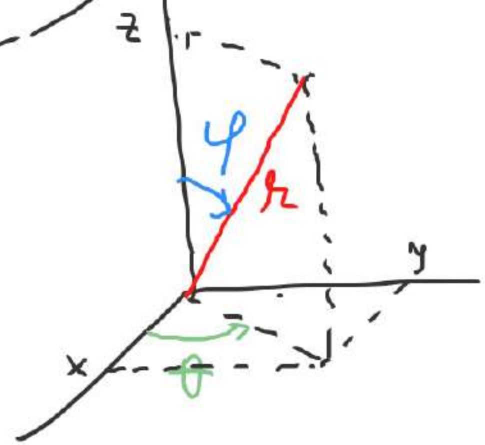
5) $I = \iiint\limits_V \sqrt{x^2+y^2+z^2}\, dx\,dy\,dz \ , \ V = \left\{ (x,y,z) \,\middle|\, x^2+y^2+z^2 \le 1,\ x^2+y^2 \le z^2,\ z \ge 0 \right\}$

$$\begin{cases} x = r\sin\varphi\cos\theta \\ y = r\sin\varphi\sin\theta \\ z = r\cos\varphi \end{cases} \qquad x^2+y^2 = r^2\sin^2\varphi$$



$$(x,y,z)\in V \Longleftrightarrow \begin{cases} r^2 \le 1 \\ r^2\sin^2\varphi \le r^2\cos^2\varphi \\ r\cos\varphi \ge 0 . \end{cases}$$

$$\Longleftrightarrow \begin{cases} 0 \le r \le 1 \\ 0 \le \theta \le 2\pi \\ \cos\varphi \ge 0 \\ \sin\varphi \le \cos\varphi \end{cases} \Longleftrightarrow \begin{cases} r \in [0,1] \\ \theta \in [0, 2\pi] \\ \varphi \in [0, \frac{\pi}{4}] \end{cases}$$



$$dx\,dy\,dz = r^2 \sin\varphi \, dr\, d\theta\, d\varphi.$$

$$I = \iiint\limits_{[0,1]\times[0,2\pi]\times[0,\frac{\pi}{4}]} r\cdot r^2 \sin\varphi \, dr\, d\theta\, d\varphi = \int_0^1 \left(\int_0^{2\pi} \left(\int_0^{\frac{\pi}{4}} r^3 \sin\varphi \, d\varphi \right) d\theta \right) dr$$

$$= \int_0^1 \left(\int_0^{2\pi} -r^3 \cos\varphi \Big|_0^{\pi/4} d\theta \right) dr = \int_0^1 \left(\int_0^{2\pi} r^3 \left(1 - \frac{\sqrt{2}}{2}\right) d\theta \right) dr$$

$$= \int_0^1 2\pi r^3 \left(1 - \frac{\sqrt{2}}{2}\right) dr = 2\pi\left(1 - \frac{\sqrt{2}}{2}\right)\cdot\frac{1}{4} = \frac{\pi}{2}\left(1 - \frac{\sqrt{2}}{2}\right)$$

Exerciții

1) $\iint_D (1+x)\,dx\,dy$, $D=\{(x,y)\in\mathbb{R}^2 \mid x^2+y^2\le 2y,\ y\le 2-x\}$

2) $\iint_D (x+xy)\,dx\,dy$, D este trapezul determinat de $A(1,0)$, $B(5,0)$, $C(3,4)$, $D(1,4)$

3) $\iint_D (x^2+y^2+xy)\,dx\,dy$, $D=\{(x,y)\in\mathbb{R}^2 \mid 4\le x^2+y^2\le 9,\ 0\le x\le y\}$

4) $\iint_D (x+y)xy\,dx\,dy$, D este limitat de dreptele $x+y=-3$, $x+y=3$, $x-y=1$, $x-y=-1$

5) Aflați aria (adică măsura Jordan) următoarelor mulțimi

i) $D = \{(x,y) \in \mathbb{R}^2 \mid x^2+y^2 \leq 4,\ y \leq x\}$

ii) $D = \{(x,y) \in \mathbb{R}^2 \mid 3 \leq 2x+2y \leq 4,\ y^2 \leq 2x\}$

iii) $D = \{(x,y) \in \mathbb{R}^2 \mid (x^2+y^2)^2 \leq a^2(x^2-y^2)\},\ a>0.$

6) $\displaystyle\iint_D \ln(x^2+y^2)\,dx\,dy$, $D = \{(x,y) \in \mathbb{R}^2 \mid 1 \leq x^2+y^2 \leq 4,\ y \leq \sqrt{3}x\}$

7) $\displaystyle\iint_D \sqrt{xy}\,dx\,dy$, D este mărginit de curbele

$y^2 = x,\ y^2 = 8x,\ xy = 1,\ xy = 8$.

9) $\iiint\limits_V x^2\,dx\,dy\,dz$, $V=\left\{(x,y,z)\in\mathbb{R}^3 \;\middle|\; z^2 \le \frac{x^2}{4}+\frac{y^2}{9} \le 2z\right\}$

10) $\iiint\limits_V z\,dx\,dy\,dz$, $V=\left\{(x,y,z)\in\mathbb{R}^3 \;\middle|\; x^2+y^2 \le 4z,\ 0\le z\le 1\right\}$

11) $\iiint\limits_V xyz\,dx\,dy\,dz$, $V=\left\{(x,y,z)\in\mathbb{R}^3 \;\middle|\; 9\le x^2+y^2 \le z^2,\ 0\le z\le 5\right\}$

12) Calc. volumul (adică masura Jordan) urmatoarelor multimi

i) V este mărginit de suprafetele: $x^2+y^2+z^2=4$ si $x^2+y^2=3z$

ii) V este mărginit de suprafetele: $x^2+y^2=4z$, $x^2+y^2=4x$, $z=0$