Aplicații liniare

$(V_i, +, \cdot)_{/\mathbb{K}}$ sp vect, $i=\overline{1,2}$

$f: V_1 \to V_2$ aplicație liniară $\Leftrightarrow$ (1) $f(x+y) = f(x) + f(y)$
(2) $f(\alpha x) = \alpha f(x)$
$\forall \alpha \in \mathbb{K}, \forall x, y \in V_1$

$\Leftrightarrow f(\alpha x + \beta y) = \alpha f(x) + \beta f(y)$
$\forall \alpha, \beta \in \mathbb{K}, \forall x, y \in V_1$

Prop $f: V_1 \to V_2$ liniară

a) f inj $\Leftrightarrow$ $\operatorname{Ker} f = \{x \in V_1 \mid f(x) = 0_{V_2}\} = \{0_{V_1}\}$

b) f surj $\Leftrightarrow$ $\dim \operatorname{Im} f = \dim V_2$.

Teorema dimensiunii

$f: V_1 \to V_2$ apl. liniară

$\Rightarrow \dim V_1 = \dim \operatorname{Ker} f + \dim \operatorname{Im} f$

Dem. $\operatorname{Ker} f \subset V_1$ subsp vect $\quad \dim \operatorname{Ker} f = p$, $p \le n = \dim V_1$

$R_0 = \{e_1, \dots, e_p\}$ reper în $\operatorname{Ker} f$.

Extindem R_0 la un reper $R = \{e_1, \dots, e_p, \underline{e_{p+1}, \dots, e_n}\}$ în V_1

Dem că $R' = \{f(e_{p+1}), \dots, f(e_n)\}$ reper în $\operatorname{Im} f$

① R' este SLI

$$\forall a_{p+1}, \dots a_n \in \mathbb{K} \text{ a.î. } \sum_{j=p+1}^{n} a_j f(e_j) = 0_{V_2} \overset{?}{\Rightarrow} a_j = 0 \quad \forall j = \overline{p+1, n}$$

$\overset{f \text{ lin}}{=}$

$$f\Big(\underbrace{\sum_{j=p+1}^{n} a_j e_j}\Big) \in \operatorname{Ker} f = \langle R_0 \rangle$$

$$\sum_{j=p+1}^{n} a_j e_j = \sum_{i=1}^{p} a_i e_i \Rightarrow \sum_{i=1}^{p} a_i e_i - \sum_{j=p+1}^{n} a_j e_j = 0_{V_1}$$

$\xRightarrow[SLI]{\mathcal{R} \text{ reper în } V_1}$ $a_i = 0,\ i=\overline{1,p}$

$a_j = 0,\ j=\overline{p+1,n} \Rightarrow \mathcal{R}'$ este SLI.

② $\mathcal{R}'$ este SG pt $\operatorname{Im} f$ i.e $\operatorname{Im} f = \langle \mathcal{R}' \rangle$

$\forall y \in \operatorname{Im} f \Rightarrow \exists x \in V_1 = \langle \mathcal{R} \rangle$ aî $f(x) = y$

$$f\left(\sum_{i=1}^{p} a_i e_i + \sum_{j=p+1}^{n} a_j e_j\right) = y \xRightarrow{f\text{ lin}} \sum_{j=p+1}^{n} a_j f(e_j) = y$$

$$\underbrace{f\Big(\underbrace{\sum_{i=1}^{p} a_i e_i}_{\in \operatorname{Ker} f}\Big)}_{= 0_{V_2}} + f\left(\sum_{j=p+1}^{n} a_j e_j\right)$$

$\Downarrow$

$\mathcal{R}'$ e SG

În concluzie: $\mathcal{R}'$ este reper în $\operatorname{Im} f$.

$\dim \operatorname{Im} f = |\mathcal{R}'| = n - p = \dim V_1 - \dim \operatorname{Ker} f$

$\Rightarrow \dim V_1 = \dim \operatorname{Ker} f + \dim \operatorname{Im} f$

Prop $f : V_1 \to V_2$ liniară

a) f inj $\Leftrightarrow \operatorname{Ker} f = \{0_{V_1}\} \Leftrightarrow \dim V_1 = \dim \operatorname{Im} f$

b) f surj $\Leftrightarrow \dim \operatorname{Im} f = \dim V_2 \Leftrightarrow$

$\dim V_1 = \dim \operatorname{Ker} f + \dim V_2$

c) f bij $\Leftrightarrow \dim V_1 = \dim V_2$.

Teoremă $V_1 \simeq V_2$ (sp. vect. izomorfe) $\Leftrightarrow \dim V_1 = \dim V_2$

Dem

"$\Rightarrow$" $V_1 \simeq V_2 \quad \exists f : V_1 \to V_2$ izomorfism $\xRightarrow{\text{Prop}} \dim V_1 = \dim V_2$

$-1^{2+2} \begin{vmatrix} 1 & 3 \\ 1 & d \end{vmatrix} =$

1^{3+2}

3 0
-1 1
3 0

$2 \Leftarrow$ " $\dim V_1 = \dim V_2 = n$, $\mathcal{R}_1 = \{e_1, .., e_n\}$ reper în V_1

$\mathcal{R}_2 = \{e'_1, .., e'_n\}$ reper în V_2.

Construim $f : V_1 \longrightarrow V_2$ liniară $f(e_i) = e'_i$, $i = \overline{1,n}$

Extindem f prin liniarizare

$$f(x) = f\left(\sum_{i=1}^{n} x_i e_i\right) \underset{lin}{=} \sum_{i=1}^{n} x_i \underbrace{f(e_i)}_{e'_i} = x'$$

f bij $\Leftrightarrow$ $\forall\, x' \in V_2$, $\exists!\, x \in V_1$ aî $f(x) = x'$

$x' = \sum_{i=1}^{n} x_i e'_i$, $x = \sum_{i=1}^{n} x_i e_i$ (sau cf Prop)

f izom. de sp vect.

Ex $f : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$, $f(x) = (x_1, x_1 + x_2 + x_3, x_1 + x_2 + x_3)$

$\operatorname{Im} f = ?$ Precizați un reper în $\operatorname{Im} f$.

SOL $\{e_1, e_2, e_3\}$ reperul canonic în $\mathbb{R}^3$

M1 $\operatorname{Ker} f = \{x \in \mathbb{R}^3 \mid f(x) = 0_{\mathbb{R}^3}\} = S(A) = \{(0, -x_3, x_3) \mid x_3 \in \mathbb{R}\}$

$$\begin{cases} x_1 = 0 \\ x_1 + x_2 + x_3 = 0 \\ x_1 + x_2 + x_3 = 0 \end{cases} \qquad A = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \qquad = \langle \{(0, -1, 1)\} \rangle$$

$\dim \operatorname{Ker} f = 3 - rg\, A = 1$

$$\begin{cases} x_1 = 0 \\ x_2 = -x_3 \end{cases}$$

$\mathcal{R}_0 = \{(0, -1, 1)\}$ reper în $\operatorname{Ker} f$

Extindem $\mathcal{R}_0$ la un reper în $\mathbb{R}^3$

$\mathcal{R} = \mathcal{R}_0 \cup \underline{\{e_3, e_1\}}$ reper în $\mathbb{R}^3$

$$rg\begin{pmatrix} 0 & 0 & 1 \\ -1 & 0 & 0 \\ 1 & 1 & 0 \end{pmatrix} = 3$$

$\mathcal{R}' = \{f(e_3), f(e_1)\}$ reper în $\operatorname{Im} f$

$f(e_3) = f(0,0,1) = (0,1,1)$; $f(e_1) = f(1,0,0) = (1,1,1)$

M2 $\operatorname{Im} f = \{y \in \mathbb{R}^3 \mid \exists\, x \in \mathbb{R}^3 \text{ aî } f(x) = y\}$

$$(*) \begin{cases} x_1 = y_1 \\ x_1 + x_2 + x_3 = y_2 \\ x_1 + x_2 + x_3 = y_3 \end{cases} \qquad A = \left(\begin{array}{ccc|c} 1 & 0 & 0 & y_1 \\ 1 & 1 & 1 & y_2 \\ 1 & 1 & 1 & y_3 \end{array}\right)$$

$(*)$ SC $\Leftrightarrow$ $rg A = rg \bar{A}$

$\Delta_c = 0$

$$\Delta_c = \begin{vmatrix} 1 & 0 & y_1 \\ 1 & 1 & y_2 \\ 1 & 1 & y_3 \end{vmatrix} = \begin{vmatrix} 1 & 0 & y_1 \\ 1 & 1 & y_2 \\ 0 & 0 & y_3 - y_2 \end{vmatrix} = y_3 - y_2 .$$

$$\operatorname{Im} f = \{ y \in \mathbb{R}^3 \mid y_3 - y_2 = 0 \} = \{ (y_1, y_2, y_2) \in \mathbb{R}^3 \mid y_1, y_2 \in \mathbb{R} \}$$

$$(y_1, y_2, y_2) = y_1(1,0,0) + y_2(0,1,1)$$

$$\operatorname{Im} f = S(A')$$

$\{(1,0,0),(0,1,1)\}$ reper în $\operatorname{Im} f$

$$\operatorname{rg}\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{pmatrix} = 2 \quad \text{sau} \quad \dim \operatorname{Im} f = 2 = 3 - \underbrace{1}_{\operatorname{rg} A'} = 2 \qquad A' = (0,-1,1)$$

sau $\dim \operatorname{Im} f = 3 - \dim \operatorname{Ker} f = 3 - 1 = 2$

Scanat cu CamScanner

Prop $f: V_1 \to V_2$ liniară

a) f inj $\Leftrightarrow$ f transformă $\forall$ SLI din V_1 într-un SLI din V_2.

b) f surj $\Leftrightarrow$ — // — $\forall$ SG din V_1 — // — SG din V_2

c) f bij $\Leftrightarrow$ — // — $\forall$ reper din V_1 — // — reper din V_2

Dem

a) „$\Rightarrow$" Ip: f inj.

$S = \{v_1, \dots, v_n\}$ SLI din V_1 $\overset{?}{\Rightarrow}$ $f(S) = \{f(v_1), \dots, f(v_n)\}$ SLI în V_2.

$\sum_{i=1}^{n} a_i f(v_i) = 0_{V_2} \overset{f \text{ lin}}{\Rightarrow} f\left(\sum_{i=1}^{n} a_i v_i\right) = 0_{V_2} \Rightarrow \sum_{i=1}^{n} a_i v_i \in \operatorname{Ker} f = \{0_{V_1}\}$

$\Rightarrow \sum_{i=1}^{n} a_i v_i = 0_{V_1} \overset{S \text{ e SLI}}{\Rightarrow} a_i = 0, \forall i = \overline{1,n}$

„$\Leftarrow$" Ip. prin abs $\exists x \in \operatorname{Ker} f$, $x \neq 0_{V_1}$ $\Rightarrow f(x) = 0_{V_2}$

$\{x\}$ e SLI în $V_1 \overset{ip}{\Rightarrow} \{f(x)\}$ e SLI în $V_2 \Rightarrow f(x) \neq 0_{V_2}$ Contrad.

Ip. este falsă $\Rightarrow \operatorname{Ker} f = \{0_{V_1}\} \Rightarrow f$ inj

b) „$\Rightarrow$" Ip f surj $\Rightarrow \dim \operatorname{Im} f = \dim V_2$.

S SG în $V_1 \Rightarrow V_1 = \langle S \rangle$

f surj $\Rightarrow V_2 = f(V_1) = f(\langle S \rangle) = \langle f(S) \rangle \Rightarrow f(S)$ SG pt V_2

„$\Leftarrow$" S e SG pt $V_1 \Rightarrow f(S)$ S.G pt V_2.

$V_2 = \langle f(S) \rangle = f(\langle S \rangle) = f(V_1) \Rightarrow f$ surj.

c) Din a), b)

f bij $\Leftrightarrow [\forall R_1$ reper în $V_1 \Rightarrow f(R_1)$ reper în $V_2]$

Scanat cu CamScanner

Matricea asociată unei aplicații liniare

$f: V_1 \to V_2$ apl. liniară

$R_1 = \{e_1, .., e_n\} \xrightarrow{A} R_2 = \{e'_1, .., e'_m\}$ repere în V_1, resp V_2

$\dim V_1 = n$, $\dim V_2 = m$.

$A = [f]_{R_1, R_2}$, $A \in \mathcal{M}_{m,n}(\mathbb{K})$

$$f(e_i) = \sum_{j=1}^{m} a_{ji} e'_j, \quad \forall i = \overline{1,n}$$

$$f(x) = f\left(\sum_{i=1}^{n} x_i e_i\right) = \sum_{i=1}^{n} x_i \underbrace{f(e_i)}_{\sum_{j=1}^{m} a_{ji} e'_j} =$$

$$= \sum_{j=1}^{m}\left(\sum_{i=1}^{n} a_{ji} x_i\right) e'_j$$

$$f(x) = y = \sum_{j=1}^{m} y_j e'_j$$

$$\Rightarrow y_j = \sum_{i=1}^{n} a_{ji} x_i \quad \forall j = \overline{1,m}$$

$$\begin{pmatrix} y_1 \\ \vdots \\ y_m \end{pmatrix} = \begin{pmatrix} a_{11} & \dots & a_{1n} \\ \vdots & & \\ a_{m1} & \dots & a_{mn} \end{pmatrix} \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \qquad Y = AX$$

Teorema de caract. a apl. lin

$f: V_1 \to V_2$ apl. liniară $\Leftrightarrow \exists A \in \mathcal{M}_{m,n}(\mathbb{K})$ a.î. coord. lui x în rap cu R_1 și coord lui $y = f(x)$ în rap cu R_2 verifică $Y = AX$.

$A' = [f]_{R'_1, R'_2}$

OBS

$$\begin{array}{ccc} R_1 = \{e_1, .., e_n\} & \xrightarrow{A} & R_2 = \{e'_1, .., e'_m\} \\ \downarrow C & & \downarrow D \\ R'_1 = \{\bar{e}_1, .., \bar{e}_n\} & \xrightarrow{A'} & R'_2 = \{\bar{e}'_1, .., \bar{e}'_m\} \end{array}$$

$$\boxed{A' = D^{-1} A C}$$

$\operatorname{rg} A' = \operatorname{rg} A$ invariant

În particular, $f \in End(V)$

$$\begin{array}{ccc} \mathcal{R}=\{e_1,..,e_n\} & \xrightarrow{A} & \mathcal{R}=\{e_1,..,e_n\} \\ C \downarrow & & \downarrow C \\ \mathcal{R}'=\{e_1',..,e_n'\} & \xrightarrow{A'} & \mathcal{R}'=\{e_1',..,e_n'\} \end{array}$$

$A=[f]_{\mathcal{R},\mathcal{R}}$

$A'=[f]_{\mathcal{R}',\mathcal{R}'}$

$A'=C^{-1}AC$

$C \in GL(n,\mathbb{K})$

Exemplu $f:\mathbb{R}^2 \to \mathbb{R}^2$, $f(x)=(x_1+x_2, 2x_2)$

$\mathcal{R}_0=\{e_1,e_2\}$ reperul canonic în $\mathbb{R}^2$

$\mathcal{R}'=\{e_1'=e_1-2e_2, e_2'=e_1+e_2\}$ reper în $\mathbb{R}^2$

a) $A=[f]_{\mathcal{R}_0,\mathcal{R}_0}$; b) $A'=[f]_{\mathcal{R}',\mathcal{R}'}$

Sol a) $f(e_1)=f(1,0)=(1,0)=e_1=\boxed{1}e_1+\boxed{0}\cdot e_2$ $\quad A=\begin{pmatrix}1&1\\0&2\end{pmatrix}$

$f(e_2)=f(0,1)=(1,2)=(1,0)+(0,2)=1\cdot e_1+2\cdot e_2$

$f(x)=y \Leftrightarrow AX=Y$

$A=\begin{pmatrix}a_{11}&a_{12}\\a_{21}&a_{22}\end{pmatrix}$; $\begin{pmatrix}1&1\\0&2\end{pmatrix}\begin{pmatrix}x_1\\x_2\end{pmatrix}=\begin{pmatrix}x_1+x_2\\2x_2\end{pmatrix}$

b) $f(e_1')=f(1,-2)=\underline{(-1,-4)} \overset{\overset{A}{\shortparallel}}{=} ae_1'+be_2'=a(1,-2)+b(1,1)$
$=\underline{(a+b,-2a+b)}$

$\begin{cases}a+b=-1\\-2a+b=-4\end{cases}$ $\ominus$

$3a\ \ \ =3$

$\begin{cases}a=1\\b=-2\end{cases}$ $\quad A'=\begin{pmatrix}1&0\\-2&2\end{pmatrix}$

$f(e_2')=f(1,1)=\underline{(2,2)}=ce_1'+de_2'=\underline{(c+d,-2c+d)}$

$\begin{cases}c+d=2\\-2c+d=2\end{cases}$ $\ominus$

$3c\ \ \ =0$

$c=0$
$d=2$

OBS $A'=C^{-1}AC$

$\mathcal{R}_0=\{e_1,e_2\} \longrightarrow \mathcal{R}'=\{\overset{e_1'}{\overset{\shortparallel}{e_1-2e_2}}, \overset{e_2'}{\overset{\shortparallel}{e_1+e_2}}\}$

$C=\begin{pmatrix}1&1\\-2&1\end{pmatrix}$

Prop $f: V_1 \to V_2$ apl. lin. $A = [f]_{R_1, R_2}$ R_1 = reper, R_2 = reper

a) f inj $\Leftrightarrow \dim V_1 = rg A$

b) f surj $\Leftrightarrow \dim V_2 = rg A$ $f(x) = y$

c) f bij $\Leftrightarrow A \in GL(n, \mathbb{K})$ $Y = AX$

Dem a) f inj $\Leftrightarrow Ker f = \{0_{V_1}\}$

T. dim: $\dim V_1 = \dim Ker f + \dim Im f$. $\dim Ker f$

$Ker f = \{x \in V_1 \mid AX = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}\} \Rightarrow \dim V_1 - rg A$

$Ker f = \{0_{V_1}\} \Leftrightarrow \dim V_1 = rg A$

b) f surj $\Leftrightarrow \dim Im f = \dim V_2$

$\dim V_1 = \dim V_1 - rg A + \dim V_2 \Leftrightarrow \dim V_2 = rg A$.

c) f bij $\Leftrightarrow \dim V_1 = \dim V_2 = rg A = n_{max} \Leftrightarrow A \in GL(n, \mathbb{K})$

OBS

a) $V_1 \xrightarrow{f} V_2 \xrightarrow{g} V_3$, h: $V_1 \to V_3$ f, g lin $\Rightarrow h$ lin.

$A_h = A_{g \circ f} = A_g \cdot A_f$

b) $V \xrightarrow{f} V \xrightarrow{f^{-1}} V$, id_V ; $V \xrightarrow{f^{-1}} V \xrightarrow{f} V$, id_V.

$I_n = A_{f^{-1}} \cdot A_f = A_f \cdot A_{f^{-1}}$ $(A_f)^{-1} = A_{f^{-1}}$.

27. 1113) $f:\mathbb{R}^3 \to \mathbb{R}^3$, $f(x) = (2x_1 + 2x_2, x_1 + x_3, x_1 + 3x_2 - 2x_3)$

a) f liniară ; f nu e izomorfism

b) $f|_{V'} : V' \to V''$ izomorfism.

$V' = \{x \in \mathbb{R}^3 \mid x_1 + x_2 - x_3 = 0\}$, $V'' = \{x \in \mathbb{R}^3 \mid 3x_1 - 4x_2 - 2x_3 = 0\}$

SOL

a) $A = [f]_{R_0, R_0}$ $\qquad f(x) = y \iff AX = Y$

$A = \begin{pmatrix} 2 & 2 & 0 \\ 1 & 0 & 1 \\ 1 & 3 & -2 \end{pmatrix}$ $\qquad \det A = 0 \Rightarrow f$ nu e bij.

b) $V' = \{(x_1, x_2, x_1 + x_2) \mid x_1, x_2 \in \mathbb{R}\}$

$= x_1 \underbrace{(1,0,1)}_{e_1'} + x_2 \underbrace{(0,1,1)}_{e_2'}$

$R' = \{e_1', e_2'\}$ reper în V'

$\dim V' = 2$

$\text{rg}\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{pmatrix} = 2.$

$f(e_1') = f(1,0,1) = \underbrace{(2,2,-1)}_{u} \in V''$ $\qquad 3\cdot 2 - 4\cdot 2 - 2(-1) = 8-8=0$

$f(e_2') = f(0,1,1) = \underbrace{(2,1,1)}_{v} \in V''$ $\qquad 3\cdot 2 - 4\cdot 1 - 2\cdot 1 = 6-6=0$

$\text{rg}\begin{pmatrix} 2 & 2 \\ 2 & 1 \\ -1 & 1 \end{pmatrix} = 2$ $\qquad \left.\begin{array}{l} \{u,v\} \text{ SLI} \\ \dim V'' = 2 \end{array}\right| \Rightarrow f(R') = \{u,v\}$ reper în V''

$f|_{V'} : V' \to V''$ $\qquad f|_{V'}$ izom sp vect.