Seminarul 3

Sisteme. Spații vectoriale. SLI. SLD. SG. Baze

1. $\begin{cases} x+y+mz-t=0 \\ 2x+y-z+t=0 \\ 3x-y-z-t=0 \\ mx-2y-2t=0 \end{cases}$

$m=?$ a.î. sist. are și sol. nenule

$$\bar{A}=\left(\begin{array}{cccc|c} 1 & 1 & m & -1 & 0 \\ 2 & 1 & -1 & 1 & 0 \\ 3 & -1 & -1 & -1 & 0 \\ m & -2 & 0 & -2 & 0 \end{array}\right)$$

$SCN \Leftrightarrow \det(A)=0$

$$\det(A)=\begin{vmatrix} 1 & 0 & m & -1 \\ 2 & 2 & -1 & 1 \\ 3 & 0 & -1 & -1 \\ m & 0 & 0 & -2 \end{vmatrix} = 2\begin{vmatrix} 1 & m & -1 \\ 3 & -1 & -1 \\ m & 0 & -2 \end{vmatrix} = 4-4m = 4(1-m)$$

$\det(A)=0 \Leftrightarrow m=1$

$m\,x\ 2y\ 2t$

(2) $\begin{cases} 3x+2y+5z+4t=-1 \\ 2x+y+3z+3t=0 \\ x+2y+3z=-3 \end{cases}$ rez. cu G. J.

$\bar{A}=(A|B)=$

$$=\left(\begin{array}{cccc|c} 3 & 2 & 5 & 4 & -1 \\ 2 & 1 & 3 & 3 & 0 \\ 1 & 2 & 3 & 0 & -3 \end{array}\right) \sim \left(\begin{array}{ccccc} 1 & 2 & 3 & 0 & -3 \\ 2 & 1 & 3 & 3 & 0 \\ 3 & 2 & 5 & 4 & -1 \end{array}\right) \underset{L_3'=\dots}{\overset{L_2'=L_2-2L_1}{\sim}} \left(\begin{array}{ccccc} 1 & 2 & 3 & 0 & -3 \\ 0 & -3 & -3 & 3 & 6 \\ 0 & -4 & -4 & 4 & 8 \end{array}\right) \sim$$

$$\sim \left(\begin{array}{cccc|c} 1 & 2 & 3 & 0 & -3 \\ 0 & 1 & 1 & -1 & -2 \\ 0 & 1 & 1 & -1 & -2 \end{array}\right) \sim \left(\begin{array}{ccccc} 1 & 2 & 3 & 0 & -3 \\ 0 & 1 & 1 & -1 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right) \Rightarrow rg\,\bar{A}=rg\,A=2$$

$\Rightarrow$ SC dublu N

$\begin{cases} x+2y+3z=-3 \\ y+z-t=-2 \end{cases} \Rightarrow \begin{cases} y=-2-\alpha+\beta \\ \cancel{x=-3+4} \\ x=-3+4+2t-2\beta-3\alpha \end{cases} \Rightarrow$

$\Rightarrow (x,y,z,t)\in\{(-\alpha-2\beta+1,\ -2-\alpha+\beta,\ \alpha,\ \beta)\mid \alpha,\beta\in\mathbb{R}\}$

3. $(\mathbb{R}^3,+,\cdot)_{|\mathbb{R}}$

$S=\{u=(1,2,3),\ v=(2,3,1),\ w=(m+3,\ m+1,\ m+2)\}$

$m\in\mathbb{R}$

$m=?$ a.î. a) $S=SLI$
b) $S=SLD$

a) $S=SLI \Leftrightarrow [(\forall)\ a,b,c\in\mathbb{R}$ a.î. $au+bv+cw=0_{\mathbb{R}^3} \Rightarrow a=b=c=0]$

$a(1,2,3)+b(2,3,1)+c(m+3,\ m+1,\ m+2)=(0,0,0)$

$\begin{cases} a+2b+(m+3)c=0 \\ 2a+3b+(m+1)c=0 \\ 3a+b+(m+2)c=0 \end{cases}$

$$A=\begin{pmatrix} 1 & 2 & m+3 \\ 2 & 3 & m+1 \\ 3 & 1 & m+2 \end{pmatrix}$$

$$\Delta=\det(A)=\begin{vmatrix} 1 & 2 & m+6 \\ 2 & 3 & m+6 \\ 3 & 1 & m+6 \end{vmatrix}=(m+6)\begin{vmatrix} 1 & 2 & 1 \\ 2 & 3 & 1 \\ 3 & 1 & 1 \end{vmatrix}=(m+6)\begin{vmatrix} 1 & 2 & 1 \\ 1 & 1 & 0 \\ 2 & -1 & 0 \end{vmatrix}=(m+6)\begin{vmatrix} 1 & 1 \\ 2 & 1 \end{vmatrix}=-3(m+6)$$

$\Rightarrow S=SLI \Leftrightarrow m\in\mathbb{R}\setminus\{-6\}$
$(rg(A)=\max)$

$S=SLD \Leftrightarrow m=6$

4. $(\mathbb{R}_2[X], +, \cdot)/\mathbb{R}$

$B = \{v_1 = 2X^2 - 3X,\ v_2 = X+1,\ v_3 = -X^2 + 4\}$

$v_1 \equiv (0,-3,2)$, $v_2 \equiv (1,1,0)$, $v_3 \equiv (4,0,-1)$

a) Dem. că B = bază

b) $S = \{v_1' = X+3,\ v_2' = X^2 - 2X,\ v_3' = \alpha X^2 - 6\}$

$\alpha = ?$ a.î. S = SLi / SLD

Obs

$p = a_0 + a_1 X + a_2 X^2 \equiv (a_0, a_1, a_2) \in \mathbb{R}^3$

$p \in \mathbb{R}_2[X]$

a) B = bază $\iff$ 1. B = SLi
2. B = SG

$$A = \begin{pmatrix} 0 & 1 & 4 \\ -3 & 1 & 0 \\ 2 & 0 & -1 \end{pmatrix}$$

$B = SLi \iff \det(A) \neq 0$

$$\det(A) = \begin{vmatrix} 8 & 1 & 4 \\ -3 & 1 & 0 \\ 0 & 0 & -1 \end{vmatrix} = -\begin{vmatrix} 8 & 1 \\ -3 & 1 \end{vmatrix} = -11 \neq 0$$

$\Rightarrow B = SLi$

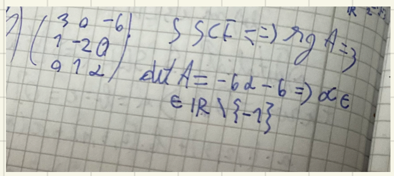
$\left.\begin{array}{l} \Rightarrow B = SLi \\ \dim_{\mathbb{R}} \mathbb{R}^2[X] = 3 = |B| \end{array}\right\} \Rightarrow B$ = bază

Prop

$(V, +, \cdot)/\mathbb{K}$ sp. vect., $\dim_{\mathbb{K}} V = n$

$S = \{v_1, v_2, \dots, v_n\}$ UAE:

1) S - bază
2) S - SLi
3) S - SG

b) 

5. $(M_2(\mathbb{R}), +, \cdot)/\mathbb{R}$, $S = \left\{ E_1 = \begin{pmatrix} 1 & 0 \\ \alpha & 0 \end{pmatrix}, E_2 = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}, E_3 = \begin{pmatrix} 0 & 1 \\ 0 & -3 \end{pmatrix}, E_4 = \begin{pmatrix} \alpha & 1 \\ 0 & -2 \end{pmatrix} \right\}$

$E_1 \equiv (1,0,\alpha,0)$, $E_2 \equiv (1,-1,0,1)$, $E_3 \equiv (0,1,0,-3)$, $E_4 \equiv (\alpha,1,0,-2)$

$\alpha = ?$ a.î. S = SLi

Obs

$M = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \equiv (a_{11}, a_{12}, a_{21}, a_{22}) \in \mathbb{R}^4$

$M \in M_2(\mathbb{R})$

$$A = \begin{pmatrix} 1 & 1 & 0 & \alpha \\ 0 & -1 & 1 & 1 \\ \alpha & 0 & 0 & 0 \\ 0 & 1 & -3 & -2 \end{pmatrix} \quad \det(A) = \alpha \begin{vmatrix} 1 & 0 & \alpha \\ -1 & 1 & 1 \\ 1 & -3 & -2 \end{vmatrix} = \alpha \begin{vmatrix} 0 & 3 & \alpha+2 \\ 0 & -2 & -1 \\ 1 & -3 & -2 \end{vmatrix} = \alpha \begin{vmatrix} 3 & \alpha+2 \\ -2 & -1 \end{vmatrix} = \alpha(-3 + 2\alpha + 4) = \alpha(1 + 2\alpha)$$

$S = SLi \iff \det(A) \neq 0$
$(\mathrm{rg}(A) = 4) \iff \alpha \in \mathbb{R} \setminus \{0, -\frac{1}{2}\}$

6. $(\mathbb{R}^3, +, \cdot)/\mathbb{R}$

$$W = \left\{ x \in \mathbb{R}^3 \,\middle|\, \begin{cases} x_1 + x_2 - x_3 = 0 \\ 3x_1 - 2x_2 + 2x_3 = 0 \\ 6x_1 + x_2 - x_3 = 0 \end{cases} \right\}$$

Prec. o bază; dimensiunea

$$A = \begin{pmatrix} 1 & 1 & -1 \\ 3 & -2 & 2 \\ 6 & 1 & -1 \end{pmatrix}, W = S(A)$$

$\dim(S(A)) = 3 - \mathrm{rang}(A)$

$\det(A) = 0$ $(C_2 = -C_3)$

$\Delta_p = \begin{vmatrix} 1 & 1 \\ 3 & -2 \end{vmatrix} = -5 \Rightarrow \mathrm{rg}(A) = 2$

$\Rightarrow \dim(S(A)) = 1$

$$\begin{cases} x_1 + x_2 = \alpha \\ 3x_1 - 2x_2 = -2\alpha \\ 6x_1 + x_2 = \alpha \end{cases}$$

$x_1 = 0,\ x_2 = \alpha$

$(x_1, x_2, x_3) \in \{(0, \alpha, \alpha) / \alpha \in \mathbb{R}\} = \{\alpha(0,1,1) / \alpha \in \mathbb{R}\}$

$\left.\begin{array}{l} \Rightarrow B = \{0,1,1\} \text{ S.G. pt. } A \\ \text{Dar } B = SLi \end{array}\right\} \Rightarrow B$ bază în $S(A)$

7.

$(\mathbb{R}_3[X], +, \cdot)_{/\mathbb{R}}$

$V = \{ v_1 = X^3 + 2X - 1,\ v_2 = 2X^2 + 1,\ v_3 = X^3 - X \}$

Verif. dacă $P = 2X^2 + 3X$, $Q = X + 1 \in \langle V \rangle$

$\langle V \rangle = \{ a_1 v_1 + a_2 v_2 + a_3 v_3 \mid a_i \in \mathbb{R},\ i = \overline{1,3} \}$

$P = 2X^2 + 1 + 2X - 1 - X^3 = 2X^2 + 3X$

$X + 1 = a_1 v_1 + a_2 v_2 + a_3 v_3 = X^3 (a_1 + a_3) + X^2 (2a_2) + X(2a_1 - a_3) - a_1 + a_2$

$$\begin{cases} a_1 + a_3 = 0 \\ 2a_2 = 0 \Rightarrow a_2 = 0 \\ 2a_1 - a_3 = 1 \Rightarrow a_3 = -3 \\ -a_1 + a_2 = 1 \Rightarrow a_1 = -1 \end{cases}$$

dar $a_1 + a_3 \neq 0 \Rightarrow$ S.I. $\Rightarrow Q \notin \langle V \rangle$

8. $(\mathcal{F}_{(\mathbb{R})} = \{ f : \mathbb{R} \to \mathbb{R} \mid f = \text{func.} \}, +, \cdot)_{/\mathbb{R}}$

a) $S = \{ f_1, f_2, f_3 \}$ = SLI?

$f_1(x) = 1$, $f_2(x) = \sin x$, $f_3(x) = \cos x$

Fie $a_1, a_2, a_3 \in \mathbb{R}$ a.î. $a_1 f_1 + a_2 f_2 + a_3 f_3 = 0_{\mathcal{F}(\mathbb{R})}$

$a_1 + a_2 \sin x + a_3 \cos x = 0\ (\forall)\ x \in \mathbb{R}$

$x = 0 \Rightarrow a_1 + a_3 = 0$ $\} \Rightarrow a_1 = a_3 = 0$

$x = \frac{\pi}{2} \Rightarrow a_1 + a_2 = 0$

$x = \pi \Rightarrow a_1 - a_3 = 0$ $\quad \Rightarrow a_2 = 0$

$\Rightarrow$ SLI

b) $S' = \{ g_1, g_2, g_3 \}$ SLI

$g_1(x) = \sin x$

$g_2(x) = \sin 2x$

$g_3(x) = \sin 3x$

Fie $a_1, a_2, a_3 \in \mathbb{R}$ a.î. $a_1 g_1 + a_2 g_2 + a_3 g_3 = 0 \Rightarrow \sin x + \sin 2x + \sin 3x = 0$

$$\begin{cases} x = \frac{\pi}{2} \Rightarrow a_1 - a_3 = 0 \Rightarrow a_3 = a_1 \\ x = \frac{\pi}{4} \Rightarrow a_1 \frac{\sqrt{2}}{2} + a_2 + a_3 \frac{\sqrt{2}}{2} = 0 \Rightarrow \sqrt{2} a_1 + a_2 = 0 \\ x = \frac{\pi}{6} \Rightarrow \frac{a_1}{2} + \frac{\sqrt{3}}{2} a_2 + a_3 = 0 \Rightarrow \sqrt{3} a_1 + a_2 = 0 \end{cases}$$

$\Rightarrow a_1 = a_2 = 0$

$\Rightarrow$ SLI