Seminar 2

Polinom caracteristic. Th. H-C. Th. Laplace. Sisteme

1 a) $A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}$

$A^{-1} = ?$

$A^3 - \sigma_1 A^2 + \sigma_2 A - \sigma_3 I_3 = O_3$

$\sigma_1 = Tr(A) = 2$

$\sigma_2 = \begin{vmatrix} 1 & 1 \\ 0 & 0 \end{vmatrix} + \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} + \begin{vmatrix} 0 & 1 \\ 1 & 1 \end{vmatrix} = 0+1-1=0$

$\sigma_3 = \det(A) = \begin{vmatrix} 0 & 1 \\ 1 & 1 \end{vmatrix} = -1$

$A^3 - 2A^2 + 0 \cdot A + I_3 = O_3 \,/ \cdot A^{-1}$

$A^{-1} = -A^2 + 2A$

$A^2 = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 2 \end{pmatrix}$

$A^{-1} = -\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 2 \end{pmatrix} + 2\begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 & -1 \\ 0 & -1 & 1 \\ 0 & 1 & 0 \end{pmatrix}$

3. $A = \begin{pmatrix} 0 & 1 & 1 & 0 \\ -1 & 2 & 0 & 1 \\ -1 & 0 & -2 & 1 \\ 0 & -1 & -1 & 0 \end{pmatrix}$

a) polinomul caracteristic

b) A^{100} (Th. H-C)

$P_A(x) = \det(A - xI_4) = x^4 - \sigma_1 x^3 + \sigma_2 x^2 - \sigma_3 x + \sigma_4 = 0$

$\sigma_1 = Tr(A) = 0$

$\sigma_2 = \begin{vmatrix} 0 & 1 \\ -1 & 2 \end{vmatrix} + \begin{vmatrix} 0 & 1 \\ -1 & 2 \end{vmatrix} + \begin{vmatrix} 0 & 0 \\ 0 & 0 \end{vmatrix} + \begin{vmatrix} 2 & 0 \\ 0 & -2 \end{vmatrix} + \begin{vmatrix} 2 & 1 \\ -1 & 0 \end{vmatrix} + \begin{vmatrix} -2 & 1 \\ -1 & 0 \end{vmatrix} = 1+1+0-4+1+1=0$

$\sigma_3 = \begin{vmatrix} 0 & 1 & 1 \\ -1 & 2 & 0 \\ -1 & 0 & -2 \end{vmatrix} + \begin{vmatrix} 0 & 1 & 0 \\ -1 & 2 & 1 \\ 0 & -1 & 0 \end{vmatrix} + \begin{vmatrix} 0 & 1 & 0 \\ -1 & -2 & 1 \\ 0 & -1 & 0 \end{vmatrix} + \begin{vmatrix} 2 & 0 & 1 \\ 0 & -2 & 1 \\ -1 & -1 & 0 \end{vmatrix} =$

$= \begin{vmatrix} 0 & 0 & 1 \\ -1 & 2 & 0 \\ -1 & 2 & -2 \end{vmatrix} + \begin{vmatrix} -1 & 1 \\ 0 & 0 \end{vmatrix} - \begin{vmatrix} -1 & 1 \\ 0 & 6 \end{vmatrix} + \begin{vmatrix} 2 & 0 & 1 \\ -2 & -2 & 0 \\ -1 & -1 & 0 \end{vmatrix} = 0$

$\sigma_4 = 0$

$x^4 = 0 \Rightarrow A^4 = O_4$

$\Rightarrow A^{100} = A^{4 \cdot 25} = O_4^{25} = O_4$

4. $A = \begin{pmatrix} 1 & -1 \\ 2 & 0 \end{pmatrix}$, $B = A^4 - 3A^3 + 3A^2 - 2A + 8I_2$

M_1 $a, b \in \mathbb{R}$, a.î. $B = aA + bI_2$

H-C : $A^2 - Tr(A)A + \det(A) I_2 = O_2$, unde $Tr(A) = 1$, $\det(A) = 2$

$A^2 = A - I_2 \Rightarrow A^3 = A^2 - 2A = -A - 2I_2 \Rightarrow A^4 = A^3 - 2A^2 = -3A + 2I_2$

$B = -3A + 2I_2 - 3(-A - 2I_2) + 3(A - 2I_2) - 2A + 8I_2 = A + 10I_2$

$\Rightarrow a = 1$, $b = 10$

M_2

$P = X^4 - 3X^3 + 3X^2 - 2X + 8$

$Q = X^2 - X + 2$

$P = Q \cdot C + R$, $R = X + 10$

$B = P(A) = O + A + 10I_2$

5.

$$A = \begin{pmatrix} 1 & 1 & 2 & 3 \\ 1 & 1 & 3 & 4 \\ 2 & 5 & 0 & 1 \\ 4 & 1 & 0 & 2 \end{pmatrix}$$

l_2, l_3 fixate

$$\det(A) = (-1)^{2+3+1+2} \begin{vmatrix} 1 & 1 \\ 2 & 5 \end{vmatrix} \begin{vmatrix} 2 & 3 \\ 0 & 2 \end{vmatrix} + (-1)^{2+3+1+3} \begin{vmatrix} 1 & 3 \\ 2 & 0 \end{vmatrix} \begin{vmatrix} 1 & 3 \\ 1 & 2 \end{vmatrix} + (-1)^{2+3+1+4} \begin{vmatrix} 1 & 4 \\ 2 & 1 \end{vmatrix} \begin{vmatrix} 1 & 2 \\ 1 & 0 \end{vmatrix}$$

$$+ (-1)^{2+3+2+3} \begin{vmatrix} 1 & 3 \\ 5 & 0 \end{vmatrix} \begin{vmatrix} 1 & 3 \\ 4 & 2 \end{vmatrix} + (-1)^{2+3+2+4} \begin{vmatrix} 1 & 4 \\ 5 & 1 \end{vmatrix} \begin{vmatrix} 1 & 2 \\ 4 & 0 \end{vmatrix} + (-1)^{2+3+3+4} \begin{vmatrix} 3 & 4 \\ 0 & 1 \end{vmatrix} \begin{vmatrix} 1 & 1 \\ 4 & 1 \end{vmatrix} =$$

$$= 3 \cdot 4 - 6 + 14 + 150 - 19 \cdot 8 - 9 =$$

$$= 6 + 14 + 150 - 152 - 9 = 20 - 2 - 9 = 9$$

10) $A = \begin{pmatrix} 1 & 2 \\ 0 & 2 \end{pmatrix}$ $\quad A^n = ?$ (H-C.) $\quad -1 \cdot (-1-8)$ $\quad 9$

$x_1 = 1 \quad y_1 = 0$

$$A^2 - \mathrm{Tr}(A) A + \det A \, I_2 = O_2 \Rightarrow A^2 = 3A - 2I_2$$

$$\begin{cases} x_1 = 1 & y_1 = 0 \\ x_2 = 3 & y_2 = -2 \end{cases} \qquad A^{n+1} = A^n \cdot A = x_{n+1} A + y_{n+1} I_2$$

$$= \underbrace{x_n (3A - 2I_2) + y_n A}_{(3x_n + y_n) A - 2x_n I_2}$$

$$\begin{cases} x_{n+1} = 3x_n + y_n \\ y_{n+1} = -2x_n \Rightarrow y_n = -2x_{n-1} \end{cases}$$

$$\Rightarrow x_{n+1} = 3x_n - 2x_{n-1} \Rightarrow x_{n+1} - 3x_n + 2x_{n-1} = 0$$

$$t^2 - 3t + 2 = 0 \quad t_1 = 1 \quad t_2 = 2$$

$$\Rightarrow x_n = c_1 t_1^n + c_2 t_2^n \Rightarrow \begin{matrix} 1 = c_1 + 2c_2 \\ 3 = c_1 + 4c_2 \end{matrix}$$

$\Rightarrow c_2 = 1 \quad c_1 = -1$

$$A^n = (-1 + 2^n) A + (2 - 2^n) I_2$$

$$y_n = -2(-1 + 2^{n-1})$$

11. $X^{2024} = A = \begin{pmatrix} 2 & -4 \\ 1 & 2 \end{pmatrix}$

a) $X \in \mathcal{M}_2(\mathbb{R})$, $X = ?$

b) $X \in \mathcal{M}_2(\mathbb{C})$, nr. de sol.?

a) $\det(A) = 0 \Rightarrow \det(X) = 0$

$\overset{C-H}{\Longrightarrow} X^2 - Tr(X) \cdot X = O_2 \Leftrightarrow X^2 = Tr(X) X$

$\Rightarrow X^{2024} = Tr(X)^{2023} X \;/ Tr$

$\Rightarrow Tr(A) = Tr(X)^{2024}$

$\Rightarrow 4 = Tr(X)^{2024}$

a) $Tr(X) = \pm \sqrt[1012]{2}$

$X = \frac{1}{Tr(X)^{2023}} \cdot A = \frac{1}{(\pm\sqrt[1012]{2})^{2023}} \begin{pmatrix} 2 & -4 \\ -1 & 2 \end{pmatrix}$

b) Ec. are 2024 de sol. complexe

$\Rightarrow X^{2024} = A$ are 2024 sol. în $\mathcal{M}_2(\mathbb{C})$

~~$\Rightarrow 4 = Tr(X)^{2024}$ $\Rightarrow Tr(X) = \pm\sqrt[1012]{2} \Rightarrow$~~

~~$\Rightarrow X = \frac{1}{(Tr(X))^{2023}} \cdot A = \frac{1}{\sqrt[1012]{2}^{\,2023}} \cdot \begin{pmatrix} 2 & -4 \\ -1 & 2 \end{pmatrix}$~~

~~b) ec. are 2024 sol. complexe $\Rightarrow X^{2024}$ are~~

~~$= A$ 2024 de sol. în $\mathcal{M}_2(\mathbb{C})$~~

I_2

$\triangle ABC$ a, b, c = lung. laturilor

$\begin{cases} ay + bx = c \\ cx + az = b \\ bz + cy = a \end{cases}$

sist. $\forall \triangle ABC$ are sol. unică $(x_0, y_0, z_0) \in (-1, 1)$

$A = \underbrace{\left(\begin{matrix} b & a & 0 \\ c & 0 & a \\ 0 & c & b \end{matrix}\,\middle|\,\begin{matrix} c \\ b \\ a \end{matrix}\right)}_{\bar{A}}$

$\Delta = \det A = b \cdot \begin{vmatrix} 0 & a \\ c & b \end{vmatrix} - a \begin{vmatrix} c & a \\ 0 & b \end{vmatrix}$

$= -bac - acb = -2abc \neq 0$

$(b^2+c^2-a^2)$

analog $y = \cos B$ $z = \cos C$ $\Rightarrow x, y, z \in (-1, 1)$

$(a^2 = b^2 + c^2 - 2bc \cos A)$ Pitagora

$$\begin{cases} x + 2y = m+1 \\ 2x - 3y = m-1 \\ mx + y = 3 \end{cases}, \quad m \in \mathbb{R}$$

$$A = \left(\begin{array}{cc|c} 1 & 2 & m+1 \\ 2 & -3 & m-1 \\ m & 1 & 3 \end{array}\right)$$

$rg(A) = 2$

$rg(\bar{A}) = 3$

$$\Delta_p = \begin{vmatrix} 1 & 2 \\ 2 & -3 \end{vmatrix} \neq 0$$

$$\Delta_c = \begin{vmatrix} 1 & 2 & m+1 \\ 2 & -3 & m-1 \\ m & 1 & 3 \end{vmatrix} = \begin{vmatrix} 1-2m & 0 & m-5 \\ 2+3m & 0 & m+8 \\ m & 1 & 3 \end{vmatrix} = -1 \begin{vmatrix} 1-2m & m-5 \\ 2+3m & m+8 \end{vmatrix} = -\left((1-2m)(m+8) - (m-5)(2+3m)\right) =$$

$$= -\left(m + 8 - 2m^2 - 16m - 2m - 3m^2 + 10 + 15m\right) =$$

$$= 5m^2 + 2m - 18 \neq 0$$

$$\Delta = 364$$

$$m_{1,2} = \frac{-2 \pm 2\sqrt{91}}{10}$$