f de $n+1$ ori derivabilă pe I, $x_0, x \in I$

Atunci există $\theta \in (x_0, x)$ sau $\theta \in (x, x_0)$ ai

$$f(x) = T_n(x) + \frac{f^{(n+1)}(\theta)}{(n+1)!}(x-x_0)^{n+1}$$ - formula lui Taylor cu restul lui Lagrange.

$$T_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!}(x-x_0)^k.$$

$k=0$: $f(x) = f(x_0) + f'(\theta)(x-x_0)$.

1) $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + \frac{x}{1!} + \dots + \frac{x^n}{n!} + \dots \quad , \forall x \in \mathbb{R}.$

$f(x) = e^x$

$f^{(n)}(x) = e^x . \qquad f^{(n)}(0) = 1.$

$T_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(0)}{k!} \cdot x^k = 1 + \frac{x}{1!} + \dots + \frac{x^n}{n!}$ - pol. Taylor de grad n asociat lui f în 0

$f(x) = T_n(x) + R_n(x)$

Fie $x \in \mathbb{R}$. Există $\theta_x \in (0, x)$ sau $\theta_x \in (x, 0)$ a.î.

$$f(x) = T_n(x) + \frac{f^{(n)}(\theta_x)}{(n+1)!} \cdot x^{n+1} = T_n(x) + \frac{e^{\theta_x}}{(n+1)!} x^{n+1}.$$

$$R_n(x) = e^{\theta_x} \cdot \frac{x^{n+1}}{(n+1)!} \qquad 0 < \theta_x < x \text{ sau } x < \theta_x < 0.$$

$$|\theta_x| < |x|$$

$$\left| \frac{e^{\theta_x}}{(n+1)!} x^{n+1} \right| \leq \frac{e^{|\theta_x|}}{(n+1)!} |x|^{n+1} < \frac{e^{|x|}}{(n+1)!} |x|^{n+1}$$

$$\lim_{n\to\infty} \frac{e^{|x|}}{(n+1)!} |x|^{n+1} = 0.$$

$$\lim_{n\to\infty} R_n(x) = 0 \Longrightarrow \lim_{n\to\infty} T_n(x) = f(x), \ \forall x \in \mathbb{R}.$$

$$\Rightarrow e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} + \cdots, \ \forall x \in \mathbb{R}.$$

$$e^{3x} = \sum_{n=0}^{\infty} \frac{3^n \cdot x^n}{n!}, \quad x \in \mathbb{R}.$$

Exerciţiu. $\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$ $\quad \forall x \in \mathbb{R}$

$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$$

2) Dezvoltaţi în serie de puteri ale lui x funcţia

$f(x) = \ln(1+x), \quad x > -1$

şi precizaţi intervalul pe care este valabilă dezvoltarea.

Soluţie. $f'(x) = \frac{1}{1+x} = 1 - x + x^2 + \dots + (-1)^n x^n + \dots$

$\forall x \in (-1, 1).$

Integrand termen cu termen obtinem:

$$f(x) + C = x - \frac{x^2}{2} + \frac{x^3}{3} + \dots + (-1)^n \cdot \frac{x^{n+1}}{n+1} + \dots, \ \forall x \in (-1,1)$$

$x=0$. $\left.\begin{matrix} f(0) + C = 0 \\ f(0) = 0 \end{matrix}\right\} \Longrightarrow C = 0.$

Deci $f(x) = \ln(1+x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{n+1}}{n+1} \ \forall x \in (-1,1)$.

Pt $x=1$ seria $\sum_{n=0}^{\infty} \frac{(-1)^n}{n+1}$ este convergentă.

T Abel: $\Longrightarrow \ln 2 = \lim_{\substack{x \to 1 \\ x<1}} \ln(1+x) = \lim_{\substack{x \to 1 \\ x<1}} f(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n+1}$

Deci

$$\ln 2 = \sum_{n=0}^{\infty} \frac{(-1)^n}{n+1} = 1 - \frac{1}{2} + \frac{1}{3} + \cdots\cdots$$

şi

$$\ln(1+x) = \sum_{n=0}^{\infty} \frac{(-1)^n \, x^{n+1}}{n+1}, \quad \forall x \in (-1,1].$$

$$\overline{\ln(1-x)} = -\sum_{n=0}^{\infty} \frac{x^{n+1}}{n+1}, \quad \forall x \in [-1,1).$$

$$(1+x)^{\alpha} = 1 + \frac{\alpha}{1!}x + \frac{\alpha(\alpha-1)}{2!}x^2 + \dots + \frac{\alpha(\alpha-1)\dots(\alpha-n+1)}{n!}x^n + \dots$$

$$\forall x \in (-1,1)$$

$$\frac{1}{\sqrt{1+x}} = (1+x)^{-\frac{1}{2}} = 1 - \frac{\frac{1}{2}}{1!}x + \frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{2!}x^2 + \dots$$

$$+ \frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\dots\left(-\frac{1}{2}-n+1\right)}{n!}x^n + \dots, \forall x \in (-1,1)$$

$$= 1 - \frac{1}{2}x + \dots + \frac{(-1)^n \cdot 1 \cdot 3 \cdot 5 \dots (2n-1)}{2 \cdot 4 \cdot 6 \dots 2n}x^n + \dots, \forall x \in (-1,1)$$

(*)

$$-\frac{1}{2} - n + 1 = \frac{-1-2n+2}{2} = \frac{-2n+1}{2} = -\frac{2n-1}{2}$$

Facem $x \longmapsto -x$ în $(*)$

$$(**) \quad \frac{1}{\sqrt{1-x}} = 1 + \frac{1}{2}x + \cdots \frac{1\cdot 3\cdot 5\cdots(2n-1)}{2\cdot 4\cdot 6\cdots(2n)}x^n + \cdots, \quad \forall x \in (-1,1)$$

3) Dezvoltați în serie de puteri ale lui x, fct. $\arcsin x$.

Soluție:

$$f(x) = \arcsin x, \quad x \in [-1,1].$$

$$f'(x) = \frac{1}{\sqrt{1-x^2}}, \quad \forall x \in (-1,1)$$

Făcând $x \longrightarrow x^2$ în $(**)$ avem.

$$f'(x) = \frac{1}{\sqrt{1-x^2}} = 1 + \frac{1}{2}x^2 + \cdots + \frac{1\cdot 3\cdots(2n-1)}{2\cdot 4\cdots(2n)}x^{2n} + \cdots, \quad x \in (-1,1).$$

Integrand termen cu termen obținem că

$$f(x)+C = x+\frac{x^3}{6}+\dots+\frac{1\cdot 3\cdot 5\cdots(2n-1)}{2\cdot 4\cdot 6\cdots(2n)}\cdot\frac{x^{2n+1}}{2n+1}+\dots$$

$\forall x\in(-1,1)$

unde $C\in\mathbb{R}$.

$x=0$. $f(0)+C=0 \Rightarrow C=0$.

Deci

$$\arcsin x = x+\frac{x^3}{6}+\dots+\frac{1\cdot 3\cdot 5\cdots(2n-1)}{2\cdot 4\cdot 6\cdots(2n)}\cdot\frac{x^{2n+1}}{2n+1}+\dots$$

$\forall x\in(-1,1)$

Pt $x=1$ obținem seria

$$1+\sum_{n=1}^{\infty}\frac{1\cdot 3\cdots(2n-1)}{2\cdot 4\cdots(2n)}\cdot\frac{1}{2n+1}$$

$$\sum_{n=1}^{\infty} \frac{1\cdot 3 \cdots (2n-1)}{2\cdot 4 \cdots 2n} \cdot \frac{1}{2n+1}$$

$$\sqrt{1\cdot 3} < \frac{1+3}{2} = 2.$$

$$\sqrt{3\cdot 5} < \frac{3+5}{2} = 4.$$

$$\vdots$$

$$\sqrt{(2n-1)(2n+1)} < 2n.$$

$$1\cdot 3\cdot 5 \cdots (2n-1)\sqrt{2n+1} < 2\cdot 4\cdot 6 \cdots 2n.$$

$$a_n = \frac{1\cdot 3 \cdots (2n-1)}{2\cdot 4 \cdots (2n)} \cdot \frac{1}{2n+1} < \frac{1}{\sqrt{2n+1}} \cdot \frac{1}{(2n+1)} = \frac{1}{\sqrt{(2n+1)^3}}$$

$$\sum \frac{1}{\sqrt{(2n+1)^3}} \sim \sum \frac{1}{n^{3/2}} \text{ convergentă.}$$

Deci $\sum \frac{1\cdot 3 \cdots (2n-1)}{2\cdot 4 \cdots 2n} \cdot \frac{1}{2n+1}$ este convergentă.

T. Abel.

$$\frac{\pi}{2} = \lim_{\substack{x\to 1\\ x<1}} \arcsin x = 1 + \sum_{n=1}^{\infty} \frac{1\cdot 3 \cdots (2n-1)}{2\cdot 4 \cdots 2n} \cdot \frac{1}{2n+1}$$

Asadar pt $\forall\, x \in [-1,1]$, avem.

$$\arcsin x = x + \frac{x^3}{6} + \cdots + \frac{1\cdot 3 \cdots (2n-1)}{2\cdot 4 \cdots 2n} \cdot \frac{x^{2n+1}}{2n+1} + \cdots$$

Multimea de conv. a seriei (1) este $\left(-\frac{1}{2}, \frac{1}{2}\right)$.

Multimea de conv a seriei

$$\sum_{n=0}^{\infty} \frac{n \cdot 2^n}{n+1}(x-1)^n$$

este $A = \left\{ x \in \mathbb{R} \;\middle|\; x-1 \in \left(-\frac{1}{2}, \frac{1}{2}\right) \right\} = \left(\frac{1}{2}, \frac{3}{2}\right)$.

5) Găsiți mulțimea de convergență a seriei

$$\sum_{n=0}^{\infty} \frac{2^n \cdot n}{n+1} (x-1)^n.$$

$$\sum_{n=0}^{\infty} \frac{2^n \cdot n}{n+1} \cdot y^n. \qquad (1)$$

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \frac{2^{n+1}(n+1)}{n+2} \cdot \frac{n+1}{n \cdot 2^n} = 2$$

Raza de conv. a seriei (1) este $R = \frac{1}{2}$.

$y = \frac{1}{2}$: $\sum_{n=0}^{\infty} \frac{n}{n+1}$ divergentă.

$y = -\frac{1}{2}$ $\sum_{n=0}^{\infty} \frac{(-1)^n n}{n+1}$ divergentă

Norma euclidiana pe $\mathbb{R}^n$; $\|\cdot\|$.

$x = (x_1, x_2, \dots, x_n)$

$\|x\| = \sqrt{x_1^2 + x_2^2 + \dots + x_n^2}$.

1) $\|x\| \geq 0$, $\|x\| = 0 \Leftrightarrow x = 0$ $\left(0 = (0, 0, \dots, 0)\right)$

2) $\|\alpha x\| = |\alpha| \cdot \|x\|$, $\forall \alpha \in \mathbb{R}, \forall x \in \mathbb{R}^n$

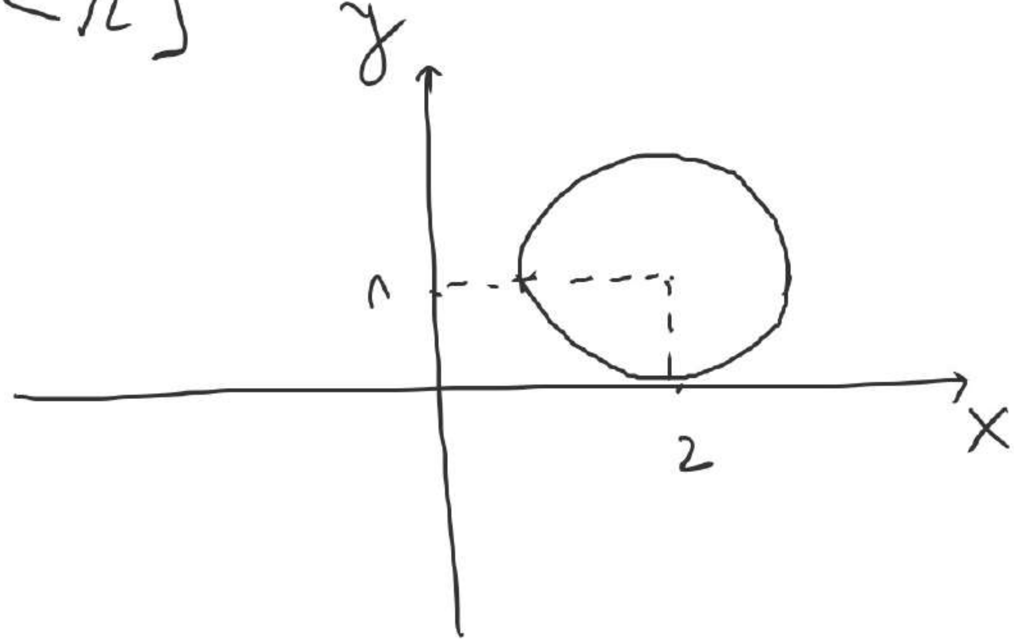
3) $\|x+y\| \leq \|x\| + \|y\|$, $\forall x, y \in \mathbb{R}^n$

d - dist. euclidiana pe $\mathbb{R}^n$.

$$d(x,y) = \|x-y\| = \sqrt{(x_1-y_1)^2 + \dots + (x_n-y_n)^2}$$

$x = (x_1, x_2, \dots, x_n)$; $y = (y_1, y_2, \dots, y_n)$

$$B_r(a) = \{ x \in \mathbb{R}^n \mid d(x,a) < r \}$$
$$= \{ x \in \mathbb{R}^n \mid \|x - a\| < r \}$$



$n = 2.$

$B_1(a)$, $a = (2,1)$.

$$\{ (x,y) \in \mathbb{R}^2 \mid (x-2)^2 + (y-1)^2 < 1 \}$$

$D \subset \mathbb{R}^n$ deschisă dacă:

$\forall x \in D$ există $\exists r > 0$ a.î. $B_r(x) \subset D$.

Exerciții: $B_a(r)$ este o mulțime deschisă.

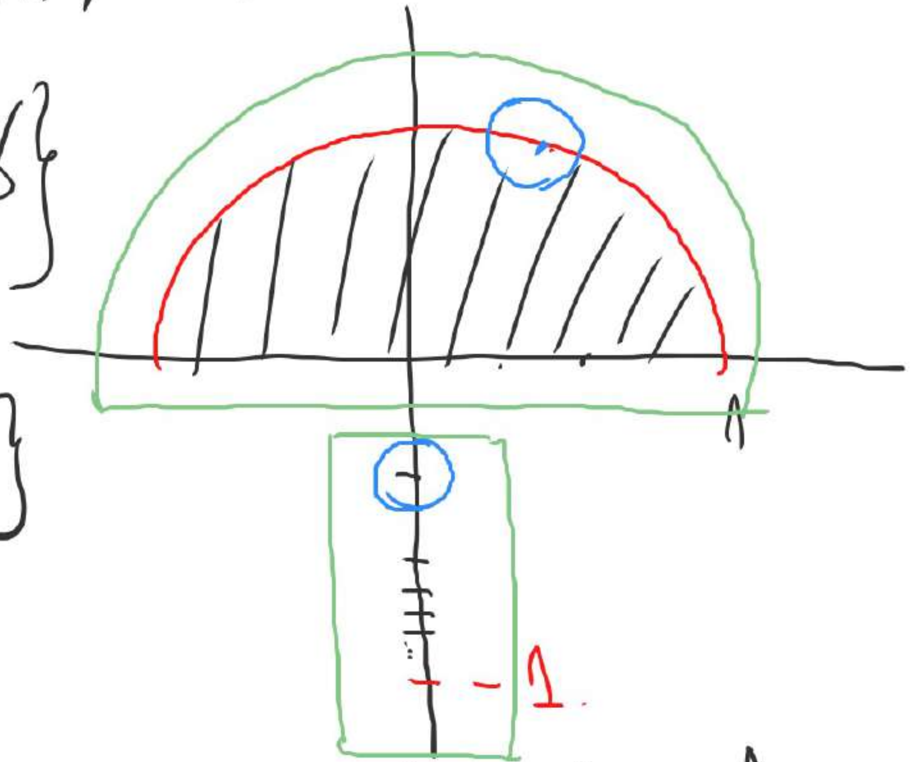
Fie $A = \{(x,y) \in \mathbb{R}^2 \mid x^2+y^2 < 1, y \geq 0\} \cup \{(0, -\frac{n}{n+1}) \mid n \in \mathbb{N}\}$

$\mathring{A} = ?$ $\bar{A} = ?$ $A' = ?$ $Fr(A) = ?$

$\bar{A} = \{\underset{(x,y)}{\overset{\shortparallel}{z}} \in \mathbb{R}^2 \mid \forall V \in \mathcal{V}(z), V \cap A \neq \emptyset\}$

$\mathring{A} = \{z \in \mathbb{R}^2 \mid \exists V \in \mathcal{V}(z), V \subset A\}$

$= \{z \in \mathbb{R}^2 \mid A \in \mathcal{V}(z)\}$



—— - nu sunt pct. din A.

$$A' = \{ z \in \mathbb{R}^2 \mid \forall V \in \mathcal{V}(z),\ V \cap (A \setminus \{z\}) \neq \emptyset \}$$

$$Fr(A) = \bar{A} \setminus \mathring{A} = \bar{A} \cap C\mathring{A} = \bar{A} \cap \overline{CA}.$$

$$\mathring{A} = \{ (x,y) \in \mathbb{R}^2 \mid x^2+y^2 < 1,\ y > 0 \}$$

$$\bar{A} = \{ (x,y) \in \mathbb{R}^2 \mid x^2+y^2 \leq 1,\ y \geq 0 \} \cup \left\{ \left(0, -\frac{n}{n+1}\right),\ n \in \mathbb{N} \right\} \cup \{(0,-1)\}$$

$$A' = \{ (x,y) \in \mathbb{R}^2 \mid x^2+y^2 \leq 1,\ y \geq 0 \} \cup \{(0,-1)\}$$

$\mathring{A} \neq A \Rightarrow A$ nu este închisă

$\bar{A} \neq A \Rightarrow A$ nu este deschisă.

A este neconexă dacă

$\exists$ $D_1, D_2 \subset \mathbb{R}^2$ deschise a.î.

$$D_1 \cap A \neq \emptyset, \quad D_2 \cap A \neq \emptyset, \quad A \subset D_1 \cup D_2$$

$$D_1 \cap D_2 \cap A = \emptyset$$

$$D_1 = \{(x,y) \in \mathbb{R}^2 \mid x^2+y^2 < 1{,}1 \ ; \ y > -0{,}1\}$$

$$D_2 = \left(-\frac{1}{2}, \frac{1}{2}\right) \times \left(-\frac{1}{3}, -2\right)$$