$\overline{\iint_A} f(x,y)\,dx\,dy = \inf\{ S_\Delta(f) \mid \Delta \text{ desc Jordan a lui } A\}$
$= \inf\{\mu(A) \mid \Delta \text{ desc Jordan a lui } A\} = \mu(A).$

$\underline{\iint_A} f(x,y)\,dx\,dy = \sup\{ s_\Delta(f) \mid \Delta \text{ desc Jordan a lui } A\}$
$= \sup\{0 \mid \Delta \text{ desc Jordan a lui } A\} = 0$

$\underline{\iint_A} f(x,y)\,dx\,dy \neq \overline{\iint_A} f(x,y)\,dx\,dy$

$\Rightarrow f$ nu e integr. R. $\square$

11.01.2023

## Seminar 13

1) Det $\int_{-\infty}^{\infty} \frac{x}{1+x^4}\,dx$

Sol: $\int_{-\infty}^{0} \frac{x}{1+x^4}\,dx = \frac{1}{2}\int_{-\infty}^{0} \frac{(x^2)'}{1+(x^2)^2}\,dx = \frac{1}{2}\cdot \lim_{c\to-\infty}\int_c^0 \frac{(x^2)'}{1+(x^2)^2}\,dx$

$= \frac{1}{2}\cdot \lim_{c\to-\infty} \operatorname{arctg}(x^2)\Big|_c^0 = \frac{1}{2}\cdot\lim_{c\to-\infty}\left(0-\operatorname{arctg}(c^2)\right) = \frac{-1}{2}\cdot\frac{\pi}{2} = \frac{-\pi}{4}$

$\int_0^{\infty} \frac{x}{1+x^4}\,dx = \frac{1}{2}\int_0^{\infty}\frac{(x^2)'}{1+(x^2)^2}\,dx = \frac{1}{2}\lim_{d\to\infty}\int_0^d \frac{(x^2)'}{1+(x^2)^2}\,dx =$

$= \frac{1}{2}\lim_{d\to\infty}\left(\operatorname{arctg}(x^2)\right)\Big|_0^d = \frac{1}{2}\lim_{d\to\infty}\left(\operatorname{arctg}(d^2)\right) = \frac{\pi}{4}$

$\int_{-\infty}^{\infty}\frac{x}{1+x^4}\,dx = \int_{-\infty}^{0}\frac{x}{1+x^4}\,dx + \int_0^{\infty}\frac{x}{1+x^4}\,dx = \frac{-\pi}{2} + \frac{\pi}{2} = 0$

2) Stud. convergența (natura) următ. integrale improprii:

a) $\int_1^{\infty} \frac{1}{x^4+1} dx$

Fie $f, g : [1, +\infty) \to [0, \infty)$, $f(x) = \frac{1}{x^4+1} dx$

$g(x) = \frac{1}{x^4}$

Avem $0 \le f(x) \le g(x)$, $\forall x \in [1, \infty)$

$\int_1^{\infty} g(x) dx = \lim_{c \to \infty} \int_1^c \frac{1}{x^4} dx = \lim_{c \to \infty} \int_1^c \left(-\frac{1}{3x^3}\right)' dx =$

$= \lim_{c \to \infty} -\frac{1}{3x^3}\Big|_1^c = \lim_{c \to \infty} \left(-\frac{1}{3c^3} + \frac{1}{3}\right) = \frac{1}{3}$.

Deci $\int_1^{\infty} g(x) dx$ e conv.

Conf. crit de comp. cu ineg. avem că $\int_1^{\infty} f(x) dx$ e conv.

b) $\int_2^{\infty} \frac{1}{\sqrt{x}-1} dx$

Fie $f, g : [2, \infty) \to (0, \infty)$; $f(x) = \frac{1}{\sqrt{x}-1}$, $g(x) = \frac{1}{\sqrt{x}}$

$\lim_{x \to \infty} \frac{f(x)}{g(x)} = \lim_{x \to \infty} \frac{\sqrt{x}}{\sqrt{x}-1} = 1 \in (0, \infty)$.

Cf Crit de comp. cu limită avem că

$\int_2^{\infty} f(x) dx \sim \int_2^{\infty} g(x) dx$

$\int_2^{\infty} g(x) dx = \lim_{d \to \infty} \int_2^d \frac{1}{\sqrt{x}} dx = \lim_{d \to \infty} \int_2^d x^{-\frac{1}{2}} dx =$

$= \lim_{d \to \infty} \frac{x^{\frac{1}{2}}}{\frac{1}{2}}\Big|_2^d = \lim_{d \to \infty} \left(2\sqrt{d} - 2\sqrt{2}\right) = +\infty$.

Deci $\int_2^{\infty} g(x) dx$ e div.

Cf Crit de comp cu limită avem că $\int_2^{\infty} f(x) dx$ e div $\square$

T: c) $\int_1^\infty \frac{1}{\sqrt{x}+1}\,dx$

d) $\int_1^\infty \sin\frac{1}{x^{11}}\,dx$

<u>Sol</u>: $\frac{1}{x^{11}} \in (0,1] \subset \left(0,\frac{\pi}{2}\right) \quad \forall x \in [1,\infty) \Rightarrow \sin\frac{1}{x^{11}} > 0 \quad \forall x \in [1,\infty)$

Fie $f: [1,\infty) \to (0,\infty)$, $f(x) = \sin\frac{1}{x^{11}}$.

$x \longmapsto \sin x$ crescătoare, $x \in \left(0,\frac{\pi}{2}\right)$, $\sin x \in \mathbb{R}$

$x \longmapsto \frac{1}{x^{11}}$ descrescătoare, $x \in [1,\infty)$, $\frac{1}{x^{11}} \in \mathbb{R}$

$\Rightarrow f$ descrescăt

Cf Crit. integral al lui Cauchy avem că

$\int_1^\infty f(x)\,dx \sim \sum_{n=1}^\infty f(n)$

$\sum_{n=1}^\infty f(n)\,dx = \sum_{n=1}^\infty \sin\frac{1}{n^{11}}$

Fie $x_n = \sin\frac{1}{x^{11}}$ $\forall n \in \mathbb{N}^*$ și $y_n = \frac{1}{n^{11}}$, $\forall n \in \mathbb{N}^*$

$\lim_{n\to\infty} \frac{x_n}{y_n} = \lim_{n\to\infty} \frac{\sin\frac{1}{n^{11}}}{\frac{1}{n^{11}}} = 1 \in (0,\infty)$

Cf Crit de comp cu limită (pt serii) avem că:

$\sum_{n=1}^\infty x_n \sim \sum_{n=1}^\infty y_n$

$\sum_{n=1}^\infty y_n = \sum_{n=1}^\infty \frac{1}{n^{11}}$ conv (serie arm. gen. cu $\alpha = 11$)

Deci $\sum_{n=1}^\infty f(n)$ e conv $\Rightarrow \int_1^\infty f(x)\,dx$ conv

3) Folosind fct. $\Gamma$ și $\beta$ det:

a) $\int_0^\infty e^{-x^2} dx = \frac{1}{2}\int_0^\infty t^{\frac{1}{2}} \cdot e^{-t} dt = \frac{1}{2}\int_0^\infty t^{\frac{1}{2}-1} \cdot e^{-t} dt =$

$t = x^2 \Rightarrow x = \sqrt{t}$

$dx = \frac{1}{2}\sqrt{t}\, dt$

$x = 0 \Rightarrow t = 0$

$x \to \infty \Rightarrow t \to \infty$

$= \frac{1}{2}\Gamma\left(\frac{1}{2}\right) = \frac{\sqrt{\pi}}{2}$

T b) $\int_{-\infty}^{\infty} e^{-x^2} dx$

c) $\int_0^\infty x^6 \cdot e^{-2x} dx = \frac{1}{2}\int_0^\infty \frac{t^6}{2^6} e^{-t} dt = \frac{1}{2^7}\int_0^\infty t^6 e^{-t} dt =$

$2x = t \Rightarrow x = \frac{t}{2}$

$dx = \frac{1}{2} dt$

$x = 0 \Rightarrow t = 0$

$x \to \infty \Rightarrow t \to \infty$

$= \frac{1}{2^7}\int_0^\infty t^{7-1} e^{-t} dt = \frac{\Gamma(7)}{2^7} = \frac{6!}{2^7}$ $\square$

d) $\int_0^\infty \sqrt{x} \cdot e^{-x^3} dx = \frac{1}{3}\int_0^\infty t^{\frac{1}{6}} \cdot e^{-t} \cdot t^{-\frac{2}{3}} dt =$

$x^3 = t \Rightarrow x = t^{\frac{1}{3}}$

$dx = \frac{1}{3} \cdot t^{-\frac{2}{3}}$

$x = 0 \Rightarrow t = 0$

$x \to \infty \Rightarrow t \to \infty$

$=$

e) $\int_0^2 \frac{x^2}{\sqrt{2-x}}\,dx = \int_0^2 \frac{x^2}{\sqrt{2\cdot\left(1-\frac{x}{2}\right)}}\,dx =$

$\frac{x}{2} = t \Rightarrow x = 2t$

$dx = 2\,dt$

$x = 0 \Rightarrow t = 0$

$x \to 2 \Rightarrow t \to 1$

$= \int_0^1 \frac{4t^2}{\sqrt{2(1-t)}}\,2\,dt = \frac{8}{\sqrt{2}}\int_0^1 t^2(1-t)^{-\frac{1}{2}}\,dt =$

$= \frac{8}{\sqrt{2}}\int_0^1 t^{3-1}(1-t)^{\frac{1}{2}-1}\,dt = \frac{8}{\sqrt{2}}\,B\left(3,\frac{1}{2}\right)$

$B\left(3,\frac{1}{2}\right) = \frac{\Gamma(3)\cdot\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(3+\frac{1}{2}\right)} = \frac{2!\cdot\sqrt{\pi}}{\Gamma\left(3+\frac{1}{2}\right)} = \frac{2\sqrt{\pi}}{\Gamma\left(3+\frac{1}{2}\right)}.$

$\Gamma\left(3+\frac{1}{2}\right) = \Gamma\left(1+2+\frac{1}{2}\right) = \left(2+\frac{1}{2}\right)\Gamma\left(2+\frac{1}{2}\right) = \frac{5}{2}\Gamma\left(1+1+\frac{1}{2}\right)$

$= \frac{5}{2}\left(1+\frac{1}{2}\right)\Gamma\left(1+\frac{1}{2}\right) = \frac{5}{2}\cdot\frac{3}{2}\cdot\frac{1}{2}\,\Gamma\left(\frac{1}{2}\right) = \frac{15\sqrt{\pi}}{8}.$

Deci $B\left(3,\frac{1}{2}\right) = \frac{2\sqrt{\pi}}{\frac{15\sqrt{\pi}}{8}} = \frac{16}{15}$ și

$\int_0^2 \frac{x^2}{\sqrt{2-x}}\,dx = \frac{8}{\sqrt{2}}\cdot\frac{16}{15} = \frac{64\sqrt{2}}{15}$

f) $\int_0^{\frac{\pi}{2}} (\sin t)^{\frac{5}{2}}\cdot(\cos t)^{\frac{3}{2}}\,dt$

Sol: $\begin{cases} 2x-1 = \frac{5}{2} \Rightarrow x = \frac{7}{4} \\ 2y-1 = \frac{3}{2} \Rightarrow y = \frac{5}{4} \end{cases}$

$\int_0^{\frac{\pi}{2}} (\sin t)^{\frac{5}{2}}\cdot(\cos t)^{\frac{3}{2}}\,dt = \frac{1}{2}\cdot\underbrace{2\int_0^{\frac{\pi}{2}} (\sin t)^{2\cdot\frac{7}{4}-1}\cdot(\cos t)^{2\cdot\frac{5}{4}-1}\,dt}_{= B\left(\frac{7}{4},\frac{5}{4}\right)}$

$= \frac{1}{2}\cdot B\left(\frac{7}{4},\frac{5}{4}\right)$

$$B\left(\frac{7}{4},\frac{5}{4}\right)=\frac{\Gamma\left(\frac{7}{4}\right)\Gamma\left(\frac{5}{4}\right)}{\Gamma\left(\frac{7}{4}+\frac{5}{4}\right)}=\frac{\Gamma\left(\frac{7}{4}\right)\Gamma\left(\frac{5}{4}\right)}{\Gamma(3)}=\frac{\Gamma\left(\frac{7}{4}\right)\cdot\Gamma\left(\frac{5}{4}\right)}{2!}.$$

$$\Gamma\left(\frac{7}{4}\right)=\Gamma\left(1+\frac{3}{4}\right)=\frac{3}{4}\Gamma\left(\frac{3}{4}\right)$$

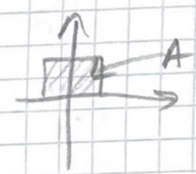
$$\Gamma\left(\frac{5}{4}\right)=\Gamma\left(1+\frac{1}{4}\right)=\frac{1}{4}\Gamma\left(\frac{1}{4}\right)$$

$$\Gamma\left(\frac{7}{4}\right)\cdot\Gamma\left(\frac{5}{4}\right)=\frac{3}{4}\cdot\Gamma\left(\frac{3}{4}\right)\cdot\frac{1}{4}\cdot\Gamma\left(\frac{1}{4}\right)=\frac{3}{16}\cdot\Gamma\left(1-\frac{1}{4}\right)\cdot\Gamma\left(\frac{1}{4}\right)$$

$$=\frac{3}{16}\cdot\frac{\pi}{\sin\left(\pi\cdot\frac{1}{4}\right)}=\frac{3}{16}\cdot\frac{\pi}{\frac{\sqrt{2}}{2}}=\frac{6\pi}{16\sqrt{2}}=\frac{3\pi\sqrt{2}}{16}.$$

$$B\left(\frac{7}{4},\frac{5}{4}\right)=\frac{3\pi\sqrt{2}}{16}\cdot\frac{1}{2}=\frac{3\pi\sqrt{2}}{32}.$$

4) Det. a) $\iint_A y\,dx\,dy$, unde $A=[-1,1]\times[0,2]$.




Sol: $A=[-1,1]\times[0,2]$ mult. compactă

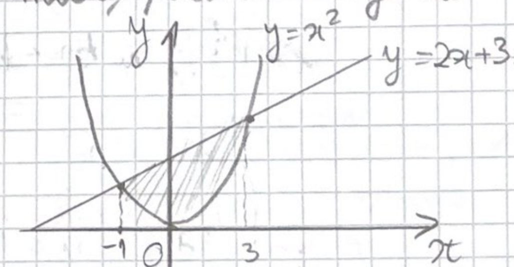
Fie $f:\mathbb{R}\to\mathbb{R}$, $f(x,y)=y$.

$f$ continuă

$$\iint_A y\,dx\,dy=\int_{-1}^{1}\left(\int_0^2 y\,dy\right)dx=\int_{-1}^{1}\left(\frac{y^2}{2}\Big|_{y=0}^{y=2}\right)dx=$$

$$=\int_{-1}^{1}\frac{2^2-0^2}{2}\,dx=\int_{-1}^{1}2\,dx=2x\Big|_{-1}^{1}=4. \quad \square$$

b) $\iint_A x\,dx\,dy$ unde $A$ e mult. plană mărg. de $y=x^2$ și $y=2x+3$




Sol: Det punctele de intersecție dintre $y=x^2$ și $y=2x+3$.

$$\begin{cases} y=x^2 \\ y=2x+3 \end{cases} \Rightarrow x^2=2x+3$$

$$\Rightarrow x^2-2x-3=0$$

$$\Delta=4+12=16 \qquad \sqrt{\Delta}=4$$

$x_1 = \frac{2+4}{2} = 3 \Rightarrow y_1 = 9$

$x_2 = \frac{2-4}{2} = -1 \Rightarrow y_2 = 1$

$A = \{(x,y) \in \mathbb{R}^2 \mid x \in [-1,3],\ x^2 \le y \le 2x+3\}$

Fie $\alpha, \beta : [-1,3] \to \mathbb{R}$, $\alpha(x) = x^2$,
$\beta(x) = 2x+3$

$\alpha, \beta$ cont.
$A \in \mathcal{J}(\mathbb{R}^2)$ și A compactă
Fie $f : A \to \mathbb{R}$, $f(x,y) = x$
f cont

$$\iint_A f(x,y)\,dx\,dy = \int_{-1}^{3} \left( \int_{x^2}^{2x+3} x\,dy \right) dx = \int_{-1}^{3} \left( xy \Big|_{y=x^2}^{y=2x+3} \right) dx$$

$$= \int_{-1}^{3} x(2x+3-x^2)\,dx = \int_{-1}^{3} (-x^3+2x^2+3x)\,dx =$$

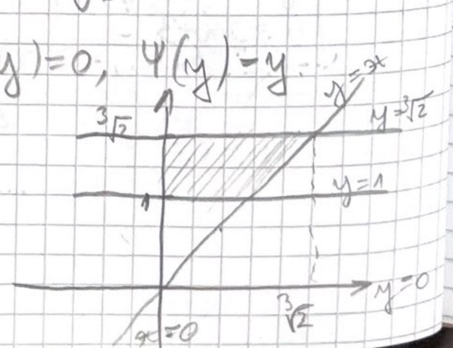
$$= -\frac{x^4}{4}\Big|_{-1}^{3} + 2\cdot\frac{x^3}{3}\Big|_{-1}^{3} + 3\cdot\frac{x^2}{2}\Big|_{-1}^{3} = \frac{32}{3}$$

T c) $\iint_A x\,dx\,dy$, unde A e mulț. plană limitată de $y = -x^2 - x + 2$ și $y = x - 1$

d) $\iint_A x\,dx\,dy$, unde A e mulț. plană mărg. de $x=0$, $y=1$, $y=\sqrt[3]{2}$, $y=x$

$A = \{(x,y) \in \mathbb{R}^2 \mid y \in [1, \sqrt[3]{2}],\ 0 \le x \le y\}$

Fie $\varphi, \psi : [1, \sqrt[3]{2}] \to \mathbb{R}$, $\varphi(y) = 0$, $\psi(y) = y$
$\varphi, \psi$ cont
$A \in \mathcal{J}(\mathbb{R}^2)$ și compactă



$f: A \to \mathbb{R}$, $f(x,y) = x$ $\quad$ $f$ cont

$$\iint_A f(x,y)\,dx\,dy = \int_1^{\sqrt[3]{2}} \left(\int_0^y x\,dx\right) dy = \int_1^{\sqrt[3]{2}} \frac{x^2}{2}\Big|_0^y dy$$

$$= \int_1^{\sqrt[3]{2}} \frac{y^2}{2}\, dy = \frac{y^3}{6}\Big|_1^{\sqrt[3]{2}} = \frac{1}{6}$$

## Curs 14

16.01.2024

### Teorema de permutare a limitei cu integrala (Cazul multidimensional)

Fie $p \in \mathbb{N}^*$, $\emptyset \neq A \in \mathcal{J}(\mathbb{R}^p)$; $f_n, f: A \to \mathbb{R}$, $\forall n \in \mathbb{N}$ a.î.

1) $f_n$ integr. Riemann și mărg (pe A)
2) $f_n \xrightarrow[n\to\infty]{u} f$

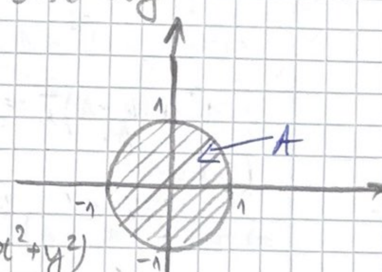
Atunci $f$ este integr. Riemann și mărg. și

$$\lim_{n\to\infty} \int_A f_n(x)\,dx = \int_A f(x)\,dx.$$

Ex: Fie $A = B[(0,0),1] = \overline{B}((0,0),1) = \{(x,y)\in\mathbb{R}^2 \mid x^2+y^2 \le 1\}$

Det $\displaystyle\lim_{n\to\infty} \iint_A \frac{\cos(n(x+y)) + 2(x^2+y^2)}{n^2+nx^2+y^2}\,dx\,dy$

Sol:




A convexă și mărg $\Rightarrow A \in \mathcal{J}(\mathbb{R}^2)$

A compactă

Fie $f_n: A \to \mathbb{R}$, $f_n(x,y) = \dfrac{\cos(n(x+y)) + 2(x^2+y^2)}{n^2+nx^2+y^2}$ $\quad \forall n \in \mathbb{N}^*$

$f_n$ cont $(\forall) n \in \mathbb{N}^*$
$A \in \mathcal{J}(\mathbb{R}^2)$ și A compactă $\Bigg|\Rightarrow$ $f_n$ integrabilă Riemann pe A și $f_n$ mărginită $\forall n \in \mathbb{N}^*$

CS: Fie $(x,y) \in A$.

$$0 \le |f_n(x,y)| = \frac{|\cos(n(x+y)) + 2(x^2+y^2)|}{n^2+nx^2+y^2} \le \frac{|\cos(n(x+y))| + 2|x^2+y^2|}{n^2+nx^2+y^2}$$

$$\le \frac{1+2}{n^2} = \frac{3}{n^2} \quad \forall n \in \mathbb{N}^*$$

Avem $0 \le |f_n(x,y)| = \frac{3}{n^2}$ $\quad \forall n \in \mathbb{N}^*$

Deci $\lim_{n\to\infty} |f_n(x,y)| = 0$. Deci $\lim_{n\to\infty} f_n(x,y) = 0$