### Prony, Padé, and Linear Prediction for Interpolation and Approximation in the Time and Frequency Domain Design of IIR Digital Filters and in Parameter Identification

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### 1 Introduction

Model based signal processing or signal analysis or signal representation has a rather different point of view from the more traditional filtering and algorithm based approaches. However, in all of these, the names of Prony, Padé, and linear prediction come up. This note examines these ideas with the goal of showing they are all based on the same principles and all can be extended and generalized.

Prony originally posed the problem of separating the individual characteristics of gases from measurements made on a combination of them in 1795 [1]. We pose this problem using modern terminology and notation as follows. A signal is modeled as y(n) which consists of equally spaced samples of x(t) which is a finite sum of N exponentials,

$$x(t) = \sum_{k=1}^{N} K_k e^{\alpha_k t}.$$
 (1)

The sampling is described by

$$y(n) = x(Tn) = \sum_{k=1}^{N} K_k e^{\alpha_k Tn} = \sum_{k=1}^{N} K_k \lambda_k^n$$
 (2)

where  $\lambda_k = e^{\alpha_k T}$ . Prony's problem is to calculate the 2N parameters,  $K_k$  and  $\alpha_k$  (or  $\lambda_k$ ), from the samples, y(n).

Prony's original approach [1] to solving this problem is straight-forward but awkward, requiring two different procedures for even order and odd order systems. A more flexible and general method can be derived using the z-transform of Prony's formulation.

Padé's method [2] is a rational approximation of a polynomial. It has been shown [3] to be the z-transformed version of Prony's method. The z-transform of (2) can be written as an  $N^{th}$  order rational function of z

$$Y(z) = \frac{b_0 + b_1 z^{-1} + \dots + b_M z^{-M}}{1 + a_1 z^{-1} + \dots + a_N z^{-N}}$$
(3)

with M = N-1. The problem is to find the 2N parameters,  $b_k$  and  $a_k$ , from the appropriate samples y(n). This can be expressed as a matrix convolution and solved [4, 5]. The equivalence of Prony's method and Padé's method is discussed in [3]. Soewito made use of this equivalence [6] as did Vargas [7].

Linear prediction uses a model of the signal which assumes that a particular value of the signal can be expressed as a weighted linear combination of the N previous values of the signal [8, 9]. Finding those weights turns out to be exactly the same problem as finding the denominator coefficients in (3) and equivalent to finding the  $\alpha_k$  or  $\lambda_k$  in (2).

These methods can all be extended from an interpolation problem to a least equation error method by using more samples y(n) than the number of unknown coefficients,  $a_n$  and  $b_n$  or  $K_k$  and  $\alpha_k$ . This is discussed in [4, 5]. Iterative methods have been developed which solve an iterative reweighted least equation error to achieve a least solution error result [10, 11, 12]. McClellan has shown this to be equivalent to a well known adaptive filtering method [13].

The basic idea behind these three methods is based on the elimination or minimization of the *equation error*, which is a linear problem, rather than the usual minimization of the *solution error*, which is a nonlinear problem. Some of these formulations also uncouple the calculation of the numerator and the denominator coefficients [12, 4]

It is possible to pose the problem in the frequency domain and also minimize the equation error [14, 5]. One advantage to doing that is better conditioning from the uniform sampling of the frequency response between zero and  $\pi$  rather than using the K samples at the beginning of an infinite time interval [5, 15, 16, 17] and to deal with the case where the design criteria are given in the frequency domain.

Prony's method (or Padé's method or linear prediction) in both the time and frequency domain is not only important for its own sake, but also as a method to start iterative methods or being the base of iterative methods that minimize other error criteria. There is much still to be discovered about this important, interesting, and powerful set of methods.

### Literature on Prony's Method and Extensions

Original [1, 2], Linear Algebra [18] Basic [19, 20, 3, 4, 21, 22, 23, 24, 25] Related [26, 27, 28, 29, 30] [31, 32, 33, 34, 35, 36] Applications[37, 38, 39, 40, 41, 42, 43, 44, 39] Iteration [45, 10, 46, 47, 48, 13, 49, 50]

### 2 The Exact Solution or the Interpolation Problem in the Time Domain (Classical Prony)

The formulation of the time-domain IIR filter design problem and the methods for its solution are a version of Prony's (actually Padé's) method.

The z-transform transfer function for an IIR filter [5] is given by

$$H(z) = \frac{B(z)}{A(z)} = \frac{b_0 + b_1 z^{-1} + \dots + b_M z^{-M}}{1 + a_1 z^{-1} + \dots + a_N z^{-N}} = h_0 + h_1 z^{-1} + h_2 z^{-2} + \dots$$
(4)

In the time domain, this becomes a convolution

$$\sum_{n=0}^{N} a_n \ h_{i-n} = b_i \tag{5}$$

where  $a_0 = 1$  and  $i = 0, 1, \dots, M$  which can be expressed in matrix form by

$$\begin{bmatrix} b_0 \\ b_1 \\ \vdots \\ b_M \\ 0 \\ \vdots \\ 0 \end{bmatrix} = \begin{bmatrix} h_0 & 0 & 0 & \cdots & 0 \\ h_1 & h_0 & 0 & & \\ h_2 & h_1 & h_0 & & \\ \vdots & & & \vdots \\ h_L & \cdots & & h_0 \end{bmatrix} \begin{bmatrix} 1 \\ a_1 \\ \vdots \\ a_N \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

$$(6)$$

Note that the  $h_n$  in (6) are the infinitely enduring causal impulse response values of the IIR filter, not the aliased version of it used in [17] and (31). A more compact matrix notation is

$$\begin{bmatrix} \mathbf{b} \\ \mathbf{0} \end{bmatrix} = \begin{bmatrix} \mathbf{H} \end{bmatrix} \begin{bmatrix} \mathbf{a} \\ \mathbf{0} \end{bmatrix} \tag{7}$$

where **H** is (L+1) by (L+1), **b** is length-(M+1), and **a** is length-(N+1). Because the lower L-N terms of the right-hand vector of (6) are zero, the **H** matrix can be reduced by deleting the right-most L-N columns to give  $\mathbf{H}_0$  which causes (7) to become

$$\begin{bmatrix} \mathbf{b} \\ \mathbf{0} \end{bmatrix} = \begin{bmatrix} \mathbf{H}_0 \end{bmatrix} \begin{bmatrix} \mathbf{a} \end{bmatrix} \tag{8}$$

Because the first element of  $\mathbf{a}$  is unity, it is partitioned to remove the unity term and the remaining length-N vector is denoted  $\mathbf{a}^*$ . The simultaneous equations represented by (8) are uncoupled by further partitioning of the  $\mathbf{H}$  matrix as shown in

$$\begin{bmatrix} \mathbf{b} \\ \mathbf{0} \end{bmatrix} = \begin{bmatrix} \mathbf{H}_1 \\ \mathbf{h}_1 & \mathbf{H}_2 \end{bmatrix} \begin{bmatrix} 1 \\ \mathbf{a}^* \end{bmatrix} \tag{9}$$

where  $\mathbf{H_1}$  is (M+1) by (N+1),  $\mathbf{h_1}$  is length-(L-M), and  $\mathbf{H_2}$  is (L-M) by N. The lower (L-M) equations are written

$$\mathbf{0} = \mathbf{h}_1 + \mathbf{H}_2 \mathbf{a}^* \tag{10}$$

or

$$\mathbf{h}_1 = -\mathbf{H}_2 \mathbf{a}^* \tag{11}$$

which must be solved for  $\mathbf{a}^*$ . The upper M+1 equations of (9) are written

$$\mathbf{b} = \mathbf{H}_1 \mathbf{a} \tag{12}$$

which allows the calculation of  $\mathbf{b}$ .

If L = N + M,  $\mathbf{H}_2$  is square. If  $\mathbf{H}_2$  is nonsingular, (11) can be solved exactly for the denominator coefficients in  $\mathbf{a}^*$ , which are augmented by the unity term to give  $\mathbf{a}$ . From (12), the numerator coefficients in  $\mathbf{b}$  are found. If  $\mathbf{H}_2$  is singular [51] and there are multiple solutions, a lower order problem can be posed. If there are no solutions, the methods of the next section can be used and/or the assumed order increased.

Note that any trade-off in the order of numerator and denominator can be prescribed. If the filter is in fact an FIR filter, **a** is unity and  $\mathbf{a}^*$  does not exist. Under these conditions, (12) states that  $b_n = h_n$ , which is one of the cases of FIR frequency sampling covered in Section 3.1 of [5]. Also note that there is no control over the stability of the filter designed by this method.

The IIR digital filter implemented using the denominator coefficients found in (11) and the numerator coefficients found in (12) will have an impulse response h(n) that exactly matches the desired values given in h(n). The designed filter will have an impulse response that interpolates the first L+1 terms of the given h(n) but says nothing about the response after that. This is true if the total number of unknown filter coefficients, a(k) and b(k) are equal to the number of given values h(n) of the desired impulse response.

## 3 An Approximate Solution: the Least Equation Error Problem in the Time Domain

In order to obtain better practical filter designs or parameter identification, the interpolation scheme of the previous section is extended to give an approximation design method [5]. It should be noted at the outset that the method developed in this section minimizes an equation-error measure and not the usual solution or signal error measure.

The number of samples specified, L+1, will be made larger than the number of filter coefficients, M+N+1. This means that  $\mathbf{H}_2$  is rectangular and, therefore, (7) cannot in general be satisfied. To formulate an approximation problem, a length-(L+1) error vector  $\varepsilon$  is introduced in (7) and (8) to give

$$\begin{bmatrix} \mathbf{b} \\ \mathbf{0} \end{bmatrix} = \begin{bmatrix} \mathbf{H}_0 \end{bmatrix} \begin{bmatrix} \mathbf{a} \end{bmatrix} + [\varepsilon] \tag{13}$$

Equation (11) becomes

$$\mathbf{h}_1 - \varepsilon = -\mathbf{H}_2 \mathbf{a}^* \tag{14}$$

where now  $\mathbf{H}_2$  is rectangular with (L-M) > N. Using the same methods as used to derive (11), the error  $\varepsilon$  is minimized in a least-squared error sense by the solution of the normal equations [51, 52] which occurs when the error is orthogonal to the impulse response.

$$\mathbf{H}_2^T \mathbf{h}_1 = -\mathbf{H}_2^T \mathbf{H}_2 \mathbf{a}^* \tag{15}$$

Alternatively, this equation can also be derived by taking the gradient of the squared error and setting it equal to zero (necessary condition for a minimum).

If the equations are not singular, the optimal solution is

$$\mathbf{a}^* = -[\mathbf{H}_2^T \mathbf{H}_2]^{-1} \mathbf{H}_2^T \mathbf{h}_1. \tag{16}$$

which uses the so-called pseudo-inverse [51, 53]. If the normal equations are singular, the pseudo-inverse can be used to obtain a minimum norm or reduced order solution.

The numerator coefficients are found by the same techniques as before in (12)

$$\mathbf{b} = \mathbf{H}_1 \mathbf{a} \tag{17}$$

which results in the upper M+1 terms in  $\varepsilon$  being zero and the total squared equation error  $(L_2 \text{ norm})$  being minimum.

As is true for least-squared-error optimal design of FIR filters, (15) is can be numerically ill-conditioned and (16) should not be used directly to solve for  $\mathbf{a}^*$ . Special algorithms such as those used by Matlab and LINPACK [54] should be employed.

# 4 General Optimal Design of Zeros of Transfer Function

Given the pole locations designed by minimizing the equation error from (12) or from any other source, the zero locations (numerator coefficients) can be separately designed to interpolate certain points, to further minimize the equation error, or to minimize the the solution error.

Reformulate (15) to have the form

$$\begin{bmatrix} b_0 \\ b_1 \\ \vdots \\ b_M \\ 0 \\ \vdots \\ 0 \end{bmatrix} + [e] = \begin{bmatrix} a_0 & 0 & 0 & \cdots & 0 \\ a_1 & a_0 & 0 & & \\ a_2 & a_1 & a_0 & & \\ \vdots & & & & \vdots \\ a_{N-1} & & & & \\ 0 & & \cdots & & a_0 \end{bmatrix} \begin{bmatrix} h_0 \\ h_1 \\ \vdots \\ h_{K-1} \\ 0 \\ \vdots \\ 0 \end{bmatrix} = Ah$$
 (18)

where A is a K by K lower triangular matrix which is therefore nonsingular. The measure of the solution error is the difference between the actual and desired impulse response:

$$e(n) = h_d(n) - h(n). (19)$$

This can be written as a K by 1 vector and becomes

$$e = h_d - h. (20)$$

giving

$$\begin{bmatrix} \mathbf{b} \\ \mathbf{0} \end{bmatrix} = \begin{bmatrix} \mathbf{A} \end{bmatrix} \begin{bmatrix} \mathbf{h_d} \end{bmatrix} + [\varepsilon] \tag{21}$$

In terms of the two measures of error, we have

$$\begin{bmatrix} \mathbf{b} \\ \mathbf{0} \end{bmatrix} + [\varepsilon] = \begin{bmatrix} \mathbf{A} \end{bmatrix} \begin{bmatrix} \mathbf{h_d} \end{bmatrix} and \begin{bmatrix} \mathbf{b} \\ \mathbf{0} \end{bmatrix} = \begin{bmatrix} \mathbf{A} \end{bmatrix} \begin{bmatrix} \mathbf{h_d} + \mathbf{e} \end{bmatrix}$$
 (22)

If a is found to minimize the equation error according to (6) or (7), then a consistent b can be found to minimize the equation error with (8). However, one might want to mix the criteria and find the b that minimizes the solution error  $||e||_2$ . From (22) we have

$$h_d + e = A^{-1} \begin{bmatrix} \mathbf{b} \\ \mathbf{0} \end{bmatrix} = D \begin{bmatrix} \mathbf{b} \\ \mathbf{0} \end{bmatrix}$$
 (23)

where  $D = A^{-1}$ . Partitioning D gives

$$H_d + e = [\mathbf{D}_1 \mathbf{D}_2] \begin{bmatrix} \mathbf{b} \\ \mathbf{0} \end{bmatrix} = D_1 \mathbf{b}$$
 (24)

with  $D_1$  being KxM.

The numerator coefficients that minimize the solution error norm is the solution to the following normal equations

$$D_1^T D_1 b = D_1^T h_d. (25)$$

This formulation can be generalized [4, 5] to allow equality constraints and a variety of other specifications.

# 5 Prony's Method in the Frequency Domain gives the Frequency-Sampling Design of IIR Filters

In this section a frequency-sampling design method is developed such that the frequency response of the IIR filter will interpolate or pass through the given samples of a desired response. This development is parallel to that used for Prony's method in the time domain. Since a causal IIR filter cannot have linear phase, the sampled response must contain both magnitude and phase. The extension of the frequency-sampling method to a LS-error approximation can be done as for the FIR filter [5]. The method presented in this section

uses a criterion based on the equation error rather than the more common error between the actual and desired frequency response. Nevertheless, it is a useful noniterative design method. Finally, a general discussion of iterative design methods for least-squares frequency response error will be given.

The method for calculating samples of the frequency response of an IIR filter can be reversed to design a filter much the same way it was for the FIR filter using frequency sampling [5]. The z-transform transfer function for an IIR filter is given by

$$H(z) = \frac{B(z)}{A(z)} = \frac{b_0 + b_1 z^{-1} + \dots + b_M z^{-M}}{1 + a_1 z^{-1} + \dots + a_N z^{-N}}.$$
 (26)

The frequency response of the filter is given by setting  $z = e^{-j\omega}$ . Using the notation

$$H(\omega) = H(z)|_{z=e^{-j\omega}}. (27)$$

Equally-spaced samples of the frequency response are chosen so that the number of samples is equal to the number of unknown coefficients in (26). These L + 1 = M + N + 1 samples of this frequency response are given by

$$H_k = H(\omega_k) = H(\frac{2\pi k}{L+1}) \tag{28}$$

and can be calculated from the length-(L+1) DFTs of the numerator and denominator (padded with zeros to the proper length).

$$H_k = \frac{\mathcal{DFT}\{b_n\}}{\mathcal{DFT}\{a_n\}} = \frac{B_k}{A_k}$$
 (29)

where the indicated division is term-by-term division for each value of k. Multiplication of both sides of (29) by  $A_k$  gives

$$B_k = H_k A_k \tag{30}$$

If the length-(L+1) inverse DFT of  $H_k$  is denoted by the length-(L+1) sequence  $h_n$ , equation (30) becomes cyclic convolution which can be expressed in matrix form by

$$\begin{bmatrix} b_0 \\ b_1 \\ \vdots \\ b_M \\ 0 \\ \vdots \\ 0 \end{bmatrix} = \begin{bmatrix} h_0 & h_L & h_{L-1} & \cdots & h_1 \\ h_1 & h_0 & h_L & & & \\ h_2 & h_1 & h_0 & & & \\ \vdots & & & & \vdots \\ h_L & & \cdots & & h_0 \end{bmatrix} \begin{bmatrix} 1 \\ a_1 \\ \vdots \\ a_N \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$
(31)

Note that the  $h_n$  in (31) are not the impulse response values of the filter (they are an aliased version of it) as used in the FIR case or in (6). Using the same approach as used for Prony's method in the time domain, a more compact matrix notation is

$$\begin{bmatrix} \mathbf{b} \\ \mathbf{0} \end{bmatrix} = \begin{bmatrix} \mathbf{H} \end{bmatrix} \begin{bmatrix} \mathbf{a} \\ \mathbf{0} \end{bmatrix} \tag{32}$$

where **H** is (L+1) by (L+1), **b** is length-(M+1), and **a** is length-(N+1). Because the lower L-N terms of the right-hand vector of (31) are zero, the **H** matrix can be reduced by deleting the right-most L-N columns to give  $\mathbf{H}_0$  which causes (32) to become

$$\begin{bmatrix} \mathbf{b} \\ \mathbf{0} \end{bmatrix} = \begin{bmatrix} \mathbf{H}_0 \end{bmatrix} \begin{bmatrix} \mathbf{a} \end{bmatrix} \tag{33}$$

Because the first element of  $\mathbf{a}$  is unity, it is partitioned to remove the unity term and the remaining length-N vector is denoted  $\mathbf{a}^*$ . The simultaneous equations represented by (33) are uncoupled by further partitioning of the  $\mathbf{H}$  matrix as shown in

$$\begin{bmatrix} \mathbf{b} \\ \mathbf{0} \end{bmatrix} = \begin{bmatrix} \mathbf{H}_1 \\ \mathbf{h}_1 & \mathbf{H}_2 \end{bmatrix} \begin{bmatrix} 1 \\ \mathbf{a}^* \end{bmatrix}$$
 (34)

where  $\mathbf{H_1}$  is (M+1) by (N+1),  $\mathbf{h_1}$  is length-(L-M), and  $\mathbf{H_2}$  is (L-M) by N. The lower (L-M) equations are written

$$\mathbf{0} = \mathbf{h}_1 + \mathbf{H}_2 \mathbf{a}^* \tag{35}$$

or

$$\mathbf{h}_1 = -\mathbf{H}_2 \mathbf{a}^* \tag{36}$$

which must be solved for  $\mathbf{a}^*$ . The upper M+1 equations of (10) are written

$$\mathbf{b} = \mathbf{H}_1 \mathbf{a} \tag{37}$$

which allows the calculation of **b**.

If L = N + M,  $\mathbf{H}_2$  is square. If  $\mathbf{H}_2$  is nonsingular, (36) can be solved exactly for the denominator coefficients in  $\mathbf{a}^*$ , which are augmented by the unity term to give  $\mathbf{a}$ . From (37), the numerator coefficients in  $\mathbf{b}$  are found. If  $\mathbf{H}_2$  is singular [51, 53] and there are multiple solutions, a lower order problem can be posed. If there are no solutions, the approximation methods must be used and/or the assumed order increased.

Note that any order numerator and denominator can be prescribed. If the filter is in fact an FIR filter, **a** is unity and  $\mathbf{a}^*$  does not exist. Under these conditions, (37) states that  $b_n = h_n$ , which is one of the cases of FIR frequency sampling covered [5]. Also note that when there is a non-trivial denominator, there is no control over the stability of the filter designed by this method.

This approach uses of the DFT therefore does not allow the possibility of unequally spaced frequency samples as was possible for FIR filter design.

The frequency-sampling design of IIR filters is somewhat more complicated than for FIR filters because of the requirement that  $\mathbf{H}_2$  be nonsingular. As for the FIR filter, the samples of the desired frequency response must satisfy the conditions to insure that  $h_n$  are real. The power of this method is its ability to interpolate arbitrary magnitude and phase

specification. In contrast to most direct IIR design methods, this method does not require any iterative optimization with the accompanying convergence problems.

As with the FIR version, because this design approach is an interpolation method rather than an approximation method, the results may be poor between the interpolation points. This usually happens when the desired frequency-response samples are not compatable with what an IIR filter can achieve. One solution to this problem is the same as for the FIR case [5], the use of more frequency samples than the number of filter coefficients and the definition of an approximation error function that can be minimized. Another solution is choose another desired frequency that is closer to what can be achieved. There is also no simple restriction that will guarantee stable filters. If the frequency-response samples are consistent with an unstable filter, that is what will be designed.

A complication for the this approach to the design of IIR filters is the need to specify both the magnitude and phase of the desired frequency response. This is not a problem for the time-domain design of IIR filters or for the frequency-domain design of linear phase FIR filters, but for the frequency domain design of IIR filters, samples of the desired frequency response are complex numbers which means both the magnitude and the phase must be specified.

### 6 Discrete Least-Squared Equation-Error IIR Filter Design in the Frequency Domain

Using the same approach as used for the time-domain, the interpolation scheme is extended to give an approximation design method in the frequency domain [5, 55]. It should again be noted at the outset that the method developed in this section minimizes an equation-error measure and not the usual frequency-response error measure.

The number of frequency samples specified, L+1, is made larger than the number of filter coefficients, M+N+1. This means that  $\mathbf{H}_2$  is rectangular and, therefore, (36) cannot in general be satisfied. To formulate an approximation problem, a length-(L+1) error vector  $\varepsilon$  is introduced in (33) and (34) to give

$$\begin{bmatrix} \mathbf{b} \\ \mathbf{0} \end{bmatrix} = \begin{bmatrix} \mathbf{H}_0 \end{bmatrix} \begin{bmatrix} \mathbf{a} \end{bmatrix} + [\varepsilon] \tag{38}$$

Equation (36) becomes

$$\mathbf{h}_1 - \varepsilon = -\mathbf{H}_2 \mathbf{a}^* \tag{39}$$

where now  $\mathbf{H}_2$  is rectangular with (L-M) > N. Using the same methods as used to derive (36), the error  $\varepsilon$  is minimized in a least-squared error sense by the solution of the normal equations [51]

$$\mathbf{H}_2^T \mathbf{h}_1 = -\mathbf{H}_2^T \mathbf{H}_2 \mathbf{a}^* \tag{40}$$

If the equations are not singular, the solution is

$$\mathbf{a}^* = -[\mathbf{H}_2^T \mathbf{H}_2]^{-1} \mathbf{H}_2^T \mathbf{h}_1. \tag{41}$$

If the normal equations are singular, the pseudo-inverse [51, 53] can be used to obtain a minimum norm or reduced order solution.

The numerator coefficients are found by the same techniques as before in (37)

$$\mathbf{b} = \mathbf{H}_1 \mathbf{a} \tag{42}$$

which results in the upper M+1 terms in  $\varepsilon$  being zero and the total squared equation error being minimum.

As is true for the least squared error design of FIR filters, (40) is often numerically ill-conditioned and (41) should not be used to solve for  $\mathbf{a}^*$ . Special algorithms such as those used by Matlab and LINPACK [56, 54] should be employed.

The error  $\varepsilon$  defined in (38) can better be understood by considering the frequency-domain formulation. Taking the DFT of (38) gives

$$B_k = H_k A_k + \varepsilon \tag{43}$$

where  $\varepsilon$  is the error in trying to satisfy (33) when the equations are over-specified. This can be reformulated in terms of  $\mathcal{E}$ , the difference between the frequency response samples of the designed filter and the desired response samples, by dividing (33) by  $A_k$  to give

$$\mathcal{E}_k = \frac{B_k}{A_k} - H_k = \frac{\varepsilon_k}{A_k} \tag{44}$$

where  $\mathcal{E}$  is the error in the solution of the approximation problem, and  $\varepsilon$  is the error in the equations defining the problem. The usual statement of a frequency-domain approximation problem is in terms of minimizing some measure of  $\mathcal{E}$ , but that results in solving nonlinear equations. The design procedure developed in this section minimizes the squared error  $\varepsilon$ , thus only requiring the solution of linear equations. There is an important relation between these problems. Equation (44) shows that minimizing  $\varepsilon$  is the same as minimizing  $\mathcal{E}$  weighted by A. However, A is unknown until after the problem is solved.

Although this is posed as a frequency-domain design method, the method of solution for both the interpolation problem and the LS equation-error problem is the same as the time-domain Prony's method, discussed in Section 7.5 of reference [5].

## 7 General Optimal Design of Zeros of Transfer Function

Given the pole locations designed by minimizing the equation error, the zero locations can be separately designed to interpolate certain points, to further minimize the equation error, or to minimize the the solution error. This is similar to what was said earlier for the time-domain formulation.

### 8 Coupled and Uncoupled Formulation of IIR Filter Design

The formulation of both the time-domain and frequency-domain design problems in (16) and (36) uncouple the calculation of the denominator coefficients (poles) from the calculation of the numerator coefficients (zeros) but it assumes an equal spacing of the frequency domain samples (the DFT us used) and the psuedo inverse allows no error weighting. Soewito [6] uses a different formulation which allows arbitrary location of the frequency samples and weights in the minimization of the squared equation error, but it couples the calculation of the numerator and denominator coefficients.

#### 9 Comments

Numerous modifications and extensions can be made to this method. If the desired frequency response is close to what can be achieved by an IIR filter, this method will give a design approximately the same as that of a true least-squared solution-error method. It can be shown that  $\varepsilon = 0 \leftrightarrow \mathcal{E} = 0$ . In some cases, improved results can be obtained by estimating  $A_k$  and using that as a weight on  $\varepsilon$  to approximate minimizing  $\mathcal{E}$ . There are iterative methods based on solving (41) and (42) to obtain values for  $A_k$ . These values are used as weights on  $\varepsilon$  to solve for a new set of  $A_k$  used as a new set of weights to solve again for  $A_k$  [5]. The solution of (41) and (42) is sometimes used to obtain starting values for other iterative optimization algorithms that need good starting values for convergence.

To illustrate this design method a sixth-order lowpass filter was designed with 41 frequency samples to approximate. The magnitude of those less than 0.2 Hz is one and of those greater than 0.2 is zero. The phase was experimentally adjusted to result in a good magnitude response. The design was performed with Program 9 in the appendix of [5] and the frequency response is shown in Figure 7-33 of [5].

In this section an LS-error approximation method was posed to design IIR filters. By using an equation-error rather than a solution-error criterion, a problem resulted that required only the solution of simultaneous linear equations.

Like the FIR filter version, the IIR frequency sampling design method and the LS equation-error extension can be used for complex approximation and, therefore, can design with both magnitude and phase specifications.

As noted earlier, if the desired frequency-response samples are close to what an IIR filter of the specified order can achieve, this method will produce a filter very close to what a true least-squared error method would. However, when the specifications are not consistent with what can be achieved and the approximating error is large, the results can be very poor and in some cases, unstable. It is often difficult to set realistic phase response specifications. With this method, it is even more important to have a design environment that will allow easy trial-and-error procedure.

Other works on this problem are [57, 58, 59, 13, 15]. Other references can be found in [5, 57]. The Matlab command invfreqz() which is an inverse to the freqz() command gives a similar or, perhaps, the same result as the method described in this note but uses a

different formulation [14, 16]. [60, 61, 62] A particularly interesting new Matlab program is iirlpnorm.m which allows the design of IIR filters with different degree numerator and denominators. It also allows use of an  $L_p$  error minimization. Unfortunately, it requires both the zeros and poles to be inside the unit circle. The poles of the transfer function should be inside the unit circle to unsure stability but allowing some zeros to be outside the unit circle (as in the linear phase FIR filter) can improve the phase response. [63] Soewito [6] and Jackson [64] formulate an iterative algorithm where the magnitue and phase are separately updated to give an optimal magnitue approximation.

### 10 Alternative Frequency Domain Formulation

The use of the DFT to reformulate the time domain statement of Prony's method into the frequency domain uncouples the calculation of the  $a_k$  and  $b_k$  but requires equally spaced samples of the desired frequency response. An alternative formulation used by Soewito [6] requires the calculation of the  $a_k$  and  $b_k$  together but allows somewhat arbitrary frequency sample spacing.

### 11 Iterative Algorithms using Prony's Methods

[50, 65, 66, 67, 7, 6]

### 12 Other Optimal IIR Filter Design Methods

Martinez/Parks design [68], Jackson's improvement [69], others [70, 71, 72, 73, 74, 75, 76, 57, 58, 77] [78, 79, 80, 81, 82, 83, 84, 85, 86, 6]

### 13 Summary

This note has developed a time-domain and a frequency-domain method to design an IIR digital filter that interpolates desired samples or that gives an optimal, least squared equation error approximation. These methods are directly related to Prony, Padé, and linear prediction. In addition, they can be used to obtain good starting values for iterative algorithms or iterated themselves to obtain optimal approximations with other criteria [14, 11, 10, 5, 87].

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