

# Neutrino Physics Note

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## 1 Quantum Kinetic Equations for Neutrino

### 1.1 Wigner phase-space density operator

The momentum expansion of Dirac fermionic fields (promote to operators):

$$\begin{aligned}\psi(x) &= \sum_s \int \frac{d^3p}{(2\pi)^3} a_{\mathbf{p}}^s u^s(p) e^{-ip \cdot x} + b_{\mathbf{p}}^{s\dagger} v^s(p) e^{ip \cdot x} \\ \psi^\dagger(x) &= \sum_s \int \frac{d^3p}{(2\pi)^3} a_{\mathbf{p}}^{s\dagger} u^{s\dagger}(p) e^{ip \cdot x} + b_{\mathbf{p}}^s v^{s\dagger}(p) e^{-ip \cdot x}.\end{aligned}$$

They obey the Dirac equation:  $(i\gamma^\mu \partial_\mu - m)\psi(x) = 0$

A Fourier transformation connects momentum and position space representation of the operators:

$$\begin{aligned}a_{\mathbf{p}}^s &= \int d^3x e^{ip \cdot x} u^s(p) \psi(x) & a_{\mathbf{p}}^{s\dagger} &= \int d^3x e^{-ip \cdot x} \psi^\dagger(x) u^{s\dagger}(p) \\ b_{\mathbf{p}}^s &= \int d^3x e^{ip \cdot x} \psi^\dagger(x) v^s(p) & b_{\mathbf{p}}^{s\dagger} &= \int d^3x e^{-ip \cdot x} v^{s\dagger}(p) \psi(x)\end{aligned}$$

using the normalization:  $u^{s\dagger}(p) u^{s'}(p) = v^{s\dagger}(p) v^{s'}(p) = \delta_{ss'}$ .

\*Note: The normalization can be up to some factors like  $2E_{\mathbf{p}}$  or  $\frac{E_{\mathbf{p}}}{m}$ . It all depends on which factor is more convenient for establishing the theory.

According to the spin-statistics theorem, at equal time the Pauli exclusion principle is implemented by anti-commutation relations of the field operators:

$$[\psi(\mathbf{x}, t), \psi(\mathbf{x}', t)] = \delta(\mathbf{x} - \mathbf{x}').$$

The creation and annihilation operators then automatically satisfy the relation:

$$\{a_{\mathbf{p}}^s, a_{\mathbf{p}'}^{s'\dagger}\} = \{b_{\mathbf{p}}^s, b_{\mathbf{p}'}^{s'\dagger}\} = (2\pi)^3 \delta_{ss'} \delta^3(\mathbf{p} - \mathbf{p}').$$

**Proof:**

$$\{a_{\mathbf{p}}^s, a_{\mathbf{p}'}^{s'\dagger}\} = \int d^3x d^3x' e^{i(p \cdot x - p' \cdot x')} u^{s\dagger}(p) u^{s'}(p') \{\psi(\mathbf{x}, t), \psi^\dagger(\mathbf{x}', t)\} = \int d^3x e^{i(p-p') \cdot x} u^{s\dagger}(p) u^{s'}(p')$$

Similarly for the antifermion operators.

Now, consider the left-handed massless neutrino field (with spin  $s = \frac{1}{2}$ ):

$$\begin{aligned}\psi(x) &= \int \frac{d^3p}{(2\pi)^3} \left( a_{\mathbf{p}}(t) u_{\mathbf{p}} + b_{-\mathbf{p}}^\dagger(t) v_{-\mathbf{p}} \right) e^{i\mathbf{p} \cdot \mathbf{x}} \\ \psi^\dagger(x) &= \int \frac{d^3p}{(2\pi)^3} \left( a_{\mathbf{p}}^\dagger(t) u_{\mathbf{p}}^\dagger + b_{-\mathbf{p}}(t) v_{-\mathbf{p}}^\dagger \right) e^{-i\mathbf{p} \cdot \mathbf{x}} \\ u_{\mathbf{p}} &: \text{negative-helicity fermionic spinor} \\ v_{\mathbf{p}} &: \text{positive-helicity fermionic spinor}\end{aligned}$$

## 2 Numerical Specification

The e.o.m. for the mixed state of neutrinos described by mean-field density matrix  $\varrho(t, \mathbf{x}, \mathbf{p})$ :

$$\left( \frac{\partial}{\partial t} + \mathbf{v} \cdot \frac{\partial}{\partial \mathbf{x}} + \mathbf{f} \cdot \frac{\partial}{\partial \mathbf{p}} \right) \varrho(t, \mathbf{x}, \mathbf{p}) = i[\varrho(t, \mathbf{x}, \mathbf{p}), \mathcal{H}(t, \mathbf{x}, \mathbf{p})] + C[\varrho].$$

$$\begin{aligned} \varrho(t, \mathbf{x}, \mathbf{p}) &= \frac{f_{\nu_e} + f_{\nu_x}}{2} \mathcal{I} + G_\nu(\mathbf{p}) \rho(t, \mathbf{x}, \hat{\mathbf{p}}) \\ \mathcal{H}_{\nu\nu} &= \sqrt{2} G_F \int \frac{d^3 \mathbf{q}}{(2\pi)^3} (1 - \hat{\mathbf{p}} \cdot \hat{\mathbf{q}}) (\varrho_{\mathbf{q}}(t, \mathbf{x}) - \bar{\varrho}_{\mathbf{q}}(t, \mathbf{x})). \end{aligned}$$

We may integrate out the energy  $|\mathbf{q}| = \varepsilon_\nu$ , and define the angular ELN distribution as:

$$g_\nu(\hat{\mathbf{p}}) = \frac{1}{n_{\nu_e}} \int \frac{\varepsilon_\nu^2 d\varepsilon_\nu}{2\pi^2} (G_\nu(\mathbf{p}) - G_{\bar{\nu}}(\mathbf{p})).$$

So that the interaction Hamiltonian becomes:

$$H_{\nu\nu}(t, \mathbf{x}, \hat{\mathbf{p}}) = \mu \int \frac{d\hat{\mathbf{q}}}{4\pi} (1 - \hat{\mathbf{p}} \cdot \hat{\mathbf{q}}) \left( g_\nu(\hat{\mathbf{q}}) \rho_{\hat{\mathbf{q}}} - g_{\bar{\nu}}(\hat{\mathbf{q}}) \bar{\rho}_{\hat{\mathbf{q}}} \right), \quad \mu = \sqrt{2} G_F n_{\nu_e}.$$

Given the explicit form:

$$\begin{aligned} \hat{\mathbf{p}} = \mathbf{v} &= (\sqrt{1 - v_z^2} \cos \varphi, \sqrt{1 - v_z^2} \sin \varphi, v_z) \\ \int d\hat{\mathbf{q}} &= \int_{-1}^1 dv_z \int_0^{2\pi} d\varphi \\ \mathbf{v} \cdot \nabla &= \sqrt{1 - v_z^2} \left( \cos \varphi \frac{\partial}{\partial x} + \sin \varphi \frac{\partial}{\partial y} \right) + v_z \frac{\partial}{\partial z}. \end{aligned}$$

### 2.1 Time Independent Hamiltonian

$$\begin{aligned} \left( \frac{\partial}{\partial t} + \mathbf{v} \cdot \frac{\partial}{\partial \mathbf{x}} \right) \varrho(t, \mathbf{x}, \mathbf{p}) &= i[\varrho(t, \mathbf{x}, \mathbf{p}), \mathcal{H}(t, \mathbf{x}, \mathbf{p})], \quad \varrho(t=0, \mathbf{x}) = f(\mathbf{x}) \\ \mathcal{H} &= \begin{pmatrix} -\cos 2\theta & \sin 2\theta \\ \sin 2\theta & \cos 2\theta \end{pmatrix} \\ \Rightarrow \varrho(t, \mathbf{x}) &= e^{-i\mathcal{H}t} f(\mathbf{x} - \mathbf{v}t) e^{i\mathcal{H}t}. \end{aligned}$$

Given the explicit form of initial condition, the solution is

$$e^{i\mathcal{H}t} = \begin{pmatrix} e^{it} \sin^2 \theta + e^{-it} \cos^2 \theta & i \sin t \sin 2\theta \\ i \sin t \sin 2\theta & e^{it} \cos^2 \theta + e^{-it} \sin^2 \theta \end{pmatrix}, \quad f(\mathbf{x}) = \begin{pmatrix} f_{ee}(\mathbf{x}) & 0 \\ 0 & f_{xx}(\mathbf{x}) \end{pmatrix}$$

$$\begin{aligned} \varrho_{ee}(t, \mathbf{x}) &= f_{ee}(\mathbf{x} - \mathbf{v}t) \left( \cos^4(\theta) + 2 \cos(2t) \cos^2(\theta) \sin^2(\theta) + \sin^4(\theta) \right) + f_{xx}(\mathbf{x} - \mathbf{v}t) \sin^2 t \sin^2 2\theta \\ \varrho_{ex}(t, \mathbf{x}) &= \varrho_{xe}^* = 2i \left( f_{ee}(\mathbf{x} - \mathbf{v}t) - f_{xx}(\mathbf{x} - \mathbf{v}t) \right) \sin(t) \sin(\theta) \cos(\theta) \left( e^{it} \cos^2(\theta) + e^{-it} \sin^2(\theta) \right) \\ \varrho_{xx}(t, \mathbf{x}) &= f_{xx}(\mathbf{x} - \mathbf{v}t) \left( \cos^4(\theta) + 2 \cos(2t) \cos^2(\theta) \sin^2(\theta) + \sin^4(\theta) \right) + f_{ee}(\mathbf{x} - \mathbf{v}t) \sin^2(t) \sin^2(2\theta). \end{aligned}$$

## 2.2 Numerical Approach in 1+2+2 dimensions

- $\partial_t \varrho = -\mathbf{v} \cdot \nabla \varrho + i[\varrho, H] \equiv g(t, \varrho)$
- $\mathbf{v} \cdot \nabla \varrho \rightarrow \frac{v_x}{12dx} (\varrho_{i-2,j} - 8\varrho_{i-1,j} + 8\varrho_{i+1,j} - \varrho_{i+2,j}) + \frac{v_z}{12dz} (\varrho_{i,j-2} - 8\varrho_{i,j-1} + 8\varrho_{i,j+1} - \varrho_{i,j+2})$
- index :  $(j + g_z) * (N_x + 2 * g_x) + (i + g_x)$   
 $- i = 0 \sim N_x - 1, j = 0 \sim N_y - 1$
- $g_x, g_z$  are number of grids in ghost zones
- RK4 -  $\varrho^{n+1} = \varrho^n + \frac{dt}{6} (g_0 + 2g_1 + 2g_2 + g_3) + O(dt^5)$ 
  - $g_0 = g(t_n, \varrho^n)$
  - $g_1 = g(t_n + \frac{dt}{2}, \varrho^n + \frac{dt}{2} g_0)$
  - $g_2 = g(t_n + \frac{dt}{2}, \varrho^n + \frac{dt}{2} g_1)$
  - $g_3 = g(t_n + dt, \varrho^n + dt g_2)$

## 2.3 Numerical Setup

Consider two-flavor system, and impose the translation symmetry on both  $x$  and  $y$  dimensions, the e.o.m. for neutrino and antineutrino:

$$(\partial_t + v_z \partial_z) \rho(t, z, v_z) = i[\rho(t, z, v_z), \mathcal{H}(t, z, v_z)]$$

$$(\partial_t + v_z \partial_z) \bar{\rho}(t, z, v_z) = i[\bar{\rho}(t, z, v_z), \bar{\mathcal{H}}(t, z, v_z)]$$

$$\text{with } \rho(t, z, v_z) = \begin{pmatrix} \rho_{ee} & \rho_{ex} \\ \rho_{ex}^* & \rho_{xx} \end{pmatrix}, \bar{\rho}(t, z, v_z) = \begin{pmatrix} \bar{\rho}_{ee} & \bar{\rho}_{ex} \\ \bar{\rho}_{ex}^* & \bar{\rho}_{xx} \end{pmatrix}$$

Ignoring the MSW effect, the Hamiltonian would be:

$$\mathcal{H}(t, z; v_z) = \begin{pmatrix} -\cos 2\theta & \sin 2\theta \\ \sin 2\theta & \cos 2\theta \end{pmatrix} + \mu \int_0^1 dv'_z (1 - v_z v'_z) [\rho(t, z; v'_z) - \bar{\rho}^*(t, z; v'_z)] \equiv \mathcal{H}_{vac} + \mathcal{H}_{\nu\nu},$$

$$\bar{\mathcal{H}}(t, z; v_z) = \begin{pmatrix} -\cos 2\theta & \sin 2\theta \\ \sin 2\theta & \cos 2\theta \end{pmatrix} - \mu \int_0^1 dv'_z (1 - v_z v'_z) [\rho^*(t, z; v'_z) - \bar{\rho}(t, z; v'_z)] \equiv \bar{\mathcal{H}}_{vac} + \bar{\mathcal{H}}_{\nu\nu}$$

The finite difference index:

	$t$	$z$	$v_z$
step	$\delta t$	$\delta z$	$\delta v_z$
numbers	$N_t$	$N_z$	$N_{v_z}$
index	$i_1$	$i_2$	$k$

Let  $\rho(t, z, v_z)$  be  $\rho_{i_1, i_2}^k$  in the discretization and the interaction Hamiltonian

$$\mathcal{H}_{\nu\nu, i_1, i_2}^k = \frac{\mu}{N} \sum_{k'=0}^N \left(1 - \frac{kk'}{N^2}\right) (\rho_{i_1, i_2}^{k'} - \bar{\rho}_{i_1, i_2}^{*k'})$$

$$\bar{\mathcal{H}}_{\nu\nu, i_1, i_2}^k = -\frac{\mu}{N} \sum_{k'=0}^N \left(1 - \frac{kk'}{N^2}\right) (\rho_{i_1, i_2}^{*k'} - \bar{\rho}_{i_1, i_2}^{k'})$$

The above Courant-Friedrichs-Lewy (CFL) stability criterion:

$$c \equiv \frac{|v_z| \delta t}{\delta z} \leq 1.$$