Neutrino Physics Note

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1 Quantum Kinetic Equations for Neutrino

1.1 Wigner phase-space density operator

The momentum expansion of Dirac fermionic fields (promote to operators):

$$\psi(x) = \sum_s \int \frac{d^3p}{(2\pi)^3}\, a^s_{\mathbf{p}} u^s(p) e^{-ip\cdot x} + b^{s\dagger}_{\mathbf{p}} v^s(p) e^{ip\cdot x}.$$

$$\psi^\dagger(x) = \sum_s \int \frac{d^3p}{(2\pi)^3}\, a^{s\dagger}_{\mathbf{p}} u^{s\dagger}(p) e^{ip\cdot x} + b^{s}_{\mathbf{p}} v^{s\dagger}(p) e^{-ip\cdot x}.$$

They obey the Dirac equation: $(i\gamma^\mu \partial_\mu - m)\psi(x) = 0$

A Fourier transformation connects momentum and position space representation of the operators:

$$a^s_{\mathbf{p}} = \int d^3x\, e^{ip\cdot x} u^{s\dagger}(p)\psi(x) \qquad a^{s\dagger}_{\mathbf{p}} = \int d^3x\, e^{-ip\cdot x} \psi^\dagger(x) u^s(p)$$
$$b^s_{\mathbf{p}} = \int d^3x\, e^{ip\cdot x} \psi^\dagger(x) v^s(p) \qquad b^{s\dagger}_{\mathbf{p}} = \int d^3x\, e^{-ip\cdot x} v^{s\dagger}(p)\psi(x)$$

using the normalization: $u^{s\dagger}(p)u^{s'}(p) = v^{s\dagger}(p)v^{s'}(p) = \delta_{ss'}$.

*Note: The normalization can be up to some factors like $2E_{\mathbf{p}}$ or $\frac{E_{\mathbf{p}}}{m}$. It all depends on which factor is more convenient for establishing the theory.

According to the spin-statistics theorem, at equal time the Pauli exclusion principle is implemented by anti-commutation relations of the field operators:

$$\big[\psi(\mathbf{x}, t), \psi(\mathbf{x}', t)\big] = \delta(\mathbf{x} - \mathbf{x}').$$

The creation and annihilation operators then automatically satisfy the relation:

$$\left\{a^s_{\mathbf{p}}, a^{s'\dagger}_{\mathbf{p}'}\right\} = \left\{b^s_{\mathbf{p}}, b^{s'\dagger}_{\mathbf{p}'}\right\} = (2\pi)^3 \delta_{ss'} \delta^3(\mathbf{p} - \mathbf{p}').$$

Proof:

$$\left\{a^s_{\mathbf{p}}, a^{s'\dagger}_{\mathbf{p}'}\right\} = \int d^3x d^3x'\, e^{i(p\cdot x - p'\cdot x')} u^{s\dagger}(p) u^{s'}(p') \left\{\psi(\mathbf{x}, t), \psi^\dagger(\mathbf{x}', t)\right\} = \int d^3x e^{i(p-p')\cdot x} u^{s\dagger}(p) u^{s'}(p')$$

Similarly for the antifermion operators.

Now, consider the left-handed massless neutrino field (with spin $s = \frac{1}{2}$):

$$\psi(x) = \int \frac{d^3p}{(2\pi)^3} \Big(a_{\mathbf{p}}(t) u_{\mathbf{p}} + b^\dagger_{-\mathbf{p}}(t) v_{-\mathbf{p}}\Big) e^{i\mathbf{p}\cdot\mathbf{x}}$$
$$\psi^\dagger(x) = \int \frac{d^3p}{(2\pi)^3} \Big(a^\dagger_{\mathbf{p}}(t) u^\dagger_{\mathbf{p}} + b_{-\mathbf{p}}(t) v^\dagger_{-\mathbf{p}}\Big) e^{-i\mathbf{p}\cdot\mathbf{x}}$$

$u_{\mathbf{p}}$: negative-helicity fermionic spinor

$v_{\mathbf{p}}$: positive-helicity fermionic spinor

2 Numerical Specification

The e.o.m. for the mixed state of neutrinos described by mean-field density matrix $\varrho(t, \mathbf{x}, \mathbf{p})$:

$$\left(\frac{\partial}{\partial t} + \mathbf{v} \cdot \frac{\partial}{\partial \mathbf{x}} + \mathbf{f} \cdot \frac{\partial}{\partial \mathbf{p}}\right) \varrho(t, \mathbf{x}, \mathbf{p}) = i[\varrho(t, \mathbf{x}, \mathbf{p}), \mathcal{H}(t, \mathbf{x}, \mathbf{p})] + C[\varrho].$$

$$\begin{aligned}
\varrho(t, \mathbf{x}, \mathbf{p}) &= \frac{f_{\nu_e} + f_{\nu_x}}{2} \mathcal{I} + G_\nu(\mathbf{p}) \rho(t, \mathbf{x}, \hat{\mathbf{p}}) \\
\mathcal{H}_{\nu\nu} &= \sqrt{2} G_F \int \frac{d^3\mathbf{q}}{(2\pi)^3} (1 - \hat{\mathbf{p}} \cdot \hat{\mathbf{q}}) \big(\varrho_{\mathbf{q}}(t, \mathbf{x}) - \bar{\varrho}_{\mathbf{q}}(t, \mathbf{x})\big).
\end{aligned}$$

We may integrate out the energy $|\mathbf{q}| = \varepsilon_\nu$, and define the angular ELN distribution as:

$$g_\nu(\hat{\mathbf{p}}) = \frac{1}{n_{\nu_e}} \int \frac{\varepsilon_\nu^2 d\varepsilon_\nu}{2\pi^2} \big(G_\nu(\mathbf{p}) - G_{\bar{\nu}}(\mathbf{p})\big).$$

So that the interaction Hamiltonian becomes:

$$H_{\nu\nu}(t, \mathbf{x}, \hat{\mathbf{p}}) = \mu \int \frac{d\hat{\mathbf{q}}}{4\pi} (1 - \hat{\mathbf{p}} \cdot \hat{\mathbf{q}}) \Big(g_{\nu)}(\hat{\mathbf{q}}) \rho_{\hat{\mathbf{q}}} - g_{\bar{\nu}}(\hat{\mathbf{q}}) \bar{\rho}_{\hat{\mathbf{q}}}\Big), \ \mu = \sqrt{2} G_F n_{\nu_e}.$$

Given the explicit form:

$$\begin{aligned}
\hat{\mathbf{p}} = \mathbf{v} &= (\sqrt{1 - v_z^2} \cos\varphi, \sqrt{1 - v_z^2} \sin\varphi, v_z) \\
\int d\hat{\mathbf{q}} &= \int_{-1}^{1} dv_z \int_0^{2\pi} d\varphi \\
\mathbf{v} \cdot \nabla &= \sqrt{1 - v_z^2} \Big(\cos\varphi \frac{\partial}{\partial x} + \sin\varphi \frac{\partial}{\partial y} \Big) + v_z \frac{\partial}{\partial z}.
\end{aligned}$$

2.1 Time Independent Hamiltonain

$$\begin{gathered}
\left(\frac{\partial}{\partial t} + \mathbf{v} \cdot \frac{\partial}{\partial \mathbf{x}}\right) \varrho(t, \mathbf{x}, \mathbf{p}) = i[\varrho(t, \mathbf{x}, \mathbf{p}), \mathcal{H}(t, \mathbf{x}, \mathbf{p})], \ \varrho(t = 0, \mathbf{x}) = f(\mathbf{x}) \\
\mathcal{H} = \begin{pmatrix} -\cos 2\theta & \sin 2\theta \\ \sin 2\theta & \cos 2\theta \end{pmatrix} \\
\Rightarrow \varrho(t, \mathbf{x}) = e^{-i\mathcal{H}t} f(\mathbf{x} - \mathbf{v}t) e^{i\mathcal{H}t}.
\end{gathered}$$

Given the explicit form of initial condition, the solution is

$$e^{i\mathcal{H}t} = \begin{pmatrix} e^{it} \sin^2\theta + e^{-it} \cos^2\theta & i \sin t \sin 2\theta \\ i \sin t \sin 2\theta & e^{it} \cos^2\theta + e^{-it} \sin^2\theta \end{pmatrix}, f(\mathbf{x}) = \begin{pmatrix} f_{ee}(\mathbf{x}) & 0 \\ 0 & f_{xx}(\mathbf{x}) \end{pmatrix}$$

$$\begin{aligned}
\varrho_{ee}(t, \mathbf{x}) &= f_{ee}(\mathbf{x} - \mathbf{v}t) \Big(\cos^4(\theta) + 2\cos(2t) \cos^2(\theta) \sin^2(\theta) + \sin^4(\theta) \Big) + f_{xx}(\mathbf{x} - \mathbf{v}t) \sin^2 t \sin^2 2\theta \\
\varrho_{ex}(t, \mathbf{x}) &= \varrho_{xe}^* = 2i \Big(f_{ee}(\mathbf{x} - \mathbf{v}t) - f_{xx}(\mathbf{x} - \mathbf{v}t) \Big) \ \sin(t) \sin(\theta) \cos(\theta) \Big(e^{it} \cos^2(\theta) + e^{-it} \sin^2(\theta) \Big) \\
\varrho_{xx}(t, \mathbf{x}) &= f_{xx}(\mathbf{x} - \mathbf{v}t) \Big(\cos^4(\theta) + 2\cos(2t) \cos^2(\theta) \sin^2(\theta) + \sin^4(\theta) \Big) + f_{ee}(\mathbf{x} - \mathbf{v}t) \sin^2(t) \sin^2(2\theta).
\end{aligned}$$

2.2 Numerical Approach in 1+2+2 dimensions

- $\partial_t \varrho = -\mathbf{v} \cdot \nabla \varrho + i[\varrho, H] \equiv g(t, \varrho)$
- $\mathbf{v} \cdot \nabla \varrho \to \frac{v_x}{12dx} \left(\varrho_{i-2,j} - 8\varrho_{i-1,j} + 8\varrho_{i+1,j} - \varrho_{i+2,j}\right) + \frac{v_z}{12dz} \left(\varrho_{i,j-2} - 8\varrho_{i,j-1} + 8\varrho_{i,j+1} - \varrho_{i,j+2}\right)$
- index : $(j + g_z) * (N_x + 2 * g_x) + (i + g_x)$
- g_x, g_z are number of grids in ghost zones
- RK4 – $\varrho^{n+1} = \varrho^n + \frac{dt}{6} \left(g_0 + 2g_1 + 2g_2 + g_3\right) + O(dt^5)$
  - $g_0 = g(t_n, \varrho^n)$
  - $g_1 = g(t_n + \frac{dt}{2}, \varrho^n + \frac{dt}{2} g_0)$
  - $g_2 = g(t_n + \frac{dt}{2}, \varrho^n + \frac{dt}{2} g_1)$
  - $g_3 = g(t_n + dt, \varrho^n + dt g_2)$

2.3 Numerical Setup

Consider two-flavor system, and impose the translation symmetry on both x and y dimensions, the e.o.m. for neutrino and antineutrino:

$$(\partial_t + v_z \partial_z)\, \rho(t, z, v_z) = i[\rho(t, z, v_z), \mathcal{H}(t, z, v_z)]$$

$$(\partial_t + v_z \partial_z)\, \overline{\rho}(t, z, v_z) = i\left[\overline{\rho}(t, z, v_z), \overline{\mathcal{H}}(t, z, v_z)\right]$$

$$\text{with } \rho(t, z, v_z) = \begin{pmatrix} \rho_{ee} & \rho_{ex} \\ \rho^*_{ex} & \rho_{xx} \end{pmatrix}, \ \overline{\rho}(t, z, v_z) = \begin{pmatrix} \bar{\rho}_{ee} & \bar{\rho}_{ex} \\ \bar{\rho}^*_{ex} & \bar{\rho}_{xx} \end{pmatrix}$$

Ignoring the MSW effect, the Hamiltonian would be:

$$\mathcal{H}(t, z; v_z) = \begin{pmatrix} -\cos 2\theta & \sin 2\theta \\ \sin 2\theta & \cos 2\theta \end{pmatrix} + \mu \int_0^1 dv'_z (1 - v_z v'_z)[\rho(t, z; v'_z) - \bar{\rho}^*(t, z; v'_z)] \equiv \mathcal{H}_{vac} + \mathcal{H}_{\nu\nu},$$

$$\overline{\mathcal{H}}(t, z; v_z) = \begin{pmatrix} -\cos 2\theta & \sin 2\theta \\ \sin 2\theta & \cos 2\theta \end{pmatrix} - \mu \int_0^1 dv'_z (1 - v_z v'_z)[\rho^*(t, z; v'_z) - \bar{\rho}(t, z; v'_z)] \equiv \overline{\mathcal{H}}_{vac} + \overline{\mathcal{H}}_{\nu\nu}$$

The finite difference index:

	t	z	v_z
step	δt	δz	δv_z
numbers	N_t	N_z	N_{v_z}
index	i_1	i_2	k

Let $\rho(t, z, v_z)$ be $\rho^k_{i_1,i_2}$ in the discretization and the interaction Hamiltonian

$$\mathcal{H}^k_{\nu\nu,i_1,i_2} = \frac{\mu}{N} \sum_{k'=0}^{N} \left(1 - \frac{kk'}{N^2}\right) \left(\rho^{k'}_{i_1,i_2} - \bar{\rho}^{*k'}_{i_1,i_2}\right).$$

$$\overline{\mathcal{H}}^k_{\nu\nu,i_1,i_2} = -\frac{\mu}{N} \sum_{k'=0}^{N} \left(1 - \frac{kk'}{N^2}\right) \left(\rho^{*k'}_{i_1,i_2} - \bar{\rho}^{k'}_{i_1,i_2}\right).$$

The above Courant-Friedrichs-Lewy (CFL) stability criterion:

$$c \equiv \frac{|v_z|\delta t}{\delta z} \leq 1.$$