Neutrino Physics Note

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1 Quantum Kinetic Equations for Neutrino

1.1 Wigner phase-space density operator

The momentum expansion of Dirac fermionic fields (promote to operators):

$$\psi(x) = \sum_s \int \frac{d^3p}{(2\pi)^3}\, a^s_{\mathbf{p}} u^s(p) e^{-ip\cdot x} + b^{s\dagger}_{\mathbf{p}} v^s(p) e^{ip\cdot x}.$$

$$\psi^\dagger(x) = \sum_s \int \frac{d^3p}{(2\pi)^3}\, a^{s\dagger}_{\mathbf{p}} u^{s\dagger}(p) e^{ip\cdot x} + b^{s}_{\mathbf{p}} v^{s\dagger}(p) e^{-ip\cdot x}.$$

They obey the Dirac equation: $(i\gamma^\mu\partial_\mu - m)\psi(x) = 0$

A Fourier transformation connects momentum and position space representation of the operators:

$$a^s_{\mathbf{p}} = \int d^3x\ e^{ip\cdot x} u^{s\dagger}(p)\psi(x) \qquad a^{s\dagger}_{\mathbf{p}} = \int d^3x\ e^{-ip\cdot x} \psi^\dagger(x) u^s(p)$$
$$b^s_{\mathbf{p}} = \int d^3x\ e^{ip\cdot x} \psi^\dagger(x) v^s(p) \qquad b^{s\dagger}_{\mathbf{p}} = \int d^3x\ e^{-ip\cdot x} v^{s\dagger}(p)\psi(x)$$

using the normalization: $u^{s\dagger}(p)u^{s'}(p) = v^{s\dagger}(p)v^{s'}(p) = \delta_{ss'}$.

*Note: The normalization can be up to some factors like $2E_{\mathbf{p}}$ or $\frac{E_{\mathbf{p}}}{m}$. It all depends on which factor is more convenient for establishing the theory.

According to the spin-statistics theorem, at equal time the Pauli exclusion principle is implemented by anti-commutation relations of the field operators:

$$\big[\psi(\mathbf{x}, t), \psi(\mathbf{x}', t)\big] = \delta(\mathbf{x} - \mathbf{x}').$$

The creation and annihilation operators then automatically satisfy the relation:

$$\left\{a^s_{\mathbf{p}}, a^{s'\dagger}_{\mathbf{p}'}\right\} = \left\{b^s_{\mathbf{p}}, b^{s'\dagger}_{\mathbf{p}'}\right\} = (2\pi)^3 \delta_{ss'}\delta^3(\mathbf{p} - \mathbf{p}').$$

Proof:

$$\left\{a^s_{\mathbf{p}}, a^{s'\dagger}_{\mathbf{p}'}\right\} = \int d^3x d^3x'\ e^{i(p\cdot x - p'\cdot x')} u^{s\dagger}(p) u^{s'}(p') \left\{\psi(\mathbf{x}, t), \psi^\dagger(\mathbf{x}', t)\right\} = \int d^3x e^{i(p-p')\cdot x} u^{s\dagger}(p) u^{s'}(p')$$

Similarly for the antifermion operators.

Now, consider the left-handed massless neutrino field (with spin $s = \frac{1}{2}$):

$$\psi(x) = \int \frac{d^3p}{(2\pi)^3} \Big(a_{\mathbf{p}}(t) u_{\mathbf{p}} + b^\dagger_{-\mathbf{p}}(t) v_{-\mathbf{p}}\Big) e^{i\mathbf{p}\cdot\mathbf{x}}$$
$$\psi^\dagger(x) = \int \frac{d^3p}{(2\pi)^3} \Big(a^\dagger_{\mathbf{p}}(t) u^\dagger_{\mathbf{p}} + b_{-\mathbf{p}}(t) v^\dagger_{-\mathbf{p}}\Big) e^{-i\mathbf{p}\cdot\mathbf{x}}$$

$u_{\mathbf{p}}$: negative-helicity fermionic spinor

$v_{\mathbf{p}}$: positive-helicity fermionic spinor

1.2 Time Independent Hamiltonain

$$\left(\frac{\partial}{\partial t} + \mathbf{v}\cdot\frac{\partial}{\partial \mathbf{x}}\right)\varrho(t,\mathbf{x},\mathbf{p}) = i[\varrho(t,\mathbf{x},\mathbf{p}),\mathcal{H}(t,\mathbf{x},\mathbf{p})],\ \varrho(t=0,\mathbf{x}) = f(\mathbf{x})$$
$$\mathcal{H} = \begin{pmatrix} -\cos 2\theta & \sin 2\theta \\ \sin 2\theta & \cos 2\theta \end{pmatrix}$$
$$\Rightarrow \varrho(t,\mathbf{x}) = e^{-i\mathcal{H}t} f(\mathbf{x}-\mathbf{v}t) e^{i\mathcal{H}t}.$$

Given the explicit form of initial condition, the solution is

$$e^{i\mathcal{H}t} = \begin{pmatrix} e^{it}\sin^2\theta + e^{-it}\cos^2\theta & i\sin t\sin 2\theta \\ i\sin t \sin 2\theta & e^{it}\cos^2\theta + e^{-it}\sin^2\theta \end{pmatrix},\ f(\mathbf{x}) = \begin{pmatrix} f_{ee}(\mathbf{x}) & 0 \\ 0 & f_{xx}(\mathbf{x}) \end{pmatrix}$$

$$\begin{aligned}
\varrho_{ee}(t,\mathbf{x}) &= f_{ee}(\mathbf{x}-\mathbf{v}t)\Big(\cos^4(\theta) + 2\cos(2t)\cos^2(\theta)\sin^2(\theta) + \sin^4(\theta)\Big) + f_{xx}(\mathbf{x}-\mathbf{v}t)\sin^2 t\sin^2 2\theta \\
\varrho_{ex}(t,\mathbf{x}) &= \varrho^*_{xe} = 2i\Big(f_{ee}(\mathbf{x}-\mathbf{v}t) - f_{xx}(\mathbf{x}-\mathbf{v}t)\Big)\ \sin(t)\sin(\theta)\cos(\theta)\Big(e^{it}\cos^2(\theta) + e^{-it}\sin^2(\theta)\Big) \\
\varrho_{xx}(t,\mathbf{x}) &= f_{xx}(\mathbf{x}-\mathbf{v}t)\Big(\cos^4(\theta) + 2\cos(2t)\cos^2(\theta)\sin^2(\theta) + \sin^4(\theta)\Big) + f_{ee}(\mathbf{x}-\mathbf{v}t)\sin^2(t)\sin^2(2\theta).
\end{aligned}$$

2 Numerical Specification

The e.o.m. for the mixed state of neutrinos described by mean-field density matrix $\varrho(t,\mathbf{x},\mathbf{p})$:

$$\left(\frac{\partial}{\partial t} + \mathbf{v}\cdot\frac{\partial}{\partial \mathbf{x}} + \mathbf{f}\cdot\frac{\partial}{\partial \mathbf{p}}\right)\varrho(t,\mathbf{x},\mathbf{p}) = i[\varrho(t,\mathbf{x},\mathbf{p}),\mathcal{H}(t,\mathbf{x},\mathbf{p})] + C[\varrho].$$

2.1 Numerical Setup

Consider two-flavor system, and impose the translation symmetry on both x and y dimensions, the e.o.m. for neutrino and antineutrino:

$$(\partial_t + v_z\partial_z)\,\rho(t,z,v_z) = i[\rho(t,z,v_z),\mathcal{H}(t,z,v_z)]$$

$$(\partial_t + v_z\partial_z)\,\overline{\rho}(t,z,v_z) = i\left[\overline{\rho}(t,z,v_z),\overline{\mathcal{H}}(t,z,v_z)\right]$$

$$\text{with } \rho(t,z,v_z) = \begin{pmatrix} \rho_{ee} & \rho_{ex} \\ \rho^*_{ex} & \rho_{xx} \end{pmatrix},\ \overline{\rho}(t,z,v_z) = \begin{pmatrix} \bar\rho_{ee} & \bar\rho_{ex} \\ \bar\rho^*_{ex} & \bar\rho_{xx} \end{pmatrix}$$

Ignoring the MSW effect, the Hamiltonian would be:

$$\mathcal{H}(t,z;v_z) = \begin{pmatrix} -\cos 2\theta & \sin 2\theta \\ \sin 2\theta & \cos 2\theta \end{pmatrix} + \mu\int_0^1 dv'_z(1-v_zv'_z)[\rho(t,z;v'_z) - \bar\rho^*(t,z;v'_z)] \equiv \mathcal{H}_{vac} + \mathcal{H}_{\nu\nu},$$

$$\overline{\mathcal{H}}(t,z;v_z) = \begin{pmatrix} -\cos 2\theta & \sin 2\theta \\ \sin 2\theta & \cos 2\theta \end{pmatrix} - \mu\int_0^1 dv'_z(1-v_zv'_z)[\rho^*(t,z;v'_z) - \bar\rho(t,z;v'_z)] \equiv \overline{\mathcal{H}}_{vac} + \overline{\mathcal{H}}_{\nu\nu}$$

The finite difference index:

	t	z	v_z
step	δt	δz	δv_z
	M_1	M_2	N
index	i_1	i_2	k

Let $\rho(t,z,v_z)$ be $\rho^k_{i_1,i_2}$ in the discretization and the interaction Hamiltonian

$$\mathcal{H}^k_{\nu\nu,i_1,i_2} = \frac{\mu}{N}\sum_{k'=0}^{N}\left(1-\frac{kk'}{N^2}\right)\left(\rho^{k'}_{i_1,i_2}-\bar{\rho}^{*k'}_{i_1,i_2}\right).$$

$$\overline{\mathcal{H}}^k_{\nu\nu,i_1,i_2} = -\frac{\mu}{N}\sum_{k'=0}^{N}\left(1-\frac{kk'}{N^2}\right)\left(\rho^{*k'}_{i_1,i_2}-\bar{\rho}^{k'}_{i_1,i_2}\right).$$

Algorithm: Lax-Wendroff Method:

Using the center space discretization

$$\partial_z\rho \to \frac{\rho_{i_1,i_2+1}-\rho_{i_1,i_2-1}}{2\delta z}.$$

$$\rho(t+\delta t,z) = \rho(t,z) + \delta t\,\frac{\partial\rho(t,z)}{\partial t} + \frac{1}{2}\delta t^2\,\frac{\partial^2\rho(t,z)}{\partial t^2} + O(\delta t^3).$$

Express the density matrix at i_1+1-th time in terms of that at i_1

$$\rho^k_{i_1+1,i_2} = \rho^k_{i_1,i_2} + F_{\rm trnspt}(\rho^k_{i_1}) + F_{\rm osc}(\rho^k_{i_1}).$$

$$F_{\rm trnspt}(\rho^k_{i_1}) = -\frac{c}{2}(\rho^k_{i_1,i_2+1}-\rho^k_{i_1,i_2-1}) + \frac{c^2}{2}(\rho^k_{i_1,i_2+1}-2\rho^k_{i_1,i_2}+\rho^k_{i_1,i_2-1}).$$

$$F_{\rm osc}(\rho^k_{i_1}) = F_1(\rho^k_{i_1}) + F_2(\rho^k_{i_1}).$$

$$F_1(\rho^k_{i_1}) = i\delta t[\rho^k_{i_1,i_2},\mathcal{H}^k_{i_1,i_2}] - \frac{ic\delta t}{2}[\rho^k_{i_1,i_2+1}-\rho^k_{i_1,i_2-1},\mathcal{H}^k_{i_1,i_2}] - \frac{\delta t^2}{2}[[\rho^k_{i_1,i_2},\mathcal{H}^k_{i_1,i_2}],\mathcal{H}^k_{i_1,i_2}].$$

$$F_2(\rho^k_{i_1}) = -\frac{1}{4}ic\delta t\left[\rho^k_{i_1,i_2},\mathcal{A}\right] + \frac{1}{2}i\delta t^2\left[\rho^k_{i_1,i_2,},\mathcal{B}\right] \to \text{interaction}.$$

$$\mathcal{H}^k_{i_1,i_2} = \mathcal{H}_{\rm vac} + \mathcal{H}^k_{\nu\nu,i_1,i_2}.$$

$$\mathcal{A} = \frac{\mu}{N}\sum_{k'=0}^{N}\left(1-\frac{kk'}{N^2}\right)\left((\rho^{k'}_{i_1,i_2+1}-\rho^{k'}_{i_1,i_2-1})-(\bar{\rho}^{*k'}_{i_1,i_2+1}-\bar{\rho}^{*k'}_{i_1,i_2-1})\right) \sim (\ \partial_z\mathcal{H}\ ).$$

$$\mathcal{B} = \frac{\mu}{N}\sum_{k'=0}^{N}\left(1-\frac{kk'}{N^2}\right)\left((i\left[\rho^{k'}_{i_1,i_2},\mathcal{H}^{k'}_{i_1,i_2}\right]-\rho^{k'}_{i_1,i_2+1}+\rho^{k'}_{i_1,i_2-1})-(i\left[\bar{\rho}^{*k'}_{i_1,i_2},\overline{\mathcal{H}}^{*k'}_{i_1,i_2}\right]-\bar{\rho}^{*k'}_{i_1,i_2+1}+\bar{\rho}^{*k'}_{i_1,i_2-1})\right) \sim (\ \partial_t\mathcal{H}\ ).$$

The above Courant-Friedrichs-Lewy (CFL) stability criterion:

$$c \equiv \frac{|v_z|\delta t}{\delta z} \leq 1.$$