Homework 3

Consider the following 2-D Laplace Equation:

$$\Delta u = (u_{xx} + u_{yy}) = 0$$

with the boundary condition:

$$\begin{gathered} u(0, y) = 0 \\ u(2\pi, y) = 0 \\ u(x, 0) = \sin(2x) + \sin(5x) + \sin(7x) \\ u(x, 2\pi) = 0 \end{gathered}$$

For the discretization, we have

$$\frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{\Delta x^2} + \frac{u_{i,j+1} - 2u_{i,j} + u_{i,j-1}}{\Delta y^2} = 0$$

Because of the given set of mesh grids $\Delta x = \Delta y = \frac{2\pi}{20}$, the Jacobi iterative method can be derived as:

$$u_{i,j}^{k+1} = \frac{u_{i-1,j}^k + u_{i+1,j}^k + u_{i,j-1}^k + u_{i,j+1}^k}{4}$$

The Gauss-Seidel iterative method can be derived as:

$$u_{i,j}^{k+1} = \frac{u_{i-1,j}^{k+1} + u_{i+1,j}^k + u_{i,j-1}^{k+1} + u_{i,j+1}^k}{4}$$

For each grid (i, j), the residual can be calculated as (use the Jacobi for example):

$$r_{i,j}^k = \frac{1}{\Delta x^2}\left(u_{i-1,j}^{k-1} - 2u_{i,j}^{k-1} + u_{i+1,j}^{k-1}\right) + \frac{1}{\Delta y^2}\left(u_{i,j-1}^{k-1} - 2u_{i,j}^{k-1} + u_{i,j+1}^{k-1}\right)$$

For the SOR iterative method, the iterative equation is shown as follows:

$$u_{i,j}^{k+1} = (1 - \omega)u_{i,j}^k + \omega\tilde{u}$$

$\tilde{u}$ is the updated point obtained by Jacobi or Gauss-Seidel.

The code and results are attached at the end.