Chapter 5
Vulnerability of Interdependent Networks and Networks of Networks

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Abstract Networks interact with one another in a variety of ways. Even though increased connectivity between networks would tend to make the system more robust, if dependencies exist between networks, these systems are highly vulnerable to random failure or attack. Damage in one network causes damage in another. This leads to cascading failures which amplify the original damage and can rapidly lead to complete system collapse.

Understanding the system characteristics that lead to cascading failures and support their continued propagation is an important step in developing more robust systems and mitigation strategies. Recently, a number of important results have been obtained regarding the robustness of systems composed of random, clustered and spatially embedded networks.

Here we review the recent advances on the role that connectivity and dependency links play in the robustness of networks of networks. We further discuss the dynamics of cascading failures on interdependent networks, including cascade lifetime predictions and explanations of the topological properties which drive the cascade.

5.1 Background: From Single Networks to Networks of Networks

As the ability to measure complex systems evolved, driven by enhanced digital storage and computation abilities in the 1990s, researchers discovered that network topology is important and not trivial. New structures were observed and new

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models proposed to explain them. Scale-free networks dominated by hubs [1, 2], small-world networks which captured the familiar "six degrees of separation" idea [3, 4], ideas of communities and clustering, and countless other variations [5, 6] were discovered and analyzed. Network topologies were shown to be very different from the abstractions of classical graph theory [7–9] in many real systems and yet important calculations, predictions and measurements could still be executed. Looking to the topology of networks provided new insights into epidemiology [10], marketing [11], percolation [12], traffic [13], and climate studies [14, 15] amongst many others.

One of the most important properties of a network that was studied was its vulnerability to the failure of a subset of its nodes. Utilizing percolation theory, network robustness can be studied via the fraction of nodes in its largest connected component P_∞ which is taken as a proxy for functionality of the network [16, 17]. Consider, for example, a telephone network composed of telephone lines and retransmitting stations. If $P_\infty \sim 1$ (the entire system), then there is a high level of connectivity in the system and information from one part of the network is likely to reach any other part. If, however, $P_\infty \sim 0$, then information in one part cannot travel far and the network must be considered nonfunctional. Even if $P_\infty \sim 1$, some nodes may be detached from the largest connected component and those nodes are considered nonfunctional. We use the term *giant connected component* (GCC) to refer to P_∞ when it is of order 1. Percolation theory is concerned with determining $P_\infty(p)$ after a random (or targeted) fraction $1 - p$ of nodes (or edges) are disabled in the network. Typically, $P_\infty(p)$ undergoes a second-order transition at a certain value p_c: for $p > p_c$, $P_\infty(p) > 0$ and it approaches zero as $p \to p_c$ but for $p < p_c$, $P_\infty(p) \equiv 0$. Thus there is a discontinuity in the derivative $P'_\infty(p)$ at p_c even though the function itself is continuous. It is in this sense that the phase transition is described as second-order [18]. It was shown, for example, that scale-free networks (SF)–which are extremely ubiquitous in nature–have $p_c = 0$ as long as the degree distribution has a sufficiently long tail [12]. This is in marked contrast to Erdős-Rényi (ER) networks ($p_c = 1/\langle k \rangle$) and 2D square lattices ($p_c \approx 0.5927$ [17]) and helps to explain the surprising robustness of many systems (e.g. the internet) with respect to random failures [12, 19].

However, in reality, networks rarely appear in isolation. In epidemiology, diseases can spread within populations but can also transition to other populations, even to different species. In transportation networks, there are typically highway, bus, train and airplane networks covering the same areas but behaving differently [20]. Furthermore, the way in which one network affects another is not trivial and often specific nodes in one network interact with specific nodes in another network. This leads to the concept of interacting networks in which links exist between nodes within a single network as well as across networks. Just as ideal gases–which by definition are comprised of non-interacting particles–lack emergent critical phenomena such as phase transitions, we will see that the behavior of interacting networks has profound emergent properties which do not exist in single networks.

Since networks interact with one another selectively (and not generally all networks affecting all other networks), we can describe *networks of networks* (NoN) with topologies between networks that are similar to the topology of nodes in a single network.

Multiplex networks are interconnected networks in which the identity of the nodes is the same across different networks but the links are different [21–23]. Multiplex networks were first introduced to describe a person who participates in multiple social networks [24]. For instance, the networks of phone communication and email communication between individuals will have different topologies and different dynamics though the actors will be the same [25]. Also, each online social network shares the same individuals though the network topologies will be very different depending on the community which the social network represents.

When discussing networks of networks, a natural question is: why describe this phenomenon as a "interconnected networks?" If we are dealing with a set of nodes and links then no matter how it is partitioned it is still a network. Each description of interacting networks will answer this question differently but any attempt to describe a network of networks will be predicated on a claim that more is different—that by splitting the overall system into component networks, new phenomena can be uncovered and predicted. One way of describing the interaction between networks which yields qualitatively new phenomena is *interdependence*. This concept has been studied in the context of critical infrastructure and been formalized in several engineering models [26, 27] (see Fig. 5.1). However, as a theoretical property of interacting networks, interdependence was first introduced in a seminal study by Buldyrev et al. in 2010 [28]. This review will focus on the theoretical framework and wealth of new phenomena discovered in interdependent networks. Some parts of this review first appeared in the proceedings of NDES 2014 [29].

5.2 Interdependence: Connectivity and Dependency Links

The fundamental property which characterizes interdependent networks is the existence of two qualitatively different kinds of links: *connectivity* links and *dependency* links [28, 30–32] (see Fig. 5.1). The connectivity links are the links which we are familiar with from single network theory and they connect nodes within the same network. They typically represent the ability of some quantity (information, electricity, traffic, disease etc.) to flow from one node to another. From the perspective of percolation theory, if a node has multiple connectivity links leading to the GCC, it will only fail if all of those links cease to function. Dependency links, on the other hand, represent the idea that for a node to function, it requires support from another node which, in general, is in another network. In such a case, if the supporting node fails, the dependent node will also fail–even if it is still connected to the GCC in its network. If one network *depends on* and *supports* another network, we describe that pair of networks as interdependent. Interdependence is a common feature of critical infrastructure (see Fig. 5.1) and

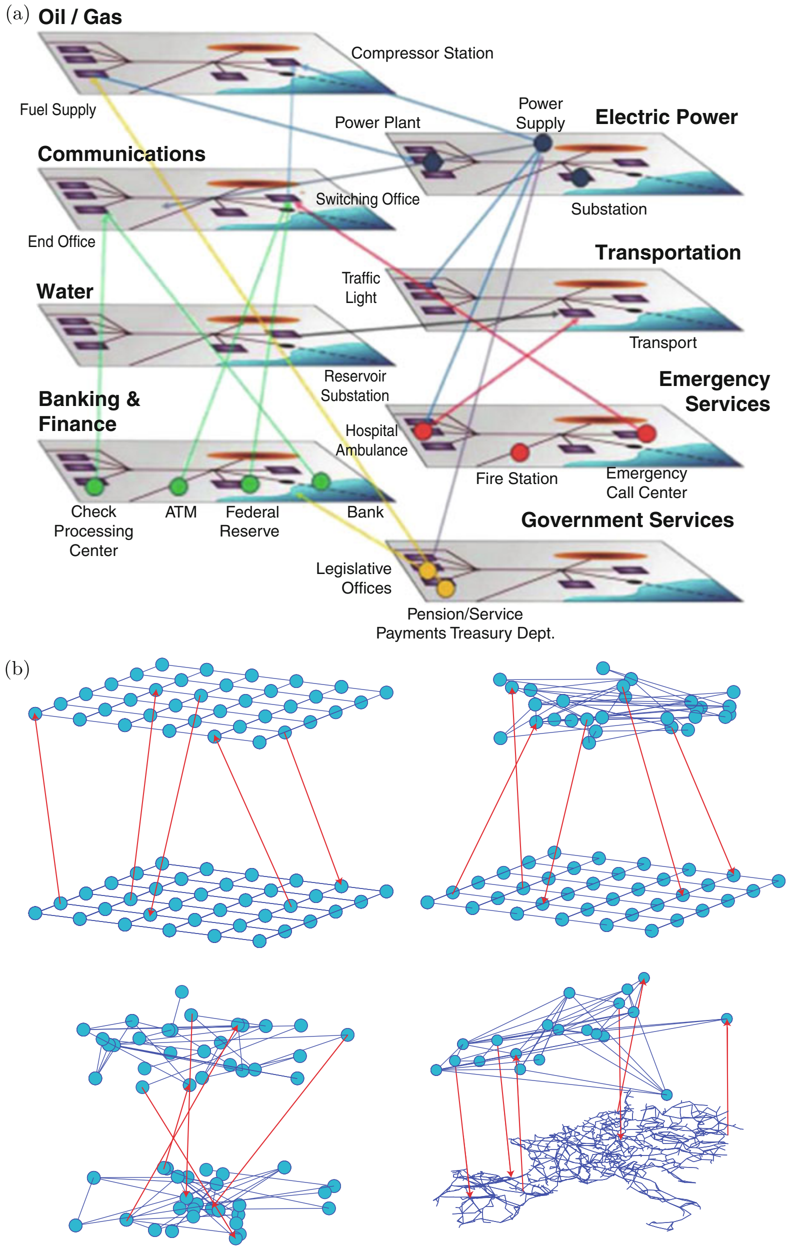

Fig. 5.1 An example of interdependent critical infrastructure systems and several modelled interdependent networks. (**a**) Schematic representation of interdependent critical infrastructure networks after [33]. (**b**) Illustration of interdependent networks composed of connectivity links (in *blue*, within the networks) and dependency links (in *red*, between the networks). Clockwise from upper-left: coupled lattices, a lattice coupled with a random regular (RR) network, two coupled RR networks and an RR network coupled to a real-world power grid (After [34])

many multiplex networks. Often whatever causes a node to stop functioning in one layer will also disable it in other layers. Indeed, the percolation properties of interdependent networks describe the typical behavior in multiplex networks as well [28]. The properties of interdependence can affect a network's function in a variety of ways but here we focus on the response of a network of interdependent networks to the failure of a subset of its nodes using the tools of percolation theory [6]. We refer the reader to recent general reviews for other descriptions of interacting networks [24, 35, 36].

Percolation on a single network is an instantaneous process but on a system of interdependent networks, the removal of a random fraction $1 - p$ of the nodes initiates a cascading failure in the following sense. Consider percolation on two interdependent networks A and B for which every node in A depends on exactly one node in B and vice versa. If we remove a fraction $1-p$ of the nodes in A, other nodes in A which were connected to the GCC via the removed nodes will also be disabled, leaving a new GCC of size $P_\infty(p) < p$. Since all of the nodes in B depend on nodes in A, a fraction $1-P_\infty(p)$ of the nodes in B will now be disabled via their dependency links. This will lead, in turn, to more nodes being cut off from the GCC in B and the new GCC in B will be smaller yet. This will lead to more damage in A due to the dependency links from B to A. This process of percolation and dependency damage accumulating iteratively continues until no more nodes are removed from iteration to iteration. This cascading failure is similar to the cascades described in flow and overload models on networks and the cascading failures in power grids which are linked to blackouts [37, 38]. The cascade triggered by a single node removal has been called an "avalanche"[39] and the critical properties of this process have been studied extensively [39, 40].

5.3 Interdependent Random Networks

This cascading failure was shown to lead to abrupt first-order transitions in systems of interdependent ER and SF networks that are qualitatively very different from the transitions in single networks (see Fig. 5.2). Furthermore, p_c of a pair of ER networks was shown to increase from $1/\langle k\rangle$ to $2.4554/\langle k\rangle$. Surprisingly, it was found that scale-free networks, which are extremely robust to random failure on their own [12, 19], become more vulnerable than equivalent ER networks when they are fully interdependent and for any $\lambda > 2$, $p_c > 0$. In general, a broader degree distribution leads to a higher p_c [28]. This is because the hubs in one network, which are the source of the stability of single scale-free networks, can be dependent on low degree nodes in the other network and are thus vulnerable to random damage via dependency links. These results were first demonstrated using the generating function formalism [28, 41], though it has recently been shown that the same results can be obtained using the cavity method [42].

After the first results on interdependent networks were published in 2010 [28], the basic model described above was expanded to cover more diverse systems. One

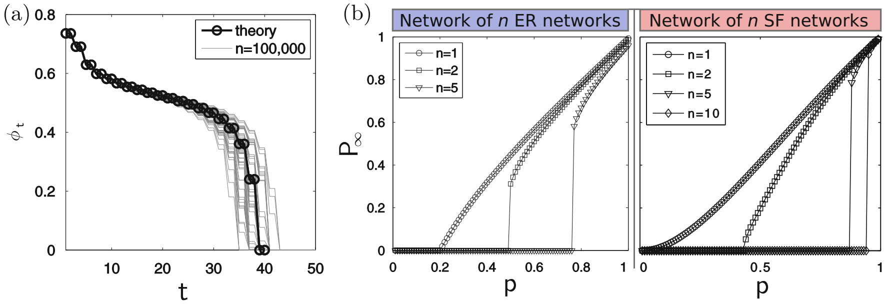

Fig. 5.2 Percolation of a network of interdependent random networks. (**a**) The fraction of viable nodes at time t for a NoN composed of 5 ER networks. The *gray lines* represent individual realizations and the *black line* is calculated analytically. After [51]. (**b**) Percolation in a NoN of ER and SF networks. Shown here is the effect of increasing the number of networks n for tree-like NoNs composed of ER and SF networks (After [51])

striking early result was that if less than an analytically calculable critical fraction q_c of the nodes in a system of two interdependent ER networks are interdependent, the phase transition reverts to the familiar second-order transition [30]. However, for scale-free networks, reducing the fraction of interdependent nodes leads to a hybrid transition, where a discontinuity in P_∞ is followed by a continuous decline to zero, as p decreases [43]. A similar transition was found when connectivity links between networks (which were first introduced in [44]) are combined with dependency links [45]. It has also been shown that the same cascading failures emerge from systems with connectivity and dependency links within a single network [46–48].

The assumption that each node can depend on only one node was relaxed in [49] and it was shown that even if a node has many redundant dependency links, the first-order transition described above can still take place. If dependency links are assigned randomly, a situation can arise in which a chain of dependency links can be arbitrarily long and thus a single failure can propagate through the entire system. To avoid this scenario, most models for interdependent networks assume uniqueness or "no feedback" which limits the length of chains of dependency links [50, 51]. For a pair of fully interdependent networks, this reduces to the requirement that every dependency link is bidirectional. Under partial dependency, this assumption is not necessary and the differences between systems with and without feedback have also been studied [34, 51].

Though both the connectivity and dependency links were treated as random and uncorrelated in Refs. [28, 30, 31, 50–52], the theory of interdependent networks has been expanded to more realistic cases. Assortativity of connectivity links was shown to decrease overall robustness [53]. Assortativity of dependency links was treated numerically [54], analytically for the case of full degree-degree correlation [55] and analytically for the general case of degree-degree correlations with connectivity or dependency links using the cavity method [42]. Interestingly, if a fraction α of the highest degree nodes are made interdependent in each network, a three-phase system with a tricritical point emerges in the α–p plane [56]. If the system is a multiplex network, there may be overlapping links, i.e., two nodes which are linked in one layer may have a tendency to be linked in other layers [57–59]. In interdependent networks this phenomenon is referred to as intersimilarity [54, 60]. Clustering, which has a negligible effect on the robustness of single networks [61], was shown to substantially reduce the robustness of interdependent networks [43, 62] and networks of networks [63].

In the following subsections, we highlight some significant methodological approaches and results from the study of cascading failures in two interdependent networks (Sect. 5.3) and from the study of networks of n interdependent networks (Sect. 5.3).

Cascading Failures in Coupled Networks

Consider two networks A and B which are partially dependent in the sense that only a fraction q_A (q_B) of the nodes in A (B) are dependent, the rest being autonomous [30]. If a fraction $1 - p$ of the nodes in A are removed, we define ψ'_t (ϕ'_t) as the fraction of viable nodes at time t in network A (B). Of those nodes, the fraction which are part of the GCC is given by $\phi_t = \phi'_t g_A(\phi'_t)$ ($\psi_t = \psi'_t g_B(\psi'_t)$). By tracing the value of ψ'_t and ϕ'_t, we can measure and predict the dynamics of the cascading failure in a system of interdependent networks. The function $g_i(p)$ can be determined analytically for ER, SF and indeed for a random network with an arbitrary degree distribution using generating functions. Using this, we can predict the size of the giant component in both networks at every time t:

$$
\begin{aligned}
\psi'_1 &\equiv p \\
\phi'_1 &= 1 - q_B[1 - p g_A(\psi'_1)] \\
\psi'_t &= p(1 - q_A[1 - g_B(\phi'_{t-1})]) \\
\phi'_t &= 1 - q_B[1 - p g_A(\psi'_{t-1})]
\end{aligned}
\tag{5.1}
$$

Since the steady state is defined as the configuration for which $\psi_t = \psi_{t-1}$ and $\phi_t = \phi_{t-1}$ we obtain a system of two equations and two unknowns:

$$
\begin{aligned}
\psi'_\infty &= p(1 - q_A[1 - g_B(\phi'_\infty)]) \\
\phi'_\infty &= 1 - q_B[1 - p g_A(\psi'_\infty)]
\end{aligned}
\tag{5.2}
$$

Depending on q_i and g_i, the size of the GCC in each network will either approach zero as $p \to p_c$ in which case there will be a second-order transition or will abruptly jump to zero and there will be a first-order transition. Results of these calculations are shown in Fig. 5.2. The cascade "plateau" emerges from the analytic predictions as well as simulations. We discuss this phenomenon in greater detail in Sect. 5.4. Letting $q_A = q_B = 1$ and $g_A = g_B = g$ we recover the results from [28].

Results for Networks of Interdependent Networks

In a series of articles, Gao et al. extended the theory of pairs of interdependent networks to networks of interdependent networks with general topologies [31, 50–52]. Within this framework, analytic solutions for a number of key percolation quantities were presented including size of the GCC at each time-step t (see Fig. 5.2), the size of the GCC at steady state (see Fig. 5.2), p_c and other values.

The NoN topologies which were solved analytically include: a tree-like NoN of ER, SF or random regular (RR) networks ($q = 1$), a loop-like NoN of ER, SF or RR networks ($q \leq 1$), a star-like NoN of ER networks ($q \leq 1$) and a RR NoN of ER, SF or RR networks ($q \leq 1$). For tree-like NoNs, it was found [31, 52] that the number of networks in the NoN (n) affects the overall robustness but the specific topology of the NoN does not. In contrast, for a RR NoN the number of networks n does not affect the robustness but the degree of each network within the NoN (m) does [50, 51]. Because the topology of the loop-like and RR NoNs allows for chains of dependency links going throughout the system, there exists a quantity q_{max} above which the system will collapse with the removal of a single node, even if each network is highly connected ($p = 1$).

In a NoN, each node is a network and pairs of networks are considered linked if dependency links exist between then. We define a “NoN adjacency matrix” Q with elements q_{ij} defined as the fraction of nodes in network i that depend on nodes in network j. Recently, it was shown that the formalism developed for analytically solvable networks can be applied to NoNs for which the percolation profile of the individual networks is known only numerically [64].

For a tree-like NoN formed of n ER networks [50, 52] the size of the GCC is obtained from the self-consistent solution to

$$P_\infty = p[1 - e^{-\langle k \rangle P_\infty}]^n \tag{5.3}$$

For $n = 1$ this is the familiar second-order transition for a single ER network [7–9] but for $n = 2$ (as in [28]) or greater, there is a discontinuity in P_∞ and the transition is first-order as shown in Fig. 5.2. For fully interdependent, tree-like NoNs, the robustness decreases as n increases but is not impacted by the specific topology of the NoN. For case of partially interdependent ER networks, the specific topology does influence the robustness as shown analytically for the special case of a star-like NoN in [31]. Similar results have been obtained for trees of RR networks [50, 52] and SF networks [51].

For a loop-like NoN of partially interdependent ER networks [31, 50] the number of networks also does not affect the robustness and the GCC can be calculated as

$$P_\infty = p(1 - e^{-\langle k \rangle P_\infty})(qP_\infty - q + 1), \tag{5.4}$$

which recovers the familiar result for single networks if $q = 0$.

A more thorough analysis of the influence of loops in NoNs appears in a random-regular network of ER networks (RR NoN of ERs). Such a system can exhibit first or second order phase transitions depending on the value of q [51]. For $q < q_c$, the transition is second-order and takes place at $p = p_c^{\mathrm{II}}$. For $q_c < q < q_{max}$, the transition is first-order and takes place at $p = p_c^{\mathrm{I}}$. Above q_{max}, the feedback loops enabled by the NoN topology lead to spontaneous collapse, even in a fully connected network. The mutual GCC for an RR NoN of ERs is

$$P_\infty = \frac{p}{2^m}\left(1 - e^{-kP_\infty}\right)\left(1 - q + \sqrt{(1-q)^2 + 4qP_\infty}\right)^m \tag{5.5}$$

from which we can derive

$$p_c^{\mathrm{II}} = \frac{1}{\langle k \rangle (1-q)^m} \tag{5.6}$$

and

$$q_c = \frac{k + m - \sqrt{m^2 + 2km}}{k}. \tag{5.7}$$

The values of q_{max} and p_c^{I} can also be derived analytically from Eq. (5.5), but require several intermediate results. We refer the interested reader to the original derivation in Gao et al. [51].

In light of these results, we can now see that single network percolation is simply a limiting case of NoN percolation theory. These results have been recently reviewed in [32] and [65].

5.4 Critical Dynamics and the Cascade "Plateau"

Because the phase transition in interdependent networks is characterized by a cascading process, it has a duration which is determined by the number of iterations and is referred to in the literature as *NOI* or τ. In random networks at criticality, the size of the GCC decreases from iteration to iteration via an initial quick drop in size, followed by a long period of very little change (the "plateau") and finally a fast collapse, see Fig. 5.2. A similar plateau with different scaling behavior appears in spatially embedded networks with random dependency links, as discussed in [66].

Explaining the critical properties of this plateau is an important result of recent research [40].

Zhou et al. [40] showed that at the critical point of a given realization, p_c, the duration of the plateau, and thus the cascade (τ) scales with system size as

$$\tau \sim N^{1/3} \tag{5.8}$$

in ER networks and thus diverges in the thermodynamic limit. The value of the scaling exponent can be derived directly from universality arguments alone. If we consider that ER networks are describable with mean-field theory, they are thus in the same universality class as 6-dimensional lattice percolation (i.e., $N = L^6 \leftrightarrow L = N^{1/6}$). If we consider the branching process which characterizes the bulk of the cascading failure (Fig. 5.3), we can see that its duration will scale linearly with the number of steps (l) required for a random walker to traverse an ER network at criticality (i.e. $\tau \sim l$). Absent interactions as is the case for mean field theory, this scales with linear system size as

$$l \sim L^2 \rightarrow \tau \sim N^{1/3}. \tag{5.9}$$

Combining Eq. (5.9) with the fact that the upper critical dimension for percolation is 6, we recover Eq. (5.8). A different derivation of this result was previously published by Zhou et al. [40]. Buldyrev et al. [28] showed that for the *mean* p_c, $\tau \sim N^{1/4}$. Zhou et al. [40] developed a theory to explain the relation of this result to Eq. (5.8) that was found for single realizations.

Recent work by Zhou et al. [40] has shed new light on the dynamics of the plateau formation at criticality (Figs. 5.2 and 5.3). When a node a_i in network A fails, it will typically cause a node to fail in network B. This may lead to further damage in B due to percolation and that damage will cause the failure of a (possibly empty) set $a' \subset A$ due to the dependency links. Thus at each iteration, there is a branching process of induced damage in each network.

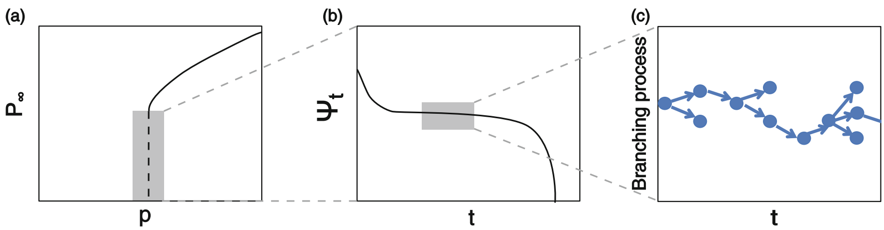

Fig. 5.3 (**a**) The first order transition in a system of interdependent networks is characterized by an abrupt drop in the size of the GCC. (**b**) On closer inspection, this jump is the product of a cascade of failures, the bulk of which is dominated by a "plateau" during which the GCC changes very little. (**c**) The plateau can be analyzed in terms of a branching process where the branching factor at time t, η_t, describes the number of failures at time t relative to the step before. During the plateau, $\eta_t \approx 1$ due to a balance of competing processes, as described in the text (After [40])

The researchers in [40] examined s_t, the number of nodes which failed at time t from the root node. They then defined the branching factor, η_t, as s_{t+1}/s_t. It was shown that η_t goes through three phases. When t is small, $\eta_t < 1$ and the branching process is decaying. This is due to the fact that the dependency damage which A carries to B causes less percolative damage in B and thus less dependency damage back in A. If the network was not also becoming more dilute in the process, then the branching process would decay and stop quickly. Indeed, for $p > p_c$ this is what happens. However, when $p = p_c$, the process continues for an infinite amount of time (in the thermodynamic limit). This is because although $\eta_t < 1$, the network becomes weaker each time nodes are removed and at criticality these processes are exactly balanced. Thus the plateau stage is a second-order percolation transition caused spontaneously by a perfect matching between the dilution of the network (which would tend to amplify the damage) and the decreasing damage due to percolation (which would decrease the damage). During this stage the branching factor is $\eta_t \approx 1$. However, due to the finite size of the system, the network eventually becomes sufficiently dilute for $\eta_t > 1$. At this point, η_t grows exponentially and the entire system collapses within a few steps.

5.5 Spatially Embedded Interdependent Networks

One of the most compelling motivations for developing a theory of interdependent networks is that many critical infrastructure networks depend on one another to function [26, 27]. Essentially all critical infrastructure networks depend on electricity to function, which is why threats like electromagnetic pulses are taken so seriously (see Fig. 5.1, Ref. [33]). The power grid itself, though, requires synchronization and control which it can only receive when the communication network is operational. One of the largest blackouts in recent history, the 2003 Italy blackout, was determined to have been caused by a cascading failure between electrical and communications networks [67].

In contrast to abstract networks, all infrastructure networks are embedded in space [20]. The nodes (e.g., power stations, communication lines, retransmitters etc.) occupy specific positions in a 2D plane and the fact that the cost of links increases with their length leads to a topology that is markedly different from random networks [68]. Thus infrastructure networks will tend to be approximately planar and the distribution of geographic link distances will be exponential with a characteristic length [69]. From universality principles, all such networks are expected to have the same general percolation behavior as standard 2D lattices [16, 69]. As such, the first descriptions of spatially embedded interdependent networks were modelled with square lattices [34, 64, 70–72] and the results have been verified on synthetic and real-world power grids [34, 71].

Analytic descriptions of percolation phenomena require the network to be "locally tree-like" and in the limit of large systems, this assumption is very accurate for random networks of arbitrary degree distribution [41]. However, lattices and other spatially embedded networks are not even remotely tree-like and analytic results on percolation properties are almost impossible to obtain [16, 17]. Therefore most of the results on spatially embedded networks are based on numerical simulations.

One of the few major analytic results for spatially embedded systems is that for interdependent lattices, if there is no restriction on the length of the dependency links then any fraction of dependency leads to a first-order transition ($q_c = 0$). In [34], it was shown that the critical fraction q_c for which the system transitions from the first-order regime to the second order regime must fulfill:

$$1 = p_c^{\star} q_c P'_{\infty}(p_c) \tag{5.10}$$

in which $p_c^{\star}$ is the percolation threshold in the system of interdependent lattices, p_c is the percolation threshold in a single lattice and $P'_{\infty}(p)$ is the derivative of $P_{\infty}(p)$ for a single lattice. Since as $p \to p_c$, $P_{\infty}(p) = A(x - p_c)^{\beta}$ and for 2D lattices $\beta = 5/36$ [73], $P'_{\infty}(p)$ diverges as $p \to p_c$ and the only way to fulfill Eq. (5.10) is if $q_c = 0$. From universality arguments, all spatially embedded networks in $d < 6$ have $\beta < 1$ [16, 17, 69] and thus all systems composed of interdependent spatially embedded networks (in $d < 6$) with random dependency links will have $q_c = 0$. In Fig. 5.1, all of the configurations shown except the RR-RR system have $q_c = 0$.

If the dependency links are of limited length, the percolation behavior is surprisingly complex and a new spreading failure emerges. Li et al. [70] introduced the parameter r, called the "dependency length," to describe the fact that in most systems of interest the dependency links, too, will likely be costly to create and, like the connectivity links, will tend to be shorter than a certain characteristic length. In this model, dependency links between networks are selected at random but are always of length less than r (in lattice units). If $r = 0$, the system of interdependent lattices behaves identically to a single lattice. If $r = \infty$, the dependency links are unconstrained and purely random as in [34]. Li et al. [70] found that p_c as a function of r shows a sharp maximum at $r_c = 8$ which is explained by the correlation length of percolation. Moreover, as long as r is below a critical length r_c, the transition is second-order but for $r > r_c$ the transition is first order (See Fig. 5.4). The first-order transition for spatially embedded interdependent networks is unique in that it is characterized by a spreading process. Once damage of a certain size emerges at a given place on the lattice, it will begin to propagate outwards and destroy the entire system (See Fig. 5.4).

If the dependency is reduced from $q = 1$ to lower values, it is found that r_c increases and diverges at $q = 0$, consistent with the result from [34] that $q_c = 0$ for $r = \infty$ [72] (See Fig. 5.4).

Recently, the framework developed in [31, 50–52] was extended to general networks of spatially embedded networks in [64]. There they developed a theory for a network of spatially embedded interdependent networks with $r = \infty$ and presented simulation results for the case of finite r.

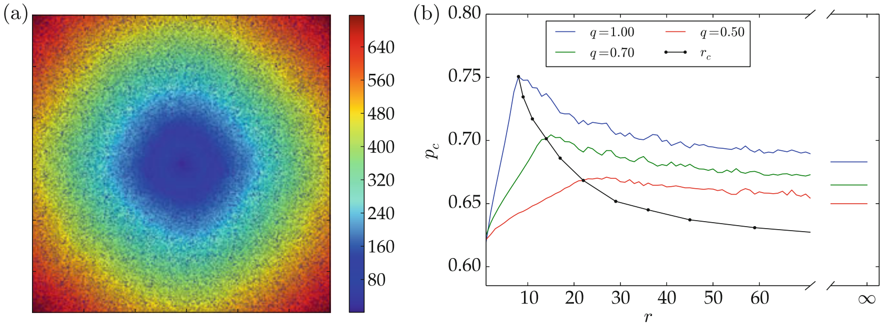

Fig. 5.4 Percolation of spatially embedded networks. (**a**) A snapshot of one lattice in a pair of interdependent lattices with nodes colored according to the time-step in which the node failed. The regularity of the color-change reflects the constant speed of the spreading failure in space (Generated for $q = 1, r = 11, L = 2900$). (**b**) The effect of r and q on p_c. As r increases, p_c increases until r reaches r_c. At that point the transition becomes first-order and p_c starts decreasing until it reaches its asymptotic value at $r = \infty$ (Both after [72])

Though the generating function for a square lattice is not analytically solvable, we do know how P_∞ behaves as a function of p for a single lattice. In [34, 70], that information was utilized to derive the theoretical mutual giant connected component for a system of two interdependent lattices. Shekhtman et al. [64] extended that theory to the case of a network composed of n lattices. Specifically, three main cases were solved: a treelike fully dependent network of lattices, a starlike partially dependent network of lattices, and a random-regular partially dependent network of lattices. Similar to the case of networks of random networks, the robustness of fully dependent tree-like spatially embedded NoNs are affected by n but not by the topology of the tree while RR NoNs are affected by m (the number of networks that each network depends on) but not by n [64]. Furthermore, the theory derived in [64] can be used to find the mutually giant connected component of any system of interdependent networks where we know the percolation profile of the individual networks.

For the case of random-regular networks there exists a certain fraction of interdependence, q_{max}, for which removing even a single node, i.e. $p \to 1$, causes the entire system to collapse [51] (see Sect. 5.3). In networks of lattices, this fraction decreases rapidly and for $m \geq 15$ only 10 % of nodes need to be interdependent for the entire system to collapse after a single node is removed [64].

The extension of analytical results from random networks to spatially embedded networks is possible only for the case in which the dependency links are purely random ($r = \infty$). As mentioned above, two fully interdependent lattices undergo a first-order transition only when $r > r_c \approx 8$ [66, 70]. This requires nodes to be dependent on their eighth nearest neighbors, which may be unlikely for a real system. Shekhtman et al. [64] showed that r_c decreases significantly as n increases (for treelike networks) and as m increases (for random-regular networks) (Fig. 5.5). They further observed that for $m \geq 15$, q_{max} is approximately

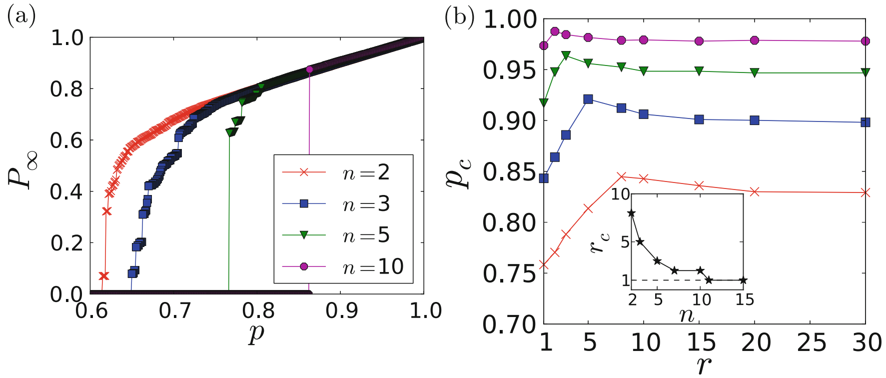

Fig. 5.5 Percolation of interdependent spatially embedded networks. (**a**) Here we observe that for fully dependent treelike NoNs with $r = 2$ the transition becomes first order as the number of networks increases. (**b**) The transition becomes first order where the p_c-r curve reaches a max. This occurs for smaller value of r as n increases. In the inset we show how r_c decreases as n increases (After [64])

independent of r. In this case, even systems with short dependency links (low r) and small fractions of dependent nodes q can collapse when only a single node is removed.

The model of spatially embedded interdependent networks was extended to the case where the ability to provide support to a node in another network requires dynamic functionality in the form of the flow of current and not just connectivity to the giant component. Process-based dependency leads to more vulnerable systems than structural dependency as described in other models. Also, the current-based model suggests that the ideas of interdependent networks can be utilized for new kinds of sensors [74].

Recently, spatially embedded interdependent networks have been modeled as multiplex networks with connectivity links of characteristic geographic length [75]. In this model, the connectivity links in each layer have lengths which are distributed exponentially. Instead of nodes in one network depending on nodes in another, each node has links in multiple networks and requires connectivity in each layer to function. This model exhibits first or second-order transitions, depending on the characteristic length of the connectivity links.

5.6 Attack and Defense of Interdependent Networks

Due to their startling vulnerabilities with respect to random failures, it is of particular interest to understand how non-random attacks affect interdependent networks and how to improve the robustness of interdependent networks through topological changes. Huang et al. [76] studied tunable degree-targeted attacks on interdependent networks. They found that even attacks which only affected low-degree nodes caused severe damage because high-degree nodes in one network can depend on low-degree nodes in another network. This framework was later expanded to general networks of networks [77].

Since high degree nodes in one network which depend on low degree nodes in another network can lead to extreme vulnerability, there have been several attempts to mitigate this vulnerability by making small modifications to the inter-network topology. Schneider et al. [78] demonstrated that selecting autonomous nodes by degree or betweenness can greatly reduce the chances of a catastrophic cascading failure. Valdez et al. have also obtained promising results by selecting a small fraction of high-degree nodes and making them autonomous [79]. These mitigation strategies are methodologically related to the intersimilarity/overlap studies discussed above [54, 60].

The theory of stochastic block models [80, 81] has been generalized to model interdependent networks and networks of networks. Using this framework, it was found that the optimal topological configuration which balances construction cost and robustness to random failure for random and interdependent networks is a core-periphery topology [82].

As we have seen, cascading failures are dynamic processes and the overall cascade lifetime can indeed be very long [40, 66]. The slowness of the process opens the door for "healing" methods allowing the dynamic recovery of failed nodes in the midst of the cascade. Recently, a possible healing mechanism along these lines has been proposed and analyzed [83].

When considering infrastructure or other spatially embedded networks, not only is the network embedded in space but failures are also expected to be geographically localized. For instance, natural disasters can disable nodes across all networks in a given area while EMP or biological attacks can disable the power grid or social network only in a given area. Geographically localized attacks of this sort have received attention in the context of single network percolation on specific networks [84] and flow-based cascading failures [85]. However, the existence of dependency between networks leads to surprising new effects. Recently, Berezin et al. [71] have shown that spatially embedded networks with dependencies can exist in three phases: stable, unstable and metastable (See Fig. 5.6). In the metastable phase, the system is robust with respect to random attacks–even if finite fractions of the system are removed. However, if all of the nodes within a critical radius r_h^c fail, it causes a cascading failure which spreads throughout the system and destroys it (See Fig. 5.6). Significantly, the value of r_h^c does not scale with system size and thus, in the limit of large systems, it constitutes a zero-fraction of the total system. A method of localized attacks on random networks was also recently studied in Shao et al. [86].

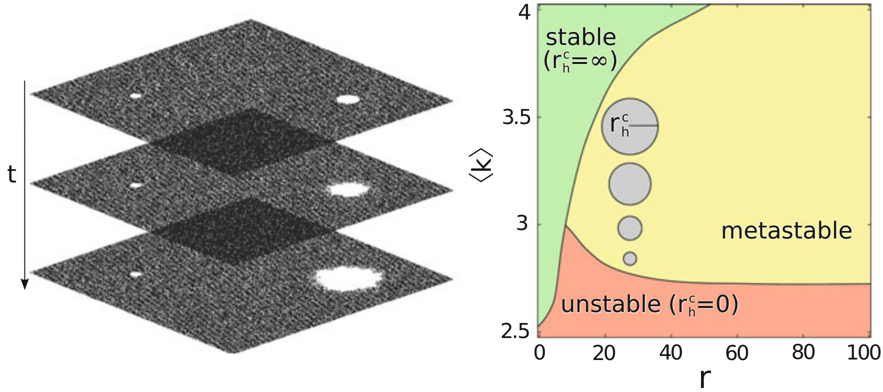

Fig. 5.6 Geographically localized attacks on interdependent networks. (**a**) The hole on the left is below r_h^c and stays in place while the hole on the right is larger than r_h^c and propagates through the system. (**b**) The phase space of localized attacks on interdependent networks. The increasing *gray circles* represent the dependence of r_h^c on $\langle k \rangle$ (Both after [71])

5.7 Applications of Networks of Networks

Many of the fields for which networks were seen as relevant models have been re-evaluated in light of the realization that interacting networks behave differently than single networks. Epidemics on interdependent and interconnected networks have received considerable attention [25, 87–90]. Economic networks composed of individuals, firms and banks all interact with one another and are susceptible to large scale cascading failures [91–93]. Interacting networks have also been found in physiological systems [94], ecology [95] and climate studies [96]. Recently, Reis et al. published an important step connecting interacting networks with fMRI measurements of brain activity [97]. Multilevel transportation networks have also been studied from the perspective of interacting networks [98]. Recently a framework for optimal recovery of interdependent networks was developed by Majdandzic et al. [99].

The breadth of applications of networks of networks is too great to address here and we refer the reader to recent reviews for more thorough treatment of applications [24, 35].

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