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MORPHISM AMONG NETWORKS OF NETWORKS[*†]

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Abstract. A network is a dominant technological structure of the present century (for example a World Wide Web graph, each web page is a node and each (hyper)link is a directed edge; communication tasks in distributed systems, the tasks themselves may be nested in a hierarchy; service primitives between adjacent protocol entities, etc.). All these networks are graphs with nodes and links only. We consider the models and examples for the next step: the different categorical concepts of a 'network of networks'. In particular: (a) each node is a host network (1-network or 1-graph) and super-links are analogous to a graph-functor, *i.e.* this is $(1,1)$-network; (b) 2-network where there are 2-links among 1-links. The general notion of network-morphism is proposed.

Streszczenie. Sieć obecnie jest dominującą strukturą technologiczną, jak na przykład, światowa pajęczyna-tkanina World Wide Web, znana jako www,

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geometry [Oziewicz & Vázquez Coutiño 2001].

We are interested in the category of directed graphs and their morphisms [Lawvere 1989]. The categorical aspects of graphs is unfamiliar outside of category theory. The value of mathematics for other subjects is that the category theory, invented by Eilenberg and Mac Lane in 1945, has developed *language* for the study of structures, language for the formulation of problems, language for development of methods of calculation and deduction, and a language for discovering and exploiting analogies between various inter-disciplinary fields of science. The attraction of category theory is that the same conceptual algebraic tool=language is applicable in a variety of multi-disciplinary scientific areas. Study of the structure (process) involves studying the pre-categories ($\equiv$ graphs) and pre-functors $\equiv$ graph-morphisms. The main idea is to consider pre-categories, pre-functors, pre-natural-transformations, pre-etc., as the most primitive initial ingredients, that could, and must be, developed *before* the category theory itself. We call such an idea a theory of 'multigraphs of multigraphs'. We consider the multigraphs of multigraphs as the best introduction to category theory itself, and as the fruitful and hopeful direction for further studies. For the most elementary introduction into theory of multigraphs-of-multigraphs we refer to forthcoming textbook in Spanish [Oziewicz & Velázquez-Quesada 2008].

A (meta)graph (not necessarily directed) consists of two collections, denoted respectively by Vertices or G_0, and Edges or G_1, together with two or three maps among these collections. An element of V is variously called vertex, agent, object, node, state, 0-cell. An element of E is variously called edge, arrow, link, transition, information channel, 1-cell. An element of G_0 is also said to be 0-morphism or 0-transition. An element of G_1 is said to be 1-morphism or 1-transition. Two maps (operations) are: domain/source $s : G_1 \to G_0$ and codomain/target $t : G_1 \to G_0$, [Topentcharov 1967; Burroni 1981; Mac Lane 1971, 1998],

$$\begin{tikzcd} V/G_0 & E/G_1 \arrow[l, bend right, "\text{source}"'] \arrow[l, bend left, "\text{target}"] \end{tikzcd}$$

Figure 1: A graph with s =source and t =target

For a cell $f \in G$, the value sf is said to be a domain or a source of f, and tf is a co-domain or target of f. Arrows/edges/links/1-cells can be weighted, labeled, and/or attribute-dependent.

There are many important examples of the simplest networks, $V \leftleftarrows E \simeq V \times V$. Such a network can be coined in analogy to names introduced in the category of groupoids [Brown 1981], as a pair network, or coarse network, or banal network.

The dynamic networks are growing in the discrete time, or expiring, death due to attack, accidents, or intrusions. The number of nodes is growing or passing away being isolated. A time-dependent out-degree of a cell $x \in G$ is a cardinality of a fiber $s^{-1}x$. An in-degree of x is a cardinality of $t^{-1}x$.

The web graph is a directed graph, as well as many other real-life graphs and real-world networks. There is no reliable way of converting the directed (web) graph into an undirected graph. For example, the information channel needs not to be symmetric and indirected.

Deng Hong-zhong et al. [2007], consider a dynamic network of agents, with the time-variable number of alive agents (agent-vertex is said to be dead if all his information channels are destroyed). The Deng dynamic network assume that each vertex-agent x possesses an attack-destruction probability $p(x)$, such that, $\log p(x) \simeq s^{-1}x \simeq t^{-1}x$, and that each agent x possesses also an another probability of escape. This allows to investigate the survivability of a dynamic network, both by analytic method, and by simulation. As a result of some cycles of the attack process, one or more agents are isolated from the network, some information channels are destroyed, network is disentangled. In a framework of the present paper, it would be desirable to see the Deng dynamic network, in terms of a morphism among graphs, *i.e.* in terms of a special kind of $(1,1)$-graph, see Dynamic network 5, and Figure 5.

For the present consideration, there is no harm in including also loop arrows for all nodes. An arrow/0-cell $\in G_1$ is said to be a loop if

$$\text{source(arrow)} = \text{target(arrow)}.$$

This means that exists an injective section $i : V/G_0 \hookrightarrow E/G_1$, such that $s \circ i = \text{id} = t \circ i$. If this is the case, a graph is said variously to be with-units, nested, reflexive, globular, one-sorted [Korczyński 2001, Korczyński et al. 2003, Oziewicz 2003].

Definition 1 (Globular graph). *Let $\{s,t\}$ be a two-letter (ordered) alphabet of a free semigroup $\{s,t\}^*$, $w \in \{s,t\}^*$ be the homogeneous word in this alphabet and $|w| \in \mathbb{N}$ be a length of w. A globular (directed) graph (a 1-graph) is defined by the globular relations*

$$\forall\, |w| = 1, \quad s \cdot w = w, \quad \& \quad t \cdot w = w. \tag{1}$$

gdzie każda strona jest węzłem, i każda (hiper)-więźń-ogniwo jest krawędzi skierowaną. Inne przykłady: zadania komunikacyjności w układach rozłożonych, takie zadania najczęściej są zagnieżdzone hierarchicznie, obsługa połącze pomiędzy sąsiadującymi pod-układami, i inne. Wszystkie takie sieci technologiczne są jedno-grafami, które posiadają jedynie węzły (to jest zero-graf) krawędzie (jedno-morfizmy). W niniejszej pracy rozpatrujemy modele matematyczne i przykłady następnego kroku złożoności: w ramach teorii kategori rozpatrujemy różnorodne koncepcje sieci-sieci i wielo-sieci złożonych z wielo-sieci. W szczególności: (a) Każdy węzeł jest gospodarzem własnej sieci (tc znaczy że każdy węzeł jest jedno-siecią lub jedno-grafem), natomiast krawędzie są analogiem endo-funktora w kategorii 1-grafów. Taką strukturę nazywamy $(1,1)$-grafem. (b) Dwu-sieć (2-graf, 2-sieć), gdzie są dwu-krawędzie (powierzchnie) między jedno-krawędziami. Proponujemy najogólniejsze pojęcie morfizmu między wielo-sieciami. Zaznaczamy, że grafy etykietowane (z etykietowanymi krawędziami), znane w literaturze jako wielografy (multigrafy) są automatycznie włączone w naszą ogólną teorię. Jednak nazwy 'wielograf' i 'wielograf-wielografów' w pracy niniejszej używamy w innym, szerszym znaczeniu, które jest zupełnie niezależne od etykietowania.

1. Network

The dominant technological structure of the present century is a network (a graph) like for example, the World Wide Web graph, Internet, neural networks, phone calls, linguistics networks, networks in natural languages, computer circuits, etc. See for example www.nd.edu/networks or the publications in http://www.internetmathematics.org, [Stark 1995, Turner 1997], etc. All these networks are directed or undirected graphs with the nodes and links (edges) only.

A graph is a category without a composition of edges, without composition of the information channels. Graphs convey more information than letters and lead to drawing pictures, which visualize and display the relationships between various operations, processes, organizations, functors, natural transformations, etc. A graph underlines a category as stressed already by Mac Lane [1971, 1998], Burroni [1981], and by Lawvere [1989]. Graphs are known as pre-categories and they provide a very useful concept in category theory, in computational science (integrating science with computation), in deformation theory, braided logic [Chávez Rodríguez et al. 2001], etc. We also found earlier an intriguing application of the graphical operad to categorical differential for example [Leinster 1998, Brown and Mosa 1999]), tri-categories [Power 1995], etc, versus theory of 2-categories [Power 1990], ..., n-categories [Steiner 1998, Bespalov 2005], and multi-groupoids [Simpson 1998]. At this stage it is not yet crystal clear how these structures are related to our concept of the $(n_1, n_2, \ldots, n_m)$-graph, for a finite or infinite ordered sequence of natural numbers, $n_i \in \mathbb{N}$. This paper is a pointer for clarifying these not trivial relations.

Is intriguing also the concept of V-category and V-graph introduced by Wolff, [Wolff 1974].

2. From Network to Multi-network

A multi-network G is a sequence of collections of i-cells/i-morphisms $\{G_i, i \in \mathbb{N}\}$, a sequence of surjective sources $\{s_i \equiv s|G_i\}$ and targets $\{t_i \equiv t|G_i\}$ as displayed on Figure .

$$\begin{tikzcd} G_0 & G_1 \arrow[l, bend right, "s"'] \arrow[l, bend left, "t"] & G_2 \arrow[l, bend right, "s"'] \arrow[l, bend left, "t"] & \cdots \arrow[l, bend right, "s"'] \arrow[l, bend left, "t"] \end{tikzcd}$$

Figure 2: Multi-graph, or ∞-graph, s =source and t =target

A 0-cell in G_0 is variously also called an object, 0-morphism, node; 1-cell - a morphism, 1-morphism, an arrow, a functor. A 2-cell is called a 2-morphism, morphism of morphism, natural transformation, etc.

A reflexive/globular 2-network, $s \circ i = \mathrm{id} = t \circ i$, is displayed on Figure .

$$\begin{tikzcd} G_0 \arrow[r, "i"'] & G_1 \arrow[l, bend right, "s"'] \arrow[l, bend left, "t"] \arrow[r, "i"'] & G_2 \arrow[l, bend right, "s"'] \arrow[l, bend left, "t"] \end{tikzcd}$$

Figure 3: Reflexive 2-network

Note suggested by anonymous Referee. Referee noted that the term 'multigraph' is used in the literature as 'a graph with labeled or etiquettes edges', and not as it is defined in the present paper on Figures -. Within the terminology used in the present paper, 'a graph with labeled edges' is a particular example of an 1-graph. Referee suggest to use another name, multi-dimensional graph, or higher-level graph. Our terminology 'multigraph', and in particular 2-graph on Figure , is motivated by terminology used in category theory, where multicategory means a multigraph (as it is used in the present paper) with an appropriate algebraic structure of partial associative and commuting compositions obeying the Godement rules. With respect to the Referee terminological suggestion, we wish to point out, that our formalism is not restricted to the single natural number sequence, 0-graph, 1-graph, 2-graph, 3-graph, ..., n-graph, where the suggested names, higher-dimensional graph, would be most appropriate. Whereas our primary aim is the multi natural number sequence, two-dimensional sequence, like, $(0,1)$-graph, $(1,1)$-graph, $(1,2)$-graph, $(1,1,1)$-graph, etc. Because each graph category, in our paper, is given by a finite sequence of natural numbers, and not by a single natural number, therefore we decided to avoid introduction of specific terminology for each case, and instead, we call every such graph as 'a multigraph of multigraphs'.

Type. A type of a cell $f \in G$ is an ordered pair $(sf, tf) \in G \times G$. By definition all 0-cells are of the same type $(\emptyset, \emptyset)$,

$$G \xrightarrow{\text{type}} G \times G.$$

Let $x, y \in G_i \times G_i$, then we denote by $G_i(x,y)$ the collection of all $(i+1)$-cells of the fixed type (x,y),

$$G_i(x,y) \equiv \{f \in G_{i+1}, \operatorname{type} f = (x,y)\} \quad \subset \quad G_{i+1}.$$

Therefore, by abuse of notation used in a category theory, a symbol of a graph=pre-category G (like a category symbol cat) denotes also a bi-map similar to internal hom-set bi-functor

$$\begin{aligned} G \times G \quad &\xrightarrow{\quad G \quad} \quad 2^G \cup \emptyset, \\ G_i \times G_i \ni x, y &\xrightarrow{\ G_i\ } G_i(x,y) \subset G_{i+1}, \\ z \in (G \circ \operatorname{type})z, &\qquad \operatorname{type} \circ G \equiv \operatorname{id}_{G \times G}. \end{aligned}$$

The following axiom is assumed: if $\operatorname{type} x \neq \operatorname{type} y$, then $G(x,y) = \emptyset$.

An alternative (better?) notation for such fiber would be $G \simeq \operatorname{type}^{-1}$ or $(s,t)^{-1}$.

Definition 2 (n-graph). *A multigraph G for which there exists $n \in \mathbb{N}$ such that $\forall\, i \in \mathbb{N}$, $G_{i+n} \simeq G_n$, is said to be an n-graph. Hence*

$$s_{i+n} = t_{i+n} \quad \textit{and} \quad i_{n+i} \circ s_{n+i} = \operatorname{id}_{G_{n+i+1}}.$$

A 0-graph is a synonym of a set/collection. A 1-graph is the same as a graph $V/G_0 \leftleftarrows E/G_1$. In the Street terminology [1998, p. 100] what we call an n-graph is said to be an n-skeletal globular object.

Definition 3 (Opposite multigraph). *Let G be n-graph. Then $\forall\, i \geq 1$, i-opposite n-graph G_i^{opp} is defined by 'reversing' i-cells, $s_i^{opp} = t_i$, $t_i^{opp} = s_i$, and unchanged j-cells for $j \neq i$, $s_{j\neq i}^{opp} = s_j$, $t_{j\neq i}^{opp} = t_j$.*

Definition 4 (Skeletal graph). *An n-graph $G \equiv \{G_i\}$, such that $\forall\, i \in \mathbb{N}$, $|G_i(x, y)| = 1$, $\forall\, x, y$, such that* type $x =$ type y, *is said to be a skeletal graph*.

3. Morphism among multigraphs-of-multigraphs

A 1-graph $G = (G_0 \leftleftarrows G_1)$ whose every 0-cell is a graph, and every 1-cell is a graph morphism, is said to be a graph of graphs. In particular, $(0, 1)$-graph is a collection of 1-graphs, and a 1-graph whose every 0-cell is a 1-graph and whose every 1-cell is a 1-graph morphism, is said to be a $(1, 1)$-graph. Therefore we interpret $(1, 0)$-graph as a more particular concept that a general 1-graph.

We begin this section by defining a morphism among two k-graphs, $k \in \mathbb{N}$. And then we show how this leads to the concept of $(1, k)$-graph. The cases of $(2, k)$-graph, and $(3, k)$-graph, are then considered in the separate sub-sections, in order to smoothly prepare the reader for accepting the rather general case of (m, n)-graph for $m, n \in \mathbb{N}$. The even more involved next step, (k, l, m)-graph, for arbitrary triple of natural numbers, $k, l, m \in \mathbb{N}$, is left as the exercise.

Let $E \equiv (E_0 \leftleftarrows E_1 \leftleftarrows \cdots \leftleftarrows E_k)$ and $F \equiv (F_0 \leftleftarrows F_1 \leftleftarrows \cdots \leftleftarrows F_k)$ be two k-graphs. A k-graph morphism $f : E \to F$, $f \in \text{graph}(E, F)$, is a collection of functions $f_i : E_i \to F_i$, intertwining the source s and the target t maps (in this case a morphism is said to be totally covariant),

$$s \circ f = f \circ s \quad \text{and} \quad t \circ f = f \circ t,$$

$$\begin{array}{ccccc} E_0 & \leftleftarrows & E_1 & \leftleftarrows & E_2 \dots \\ \downarrow f_0 & & \downarrow f_1 & & \downarrow f_2 \\ F_0 & \leftleftarrows & F_1 & \leftleftarrows & F_2 \dots \end{array}$$

Figure 4: Totally covariant k-morphism

Analogously for $n \in \mathbb{N}$ we define a globular n-graph by the following globular relations in the free monoid $\{s,t\}^*$,

$$\forall\, |w| = n \in \mathbb{N}, \quad s \cdot w = w, \quad \& \quad t \cdot w = w. \tag{2}$$

Note. One can recover a nested graph, $V \leftleftarrows E$, from a globular graph by setting $V \equiv \operatorname{im} s = \operatorname{im} t$. A nested graph is a graph with an injection

$$V/G_0 \overset{\iota}{\hookrightarrow} E/G_1, \tag{3}$$

$$\begin{tikzcd} V/G_0 \arrow[r, "\iota"'] & E/G_1 \arrow[l, bend right, "s"'] \arrow[l, bend left, "t"] \end{tikzcd} \quad \text{such that} \quad s \circ \iota = \mathrm{id} = t \circ \iota. \tag{4}$$

1.1. Aim of this Note

It was observed by Bharat et al. [2001], that a web-graph is a 2-level structure. We propose to consider the models and examples for the next step: the different categorical concepts of a 'network of networks' or an hierarchical network. For $m \in \mathbb{N}$ we are introducing a concept of a m-network, and moreover for $m, n \in \mathbb{N}$ a fine concept of (m,n)-network. In particular, our aim is to elucidate the differences and similarities among the following algebraic hierarchical structures.

(a) Each node is a usual (connected) host network (1-network or 1-graph) and super-links are analogous to a graph-functor. Such structure we call $(1,1)$-network or $(1,1)$-graph, or 'graph of host-graphs'.

(b) 2-network (2-graph) where there are 2-links among specified 1-links of the same type.

We consider a graph, denoted by 1-graph, graph of graphs denoted by $(1,1)$-graph, multi-graph denoted by m-graph, multi-graph of multi-graphs denoted by (m,n)-graph, and iterations, $(n_1,\dots,n_k)$-graph for $\forall\, n_i \in \mathbb{N}$. The main conceptual and technical result of this note is summarized on Figure 7.

Each of the above network structure can be a carrier for a categorical structure for the appropriate non-unique associative and mutually commutative partial compositions, known as the Godement rules, or as the interchange law, [Marcinek and Oziewicz 2001].

We must stress that since the lectures by [Bénabou 1967], there are extensive studies in the theory of double-categories (invented by Bénabou, see

3.1. 2-graph of k-graphs $\equiv$ $(2,k)$-graph

Let G be a 2-graph such that every 0-cell is a k-graph and every 1-cell is a k-graph morphism. A $(0,2)$-graph is a collection G_0 whose every element is a 2-graph. A $(2,0)$-graph is not the same as a 2-graph, *i.e.* a 2-graph $G_0 \leftleftarrows G_1 \leftleftarrows G_2$ whose every 0-cell in G_0 is a 0-graph$\simeq$set, 1-cells are mappings and 2-cells are operators.

We need to define 2-cell as a k-graph natural transformation. The following definition generalizes the notion of the natural transformation of (pre)functors from the case 2-graph of $(k=1)$-graphs invented by Eilenberg and MacLane [1945], to the case of arbitrary natural number $k \in \mathbb{N}$.

Let $f, g \in G_0(E,F) \subset G_1$. This means

$$G_0(E,F)(f,g) \quad \subset \quad G_1(f,g) \quad \subset \quad G_2.$$

A k-graph natural transformation $t \in \operatorname{nat}(f,g) \equiv G_1(f,g) \subset G_2$, is a collection of $(i+1)$-cells-valued map intertwining f with g,

$$\begin{tikzcd}
\cdots & E_i \arrow[l, shift left] \arrow[l, shift right] \arrow[d, shift left] \arrow[d, shift right] \arrow[dr, "t"] & E_{i+1} \arrow[l, shift left] \arrow[l, shift right] \arrow[d, shift left] \arrow[d, shift right] \arrow[dr, "t"] & E_{i+2} \arrow[l, shift left] \arrow[l, shift right] \arrow[d, shift left] \arrow[d, shift right] \arrow[dr, "t"] & \cdots \\
\cdots & F_i \arrow[l, shift left] \arrow[l, shift right] & F_{i+1} \arrow[l, shift left] \arrow[l, shift right] & F_{i+2} \arrow[l, shift left] \arrow[l, shift right] & \cdots
\end{tikzcd}$$

$$\begin{aligned} E_i \ni x &\xmapsto{t} tx \in F_i(fx, gx) \subset F_{i+1}, \\ \forall a \in E_i(x,y) \subset E_{i+1}, &\quad (ga)\circ(tx) = (ty)\circ(fa) \quad \in F_{i+1}. \end{aligned} \tag{6}$$

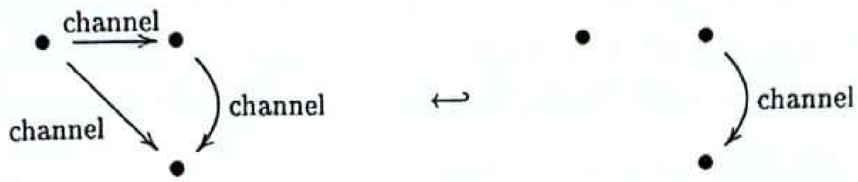
Figure 6: A k-graph natural transformation

The inter-twining condition (6) is referred as the natural condition.

Let s_F denote F-source map. Then $s_F \circ t = f = s_G t$. Therefore s_G is given be push-forward, $s_G = (s_F)_*$.

The k-graph natural transformation $t \in \operatorname{nat}(f,g)$ for the case of a 'graph of 1-graphs' is analogous to a natural transformation of functors. A natural transformation $t \in \operatorname{nat}(f,g)$ can also be illustrated by means of a commutative

5 Dynamic network. Consider $(1,1)$-graph, where 0-cells (networks of agents) are labeled by discrete time, $t \in \mathbb{N}$, $\ldots, \hookleftarrow N_t \hookleftarrow N_{t+1} \hookleftarrow, \ldots$, and each 1-cell is an attack process that split a network N_t into a sum (categorical coproduct) N_{t+1}. In this case, when an object split into a sum of objects, there is an injective morphism from the result of an attack, N_{t+1} into a domain of an attack N_t, as it is illustrated on Figure 5.



(5)

Figure 5: On the right one agent, with empty set of communication channels, is dead after an attack

Deng Hong-zhong et al. [2007], consider a dynamic network of agents, under the attack process, and study the survivability of such dynamic network. We propose to see the Deng dynamic network, as an example of $(1,1)$-graph, Figure 5. This allow to extend conceptually the dynamic network, to dynamic network of networks.

A graph-morphism can also be coined as a pre-functor. One can ask: what is a good notion of adjunction, of an adjoint pre-functor? An adjunction of a functor is a basic categorical concept due to Kan [1958]. The graph-morphisms $f \in \text{graph}(E,F)$ and $g \in \text{graph}(F,E)$ are said to be mutually adjoint (to be in adjunction) if for $\forall i \in \mathbb{N}$ there is a bijection of collections of cells, this is analogous to the Definition given by MacLane [1998, p. 80],

$$E \underset{f}{\overset{g}{\rightleftarrows}} F, \qquad F_i(fx,y) \simeq E_i(x,gy).$$

Set $E, F \in G_0$ to be 0-cells, and $\text{graph}(E,F) \subset G_1$, then $G \equiv (G_0 \leftleftarrows G_1)$ is a graph of k-graphs, that we denote by $(1,k)$-graph. This means that every 0-cell of G is a k-graph and every 1-cell of $(1,k)$-graph G is a k-graph morphisms, $f \in G_0(E,F) \equiv \text{graph}(E,F) \subset G_1$.

A k-graph modification $\mu \in G_2(s,t)$ is a collection of $(i+2)$-cells $\{\mu x \in F_{i+1}(sx,tx), \forall x \in E_i\}$, intertwining s with t, such that $\forall i \in \mathbb{N}$ the following equations hold

$$\left.\begin{array}{l}(ga)\circ(sx)=(sy)\circ(fa),\\(gb)\circ(sx)=(sy)\circ(fb),\\(ga)\circ(tx)=(ty)\circ(fa),\\(gb)\circ(tx)=(ty)\circ(fb).\end{array}\right\}\quad\Longleftrightarrow\quad s,t\in\mathrm{nat}(f,g),$$
$$(g\alpha)\circ(\mu x)=(\mu y)\circ(f\alpha)\quad\Longleftrightarrow\quad \mu\in\mathrm{modi}(s,t). \tag{7}$$

Here we have the general sequence of multi-morphisms,

$$\begin{aligned}G_0(E,F)(f,g)(s,t)(\mu,\nu)\quad&\subset\quad G_1(f,g)(s,t)(\mu,\nu)\\&\subset\quad G_2(s,t)(\mu,\nu)\quad\subset\quad G_3(\mu,\nu)\quad\subset\quad G_4.\end{aligned}\tag{8}$$

A 'modification of modifications' $A \in G_3(\mu,\nu) \subset G_4$ is a collection of $(i+3)$-cells-valued map $\{Ax \in F_{i+2}(\mu x, \nu x)\}$, intertwining μ with ν, such that $\forall i \in \mathbb{N}$, analogous for (7), certain set relations hold.

3.3. m-graph of n-graphs $\equiv$ (m,n)-graph

Let G be a m-graph such that every 0-cell, $E, F \in G_0$, is an n-graph, every 1-cell $f \in G_0(E,F) \subset G_1$ is a n-graph morphisms, every 2-cell $t \in G_1(f,g) \subset G_2$, where $f, g \in G_0(E,F)$, is n-graph natural transformation,

$$G_0(E,F)(f,g)\quad\subset\quad G_1(f,g)\quad\subset\quad G_2,$$

every 3-cell is a n-graph modification, every 4-cell is a n-graph modifications of modifications, etc. Then an m-graph G is said to be an 'm-graph of n-graphs' or (m,n)-graph.

To be more specific, one can ask: what must be a notion of a k-cell $\mu \in G_k$ for every $k \leq m$ in a given (m,n)-graph G? The previous considerations allow the following concluding concept. A k-cell $\mu \in G_k$ must be a collection of $(i+k-1)$-cells in every n-graph $F \in G_0$ for $i+k-1 \leq n$,

Evidently, for the maps on Figure 7 must hold the relations analogous to equation (7).

The above concept of a network of networks, or (m,n)-graph, needs a notion of a (m,n)-(hyper)morphism of (m,n)-graphs. We will not consider this in detail in the present note.

Let $m,n,k \in \mathbb{N}$ be natural numbers. Let G be an m-graph such that every 0-cell is a n-graph of k-graphs. Then G is said to be a (m,n,k)-graph.

$$\forall \mu_k \in G_k, \quad E_i \xrightarrow{\mu_k \in G_k} F_{i+k-1},$$

$$\begin{tikzcd}
\ldots \leftleftarrows E_i \leftleftarrows \ldots \arrow[d, "\mu_1 \in G_1"'] \arrow[drr, "\mu_2 \in G_2"'] \arrow[drrrr, "\mu_k \in G_k"] & & & & & \in G_0 \\
\ldots \leftleftarrows F_i \leftleftarrows & & F_{i+1} \leftleftarrows \ldots & & \leftleftarrows F_{i+k-1} \leftleftarrows & \in G_0
\end{tikzcd}$$

Figure 7: An (m, n)-graph $G \equiv (G_0 \leftleftarrows G_1 \leftleftarrows \ldots \leftleftarrows G_m)$.

Every 1-cell in G_1 must be a morphism ($\simeq (n, k)$-functor) of (n, k)-graphs. By iteration one can define a most subtle structure, namely an $(n_1, \ldots, n_k)$-graph.

4. Conclusion

The study of the categorical aspects of networks leads to new insights. A category is a reflexive directed graph with additional multiplication [Mac Lane 1971, 1998, Lawvere 1989]. An n-graph is a carrier of n-category and there is a forgetful functor from n-category to his n-graph. An n-graph as a carrier can be also called an n-pre-category. A skeletal n-graph admits exactly the unique structure of a free n-category.

The notion of a graph of graphs underlines the theory of n-categories. There has been extensive research in pure mathematics on this subject, see for example [Baez 1997, Batanin 1998, Simpson 1998, Leinster 2000]. A 'category of categories' can be seen as an extra structure over graph of graphs; an n-category is an n-graph equipped with the partial compositions of cells, or even with the grafting of cells in case a family of 0-cells is a strict monoid [Oziewicz & Vázquez Coutiño 2001, Oziewicz 2003].

Another important example is an n-groupoid, defined as an n-category with an inversion [Brown 1981, 1992; Simpson 1998]. We are proposing a structure of the $(1, 1)$-groupoid to replace the Lorentz relativity group, in the special relativity theory [Oziewicz 2007].

In the present note, we have developed the conceptual notion of a graph of graphs, that can be used for refinement of the hierarchical structures and/or 'network of networks'. In particular, we believe that a conceptual hierarchical structure 'graph of graphs' can be useful for description of the networks where the links are attribute-dependent and therefore are natural 'higher' links between attributes leading to multi-level systems, [Obtułowicz 2001; Korczyński, Cruz and Oziewicz 2003; Cruz and Oziewicz 2004]. Consider for example: OSI allows the possibility of interface data units to transfer fragments of a service data unit across a service; interface data units might be regarded as an implementation matter and thus not at the same level of abstraction as a service data unit, etc. [Cruz et al. 2006, 2007].

Besides it is hard to believe that real-life technological structures must be reduced to networks without hierarchical structures. There is an open phenomenological problem to investigates the concrete examples of the networks of networks in interdisciplinary sciences and in technology.

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diagram with pull-back and push-forward,

$$\begin{tikzcd} E(x,y) \arrow[r, "f"] \arrow[d, "g"'] & F(fx,fy) \arrow[d, "(ty)_*"] \\ F(gx,gy) \arrow[r, "(tx)^*"] & F(fx,gy) \end{tikzcd}$$

Therefore, $G_0(E,F) \subset G_1$, can be seen as an 1-graph (= pre-functor graph): vertices are the morphisms of graphs, $f,g \in \mathrm{graph}(E,F)$, and edges are natural transformations $\mathrm{nat}(f,g) \subset G_2$.

3.2. 3-graph of k-graphs $\equiv$ $(3,k)$-graph

Eilenberg and MacLane when in 1945 introduced the category theory, a concept of a category ($\simeq$ 0-cell as a 1-graph) motivated by a need to define a functor ($\simeq$ 1-cell), and a functor as a necessary concept in order to define a natural transformation ($\simeq$ 2-cell). Bénabou in 1967, 22 years later on, found that these concepts need the next step and introduced a notion of a modification ($\simeq$ 3-cell). In this subsection we are generalizing a notion of the Bénabou modification (for 3-graph of 1-graphs) to the case of 3-graph of k-graphs for arbitrary natural number $k \in \mathbb{N}$.

We wish to define a graph, $G = (G_0 \leftleftarrows G_1 \leftleftarrows G_2 \leftleftarrows G_3)$, as a 3-graph of k-graphs, *i.e.* as $(3,k)$-graph. This means that every 0-cell, $E,F \in G_0$, is a k-graph. Assume that 1-cells are the 'k-functors' $f,g \in G_0(E,F)$, and 2-cells $s,t \in G_1(f,g) \simeq \mathrm{nat}(f,g)$ are 'k-transformations', as defined in Figure 6. We need to define explicitly a 3-cell, $\mu \in G_2(s,t) \subset G_3$, as a k-graph modification.

A k-graph modification $\mu \in \mathrm{modi}(s,t) \equiv G_2(s,t) \subset G_3$, is a collection of $(i+2)$-cells-valued map, intertwining s with t,

$$\begin{gathered} E_i \ni x \overset{\mu}{\longmapsto} \mu x \in F_{i+1}(sx,tx) \subset F_{i+2}, \\ \forall\, \alpha \in E_i(a,b), \quad (g\alpha)\circ(\mu x) = (\mu y)\circ(f\alpha) \in F_{i+1}. \end{gathered}$$

To be more explicit let us fix the following notation, for every $i \in \mathbb{N}$,

$$\begin{gathered} a,b \in E_i(x,y), \\ \alpha \in E_i(x,y)(a,b) \quad \subset \quad E_{i+1}(a,b) \quad \subset \quad E_{i+2}. \end{gathered}$$

This gives the following images in a k-graph $F \in G_0$,

$$\begin{array}{rcl} \text{surfaces:} & fx,\ gx,\ fy,\ gy & \in F_i, \\ \text{arrows:} & sx, tx, sy, ty, fa, fb, ga, gb & \in F_{i+1}, \\ \text{nodes:} & \mu x,\ \mu y,\ f\alpha,\ g\alpha & \in F_{i+2}. \end{array}$$