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Unit 4: Linear Transformation (LT)

Let V and U are vector space, then the relation $T: V \longrightarrow U$ said to be linear transformation

if ① for all $u, v \in V$ (domain) $T(u+v) = T(u) + T(v)$

② for all $u \in V$ (domain), for all $\alpha \in R$

$$T(\alpha u) = \alpha T(u)$$

NOTE: If $V = U$ in above definition then it called linear operator.

Matrix Representation (MR) of Linear transformation (LT)

Let $T: V \longrightarrow U$ be LT, $\dim V = n$ $\dim U = m$

then order of MR of LT is $\boxed{m \times n}$

Regular (non-singular) and irregular (Singular) LT

Let, $T: V \longrightarrow U$ be LT and 'A' be MR of LT

then ① 'T' is regular iff $|A| \neq 0$.

∴ If 'T' is regular then T^{-1} exist and $T^{-1} = A^{-1}$

② 'T' is irregular if $|A| = 0$

∴ If 'T' is irregular then T^{-1} does not exist.

Q. Which of the following are Linear transformation and also write down matrix representation and Hence decide regular or irregular.

1) $T: R^2 \longrightarrow R^3$ $T(x,y) = (x+y, x, 0)$

⇒ Check for LT,

① $u = (x_1, y_1) \in V = R^2$

$v = (x_2, y_2) \in V = R^2$

$T(u+v) = T((x_1, y_1) + (x_2, y_2))$

$= T(x_1 + x_2, y_1 + y_2)$

$T(u+v) = (x_1 + x_2 + y_1 + y_2, x_1 + x_2, 0)$ — ①

$$T(u)+T(v) = T(x_1,y_1) + T(x_2,y_2)$$
$$= (x_1+y_1, x_1, 0) + (x_2+y_2, x_2, 0)$$
$$= (x_1+y_1+x_2+y_2, x_1+x_2, 0+0) \quad -②$$

① = ②

$$T(u+v) = T(u) + T(v)$$

② $\alpha \in R$, $u=(x,y) \in V=R^2$

$$T(\alpha u) = T(\alpha(x,y)) = T(\alpha x, \alpha y)$$
$$= (\alpha x+\alpha y, \alpha x, 0)$$
$$= \alpha(x+y, x, 0) = \alpha T(x,y)$$
$$= \alpha T(u)$$

∴ 'T' is LT.

2) $T: R^3 \to R^2$

$$T(x,y,z) = (x+2y, z)$$

⇒ ① $u=(x_1,y_1,z_1) \in V=R^3$

$v=(x_2,y_2,z_2) \in V=R^3$

$$T(u+v) = T((x_1,y_1,z_1)+(x_2,y_2,z_2))$$
$$= T(x_1+x_2, y_1+y_2, z_1+z_2)$$
$$= ((x_1+x_2)+2(y_1+y_2), z_1+z_2) \quad -①$$

$$T(u)+T(v) = T(x_1,y_1,z_1) + T(x_2,y_2,z_2)$$
$$= (x_1+2y_1, z_1) + (x_2+2y_2, z_2)$$
$$= (x_1+2y_1+x_2+2y_2, z_1+z_2) \quad -②$$

① = ②, ∵ $T(u+v) = T(u)+T(v)$

② $\alpha \in R$, $u=(x,y,z) \in V=R^3$

$$T(\alpha u) = T(\alpha(x,y,z)) = T(\alpha x, \alpha y, \alpha z)$$
$$= (\alpha x + 2\alpha y, \alpha z)$$
$$= \alpha(x+2y, z) = \alpha T(x,y,z) = \alpha T(u)$$

∴ 'T' is LT

If Domain = codomain then it is linear operator.

3) $T: R^2 \to R^2$

$T(x, y) = (x, y)$

$\Rightarrow$ ① $u = (x_1, y_1) \in V = R^2$

$v = (x_2, y_2) \in V = R^2$

$T(u+v) = T((x_1, y_1) + (x_2, y_2)) = T(x_1 + x_2, y_1 + y_2)$

$= (x_1 + x_2, y_1 + y_2)$ — ①.

$T(u) + T(v) = T(x_1, y_1) + T(x_2, y_2)$

$= (x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2)$ — ②.

① = ② $\therefore T(u+v) = T(u) + T(v)$.

② $\alpha \in R$, $u = (x, y) \in V = R^2$

$T(\alpha u) = T(\alpha(x, y)) = T(\alpha x, \alpha y)$

$= (\alpha x, \alpha y) = \alpha(x, y) = \alpha T(x, y)$

$= \alpha T(u)$.

$\therefore$ 'T' is LT.

4) $T: R^2 \to R^2$ $T(x, y) = (x, y+1)$

$\Rightarrow$ 1) $u = (x_1, y_1) \in V = R^2$

$v = (x_2, y_2) \in V = R^2$

$T(u+v) = T((x_1, y_1) + (x_2, y_2)) = T(x_1 + x_2, y_1 + y_2)$

$= (x_1 + x_2, y_1 + y_2 + 1)$ — ①

$T(u) + T(v) = T(x_1, y_1) + T(x_2, y_2)$

$= (x_1, y_1 + 1) + (x_2, y_2 + 1)$

$= (x_1 + x_2, y_1 + y_2 + 2)$. — ②.

① $\neq$ ②

$\therefore$ 'T' is not LT.

(5M)

Q2) Write down Matrix Representation of given LT and Hence check whether it is Regular. If Yes, then find Inverse of given LT.

1) $T: R^2 \to R^2$ $\quad T(x,y) = (x+y,\ 2x+3y)$

$\Rightarrow$ $V \quad U$

$\dim V = 2, \quad \dim U = 2$

$\because$ order of MR of LT is $\dim U \times \dim V \Rightarrow 2\times2$ matrix.

i.e 'A' is 2×2 matrix.

$\therefore$ MR of LT is $'A' = \begin{bmatrix} 1 & 1 \\ 2 & 3 \end{bmatrix} \begin{matrix} \to x+y \\ \to 2x+3y \end{matrix}$

OR

Std. Basis of $V = R^2$ is $\{(1,0), (0,1)\}$
$\qquad e_1 \quad e_2$

$T(e_1) = T(1,0) = (1+0,\ 2(1)+3(0)) = (1,2)$

$T(e_2) = T(0,1) = (0+1,\ 2(0)+3(1)) = (1,3)$

$\therefore$ MR of LT is $A = \begin{bmatrix} 1 & 1 \\ 2 & 3 \end{bmatrix}$

$|A| = 1 \neq 0$. $\therefore$ 'T' is regular (non-singular)

$\therefore$ 'T' is Bijective (one-one and onto)

$\therefore T^{-1}$ exist

$\therefore T^{-1} = A^{-1} = \frac{1}{|A|}\begin{bmatrix} 3 & -1 \\ -2 & 1 \end{bmatrix} = \begin{bmatrix} 3 & -1 \\ -2 & 1 \end{bmatrix}_{2\times2}$ $(m\times n)$

$\therefore T^{-1}: R^2 \to R^2$

$T^{-1}(x,y) = (3x-y,\ -2x+y)$

2) $T: R^2 \to R^2$ $\quad T(x,y) = (x+y,\ 0)$

$\Rightarrow$ $\dim V = 2, \quad \dim U = 2$

order of MR of LT is 2×2 matrix.

$\therefore$ MR of LT is 'A' = $\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$

$|A| = 0$. $\therefore$ 'T' is irregular (singular)

$\therefore$ 'T' is not Bijective

$\therefore$ T^{-1} does not exist.

3) $T: R^3 \to R^2$ $\quad T(x, y, z) = (x+y, y-z)$

$\Rightarrow$ dim V = 3, dim U = 2

Order of MR of LT is dim U × dim V

i.e 'A' is 2×3 matrix

dim U ≠ dim V
↳ irregular.

$\therefore$ MR of LT is $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & -1 \end{bmatrix}_{2\times3}$

$\therefore$ |A| does not exist

'A' is irregular.

$\therefore$ T^{-1} does not exist.

NOTE: $\boxed{1}$ Dimension of ~~MR~~ domain is not equal to dimension of codomain then 'T' is always irregular.

$\boxed{2}$ MR is LT and LT is MR

If we have to check any property of LT then check same property of MR.

Kernal of LT

let $T: V \to U$ be LT, 'A' be MR of LT

then

$\text{Ker}(T) = \{ v \in V \text{ (domain)} \mid T(v) = e \}$ → AX=0 system

Where, 'e' is additive identity of codomain.

i.e. ker(T) = We have to find element of domain whose image is additive identity of codomain.

If $\dim = 0$ then Basis $= \phi$.

NOTE:

① $\ker(T)$ is subspace of domain

② $\ker(T) = \text{Null}(A)$, where 'A' is MR of LT.

③ $\dim(\ker(T))$ = no. of free variable

or $\dim V - \rho(A)$

④ Basis $(\ker(T))$ = Basis $(\text{Null}(A))$

Image or Range of LT

let $T: V \to U$ be LT, 'A' be MR of LT

then, $\text{Im}(T) = \left\{ u \in U \text{ (codomain)} \;\middle|\; \begin{array}{l} \text{There exist } v \in V \\ \text{such that } T(v) = u \end{array} \right\} \to AX = B$

i.e $\text{Im}(T)$ = We have to find element of codomain which has preimage in domain.

i.e $\text{Im}(T) = \text{Col}(A)$

NOTE: ① $\text{Im}(T)$ is subspace of codomain.

② $\dim(\text{Im}(T)) = \rho(A)$

③ Basis $(\text{Im}(T))$ = Basis $(\text{Col}(A))$

Q. find dimension and Basis of kernal and Image.

$T: R^2 \to R^3 \quad T(x, y) = (x+y, x-y, 3y)$

$\Rightarrow \ker(T) = \{ v \in V \mid T(v) = e \}$

$\ker(T) = \{ v = (x,y) \in V = R^2 \mid T(x,y) = (0,0,0) \}$

$\ker(T) = \{ (x,y) \mid (x+y, x-y, 3y) = (0,0,0) \}$ $AX = 0$

$\ker(T) = \left\{ (x,y) \;\middle|\; \begin{array}{l} x+y=0 \\ x-y=0 \\ 3y=0 \end{array} \right\}$ $AX = 0$

$m = 3$

$n = 2$

$$[A|B] = \left[\begin{array}{cc|c} 1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 3 & 0 \end{array}\right]$$

$R_2 - R_1$ $$[A|B] = \left[\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & -2 & 0 \\ 0 & 3 & 0 \end{array}\right]$$

$2R_3 + 3R_2$ $$[A|B] = \left[\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 0 \end{array}\right] \rightarrow \text{REF.}$$

$\rho(A) = 2$, $\rho(A|B) = 2$, $n = 2$

$AX = 0$ have trivial solⁿ.

$\therefore x = 0,\ y = 0$.

$\therefore \ker(T) = \{(0,0)\}$

① $\ker(T)$ is subspace of domain $V = R^2$

② $\dim(\ker(T)) = n - \rho(A) = 2 - 2 = 0$.

③ Basis of $\ker(T) = \phi$

Now, $Im(T) = col(A)$, 'A' is MR of LT

$$A = \begin{bmatrix} 1 & 1 \\ 1 & -1 \\ 0 & 3 \end{bmatrix}$$ convert 'A' into REF

$R_2 - R_1$ $$A = \begin{bmatrix} 1 & 1 \\ 0 & -2 \\ 0 & 3 \end{bmatrix}$$

$2R_3 + 3R_2$ $$A = \begin{bmatrix} 1 & 1 \\ 0 & -2 \\ 0 & 0 \end{bmatrix} \rightarrow \text{REF}$$

$\rho(A) = 2$

① $Im(T)$ is subspace of codomain $U = R^3$

② $\dim(Im(T)) = \rho(A) = 2$

③ Basis of $Im(T) = \{(1,1,0)\ (1,-1,3)\}$

Q. Find Dimension and Basis of kernal and Image.

$T: R^2 \to R^2 \quad T(x,y) = (x+y, \ 2x+2y)$.

$\Rightarrow \ker(T) = \{ v \in V \mid T(v) = e \}$

$\ker(T) = \{ v = (x,y) \in V = R^2 \mid T(x,y) = (0,0) \}$

$\ker(T) = \{ (x,y) \mid (x+y, \ 2x+2y) = (0,0) \}$ $\quad AX=0$

$\ker(T) = \left\{ (x,y) \ \middle| \ \begin{matrix} x+y=0 \\ 2x+2y=0 \end{matrix} \right\}$ $\quad AX=0$, $m=2$, $n=2$

$$[A|B] = \left[\begin{array}{cc|c} 1 & 1 & 0 \\ 2 & 2 & 0 \end{array}\right]$$

$R_2 \to R_2 - 2R_1$

$$[A|B] = \begin{array}{c} \begin{array}{cc} x & y \end{array} \\ \left[\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right] \end{array} \to \text{REF}$$

$\rho(A) = 1, \quad \rho(A|B) = 1$

$n = 2$. $\quad AX=0$ have

$\rho(A) = \rho(A|B) < n$ (Infinitely many non-zero soln)

Put, $y = t$ $\quad$ (x is LE, y is free)

$x+y=0 \Rightarrow x+t=0$

$x = -t, \ y = t$.

$\therefore \ker(T) = \{ (-t, t) \mid t \in R \}$

① $\ker(T)$ is subspace of domain, $V = R^2$

② $\dim(\ker(T)) = n - \rho(A) = 2 - 1 = 1$

③ Basis of $\ker(T) = \{(-1, 1)\}$.

Now, $Im(T) = col(A)$

$A = \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}$ $\quad$ Convert 'A' into REF

$R_2 \to R_2 - 2R_1$ $\quad A = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \to$ REF

$\rho(A) = 1$

① $Im(T)$ is subspace of codomain $U = R^2$

② $\dim(Im(T)) = \rho(A) = 1$

③ Basis of $Im(T) = \{(1,2)\}$

$T : R^2 \rightarrow R^2$

$T(x,y) = (0,0)$

$\ker(T) = \{ v \in V \mid T(v) = e\}$

$\ker(T) = \{ v = (x,y) \in V = R^2 \mid T(x,y) = (0,0)\}$

$\ker(T) = \{(x,y) \mid (0,0) = (0,0)\}$ $AX = 0$.

$\ker(T) =$

$$\left\{ (x,y) \;\middle|\; \begin{matrix} 0x + 0y = 0 \\ 0x + 0y = 0 \end{matrix} \right\}$$

$AX = 0$

$m = 2$

$$[A \mid B] = \left[\begin{array}{cc|c} 0 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right] \longrightarrow \text{REF}$$

$\rho(A) = 0$, $\rho(A|B) = 0$, $n = 2$

$\rho(A) = \rho(A|B) < n$. (Infinitely many non-zero sol^n)

x, y are free

$x = t_1$, $y = t_2$

$\ker(T) = \{(t_1, t_2) \mid t_1, t_2 \in R\}$

$\ker(T) = R^2$

$$\begin{bmatrix} t_1 \\ t_2 \end{bmatrix} = t_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + t_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

① $\ker(T)$ is subspace of domain $V = R^2$

② $\dim(\ker(T)) = n - \rho(A) = 2$

③ Basis of $\ker(T) = \{(1,0)\ (0,1)\}$

Now, $Im(T) = col(A)$

$$A = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

$\rho(A) = 0$

① $Im(T)$ is subspace of codomain, $U = R^2$

② $\dim(Im(T)) = \rho(A) = 0$

③ Basis of $Im(T) = \phi$

$Im(T) = \{(0,0)\}$

$T: R^2 \to R^2 \qquad T(x,y) = (x,y)$

$ker(T) = \{ v \in V \mid T(v) = e \}$

$ker(T) = \{ (x,y) \mid (x,y) = (0,0) \}$ $\quad AX = 0$

$ker(T) = \left\{ (x,y) \;\middle|\; \begin{array}{l} x + 0y = 0 \\ 0x + y = 0 \end{array} \right\}$ $\quad AX = 0, \ m = 2, \ n = 2$

$[A|B] = \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \end{array}\right] \rightarrow REF$

$\rho(A) = 2, \quad \rho(A|B) = 2, \quad n = 2.$

$AX = 0$ have trivial soln.

$\therefore \ x = 0, \ y = 0$

$ker(T) = \{(0,0)\}$

① $ker(T)$ is subspace of domain $V = R^2$

② $\dim(ker(T)) = n - \rho(A) = 2 - 2 = 0$

③ Basis of $ker(T) = \phi$

Now, $Im(T) = col(A)$

$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$\rho(A) = 2$

① $Im(T)$ is subspace of codomain, $U = R^2$

② $\dim(Im(T)) = \rho(A) = 2$

③ Basis of $Im(T) = \{(1,0)\ (0,1)\}$

$Im(T) = R^2$

Orthogonal Matrix:

Let 'A' be $n\times n$ matrix then 'A' is orthogonal matrix iff ① $A^T = A^{-1}$

or ② $AA^T = I_{n\times n}$ and $A^TA = I_{n\times n}$

NOTE: ① If 'A' is orthogonal matrix then $|A| = \pm 1$ but not conversly.

Orthogonal LT:

Let $T: V \to V$ be LT, 'A' be MR of LT then 'T' is orthogonal LT iff 'A' is orthogonal matrix.

Composition of LT:

Let $T_1 : V \to U$ and $T_2 : U \to W$ both are LT

and $\dim V = n$, $\dim U = m$, $\dim W = p$

Then $T_1 . T_2$ and $T_2 . T_1$ also LT if defined.

Let 'A' be MR of T_1 and 'B' be MR of T_2

Then MR of $T_1 . T_2 = AB$ and MR of $T_2 . T_1 = BA$

* One to one and onto LT

let $T: V \to U$ be LT, then

① 'T' is one to one if $\ker(T) = \{e\}$

i.e $\dim(\ker(T)) = 0$

Where, 'e' is additive identity of domain.

② 'T' is onto if $Im(T) = $ Codomain
i.e. $\dim(Im(T)) = \dim(\text{codomain})$

* Rank Nullity Theorem

Let, $T: V \rightarrow U$ be LT
$\dim V = n$; $\dim U = m$
'A' be MR of LT
Then,

$$\dim V = \dim(Im(T)) + \underbrace{\dim(\ker(T))}_{\text{Nullity}}$$

$n = \rho(A) +$ no. of free variable.

NOTE:
In term of matrix Rank Nullity Theorem is
Let 'A' be $m \times n$ matrix.
Then, $n = \rho(A) +$ no. of free variables.

Q. Which of the following are orthogonal Matrix.

1) $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}_{2\times2}$

$\Rightarrow A^T = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $A^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ $\therefore A^T = A^{-1}$

$\therefore$ 'A' is orthogonal matrix.

OR

$AA^T = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I_{2\times2}$

$A^TA = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I_{2\times2}$

$AA^T = A^TA = I_{2\times2}$ $\therefore$ 'A' is orthogonal matrix

2) $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$

$\Rightarrow$ ① $A^T = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}$, $A^{-1} = \frac{-1}{2}\begin{bmatrix} 4 & -2 \\ -3 & 1 \end{bmatrix}$, $A^T \neq A^{-1}$.

OR ② $AA^T = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix} = \begin{bmatrix} 5 & 11 \\ 11 & 25 \end{bmatrix} \neq I$

OR

③ We know that, If 'A' is orthogonal matrix then its $|A| = \pm 1$.

let, 'A' is orthogonal matrix,

$|A| = 4 - 6 = -2 \neq \pm 1$.

$\therefore$ our assumption is wrong

$\therefore$ 'A' is not orthogonal matrix.

(solve any one of above).

Q. check whether given LT is orthogonal.

$T: R^2 \to R^2 \quad T(x, y) = (x+y, 0)$

$\Rightarrow$ 'T' is orthogonal iff it's MR is orthogonal.

MR of LT or 'A' = $\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$.

$AA^T = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix} \neq I_{2\times 2}$.

$\because$ 'A' is not orthogonal matrix.

$\therefore$ 'T' is not orthogonal LT.

Q. Write down Matrix representation of $T_1 . T_2$ and $T_2 . T_1$ if defined and Hence decide whether given composition is regular and orthogonal.

$f: A \to B$, $g: B \to C$

$f.g$ $\leftarrow$ 'g' from $B \to C$ and 'f' from $A \to B$.

'f' starts from 'C' then only it is defined.

$\therefore$ f.g is not defined.

$g.f$ $\leftarrow$ similarly, 'f' from $A \to B$ and 'g' from $B \to C$

'g' starts from 'B' that is, it is defined.

$\therefore$ g.f is defined.

$T_1 : R^2 \to R^3$ $\quad T_1(x,y) = (x+y,\ x-y,\ x)$

$T_2 : R^3 \to R^2$ $\quad T_2(x,y,z) = (x+y,\ y-z)$

$\Rightarrow$ $T_1 . T_2 : R^3 \to R^3$

$$(T_1 . T_2)(x,y,z) = T_1(T_2(x,y,z))$$
$$= T_1(x+y,\ y-z)$$
$$= (x+y+y-z,\ x+y-y+z,\ x+y)$$
$$(T_1 . T_2)(x,y,z) = (x+2y-z,\ x+z,\ x+y)$$

$T_1 . T_2 : R^3 \to R^3$

$$T_1 . T_2(x,y,z) = (x+2y-z,\ x+z,\ x+y)$$

$\therefore$ MR of $T_1 . T_2 = \begin{bmatrix} 1 & 2 & -1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}_{3\times 3}$

$\therefore$ MR of T_1 is $A = \begin{bmatrix} 1 & 1 \\ 1 & -1 \\ 1 & 0 \end{bmatrix}_{3\times 2}$

MR of T_2 is $B = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & -1 \end{bmatrix}_{2\times 3}$

$\therefore$ MR of $T_1 . T_2$ is $AB = \begin{bmatrix} 1 & 1 \\ 1 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & -1 \end{bmatrix}$

$$= \begin{bmatrix} 1 & 2 & -1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

$|AB| = 1(-1) - 2(-1) - 1(1) = -1 + 2 - 1 = 0$

$\therefore$ $T_1 . T_2$ is irregular but not orthogonal.

Q. For the given LT write down MR, check whether it is one-one onto or bijective. And Hence ~~Null~~ verify Rank Nullity theorem.

$T: R^3 \to R^2$ $\quad T\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -x-y-2z \\ -x+2y+3z \end{bmatrix}$

⇒ Matrix representation is, $\quad \dim V = 3, \ \dim U = 2$

$A = \begin{bmatrix} -1 & -1 & -2 \\ -1 & 2 & 3 \end{bmatrix}$ $\quad$ (n, m)

order $= m \times n = 2 \times 3$

$A = \begin{bmatrix} -1 & -1 & -2 \\ -1 & 2 & 3 \end{bmatrix}$

$R_2 \to R_2 - R_1$ $\quad A = \begin{bmatrix} -1 & -1 & -2 \\ 0 & 5 & 5 \end{bmatrix}$ $\quad \rho(A) = 2$.

It is irregular. $(\because |A| \neq 0)$.

or. $\dim V \neq \dim U$ then it is always irregular.

We know that, given LT is one-one iff $\dim(\text{kernal}) = 0$

$\therefore$ by rank Nullity theorem, $\dim(\text{domain}) = \dim(\text{Image}) + \dim(\ker)$

$\therefore \quad 3 = 2 + \dim(\text{kernal})$

$\dim(\text{kernal}) = 1$

$\therefore$ 'T' is not one-one.

'T' is onto iff $\dim(\text{Image}) = \dim$ ~~(kernal)~~ (codomain)

$2 = 2$

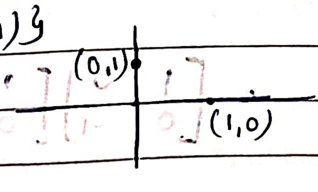
$\therefore$ 'T' is onto

Geometric Linear Transformations in R^2.

* Reflection:

→ Type 1: Reflection about x-axis.

Std basis of R^2: $\{(1,0)\ (0,1)\}$

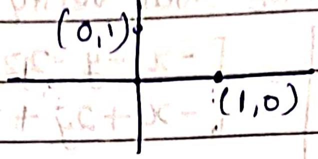


$(1,0) \to (1,0)$ $\quad \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

$(0,1) \to (0,-1)$

→ Type 2: Reflection about Y-axis.

$\{(1,0)\ (0,1)\}$



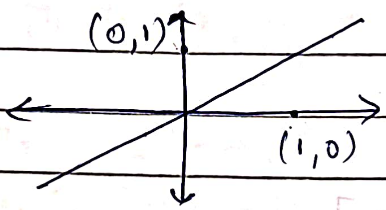
$(1,0) \to (-1,0)$

$(0,1) \to (0,1)$

$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$

→ Type 3: Reflection about the line $y = x$.

$\{(1,0)\ (0,1)\}$



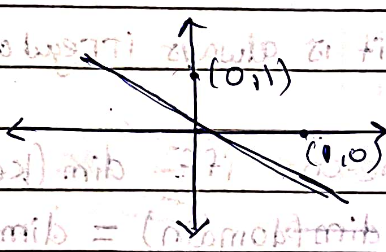
$(1,0) \to (0,1)$

$(0,1) \to (1,0)$

$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

→ Type 4: Reflection about the line $y = -x$.

$\{(1,0)\ (0,1)\}$



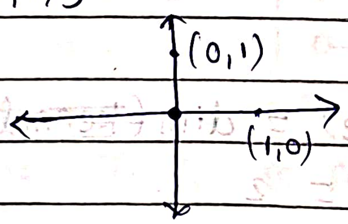
$(1,0) \to (0,-1)$

$(0,1) \to (-1,0)$

$\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$

→ Type 5: Reflection through origin.

$\{(1,0)\ (0,1)\}$



$(1,0) \to (-1,0)$

$(0,1) \to (0,-1)$

$\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$

Q. S: square with vertices $\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

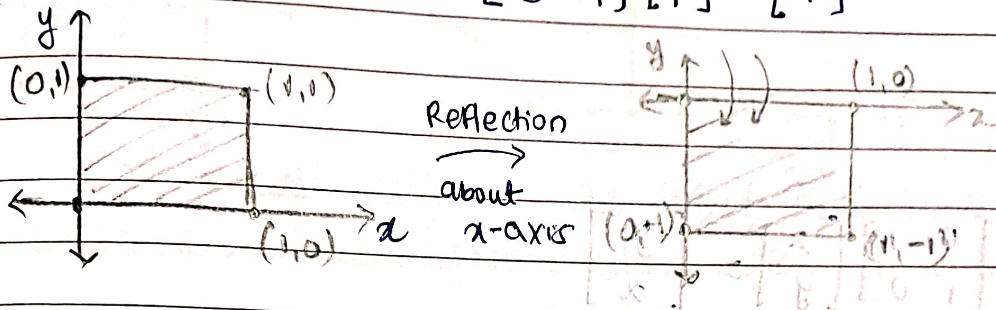
⇒ 1) Reflection about X-axis.

$A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$,

$$\begin{bmatrix}1 & 0\\0 & -1\end{bmatrix}\begin{bmatrix}0\\1\end{bmatrix}=\begin{bmatrix}0\\-1\end{bmatrix}, \quad \begin{bmatrix}1 & 0\\0 & -1\end{bmatrix}\begin{bmatrix}1\\1\end{bmatrix}=\begin{bmatrix}1\\-1\end{bmatrix}$$

Given,



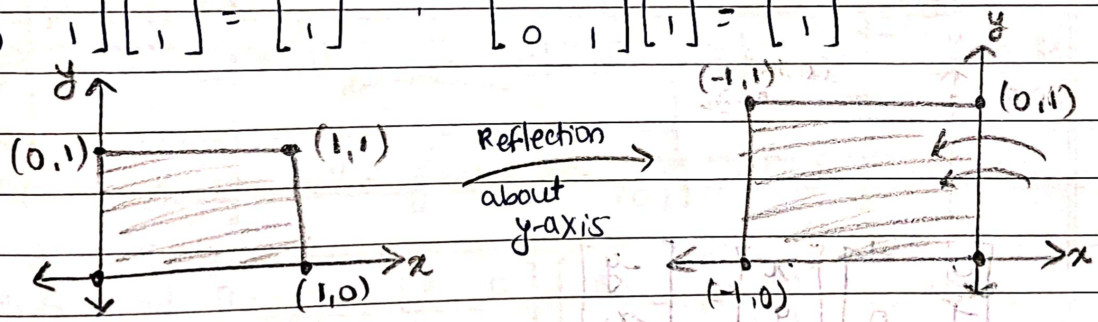
$$\begin{bmatrix}1 & 0\\0 & -1\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix}=\begin{bmatrix}x\\-y\end{bmatrix}$$

2) Reflection about y-axis: $A=\begin{bmatrix}-1 & 0\\0 & 1\end{bmatrix}$

$$\begin{bmatrix}-1 & 0\\0 & 1\end{bmatrix}\begin{bmatrix}0\\0\end{bmatrix}=\begin{bmatrix}0\\0\end{bmatrix}, \quad \begin{bmatrix}-1 & 0\\0 & 1\end{bmatrix}\begin{bmatrix}1\\0\end{bmatrix}=\begin{bmatrix}-1\\0\end{bmatrix}$$

$$\begin{bmatrix}-1 & 0\\0 & 1\end{bmatrix}\begin{bmatrix}0\\1\end{bmatrix}=\begin{bmatrix}0\\1\end{bmatrix}, \quad \begin{bmatrix}-1 & 0\\0 & 1\end{bmatrix}\begin{bmatrix}1\\1\end{bmatrix}=\begin{bmatrix}-1\\1\end{bmatrix}$$

Given



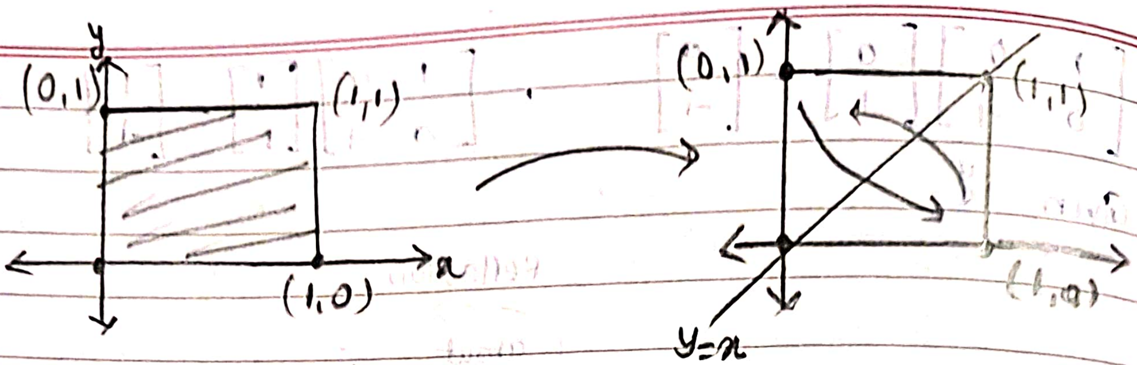
$$\begin{bmatrix}-1 & 0\\0 & 1\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix}=\begin{bmatrix}-x\\y\end{bmatrix}$$

3) Reflection about the line $y=x$ $A=\begin{bmatrix}0 & 1\\1 & 0\end{bmatrix}$

$$\begin{bmatrix}0 & 1\\1 & 0\end{bmatrix}\begin{bmatrix}0\\0\end{bmatrix}=\begin{bmatrix}0\\0\end{bmatrix}, \quad \begin{bmatrix}0 & 1\\1 & 0\end{bmatrix}\begin{bmatrix}1\\0\end{bmatrix}=\begin{bmatrix}0\\1\end{bmatrix}$$

$$\begin{bmatrix}0 & 1\\1 & 0\end{bmatrix}\begin{bmatrix}0\\1\end{bmatrix}=\begin{bmatrix}1\\0\end{bmatrix}, \quad \begin{bmatrix}0 & 1\\1 & 0\end{bmatrix}\begin{bmatrix}1\\1\end{bmatrix}=\begin{bmatrix}1\\1\end{bmatrix}$$

Given,



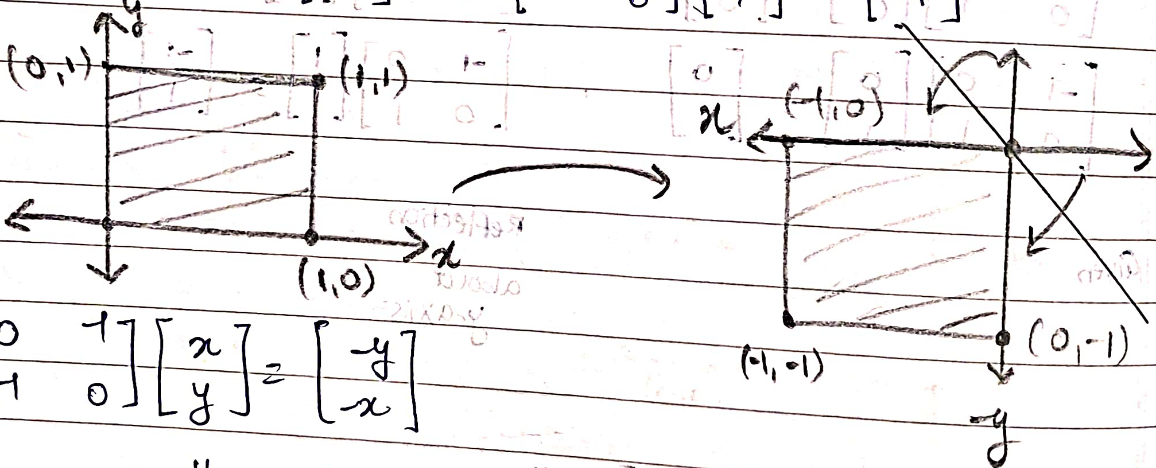
$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} y \\ x \end{bmatrix}$$

4) Reflection about line, $y=-x$ $A = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$

$$\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$$

$$\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \end{bmatrix}$$

Given,

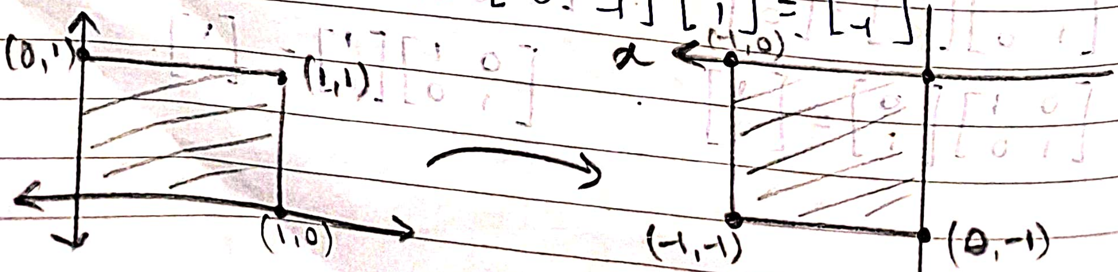


$$\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -y \\ -x \end{bmatrix}$$

5) Reflection ~~about~~ through origin. $A = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$

$$\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \end{bmatrix}, \quad \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \end{bmatrix}$$



$$\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -x \\ -y \end{bmatrix}$$

* Scaling.

A transformation on an object that results in contraction or dilation (stretching) is called a "scaling".

1) Horizontal scaling (in X-dir^n) is $A = \begin{bmatrix} K & 0 \\ 0 & 1 \end{bmatrix}$, $K>1$

2) Vertical scaling (in Y-dir^n) is $A = \begin{bmatrix} 1 & 0 \\ 0 & K \end{bmatrix}$, $K>1$

3) Scaling in Both dir^n 'X' and 'Y' is $A = \begin{bmatrix} K & 0 \\ 0 & K \end{bmatrix}$, $K>1$

4) Compression $(0<K<1)$ in 'X' dir^n is $A = \begin{bmatrix} K & 0 \\ 0 & 1 \end{bmatrix}$, $0<K<1$

5) Compression $(0<K<1)$ in 'Y' dir^n is $A = \begin{bmatrix} 1 & 0 \\ 0 & K \end{bmatrix}$, $0<K<1$

Note: 'K' is called 'factor'

→ •) Horizontal scaling. $\begin{bmatrix} K & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} Kx \\ y \end{bmatrix}$, $K>1$.

•) Vertical scaling $\begin{bmatrix} 1 & 0 \\ 0 & K \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ Ky \end{bmatrix}$, $K>1$

•) Scaling in both dir^n 'X' & 'Y', $\begin{bmatrix} K & 0 \\ 0 & K \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} Kx \\ Ky \end{bmatrix}$, $K>1$.

•) Compression $(0<K<1)$ in 'X' dir^n $\begin{bmatrix} K & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} Kx \\ y \end{bmatrix}$, $0<K<1$

•) Compression $(0<K<1)$ in 'Y' dir^n $\begin{bmatrix} 1 & 0 \\ 0 & K \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ Ky \end{bmatrix}$, $0<K<1$

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* **Shearing:** The transformation which produces the visual effect of slanting is called 'Shearing'.

1) Shearing in X-dirⁿ $A = \begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix}$, $k \in R$.

$$\begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x + ky \\ y \end{bmatrix}$$

2) Shearing in Y-dirⁿ, $A = \begin{bmatrix} 1 & 0 \\ k & 1 \end{bmatrix}$, $k \in R$

$$\begin{bmatrix} 1 & 0 \\ k & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ kx + y \end{bmatrix}$$

* **Rotation:**
The transformation which rotates given a point in R^2 through an angle 'θ' is called Rotation.

1) Rotation matrix for anticlockwise by angle 'θ'. is
$$A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

2) Rotation Matrix for clockwise by angle 'θ' is
$$A = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

* **Projection:**
1) Projection on x-axis is $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$
2) Projection on y-axis is $\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$

Find the matrix of the transformation $T: R^3 \to R^2$ which produces the effect of shear of factor 3 along +ve y-dirⁿ followed by reflection about $y = -x$ followed by clockwise rotation through an angle 30°. Hence find the image of $u = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ under the transformation T.

Shear factor 3, along +ve y-dirⁿ, $A = \begin{bmatrix} 1 & 0 \\ k & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix}$

Reflection about $y = -x$, $B = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$

clockwise rotation through 30°, $C = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} = \begin{bmatrix} \sqrt{3}/2 & 1/2 \\ -1/2 & \sqrt{3}/2 \end{bmatrix}$

Required transformation is CBA =

$$\begin{bmatrix} \sqrt{3}/2 & 1/2 \\ -1/2 & \sqrt{3}/2 \end{bmatrix} \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix} = \begin{bmatrix} \sqrt{3}/2 & 1/2 \\ -1/2 & \sqrt{3}/2 \end{bmatrix} \begin{bmatrix} -3 & -1 \\ -1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} -\frac{3\sqrt{3}}{2} - \frac{1}{2} & -\sqrt{3}/2 \\ \frac{3}{2} - \frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix}$$

Image of $u = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ is

$$CBAu = \begin{bmatrix} \frac{-3\sqrt{3}-1}{2} & \frac{-\sqrt{3}}{2} \\ \frac{3-\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -\frac{\sqrt{3}}{2} \\ 1/2 \end{bmatrix}$$ → Image of u. //