Unit 5

Eigen Value And Eigen Vector

Let 'A' be $n \times n$ matrix, then the number $\boxed{\lambda \in R}$, said to be eigen value of 'A' if there exist non-zero vector $v \in R^n$ such that $\boxed{Av = \lambda v}$ $(v \neq 0)$

let, $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}_{2\times 2}$ $\therefore n = 2$

characteristic eq^n of 'A' is $\Delta(\lambda) = \lambda^2 - \lambda(\text{Trace } A) + \det(A) = 0$ (Trace: add^n of diagonal element)

i.e. $(\Delta\lambda) = \lambda^2 - \lambda(\text{Trace } A) + \det(A) = 0$

$= \lambda^2 - \lambda(a+d) + (ad - bc) = 0$ — ①

roots of eq^n ① are called eigen values of 'A'.

let, λ_1 and λ_2 are eigen value of 'A'.

Now, Eigen vector V_i of 'A' corresponding to Eigen value λ_i is basis of null space

$$\left((A - \lambda_i)\, I_{2\times 2}\right)\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \qquad (A - \lambda_i I_{2\times 2})\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

NOTE:

Characteristic equation also denoted by $|A - \lambda I| = 0$

Q. find eigen value and eigen vector of $A = \begin{bmatrix} 3 & -4 \\ 2 & -6 \end{bmatrix}_{2\times 2}$

⇒ $n = 2$

characteristic eq^n of 'A' is

$\Delta\lambda = \lambda^2 - \lambda(\text{Trace } A) + \det(A) = 0$

$\Delta\lambda = \lambda^2 - \lambda(3 + (-6)) + (-18 + 8) = 0$

$= \lambda^2 + 3\lambda - 10 = 0$

$\lambda_1 = -5$, $\lambda_2 = 2$ are eigen value of 'A'.

Eigen vector v_1 of 'A' corresponding to eigen value $\lambda_1 = -5$ is basis of null space $(A - \lambda_i) I_{2\times 2}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

$$\left(\begin{bmatrix}3 & -4\\2 & -6\end{bmatrix} - (-5)\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}\right)\begin{bmatrix}x\\y\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix}$$

$$\Rightarrow \begin{bmatrix}8 & -4\\2 & -1\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix} \rightarrow \underline{AX=0} \text{ system.}$$

$$[A|B] = \left[\begin{array}{cc|c}8 & -4 & 0\\2 & -1 & 0\end{array}\right]$$

$R_2 \rightarrow 4R_2 - R_1$ $\quad [A|B] = \left[\begin{array}{cc|c}8 & -4 & 0\\0 & 0 & 0\end{array}\right] \rightarrow$ REF

$\rho(A) = 1$, $\rho(A|B) = 1$, $n = 2$

$\therefore \rho(A) = \rho(A|B) < n$ (Infinitely many non-zero solutions)

Always we get infinitely many non-zero solⁿ. If we get trivial solⁿ that means we make any mistake in calculⁿ.

'y' is free

put, $y = t$. from REF, $8x - 4y = 0$

$\Rightarrow 8x = 4t \Rightarrow x = t/2$

$$\begin{bmatrix}x\\y\end{bmatrix} = \begin{bmatrix}t/2\\t\end{bmatrix} = t\begin{bmatrix}1/2\\1\end{bmatrix}$$

$\therefore v_1 = \begin{bmatrix}1/2\\1\end{bmatrix}$ or $v_1 = \begin{bmatrix}1\\2\end{bmatrix}$ is eigen vector of 'A' corresponding to $\lambda_1 = -5$

Now, eigen vector v_2 of 'A' corresponding to eigen value $\lambda_2 = 2$ is basis of null space $(A - \lambda_2 I_{2\times2})\begin{bmatrix}x\\y\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix}$

$$\left(\begin{bmatrix}3 & -4\\2 & -6\end{bmatrix} - 2\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}\right)\begin{bmatrix}x\\y\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix}$$

$$\begin{bmatrix}1 & -4\\2 & -8\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix} \quad AX=0$$

$$[A|B] = \left[\begin{array}{cc|c}1 & -4 & 0\\2 & -8 & 0\end{array}\right]$$

$R_2 \rightarrow R_2 - 2R_1$ $\quad [A|B] = \left[\begin{array}{cc|c}1 & -4 & 0\\0 & 0 & 0\end{array}\right] \rightarrow$ REF

$\rho(A) = 1$, $\rho(A|B) = 1$, $n = 2$

$\rho(A) = \rho(A|B) < n$ (Infinitely many non-zero solⁿ)

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'y' is free,

Put, $y = t$. From REF, $x - 4y = 0 \Rightarrow x - 4t = 0$

$x = 4t$.

$\therefore \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 4t \\ t \end{bmatrix} = t\begin{bmatrix} 4 \\ 1 \end{bmatrix}$

$\therefore v_2 = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$ is eigen vector of 'A' corresponding to $\lambda_2 = 2$.

$\therefore$ Eigen values of given matrix are $\lambda_1 = -5$ & $\lambda_2 = 2$ and Eigen vectors are $v_1 = \begin{bmatrix} 1/2 \\ 1 \end{bmatrix}$ or $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ & $v_2 = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$

$A = \begin{bmatrix} 5 & -1 \\ 1 & 3 \end{bmatrix}_{2\times 2}$ $n = 2$.

Characteristics eqn of 'A' is,

$\Delta\lambda = \lambda^2 - \lambda(\text{Trace } A) + \det(A) = 0$

$= \lambda^2 - \lambda(8) + (15+1) = 0 \Rightarrow \lambda^2 - 8\lambda + 16 = 0$

$\lambda_1 = 4, \lambda_2 = 4$ are eigen values of 'A'.

Eigen vector v of 'A' corresponding to eigen value $\lambda_1 = \lambda_2 = 4$ is basis of null space $(A - (\lambda_1 = \lambda_2) I_{2\times 2})\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

Shortcut: Non-diagonal elements as it is and on place of diagonal elements $\rightarrow$ diagonal element $-$ corresp. eigen value.

$\begin{bmatrix} 5-4 & -1 \\ 1 & 3-4 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \longrightarrow AX = 0$.

$[A|B] = \left[\begin{array}{cc|c} 1 & -1 & 0 \\ 1 & -1 & 0 \end{array}\right]$

$R_2 \rightarrow R_2 - R_1$ $[A|B] = \left[\begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right] \rightarrow$ REF.

$\rho(A) = 1$, $\rho(A|B) = 1$, $n = 2$.

$\rho(A) = \rho(A|B) < n$. (Infinite non-zero soln) 'y' is free.

Put, $y=t$ from REF, $x-y=0$
$\Rightarrow x=t$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} t \\ t \end{bmatrix} = t\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$v = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is eigen vector of 'A' corresponding to $\lambda = 4$.

Q. $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $n=2$.

$\Rightarrow$ characteristic eq^n of A is

$\Delta\lambda = \lambda^2 - \lambda(1+1) + (1-0) = 0$

$\Rightarrow \lambda^2 - 2\lambda + 1 = 0$

$\lambda^2 - \lambda - \lambda + 1 = 0 \Rightarrow \lambda(\lambda-1) - 1(\lambda-1) = 0$

$\lambda_1 = 1$ and $\lambda_2 = 1$ are eigen values of 'A'.

Eigen vector V of 'A' corresponding to eigen value $\lambda_1 = \lambda_2 = 1$ is basis of null space $(A - \lambda I_{n\times n})\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$$\left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - 1\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$[A|B] = \left[\begin{array}{cc|c} 0 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right] \rightarrow \text{REF}.$$

$\rho(A) = 0$, $\rho(A|B) = 0$, $n=2$.

Infinite many non-zero sol^n.

x, y both free.

Put, $x = t_1$ and $y = t_2$.

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} t_1 \\ t_2 \end{bmatrix} = t_1\begin{bmatrix} 1 \\ 0 \end{bmatrix} + t_2\begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$v_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $v_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ are eigen vectors of 'A' corresponding to $\lambda_1 = \lambda_2 = 1$.

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NOTE: 1) If λ_1 and λ_2 are distinct then
$n =$ eigen values = eigen vectors.

2) If λ_1 and λ_2 are similar then,
$n =$ eigen values but eigen vectors $\leq n$.
(observe the previous 3 questions)

Q. $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}_{2\times2}$ $n=2$

$\Rightarrow$ characteristic eqn of 'A' is

$\Delta\lambda = \lambda^2 - \lambda(1+1) + (1-1) = 0.$

$= \lambda^2 - 2\lambda + 0 = 0$, $\Rightarrow \lambda^2 - 2\lambda = 0$

$\lambda(\lambda-2) = 0 \Rightarrow \lambda_1 = 0, \lambda_2 = 2.$

$\lambda_1 = 0, \lambda_2 = 2$ are eigen values of 'A'.

Eigen vector 'v_1' of 'A' corresponding to eigen value $\lambda_1 = 0$ is basis of null space $(A - \lambda_1 I_{2\times2})\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\left(\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} - 0\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\Rightarrow \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$[A|B] = \left[\begin{array}{cc|c} 1 & 1 & 0 \\ 1 & 1 & 0 \end{array}\right]$ $R_2 \to R_2 - R_1$

$[A|B] = \left[\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right] \to$ REF.

$\rho(A) = 1$, $\rho(A|B) = 1$, $n = 2$

Infinite many non-zero soln.

'y' is free. put, $y = t$

from REF, $x + y = 0.$

$\Rightarrow x = -t.$

$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -t \\ t \end{bmatrix} = t\begin{bmatrix} -1 \\ 1 \end{bmatrix}$. $\therefore v_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ is eigen vector of 'A' corresponding to $\lambda_1 = 0$.

Eigen vector v_2 of 'A' corresponding eigen value $\lambda_2 = 2$ is basis of Null space $(A - \lambda_2 I_{2\times2})\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$$\left(\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} - 2\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$[A|B] = \left[\begin{array}{cc|c} -1 & 1 & 0 \\ 1 & -1 & 0 \end{array}\right]$$

$R_2 \to R_2 + R_1$ $\quad [A|B] = \left[\begin{array}{cc|c} -1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right] \to$ REF.

$\rho(A) = 1$, $\rho(A|B) = 1$, $n = 2$.

Infinite many non-zero solⁿ.

y = free. Put, $y = t$.

from REF, $-x + y = 0$.

$x = t$.

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} t \\ t \end{bmatrix} = t\begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

$\Rightarrow v_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is eigen vector of 'A' correspondi[ng] to $\lambda_2 = 2$

Q. $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ $\quad n = 2$.

$\Rightarrow$ Characteristic eqⁿ of 'A' is

$$\Delta\lambda = \lambda^2 - \lambda(0+0) + 0 = 0.$$

$\lambda^2 = 0$ $\quad \Rightarrow \lambda_1 = 0$, $\lambda_2 = 0$. are eigen values of 'A'.

Eigen vector v of 'A' corresponding to eigen value $\lambda_1 = \lambda_2 = 0$ is basis of null space

$$\left(\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} - 0\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

$$\Rightarrow \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$[A|B] = \left[\begin{array}{cc|c} 0 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right] \rightarrow REF.$

x, y are free. Put, $x = t_1$ and $y = t_2$.

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} t_1 \\ t_2 \end{bmatrix} = t_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + t_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$\therefore v_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $v_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ are eigen vector of 'A' corresponding to $\lambda_1 = \lambda_2 = 0$

Eigen Value and Eigen vector of 3x3 matrix:

Let, $A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}_{3\times3}$ $n=3$

Characteristic eq^n of 'A' is $\Delta(\lambda) = $

$$\lambda^3 - \lambda^2 (\text{Trace}(A)) + \lambda \left\{ \underbrace{\begin{vmatrix} e & f \\ h & i \end{vmatrix}}_{\text{Cofactor of } a_{11}} + \underbrace{\begin{vmatrix} a & c \\ g & i \end{vmatrix}}_{\text{Cofactor of } a_{22}} + \underbrace{\begin{vmatrix} a & b \\ d & e \end{vmatrix}}_{\text{Cofactor of } a_{33}} \right\} - \det(A) = 0$$

root of these eq^n are eigen value of 'A'.

$\therefore$ let $\lambda_1, \lambda_2, \lambda_3$ are eigen value of 'A'.

Now, eigen vector v_i of 'A' corresponding to eigen value λ_i is basis of null space of

$$(A - \lambda_i I_{3\times3}) \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

det of diagonal matrix = Product of diagonal element.

Q. find eigen value and eigen vector of $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}_{3\times3}$

$\Rightarrow$ $n = 3$.

Characteristic eqn of 'A',

$\Delta(\lambda) = \lambda^3 - \lambda^2(1+2+3) + \lambda\left\{\begin{vmatrix} 2 & 0 \\ 0 & 3 \end{vmatrix} + \begin{vmatrix} 1 & 0 \\ 0 & 3 \end{vmatrix} + \begin{vmatrix} 1 & 0 \\ 0 & 2 \end{vmatrix}\right\}$

$- (1\times2\times3) = 0.$

$\Delta(\lambda) = \lambda^3 - 6\lambda^2 + \lambda\{6+3+2\} - 6 = 0$

$\Rightarrow \lambda^3 - 6\lambda^2 + 11\lambda - 6 = 0.$

$\lambda_1 = 1$, $\lambda_2 = 2$, $\lambda_3 = 3$ are eigen values of 'A'.

Eigen vector v_1 of 'A' corresponding eigen value $\lambda_1 = 1$ is basis of Null $\cancel{(A)}$ $(A - \lambda_1 I_{3\times3})\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

Non-diagonal elements as it is. subtract corresponding eigen value from each diagonal element.

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$[A|B] = \left[\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \end{array}\right]$$

$R_2 \leftrightarrow R_1$ and $R_2 \leftrightarrow R_3$ $[A|B] = \left[\begin{array}{ccc|c} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \rightarrow$ REF.

'x' is free. let, $x = t$, then $y = 0$, $z = 0$.

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} t \\ 0 \\ 0 \end{bmatrix} = t\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$\therefore v_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ is eigen vector of 'A' corres. to $\lambda_1 = 1$.

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Eigen vector v_2 of 'A' corres. eigen value $\lambda_2 = 2$ is basis of Null $(A - \lambda_2 I_{3\times3})\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$$\left(\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} - 2\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right)\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix},$$

$$\Rightarrow [A|B] = \left[\begin{array}{ccc|c} -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right] \Rightarrow R_2 \leftrightarrow R_3 \Rightarrow [A|B] = \left[\begin{array}{ccc|c} -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

REF $\rightarrow$

'y' is free, Put $y = t$. then $x = 0$, $z = 0$.

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ t \\ 0 \end{bmatrix} = t\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \quad \therefore v_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$ is eigen vector of 'A' corresp. to $\lambda_2 = 2$.

Eigen vector v_3 of 'A' corresp. eigen value $\lambda_3 = 3$ is basis of Null $(A - \lambda_3 I_{3\times3})\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$$\left(\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} - 3\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right)\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} -2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$[A|B] = \left[\begin{array}{ccc|c} -2 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \rightarrow \text{REF}.$$

'z' is free. let, $z = t$.

then, $x = 0$, $y = 0$.

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$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ t \end{bmatrix} = t \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

$v_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ is eigen vector of 'A' corresponding to $\lambda_3 = 3$.

$\therefore$ Eigen vectors of given 'A' matrix corresponding $\lambda_1 = 1$, $\lambda_2 = 2$ and $\lambda_3 = 3$ are. $v_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $v_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, $v_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

Q. $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}_{3\times3}$ $n=3$.

$\Rightarrow$ characteristic eqn of 'A'.

$$\Delta(A) = \lambda^3 - \lambda^2(1+1+1) + \lambda \left\{ \begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix} + \begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix} + \begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix} \right\}$$

$$- \det(A) = 0$$

$$\Rightarrow \lambda^3 - 3\lambda^2 + 0\lambda - 0 = 0$$

$$\lambda^3 - 3\lambda^2 = 0. \Rightarrow \lambda^2(\lambda - 3) = 0$$

$\lambda_1 = 0$, $\lambda_2 = 0$, $\lambda_3 = 3$ are eigen values of 'A'.

Eigen vector v of 'A' corresponding eigen value $\lambda_1 = \lambda_2 = 0$ is basis of Null $(A - \lambda I_{3\times3}) \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$$\left(\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} - 0 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right) \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

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$$[A|B] = \left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \end{array}\right]$$

$R_2 \to R_2 - R_1$
$R_3 \to R_3 - R_1$
$$[A|B] = \left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

y, z are free. let, $y = t_1$ and, $z = t_2$. then, $x = -t_1 - t_2$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -t_1 - t_2 \\ t_1 \\ t_2 \end{bmatrix} = t_1 \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + t_2 \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

$\therefore$ $v_1 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$ and $v_2 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$ are eigen vectors of 'A' corresp. to $\lambda = 0$.

Eigen vector v_3 of 'A' corresponding eigen value $\lambda_3 = 3$ is basis of Null $(A - \lambda_3 I_{3\times3})$ $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$$\left(\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} - 3 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right) \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow [A|B] = \left[\begin{array}{ccc|c} -2 & 1 & 1 & 0 \\ 1 & -2 & 1 & 0 \\ 1 & 1 & -2 & 0 \end{array}\right]$$

$R_2 \to 2R_2 + R_1$
$R_3 \to 2R_3 + R_1$
$$[A|B] = \left[\begin{array}{ccc|c} -2 & 1 & 1 & 0 \\ 0 & -3 & 3 & 0 \\ 0 & 3 & -3 & 0 \end{array}\right]$$

$R_3 \to R_3 + R_2$
$$[A|B] = \left[\begin{array}{ccc|c} -2 & 1 & 1 & 0 \\ 0 & -3 & 3 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \to \text{REF}.$$

'z' is free $\Rightarrow$ let, $z = t$.

then from REF, $-3y + 3t = 0$ and $-2x + t + t = 0$

$y = t$. $\qquad x = t$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} t \\ t \\ t \end{bmatrix} = t \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$v_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ is eigen vector of 'A' corresp. to $\lambda_3 = 3$.

Q. $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ $n = 3$

Characteristic eq^n of 'A'.

$\Rightarrow$ $\Delta(\lambda) = \lambda^3 - \lambda^2(1+1+1) + \lambda \left\{ \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} + \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} + \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} \right\}$

$- \det(A) = 0$

$\Rightarrow \lambda^3 - 3\lambda^2 + 3\lambda - 1 = 0$.

$\Rightarrow (\lambda - 1)^3 = 0$.

$\therefore \lambda_1 = 1, \lambda_2 = 1, \lambda_3 = 1$ are eigen values of 'A'.

Eigen Vector 'v' of 'A' corresponding eigen value $\lambda_1 = \lambda_2 = \lambda_3 = 1$ is basis of Null $(A - \lambda I_{3\times3}) \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$$\left(\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - 1 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right) \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\Rightarrow$ $[A|B] \left[\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \rightarrow$ REF.

x, y, z are free.

let, $x = t_1$, $y = t_2$, $z = t_3$

$[A|B] = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} t_1 \\ t_2 \\ t_3 \end{bmatrix} = t_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + t_2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + t_3 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

$v_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, v_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, v_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ are eigen vectors of 'A' corresponding eigen value $\lambda_1 = \lambda_2 = \lambda_3 = 1$.

NOTE: 1) If Matrix is symmetric, $(A^T = A)$
then $n =$ no. of Eigen value $=$ no. of Eigen vector
↳ repeated or similar, nothing matters.

2) If matrix is non-symmetric, then
- if eigen values are distinct then
$n =$ no. of Eigen value $=$ no. of Eigen Vector.
- If eigen values are similar (or repeated) then,
$n =$ no. of Eigen value and no. of Eigen vector $\leq n$.

* Shortcut to find Eigen Value and Eigen Vector.
Let 'A' be $n \times n$ matrix, $\lambda_1, \lambda_2, \ldots \lambda_n$ are eigen vector of 'A'.

1. Addition of all eigen value = Trace of given matrix.
2. Product of all eigen value = det of given matrix.
3. If det of 'A' is non-zero $(|A| \neq 0)$ then $1/\lambda$ is eigen value of A^{-1}.
4. Eigen value of 'A' and A^T are same, but eigen vector may be different.
5. * If given matrix is diagonal or upper triangular or lower triangular then its eigen value are diagonal element of given matrix.

⑥ If 'v' is the eigen vector of 'A' then same 'v' is a eigen vector of A^m corresponding to eigen value λ^m.

⑦ If 'λ' is eigen value of 'A' and $|A| \neq 0$ then $1/\lambda$ is eigen value of A^{-1}.

Q. find eigen value and eigen vector of A^5, A^{-1}, A^{-2}

$\Rightarrow$ $A = \begin{bmatrix} 3 & -4 \\ 2 & -6 \end{bmatrix}$ $n = 2$. shortcut ① & shortcut ②.

$\lambda_1 + \lambda_2 = 3 + (-6) \Rightarrow \lambda_1 + \lambda_2 = -3$. — ①

$\lambda_1 \lambda_2 = |A| = (-6)(3) - (-4)(2) = -18 + 8 = -10$. — ②

from ① and ②

$\lambda_1 = -5$, $\lambda_2 = 2$.

$v_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$, $v_2 = \begin{bmatrix} 1 \\ 4 \end{bmatrix}$

① eigen value of A^5 are $(-5)^5$, $(2)^5$

$\Rightarrow \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 4 \end{bmatrix}$

② A^{-1} eigen value, $\begin{bmatrix} 1 \\ -5 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

③ eigen value of $A^{-2} = \frac{1}{A^2} = \frac{1}{(-5)^2}, \frac{1}{(2)^5}$

Algebraic Multiplicity (AM) of eigen value.

How many times that eigen value occurs is called as 'AM'.

Geometric Multiplicity (GM) of eigen value

How many eigen vector we get for given eigen value is called as 'GM'.

NOTE: GM $\leq$ AM.

* Eigen Space

Total number of eigen vector ~~of~~ for given matrix is called as Eigen space.

Diagonalization of Matrix.

Let 'A' be $n \times n$ matrix. $\lambda_1, \lambda_2, \ldots, \lambda_n$ are eigen value of 'A'. Then 'A' is diagonalizable if

① 'A' have 'n' no. of linearly independent eigenvector.

ie $n =$ Total no. of eigen vector of 'A'.

OR

② AM = GM for all eigen value.

OR

③ There exist invertible Matrix, $P = [v_1, v_2, \ldots v_n]$ (eigen vector of 'A')

such that $P^{-1}AP = D$

Where, 'D' is diagonal matrix whose diagonal element are eigen value of 'A'.

NOTE: 'P' called Modal matrix.

'D' called spectral matrix.

Q. Check whether given matrix is diagonalizable? If yes, then diagonalize it.

1) $A = \begin{bmatrix} 3 & -4 \\ 2 & -6 \end{bmatrix} \Rightarrow \lambda_1 = -5, \quad \lambda_2 = 2 \quad n = 2$

$v_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \quad v_2 = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$

$n = 2$, Total no. of eigen vector $= 2$.

$\therefore n =$ Total no. of eigen vector.

$\therefore$ A is diagonalizable

* Diagonalization of symmetric matrix using orthogonal diagonalization

Let, 'A' be $n \times n$ symmetric matrix. $v_1, v_2, \ldots v_n$ are eigen vector of A corr. to eigen value $\lambda_1, \lambda_2, \ldots \lambda_n$. 'A' is always diagonalizable.

Additional step:

If all eigen vector of 'A' are mutually orthogonal.

i.e. $\langle v_i, v_j \rangle = 0$, for all i, j

$\therefore$ let, $V = R^n$, use std I.P on R^n i.e

$\langle (x_1, x_2, \ldots x_n) \cdot (y_1, y_2, y_3, \ldots y_n) \rangle = x_1 y_1 + x_2 y_2 + \ldots + x_n y_n$

($V = R^n \rightarrow$ where 'n' is order of matrix)

- Case I: Yes, then $P = [v_1, v_2, \ldots v_n]$

$\therefore P^{-1} A P = D$

- Case II: No, then make all eigen vector mutually orthonormal using Gram-Schmidt process.

let, $V = R^n$, use std IP on R^n,

$B = \{v_1, v_2, \ldots, v_n\}$ Basis of $V = R^n$.

$\therefore$ We get orthonormal Basis $= \left\{ \frac{w_1}{\|w_1\|}, \frac{w_2}{\|w_2\|}, \frac{w_3}{\|w_3\|} \right\}$

$\therefore P = \left[\frac{w_1}{\|w_1\|}, \frac{w_2}{\|w_2\|}, \ldots, \frac{w_n}{\|w_n\|} \right] \longrightarrow$ orthogonal matrix

$\therefore P^{-1} A P = D$

Q. Check whether given matrix is diagonalizable. If yes, then diagonalize it
OR Using orthogonal diagonalization, diagonalize the given matrix.

① $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $n = 2$

$\Rightarrow \lambda_1 = 1, \lambda_2 = 1$

$v_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, v_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

Matrix is symmetric.

$\therefore$ Check all e.vector are mutually orthogonal

$\therefore V = R^n = R^2$, use std IP on R^2.

$\langle (x_1, x_2)(y_1, y_2) \rangle = x_1 y_1 + x_2 y_2$

$\langle v_1, v_2 \rangle = \langle (1,0)(0,1) \rangle = 0$

$\therefore$ all e.vector are mutually orthogonal.

$\therefore P = [v_1, v_2] = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ $\therefore P^{-1} A P = D$

② $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}_{3 \times 3}$, $n = 3$

$\Rightarrow \lambda_1 = \lambda_2 = 0, \lambda_3 = 3$

$v_1 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, v_2 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, v_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

$\langle v_1, v_2 \rangle = 1$, $\langle v_2, v_3 \rangle = 0$, $\langle v_1, v_3 \rangle = 0$

$\therefore$ all e.vectors are not mutually orthogonal. $\therefore$ Use Gram Schmidt process

$\therefore$ Basis, $B = \left\{ \underset{v_1}{\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}}, \underset{v_2}{\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}}, \underset{v_3}{\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}} \right\}$

find $\frac{w_1}{\|w_1\|}, \frac{w_2}{\|w_2\|}, \frac{w_3}{\|w_3\|}$

$w_1 = v_1 = (-1,1,0)$

$w_2 = v_2 - \frac{\langle v_2, w_1\rangle}{\|w_1\|^2} w_1$

$\langle v_2, w_1\rangle = \langle(-1,0,1)(-1,1,0)\rangle = 1$

$\|w_1\| = \sqrt{\langle(-1,1,0)(-1,1,0)\rangle} = \sqrt{2}$

$w_2 = (-1,0,1) - \frac{1}{2}(-1,1,0)$

$= (-1,0,1) - \left(\frac{-1}{2}, \frac{1}{2}, 0\right)$

$w_2 = \left(\frac{-1}{2}, \frac{-1}{2}, 1\right)$, $\|w_2\| = \sqrt{3/2}$

$w_3 = v_3 - \frac{\langle v_3, w_2\rangle}{\|w_2\|^2} w_2 - \frac{\langle v_3, w_1\rangle}{\|w_1\|^2} w_1$

$\langle v_3, w_2\rangle = \langle(1,1,1)(-1/2,-1/2,1)\rangle = 0$

$\langle v_3, w_1\rangle = \langle(1,1,1)(-1,1,0)\rangle = 0$

$w_3 = (1,1,1)$

$\frac{w_1}{\|w_1\|} = \frac{(-1,1,0)}{\sqrt{2}}$, $\frac{w_2}{\|w_2\|} = \frac{(-1,0,1)}{\sqrt{3/2}}$

$\frac{w_3}{\|w_3\|} = \frac{(1,1,1)}{\sqrt{3}}$

orthogonal e.vector $= \{(-1,1,0)(-1/2,-1/2,1), (1,1,1)\}$

orthonormal e.vector $=$

$\left\{\frac{(-1,1,0)}{\sqrt{2}}, \frac{(-1/2,-1/2,1)}{\sqrt{3/2}}, \frac{(1,1,1)}{\sqrt{3}}\right\}$

$P = \begin{bmatrix} \frac{w_1}{\|w_1\|}, & \frac{w_2}{\|w_2\|}, & \frac{w_3}{\|w_3\|} \end{bmatrix} =$

$\begin{bmatrix} -1/\sqrt{2} & -\sqrt{1/6} & 1/\sqrt{3} \\ 1/\sqrt{2} & -\sqrt{1/6} & 1/\sqrt{3} \\ 0 & \sqrt{2/3} & 1/\sqrt{3} \end{bmatrix}$

$\therefore \boxed{P^{-1}AP = D}$

* Principal Axes Theorem:

- **Quadratic form in 2 variable, x & y is**

$\varphi(x,y) = ax^2 + by^2 + cxy$

$\varphi(x,y) = [x, y]\begin{bmatrix} a & c/2 \\ c/2 & b \end{bmatrix}_{2\times2}\begin{bmatrix} x \\ y \end{bmatrix}$

i.e $\varphi(x,y) = X^TAX$, where $X = \begin{bmatrix} x \\ y \end{bmatrix}$

$\therefore$ Matrix of $\varphi(x,y)$ is $A = [\ \ ]$

always symmetric.

always diagonalizable

$\therefore$ follow the procedure of orthogonal diagonalization of symmetric matrix.

$\to$ Canonical form of

$\varphi(x,y) = \lambda_1 y_1^2 + \lambda_2 y_2^2$

$\therefore \lambda_1, \lambda_2$ are e.values of 'A'.

- **Quadratic form in 3 variables**

$\varphi(x,y,z) = ax^2 + by^2 + cz^2 + dxy + eyz + fxz$

i.e $\varphi(x,y,z) = [x\ y\ z]\begin{bmatrix} a & d/2 & e/2 \\ d/2 & b & f/2 \\ e/2 & f/2 & c \end{bmatrix}_{3\times3}\begin{bmatrix} x \\ y \\ z \end{bmatrix}$

i.e $\varphi(x,y,z) = X^TAX$, $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$

$\therefore$ Matrix of $\varphi(x,y,z)$ is

$A = \begin{bmatrix} a & d/2 & e/2 \\ d/2 & b & f/2 \\ e/2 & f/2 & c \end{bmatrix}_{3\times3}$ $\to$ always symmetric always diagonalizable

$\therefore$ follow the procedure of orthogonal diagonalization of symmetric matrix.

$\therefore$ Canonical form of

$\varphi(x,y,z) = \lambda_1 y_1^2 + \lambda_2 y_2^2 + \lambda_3 y_3^2$

$\therefore \lambda_1, \lambda_2, \lambda_3$ are e.values of 'A'.

Diagonal element : coeff of x^2 & y^2.
and non diago. element : the product term (1/2) of it.

eg, ax^2+by^2+cxy.

$$\varphi(x,y) = [x \quad y]\begin{bmatrix} a & c/2 \\ c/2 & b \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$$

↳ always symm.
always diagonalizable.

Q.1.) Convert given quadratic form into matrix and write down canonical form
OR
From given matrix write down quadadic form.

① $\varphi(x,y) = x^2+y^2+2xy$

$\Rightarrow$ matrix of $\varphi(x,y)$ is $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}_{2\times 2}$.

e.value of 'A' are $\lambda_1 = 0$, $\lambda_2 = 2$

canonical form $\varphi(y_1,y_2) = \lambda_1 y_1^2 + \lambda_2 y_2^2$
$= 0y_1^2 + 2y_2^2$

② $A = \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix} \Rightarrow \varphi(x,y) = x^2+3y^2+4xy$

③ $\varphi(x,y) = x^2+2xy$

$A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}_{2\times 2}$.

④ $\varphi(x,y,z) = x^2+y^2+z^2+2xy+2yz+2xz$.

$\Rightarrow \varphi(x,y,z) = ax^2+by^2+cz^2+dxy+eyz+fxz$.

$\begin{bmatrix} a & d/2 & e/2 \\ d/2 & b & f/2 \\ e/2 & f/2 & c \end{bmatrix}$ $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$

⑤ $\varphi(x,y,z) = xy$ $A = \begin{bmatrix} 0 & 1/2 & 0 \\ 1/2 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}_{3\times 3}$

6M

Q.2 Convert quadratic form into canonical form using orthogonal diagonalization and hence identify section.
OR
Using principal Axes thm, find canonical form of given quadratic form.

① $\varphi(x,y) = 2x^2+2y^2+4xy$

$\Rightarrow$ Matrix of $\varphi(x,y) = \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix}$

↳ It is symm. and diagonalizable.

follow the procedure of diagonalizable of symmetric matrix.

$\therefore \lambda_1 = 0$, $\lambda_2 = 4$.

$v_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ $v_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

check v_1 and v_2 are mutually orthogonal.

$\therefore n = 2$, $V = R^n = R^2$

use std IP on R^2.

$\langle (x_1,x_2)(y_1,y_2)\rangle = x_1y_1 + x_2y_2$

$\langle v_1, v_2\rangle = \langle (-1,1)(1,1)\rangle$
$= -1+1 = 0$

$\therefore v_1$ and v_2 are mutually orthogonal.

$\therefore P = [v_1 \quad v_2] = \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}$

$\therefore P^{-1}AP =$

$$\frac{-1}{2}\begin{bmatrix} +1 & -1 \\ -1 & -1 \end{bmatrix}\begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix}\begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 \\ 0 & 4 \end{bmatrix}$$

$\therefore$ Canonical form is $\varphi(y_1,y_2) = \lambda_1 y_1^2 + \lambda_2 y_2^2$

$= 0y_1^2 + 4y_2^2$ ⟶ ~~circle~~ ellipse (conic section).

★ $\boxed{P^{-1}AP = D}$

↳ to find 'D', directly write e.values in diagonal posn and rest other elements '0'.

ex: $\begin{bmatrix} 0 & 0 \\ 0 & 4 \end{bmatrix}$

If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ 2

then $(\text{adj } A) = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$