Unit. II   function of two variables

$y = e^x$

$\frac{dy}{dx} = e^x \rightarrow$ change in y along x-axis

$\frac{d}{dy}(e^x) \rightarrow$ can't calculate because $y = f(x) = e^x$
dependent   Independent

$y = f(x) \rightarrow$ funⁿ of single variable
only one independent variable
corresponding will be total d/dx

$y = f(x) \rightarrow$ doesn't happened
$z = f(x, y) \rightarrow$ funⁿ of two variable → These happened

If more than one independent variable is present then derivative will be partial.

Ex. $z = x^2 + y^2$

$\frac{\partial z}{\partial x}$ = diff. given funⁿ wrt 'x' by keeping 'y' constant

$\frac{\partial z}{\partial x} = 2x + 0$

$\frac{\partial z}{\partial y}$ = diff given funⁿ wrt 'y' by keeping 'x' constant.

$\frac{\partial z}{\partial y} = 0 + 2y = 2y$

$\frac{\partial z}{\partial x} = z_x = \left(\frac{\partial z}{\partial x}\right)_y$, $\frac{\partial z}{\partial y} = z_y = \left(\frac{\partial z}{\partial y}\right)_x$

1st order partial derivative (PDE)

$\rightarrow$ $d(u+v) = du + dv$

$\rightarrow$ $d(uv) = u\,dv + v\,du$

$\rightarrow$ $d\left(\frac{u}{v}\right) = \frac{v\,du - u\,dv}{v^2}$

Q. $z = e^{2x^2y^2} \Rightarrow z_x = e^{2x^2y^2} \cdot 2(2x)y^2 = 4xy^2 e^{2x^2y^2}$

Q. $z = \sin(xy) \Rightarrow z_x = \cos(xy) \cdot y \cdot 1 = y\cos(xy)$

$z_y = x\cos(xy)$

Q. $z = e^{x^2}\sin(xy)$

$\Rightarrow z_x = e^{x^2}\cos(xy) \cdot y + \sin(xy)\, e^{x^2} \cdot 2x$

$z_y = e^{x^2}\cos(xy) \cdot x + \sin(xy) \cdot 0 = e^{x^2}\cos(xy) \cdot x$

* second order partial derivative.

$$\frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) = z_{xx}$$

Ex. $z = \sin(e^{x^2}) \Rightarrow z_x = \cos(e^{x^2}) \cdot e^{x^2} \cdot 2x$

Ex. $z = e^{2x^2}$ $\Rightarrow z_{xx} = 16x^2 e^{2x^2} + 4e^{2x^2}$

Ex. $z = \sin(xy^2) \Rightarrow z_y = \cos(xy^2) \cdot 2xy$

$z_{yy} = \cos(xy^2)\, 2x + 2xy(-\sin(xy^2) \cdot 2xy)$

$\rightarrow$ $z_{xy} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right)$ $\quad\rightarrow z_{yx} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right)$

Ex. $z = \log(x^2y)$

$z_y = \frac{1}{x^2y}(x^2) = \frac{1}{y}$, $\quad z_{xy} = 0$

z_{xy} = 1st diff wrt 'y' and then diff. wrt 'x'

z_{yx} = 1st diff. wrt 'x' and then diff wrt 'y'

$z_{xx}, z_{yy}, z_{xy}, z_{yz} \rightarrow$ 2nd order PD.

* 3rd order partial derivative.

$$z_{xxx} = \frac{\partial}{\partial x}\left(\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right)\right), \quad z_{yyy} = \frac{\partial}{\partial y}\left(\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right)\right)$$

$$z_{xy\cancel{x}} = \frac{\partial}{\partial x}\left(\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right)\right)$$

Ex. $x = r\cos\theta \Rightarrow x_r = \cos\theta$

$x_\theta = -r\sin\theta$

Ex. $z = x^2y^2 + 3x^3$

$\Rightarrow$ $z_x = 2xy^2 + 9x^2$ $\quad\quad$ $z_y = 2yx^2 + 0$

$z_{xx} = 2y^2 + 18x$ $\quad\quad$ $z_{yy} = 2x^2$

$z_{xxx} = 18$ $\quad\quad$ $z_{yyy} = 0$.

Q. If $u = \tan^{-1}\left(\frac{y}{x}\right)$ then find $z_{xx}, z_{yy}, z_{xy}, z_{yx}$

$\Rightarrow$ $\frac{d}{dx}(\tan^{-1}x) = \frac{1}{1+x^2}$

$$z_x = \frac{1}{1+\left(\frac{y}{x}\right)^2} \times \left(\frac{-y}{x^2}\right) = \frac{x^2}{x^2+y^2} \cdot \left(\frac{-y}{x^2}\right) = \frac{-y}{x^2+y^2}$$

$$z_{xx} = \frac{(x^2+y^2)\cdot(0) - (-y)\cdot(2x)}{(x^2+y^2)^2} = \cancel{x^2}\ \frac{2xy}{(x^2+y^2)^2}$$

$\boxed{Z_{xy} = Z_{yx}}$

→ When both are continuous.

→ **NOTE**: To find $\left(\frac{\partial r}{\partial x}\right)_y$, first express 'r' as a funⁿ of 'x' and 'y' both and then find r_x and r_y.

Q. If $u = \log(x^2+y^2)$ then find $u_x, u_y, u_{xx}, u_{yy}, u_{xy}$ and u_{yx}.

⇒ $u_x = \frac{1}{x^2+y^2} \times 2x = \frac{2x}{x^2+y^2}$.

$u_y = \frac{1}{x^2+y^2} \times 2y = \frac{2y}{x^2+y^2}$

$u_{xx} = \frac{(x^2+y^2)(2) - 2x(2x)}{(x^2+y^2)^2} = \frac{2x^2+2y^2-4x^2}{(x^2+y^2)^2} = \frac{2y^2-2x^2}{(x^2+y^2)^2}$

$u_{yy} = \frac{(x^2+y^2)(2) - (2y)(2y)}{(x^2+y^2)^2} = \frac{2x^2-2y^2}{(x^2+y^2)^2}$

$u_{xy} \Rightarrow u_y = \frac{2y}{x^2+y^2}$, Now diff. w r t x.

$u_{xy} = \frac{(x^2+y^2)(0) - (2y)(2x)}{(x^2+y^2)^2} = \frac{-4xy}{(x^2+y^2)^2}$

$u_{yx} \Rightarrow u_x = \frac{2x}{x^2+y^2}$, Now diff wrt y.

$u_{yx} = \frac{-4yx}{(x^2+y^2)^2}$

$\therefore \boxed{u_{xy} = u_{yx}}$

→ If given is continuous.

Q. $u = x^y + y^x$ then find $u_x, u_y, u_{xx}, u_{yy}, u_{xy}, u_{yx}$

⇒ $u_x = yx^{(y-1)} + y^x \log y$

$u_y = x^y \log x + xy^{(x-1)}$

$u_{xx} = y(y-1)x^{y-2} + \log y\,(y^x \log y)$

$= y(y-1)x^{y-2} + (\log y)^2 y^x$

$\star\ (x^3+y^3+z^3-3xyz) = (x+y+z)(x^2+y^2+z^2-xy-yz-xz)$

Date ____
Page ____

$u_{yy} = x^y \log x \cdot \log x + x(x-1)y^{x-2}$

$= x^y (\log x)^2 + x(x-1)y^{x-2}$

$u_{xy} = x^y \cdot \frac{1}{x} + \log x \cdot y\,(x)^{y-1} + xy^{x-1}\log y + y^{x-1}$

$u_{yx} = x^{y-1} + yx^{y-1}\log x + xy^{x-1}\log y + y^x \cdot \frac{1}{y}$.

If $u = \log(x^3+y^3+z^3-3xyz)$ then show that

$$\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)^2 u = \frac{-9}{(x+y+z)^2}$$

$$\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)^2 u = \left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)\left(\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}\right) \quad \text{—(1)}.$$

$$u_x = \frac{1}{x^3+y^3+z^3-3xyz}\cdot 3x^2+0+0-3yz = \frac{3x^2-3yz}{x^3+y^3+z^3-3xyz}$$

$$u_y = \frac{3y^2-3xz}{x^3+y^3+z^3-3xyz}, \quad u_z = \frac{3z^2-3xy}{x^3+y^3+z^3-3xyz}$$

$$u_x+u_y+u_z = \frac{3x^2+3y^2+3z^2-3xy-3yz-3xz}{x^3+y^3+z^3-3xyz}$$

from (1)

$$\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)\left(\underbrace{\frac{3x^2+3y^2+3z^2-3xy-3yz-3xz}{x^3+y^3+z^3-3xyz}}_{\text{let } P}\right)$$

$$\Rightarrow \left(\frac{\partial P}{\partial x}+\frac{\partial P}{\partial y}+\frac{\partial P}{\partial z}\right)$$

$$\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z} = \frac{3(x^2+y^2+z^2-xy-yz-xz)}{x^3+y^3+z^3-3xyz}$$

$$= \frac{3(x^2+y^2+z^2-xy-yz-xz)}{x^3+y^3+z^3-3xyz}\cdot\frac{(x+y+z)}{(x+y+z)}$$

$(x^3+y^3+z^3-3xyz) = (x^2+y^2+z^2-xy-yz-xz)(x+y+z)$

$$\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z} = \frac{3(x^3+y^3+z^3-3xyz)}{x^3+y^3+z^3-3xyz} \cdot \frac{1}{(x+y+z)} = \frac{3}{x+y+z}$$

putting value of $\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}$ in ① we get,

$$\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)^2 u = \frac{-9}{(x+y+z)^2}$$

Q find the value of 'n' if $u = r^n(3\cos^2\theta-1)$ satisfying the eqn $\frac{\partial}{\partial r}\left(r^2\frac{\partial u}{\partial r}\right)+\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial u}{\partial\theta}\right)=0$ — ②

$\Rightarrow$ $u = r^n(3\cos^2\theta-1)$ — ①

$$\frac{du}{dr} = nr^{n-1}(3\cos^2\theta-1)$$

$$\therefore r^2\frac{\partial u}{\partial r} = nr^2 r^{n-1}(3\cos^2\theta-1) = nr^{n+1}(3\cos^2\theta-1)$$

$$\frac{\partial}{\partial r}\left(r^2\frac{\partial u}{\partial r}\right) = \frac{\partial}{\partial r}\left(nr^{n+1}(3\cos^2\theta-1)\right)$$

$$= n(n+1)r^n(3\cos^2\theta-1)$$

$$\frac{\partial u}{\partial\theta} = -3r^n\sin 2\theta$$

$\therefore \cos 2\theta = 2\cos^2\theta-1$

$$\sin\theta\left(\frac{\partial u}{\partial\theta}\right) = -3r^n\sin\theta\sin 2\theta$$

$$\frac{\partial}{\partial\theta}\left(\sin\theta\left(\frac{\partial u}{\partial\theta}\right)\right)\Big/\sin\theta = -3r^n(6\cos^2\theta-2)$$

$\Rightarrow n(n+1)r^n(3\cos^2\theta-1) - 3r^n(6\cos^2\theta-2) = 0.$

$r^n(3\cos^2\theta-1)[n(n+1)-6] = 0.$

$n^2+n-6=0. \Rightarrow n = -3, 2$

classmate Date ____ Page ____

If, $u = \log(e^x + e^y)$, show that $\frac{\partial^2 u}{\partial x^2} \cdot \frac{\partial^2 u}{\partial y^2} = \left(\frac{\partial^2 u}{\partial x \partial y}\right)^2$

$u = \log(e^x + e^y)$

$u_x = \frac{e^x}{e^x + e^y}$, $u_y = \frac{e^y}{e^x + e^y}$

$u_{xx} = \frac{e^x e^y}{(e^x+e^y)^2}$, $u_{yy} = \frac{e^x e^y}{(e^x+e^y)^2}$

$\left(\frac{\partial u}{\partial x} \frac{\partial u}{\partial y}\right)^2 = \frac{e^x e^y}{(e^x+e^y)^2} \cdot \frac{e^x e^y}{(e^x+e^y)^2} = \frac{e^{2x} e^{2y}}{(e^x+e^y)^4} = \frac{e^{2x+2y}}{(e^x+e^y)^4}$.

Hence proved.

Euler function (Theorem) for function of several variable.

Homogeneous funⁿ : $f(x, y) = x^2 + y^2$

① put $x = tx$, $y = ty$ in given funⁿ.

$f(tx, ty) = (tx)^2 + (ty)^2 = t^2(x^2+y^2)$

$f(tx, ty) = t^2 f(x, y)$

$f(x,y) = x^2+y^2$ is Homogeneous funⁿ of degree $n=2$

② $f(x,y) = x^3 + y^3 + 3xy + x^2$

$f(tx, ty) = t^3x^3 + t^3y^3 + 3t^2xy + t^2x^2$

$= t^2(tx^3 + ty^3 + 3xy + x^2)$

$\neq t^3 f(x,y)$

Not Homogeneous funⁿ.

Homogeneous funⁿ of degree 'n' means?

A funⁿ (x, y) of two variable 'x' and 'y' is said to homogeneous funⁿ of degree 'n' if

$f(x,y) = x^n \phi\left(\frac{y}{x}\right)$ or $f(x,y) = y^n \phi\left(\frac{x}{y}\right)$

Alternatively, funⁿ (x,y) of two variables x and y is said to homogeneous funⁿ of degree 'n' if.

$f(tx, ty) = t^n f(x,y)$, where 't' is a parameter

→ $f(x,y) = \dfrac{x^2 + y^2 + 2x^1 y^1}{x^1 - y^1}$ $\dfrac{\text{Homo}}{\text{Homo}} = \dfrac{2}{1} \Rightarrow 2-1 = 1$

Degree is 1

∵ all Numerator have degree 2.
and all Denoⁿ have degree 1.
∴ They both are homogeneous.

→ $f(x,y) = \dfrac{x^2 + x^1y^1 + y^1}{x^3 + y^3}$ → (Not same degree all over Numʳ) → Not Homo

→ $f(x,y) = x^2y^2$ → Homo. of degree 4.

$x^2y^2 = x^2 \cdot y^2 = x^4\left(\dfrac{y}{x}\right)^2 = x^4 \phi\left(\dfrac{y}{x}\right)$

This is of the form $f(x,y) = x^n \phi\left(\dfrac{y}{x}\right)$

→ $x^2y^2 = y^4\left(\dfrac{x}{y}\right)^2 = y^4 \phi\left(\dfrac{x}{y}\right) \Rightarrow y^n \phi\left(\dfrac{x}{y}\right)$

→ $f(x,y) = \log(x^2+y^2)$ is not a homogeneous funⁿ.

If 'u' is a homogeneous funⁿ of 'x' and 'y' of degree 'n' then
$x^2 u_{xx} + y^2 u_{yy} + 2xy u_{xy} = n(n-1)u$

Date 21/11/24

* Homogeneous funⁿ :
$f(x,y)$ said to Homo. funⁿ if
① $f(tx, ty) = t^n f(x,y)$ then we say 'f' is Homo of degree 'n'.
OR
② $f(x,y) = x^n \phi(y/x)$
OR
③ $f(x,y) = y^n \phi(x/y)$

* Euler Theorem for funⁿ of two variables and 3 variables.
① If 'u' is Homogeneous funⁿ of degree 'n' then
ⓐ $x u_x + y u_y = nu$ ⟶ for 2 variable
ⓑ $x u_x + y u_y + z u_z = nu$ ⟶ for 3 variable

* Reduction of Euler Theorem for function of two variable.
If 'u' is Homo. funⁿ of degree 'n' and $\boxed{z = f(u)}$ then
ⓐ $x u_x + y u_y = \dfrac{n f(u)}{f'(u)}$

ⓑ $x^2 u_{xx} + y^2 u_{yy} + 2xy u_{xy} = g(u)[g'(u) - 1]$
where $g(u) = \dfrac{n f(u)}{f'(u)}$

Q. Verify Euler theorem. If $u = x^2 + y^2$.
⇒ To show $x u_x + y u_y = nu$
u is Homogeneous funⁿ of degree $n = 2$
∴ By Euler Theorem $x u_x + y u_y = nu = 2(x^2 + y^2)$
and
$u_x = 2x$ ∴ $x u_x = 2x^2$
$u_y = 2y$ ∴ $y u_y = 2y^2$
∴ $x u_x + y u_y = 2(x^2 + y^2)$

find $xu_x + yu_y$ if $u = 8(x^2+y^2)(\log x - \log y)$

$u = 8(x^2+y^2)(\log x - \log y)$

$u(tx, ty) = 8(t^2x^2 + t^2y^2)\left(\log\left(\frac{tx}{ty}\right)\right)$

$= t^2\, 8(x^2+y^2)(\log(x/y))$

$= t^2\, 8(x^2+y^2)(\log x - \log y)$

$= t^2\, u(xy)$

'u' is Homo funⁿ of degree $n=2$.

$\therefore$ By Euler Theorem, $xu_x + yu_y = nu = 2[8(x^2+y^2)(\log x - \log y)]$

If $u = e^{x^2+y^2}$ then find $x^2u_{xx} + 2xy\,u_{xy} + y^2u_{yy}$

$u = e^{x^2+y^2}$, $u(tx, ty) = e^{\cancel{(t^2x^2+t^2y^2)}} = \cancel{e^{t^2(x^2+y^2)}} = e^{t^2} \cdot e^{(x^2+y^2)} \neq t^n u(xy)$

$\therefore$ u is not Homogeneous.

$\therefore \log u = x^2+y^2$

$\therefore \log u$ is Homogeneous funⁿ of degree, $n=2$

$f(u) = \log u$

By reduction formula of Euler Then,

$x^2u_{xx} + y^2u_{yy} + 2xyu_{xy} = g(u)[g'(u)-1]$

but $g(u) = \dfrac{n f(u)}{f'(u)} = \dfrac{2\log(u)}{1/u} = 2u\log(u)$

$g(u) = 2u\log u$

$g'(u) = 2u \cdot \frac{1}{u} + 2\log u \cdot 1 = 2 + 2\log u$

$\therefore x^2u_{xx} + y^2u_{yy} + 2xy\,u_{xy} = 2u\log u[2 + 2\log u - 1]$

$= 2u\log u[1 + 2\log u]$

Q. If $u = \log\left[\dfrac{x^3+y^3}{x^2+y^2}\right]$. find ① $xu_x + yu_y$

② $x^2u_{xx} + y^2u_{yy} + 2xy\,u_{xy}$.

$\Rightarrow u = \log\left[\dfrac{x^3+y^3}{x^2+y^2}\right]$

$e^u = \dfrac{x^3+y^3}{x^2+y^2}$ $\rightarrow$ degree 3 $\quad n = 3-2 = 1$.

$\rightarrow$ degree 2

$f(u)$

↳ This is Homogeneous funⁿ with degree 1.

by reduction of Euler Theorem, we know.

① $xu_x + yu_y = n\dfrac{f(u)}{f'(u)}$

$f(u) = e^u$.

$f'(u) = e^u$. $\quad xu_x + yu_y = (1)\dfrac{e^u}{e^u} = 1$

$\therefore xu_x + yu_y = 1.$

② $x^2u_{xx} + y^2u_{yy} + 2xy\,u_{xy}$.

by redⁿ of Euler Theorem, we know

$x^2u_{xx} + y^2u_{yy} + 2xy\,u_{xy} = g(u)[g'(u) - 1]$

$g(u) = \dfrac{n\,f(u)}{f'(u)} \Rightarrow g(u) = (1)\dfrac{e^u}{e^u} = 1 \quad \therefore g'(u) = 0$.

$g(u)[g'(u)-1] = (1)[0-1] = -1$.

$\therefore x^2u_{xx} + y^2u_{yy} + 2xy\,u_{xy} = -1.$

classmate
Date
Page

If $u = \sin^{-1}\left[\sqrt{x^2+y^2}\right]$ find

① $x u_x + y u_y$

② $x^2 u_{xx} + y^2 u_{yy} + 2xy\, u_{xy}$

$u = \sin^{-1}\left[\sqrt{x^2+y^2}\right]$ check for Homogenanity.

$u(tx, ty)$

$$= \sin^{-1}\left[\sqrt{t^2x^2+t^2y^2}\right] = \sin^{-1}\left[\sqrt{t^2(x^2+y^2)}\right]$$

$$= \sin^{-1}\left[t\sqrt{x^2+y^2}\right]$$

$u(tx, ty) \neq u(x, y)$ $\therefore$ It is not Homogeneous Funⁿ.

$u = \sin^{-1}\sqrt{x^2+y^2}$.

$$u_x = \frac{1}{\sqrt{1-(x^2+y^2)}} \frac{(2x)}{2\sqrt{x^2+y^2}} = \frac{x}{\sqrt{1-(x^2+y^2)}} \cdot \frac{1}{\sqrt{x^2+y^2}}$$

$$u_y = \frac{1}{\sqrt{1-(x^2+y^2)}} \frac{2y}{2\sqrt{x^2+y^2}} = \frac{y}{\sqrt{1-(x^2+y^2)}} \cdot \frac{1}{\sqrt{x^2+y^2}}$$

$x u_x + y u_y$.

$\Rightarrow$

$$= x \cdot \frac{x}{\sqrt{1-(x^2+y^2)}} \cdot \frac{1}{\sqrt{x^2+y^2}} = \frac{x^2}{\sqrt{(1-x^2-y^2)(x^2+y^2)}}$$

$$= \frac{y^2}{\sqrt{(1-x^2-y^2)(x^2+y^2)}}$$

$\therefore x u_x + y u_y =$

$$\frac{x^2+y^2}{\sqrt{(1-x^2-y^2)(x^2+y^2)}} = \sqrt{\frac{x^2+y^2}{(1-x^2-y^2)}}$$

(2) $u_x = \dfrac{x}{\sqrt{(1-x^2-y^2)(x^2+y^2)}}$

$u_{xx} = \dfrac{\sqrt{(1-x^2-y^2)(x^2+y^2)}\,(1) - (x)\left[\dfrac{(-2x)(x^2+y^2)+(1-x^2-y^2)(2x)}{2\sqrt{(1-x^2-y^2)(x^2+y^2)}}\right]}{(1-x^2-y^2)(x^2+y^2)}$

$= \dfrac{2(1-x^2-y^2)(x^2+y^2) - 2x^2\left[(1-x^2-y^2)-(x^2+y^2)\right]}{2\sqrt{(1-x^2-y^2)(x^2+y^2)}\,.\,(1-x^2-y^2)(x^2+y^2)}$

$\Rightarrow u_{xx} = \dfrac{2(1-x^2-y^2)(x^2+y^2) - 2x^2+2x^4+2x^2y^2}{2(1-x^2-y^2)(x^2+y^2)\sqrt{(1-x^2-y^2)(x^2+y^2)}}$

Similarly, $u_{yy} =$

$u = \sin^{-1}\left[t\sqrt{x^2+y^2}\right]$

$\sin u = t\sqrt{x^2+y^2}$ $\quad n=1.$ (degree)

(1) $xu_x + yu_y = n\dfrac{f(u)}{f'(u)}$ $\qquad f(u) = \sin u$

$f'(u) = \cos u$

$xu_x + yu_y = (1)\dfrac{\sin u}{\cos u}$ $\qquad \therefore\ xu_x + yu_y = \underline{\underline{\tan u}}$

(2)

$x^2u_{xx} + y^2u_{yy} + 2xy\,u_{xy} = g(u)\left[g'(u) - 1\right]$

$g(u) = \dfrac{nf(u)}{f'(u)} = \dfrac{1\,\sin u}{\cos u} = \tan u.$

$g'(u) = \sec^2 u.$

$x^2u_{xx} + y^2u_{yy} + 2xy\,u_{xy} = \tan u\left[\sec^2 u - 1\right]$

$= \tan u\,(\tan^2 u)$

$\therefore\ x^2u_{xx} + y^2u_{yy} + 2xy\,u_{xy} = \underline{\underline{\tan^3 u.}}$

Date 22/11/24

Q. If $T = \sin\left(\dfrac{xy}{x^2+y^2}\right) + \sqrt{x^2+y^2} + \dfrac{x^2y}{x+y}$

Then find $xT_x + yT_y$

$\Rightarrow$

$u = \sin\left(\dfrac{xy}{x^2+y^2}\right)$, $V = \sqrt{x^2+y^2}$, $w = \dfrac{x^2y}{x+y}$.

$u(tx, ty) = \sin\left(\dfrac{tx \cdot ty}{t^2x^2+t^2y^2}\right) = \sin\left(\dfrac{t^2\, xy}{t^2(x^2+y^2)}\right) = u(x,y)$.

$n=0$.

'u' is Homo. funⁿ of degree, $n=0$.

V is Homo funⁿ of degree, $n=1$

w is Homo funⁿ of degree, $n = 3-1 = 2$.

$T = U + V + w$.

by Euler Theorem,

$xu_x + yu_y = nu \Rightarrow 0$.

$xV_x + yV_y = nV \Rightarrow V$

$xw_x + yw_y = nw \Rightarrow 2w$.

But, $T = u + V + w$.

$T_x = u_x + V_x + w_x$, $T_y = u_y + V_y + w_y$.

$xT_x + yT_y = xu_x + xV_x + xw_x + yu_y + yV_y + yw_y$.

$= xu_x + yu_y + xV_x + yV_y + xw_x + yw_y$

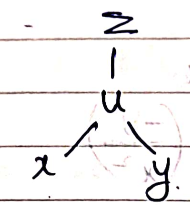
$= 0 + V + 2w$.

$\therefore xT_x + yT_y = \sqrt{x^2+y^2} + 2\left(\dfrac{x^2y}{x+y}\right)$

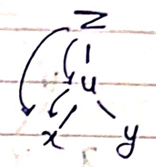
COMPOSITE FUNCTION.

① If $z = f(u)$ and $u = \phi(x\,y)$

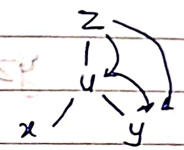
i.e, z is funⁿ of u; u is funⁿ of x and y.



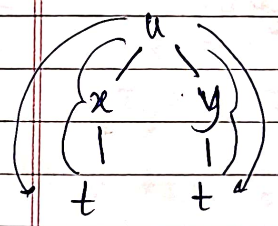
$$z_x = \frac{dz}{du} u_x = f'(u)\, u_x$$



$$z_y = \frac{dz}{du} u_y = f'(u)\, u_y$$



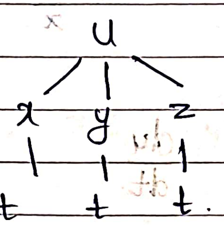
② If $u = f(x, y)$, $x = \phi(t)$, $y = \psi(t)$.



$$\frac{du}{dt} = u_x \frac{dx}{dt} + u_y \frac{dy}{dt}.$$ → Total derivative.

③

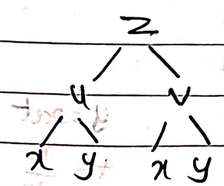
If $u = f(x, y, z)$, $x = \phi(t)$, $y = \psi(t)$, $z = G(t)$



$$\frac{du}{dt} = u_x \frac{dx}{dt} + u_y \frac{dy}{dt} + u_z \frac{dz}{dt}$$

→ Total derivative.

④ $z = f(u, v)$, $u = \phi(x\,y)$, $v = \psi(x, y)$



$$z_x = z_u u_x + z_v v_x$$

$$z_y = z_u u_y + z_v v_y.$$

Q. If $z = f(u,v)$, $u = \log(x^2+y^2)$, $v = y/x$
Then show that
$$xz_y - yz_x = (1+v^2)z_v$$

$\Rightarrow$

$$\begin{array}{ccccccc} & & & z & & & \\ & & u & & v & & \\ & x & y & & x & y. & \end{array}$$

$$z_x = z_u u_x + z_v v_x$$

$$z_x = z_u \frac{2x}{x^2+y^2} + z_v\left(\frac{-y}{x^2}\right)$$

$$yz_x = \frac{2xy}{x^2+y^2} z_u - \frac{y^2}{x^2} z_v \quad -(1).$$

$$z_y = z_u u_y + z_v v_y$$
$$= z_u \frac{2y}{x^2+y^2} + z_v\left(\frac{1}{x}\right)$$

$$xz_y = \frac{2xy}{x^2+y^2} z_u + z_v \quad -(2)$$

(2) $-$ (1),

$$\Rightarrow z_v + \frac{y^2}{x^2} z_v = (1 + v^2)z_v \quad \because v = \frac{y}{x}.$$

Q. If $u = y^2 - 4ax$, $x = at^2$, $y = 2at$ find $\frac{du}{dt}$.

$\Rightarrow$

$$\begin{array}{ccc} & u & \\ x & & y \\ | & & | \\ t & & t \end{array}$$

$$\frac{du}{dt} = u_x \frac{dx}{dt} + u_y \frac{dy}{dt}$$

$$= (-4a)(2at) + (2y)(2a)$$
$$= -8a^2t + 4ay.$$
$$= -8a^2\left(\frac{y}{2a}\right) + 4a(y)$$
$$= -4ay + 4ay = 0$$

$y = 2at$

$t = \frac{y}{2a}$

Homework Questions

① If $u = \sin^{-1}\left[\dfrac{x+y}{\sqrt{x}+\sqrt{y}}\right]$ then find ⓐ $x u_x + y u_y$

ⓑ $x^2 u_{xx} + y^2 u_{yy} + 2xy\, u_{xy}$

② If $z = f(u,v)$, $u = x\cos\theta - y\sin\theta$

$v = x\sin\theta + y\cos\theta$ $(\theta = \text{constant})$

then show that $x z_x + y z_y = u z_u + v z_v$.

$\Rightarrow$

$$\begin{tikzcd}[column sep=small] & & z \arrow[dl, dash] \arrow[dr, dash] & & \\ & u \arrow[dl, dash] \arrow[d, dash] & & v \arrow[d, dash] \arrow[dr, dash] & \\ x & y & & x & y \end{tikzcd}$$

$z_x = z_u u_x + z_v v_x.$

$= z_u(\cos\theta) + z_v(\sin\theta)$

$x z_x = x z_u\cos\theta + x z_v \sin\theta$ —①.

$z_y = z_u u_y + z_v v_y.$

$= z_u(-\sin\theta) + z_v(\cos\theta)$

$y z_y = -z_u \sin\theta \cdot y + y z_v \cos\theta$ —②.

① + ②, $x z_u\cos\theta + x z_v \sin\theta - z_u \sin\theta \cdot y + y z_v \cos\theta = x z_x + y z_y$.

Now,

$u z_u + v z_v = z_u(x\cos\theta - y\sin\theta) + z_v(x\sin\theta + y\cos\theta)$

$= z_u x\cos\theta - z_u y\sin\theta + z_v x\sin\theta + z_v y\cos\theta$..

$\therefore x z_x + y z_y = u z_u + v z_v.$

$u = \sin^{-1}\left[\dfrac{x+y}{\sqrt{x}+\sqrt{y}}\right]$

$\sin u = \dfrac{x+y}{\sqrt{x}+\sqrt{y}}$ $\begin{matrix}\to \text{degree} = 1 \\ \to \text{degree} = 1/2\end{matrix}$ $n = 1 - \frac{1}{2} = 1/2$. This is Homogeneous funⁿ.

$\sin u = f(u)$. ⓐ $x u_x + y u_y = \dfrac{n f(u)}{f'(u)}$

$f'(u) = \cos u.$

$$x u_x + y u_y = \frac{1}{2}\frac{\sin u}{\cos u} = \frac{\tan u}{2}$$

(b) $x^2 u_{xx} + y^2 u_{yy} + 2xy\, u_{xy} = g(u)\left[g'(u) - 1\right]$.

$$g(u) = n\frac{f(u)}{f'(u)} = \frac{1}{2}\frac{\sin u}{\cos u} = \frac{\tan u}{2}.$$

$$g'(u) = \frac{\sec^2 u}{2}.$$

$$\therefore x^2 u_{xx} + y^2 u_{yy} + 2xy\, u_{xy} = \frac{\tan u}{2}\left[\frac{\sec^2 u}{2} - 1\right]$$

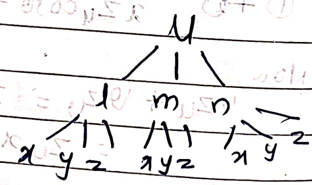
Q. If $u = f(2x - 3y,\ 3y - 4z,\ 4z - 2x)$ then find $\frac{1}{2}u_x + \frac{1}{3}u_y + \frac{1}{4}u_z$.

$\Rightarrow$ Put, $l = 2x - 3y$, $m = 3y - 4z$, $n = 4z - 2x$

$u = f(l, m, n)$. $l = \phi(x, y, z)$

$m = \psi(x, y, z)$

$n = \lambda(x, y, z)$



$$u_x = u_l l_x + u_m m_x + u_n n_x$$
$$= u_l(2) + u_m(0) + u_n(-2)$$
$$= 2u_l - 2u_n.$$

$$\therefore \frac{1}{2}u_x = u_l - u_n.$$

$$u_y = u_l l_y + u_m m_y + u_n n_y.$$
$$= u_l(-3) + u_m(3) + u_n(0)$$
$$= -3u_l + 3u_m$$

$$\therefore \frac{1}{3}u_y = u_m - u_l$$

$u_z = u_l l_z + u_m m_z + u_n n_z$

$= u_l(0) + u_m(-4) + u_n(4)$

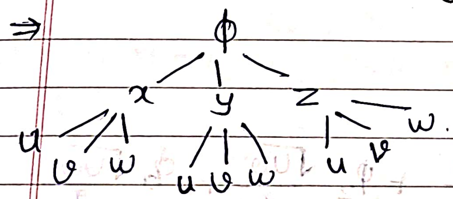
$= -4u_m + 4u_n$. $\therefore \frac{1}{4}u_z = u_n - u_m$.

$\Rightarrow \frac{1}{2}u_x + \frac{1}{3}u_y + \frac{1}{4}u_z = u_l - u_n + u_m - u_l + u_n - u_m$

$= \underline{\underline{0}}$

Q. If $\phi = f(x, y, z)$, $x = \sqrt{vw}$, $y = \sqrt{uw}$, $z = \sqrt{uv}$.
prove that, $x\phi_x + y\phi_y + z\phi_z = u\phi_u + v\phi_v + w\phi_w$

$\Rightarrow$



$\phi_u = \phi_x x_u + \phi_y y_u + \phi_z z_u$.

$\phi_v = \phi_x x_v + \phi_y y_v + \phi_z z_v$.

$\phi_w = \phi_x x_w + \phi_y y_w + \phi_z z_w$.

~~$x\phi_x + y\phi_y + z\phi_z = u(\phi_x x_u + \phi_y y_u +$~~

LHS. ~~$x\phi_x + y\phi_y + z\phi_z$~~.

$x = \sqrt{vw}$, $x_u = 0$, $x_v = \frac{w}{2\sqrt{vw}}$

$y = \sqrt{uw}$, $y_u = \frac{w}{2\sqrt{uw}}$, $y_v = 0$.

$z = \sqrt{uv}$, $z_u = \frac{v}{2\sqrt{uv}}$, $z_v = \frac{u}{2\sqrt{uv}}$

$x_w = \frac{v}{2\sqrt{vw}}$, $y_w = \frac{u}{2\sqrt{uw}}$, $z_w = 0$.

$\phi_u = \phi_x(0) + \phi_y\left(\frac{w}{2\sqrt{uw}}\right) + \phi_z\left(\frac{v}{2\sqrt{uv}}\right)$

$\phi_v = \phi_x\left(\frac{w}{2\sqrt{vw}}\right) + \phi_y(0) + \phi_z\left(\frac{u}{2\sqrt{uv}}\right)$

$\phi_w = \phi_x\left(\frac{v}{2\sqrt{vw}}\right) + \phi_y\left(\frac{u}{2\sqrt{uw}}\right) + \phi_z(0)$.

$$x\phi_x + y\phi_y + z\phi_z = u\phi_u + v\phi_v + w\phi_w.$$

$$\Rightarrow x\phi_x + y\phi_y + z\phi_z = u\left[\phi_y\left(\frac{w}{2\sqrt{uw}}\right) + \phi_z\left(\frac{v}{2\sqrt{uv}}\right)\right]$$

$$+ v\left[\phi_x\left(\frac{w}{2\sqrt{vw}}\right) + \phi_z\left(\frac{u}{2\sqrt{uv}}\right)\right] + w\left[\phi_x\left(\frac{v}{2\sqrt{vw}}\right) + \phi_y\left(\frac{u}{2\sqrt{uw}}\right)\right]$$

RHS,

$$\Rightarrow \phi_y\left(\frac{wu}{2\sqrt{uw}}\right) + \phi_z\left(\frac{vu}{2\sqrt{uv}}\right) + \phi_x\left(\frac{wv}{2\sqrt{wv}}\right) + \phi_z\left(\frac{uv}{2\sqrt{uv}}\right)$$

$$+ \phi_x\left(\frac{vw}{2\sqrt{vw}}\right) + \phi_y\left(\frac{uw}{2\sqrt{uw}}\right)$$

$$\Rightarrow \phi_y\frac{\sqrt{uw}}{2} + \phi_z\frac{\sqrt{uv}}{2} + \phi_x\frac{\sqrt{wv}}{2} + \phi_z\frac{\sqrt{uv}}{2} + \phi_x\frac{\sqrt{vw}}{2}$$

$$+ \phi_y\frac{\sqrt{uw}}{2}.$$

RHS, $\Rightarrow \phi_x\sqrt{vw} + \phi_y\sqrt{uw} + \phi_z\sqrt{uv}$

LHS $\Rightarrow \phi_x x + \phi_y y + \phi_z z = \phi_x\sqrt{vw} + \phi_y\sqrt{uw} + \phi_z\sqrt{uv}.$

LHS = RHS.

* Maxima and Minima or Extremum value of function of two variables x, y.

Let $f(x\ y)$ be given function.

Step 1. Calculate $p = f_x$, $q = f_y$, $r = f_{xx}$, $s = f_{xy}$, $t = f_{yy}$.

Step 2. find stationary point by solving $p=0$, $q=0$ simultaneously.

Step 3. Calculate p, q, r, s, t at stationary point

Step 4. find $rt - s^2$ at stationary point.

1. If $rt-s^2>0$ and $r<0$ at stationary point, then $f(x,y)$ has maxima at given stationary point.

2. If $rt-s^2>0$ and $r>0$ at stationary point, then $f(x,y)$ has Minima at given stationary point.

3. If $rt-s^2<0$ then neither Maxima nor minima at Stationa Point called Saddle point.

4. If $rt-s^2=0$ then further investigation Needed.

Note:
1. stationary point are always real, it complex then exclude.
2. stationary point may more than one.

Q.1 find maxima or minima of given funⁿ. Or decide max. or min.

$f(x,y) = x^2+y^2$

⇒ Step 1: $p = f_x = 2x$, $r = f_{xx} = 2$, $t = f_{yy} = 2$

$q = f_y = 2y$, $s = f_{xy} = 0$

Step 2: Standard point : $p=0 \rightarrow 2x=0 \rightarrow x=0$

$q=0 \rightarrow 2y=0 \rightarrow y=0.$

$\therefore$ Stationary point $(0,0)$ — x, y

Step 3 at stationary point $(0,0)$

$p|_{(0,0)}=0$, $q|_{(0,0)}=0$, $r|_{(0,0)}=2$.

$s|_{(0,0)}=0$, $t|_{(0,0)}=2$

Step 4 $rt - s^2$ at stationary point

$rt - s^2 = 4 - 0^2 = 4 > 0$.

$r = 2 > 0$

given funⁿ have minima at $(0,0)$ and minimum value of $f(x,y)$ is $f(0,0) = 0^2 + 0^2 = 0$

$f(x,y) = xy - x^2 - y^2 - 2x - 2y + 4$

1)

$p = f_x = y - 2x - 2$ $\quad s = f_{xy} = 1$

$q = f_y = x - 2y - 2$ $\quad t = f_{yy} = -2$

$r = f_{xx} = -2$

2) $p = 0 \rightarrow y - 2x - 2 = 0$

$q = 0 \rightarrow x - 2y - 2 = 0.$

$$\begin{array}{r} x - 2y - \cancel{2} = 0 \\ -2x - y - \cancel{2} = 0 \\ \hline x - y = 0 \end{array}$$

$x = y$. $\quad y - 2y - 2 = 0$

$-y - 2 = 0 \Rightarrow y = -2.$

$x = -2.$

$\therefore$ stationary point $(-2, -2)$
$\quad x \quad y$

3) at stationary point $(-2, -2)$

$p|_{(-2,-2)} = (-2) - 2(-2) - 2 = -2 + 4 - 2 = 0.$

$q|_{(-2,-2)} = 0.$ $\quad s = 1$

$r_{(-2,-2)} = -2.$ $\quad t = -2$

4) $rt - s^2 = (-2)(-2) - 1 = +4 - 1 = \cancel{5}\ 3 > 0$

$r = -2 < 0$

given funⁿ have maxima at $(-2,-2)$
and maximum value of $f(x,y)$ is $f(-2,-2)$

$$= (-2)(-2) \cancel{+} (-2)^2 - (-2)^2 - 2(-2) - 2(-2) + 4$$
$$= 8$$

③ $f(x,y) = x^2 + 4y^3 - 12y^2 - 36y + 2$

$\Rightarrow$ 1)

$p = f_x = 2x$ $\qquad$ $s_{xy} = f_{xy} = 0$

$q = f_y = 12y^2 - 24y - 36$

$r = f_{xx} = 2$ $\qquad$ $t = f_{yy} = 24y - 24$

2) $p = 0$ $\qquad 2x = 0 \rightarrow x = 0$

$q = 0$ $\qquad 12y^2 - 24y - 36 = 0$

$$y^2 - 2y - 3 = 0$$
$$y^2 - 3y + y - 3 = 0 \Rightarrow y(y-3) + 1(y-3) = 0$$
$$y = 3, -1$$

Stationary points are $(0,3)$ and $(0,-1)$

3) at st. point $(0,3)$

$p = 0$ $\qquad$ $r = 2$

$q = 12(9) - 24(3) - 36$ $\qquad$ $s = 0$

$= 108 - 72 - 36$ $\qquad$ $t = 24(3) - 24 = 72 - 24$

$= 108 - 108$ $\qquad$ $= 48$

$= 0$

at st. point $(0,-1)$ $\qquad$ $r = 2$

$p = 0$ $\qquad$ $s = 0$

$q = 12(1) - 24(-1) - 36$ $\qquad$ $t = 24(-1) - 24$

$= 12 + 24 - 36$ $\qquad$ $= -24 - 24$

$= 36 - 36 = 0$ $\qquad$ $= -48$

4) $rt - s^2$ at $(0,3)$

$\Rightarrow (2)(48) - (0)^2 = 96 > 0$

$r = 2 > 0$

given funⁿ have minima at $(0,3)$
and min. value of $f(x,y)$ is $f(0,3)$

$=$

$= -106$

$rt - s^2$ at $(0,-1)$

$\Rightarrow (2)(-48) - 0 = -96 < 0$

$r = 2 > 0$

given funⁿ have saddle point at $(0,-1)$

④ $f(x,y) = x^4 + y^4 - 4xy + 1$

$\Rightarrow$

1) $p = f_x = 4x^3 - 4y$ $\qquad s = f_{xy} = -4$

$q = f_y = 4y^3 - 4x$ $\qquad t = f_{yy} = 12y^2$

$r = f_{xx} = 12x^2$.

2) St. point $\quad p = 0 \rightarrow 4x^3 - 4y = 0 \rightarrow x^3 = y$

$q = 0 \rightarrow 4y^3 - 4x = 0 \rightarrow y^3 = x$

Put $y = x^3$ in $y^3 = x$. $\Rightarrow (x^3)^3 = x$

$\Rightarrow x^9 - x = 0$

$\Rightarrow x(x^8 - 1) = 0$

$\Rightarrow x((x^4)^2 - 1) = 0$

$x(x^4 - 1)(x^4 + 1) = 0$

$x((x^2)^2 - 1)(x^4 + 1) = 0 \Rightarrow x(x^2 - 1)(x^2 + 1)(x^4 + 1) = 0$

$x = 0 \longrightarrow x = 0$

$x^2 - 1 = 0 \longrightarrow x = \pm 1$

$x^2 + 1 = 0 \longrightarrow$ Imaginary

$x^4 + 1 = 0 \longrightarrow$ Imaginary

$x = 0, 1, -1$

but $y = x^3$.

If $x=0$ then $y=0$ $\Rightarrow$ Stationary points are

$x=1$ then $y=1$ $(0,0)\ (1,1)\ (-1,-1)$

$x=-1$ then $y=-1$

* Lagrange's Method of Undermined Multiplier.

let $f(x,y,z)$ be given funⁿ with given relation (condition or restriction) $\phi(x,y,z)$

Step I: Write auxilary eqⁿ $F(x,y,z) = f(x,y,z) + \lambda\phi(x,y,z)$ where 'λ' is undetermined Multiplier to be determined

Step II find F_x, F_y, F_z

Step III Eliminate 'λ' by solving $F_x=0$, $F_y=0$, $F_z=0$ together with $\phi(xyz)=0$

Step IV: from step III we get value of xyz

$\therefore$ (x,y,z) is stationary point.

NOTE: This method Just find stationary point but can't find Nature of stationary point. i.e Maxima or Minima.

~~Q.1.~~ $\cancel{f(x,y,z) = x^2y^2z^2}$

Q.1 Find maximum value of $f(x,y,z) = x^2y^3z^4$ subjected to condⁿ $x+y+z=5$.

$\Rightarrow$ $f(x,y,z) = x^2y^3z^4$, $\phi(x,y,z) = x+y+z=5$.

let $F(x,y,z) = f + \lambda\phi = x^2y^3z^4 + \lambda x + \lambda y + \lambda z - 5\lambda$

$F_x = 2xy^3z^4 + \lambda = 0$ — ①

$F_y = 3y^2x^2z^4 + \lambda = 0$ — ②

$F_z = 4z^3x^2y^3 + \lambda = 0$ — ③

Eliminate 'λ' from eqⁿ ① ② and ③

$\lambda = -2xy^3z^4$

$\lambda = -3y^2x^2z^4 \Rightarrow -2xy^3z^4 = -3y^2x^2z^4 = -4z^3x^2y^3$ — ④

$\lambda = -4z^3x^2y^3$

divide by $x^2y^3z^4$ to eqⁿ ④

$\frac{2}{x} = \frac{3}{y} = \frac{4}{z}$ consider any two eqⁿ from there

$\Rightarrow \frac{2}{x} = \frac{3}{y} \Rightarrow y = \frac{3x}{2}$

$\frac{2}{x} = \frac{4}{z} \Rightarrow z = 2x$

Now using given condⁿ, $x + y + z = 5$.

$\Rightarrow x + \frac{3x}{2} + 2x = 5$

$\Rightarrow \frac{9x}{2} = 5 \Rightarrow \boxed{x = \frac{10}{9}}$

$y = \frac{3x}{2} = \frac{3 \times 10}{9 \times 2} = \frac{5}{3} = \frac{15}{9}.$

$z = 2x = 2\left(\frac{10}{9}\right) = \frac{20}{9}$

Max. value of $f(x,y,z)$ is $f\left(\frac{10}{9}, \frac{15}{9}, \frac{20}{9}\right)$

$= \frac{2^{10} \times 5^9}{3^{15}}$

classmate

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Page ______

Q.2. Divide 24 into three parts such that continued product of first, square of second, cube of third may be maximum.

$\Rightarrow f(x,y,z) = xy^2z^3.$

$\phi(x,y,z) = x+y+z = 24.$

$F(x,y,z) = f(x,y,z) + \lambda\,\phi(x,y,z)$

$= xy^2z^3 + \lambda(x+y+z-24)$

$F(x,y,z) = xy^2z^3 + x\lambda + y\lambda + z\lambda - 24\lambda.$

$F_x = y^2z^3 + \lambda = 0$ — ①

$F_y = 2xyz^3 + \lambda = 0$ — ②

$F_z = 3xy^2z^2 + \lambda = 0$ — ③

$\lambda = -y^2z^3, \quad \lambda = -2xyz^3, \quad \lambda = -3xy^2z^2.$

$\Rightarrow -y^2z^3 = -2xyz^3 = -3xy^2z^2$

divide each by xy^2z^3,

$$\frac{-y^2z^3}{xy^2z^3} = \frac{-2xyz^3}{xy^2z^3} = \frac{-3xy^2z^2}{xy^2z^3}.$$

$$\Rightarrow \frac{1}{x} = \frac{2}{y} = \frac{3}{z}. \Rightarrow \begin{array}{l} y = 2x \\ z = 3x \end{array}$$

Now using given ~~giv~~ condn,

$x+y+z = 24.$

$x + 2x + 3x = 24 \Rightarrow \boxed{x=4.} \quad \boxed{y=8} \quad \boxed{z=12}$ — stationary point.

max value of $f(x,y,z)$ is $f(4,8,12)$.

$= 4 \times 8^2 \times 12^3$

$= 4 \times 64 \times 144 \times 12.$

$= \underline{\underline{442368}}.$

Sum of three number is constant, Prove that their product is Maximum when they are equal.

$x + y + z = c$ where 'c' is constant

$f(x,y,z) = xyz$

$F(x,y,z) = f(x,y,z) + \lambda \phi(x,y,z)$

$= xyz + \lambda(x+y+z-c)$

$= xyz + x\lambda + y\lambda + z\lambda - c\lambda$

$F_x = yz + \lambda = 0$ $\quad -yz = \lambda, \quad -xz = \lambda, \quad -xy = \lambda.$

$F_y = xz + \lambda = 0$ $\quad yz = xz = xy.$

$F_z = xy + \lambda = 0$ $\quad$ divide each by xyz,

$\frac{yz}{xyz} = \frac{xz}{xyz} = \frac{xy}{xyz} \Rightarrow \frac{1}{x} = \frac{1}{y} = \frac{1}{z}.$

$\therefore x = y = z.$

$x + y + z = c.$

$3x = c \Rightarrow x = c/3$

$\Rightarrow y = c/3$

$\Rightarrow z = c/3$

$f(x,y,z) = xyz = \frac{c}{3} \times \frac{c}{3} \times \frac{c}{3} = \frac{c^3}{27}.$

max. value is $\frac{c^3}{27}$

* Jacobian of function of 2 variable x & y.

Let, $u = f(x,y)$, $v = g(x,y)$ then Jacobian of u & v w.r.t to x and y is

$$J = \frac{\partial(uv)}{\partial(xy)} = \begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix} \quad \& \quad J' = \frac{\partial(xy)}{\partial(uv)} = \begin{vmatrix} x_u & x_v \\ y_u & y_v \end{vmatrix}$$

find x & y from u & v

* Jacobian of function of 3 variable x, y, z.

let, $u = f(x,y,z)$, $v = g(x,y,z)$, $w = h(x,y,z)$

Then,

$$J = \frac{\partial(u\,v\,w)}{\partial(x\,y\,z)} = \begin{vmatrix} u_x & u_y & u_z \\ v_x & v_y & v_z \\ w_x & w_y & w_z \end{vmatrix}$$

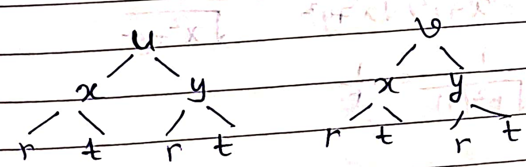
$$J' = \frac{\partial(x\,y\,z)}{\partial(u\,v\,w)} = \begin{vmatrix} x_u & x_v & x_w \\ y_u & y_v & y_w \\ z_u & z_v & z_w \end{vmatrix}$$

find x, y, z from u, v, w.

* Jacobian of composite funⁿ

let, $u = f(x,y)$, $v = g(x,y)$, $x = \phi(r,t)$, $y = \psi(r,t)$

$$J = \frac{\partial(u,v)}{\partial(r,t)} = \frac{\partial(u,v)}{\partial(x,y)} \cdot \frac{\partial(x,y)}{\partial(r,t)} = \begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix} \cdot \begin{vmatrix} x_r & x_t \\ y_r & y_t \end{vmatrix}$$



NOTE:

1. $J \cdot J' = 1$.
2. If $J \neq 0$, then given funⁿ are independent
3. If $J = 0$, then given funⁿ are dependent

Find J and J' and check whether given funⁿ are functionally dependent or functionally independent.

(1) $x = r\cos\theta$, $y = r\sin\theta$

$x = f(r,\theta)$ and $y = f(r,\theta)$

$$J = \frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} x_r & x_\theta \\ y_r & y_\theta \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix}$$

$$= r\cos^2\theta - (-r\sin^2\theta) = r(\cos^2\theta + \sin^2\theta) = \underline{\underline{r.}}$$

but 'r' and 'θ' not given,

$$J' = \frac{\partial(r,\theta)}{\partial(x,y)} = \begin{vmatrix} r_x & r_y \\ \theta_x & \theta_y \end{vmatrix}.$$

$x^2 + y^2 = r^2$. $\Rightarrow r = \sqrt{x^2+y^2}$. $\tan\theta = \frac{y}{x} \Rightarrow \theta = \tan^{-1}\left(\frac{y}{x}\right)$

$\theta = \tan^{-1}\left(\frac{y}{x}\right)$

$$J' = \begin{vmatrix} \frac{2x}{2\sqrt{x^2+y^2}} & \frac{2y}{2\sqrt{x^2+y^2}} \\ \frac{1}{1+\left(\frac{y}{x}\right)^2} \times \left(\frac{-y}{x^2}\right) & \frac{1}{1+\left(\frac{y}{x}\right)^2} \times \left(\frac{1}{x}\right) \end{vmatrix}$$

$$J' = \frac{x^2}{(x^2+y^2)\sqrt{x^2+y^2}} + \frac{y^2}{(x^2+y^2)\sqrt{x^2+y^2}} = \frac{1}{\sqrt{x^2+y^2}}$$

$$J' = \frac{1}{\sqrt{(r^2\cos^2\theta + r^2\sin^2\theta)}} = \frac{1}{\sqrt{r^2(1)}} = \frac{1}{r}.$$

$JJ' = r \times \frac{1}{r} = 1.$

$\therefore J \neq 0$, therefore, given funⁿ are independent.

Q.1)
Q.2)
Q.3)
Q.4)
⇒Q.1)
Q.3)
x
u θ

Homework.

Q.1) If $u = xyz$, $v = x^2+y^2+z^2$, $w = x+y+z$ find J.

Q.2) If $u = x^2-y^2$, $v = 2xy$, $x = r\cos\theta$, $y = r\sin\theta$, find J.

Q.3) If $x = e^{v}\cos u$, $y = e^{v}\sin u$. Then prove that $JJ' = 1$

Q.4) If $x = uv$, $y = \dfrac{u+v}{u-v}$ then show that $\dfrac{\partial(u,v)}{\partial(x,y)} = \dfrac{(u-v)^2}{4uv}$.

⇒Q.1) $u = xyz$, $v = x^2+y^2+z^2$, $w = x+y+z$.

$$J = \frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} u_x & u_y & u_z \\ v_x & v_y & v_z \\ w_x & w_y & w_z \end{vmatrix}$$

$u_x = yz$. $\quad v_x = 2x \quad w_x = 1$

$u_y = xz \quad v_y = 2y \quad w_y = 1$

$u_z = xy \quad v_z = 2z \quad w_z = 1$

$$J = \begin{vmatrix} yz & xz & xy \\ 2x & 2y & 2z \\ 1 & 1 & 1 \end{vmatrix} = yz(2y-2z) - xz(2x-2z) + xy(2x-2y)$$

$= 2zy^2 - 2yz^2 - 2zx^2 + 2xz^2 + 2yx^2 - 2xy^2$

$= 2[zy^2 - xy^2 - yz^2 + xz^2 - zx^2 + yx^2]$

$\Rightarrow J = 2y^2(z-x) + 2z^2(x-y) + 2x^2(y-z)$

$\Rightarrow J = 2zy^2 - 2yz^2 - 2zx^2 + 2xz^2 + 2yx^2 - 2xy^2$

$J = 2(x-y)(y-z)(z-x)$

Q.3) $x = e^{v}\cos u$, $y = e^{v}\tan u$.

x y

u v u v

$$J = \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} x_u & x_v \\ y_u & y_v \end{vmatrix}$$

$x_u = e^{v}(-\sin u) \quad y_u = e^{v}(\sec^2 u)$

$x_v = e^{v}\cos u \quad y_v = e^{v}\tan u$

$$J = \begin{vmatrix} -\sin u\, e^{v} & \cos u\, e^{v} \\ \sec^2 u\, e^{v} & \tan u\, e^{v} \end{vmatrix} = (-\tan u \sin u)e^{2v} - \cos u \sec^2 u\, e^{2v}$$

$= (-\tan u \sin u)e^{2v}$

$= -\sin^2 u \sec u\, e^{2v} - \sec u\, e^{2v}$

$J = -\sec u\, e^{2v}(\sin^2 u + 1)$

Q.4) $x = uv$, $\quad y = \dfrac{u+v}{u-v}$

$$\begin{array}{c} x \\ u \quad v \end{array} \qquad \begin{array}{c} y \\ u \quad v \end{array}$$

$J = \dfrac{\partial(x,y)}{\partial(u,v)}$ We have to find $\dfrac{\partial(u,v)}{\partial(x,y)} \to J'$

We know that $JJ' = 1$.

$$J = \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} x_u & x_v \\ y_u & y_v \end{vmatrix}$$

$x_u = v$. $\qquad y_u = \dfrac{(u-v)(1) - (u+v)(1)}{(u-v)^2} = \dfrac{u-v-u-v}{(u-v)^2} = \dfrac{-2v}{(u-v)^2}$

$x_v = u$.

$$y_v = \frac{(u-v)(1) - (u+v)(-1)}{(u-v)^2} = \frac{u-v+u+v}{(u-v)^2} = \frac{2u}{(u-v)^2}$$

$$J = \begin{vmatrix} x_u & x_v \\ y_u & y_v \end{vmatrix} = \begin{vmatrix} v & u \\ \dfrac{-2v}{(u-v)^2} & \dfrac{2u}{(u-v)^2} \end{vmatrix} = \frac{2uv}{(u-v)^2} + \frac{2uv}{(u-v)^2}$$

$J = \dfrac{4uv}{(u-v)^2}$ $\qquad \because JJ' = 1. \quad \Rightarrow J' = \dfrac{1}{J}$.

$$\Rightarrow J' = \frac{\partial(u,v)}{\partial(x,y)} = \frac{(u-v)^2}{4uv}.$$

Q.3) $x = e^v \cos u$, $\quad y = e^v \sin u$

$$\begin{array}{c} x \\ u \quad v \end{array} \qquad \begin{array}{c} y \\ u \quad v \end{array} \qquad J = \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} x_u & x_v \\ y_u & y_v \end{vmatrix}$$

$x_u = e^v(-\sin u)$

$x_v = e^v \cos u$.

$y_u = e^v \cos u$.

$y_v = e^v \sin u$.

$$J = \begin{vmatrix} -e^v \sin u & e^v \cos u \\ e^v \cos u & e^v \sin u \end{vmatrix}$$

$$J = -e^{2v}\sin^2 u - e^{2v}\cos^2 u = -e^{2v}(1)$$

$$J = -e^{2v}$$

$x = e^v \cos u, \quad y = e^v \sin u.$

$x^2+y^2 = e^{2v}\cos^2 u + e^{2v}\sin^2 u = e^{2v}(\cos^2 u + \sin^2 u)$
$= e^{2v}$

$x^2+y^2 = e^{2v}.$

$\ln(x^2+y^2) = 2v.$

$\Rightarrow \quad v = \dfrac{\ln(x^2+y^2)}{2}.$

$\dfrac{x}{y} = \dfrac{e^v \cos u}{e^v \sin u} = \cot u = \dfrac{x}{y}. \Rightarrow u = \cot^{-1}\left(\dfrac{x}{y}\right)$

$J' = \dfrac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix}$

$u_x = \dfrac{-1}{1+\left(\frac{x}{y}\right)^2}\left(\dfrac{1}{y}\right) = \dfrac{-y}{x^2+y^2}, \quad u_y = \dfrac{-1}{1+\left(\frac{x}{y}\right)^2}\left(\dfrac{-x}{y^2}\right)$

$u_y = \dfrac{y^2}{x^2+y^2}\left(\dfrac{x}{y^2}\right) = \dfrac{x}{x^2+y^2}$

$v_x = \dfrac{2x}{2(x^2+y^2)} = \dfrac{x}{x^2+y^2}, \quad v_y = \dfrac{y}{x^2+y^2}.$

$J' = \begin{vmatrix} -y/x^2+y^2 & +x/x^2+y^2 \\ \dfrac{x}{x^2+y^2} & \dfrac{y}{x^2+y^2} \end{vmatrix} = \dfrac{-y^2}{(x^2+y^2)^2} - \dfrac{x^2}{(x^2+y^2)^2}$

$J' = \dfrac{-x^2-y^2}{(x^2+y^2)^2} = \dfrac{-(x^2+y^2)}{(x^2+y^2)^2} = \dfrac{-1}{x^2+y^2}$

$J = -e^{2v} = -e^{2\frac{\ln(x^2+y^2)}{2}} = -(x^2+y^2)$

$\Rightarrow JJ' = -(x^2+y^2)\left(\dfrac{-1}{x^2+y^2}\right) = 1$

Hence proved

Date 30/11/24

* IMPLICIT FUNCTION:

function $f(x,y) = c$ called implicit function.
If 'y' is function of 'x' but we can't express in term of 'x' or 'x' is funⁿ of 'y' but We can't express 'x' is funⁿ of 'y'.

Ex. $x^y + y^x = c$, $y\sin x = x\cos y$, $x^3+y^3 = 3axy$

→ Total derivative of Implicit funⁿ:
let $f(x,y) = c$ be implicit funⁿ.
then,

$$\boxed{\frac{dy}{dx} = \frac{-f_x}{f_y}}$$

Q. find total derivative of implicit funⁿ.

$f(x,y) = x^3 + y^3 = 3axy$

$\Rightarrow f(x,y) = x^3 + y^3 - 3axy$

$\therefore \frac{dy}{dx} = \frac{-f_x}{f_y}$ $\Rightarrow \frac{dy}{dx} = \frac{-(3x^2 - 3ay)}{3y^2 - 3ax} = \frac{-(x^2 - ay)}{(y^2 - ax)}$

* Error ~~And~~ and Approximation.

let $f(x,y)$ is continuous funⁿ in 'x' and 'y' if δx, δy are increament in 'x' and 'y' then increament δf in 'f' is given by

$$\delta f = f(x+\delta x, y+\delta y) - f(x,y)$$

① δx is called as absolute error in 'x'.
② $\frac{\delta x}{x}$ is called as relative error in 'x'.
③ $\frac{\delta x}{x} \times 100$ is called percentege error in 'x'.

NOTE: ① If $u = f(x,y)$ then $\delta u = \frac{\partial u}{\partial x}\delta x + \frac{\partial u}{\partial y}\delta y$

② If $u = f(x,y,z)$ then

$$\delta u = \frac{\partial u}{\partial x}\delta x + \frac{\partial u}{\partial y}\delta y + \frac{\partial u}{\partial z}\delta z$$

Q.1) Find % error in calculating the area of rectangle, when error of 3% is made in measuring each of it's sides.

⇒ let a, b be the sides of rectangle and 'A' is its area.

$A = ab$

Step I: apply 'log' on both sides

$\log A = \log(ab)$

$\log A = \log(a) + \log(b)$

Step II. differentiate above eqⁿ.

$$\frac{1}{A}\delta A = \frac{1}{a}\delta a + \frac{1}{b}\delta b$$

Step III. multiply by 100

$$\frac{1}{A}\delta A \times 100 = \frac{1}{a}\delta a(100) + \frac{\delta b}{b}(100)$$

but given % error in sides is $\frac{\delta a}{a} \times 100 = 3\%$

$\frac{\delta b}{b} \times 100 = 3\%$

$\therefore \frac{\delta A}{A} \times 100 = 6\%$