# Vector Space

A set V is said to be Vector space if it satisfy 10 properties.

* Algebra of Higher dimensional vector space.

**for $R^2$:** $\boxed{1}$ $V = R^2 = xy \text{ plane} = \{(x\ y) \mid x, y \in R\}$

$u = (x_1, y_1) \in V = R^2$, $x_1, y_1 \in R$
$v = (x_2, y_2) \in V = R^2$, $x_2, y_2 \in R$

I) addition: $u + v = (x_1, y_1) + (x_2, y_2)$
$= (x_1 + x_2, y_1 + y_2) \in R^2$

II) subtraction: $u - v = (x_1, y_1) - (x_2, y_2)$
$= (x_1 - x_2, y_1 - y_2) \in R^2$

III) Scalar multiplication (SM):
$\alpha \in R$ $u = (x, y) \in V = R^2$
↓
Scalar

$\alpha \cdot u = \alpha u = \alpha(x, y) = (\alpha x, \alpha y) \in R^2$

* Additive Identity:
$u = (x, y) \in V = R^2$
$\therefore (x, y) + (0, 0) = (x, y)$
$e = (0, 0)$ is additive identity of $R^2$

**for $R^3$:**

$V = R^3 = xyz \text{ plane} = \{(x, y, z) \mid x, y, z \in R\}$

$u = (x_1, y_1, z_1) \in V = R^3$, $x_1, y_1, z_1 \in R$
$v = (x_2, y_2, z_2) \in V = R^3$, $x_2, y_2, z_2 \in R$

I) addition: $u+v = (x_1, y_1, z_1) + (x_2, y_2, z_2)$
$= (x_1+x_2, \; y_1+y_2, \; z_1+z_2) \in R^3$

II) subtraction: $u-v = (x_1, y_1, z_1) - (x_2, y_2, z_2)$
$= (x_1-x_2, \; y_1-y_2, \; z_1-z_2) \in R^3$

III) scalar multiplication (SM):
$\alpha \in R$     $u = (x_1, y_1, z_1) \in V = R^3$
$\downarrow$
scalar
$\alpha . u = \alpha u = \alpha(x_1, y_1, z_1) = (\alpha x_1, \alpha y_1, \alpha z_1) \in V = R^3$

* Additive identity
$u = (x, y, z) \in V = R^3$
$\therefore (x, y, z) + (0,0,0) = (x, y, z)$
$e = (0,0,0)$ is additive identity of $R^3$

NOTE:
$V = R^n = \{(x_1, x_2, x_3, \ldots, x_n) \mid x_i \in R\}$
$e = (\underbrace{0, 0, 0, \ldots, 0}_{n \text{ times}}) \rightarrow$ Additive identity

[2] $V = M_{2\times2} = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \;\middle|\; a, b, c, d \in R \right\}$
= set of all 2X2 matrix.

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$e = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ is additive identity of $M_{2\times2}$

**NOTE:** $V = M_{m\times n}$ = set of all $m\times n$ matrix.
$e$ = Null matrix of order $m\times n$.

$\boxed{3}$ $V = P_2(t) = \{a_0 + a_1 t + a_2 t^2 \mid a_0, a_1, a_2 \in \mathbb{R}\}$

= set of all polynomial in 't' having degree less than or equal to 2.

$a_0 + a_1 t + a_2 t^2 + 0 + 0t + 0t^2 = a_0 + a_1 t + a_2 t^2$

$e = 0 + 0t + 0t^2$ is additive identity of $P_2$

**NOTE:** $V = P_n(t) = \{a_0 + a_1 t + a_2 t^2 + \dots + a_n t^n \mid a_i \in \mathbb{R}\}$
= set of all polynomial having degree less than or equal to 'n'.
$e = 0 + 0t + 0t^2 + \dots + 0t^n$ is additive identity of $P_n$

$\boxed{4}$ $V = \overset{\text{continuous}}{C}[a,b] = \{f : [a,b] \to \mathbb{R} \mid f \text{ is continuous}\}$
= set of all continuous function on $[a,b]$
$e = \bar{0}(x)$ is additive identity of function

$\bar{0} : [a,b] \to \mathbb{R}$
$\bar{0}(x) = 0$

## * Subspace of Vector space.

Let $V$ be vector space, $W$ be non-empty subset of $V$ then $W$ is subspace of $V$ if

1. $\boxed{\text{for all}}$ $u, v \in W$, $u + v \in W$
2. $\boxed{\text{for all } \alpha \in \mathbb{R}}$, for all $u \in W$; $\alpha u \in W$
3. $e \in W$ i.e. additive identity of $V$ must belong to $W$

All three properties are satisfied then only $W$ is subspace of V
If any one property not satisfied then $W$ is not a subspace

ex: $V=R$ $W=\{5,9\}$

× ① $5+5=10 \notin W$
$9+9=18 \notin W$
$9+5=14 \notin W$
$u+v \notin W$

× ② $\alpha \in R$, $\alpha \cdot 5 \notin W$
let, $\alpha=3 \Rightarrow 3\cdot 5=15 \notin W$
$\alpha u \notin W$

× ③ $e=0 \notin W$ $e \notin W$
$W$ is not subspace of $R$.

ex: $V=R$, $W=\{0\}$

✓ ① $0+0=0 \in W$, $u+v \in W$
✓ ② $\alpha \cdot 0 = 0 \in W$, $\alpha u \in W$
✓ ③ $e=0 \in W$, $e \in W$
$W$ is subspace of $R$

Property ①: If $u, v \in W$
then check for $u+u \in W$
$v+v \in W$, $u+v \in W$
(addⁿ of itself of all elements and addⁿ with each other, all are belonging to $W$ then only Property ① satisfied)

Property ②, $\alpha u \in W$ and $\alpha \in R$.
↳ all are satisfied then only property ② satisfied.

Q. $V=R^2$ $W=\{(x,y) \mid x+y=0\}$

⇒ Check for properties.

① $u=(x_1,y_1) \in W \Rightarrow x_1+y_1=0$
$v=(x_2,y_2) \in W \Rightarrow x_2+y_2=0$

$u+v=(x_1,y_1)+(x_2,y_2)$
$=(x_1+x_2, y_1+y_2) \in W$
$x_1+x_2+y_1+y_2 = x_1+y_1+x_2+y_2$
$=0+0=0$

$u+v \in W$

* arbitary constants are used when there are multiple elements in $W$.

② $\alpha \in R$ , $u = (x, y) \in W \Rightarrow x + y = 0$

$\alpha u = \alpha(x, y) = (\alpha x, \alpha y) \in W$

$\alpha x + \alpha y = \alpha(x + y) = \alpha 0 = 0$

$\alpha u \in W$

③ $e = (0, 0) \in W$ If $x = 0$ & $y = 0$

$0 + 0 = 0$.

$\Rightarrow W$ is subspace of $R^2$

Q. $V = R^2$ $W = \{(x, y) \mid x + y = 3\}$

$\Rightarrow$ ① $u = (x_1, y_1) \in W \Rightarrow x_1 + y_1 = 3$

$v = (x_2, y_2) \in W \Rightarrow x_2 + y_2 = 3$

$u + v = (x_1, y_1) + (x_2, y_2)$

$= (x_1 + x_2, y_1 + y_2) \in W$

$x_1 + x_2 + y_1 + y_2 = x_1 + y_1 + x_2 + y_2$

$= 3 + 3 = 6$

$u + v \notin W$.

② $\alpha \in R$ , $u = (x, y) \in W \Rightarrow x + y = 3$

$\alpha u = \alpha(x, y) = (\alpha x, \alpha y) \in W$

$\alpha x + \alpha y = \alpha(x + y) = 3\alpha \neq 3$ (let, $\alpha = 5$)

$\alpha u \notin W$

③ $e = (0, 0) \notin W$ $0 + 0 \neq 3$

W is <u>not subspace</u> of $R^2$

(not necessary to write all three properties, write anyone)

Q. $V = R^2$ $W = \{(x, y) \mid x^2 + y^2 = 0\}$

$\Rightarrow$ There is no need of arbitrary constant, as only one possibility

i.e $0^2 + 0^2 = 0$

$x^2+y^2=0 \Rightarrow x=0,\ y=0$

$W=\{(0,0)\} \rightarrow$ Trivial subspace.

① $u=(0,0)$ $\quad v=(0,0)$ $\quad u+v=(0,0)\in W$

$\therefore u+v\in W$.

② $\alpha u=\alpha(0,0)=(0,0)\in W \quad \therefore \alpha u\in W$

③ $e=(0,0)\in W$

$\therefore W$ is subspace of $R^2$.

Trivial subspace:
If 'W' contain exactly <u>one</u> element and that element is additive Identity of given vector space, then it is called as Trivial subspace and it is always subspace. there is no Need to check.

$V=R^2$ ① $W_1=\{(x,y)\mid x^2-y^2=0\}$ ✗
② $W_2=\{(x,y)\mid x=2y\}$ ✓
③ $W_3=\{(x,y)\mid |x|^2+|y|^2=0\}$ ✓
④ $W_4=\{(x,y)\mid x>0\}$ ✓
⑤ $W_5=\{(x,0)\}$ ✓
⑥ $W_6=\{(0,1)\}$ ✗

$V=R^3$ ① $W_1=\{(x,y,z)\mid x+2y+3z=0\}$ ✓
② $W_2=\{(x,y,z)\mid x=2y\}$ ✓
③ $W_3=\{(x,y,z)\mid x^2-y^2=0\}$ ✗

Q.1) ① $W_1=\{(x,y)\mid x^2-y^2=0\}$ $\quad V=R^2$

$\Rightarrow$ [1] $u=(x_1,y_1)\in W \Rightarrow x_1^2-y_1^2=0$

$v=(x_2,y_2)\in W \Rightarrow x_2^2-y_2^2=0$

$u^2-v^2=(x_1^2,y_1^2)-(x_2^2,y_2^2)$

$=(x_1^2-x_2^2,\ y_1^2-y_2^2)\in W_1$

$x_1^2-x_2^2-(y_1^2-y_2^2)=x_1^2-x_2^2-y_1^2+y_2^2$

Q.1) ① $W_1 = \{(x,y) \mid x^2 - y^2 = 0\}$ , $V = R^2$

$\Rightarrow$ 1] $u = (x_1, y_1) \in W \Rightarrow x_1^2 - y_1^2 = 0$

$v = (x_2, y_2) \in W \Rightarrow x_2^2 - y_2^2 = 0$

$u + v = (x_1, y_1) + (x_2, y_2)$

$= (x_1 + x_2, y_1 + y_2) \in W$

$x_1^2 - y_1^2 + x_2^2 - y_2^2 = (x_1^2 + x_2^2) - (y_1^2 + y_2^2)$

$= (x_1^2 - y_1^2) + (x_2^2 - y_2^2) = 0 + 0 = 0$

$u + v \in W$.

2] $\alpha \in R$, $u = (x, y) \in W \Rightarrow x^2 - y^2 = 0$

$\alpha u = \alpha(x, y) = (\alpha x, \alpha y) \in W$

$\cancel{v} = \alpha x + \alpha y =$

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Q. 1] Here condition are two then solve using type.

$V = R^2$ $W = \{(x,y) \mid x^2 - y^2 = 0\}$

$y = x$, $y = -x$

$W = \{(x, -x) \quad (x, x)\}$

① $(x, -x) + (x, -x) = (2x, -2x) \in W$

$(x, x) + (x, x) = (2x, 2x) \in W$

$(x, -x) + (x, x) = (2x, 0) \notin W$.

$\therefore u + v \notin W$

'W' is not subspace of $R^2$.

Q. $V = R^3$ $W = \{(x,y,z) \mid x = y\}$

OR.

$W = \{(x, x, z) \mid x, z \in R\}$. or $W = \{(y,y,z) \mid y, z \in R\}$

$\Rightarrow$ ① $u = (x_1, y_1, z_1) \in W \Rightarrow x_1 = y_1$

$v = (x_2, y_2, z_2) \in W \Rightarrow x_2 = y_2$

$u + v = (x_1, y_1, z_1) + (x_2, y_2, z_2)$

$= (x_1 + x_2, y_1 + y_2, z_1 + z_2) \in W$

$x_1 + x_2 = y_1 + y_2$

$\therefore u + v \in W$

(2) $\alpha \in R$ , $u = (x, y, z) \in R \Rightarrow x = y$

$\alpha u = \alpha(x, y, z) = (\alpha x, \alpha y, \alpha z) \in W$

$\alpha x = \alpha y \quad \therefore \alpha u \in W$

(3) $e = (0, 0, 0) \in W \qquad 0 = 0$

$\therefore$ W is subspace of $R^3$

Q. $V = R^3 \quad W = \{(x, y, z) \mid x > 0\}$

OR

$W = \{(x, y, z) \mid x > 0, \; y, z \in R\}$

$\Rightarrow$ (3) $e = (0, 0, 0) \notin W$.

$\therefore x = 0 \not> 0$ W is not subspace of $R^3$.

(1) $u = (x_1, y_1, z_1) \in W \Rightarrow x_1 > 0$

$v = (x_2, y_2, z_2) \in W \Rightarrow x_2 > 0$

$u + v \Rightarrow x_1 + x_2 > 0$

first property satisfied.

(2) $\alpha \in R$, $u = (x, y, z) \in R$

$x > 0 \Rightarrow \alpha x > 0$.

Q. $V = M_{2\times2}$ $W = \left\{ \begin{bmatrix} a & b \\ b & c \end{bmatrix} \middle| \; a, b, c \in R \right\}$

OR

$W = \{A \in M_{2\times2} \mid A^T = A\}$

OR

W = set of all 2X2 symmetric matrix.

OR

$W = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \middle| \; b = c \right\}$

① $A, B \in W \Rightarrow A^T = A, \ B^T = B$

$A + B \in W$ iff $(A+B)^T = A+B$

LHS. $(A+B)^T = A^T + B^T = A+B$

$\therefore A+B \in W$.

② $\alpha \in R$, $A \in W \Rightarrow A^T = A$

$\alpha A \in W$ iff $(\alpha A)^T = \alpha A$.

LHS. $(\alpha A)^T = \alpha A^T = \alpha A$. $\therefore \alpha A \in W$.

③ $e = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \in W \because e^T = e$

$\Rightarrow W$ is subspace of $M_{2\times 2}$.

OR

① $u_1 = \begin{bmatrix} a_1 & b_1 \\ b_1 & c_1 \end{bmatrix} \in W$ $\cancel{\Rightarrow b_1 = c_1}$

$v_1 = \begin{bmatrix} a_2 & b_2 \\ b_2 & c_2 \end{bmatrix} \in W$

$u_1 + v_1 = \begin{bmatrix} a_1 + a_2 & b_1 + b_2 \\ b_1 + b_2 & c_1 + c_2 \end{bmatrix} \Rightarrow b_1 + b_2 = b_1 + b_2$.

$\Rightarrow W$ is subspace of $M_{2\times 2}$

$V = M_{2\times 2}$ $\quad W = \{ A \in M_{2\times 2} \mid |A| \neq 0 \}$

OR

$W$ = set of all $2\times 2$ non-singular matrix.

$W$ is not subspace of $M_{2\times 2}$

$\because e = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \notin W \quad \because |e| = 0$

Q. $V = P_2(t)$ $\quad W = \{a_0 + a_1 t + a_2 t^2 \mid a_0 = a_1\}$

$\Rightarrow$ ① $u = a_0 + a_1 t + a_2 t^2 \in W \quad \Rightarrow a_0 = a_1$

$v = b_0 + b_1 t + b_2 t^2 \in W \quad \Rightarrow b_0 = b_1$

$u + v = (a_0 + b_0) + (a_1 + b_1) t + (a_2 + b_2) t^2 \in W.$

$a_0 + b_0 = a_1 + b_1$

$\therefore u + v \in W.$

② $\alpha \in R$, $\quad u = a_0 + a_1 t + a_2 t^2 \in W \Rightarrow a_0 = a_1$

$\alpha u = \alpha a_0 + \alpha a_1 t + \alpha a_2 t^2 \in W.$

$\alpha a_0 = \alpha a_1 \quad \therefore \alpha u \in W.$

③ $e = 0 + 0t + 0t^2 \in W.$

$\therefore a_0 = 0, \; a_1 = 0, \; a_0 = a_1$

$\therefore W$ is subspace of $P_2(t)$.

Q. $V = P_2(t)$ $\quad W = \{(a_0 + a_1 t + a_2 t^2) \mid a_2 = 4\}$

$\Rightarrow$ ① $u = a_0 + a_1 t + a_2 t^2 \in W \quad \Rightarrow a_2 = 4$

$v = b_0 + b_1 t + b_2 t^2 \in W \quad \Rightarrow b_2 = 4.$

$u + v = (a_0 + b_0) + (a_1 + b_1) t + (a_2 + b_2) t^2 \notin W$

$\hookrightarrow 8 \neq 4.$

$\therefore u + v \notin W.$

② $\alpha \in R$, $\quad u = a_0 + a_1 t + a_2 t^2 \in W.$

$\alpha u = \alpha a_0 + \alpha a_1 t + \alpha a_2 t^2$

$\alpha a_2 \neq 4$ (always, $\alpha$ can be anything)

$\therefore$ Not a subspace.

③ $e = 0 + 0t + 0t^2 \in W$

$\hookrightarrow a_2 = 4.$ (given)

$4 \neq 0. \notin W.$

$\therefore W$ is not the subspace of $P_2(t)$.

# Sub-space Shortcuts.

$V = R^2$ has only 3 type of subspace.

① $W = R^2$

② $W = \{(0,0)\}$ → trivial

③ $W = \{(x,y) \mid y = mx\}$ single straight line passing through origin.

$V = R^3$ has only 4 type of subspace.

① $W = R^3$

② $W = \{(0,0,0)\}$ → Trivial.

③ $W = \{(x,y,z) \mid ax + by + cz = 0\}$ → single plane passing through origin.

④ Constructive

Examples: $V = R^2$

$W_1 = \{(x,y) \mid 4x + 4y = 0\}$ → ✓ single straight line passing through o

$W_2 = \{(x,y) \mid x = 0\}$ ✓

$W_3 = \{(x,y) \mid x^2 + y^2 = 0\}$ ✓ trivial. (only for $x=0, y=0 \Rightarrow x^2+y^2=0$)

$W_4 = \{(x,y) \mid x^2 - y^2 = 0\}$ ✗ $x = y$, $x = -y$ two lines.

$W_5 = \{(x,y) \mid |x|^2 + |y|^2 = 0\}$ ✓ $x^2 + y^2 = 0$ (trivial)

$W_6 = \{(x,y) \mid x = 6\}$ ✗ $x = 6$, $y = 0$ (Not passing through origin)

$W_7 = \{(x,y) \mid x, y \in R\}$ ✓ ($W = R^2$)

$W_8 = \{(x,y) \mid x + 2y = 0,\ 3x + 4y = 0\}$ ✓ by solving these $x = 0$ & $y = 0$ (trivial) ∴ passing through origin a single line.

It looks like they are two lines so they cannot be subspace. but by solving $x=0, y=0$ ∴ It is only one line and which is passing through origin.

# Linearly Independent (LI) and dependent (LD)

Let $V$ be Vector space $A = \{v_1, v_2, \dots, v_n\}$ are subset of $V$.

Let $c_1v_1 + c_2v_2 + \dots + c_nv_n = e$ ['e' is additive identity of V → Vector space]

($AX=0$ system in unknown $c_1, c_2, \dots c_n$)

Then ① $v_1, v_2, \dots v_n$ are LI if $AX=0$ have trivial soln.
i.e. $c_1 = c_2 = c_3 = \dots = c_n = 0$

② $v_1, v_2, \dots v_n$ are LD if $AX=0$ have infinitely many soln.
i.e atleast one $c_i \neq 0$.

## * Linear Span or spanning set of V.

Let $V$ be Vector space, $A = \{v_1, v_2, \dots v_n\}$ be subset of $V$

Let $c_1v_1 + c_2v_2 + \dots + c_nv_n = v$ (called linear combination)

($AX = B$ system in unknown $c_1, c_2, \dots c_n$)

Then

① 'A' is spanning set of V if $AX=B$ is consistent (i.e. $\rho(A) = \rho(A|B)$ without any condition)

② 'A' is not spanning set of V if $\rho(A) = \rho(A|B)$ with some condition.

## * Basis of vector space:

Let $V$ be vector space $A = \{v_1, v_2, \dots v_n\}$ be subset of $V$ then $A$ is Basis of $V$ if

① $v_1, v_2 \dots v_n$ are LI (linearly independent)

② 'A' is spanning set of V.

for Basis both condition will be satisfied. If anyone of it not satisfied then it is not Basis.

* **Dimension of Vector Space:**

Let $V$ be vector space.

$A = \{v_1, v_2, \dots, v_n\}$ be Basis of $V$ then dimension of vector space is Number of element in Basis set.

**NOTE:** ① Every vector space has infinitely many Basis set (except Trivial) but Number of element in Every basis set must be same.

② Dimension of $R^n = n$

Dimension of $M_{m \times n} = mn$

Dimension of $P_n(t) = n+1$

Dimension of $C[a,b] = \infty$

Q.1 Which of the following are linearly independent subset of R

Q.2 Which of the following are spanding set of $R^2$

Q.3 find span of 'A'.

Q.4 check whether subset 'A' is basis of $R^2$

Q.5 find dimension of $R^2$.

$V = R^2$ $\quad A = \{\underset{v_1}{(1,2)}, \underset{v_2}{(3,4)}\}$ OR $A = \left\{\underset{v_1}{\begin{bmatrix}1\\2\end{bmatrix}}, \underset{v_2}{\begin{bmatrix}3\\4\end{bmatrix}}\right\}$ → write as it is

↳ write transpose of it

⇒ 1) $c_1v_1 + c_2v_2 = e$ $\quad \because e$ for $R^2$ is $(0,0)$

$c_1(1,2) + c_2(3,4) = (0,0)$

$(c_1, 2c_1) + (3c_2, 4c_2) = (0,0)$

$(c_1 + 3c_2, 2c_1 + 4c_2) = (0,0)$

⇒ $c_1 + 3c_2 = 0$ $\quad AX = 0$ system

$2c_1 + 4c_2 = 0$ $\quad m = 2$ (Power of 'R')

$n = 2$ (no. of elements, here, $v_1$ and $v_2$

unknown = $v_1$ and $v_2$

ng set $R^2$ me jitne bhi elements hai unko $v_1$ and $v_2$ create karte hai (सकता)

Vertically → as it is
Horizontally → transpose.

$$[A|B] = \left[\begin{array}{cc|c} 1 & 3 & 0 \\ 2 & 4 & 0 \end{array}\right] \Rightarrow R_2 - 2R_1 \quad [A|B] = \left[\begin{array}{cc|c} 1 & 3 & 0 \\ 0 & -2 & 0 \end{array}\right] \rightarrow \text{RE}$$

$\rho(A) = 2$, $\rho(A|B) = 2$, $n = 2$

$\rho(A) = \rho(A|B) = n$

$\therefore$ $AX = 0$ Homo system have trivial solⁿ.

$c_1 = c_2 = 0$

$\therefore$ $v_1, v_2$ are linearly independent.

Spaning set :

Let $v = (x, y)$ be any arbitary element of $R^2$

$\therefore c_1 v_1 + c_2 v_2 = v$

$\therefore c_1 v_1 + c_2 v_2 = (x, y) \longrightarrow AX = B$ system

$m = 2$, $n = 2$, unknown $= c_1$ & $c_2$

$$\therefore [A|B] = \left[\begin{array}{cc|c} 1 & 3 & x \\ 2 & 4 & y \end{array}\right]$$

$$R_2 - 2R_1 \Rightarrow [A|B] = \left[\begin{array}{cc|c} 1 & 3 & x \\ 0 & -2 & y-2x \end{array}\right] \longrightarrow \text{Ref}$$

$\rho(A) = 2$, $\rho(A|B) = 2$

(without any condⁿ on $y - 2x$)

$\therefore \rho(A) = \rho(A|B)$ without condⁿ on $y - 2x$

$\therefore$ 'A' is spanning set of $R^2$

$\because$ 'A' is LI and spanning set of $R^2$

$\therefore$ 'A' is Basis of $R^2$

$\therefore$ Dimension of $R^2 = 2$ (elements in Basis set)

Span – किसको create करता है $R^2$ ke.

Spanning set ✓ → span(A) = V (vector space)

Spanning set ✗ → span(A) = $\{(x,y) \mid \text{cond}^n\}$

linear comb$^n$ 'V'

**NOTE:**

If 'A' set is the spanning set of 'V' then span of 'A' is equal to 'V'.

$\Rightarrow$ span(A) = V

Q. $A = \{(1,2), (2,4)\}$ $\quad V = R^2$

$\quad\quad\;\; v_1 \quad\;\; v_2$

⇒ ① for linear independent,

$c_1 v_1 + c_2 v_2 = (0,0)$ $\qquad AX = 0.$

$m = 2, \quad n = 2$

(power of R) (no. of elements i.e. $v_1$ and $v_2$)

$$[A|B] = \left[\begin{array}{cc|c} 1 & 2 & 0 \\ 2 & 4 & 0 \end{array}\right] \xrightarrow{R_2 - 2R_1} [A|B] = \left[\begin{array}{cc|c} 1 & 2 & 0 \\ 0 & 0 & 0 \end{array}\right] \rightarrow \text{REF}$$

$\rho(A) = 1, \rho(A|B) = 1, \quad n = 2$

$\rho(A) = \rho(A|B) < n$ (infinite sol$^n$).

$\therefore$ $v_1$ and $v_2$ are linearly dependent.

② for spanning set.

$c_1 v_1 + c_2 v_2 = (x,y)$

$$[A|B] = \left[\begin{array}{cc|c} 1 & 2 & x \\ 2 & 4 & y \end{array}\right] \xrightarrow{R_2 - 2R_1} [A|B] = \left[\begin{array}{cc|c} 1 & 2 & x \\ 0 & 0 & y-2x \end{array}\right]$$

↳ REF

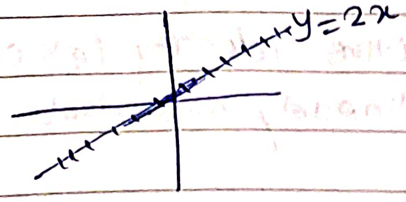
$\rho(A) = 1, \quad \rho(A|B) = 1$ if $y - 2x = 0$

$\rho(A) = \rho(A|B)$ with cond$^n$ $y - 2x = 0$.

'A' is not spanning set of $R^2$.

③ span = ?

$\text{span}(A) = \{(x,y) \mid y - 2x = 0\}$




Only the elements which on the line $y-2x=0 \Rightarrow y=2x$ only these elements are created by $V=\mathbb{R}^2$.
(These is span (A))

④ Basis: 'A' is not the Basis of $V=\mathbb{R}^2$.
as it is linearly dependent and 'A' is not the spanning set of $\mathbb{R}^2$.

⑤ Dimension: There is no dimension, as no Basis set.

## Shortcuts:

let $V$ be the vector space, 'A' be the subset of 'B' then

① If number of element in subset are more than dimension of vector space, then given subset is linearly dependent.

② If number of element in subset is equal to dimension of Vector space then write down 'A' matrix and find determinant of 'A' then ~~first statement~~

a) If $|A| \neq 0$, then subset is linearly independent.

b) If $|A| = 0$, then given subset is linearly dependent.

③ If number of element in subset are less than dimension of vector space then write down 'A' matrix and find Rank of 'A', then

a) If $\rho(A) =$ no. of element in subset then 'A' is linearly independent.

b) If $\rho(A) <$ no. of element in subset then 'A' is linearly dependent.

④ If given ~~State~~ subset contain, additive identity (e) of vector space, then given subset is linearly dependent.

⑤ If one of the element in subset, is a scalar ~~of~~ multiple of another element in subset then given subset is linearly dependent. ex. $A = \{(1,2,3)\ (2,4,6)\}$

$v_1 \quad v_2 \quad \therefore v_1 = v_2$

⑥ If given subset contain exactly one element and that element is non-zero then it is always linearly independent.

* 

> for linearly independend and spanning set:
> m = dimension of given vector space.
> n = no. of elements in subset.

Q. $V = M_{2\times 2}$ $A = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 2 & 3 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 3 & 4 \\ 5 & 0 \end{bmatrix} \begin{bmatrix} 4 & 5 \\ 6 & 7 \end{bmatrix} \right\}$

⇒ write like this.

$c_1 v_1 + c_2 v_2 + c_3 v_3 + c_4 v_4 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ $AX = 0$

$m = 4$ ($2\times 2$) → order.

$n = 4$ (no. of elements)

$[A|B] = \left[\begin{array}{cccc|c} 1 & 2 & 3 & 4 & 0 \\ 0 & 3 & 4 & 5 & 0 \\ 0 & 0 & 5 & 6 & 0 \\ 0 & 0 & 0 & 7 & 0 \end{array}\right]$ → $\rho(A) = \rho(A|B) = n.$

Trivial solⁿ.

∴ It is linearly independent.

Q. $V = P_2(t)$ $A = \{1,\ 2+3t,\ 4+5t+6t^2\}$

⇒ $c_1 v_1 + c_2 v_2 + c_3 v_3 = 0 + 0t + 0t^2$

$AX = 0$

$(P_n(t))$ $m = n+1 = 2+1 = 3$

$n = 3$ (no. of elements).

$1 = 1 + 0t + 0t^2$

$2+3t = 2 + 3t + 0t^2$

$4 + 5t + 6t^2$ as it is

$$[A|B] = \left[\begin{array}{ccc|c} 1 & 2 & 4 & 0 \\ 0 & 3 & 5 & 0 \\ 0 & 0 & 6 & 0 \end{array}\right] ; \quad \rho(A) = \rho(A|B) = n$$

$\therefore$ LI.

## * Dimension and Basis of Subspace

let 'V' be vector space, 'W' be subspace of V.

Then ① dimension W < dimension V

② dimenⁿ W = dimenⁿ V iff V = W.

## * Row space of Matrix:

Let 'A' be any $m \times n$ matrix,

Then $Row(A) = span\{R_1, R_2, \dots R_m\}$, where $R_1, R_2, \dots R_m$ are rows of 'A'. i.e, Row(A) = We have to find element of $\boxed{\mathbb{R}^n}$ which can be written as linear combination of $R_1, R_2, \dots R_m$

**NOTE:** ① Row (A) is subspace of $\boxed{\mathbb{R}^n}$

② dimension(Row(A)) = $\rho(A)$

③ Basis of Row (A) = non-zero row's of REF of 'A'.

## * Column space of Matrix:

let 'A' be any $m \times n$ matrix,

then $col(A) = span\{C_1, C_2, \dots C_n\}$ where $C_1, C_2 \dots C_n$ are column of 'A'. i.e col (A) = We have to find element of $\boxed{\mathbb{R}^m}$ which can be written as linear combination (span) of $C_1, C_2, \dots, C_n$.

**NOTE:** ① Col (A) is subspace of $\boxed{\mathbb{R}^m}$

② dimension (col(A)) = $\rho(A)$

③ Basis of Col (A) = check LE in REF, The LE present in that colm, those colm in original matrix are Basis of col (A) no. of diff. variables = dimension of 'W'.

* Null space of Matrix or $AX=0$ system:

let 'A' be any $m \times n$ matrix or $AX=0$ system of order $m \times n$ then null $(A)$ = $sol^n$ of $AX=0$

NOTE: ① Null (A) is subspace of $\underline{R^n}$

② Dimension (Null (A)) is no. of free variable or $\boxed{n-r(A)}$

Q.1 find dimension and Basis of subspace.

$V = R^2$; $W = \{(x,y) \mid x+y=0\}$

or

$W = \{(x,-x) \mid x \in R\}$

$\Rightarrow (x,-x) = x(1,-1)$

$v = c_1 v_1$

$\therefore A = \{(1,-1)\}$ is spanning set.

'A' is LI $\because$ We know that, single non-zero element is always LI.

(or $c_1 v_1 = (0,0)$) show that $c_1 = 0$ i.e Trivial $sol^n$)

$\therefore$ 'A' is Basis of 'W'.

Dimension of $W = 1$. (no. of diff. variables)

'$x$' and '$-x$' are same variable.

Q.2. $W = \{(x,y) \mid x,y \in R\}$ $V=R^2$.

$\Rightarrow (x,y) = x(1,0) + y(0,1)$

no. of diff. variables = 2 ($x$ and $y$).

$\therefore$ Dimension of W is 2.

$A = \{(1,0)(0,1)\}$ is spanning set of 'W'.

'A' is LI $\because \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1 \neq 0$.

'A' is Basis of 'W'.

$\therefore$ dim W = 2, W = V.

$V = R^2$, $W = \{(x,y) \mid x^2+y^2=0\}$.

$\Rightarrow$ i.e, $x^2+y^2=0 \Rightarrow x=0, y=0$.

$W = \{(0,0)\}$

Dim W = 0.

Basis of 'W' = $\phi$ (empty set)

(~~Basis~~ Dimension means, no. of elements in Basis set, but here Dim W = 0, but there is only one set which has zero element. i.e $\phi$ set.)

Q. $V = R^3$, $W = \{(x,y,z) \mid x=y\}$.

$\Rightarrow$ $(x, x, z)$. $\rightarrow$ two different variables.

$x(1,1,0) + z(0,0,1)$

$A = \{(1,1,0), (0,0,1)\}$ is spanning set.

Dimension of W = 2

$\hookrightarrow$ Two diff. variables ($x$ & $z$)

Q. $V = M_{2\times 2}$ $W = \left\{ \begin{bmatrix} a & b \\ b & c \end{bmatrix} \middle| a,b,c \in R \right\}$

$\Rightarrow$ $\begin{bmatrix} a & b \\ b & c \end{bmatrix} = a\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + b\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + c\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$

$A = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$ is spanning set.

Dimension of W = 3 (a, b, c are diff. variables).

Q. Find dimension and Basis of Row(A), col(A)

$$A = \begin{bmatrix} 1 & 3 & 5 \\ 1 & 3 & 0 \end{bmatrix}_{\substack{2\times 3 \\ (m\times n)}} \qquad m=2,\ n=3.$$

$\Rightarrow$ convert 'A' into REF,

$R_2 \Rightarrow R_2 - R_1$ $\quad A = \begin{bmatrix} 1 & 3 & 5 \\ 0 & 0 & -5 \end{bmatrix} \longrightarrow$ REF.

$\rho(A) = 2$

for Row(A),

① Row(A) is subspace of $R^n = R^3$

$\therefore W = \text{Row}(A) \quad V = R^3.$

② $\text{Dim}(\text{Row}(A)) = \rho(A) = 2$

③ Basis of Row(A) $= \{(1,3,5)\ (0,0,-5)\}$

④ Row(A) is 2 dimensional subspace of $R^3$.

for Col(A),

① col(A) is subspace of $R^m = R^2$.

$\therefore W = \text{col}(A),\ V = R^2$

② $\text{Dim}(\text{col}(A)) = \rho(A) = 2.$

③ Basis of col(A) $= \{(1,1)\ (5,0)\}$

$\because$ LE are $1, -5$ which are present in $1^{st}$ and $3^{rd}$ column of REF,

$\therefore$ $1^{st}$ and $3^{rd}$ colum of original matrix are Basis of col(A). $\quad [\text{col}(A) = \text{Row}(A^T)]$

④ col(A) is 2 dimensional subspace of $R^2$.

$\therefore V = W$

i.e. $\boxed{\text{col}(A) = R^2}$

Q. find dimension and Basis of Null (A).

$x + 2y + 3z = 0$ or $A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 2 & 1 \end{bmatrix}$

$x + 2y + z = 0$

$\Rightarrow$ find REF first,

$A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 2 & 1 \end{bmatrix} \Rightarrow R_2 \to R_2 - R_1$ $A = \begin{array}{c} \begin{matrix} x & y & z \end{matrix} \\ \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & -2 \end{bmatrix} \end{array} \to$ REF.

$\rho(A) = 2$, $\rho(A|B) = 2$, $n = 3$. $AX = 0$ have infinite sol$^n$.

$x, z$ are LE, $\therefore y$ is free

put $y = t$.

from REF, $x + 2y + 3z = 0$

$-2z = 0. \Rightarrow z = 0.$

$x + 2t + 0 = 0 \Rightarrow x = -2t, y = t, z = 0.$

$\therefore$ Null (A) = sol$^n$ of $AX = 0$

$\Rightarrow$ Null $(A) = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -2t \\ t \\ 0 \end{bmatrix}$

① Null (A) is subspace of $R^n = R^3$

$\therefore W = \text{Null}(A)$, $V = R^3$.

② Dimension (Null (A)) = no. of free variable or $n - \rho(A)$

$\therefore$ Dim (Null (A)) $= 3 - 2 = 1$.

③ Basis of Null (A) $= \left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} \right\}$

$\because \begin{bmatrix} -2t \\ t \\ 0 \end{bmatrix} = t \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$

for calculating Basis (Null (A)) remove the variable to out side and remaining is the Basis.

ex. $\begin{bmatrix} -r-s \\ r \\ s \end{bmatrix} = r \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + s \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$

Dim = 2 $(r, s)$.

Basis (Null (A)) $= \left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right\}$