UNIT-III : Vector Differentiation.

let $A = a_1 \bar{i} + a_2 \bar{j} + a_3 \bar{k}$, $B = b_1 \bar{i} + b_2 \bar{j} + b_3 \bar{k}$

then ① Dot product :

$$A \cdot B = a_1 b_1 + a_2 b_2 + a_3 b_3$$

i.e $A \cdot B = |A|\,|B| \cos\theta$

② cross Product : $A \times B = \begin{vmatrix} i & j & k \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$

i.e. $A \times B = |A||B| \sin\theta$

* Position Vector : 'r' is vector that point from origin to point $p(x, y, z)$ and use to locate point in space.

$\therefore \bar{r}(t) = x\,\bar{i} + y\,\bar{j} + z\,\bar{k}$

$|\bar{r}| = r = \sqrt{x^2 + y^2 + z^2}$

* Scalar point function (SPF) : is scalar quantity, ϕ depend on it's value on it's position say (x, y, z) in space then $\phi(x\,y\,z)$ called scalar point function (SPF).

eg. Temp., density, Potential etc.

* Vector point function (VPF): is vector quantity $\bar{F}$ depend on it's value on it's position say $(x\,y\,z)$ in space then $\bar{F}(x\,y\,z)$ called Vector point function (VPF).

eg. Velocity, acceleration etc.

* Vector Differentiation : let 'P' be any point on the curve whose Position vector is $\bar{r}(t)$. Then tangent vector (derivative) T at point 'p' is given by $\dfrac{d\bar{r}}{dt}$

IF 't' is Time variable then $v = \dfrac{d\bar{r}}{dt}$ (velocity)

and acceleration , $a = \dfrac{d^2\bar{r}}{dt^2}$ (acceleration)

NOTE: If $\bar{r}$ is position vector then

① $d\bar{r} = i\,dx + j\,dy + k\,dz$

② If $\bar{r} \cdot \frac{d\bar{r}}{dt} = 0$ then 'r' has constant magnitude.

③ If $\bar{r} \times \frac{d\bar{r}}{dt} = 0$ then 'r' has constant direction.

* Del operator: It is vector differential operator

$$\nabla = i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}$$

Note:

① $\nabla^2 = i\frac{\partial^2}{\partial x^2} + j\frac{\partial^2}{\partial y^2} + k\frac{\partial^2}{\partial z^2}$ → Laplacian operator.

$\nabla^2 \phi = 0$ → Laplace eqn

② $\nabla^3 = i\frac{\partial^3}{\partial x^3} + j\frac{\partial^3}{\partial y^3} + k\frac{\partial^3}{\partial z^3}$

* Gradient of $\boxed{\text{Scalar Point function}}$

let Φ be scalar point function, then

* Gradient of ϕ is $\nabla\phi = i\frac{\partial\phi}{\partial x} + j\frac{\partial\phi}{\partial y} + k\frac{\partial\phi}{\partial z}$

Notation Grad ϕ or $\nabla\Phi$

Note: ① $\nabla\Phi$ denote Normal to the surface Φ

② It is Maximum rate of change of Φ which is outword. drawn Normal to the surface

③ $\nabla\phi$ is vector quantity

$\bar{r} = xi + yj + zk$

① $\nabla \times \bar{r} = 0$ ③ $\nabla(\bar{a} \cdot \bar{r}) = \bar{a}$ ⑤ $\nabla F(r) = \frac{F'(r)}{r} \bar{r}$

② $\nabla \bar{r} = \frac{\bar{r}}{r}$ ④ $\nabla r^m = m r^{m-2} \bar{r}$ ⑥ $\nabla^2 f(r) = F''(r) + \frac{F'(r)}{r}$

Q.1. Find Grad ϕ or $\nabla \phi$, if $\phi = x^2 + y^2 + z^2$ at $(1,1,1)$

OR

Find Normal to the surface $x^2 + y^2 + z^2$ at $(1,1,1)$

$$\Rightarrow \nabla \phi = i \frac{\partial \phi}{\partial x} + j \frac{\partial \phi}{\partial y} + k \frac{\partial \phi}{\partial z}$$

$$\nabla \phi = i(2x) + j(2y) + k(2z)$$

$\therefore \nabla \phi$ at point $(1,1,1)$

$$\nabla \phi = 2i + 2j + 2k.$$

Q.2. Find grad ϕ if $\phi = \log(x^2 + y^2 + z^2)$

$$\Rightarrow \nabla \phi = i \frac{\partial \phi}{\partial x} + j \frac{\partial \phi}{\partial y} + k \frac{\partial \phi}{\partial z}$$

$$= i \left(\frac{2x}{x^2+y^2+z^2} \right) + j \left(\frac{2y}{x^2+y^2+z^2} \right) + k \left(\frac{2z}{x^2+y^2+z^2} \right)$$

$$\nabla \phi = \frac{2x}{x^2+y^2+z^2} i + \frac{2y}{x^2+y^2+z^2} j + \frac{2z}{x^2+y^2+z^2} k.$$

* ~~Divergence of Vector~~

$$\nabla \phi = \frac{2}{x^2+y^2+z^2} (xi + yj + zk)$$

$$\Rightarrow \nabla \phi = \frac{2\bar{r}}{r^2}$$

$\because \bar{r} = xi + yj + zk$

$r = \sqrt{x^2 + y^2 + z^2}$

* Divergence of $\boxed{\text{Vector point function}}$

If $\bar{F}$ is continuously differentiable vector point function then divergence of $\bar{F} = F_1 i + F_2 j + F_3 k$.

$$\nabla \cdot \bar{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$$

NOTE: ① $\nabla . F$ is scalar quantity.

② $\nabla . F$ represent any physical quantity represent the rate of outword flow through Unit volume per unit time.

*③ vector field F is solenoidal if $\nabla . \bar{F} = 0$

Q. find $\nabla . \bar{F}$ if $\bar{F} = x^2 y\, i + 2x\, j + z^2 k$ at $(1,2,3)$.

⇒ $F_1 = x^2 y$, $F_2 = zx$, $F_3 = z^2$

$$\nabla . \bar{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$$

$$= 2xy + 0 + 2z$$

$$\therefore \nabla . \bar{F} = 2xy + 2z$$

$$\nabla . \bar{F}\Big|_{(1,2,3)} = 2(1)(2) + 2(3)$$

$(1,2,3)$ → $x\ y\ z$

$$= 4 + 6$$

$$= 10$$ → Not solenoid, It has divergence.

* Curl of Vector point function.

If $\bar{F}$ is continuously differentiable vector point function then ~~divergence~~ curl of $\bar{F} = F_1 i + F_2 j + F_3 k$

$$\nabla \times \bar{F} = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ f_1 & f_2 & f_3 \end{vmatrix}$$

NOTE:

① $\nabla \times \bar{F}$ is vector quantity.

② $\nabla \times F$ represent measure angular velocity at any point of vector field.

③ vector field $\bar{F}$ is irrotational if $\nabla \times \bar{F} = 0$

$\nabla\phi$ → Normal to the surface → vector quantity

$\nabla\cdot\vec{F}$ → max. rate of change outward direction. → scalar quantity

$\nabla\times\vec{F}$ → angular velocity → vector quantity

Q. If $\vec{F} = xy^2\vec{i} + 2x^2yz\vec{j} - 3yz^2\vec{k}$. find $\nabla\cdot\vec{F}$ and $\nabla\times\vec{F}$ at the point $(1,-1,1)$

⇒ $F_1 = xy^2$, $F_2 = 2x^2yz$, $F_3 = -3yz^2$

$$\therefore \nabla\times\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ F_1 & F_2 & F_3 \end{vmatrix} = \begin{vmatrix} \hat{i} & \hat{j} & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ xy^2 & 2x^2yz & -3yz^2 \end{vmatrix}$$

$$= i\left[\frac{\partial}{\partial y}(-3yz^2) - \frac{\partial}{\partial z}(2x^2yz)\right] - j\left[\frac{\partial}{\partial x}(-3yz^2) - \frac{\partial}{\partial z}(xy^2)\right]$$

$$+ k\left[\frac{\partial}{\partial x}(2x^2yz) - \frac{\partial}{\partial y}(xy^2)\right]$$

$$= i[-3z^2 - 2x^2y] - j[0-0] + k[4xyz - 2xy]$$

$\therefore$ $\Delta\times\vec{F}$ at point $(1,-1,1)$

$$= i[-3(1) - 2(1)(-1)] - j(0) + k[4(1)(-1)(1) - 2(1)(-1)]$$

$$= i[-3+2] - j(0) + k(-4+2)$$

$$= -\hat{i} - 0\hat{j} - 2\hat{k}$$

⇒ $\Delta\times\vec{F} = -\hat{i} - 0\hat{j} - 2\hat{k}$

$\therefore$ $\vec{F}$ is rotational.

$$\nabla\cdot\vec{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z} = \frac{\partial}{\partial x}(xy^2) + \frac{\partial}{\partial y}(2x^2yz)$$

$$+ \frac{\partial}{\partial z}(-3yz^2)$$

$$= y^2 + 2x^2z - 6yz$$

$$\nabla \cdot \vec{F} = y^2 + 2x^2 z - 6yz$$

$$\nabla \cdot \vec{F}\Big|_{(1,-1,1)} = (-1)^2 + 2(1)^2 - 6(-1)(1) = 1 + 2 + 6 = \underline{\underline{9}}$$ → Not solenoid, It has divergence.

Q. Find the value of 'a' so that $\vec{F} = (x+3y)\,\hat{i} + (y-2z)\,\hat{j} + (x-az)\,\vec{k}$ is irrotational.

⇒ for irrotational. $\nabla \times \vec{F} = 0$.

$$\nabla \times \vec{F} = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ F_1 & F_2 & F_3 \end{vmatrix}$$

$F_1 = x + 3y$

$F_2 = y - 2z$

$F_3 = x - az$.

$$= \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ (x+3y) & (y-2z) & (x-az) \end{vmatrix}$$

$$\nabla \times \vec{F} = i\left[\frac{\partial}{\partial y}(x-az) - \frac{\partial}{\partial z}(y-2z)\right] - j\left[\frac{\partial}{\partial x}(x-az) - \frac{\partial}{\partial z}(x+3y)\right] + k\left[\frac{\partial}{\partial x}(y-2z) - \frac{\partial}{\partial y}(x+3y)\right]$$

$$= i[0-(-2)] - j[(1) - 0] + k[0-3]$$

$$= 2i - j - 3k$$

$$\Rightarrow 2i - j - 3k = 0i + 0j + 0k$$

If 'a' is present then we compare it with RHS.
But here, 'a' is not present in above eqn.

$\therefore a \notin R$. There is no 'a'.

$\nabla \times F = 0$.

Q. Check whether the vector $\vec{F} = (6xyz^3)\hat{i} + (3x^2 - z)\hat{j} + (3xz^2 - y)\hat{k}$ is irrotational.

$\Rightarrow$ $F_1 = 6xyz^3$, $F_2 = 3x^2 - z$, $F_3 = 3xz^2 - y$.

$\nabla \times \vec{F} = 0$ for irrotational.

$$\nabla \times F = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 6xyz^3 & 3x^2 - z & 3xz^2 - y \end{vmatrix}$$

$$= i\left[\frac{\partial}{\partial y}(3xz^2 - y) - \frac{\partial}{\partial z}(3x^2 - z)\right] - j\left[\frac{\partial}{\partial x}(3xz^2 - y) - \frac{\partial}{\partial z}(6xyz^3)\right]$$

$$+ k\left[\frac{\partial}{\partial x}(3x^2 - z) - \frac{\partial}{\partial y}(6xyz^3)\right]$$

$$= i[(-1) - (-1)] - j[3z^2 - 18xyz^2] + k[6x - 6xz^3]$$

$$= 0\hat{i} - j[3z^2 - 18xyz^2] + k[6x - 6xz^3] = 0i + 0j + 0k$$

$$\nabla \times F = 0\hat{i} - j[3z^2(1 - 6xy)] + k[6x(1 - z^3)]$$

The curl is not identically zero, $\therefore$ The vector $\vec{F}$ is not irrotational.

Q. Prove that the vector $\vec{F} = (y^2 - z^2 + 3yz - 2x)\hat{i} + (3xz + 2xy)\hat{j} + (3xy - 2xz + 2z)\hat{k}$ is both irrotational and solenoid.

$\Rightarrow$ $F_1 = y^2 - z^2 + 3yz - 2x$, $F_2 = 3xz + 2xy$

$F_3 = 3xy - 2xz + 2z$

for irrotational $\nabla \times F = 0$.

$$\nabla \times F = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ y^2 - z^2 + 3yz - 2x & 3xz + 2xy & 3xy - 2xz + 2z \end{vmatrix}$$

$$= i\left[\frac{\partial}{\partial y}(3xy - 2xz + 2z) - \frac{\partial}{\partial z}(3xz + 2xy)\right]$$

for D.D, If Question like this, along $\frac{x-1}{2} = \frac{y+1}{3} = \frac{z-1}{1}$

then $\vec{a}$ will be $2\hat{i}+3\hat{j}+\hat{k}$. These is $\vec{a}$.

If Q's like along PQ and P(2,1,3) & Q(0,3,2) the $PQ = Q - P = -2\hat{i}+2\hat{j}-\hat{k}$ $\rightarrow \vec{a}$

$$-\hat{j}\left[\frac{\partial}{\partial x}(3xy-2xz+2z) - \frac{\partial}{\partial z}(y^2-z^2+3yz-2x)\right]$$

$$+\hat{k}\left[\frac{\partial}{\partial x}(3xz+2xy) - \frac{\partial}{\partial y}(y^2-z^2+3yz-2x)\right]$$

$$= \hat{i}[(3x-3x)] - \hat{j}[(3y-2z)-(-2z+3y)] + \hat{k}[(3z+2y)-(2y+3z)]$$

$$= 0\hat{i} + 0\hat{j} + 0\hat{k}$$

$\therefore \nabla \times \bar{F} = 0$. $\therefore \bar{F}$ is irrotational.

for solenoid, $\nabla\cdot\bar{F} = 0$.

$$\nabla\cdot\bar{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$$

$$\nabla\cdot\bar{F} = \frac{\partial}{\partial x}(y^2-z^2+3yz-2x) + \frac{\partial}{\partial y}(3xz+2xy) + \frac{\partial}{\partial z}(3xy-2xz+2z)$$

$$= (-2) + (2x) + (2) - 2x$$

$$= 0.$$

$\therefore \bar{F}$ is solenoid.

NOTE: Magnitude of Maximum Directional Derivative, $DD = |\nabla\phi|$

⁂ Directional derivative (D.D) of scalar point funⁿ.

Directional derivative of scalar point function ϕ in given direction of vector $\bar{a}$ is rate of change in ϕ in direction of $\bar{a}$.

$$D.D = \frac{\nabla\phi\cdot\bar{a}}{|\bar{a}|}$$

if $a = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$

$$|\bar{a}| = \sqrt{a_1^2+a_2^2+a_3^2}$$

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Q. Prove that the vector $\vec{F} = z\hat{i} + x\hat{j} + y\hat{k}$ is solenoidal.

$\Rightarrow$ $\nabla \cdot \vec{F} = 0$, for solenoidal

$$\nabla \cdot \vec{F} = \frac{\partial}{\partial x} F_1 + \frac{\partial}{\partial y} F_2 + \frac{\partial}{\partial z} F_3 \qquad F_1 = z,\ F_2 = x,\ F_3 = y$$

$$\nabla \cdot \vec{F} = \frac{\partial}{\partial x}(z) + \frac{\partial}{\partial y}(x) + \frac{\partial}{\partial z}(y) = 0$$

Hence, the given vector $\vec{F} = z\hat{i} + x\hat{j} + y\hat{k}$ is solenoidal

Q. Check whether the vector $\vec{F} = 4y^4z^2\hat{i} + 4x^3z^2\hat{j} - 3x^2y^2\hat{k}$ is solenoidal.

$\Rightarrow$ $\vec{F} = 4y^4z^2\hat{i} + 4x^3z^2\hat{j} - 3x^2y^2\hat{k}$

$F_1 = 4y^4z^2, \quad F_2 = 4x^3z^2, \quad F_3 = -3x^2y^2$

$$\nabla \cdot \vec{F} = \frac{\partial}{\partial x} F_1 + \frac{\partial}{\partial y} F_2 + \frac{\partial}{\partial z} F_3$$

$$= \frac{\partial}{\partial x}(4y^4z^2) + \frac{\partial}{\partial y}(4x^3z^2) + \frac{\partial}{\partial z}(-3x^2y^2)$$

$$= 0 + 0 + 0 = 0$$

Hence, the given vector $\vec{F} = 4y^4z^2\hat{i} + 4x^3z^2\hat{j} - 3x^2y^2\hat{k}$ is solenoidal.

Q. If $\vec{F} = xy^2\hat{i} + 2x^2yz\hat{j} - 3yz^2\hat{k}$, find $\nabla \cdot \vec{F}$ and $\nabla \times \vec{F}$ at the point $(1, -1, 1)$

$\Rightarrow$ $\vec{F} = xy^2\hat{i} + 2x^2yz\hat{j} - 3yz^2\hat{k}$

$F_1 = xy^2, \quad F_2 = 2x^2yz, \quad F_3 = -3yz^2$

$$\nabla \cdot \vec{F} = \frac{\partial}{\partial x} F_1 + \frac{\partial}{\partial y} F_2 + \frac{\partial}{\partial z} F_3$$

$$= \frac{\partial}{\partial x}(xy^2) + \frac{\partial}{\partial y}(2x^2yz) + \frac{\partial}{\partial z}(-3yz^2)$$

$$= y^2 + 2x^2z - 6yz$$

at point $(1, -1, 1)$

$\Rightarrow$ $\nabla \cdot \vec{F} = 1 + 2(1)(1) - 6(-1)(1) = 1 + 2 + 6 = \underline{9}$

$$\nabla \times \bar{F} = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ F_1 & F_2 & F_3 \end{vmatrix} \qquad \begin{aligned} F_1 &= xy^2 \\ F_2 &= 2x^2yz \\ F_3 &= -3yz^2 \end{aligned}$$

$$\nabla \times F = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ xy^2 & 2x^2yz & -3yz^2 \end{vmatrix} = i\left[\frac{\partial}{\partial y}(-3yz^2) - \frac{\partial}{\partial z}(2x^2yz)\right] - j\left[\frac{\partial}{\partial x}(-3yz^2) - \frac{\partial}{\partial z}(xy^2)\right]$$

$$+ k\left[\frac{\partial}{\partial x}(2x^2yz) - \frac{\partial}{\partial y}(xy^2)\right]$$

$$\nabla \times F = i[(-3z^2 - 2x^2y)] - j[0 - 0] + k[4xyz - 2xy]$$

at point $(1,-1,1)$

$$\nabla \times F = i[-3 - 2(-1)] - 0j + k[4(1)(-1)(1) - 2(1)(-1)]$$

$$= i[-3+2] - 0j + k[-4+2]$$

$$\Rightarrow \nabla \times F = -i - 0j - 2k.$$

Q. find div F and curl F, where $\bar{F} = \text{grad}(x^2+y^2+z^2-3xyz)$

$\Rightarrow$ Gradient of F is $\nabla F = i\frac{\partial F}{\partial x} + j\frac{\partial F}{\partial y} + k\frac{\partial F}{\partial z}$

$$\nabla F = i\frac{\partial}{\partial x}(x^2+y^2+z^2-3xyz) + j\frac{\partial}{\partial y}(x^2+y^2+z^2-3xyz)$$

$$+ k\frac{\partial}{\partial z}(x^2+y^2+z^2-3xyz)$$

$$= i[2x-3yz] + j[2y-3xz] + k[2z-3xy]$$

$$\nabla F = (2x-3yz)i + (2y-3xz)j + (2z-3xy)k.$$

$$\nabla \cdot F = \frac{\partial}{\partial x}F_1 + \frac{\partial}{\partial y}F_2 + \frac{\partial}{\partial z}F_3$$

$$= \frac{\partial}{\partial x}(2x-3yz) + \frac{\partial}{\partial y}(2y-3xz) + \frac{\partial}{\partial z}(2z-3xy)$$

$\nabla \cdot \bar{F} = 2 + 2 + 2 = \underline{\underline{6}}$

$F_1 = 2x - 3yz$

$F_2 = 2y - 3xz$

$F_3 = 2z - 3xy.$

$$\nabla \times \bar{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ F_1 & F_2 & F_3 \end{vmatrix}$$

$$= \hat{i}\left[\frac{\partial}{\partial y}(F_3) - \frac{\partial}{\partial z}(F_2)\right] - \hat{j}\left[\frac{\partial}{\partial x}(F_3) - \frac{\partial}{\partial z}(F_1)\right] + \hat{k}\left[\frac{\partial}{\partial x}(F_2) - \frac{\partial}{\partial y}(F_1)\right]$$

$$= \hat{i}\left[\frac{\partial}{\partial y}(2z - 3xy) - \frac{\partial}{\partial z}(2y - 3xz)\right] - \hat{j}\left[\frac{\partial}{\partial x}(2z - 3xy) - \frac{\partial}{\partial z}(2x - 3yz)\right]$$

$$+ \hat{k}\left[\frac{\partial}{\partial x}(2y - 3xz) - \frac{\partial}{\partial y}(2x - 3yz)\right]$$

$$= \hat{i}\left[(-3x) - (-3x)\right] - \hat{j}\left[(-3y) - (-3y)\right] + \hat{k}\left[(-3z) - (-3z)\right]$$

$$= \hat{i}\left[-3x + 3x\right] - \hat{j}\left[-3y + 3y\right] + \hat{k}\left[-3z + 3z\right]$$

$$= 0\hat{i} + 0\hat{j} + 0\hat{k}$$

$\nabla \times \bar{F} = 0.$

vector field $\bar{F}$ is irrotational.

Q. find div (grad ϕ) and curl (grad ϕ) at $(1,1,1)$ for $\phi = x^2y^3z^4$

$\Rightarrow$ $\phi = x^2y^3z^4.$

$$\nabla\phi = \hat{i}\frac{\partial\phi}{\partial x} + \hat{j}\frac{\partial\phi}{\partial y} + \hat{k}\frac{\partial\phi}{\partial z}$$

$$= \hat{i}\frac{\partial(x^2y^3z^4)}{\partial x} + \hat{j}\frac{\partial(x^2y^3z^4)}{\partial y} + \hat{k}\frac{\partial(x^2y^3z^4)}{\partial z}$$

$$\nabla\phi = \hat{i}(2xy^3z^4) + \hat{j}(3x^2y^2z^4) + \hat{k}(4x^2y^3z^3)$$

$\therefore$ grad ϕ is $(2xy^3z^4)\hat{i} + (3x^2y^2z^4)\hat{j} + (4x^2y^3z^3)\hat{k}.$

$$\text{div}(\text{grad}\,\phi) = \frac{\partial}{\partial x}F_1 + \frac{\partial}{\partial y}F_2 + \frac{\partial}{\partial z}F_3$$

$$= \frac{\partial}{\partial x}(2xy^3z^4) + \frac{\partial}{\partial y}(3x^2y^2z^4) + \frac{\partial}{\partial z}(4x^2y^3z^3)$$

$$\text{div}(\text{grad}\,\phi) = 2y^3z^4 + 6x^2yz^4 + 12x^2y^3z^2$$

div (grad ϕ) at point (1, 1, 1).

div (grad ϕ) $= 2y^3z^4 + 6x^2yz^4 + 12x^2y^3z^2$

$= 2 + 6 + 12$

$= \underline{\underline{20}}$.

curl (grad ϕ) $= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ F_1 & F_2 & F_3 \end{vmatrix}$

$F_1 = 2xy^3z^4$

$F_2 = 3x^2y^2z^4$

$F_3 = 4x^2y^3z^3$

$= \hat{i}\left[\frac{\partial}{\partial y}(F_3) - \frac{\partial}{\partial z}(F_2)\right] - \hat{j}\left[\frac{\partial}{\partial x}(F_3) - \frac{\partial}{\partial z}F_1\right] + \hat{k}\left[\frac{\partial}{\partial x}F_2 - \frac{\partial}{\partial y}F_1\right]$

$= \hat{i}\left[\frac{\partial}{\partial y}(4x^2y^3z^3) - \frac{\partial}{\partial z}(3x^2y^2z^4)\right] - \hat{j}\left[\frac{\partial}{\partial x}(4x^2y^3z^3) - \frac{\partial}{\partial z}(2xy^3z^4)\right]$

$+ \hat{k}\left[\frac{\partial}{\partial x}(3x^2y^2z^4) - \frac{\partial}{\partial y}(2xy^3z^4)\right]$

$= \hat{i}\left[12x^2y^2z^3 - 12x^2y^2z^3\right] - \hat{j}\left[8xy^3z^3 - 8xy^3z^3\right] +$

$\hat{k}\left[6xy^2z^4 - 6xy^2z^4\right]$

$= 0\hat{i} + 0\hat{j} + 0\hat{k} = 0$.

$\therefore$ curl (grad ϕ) is irrotational at (1, 1, 1)

Q. Find directional derivative of $\phi = xy^2 + yz^2$ at (2, -1, 1) in the direction of $i + 2j + 2k$.

$\Rightarrow$ D.D $= \frac{\nabla\phi \cdot \vec{a}}{|\vec{a}|}$

given, $\vec{a} = \hat{i} + 2\hat{j} + 2\hat{k}$.

$|\vec{a}| = \sqrt{1+4+4} = \sqrt{9} = 3$.

$\nabla\phi = i\frac{\partial\phi}{\partial x} + \hat{j}\frac{\partial\phi}{\partial y} + k\frac{\partial\phi}{\partial z}$.

$$\nabla\phi = i\frac{\partial}{\partial x}(xy^2+yz^2) + j\frac{\partial}{\partial y}(xy^2+yz^2) + k\frac{\partial}{\partial z}(xy^2+yz^2)$$

$$= i(y^2) + j(2xy+z^2) + k(2zy).$$

$$\nabla\phi\Big|_{(2,-1,1)} = (-1)^2 i + (2(2)(-1)+1)j + (2\times1\times-1)k.$$

$$\nabla\phi\Big|_{(2,-1,1)} = i - 3j - 2k.$$

$$D.D = \nabla\phi\Big|_{(2,-1,1)} \cdot \frac{\vec{a}}{|\vec{a}|} = \frac{(i-3j-2k).(i+2j+2k)}{3}$$

$$D.D = \frac{1-6-4}{3} = \frac{-9}{3} = \underline{\underline{-3}}$$

Q. Find directional derivative of $\phi = x^2 - y^2 + 2z^2$ at $p(1,2,3)$ in the direction of line PQ where $Q(5,0,4)$.

$\Rightarrow$ $D.D = \dfrac{\nabla\phi.\vec{a}}{|\vec{a}|}$ given, $\vec{a} = \vec{PQ} = (5i+0j+4k) - (i+2j+3k)$

$$= 4i - 2j + k.$$

$$|\vec{a}| = \sqrt{16+4+1} = \sqrt{21}.$$

$$\nabla\phi = i\frac{\partial\phi}{\partial x} + j\frac{\partial\phi}{\partial y} + k\frac{\partial\phi}{\partial z} = i\frac{\partial}{\partial x}(x^2-y^2+2z^2) + j\frac{\partial}{\partial y}(x^2-y^2+2z^2)$$

$$+ k\frac{\partial}{\partial z}(x^2-y^2+2z^2)$$

$$\nabla\phi = (2x)i + (-2y)j + (4z)k.$$

$$\nabla\phi\Big|_{(1,2,3)} = 2i - 4j + 12k.$$

$$D.D = \frac{(2i-4j+12k).(4i-2j+k)}{\sqrt{21}} = \frac{8+8+12}{\sqrt{21}} = \frac{28}{\sqrt{21}}.$$

$$= \frac{28\times\sqrt{21}}{21} = \frac{4\sqrt{21}}{3}$$

Q. find directional derivative of $\phi = xy^2 + yz^3$ at $P(1,-1,1)$ ~~in the direction~~ along the direction Normal to the surface $x^2+y^2+z^2=9$ at $(1,2,2)$

$\Rightarrow$ $\nabla\phi = i\frac{\partial\phi}{\partial x} + j\frac{\partial\phi}{\partial y} + k\frac{\partial\phi}{\partial z}$

$= i(y^2) + j(2xy+z^2) + k(3yz^2)$

$\nabla\phi|_{(1,-1,1)} = i + (-2+1)j + \cancel{(-2)k}\ (-3)k$

$= i - j - 3k$

$\bar{a}$ = Normal to surface $\underbrace{x^2+y^2+z^2=9}_{f}$ at $(1,2,2)$

$\bar{a} = \nabla\phi|_{(1,2,2)}$

$\cancel{\bar{a} = i(y^2) + j(2xy+z^2) + k(3yz^2)}$

$\cancel{\text{at } (1,2,2) \Rightarrow \bar{a} = 4i +}$

$\bar{a} = \nabla f|_{(1,2,2)}$ $\quad \nabla f = \frac{\partial f}{\partial x}i + \frac{\partial f}{\partial y}j + \frac{\partial f}{\partial z}k$

$\bar{a} = \nabla f = \frac{\partial}{\partial x}(x^2+y^2+z^2-9)i + \frac{\partial f}{\partial y}j + \frac{\partial f}{\partial z}k$

$= (2x)i + (2y)j + (2z)k$

$\bar{a} = \nabla f|_{(1,2,2)} = 2i + 4j + 4k$

$|\bar{a}| = \sqrt{4+16+16} = \sqrt{36} = 6$

$D.D = \frac{(i - j - 3k)\cdot(2i + 4j + 4k)}{6}$

$= \frac{2-4-12}{6} = \frac{-14}{6} = \frac{-7}{3}$

Q. Find directional derivative of $\phi = 4xz^3 - 3x^2yz^2$ at $P(2,-1,2)$ along z-axis.

$\Rightarrow \nabla\phi = \hat{i}\frac{\partial\phi}{\partial x} + \hat{j}\frac{\partial\phi}{\partial y} + \hat{k}\frac{\partial\phi}{\partial z}$.

$= (4z^3 - 6xyz^2)\hat{i} + (-3x^2z^2)\hat{j} + (12xz^2 - 6x^2yz)\hat{k}$

$\nabla\phi_{(2,-1,2)} = [4(8) - 6(2)(-1)(4)]\hat{i} + (-3(4)(4))\hat{j} + [12(2)(4) - 6(4)(-1)(2)]\hat{k}$.

$= [32 + 48]\hat{i} - 48\hat{j} + (96 + 48)\hat{k}$

$\nabla\phi = 80\hat{i} - 48\hat{j} + 144\hat{k}$.

$\vec{a} = 0\hat{i} + 0\hat{j} + 1\hat{k}$. $\Rightarrow |\vec{a}| = 1$.

$D.D = \frac{\nabla\phi \cdot \vec{a}}{|\vec{a}|} = \frac{(80\hat{i} - 48\hat{j} + 144\hat{k}) \cdot (0\hat{i} + 0\hat{j} + 1\hat{k})}{1}$

$\underline{D.D = 144}$

Scalar Potential.

If $\bar{F}$ is irrotational ($\nabla \times \bar{F} = 0$) then there exist scalar function ϕ such that $\boxed{\bar{F} = \nabla\phi}$ and 'ϕ' called scalar Potential function.

let $F = F_1\hat{i} + F_2\hat{j} + F_3\hat{k}$

$\therefore \phi = \underset{\substack{y,z \\ \text{constant}}}{\int F_1\,dx} + \underset{\substack{x,z \\ \text{constant}}}{\int F_2\,dy} + \underset{\substack{x,y \\ \text{constant}}}{\int F_3\,dz}$.

Angle between two surface ϕ_1 and ϕ_2

$$\cos\theta = \frac{\nabla\phi_1 \cdot \nabla\phi_2}{|\nabla\phi_1| \cdot |\nabla\phi_2|}$$

→ for MCQ's, If you have to find scalar potential fun' then don't check for $\nabla \times F = 0$ (i.e irrotational)
It is always irrotational in MCQ's.

Q. find scalar potential function, ϕ. If $\bar{F} = (6xy + z^3)\hat{i} + (3x^2 - z)\hat{j} + (3xz^2 - y)\hat{k}$.

⇒ check $\nabla \times F = 0$ or not.

$$\nabla \times F = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ F_1 & F_2 & F_3 \end{vmatrix}$$

$F_1 = 6xy + z^3$
$F_2 = 3x^2 - z$
$F_3 = 3xz^2 - y$.

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 6xy + z^3 & 3x^2 - z & 3xz^2 - y \end{vmatrix}$$

$$= \hat{i}\left[\frac{\partial}{\partial y}(3xz^2 - y) - \frac{\partial}{\partial z}(3x^2 - z)\right] - \hat{j}\left[\frac{\partial}{\partial x}(3xz^2 - y) - \frac{\partial}{\partial z}(6xy + z^3)\right] + \hat{k}\left[\frac{\partial}{\partial x}(3x^2 - z) - \frac{\partial}{\partial y}(6xy + z^3)\right]$$

$$= \hat{i}[-1 - (-1)] + \hat{j}[3z^2 - 3z^2] + \hat{k}[6x - 6x]$$

$$= \hat{i}[-1 + 1] + \hat{j}(0) + \hat{k}(0) = 0\hat{i} + 0\hat{j} + 0\hat{k} = 0$$

$\therefore \nabla \times F = 0$. → irrotational.

$F_1 = 6xy + z^3$, $F_2 = 3x^2 - z$, $F_3 = 3xz^2 - y$

$$\phi = \int F_1\,dx + \int F_2\,dy + \int F_3\,dz.$$

$$= \int (6xy + z^3)\,dx + \int (3x^2 - z)\,dy + \int (3xz^2 - y)\,dz.$$

$$= \frac{6x^2y}{2} + xz^3 + 3x^2y - zy + \frac{3xz^3}{3} - yz$$

$$= 3x^2y + xz^3 + 3x^2y - zy + xz^3 - yz$$

⇒ $\phi = 6x^2y + 2xz^3 - 2yz$.
↳ scalar potential function

$\phi = 6x^2y + 2xz^3 - 2yz$

$\nabla\phi = \bar{F}$

$\nabla\phi = (12xy + 2z^3)\hat{i} + (6x^2 - 2z)\hat{j} + (-2y + 6xz^2)\hat{k}$

$= 2[(6xy + z^3)\hat{i} + (3x^2 - z)\hat{j} + (3xz^2 - y)]\hat{k}$

$\nabla\phi = \bar{F}$

(2 doesn't matter here ∵ It is scalar)

Q. find angle betn surfaces

$z = x^2 + y^2 - 3$ and $x^2 + y^2 + z^2 = 9$ at $(2, -1, 2)$

$\Rightarrow \phi_1 = \cancel{x^2 + y} \; z - x^2 - y^2 + 3$

$\phi_2 = x^2 + y^2 + z^2 - 9$

$\nabla\phi_1 = (-2x)\hat{i} + (-2y)\hat{j} + (1)\hat{k}$

$\nabla\phi_{1(2,-1,2)} = -4\hat{i} + 2\hat{j} + \hat{k}$

$\nabla\phi_2 = (2x)\hat{i} + (2y)\hat{j} + (2z)\hat{k}$

$\nabla\phi_{2(2,-1,2)} = 4\hat{i} - 2\hat{j} + 4\hat{k}$

$\because \cos\theta = \dfrac{\nabla\phi_1 \cdot \nabla\phi_2}{|\nabla\phi_1||\nabla\phi_2|} = \dfrac{(-4\hat{i} + 2\hat{j} + \hat{k}) \cdot (4\hat{i} - 2\hat{j} + 4\hat{k})}{\sqrt{16+4+1}\;\sqrt{16+4+16}}$

$\cos\theta = \dfrac{-16 - 4 + 4}{\sqrt{21} \times 6} = \dfrac{-16}{6\sqrt{21}} = \dfrac{-8}{3\sqrt{21}}$

$\therefore \nabla\phi_1$ and $\nabla\phi_2$ are not orthogonal ($\perp$)

$\because \nabla\phi_1 \cdot \nabla\phi_2 \neq 0$