Unit IV: # Differential Equation

n^{th} order linear differential eqⁿ with constant coefficient is

$$a_0 \frac{d^n y}{dx^n} + a_1 \frac{d^{(n-1)} y}{dx^{n-1}} + \dots + a_n y = f(x) \quad \text{——} ①$$

OR

$$a_0 D^n y + a_1 D^{n-1} y + \dots + a_n y = f(x)$$

OR

$$(a_0 D^n + a_1 D^{n-1} + \dots + a_n) y = f(x)$$

where, $D = \frac{d}{dx}$, $D^2 = \frac{d^2}{dx^2}$, ..., $D^n = \frac{d^n}{dx^n}$

* General solⁿ of ① is $y = y_c + y_p$

1) Homogeneous L.D.E : If R.H.S in eqⁿ ① equal to '0'.
2) Non-Homo L.D.E : If R.H.S in eqⁿ ① is non-zero.

* Auxillary eqⁿ of eqⁿ ① is

$$\phi(m) = a_0 m^n y + a_1 m^{n-1} y + \dots + a_n y = 0$$
$$= (a_0 m^n + a_1 m^{n-1} + \dots + a_n) y = 0$$

let, $m_1, m_2, \dots m_n$ are roots of eqⁿ ②

* Type of y_c (complementary solⁿ) :-

I) Roots are Real and Distinct.

let $m_1, m_2, \dots, m_n$ are real and distinct roots of auxillary eqⁿ.

$$\boxed{y_c(x) = c_1 e^{m_1 x} + c_2 e^{m_2 x} + \dots + c_n e^{m_n x}}$$

where, $c_1, c_2, \dots, c_n$ are constant.

Ex. roots are 1, 2 then $y_c = c_1 e^{x} + c_2 e^{2x}$

II) Roots are Real and Repeated.

let $m_1, m_2, \ldots, m_n$ are roots of auxillary eqn.
say $m_1 = m_2 = m_3$ and rest are distinct.

$$y_c(x) = C_1 e^{m_1 x} + C_2 x e^{m_1 x} + C_3 x^2 e^{m_1 x} + C_4 e^{m_4 x}$$
$$+ C_5 e^{m_5 x} + \ldots + C_n e^{m_n x}$$

Ex. roots are (1, 1) $y_c = C_1 e^x + C_2 x e^x$
roots are (1, 1, 1) $y_c = C_1 e^x + C_2 x e^x + C_3 x^2 e^x$
roots are (1, 2, 2, 3, 4)
$$y_c = C_1 e^x + C_2 e^{2x} + C_3 x e^{2x} + C_4 e^{3x} + C_5 e^{4x}$$

III) Roots are Complex and Distinct.

[always take coeff. of (i) for "b"]

let roots are $a + ib$ and $a - ib$

$$y_c(x) = e^{ax}(C_1 \cos bx + C_2 \sin bx)$$

IV) Roots are Complex and Repeated.

let roots are $a + ib$ and $a + ib$

$$y_c(x) = e^{ax}\left((C_1 + C_2 x)\cos bx + (C_3 + C_4 x)\sin bx\right)$$

Ex. ① $2 + 3i$, $2 - 3i$
$\Rightarrow a = 2$, $b = 3$
$$y_c = e^{2x}(C_1 \cos 3x + C_2 \sin 3x)$$

② $1 \pm 2i$, $1 \pm 2i$
$\Rightarrow a = 1$, $b = 2$
$$y_c = e^{x}\left((C_1 + C_2 x)\cos 2x + (C_3 + C_4 x)\sin 2x\right)$$

③ 1, 2, 3, 4, 4, $\pm i$, $2 \pm i$, $2 \pm i$

$\Rightarrow y_c = c_1 e^{x} + c_2 e^{2x} + c_3 e^{3x} + c_4 e^{4x} + c_5 x e^{4x}$

$+ e^{0x}(c_6 \cos x + c_7 \sin x) + e^{2x}\big((c_8 + c_9 x)\cos x$

$+ (c_{10} + c_{11} x)\sin x\big)$

Q.1) solve.

① $(D^6 - 6D^5 + 12D^4 - 6D^3 - 9D^2 + 12D - 4)y = 0$

$\Rightarrow$ Using synthetic method (Trial and error)

$4 \rightarrow \pm 1, \pm 2, \pm 4$

1	1	-6	12	-6	-9	12	-4
	↓	1	-5	7	1	-8	4
1	1	-5	7	1	-8	4	$\boxed{0}$
	↓	1	-4	3	4	-4	
1	1	-4	3	4	-4	$\boxed{0}$	
	↓	1	-3	0	4		
-1	1	-3	0	4	$\boxed{0}$		
	↓	-1	+4	-4			
2	1	-4	4	$\boxed{0}$			
	↓	2	4				
2	1	-2	$\boxed{0}$				
	↓						
	1	$\boxed{0}$					

Roots of the eqⁿ are 1, 1, 1, -1, 2, 2

$\therefore y_c = c_1 e^{x} + c_2 x e^{x} + c_3 x^2 e^{x} + c_4 e^{-x} + c_5 e^{2x} + c_6 x e^{2x}$

$\therefore y = y_c + y_p$

$\therefore \boxed{y = y_c}$ since, $y_p = 0$

② $(D^4 + 10D^2 + 25)y = 0$

$\Rightarrow D^2 = t$

$t^2 + 10t + 25 = 0$

$\therefore t = -5, \quad t = -5$

$\boxed{D = \pm\sqrt{5}\, i}$ $\quad$ $\boxed{D = \pm\sqrt{5}\, i}$ $\quad$ $a = 0,\ b = \pm\sqrt{5} i, \pm\sqrt{5} i$

$$y_c = e^{0x}\left(\left(c_1\cos\sqrt{5}x + c_2 x\cos\sqrt{5}x\right) + \left(c_3\sin\sqrt{5}x + c_4 x\sin\sqrt{5}x\right)\right)$$

③ $(D^3 + D^2 + D + 1)y = 0$

$\Rightarrow y_c = e^{-x} + c_2\cos x + c_3\sin x \qquad (-1, \pm i)$

④ $(D^2 - 2D + 10)y = 0$ if $y(0) = 4$, $y'(0) = 1$

called boundary value problem.

$\Rightarrow$ Roots are $(1 \pm 3i)$

$y_c = c_1 e^x \cos 3x + c_2 e^x \sin 3x$

$\therefore y = y_c + y_p = y_c$

$\therefore y(x) = y_c(x) = c_1 e^x \cos 3x + c_2 e^x \sin 3x$

$y(0) = y_c(0) = c_1 e^0 \cos(0) + c_2 e^0 \sin(0)$

$y(0) = c_1 = 4$ ——— (since, $y(0) = 4$)

$\therefore \boxed{c_1 = 4}$

$y'(x) = y_c'(x) = e^x c_1 \cos 3x - 3c_1 e^x \sin 3x$
$\qquad + c_2 e^x \sin 3x + 3c_2 e^x \cos 3x$

$\Rightarrow y'(0) = y_c'(x) = 1$

$\therefore c_1 + 3c_2 = 1$

$4 + 3c_2 = 1 \Rightarrow \boxed{c_2 = -1}$

$\therefore \boxed{y_c = 4e^x \cos 3x - e^x \sin 3x}$

Solution of Non-Homogeneous LDE with constant coefficient

let $(a_0D^n + a_1D^{n-1} + a_2D^{n-2} + \dots + a_n)y = f(x)$ — ①

where, $f(x) \neq 0$

Method I : Variation of Parameter corresponding auxillary equation of ① is

$\phi(m) = (a_0m^n + a_1m^{n-1} + \dots + a_n)y = 0$

let, $m_1, m_2, \dots m_n$ are root of auxillary equation

y_c be complimentry solution of equation ①

NOTE:

Variation of parameter method use to solve 2nd order LDE with constant coefficient.

$(a_0D^2 + a_1D + a_2)y = f(x)$

$\phi(m) = (a_0m^2 + a_1m + a_2)y = 0$

m_1, m_2 are root of $\phi(m)$.

$y_c = c_1\phi_1 + c_2\phi_2$

To find y_p (perticular solution)

$y_p = u_1\phi_1 + u_2\phi_2$

where, $u_1 = \int \frac{-\phi_2 f(x)}{W} dx$, $u_2 = \int \frac{\phi_1 f(x)\, dx}{W}$

$W = \begin{vmatrix} \phi_1 & \phi_2 \\ \phi_1' & \phi_2' \end{vmatrix}$ called Wronskian and $\boxed{W \neq 0}$

Q. $(D^2 + 3D - 10)y = e^{3x}$

⇒ corresponding auxillary eqn

$\phi(m) = (m^2 + 3m - 10)y = 0$

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root are: $m_1 = -5$, $m_2 = 2$

$y_c = c_1 e^{-5x} + c_2 e^{2x}$

$\therefore \phi_1 = e^{-5x}$, $\phi_2 = e^{2x}$

$y_p = u_1 \phi_1 + u_2 \phi_2$

$u_1 = \int \frac{-\phi_2 f(x)}{W} dx$; $u_2 = \int \frac{\phi_1 f(x) dx}{W}$

$u_1 = \int \frac{-e^{2x} \cdot e^{3x}}{7e^{-3x}} dx$, $u_1 = \int \frac{-e^{8x}}{7} dx = \frac{-1}{56} e^{8x}$

$u_2 = \int \frac{e^{-5x} e^{3x}}{7e^{-3x}} dx = \int \frac{e^{x}}{7} dx = \frac{e^{x}}{7}$

$\therefore y_p = u_1 \phi_1 + u_2 \phi_2 = \frac{-1}{56} e^{8x} \cdot e^{-5x} + \frac{1}{7} e^{x} e^{2x}$

$= \frac{-e^{3x}}{56} + \frac{e^{3x}}{7}$

$W = \begin{vmatrix} \phi_1 & \phi_2 \\ \phi_1' & \phi_2' \end{vmatrix} = \begin{vmatrix} e^{-5x} & e^{2x} \\ -5e^{-5x} & 2e^{2x} \end{vmatrix} = 2e^{-3x} + 5e^{-3x} = 7e^{-3x}$

$y = y_c + y_p$

$= c_1 e^{-5x} + c_2 e^{2x} - \frac{e^{3x}}{56} + \frac{e^{3x}}{7}$

General sol^n.

Q. $(D^2 - 1) y = \frac{2}{1 + e^{x}}$

$\Rightarrow$ Corresponding auxillary eq^n:

$\phi(m) = (m^2 - 1)y = 0$

roots are $m_1 = 1$, $m_2 = -1$

$y_c = c_1 e^{x} + c_2 e^{-x}$

$\phi_1 = e^{x}$, $\phi_2 = e^{-x}$

$$W = \begin{vmatrix} \phi_1 & \phi_2 \\ \phi_1' & \phi_2' \end{vmatrix} = \begin{vmatrix} e^{x} & e^{-x} \\ e^{x} & -e^{-x} \end{vmatrix} = -2$$

$$u_1 = \int \frac{-e^{-x} \cdot \frac{2}{1+e^{x}}}{-2} dx$$

$$u_1 = \int \frac{e^{-x}}{1+e^{x}} dx = \int \frac{1}{e^{x}(1+e^{x})} dx$$

$$= \int \frac{1}{e^{x}} - \frac{1}{1+e^{x}} dx$$

$$= \int e^{-x} dx - \int \frac{1}{e^{x}(1+e^{-x})} dx$$

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Q. $(D^2 - 6D + 9)y = \frac{e^{3x}}{x^2}$

⇒ corresponding auxillary eqn.

$\phi(m) = (m^2 - 6m + 9)y = 0$

root are $m_1 = 3$, $m_2 = 3$

$y_c(x) = c_1 e^{3x} + c_2 x e^{3x}$

$\therefore \phi_1 = e^{3x}$, $\phi_2 = xe^{3x}$

$W = \begin{vmatrix} \phi_1 & \phi_2 \\ \phi_1' & \phi_2' \end{vmatrix} = \begin{vmatrix} e^{3x} & xe^{3x} \\ 3e^{3x} & 3xe^{3x} + e^{3x} \end{vmatrix} = e^{6x}$

$u_1 = \int \frac{-\phi_2 f(x)}{W} dx = \int \frac{-xe^{3x} \cdot \frac{e^{3x}}{x^2}}{e^{6x}} dx$

$= \int \frac{dx}{x} = -\log x$

$u_2 = \int \frac{\phi_1 f(x)}{W} dx = \int \frac{e^{3x} \frac{e^{3x}}{x^2}}{e^{6x}} dx = \int \frac{dx}{x^2} = -\frac{1}{x}$

$\therefore y_p = u_1\phi_1 + u_2\phi_2$

$= -\log x \cdot e^{3x} - \frac{xe^{3x}}{x}$

$= -\log x e^{3x} - e^{3x}$

$y = y_c + y_p$

$\therefore y = c_1 e^{3x} + c_2 x e^{3x} + (-\log x e^{3x} - e^{3x})$

Q. $(D^2 - 2D + 2)y = e^x \tan x$

⇒ corresponding auxillary eqn.

$\phi(m) = (m^2 - 2m + 2)y = 0$

root are $1 \pm i$, $a = 1$, $b = 1$

$y_c(x) = c_1 e^x \cos x + c_2 e^x \sin x$

$\therefore \phi_1 = e^x \cos x$, $\phi_2 = e^x \sin x$

$W = \begin{vmatrix} \phi_1 & \phi_2 \\ \phi_1' & \phi_2' \end{vmatrix} = \begin{vmatrix} e^x \cos x & e^x \sin x \\ e^x \cos x - e^x \sin x & e^x \sin x + e^x \cos x \end{vmatrix} = e^{2x}$

$$u_1 = \int \frac{-\phi_2 f(x)}{W} dx$$

$$u_1 = \int \frac{-e^x \sin x \cdot e^x \tan x}{e^{2x}} dx = \int \frac{-\sin^2 x}{\cos x} dx$$

$$= \int \frac{-(1-\cos^2 x)}{\cos x} dx = \int \left(\frac{-1}{\cos x} + \cos x\right) dx$$

$$= \int -\sec x \, dx + \int \cos x \, dx$$

$$= -\log(\sec x + \tan x) + \sin x$$

$$u_2 = \int \frac{\phi_1 f(x)}{W} dx = \int \frac{e^x \cos x \cdot e^x \tan x}{e^{2x}} dx$$

$$= \int \sin x \, dx = -\cos x$$

$$y_p = u_1\phi_1 + u_2\phi_2 = e^x \cos x(-\log(\sec x + \tan x)) + e^x \sin x(-\cos x).$$

Q. $(D^2-1)y = e^{-x}\sin(e^{-x}) + \cos(e^{-x})$

$\Rightarrow$ corr. auxillary eqn.

$\phi(m) = (m^2-1)y = 0$

root are $m_1 = 1$, $m_2 = -1$

$y_c(x) = c_1 e^x + c_2 e^{-x}$

$\therefore \phi_1 = e^x, \quad \phi_2 = e^{-x}$

$$W = \begin{vmatrix} \phi_1 & \phi_2 \\ \phi_1' & \phi_2' \end{vmatrix} = \begin{vmatrix} e^x & e^{-x} \\ e^x & -e^{-x} \end{vmatrix} = -2$$

$$u_1 = \int \frac{-\phi_2 f(x)}{W} dx = \int \frac{-e^{-x}(e^{-x}\sin(e^{-x}) + \cos(e^{-x}))}{-2} dx$$

$$u_1 = \int \frac{e^{-2x}\sin(e^{-x})}{2} dx + \int \frac{e^{-x}\cos(e^{-x})}{2} dx$$

Put, $e^{-x} = t$

$-e^{-x} dx = dt$

$e^{-x} dx = -dt$

$$u_1 = \int \frac{-t \sin t \, dt}{2} + \int -\cos t \, dt$$

$$\int uv = u\int v - \int\left(\frac{du}{dx}\int v\, dx\right) dx \qquad \text{LIATE}$$

$$u_1 = -t\int \sin t - \int \frac{d(t)}{dt}\int \sin t\, dt\Big) dt - \frac{\sin t}{2}$$

$$= -t(-\cos t) - \int -\cos t \, dt - \frac{\sin t}{2}$$

$$u_1 = t\cos t + \sin t - \frac{\sin t}{2}$$

$$u_2 = \int \frac{\phi_1 f(x)}{W} dx$$

$$= \int \frac{e^x [e^{-x} \sin(e^{-x}) + \cos e^{-x}]}{-2} dx$$

$$= \int \frac{\sin e^{-x}}{-2} dx + \int \frac{e^x \cos e^{-x}}{-2} dx$$

Put $e^{-x} = t \Rightarrow -x = \log t$

$-e^{-x} dx = dt \Rightarrow -dx =$

* **Undetermined Coeff. Method for finding y_p**

Case I: Let $f(x) = e^{ax}$

1) If $f(x) \notin y_c$ then choice of $y_p = ke^{ax}$
2) If $f(x) \in y_c$ and suppose a is root occurs 'n' times $f(x)$ is appearing 'n' times in y_c then choice of

$$y_p = kx^n e^{ax}$$

Q. Solve the given undetermined eqn

$$(D^2 + 3D - 10)y = e^{4x}$$

⇒ Corr. auxillary eqn.

$$\phi(m) = (m^2 + 3m - 10)y = 0$$

root are $m_1 = -5$, $m_2 = 2$

$$y_c = c_1 e^{-5x} + c_2 e^{2x}$$

Here $f(x) = e^{4x}$.

$\therefore a = 4$

$f(x) = e^{4x} \notin y_c$

$\therefore$ choice of $y_p = ke^{4x}$

$\therefore D^2y + 3Dy - 10y = e^{4x}$

Now, we have to find value of 'k'.

$$\therefore D^2y_p + 3Dy_p - 10y_p = e^{4x} \quad \text{—} (1)$$

$$\therefore Dy_p = \frac{d}{dx} ke^{4x} = 4ke^{4x}$$

$$D^2y_p = \frac{d^2}{dx^2} ke^{4x} = 16ke^{4x}$$

$$\therefore (1) \Rightarrow 16ke^{4x} + 3(4ke^{4x}) - 10(ke^{4x}) = e^{4x}$$

$$\Rightarrow 18ke^{4x} = e^{4x}$$

$\therefore 18k = 1 \quad \therefore k = 1/18$

$$\therefore y_p = ke^{4x} = \frac{e^{4x}}{18}$$

(If we get value of 'k' then y_p is correc

$$\therefore y = y_c + y_p$$

$$y = c_1 e^{-5x} + c_2 e^{2x} + \frac{e^{4x}}{18}$$

Q. $(D^2 - 2D + 1)y = e^x$

⇒ corr. auxillary eqn

$\phi(m) = (m^2 - 2m + 1)y = 0$

root are $m_1 = 1$, $m_2 = 1$

$y_c = c_1 e^x + c_2 x e^x$

Here; $f(x) = e^x$ ∴ $a = 1$

∴ $f(x) = e^x \in y_c$ and $a = 1$ root occurs two times $(n=2)$

∴ choice of $y_p = kx^2 e^{ax} = kx^2 e^x$

$D^2 y - 2Dy + 1y = e^x$

Now, we have to find value of 'k'.

∴ $D^2 y_p - 2Dy_p + 1y_p = e^x$ —— ①

∴ $Dy_p = \frac{d}{dx}(kx^2 e^x) = k2xe^x + kx^2 e^x$

$D^2 y_p = \frac{d^2}{dx^2}(kx^2 e^x) = k2e^x + k2xe^x + k2xe^x + kx^2 e^x$

$= 2ke^x + k4xe^x + kx^2 e^x$

∴ ① ⇒ $(2ke^x + 4kxe^x + kx^2 e^x) - 2(2kxe^x + kx^2 e^x) + kx^2 e^x = e^x$

⇒ $2ke^x = e^x$

⇒ $2k = 1$ ⇒ $k = 1/2$.

$y_p = \frac{1}{2} x^2 e^x$

$y = y_c + y_p$

$= c_1 e^x + c_2 x e^x + \frac{1}{2} x^2 e^x$

Case II: If $f(x) = \sin ax$ or $\cos ax$ or $\sin ax + \cos ax$

then 1) If $f(x) \notin y_c$ then $y_p = k_1 \sin ax + k_2 \cos ax$

2) If $f(x) \in y_c$ and $\pm ai$ root occurs 'n' time in y_c

then $y_p = x^n (k_1 \sin ax + k_2 \cos ax)$

NOTE: This case is applicable only for $\pm ai$ root of y_c not $b \pm ai$ i.e. $b \neq 0$.

Q. $(D^2+1)y = \sin 3x$

$\Rightarrow$ Corr. auxillary eqn

$\phi(m) = (m^2+1) y = 0$

root are $\pm i$

$\therefore y_c = c_1 \cos x + c_2 \sin x$

Now, $f(x) = \cos 3x \notin y_c$

$\therefore$ choice of $y_p = k_1 \sin 3x + k_2 \cos 3x$

Now, $D^2 y + y = \sin 3x$

$D^2 y_p + y_p = \sin 3x$ —①

$Dy_p = \frac{d}{dx}(k_1 \sin 3x + k_2 \cos 3x) = 3k_1 \cos 3x - 3k_2 \sin 3x$

$D^2 y_p = \frac{d^2}{dx^2}(k_1 \sin 3x + k_2 \cos 3x) = -9k_1 \sin 3x - 9k_2 \cos 3x$

① $\Rightarrow (-9k_1 \sin 3x - 9k_2 \cos 3x) + k_1 \sin 3x + k_2 \cos 3x = \sin 3x$

$\Rightarrow -8k_1 \sin 3x - 8k_2 \cos 3x = \sin 3x$

$\Rightarrow -8k_1 \sin 3x = \sin 3x$ and $-8k_2 \cos 3x = 0 \cos 3x$

$\Rightarrow -8k_1 = 1$ Provided $\sin 3x \neq 0$

$k_1 = -1/8$

$-8k_2 \cos 3x = 0 \cos 3x$, provided $\cos 3x \neq 0$

$k_2 = 0.$

$y_p = \frac{-1}{8} \sin 3x$

$y = y_c + y_p$ $\Rightarrow y = c_1 \cos x + c_2 \sin x + \left(\frac{-1}{8} \sin 3x\right)$

Q. $(D^2+1)y = \sin x + \cos x$

$\Rightarrow$ corr. auxillary eqn $\phi(m) = (m^2+1)y = 0$.

root are $\pm i$

$\therefore y_c = c_1 \cos x + c_2 \sin x$

Now, $f(x) = \sin x + \cos x \in y_c$

$\therefore$ choice of $y_p = x(k_1 \sin x + k_2 \cos x)$

$D^2 y_p + y_p = \sin x + \cos x$

$Dy_p = \frac{d}{dx}(k_1 x \sin x + k_2 x \cos x)$

$= k_1(x\cos x + \sin x) + k_2(\cos x - x\sin x)$

$D^2 y_p = k_1[-x\sin x + 2\cos x] + k_2[-x\cos x - 2\sin x]$

$D^2 y_p + y_p = \sin x + \cos x$

$\Rightarrow$

$-k_1 x \sin x + 2k_1 \cos x - k_2 x \cos x - 2k_2 \sin x + xk_1 \sin x + xk_2 \cos x$

$= \sin x + \cos x.$

$\Rightarrow 2k_1 \cos x - 2k_2 \sin x = \sin x + \cos x$

$\therefore k_1 = \frac{1}{2}$ and $k_2 = \frac{-1}{2}$.

$\therefore y_p = x\left(\frac{1}{2}\sin x - \frac{1}{2}\cos x\right)$

General soln. $y = y_c + y_p$

$y = (c_1 \cos x + c_2 \sin x) + x\left(\frac{1}{2}\sin x - \frac{1}{2}\cos x\right)$.

Case III: If $f(x) = e^{ax}\cos bx$ or $e^{ax}\sin bx$ or $e^{ax}\cos bx + e^{ax}\sin bx$

Put value of 'a' and 'b' from

① If $f(x) \notin y_c$ i.e then $y_p = k_1 e^{ax}\cos bx + k_2 e^{ax}\sin bx$

② If $f(x) \in y_c$ and $a \pm ib$ occurs 'n' times then
$$y_p = x^n [k_1 e^{ax}\cos bx + k_2 e^{ax}\sin bx]$$
↳ Put value of 'a' and 'b' from $f(x)$ or from roots both are same

Q. $(D^2+1)\,y = e^{2x}\sin x$

⇒ Here, $f(x) = e^{2x}\sin x$ $\therefore a = 2,\ b = 1$

$\therefore$ roots are $2 \pm i$

Corr. auxillary eqn, $\phi(m) = (m^2+1)y = 0$

root are $\pm i$ $\therefore c \pm di$ $\therefore c = 0,\ d = 1$.

$\therefore y_c(x) = c_1\cos x + c_2\sin x$

$\therefore f(x) \notin y_c$ ($\because a \pm ib$ is not present in root of y_c i.e. $2 \pm i$)

$\therefore$ choice of $y_p = k_1 e^{2x}\cos x + k_2 e^{2x}\sin x$

$\therefore (D^2+1)\,y_p = e^{2x}\sin x$ —— ①

$\Rightarrow D^2 y_p + y_p = e^{2x}\sin x$

$$Dy_p' = \frac{d\,y_p}{dx} = 2e^{2x}k_1\cos x - k_1 e^{2x}\sin x + 2k_2 e^{2x}\sin x + k_2 e^{2x}\cos x$$
$$= (2k_1 + k_2)e^{2x}\cos x + (2k_2 - k_1)e^{2x}\sin x$$

$$D^2 y_p = \frac{d^2 y_p}{dx} = (2k_1 + k_2)\,2e^{2x}\cos x - (2k_1 + k_2)e^{2x}\sin x + (2k_2 - k_1)\,2e^{2x}\sin x + (2k_2 - k_1)e^{2x}\cos x$$

$$= (k_1 + 3k_2)e^{2x}\cos x + (-3k_1 + k_2)e^{2x}\sin x$$

① ⇒ $(k_1 + 3k_2)e^{2x}\cos x + (-3k_1 + k_2)e^{2x} + k_1 e^{2x}\cos x + k_2 e^{2x}\sin x = e^{2x}\sin x$

$\Rightarrow (2k_1 + 3k_2)\, e^{2x} \cos x + (-3k_1 + 2k_2) e^{2x} \sin x = e^{2x} \sin x$

$\Rightarrow 2k_1 + 3k_2 = 0$

$-3k_1 + 2k_2 = \cancel{0}\ 1$ $\quad \cancel{\Rightarrow 2k_2 = 3k_1 \Rightarrow k_2 = \frac{3k_1}{2}}$

$\cancel{2k_1 + 3 \times \frac{3k_1}{2} = 0 \Rightarrow \frac{10k_1}{2} = 0 \Rightarrow k_1 = 0}$

$\cancel{k_1 = 0 \text{ and } k_2 = 0}$

$3(2k_1 + 3k_2 = 0) \Rightarrow 6k_1 + 9k_2 = 0$

$2(-3k_1 + 2k_2 = 1) \Rightarrow + \; -6k_1 + 4k_2 = 2$

$13k_2 = 2 \Rightarrow \boxed{k_2 = 2/13}$

$2k_1 + 3k_2 = 0 \Rightarrow 2k_1 + 3\left(\frac{2}{13}\right) = 0$

$2k_1 = -\frac{6}{13} \Rightarrow \boxed{k_1 = -3/13}$

$\Rightarrow \therefore y_p = \frac{-3}{13} e^{2x} \cos x + \frac{2}{13} e^{2x} \sin x$

Q. ______ $= e^{2x} \sin x$

$\Rightarrow$ Here, $f(x) = e^{2x} \sin x$ $\quad \therefore a = 2,\ b = 1.$

roots are $2 \pm i$.

suppose, root of y_c are $2 \pm i$ $\quad (c \pm di)$

$\therefore y_c = c_1 e^{2x} \cos x + c_2 e^{2x} \sin x$

Now, $f(x) \in y_c$ $\quad$ ($\because$ $2 \pm i$ present in root of y_c one time)

Choice of $y_p = x^n [k_1 e^{ax} \cos bx + k_2 e^{ax} \sin bx]$

($n = 1$, only one time occure).

rest is the history..

$\therefore y_p = x\left[\frac{-3}{13} e^{2x} \cos x + \frac{2}{13} e^{2x} \sin x\right]$

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Case IV: $f(x) = x^m$ or $a_0x^n + a_1x^{n-1} + \dots + a_n$

If $f(x) \in y_c$ or $\notin y_c$ then

$y_p = k_1x^n + k_2x^{n-1} + \dots + k_n$.

Real and repeated root

Case V:

$f(x) = $ constant, say k.

If $f(x) \notin y_c$ then $y_p = k$

If $f(x) \in y_c$ i.e constant present in y_c 'n' times

then $y_p = x^n k$

Zero as root of ...

Q. $(D^2 - 2D + 1)y = x$

$\Rightarrow$ Corr. auxillary eqn

$\phi(m) = (m^2 - 2m + 1)y = 0$

roots are $m_1 = 1$, $m_2 = 1$

$\therefore y_c(x) = c_1e^x + c_2xe^x$

$f(x) = x \in y_c$.

Choice of $y_p = k_1 + k_2x$

$\therefore D^2y_p - 2Dy_p + y_p = x$

$Dy_p = k_2 \Rightarrow D^2y_p = 0$

$-2(k_2) + k_1 + k_2x = x$

$(k_1 - 2k_2) + k_2x = x$

$\therefore k_1 - 2k_2 = 0 \Rightarrow k_2 = 1$, $k_1 = 2$

$\therefore y_p = 2 + x$

$\therefore y = y_c + y_p \Rightarrow y = (c_1e^x + c_2xe^x) + (2+x)$

Q. $(D^3 - 2D^2 + D - 2)y = 5\cos 2x - 6x^2$

$\Rightarrow$ Corr. auxillary eqn.

$\phi(m) = (m^3 - 2m^2 + m - 2)y = 0$.

roots are 2, $\pm i$

$y_c(x) = c_1e^{2x} + c_2\cos x + c_3\sin x$

Here, $f(x) = 5\cos 2x - 6x^2$

$\therefore f(x) = f_1 + f_2$ or $f_1 - f_2$

If $f(x) = f_1 + f_2$ or $f_1 - f_2$

then $y_p = y_{p_1} + y_{p_2}$ or $y_{p_1} - y_{p_2}$

but if $f(x) = f_1 \times f_2$ then $y_p \neq y_{p_1} \times y_{p_2}$.

$f_1 = 5\cos 2x$, $f_2 = -6x^2$

$\therefore f_1 \notin y_c$ $\therefore y_{p_1}$ for f_1 is $y_{p_1} = k_1 \cos 2x + k_2 \sin 2x$

$\therefore f_2 \notin y_c$ $\therefore y_{p_2}$ for f_2 is $y_{p_2} = k_3 + k_4 x + k_5 x^2$

$\therefore y_p$ for $f(x)$ is $y_p = y_{p_1} + y_{p_2}$

$\therefore y_p = k_1 \cos 2x + k_2 \sin 2x + k_3 + k_4 x + k_5 x^2$.

find k_1, k_2, k_3, k_4, k_5.

Q. $(D^2+1) y = x \sin x$.

$\Rightarrow$ Corr. auxillary eqn, $\phi(m) = (m^2+1) y = 0$.

roots are $\pm i$.

$y_c = C_1 \cos x + C_2 \sin x$.

$f(x) = \underset{f_1}{x}\,\underset{f_2}{\sin x}$ $\therefore f = f_1 \times f_2$

$f_1 = x \notin y_c$ $\therefore y_{p_1} = k_1 + k_2 x$

$f_2 = \sin x \in y_c$ $\therefore y_{p_1} = k_3 x \cos x + k_4 x \sin x$

$\therefore y_p = y_{p_1} \times y_{p_2} = (k_1 + k_2 x)(k_3 x \cos x + k_4 x \sin x)$

$y_p = \underbrace{k_1 k_3}_{A} x \cos x + \underbrace{k_1 k_4}_{B} x \sin x + \underbrace{k_2 k_3}_{C} x^2 \cos x + \underbrace{k_2 k_4}_{D} x^2 \sin x$

$\boxed{y_p = A x \cos x + B x \sin x + C x^2 \cos x + D x^2 \sin x}$ $\leftarrow$ use it

rest is history.

Application of Diagonalization for soln of DE.

$$\frac{dx_1}{dt} = a_{11} x_1 + a_{12} x_2 + a_{13} x_3 + \dots + a_{1n} x_n$$

$$\frac{dx_2}{dt} = a_{21} x_1 + a_{22} x_2 + a_{23} x_3 + \dots + a_{2n} x_n$$

$$\frac{dx_n}{dt} = a_{n1} x_1 + a_{n2} x_2 + a_{n3} x_3 + \dots + a_{nn} x_n$$

Now, this system can be written as,

$$\begin{bmatrix} x_1' \\ x_2' \\ x_3' \\ \vdots \\ x_n' \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix}$$

$\therefore X'(t) = A$

'A' must be diagonalizable, hence find eigen values and eigen vec of 'A'. let, $v_1, v_2, \ldots v_n$ be the eigen vectors of 'A' corr. to eigen values $\lambda_1, \lambda_2, \ldots \lambda_n$.

soln of system is –

$$X(t) = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = c_1 v_1 e^{\lambda_1 t} + c_2 v_2 e^{\lambda_2 t} + \ldots + c_n v_n e^{\lambda_n t}$$

LCR Circuits.

Quantity	Symbol	Unit	Abbreviation
Voltage, emf or Potential	E or V	Volt	V
Resistance	R	ohm	W or Ω
Inductance	L	Henry	h
Capacitance	C	farad	f
Current	I	Ampere	amp.
Charge	Q	Coulomb	none.

Case 1.: Electrical circuit consists of inductance 'L' and capacitance 'C' in series (without applied emf). Differential eqn of the circuit is given by.

$$L\frac{DI}{dt} + \frac{Q}{C} = 0 \quad \text{or} \quad L\frac{d^2Q}{dt^2} + \frac{Q}{C} = 0$$

$$\frac{d^2Q}{dt^2} + \omega^2 Q = 0$$

CASE 2: Electrical circuit consists of inductance 'L' and capacitance 'C' and applied emf $E_0 \sin nt$. Differential eqn of the circuit is given by

$$L\frac{DI}{dt} + \frac{\varphi}{c} = E_0 \sin nt \quad \text{or} \quad L\frac{d^2\varphi}{dt^2} + \frac{\varphi}{c} = E_0 \sin nt$$

$$\frac{d^2\varphi}{dt^2} + \omega^2\varphi = E_0 \sin nt \quad \left(\because \omega^2 = \frac{1}{LC},\ E = \frac{E_0}{L}\right)$$

CASE 3: Electrical circuit consists of inductance L, resistance R and capacitance C in series without applied emf.

Differential eqn of the circuit is given by

$$L\frac{DI}{dt} + IR + \frac{\varphi}{c} = 0 \quad \text{or} \quad L\frac{d^2\varphi}{dt^2} + R\frac{d\varphi}{dt} + \frac{\varphi}{c} = 0$$

$$\frac{d^2\varphi}{dt^2} + k_1\frac{d\varphi}{dt} + \omega^2\varphi = 0 \quad \left(\because k_1 = \frac{R}{L},\ \omega^2 = \frac{1}{LC}\right)$$

CASE 4 :

Electrical circuit consists of inductance L, resistance R, and capacitance C in series with applied emf $E_0 \sin nt$

Differential eqn of the circuit is given by

$$L\frac{dI}{dt} + IR + \frac{Q}{C} = E_0 \sin nt$$

or

$$L\frac{d^2Q}{dt^2} + R\frac{dQ}{dt} + \frac{Q}{C} = E_0 \sin nt$$

$$\frac{d^2Q}{dt^2} + k_1\frac{dQ}{dt} + \omega^2 Q = E_0 \sin nt$$

$$\left(\because k_1 = \frac{R}{L},\quad \omega^2 = \frac{1}{LC},\quad E = \frac{E_0}{L}\right.$$