Unit III: Inner Product Space (IPS)

Let V be vector space then the relation $\langle\,\rangle : V\times V \to R$ said to be Inner Product if for all $u, v, w \in V$, $\alpha \in R$

① $\langle u+v, w\rangle = \langle u,w\rangle + \langle v,w\rangle$ — linearity
② $\langle u,v\rangle = \langle v,u\rangle$ — symmetry.
③ $\langle \alpha u, v\rangle = \alpha\langle u,v\rangle$
④ a) $\langle u,u\rangle \geq 0$, for all u } Positive definite.
b) $\langle u,u\rangle = 0$, iff $u=0$

Then V with $\langle\,\rangle$ called Inner Product space (IPS)

Q. Which of the following are inner product of R^2.
$V=R^2$ $\quad \langle (x_1, x_2)(y_1, y_2)\rangle = x_1y_1 + 3x_2y_2$
1 3 3 (2 4)

⇒ ① $u=(x_1, x_2)$, $v=(y_1,y_2)$, $w=(z_1,z_2) \in V=R^2$

$$\langle u+v, w\rangle = \langle (x_1,x_2)+(y_1,y_2), (z_1,z_2)\rangle$$
$$= \langle (x_1+y_1, x_2+y_2), (z_1,z_2)\rangle$$
1 2 3 4
$$= (x_1+y_1)z_1 + 3(x_2+y_2)z_2$$
$$= x_1z_1 + y_1z_1 + 3x_2z_2 + 3y_2z_2$$
$$= x_1z_1 + 3x_2z_2 + y_1z_1 + 3y_2z_2$$
$$= \langle (x_1,x_2)(z_1,z_2)\rangle + \langle (y_1,y_2),(z_1,z_2)\rangle$$
$$\langle u+v, w\rangle = \langle u,w\rangle + \langle v,w\rangle$$

② $\langle u,v\rangle = \langle (x_1,x_2)(y_1,y_2)\rangle = x_1y_1 + 3x_2y_2$
$= y_1x_1 + 3y_2x_2$
$\Rightarrow \langle (y_1,y_2)(x_1,x_2)\rangle = \langle v,u\rangle$

③ $\langle \alpha u, v\rangle = \langle \alpha(x_1 x_2), (y_1,y_2)\rangle = \langle (\alpha x_1, \alpha x_2)(y_1,y_2)\rangle$
$= \alpha x_1y_1 + 3\alpha x_2y_2$
$= \alpha(x_1y_1 + 3x_2y_2) = \alpha\langle (x_1,x_2)(y_1, y_2)\rangle$
$= \alpha\langle u, v\rangle$

(for IP all ④ properties should be satisfied, if any one of it non satisfied then it is not an IP on V.
for property 4) both parts should be satisfied)

④ a) $\langle u,u\rangle = \langle (x_1,x_2)(x_1,x_2)\rangle = x_1x_1 + 3x_2x_2$
$= x_1^2+3x_2^2 \geq 0$ (always)

$\therefore \langle u,u\rangle \geq 0$ (always)

b) $\langle u,u\rangle = 0$ iff $x_1^2+3x_2^2=0$ iff, $x_1=0, x_2=0$

$\langle u,u\rangle=0$ iff $u=0$ iff, $u=(0,0)$

$\therefore \langle(x_1,x_2)(y_1,y_2)\rangle = x_1y_1+3x_2y_2$ is IP on R^2.

Q. Which of the following are inner product of R^2.

$V=R^2$ $\langle(x_1,x_2)(y_1,y_2)\rangle = x_1y_1 - 3x_2y_2$

1 3 3(2 4)

$\Rightarrow$ $\langle(x_1,x_2)(y_1,y_2)\rangle = x_1y_1 - 3x_2y_2$ is not IP on R^2.

Property ④ a) $\langle u,u\rangle = \langle(x_1,x_2)(x_1,x_2)\rangle$
$= x_1x_1 - 3x_2x_2$
$= x_1^2-3x_2^2 \geq 0$ (not always)

Property ④ a) not satisfied. It is not IP on R^2

OR

① $u=(x_1,x_2)$, $v=(y_1,y_2)$, $w=(z_1,z_2) \in V=R^2$

$\langle u+v,w\rangle = \langle (x_1,x_2)+(y_1,y_2), (z_1,z_2)\rangle$
$= \langle (x_1+y_1, x_2+y_2), (z_1,z_2)\rangle$
$= (x_1+y_1)z_1 - 3(x_2+y_2)(z_2)$
$= x_1z_1+y_1z_1 - 3x_2z_2 - 3y_2z_2$
$= x_1z_1 - 3x_2z_2 + y_1z_1 - 3y_2z_2$
$= \langle(x_1,x_2),(z_1,z_2)\rangle + \langle(y_1,y_2),(z_1,z_2)\rangle$

$\langle u+v,w\rangle = \langle u,w\rangle + \langle v,w\rangle$ — satisfied.

② $\langle u,v\rangle = \langle(x_1,x_2),(y_1,y_2)\rangle$
$= x_1y_1 - 3(x_2y_2) = y_1x_1 - 3(y_2x_2)$
$\Rightarrow \langle(y_1,y_2)(x_1,x_2)\rangle = \langle v,u\rangle$.

③ $\langle \alpha u, v\rangle = \langle \alpha(x_1, x_2), (y_1, y_2)\rangle$
$= \langle (\alpha x_1, \alpha x_2), (y_1, y_2)\rangle$
$= \alpha x_1 y_1 - 3\alpha x_2 y_2$
$= \alpha(x_1 y_1 - 3x_2 y_2)$ * $\cancel{\alpha\langle u, v\rangle}$
$= \alpha\langle (x_1, x_2), (y_1, y_2)\rangle = \alpha\langle u, v\rangle$.

④ a) $\langle u, u\rangle \geq 0 \Rightarrow \langle (x_1, x_2)(x_1, x_2)\rangle$
$= x_1 x_1 - 3x_2 x_2 = x_1^2 - 3x_2^2$
$x_1^2 - 3x_2^2 \geq 0$ (Not always)

b) $\langle u, u\rangle = 0$ iff. $u = 0$.
$x_1^2 - 3x_2^2 = 0$ iff $x_1^2 = 3x_2^2$
↳ Not always.

$\therefore \langle (x_1, x_2)(y_1, y_2)\rangle = x_1 y_1 - 3x_2 y_2$ is <u>Not IP on R^2</u>

HW Q.1) Which of the following are inner product of R^2
$V = R^2 \quad \langle (x_1, x_2)(y_1, y_2)\rangle = x_1 y_2$

Q. $V = R^3 \quad \langle (x_1, x_2, x_3)(y_1, y_2, y_3)\rangle = x_1 y_1 + x_2 y_2$
(1 4) (2 5)

⇒ ① $u = (x_1, x_2, x_3)$, $v = (y_1, y_2, y_3)$, $w = (z_1, z_2, z_3)$
$\langle u+v, w\rangle = \langle (x_1, x_2, x_3) + (y_1, y_2, y_3), (z_1, z_2, z_3)\rangle$
$= \langle (x_1+y_1, x_2+y_2, x_3+y_3), (z_1, z_2, z_3)\rangle$
1 2 3 4 5 6
$= (x_1+y_1)z_1 + (x_2+y_2)z_2$
$= x_1 z_1 + y_1 z_1 + x_2 z_2 + y_2 z_2$
$= x_1 z_1 + x_2 z_2 + y_1 z_1 + y_2 z_2$
$= \langle (x_1, x_2, x_3)(z_1, z_2, z_3)\rangle + \langle (y_1, y_2, y_3)(z_1, z_2, z_3)\rangle$
$= \langle u, w\rangle + \langle v, w\rangle$

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② $\langle u, v\rangle = \langle (x_1, x_2, x_3)(y_1, y_2, y_3)\rangle = x_1y_1 + x_2y_2$
$= y_1x_1 + y_2x_2$
$= \langle (y_1, y_2, y_3)(x_1, x_2, x_3)\rangle$
$= \langle v, u\rangle$

③ $\langle \alpha u, v\rangle = \langle \alpha(x_1, x_2, x_3), (y_1, y_2, y_3)\rangle$
$= \langle (\alpha x_1, \alpha x_2, \alpha x_3)(y_1, y_2, y_3)\rangle = \alpha x_1 y_1 + \alpha x_2 y_2$
$= \alpha(x_1y_1 + x_2y_2) = \alpha\langle (x_1\ x_2\ x_3)(y_1\ y_2\ y_3)\rangle$
$= \alpha\langle u, v\rangle$

④ a) $\langle u, u\rangle = \langle (x_1, x_2, x_3)(x_1, x_2, x_3)\rangle = x_1x_1 + x_2x_2$
$= x_1^2 + x_2^2 \geq 0$ (always)
b) $\langle u, u\rangle = 0$ iff $x_1^2 + x_2^2 = 0$ iff $x_1 = 0, x_2 = 0$
iff. $u = (0, 0, 0)$

$\therefore \langle (x_1, x_2, x_3)(y_1, y_2, y_3)\rangle = x_1y_1 + x_2y_2$ is inner product on R^3.

Hld. Q.2) Which of the following are inner product of R^3.
$V = R^3$ $\quad \langle (x_1, x_2, x_3)(y_1, y_2, y_3)\rangle = x_1y_1 + x_2y_2 + 11.$
1 2 3 4 5 6 14 25

$\Rightarrow$ ① $u = (x_1, x_2, x_3)$, $v = (y_1, y_2, y_3)$, $w = (z_1, z_2, z_3)$
$\langle u+v, w\rangle = \langle (x_1, x_2, x_3) + (y_1, y_2, y_3), (z_1, z_2, z_3)\rangle$
$= \langle (x_1+y_1, x_2+y_2, x_3+y_3), (z_1, z_2, z_3)\rangle$
1 2 3 4 5 6

$= (x_1+y_1)z_1 + (x_2+y_2)z_2 + 11$
$= x_1z_1 + y_1z_1 + x_2z_2 + y_2z_2 + 11$
$= x_1z_1 + x_2z_2 + y_1z_1 + y_2z_2 + 11$
$= \cancel{\langle (x_1, x_2, x_3)(z_1, z_2, z_3)\rangle + \langle (y_1, y_2, y_3)(z_1, z_2, z_3)\rangle}$
$\neq$ RHS.
Not satisfying these property.

② $\langle u, v\rangle = \langle (x_1, x_2, x_3)(y_1, y_2, y_3)\rangle$
$= x_1y_1 + x_2y_2 + 11$
$= y_1x_1 + y_2x_2 + 11$
$= \langle (y_1, y_2, y_3)(x_1, x_2, x_3)$
$\cancel{\neq \langle (y_1, y_2, y_3)(x_1, x_2, x_3)}$.

Property ② satisfied.

③ $\langle \alpha u, v\rangle = \langle \alpha(x_1, x_2, x_3), (y_1, y_2, y_3)\rangle$
$= \langle(\alpha x_1, \alpha x_2, \alpha x_3), (y_1, y_2, y_3)\rangle$
$= \alpha x_1 y_1 + \alpha x_2 y_2 + 11$
$\neq$ RHS.

Not satisfying property ③

④ a) $\langle u, u\rangle = \langle (x_1, x_2, x_3), (x_1, x_2, x_3)\rangle$
$= x_1x_1 + x_2x_2 + 11$
$= x_1^2 + x_2^2 + 11 \geq 0$ (always)

b) $\langle u, u\rangle = 0$ iff $x_1^2 + x_2^2 = -11$
but it is not possible.

Property ④ Not satisfying.

$\therefore \langle (x_1, x_2, x_3), (y_1, y_2, y_3)\rangle = x_1y_1 + x_2y_2 + 11$ is not IP on R^3.

Q.1) Which of the following are inner product of R^2.
$V = R^2$ $\langle (x_1, x_2)(y_1, y_2)\rangle = x_1 y_2$

$\Rightarrow$ ① $u = (x_1, x_2)$, $v = (y_1, y_2)$, $w = (z_1, z_2)$

$\langle u+v, w\rangle = \langle (x_1, x_2) + (y_1, y_2), (z_1, z_2)\rangle$
$= \langle (x_1 + y_1), x_2 + y_2), (z_1, z_2)\rangle$
1 2 3 4
$= (x_1 + y_1) z_2$
$= x_1 z_2 + y_1 z_2$
$= \langle (x_1, x_2)(z_1, z_2)\rangle + \langle (y_1, y_2)(z_1, z_2)\rangle$

$\langle u+v, w\rangle = \langle u, w\rangle + \langle v, w\rangle$ — satisfied

② $\langle u, v\rangle = \langle (x_1, x_2), (y_1, y_2)\rangle$
$= x_1 y_2 = y_2 x_1$
$= \langle (y_1, y_2)(x_1, x_2)\rangle$
$\langle u, v\rangle = \langle v, u\rangle$ — satisfied

③ $\langle \alpha u, v\rangle = \langle \alpha(x_1, x_2), (y_1, y_2)\rangle = \langle (\alpha x_1, \alpha x_2), (y_1, y_2)\rangle$
$= \alpha x_1 y_2 = \alpha(x_1 y_2)$
$= \alpha\langle (x_1, x_2), (y_1, y_2)\rangle$
$= \alpha\langle u, v\rangle$ — satisfied

④ a) $\langle u, u\rangle \geq 0 \Rightarrow \langle (x_1, x_2), (x_1, x_2)\rangle$
$\Rightarrow x_1 x_2 \geq 0$ (Not always)
if, $x_1 < 0$ then $x_1 x_2 < 0$ or $x_2 < 0$ then $x_1 x_2 < 0$.

b) $\langle u, u\rangle = 0 \Rightarrow x_1 x_2 = 0$ (Not always)
if, x_1 or x_2 is zero then only $x_1 x_2 = 0$.
so, Property ④ Not satisfied.
$\therefore \langle (x_1, x_2)(y_1, y_2)\rangle = x_1 y_2$ is Not IP on R^2

Standard IP on Vector space.

1) $V = R^n = \{(x_1, x_2, x_3, \ldots, x_n) \mid x_i \in R\}$
$\langle (x_1, x_2, x_3, \ldots x_n)(y_1, y_2, y_3, \ldots, y_n)\rangle = x_1 y_1 + x_2 y_2 + x_3 y_3 + \ldots + x_n y_n$.

2) $V = P_n(t) = \{a_0 + a_1 t + a_2 t^2 + \ldots + a_n t^n \mid a_n \in R\}$
$\langle a_0 + a_1 t + a_2 t^2 + \ldots + a_n t^n, b_0 + b_1 t + b_2 t^2 + \ldots + b_n t^n\rangle$
$= a_0 b_0 + a_1 b_1 + a_2 b_2 + \ldots + a_n b_n$

Standard means coeff. of every element is 1.
ex. $x_1 x_2 + y_1 y_2$ etc.

3) $V = C[a,b]$ $\quad \langle f(x).g(x) \rangle = \int_a^b f(x)\, g(x)\, dx.$

4) $V = M_{m\times n} = \left\{ \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \middle| \; a_{ij} \in R \right\}$

$\langle A, B \rangle = \text{Trace.}(B^T A)$

Where, $A_{m\times n}$, $B_{m\times n}$ matrix.

i.e. $\left\langle \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} & b_{13} & \cdots & b_{1n} \\ b_{21} & b_{22} & b_{23} & \cdots & b_{2n} \\ \vdots & \vdots & \vdots & & \\ b_{m1} & b_{m2} & b_{m3} & \cdots & b_{mn} \end{bmatrix} \right.$

$= a_{11} b_{11} + a_{12} b_{12} + a_{13} b_{13} + \cdots + a_{mn} b_{mn}$

NOTE:

Other than Inner Product (IP) mentioned above, are non-standard Inner product. (If it is IP)

* 7 Point.

Let V be vector space, $u, v \in V$, $\langle\ \rangle$ IP on V # (If IP not given then use standard IP on V) then

1) Norm (Distance from origin): $\|u\| = \sqrt{\langle u, u \rangle}$, $\|v\| = \sqrt{\langle v, v \rangle}$

2) Distance between 'u' and 'v':

$$d(u, v) = \|u - v\| = \sqrt{\langle u - v, u - v \rangle}$$

3) Angle betn 'u' and 'v': $\cos\theta = \dfrac{\langle u, v \rangle}{\|u\| \times \|v\|}$

4) 'u' and 'v' are orthogonal ($\perp^r$) if $\langle u, v \rangle = 0$

5) 'u' and 'v' are ortho~~gonal~~normal if ⓐ $\langle u, v \rangle = 0$

ⓑ $\|u\| = 1$, $\|v\| = 1$

6) Projection of 'u' on 'v' :

$$\text{Projection}_v\, u \quad \text{or} \quad \text{Proj}(u, v) = \frac{\langle u, v \rangle}{\|v\|^2} v$$

7) Projection of 'v' on 'u' :

$$\text{Projection}_u\, v = \frac{\langle u, v \rangle}{\|u\|^2} u$$

8) Unit vector : $\frac{u}{\|u\|}$, $\frac{v}{\|v\|}$

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Q. find Norm, Distance, projection, Angle and check whether 'u' and 'v' are orthogonal and orthonormal.

$V = R^2$ $\quad u = (1, -1)$, $v = (2, 3)$

$\langle (x_1, x_2)(y_1, y_2) \rangle = 3x_1y_1 + x_2y_2$

$\Rightarrow$ ① $\|u\| = \sqrt{\langle u, u \rangle} = \sqrt{\langle (1,-1), (1,-1) \rangle} = \sqrt{3(1)(1) + (-1)(-1)} = 2$

$\quad x_1 \; x_2 \; y_1 \; y_2$

$\|v\| = \sqrt{\langle v, v \rangle} = \sqrt{\langle (2,3)\;(2,3) \rangle} = \sqrt{3(2)(2) + (3)(3)} = \sqrt{21}$

$\quad x_1 \; x_2 \; y_1 \; y_2$

② $d(u, v) = \|u - v\| = \|(1,-1) - (2,3)\|$

$= \|(-1, -4)\| = \sqrt{\langle (-1,-4), (-1,-4) \rangle}$

$= \sqrt{3(-1)(-1) + (-4)(-4)} = \sqrt{19}$

③ $\langle u, v \rangle = \langle (1,-1)(2,3) \rangle = 3(1)(2) + (-1)(3) = 3.$

④ $\cos\theta = \dfrac{\langle u, v \rangle}{\|u\| \times \|v\|} = \dfrac{3}{2 \times \sqrt{21}} = \dfrac{3}{2\sqrt{21}}$

④ 'u' and 'v' are orthogonal if $\langle u, v \rangle = 0$, but $\langle u, v \rangle = 3.$
so, 'u' and 'v' is not orthogonal.

⑤ $\langle u, v \rangle \neq 0$, $\|u\| = 2 \neq 1$, $\|v\| = \sqrt{21} \neq 1$

$\therefore$ 'u' and 'v' is not orthonormal.

⑥ $\text{Proj}_v u = \dfrac{\langle u, v \rangle}{\|v\|^2} v = \dfrac{3}{(\sqrt{21})^2}(2, 3) = \left(\dfrac{6}{21}, \dfrac{9}{21}\right) = \left(\dfrac{2}{7}, \dfrac{3}{7}\right)$

$\text{Proj}_u v = \dfrac{\langle u, v \rangle}{\|u\|^2} u = \dfrac{3}{(2)^2}(1, -1) = \left(\dfrac{3}{4}, \dfrac{-3}{4}\right)$

⑦ Unit vector: $\dfrac{u}{\|u\|} = \dfrac{(1,-1)}{2} = \left(\dfrac{1}{2}, \dfrac{-1}{2}\right)$, $\dfrac{v}{\|v\|} = \dfrac{(2,3)}{\sqrt{21}} = \left(\dfrac{2}{\sqrt{21}}, \dfrac{3}{\sqrt{21}}\right)$

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Q. find Norm, Distance, projection, Angle and check whether 'u' and 'v' are orthogonal and orthonormal.

$V = \mathbb{R}^2 \quad u = (1,-1), \; v = (2,3).$

⇒ If 'IP' is not given then use standard IP on $V = \mathbb{R}^2$.

i.e. $\langle (x_1, x_2)(y_1, y_2) \rangle = x_1 y_1 + x_2 y_2$.

① $\|u\| = \sqrt{\langle u, u \rangle} = \sqrt{\langle \underset{x_1 \; x_2}{(1,-1)} \; \underset{y_1 \; y_2}{(1,-1)} \rangle} = \sqrt{(1)(1) + (-1)(-1)} = \sqrt{2}$

$\|v\| = \sqrt{\langle v, v \rangle} = \sqrt{\langle (2,3)(2,3) \rangle} = \sqrt{2(2) - 13(3)} = \sqrt{13}$

② $d(u,v) = \|u - v\| = \|(-1,-4)\| = \sqrt{(-1)(-1) + (-4)(-4)} = \sqrt{17}$

③ $\langle u, v \rangle = \langle (1,-1)(2,3) \rangle = (1)(2) + (-1)(3) = -1$

Angle, $\cos\theta = \dfrac{\langle u, v \rangle}{\|u\| \times \|v\|} = \dfrac{-1}{\sqrt{2} \times \sqrt{13}} = \dfrac{-1}{\sqrt{26}}$

④ 'u' and 'v' neither orthogonal nor orthonormal.

$\because \langle u, v \rangle = -1$.

⑤ Projection of u on v: $\text{Proj}_v u = \dfrac{\langle u, v \rangle}{\|v\|^2} v$

$\text{Proj}_v u = \dfrac{-1}{(\sqrt{13})^2}(2,3) = \left(\dfrac{-2}{13}, \dfrac{-3}{13} \right)$

$\text{Proj}_u v = \dfrac{\langle u, v \rangle}{\|u\|^2} u = \dfrac{-1}{2}(1,-1) = \left(\dfrac{-1}{2}, \dfrac{1}{2} \right)$

⑥ Unit vector: $\dfrac{u}{\|u\|} = \dfrac{(1,-1)}{\sqrt{2}} = \left(\dfrac{1}{\sqrt{2}}, \dfrac{-1}{\sqrt{2}} \right)$

$\dfrac{v}{\|v\|} = \dfrac{(2,3)}{\sqrt{13}} = \left(\dfrac{2}{\sqrt{13}}, \dfrac{3}{\sqrt{13}} \right)$

Q. find Norm, Distance, Projection, angle and check whether they are orthogonal and orthonormal.

$V = M_{2\times2}$ $\quad u = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, \quad v = \begin{bmatrix} 1 & 0 \\ 3 & 2 \end{bmatrix}$ $\quad \langle u, v \rangle = 2a_1b_1 + a_4b_4$

(Non std IP)

$\Rightarrow$ ① $\|u\| = \sqrt{\langle u,u\rangle} = \sqrt{\left\langle \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \right\rangle} = \sqrt{2(1)(1)+(4)(4)} = \sqrt{18}$

$\|v\| = \sqrt{\langle v,v\rangle} = \sqrt{\left\langle \begin{bmatrix} 1 & 0 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 3 & 2 \end{bmatrix} \right\rangle} = \sqrt{2(1)(1)+(2)(2)} = \sqrt{6}$

② $d(u,v) = \|u-v\| = \left\| \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 3 & 2 \end{bmatrix} \right\| = \left\| \begin{bmatrix} 0 & 2 \\ 0 & 2 \end{bmatrix} \right\|$

$= \sqrt{\left\langle \begin{bmatrix} 0 & 2 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 0 & 2 \\ 0 & 2 \end{bmatrix} \right\rangle} = \sqrt{2(0)(0)+(2)(2)} = 2$

③ $\langle u,v\rangle = \left\langle \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 3 & 2 \end{bmatrix} \right\rangle = 2(1)(1) + (4)(2) = 2+8 = 10$

$\text{Proj}_v u = \frac{\langle u,v\rangle}{\|v\|^2} v = \frac{10}{(\sqrt{6})^2} \begin{bmatrix} 1 & 0 \\ 3 & 2 \end{bmatrix} = \frac{5}{3} \begin{bmatrix} 1 & 0 \\ 3 & 2 \end{bmatrix}$

$\text{Proj}_u v = \frac{\langle u,v\rangle}{\|u\|^2} u = \frac{10}{(\sqrt{18})^2} \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \frac{5}{9} \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$

④ for orthogonal, $\langle u,v\rangle = 0$ but $\langle u,v\rangle = 10 \neq 0$.

⑤ for orthonormal, $\langle u,v\rangle = 0$, $\|u\| = 1$, $\|v\| = 1$

but $\langle u,v\rangle = 10 \neq 0$, $\|u\| = \sqrt{18} \neq 1$, $\|v\| = \sqrt{6} \neq 1$

$\therefore$ 'u' and 'v' is neither orthogonal nor orthonormal.

⑥ Unit vector: $\frac{u}{\|u\|} = \frac{1}{\sqrt{18}} \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ and $\frac{v}{\|v\|} = \frac{1}{\sqrt{6}} \begin{bmatrix} 1 & 0 \\ 3 & 2 \end{bmatrix}$

⑦ angle, $\cos\theta = \frac{\langle u,v\rangle}{\|u\| \times \|v\|} = \frac{10}{\sqrt{18} \times \sqrt{6}} = \frac{10}{2\sqrt{17}} = \frac{5}{\sqrt{17}}$.

Q. find Norm, Distance, Projection, angle and check whether they are orthogonal and orthonormal.

$V = M_{2\times2}$ $\quad u = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, \quad v = \begin{bmatrix} 1 & 0 \\ 3 & 2 \end{bmatrix}$

$\Rightarrow$ IP not given, use std IP on $M_{2\times2}$.

$\because \left\langle \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix} \begin{bmatrix} b_1 & b_2 \\ b_3 & b_4 \end{bmatrix} \right\rangle = a_1b_1 + a_2b_2 + a_3b_3 + a_4b_4$

① $\|u\| = \sqrt{\langle u, u\rangle} = \sqrt{\left\langle \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \right\rangle} = \sqrt{(1)(1) + (2)(2) + (3)(3) + (4)(4)} = \sqrt{30}$

$\|v\| = \sqrt{\langle v, v\rangle} = \sqrt{\left\langle \begin{bmatrix} 1 & 0 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 3 & 2 \end{bmatrix} \right\rangle} = \sqrt{(1)(1) + 0 + (3)(3) + (2)(2)} = \sqrt{14}$

② $d(u, v) = \|u - v\| = \left\| \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 3 & 2 \end{bmatrix} \right\| = \left\| \begin{bmatrix} 0 & 2 \\ 0 & 2 \end{bmatrix} \right\|$

$= \sqrt{\left\langle \begin{bmatrix} 0 & 2 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 0 & 2 \\ 0 & 2 \end{bmatrix} \right\rangle} = \sqrt{0 + (2)(2) + 0 + (2)(2)} = \sqrt{8}$

③ $\langle u, v\rangle = \left\langle \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 3 & 2 \end{bmatrix} \right\rangle = (1)(1) + (2)(0) + (3)(3) + (4)(2)$
$= 1 + 0 + 9 + 8 = 18$

$\cos\theta = \dfrac{\langle u, v\rangle}{\|u\|\,\|v\|} = \dfrac{18}{\sqrt{30} \times \sqrt{14}} = \dfrac{18}{\sqrt{420}}$

④ $\text{Proj}_v u = \dfrac{\langle u, v\rangle}{\|v\|^2} v = \dfrac{18}{14} \begin{bmatrix} 1 & 0 \\ 3 & 2 \end{bmatrix} = \dfrac{9}{7} \begin{bmatrix} 1 & 0 \\ 3 & 2 \end{bmatrix}$

$\text{Proj}_u v = \dfrac{\langle u, v\rangle}{\|u\|^2} u = \dfrac{18}{30} \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \dfrac{3}{5} \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$

⑤ For orthogonal: $\langle u, v\rangle = 0$ but $\langle u, v\rangle = 18 \neq 0$.

for orthonormal, $\langle u, v\rangle = 0$, $\|u\| = 1$, $\|v\| = 1$ but $\langle u, v\rangle \neq 0$, $\|u\| \neq 1$, $\|v\| \neq 1$.

$\therefore$ 'u' and 'v' is neither orthogonal nor orthonormal.

⑥ Unit vector: $\dfrac{u}{\|u\|} = \dfrac{1}{\sqrt{30}} \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, \quad \dfrac{v}{\|v\|} = \dfrac{1}{\sqrt{14}} \begin{bmatrix} 1 & 0 \\ 3 & 2 \end{bmatrix}$

Q. $V = P_2(t)$, $u = t$, $v = 1+t$, $\langle u,v\rangle = \int_0^1 u.v\, dt$ (non-std.)

$\Rightarrow$ ① $\|u\| = \sqrt{\langle u,u\rangle} = \sqrt{\langle t,t\rangle} = \sqrt{\int_0^1 (u.u)\,dt} = \sqrt{\int_0^1 t(t)\,dt}$

$= \sqrt{\int_0^1 t^2\,dt} = \sqrt{\left[\frac{t^3}{3}\right]_0^1} = \frac{1}{\sqrt{3}}$

② $\|v\| = \sqrt{\langle v,v\rangle} = \sqrt{\langle 1+t, 1+t\rangle} = \sqrt{\int_0^1 (v.v)\,dt} = \sqrt{\int_0^1 (1+t)(1+t)}$

$= \sqrt{\int_0^1 (1+t)^2\,dt} = \sqrt{\int_0^1 (1+t^2+2t)\,dt} = \sqrt{\left[t + \frac{t^3}{3} + \frac{2t^2}{2}\right]_0^1}$

$= \sqrt{\left(1+\frac{1}{3}+1\right)} = \sqrt{\frac{3+1+3}{3}} = \sqrt{\frac{7}{3}}$

③ $d(u,v) = \|u-v\| = \|t-(1+t)\| = \|-1\| = \sqrt{\int_0^1 (-1)\,dt}$

$= \sqrt{[-t]_0^1} = \|-1\| = \sqrt{\langle -1,-1\rangle} = \sqrt{\int_0^1 (-1)(-1)\,dt} = \sqrt{\int_0^1 1\,dt} = \sqrt{[t]_0^1} = 1$

④ $\langle u,v\rangle = \cancel{\langle t, 1+t\rangle = \int} \int_0^1 u.v\,dt = \int_0^1 t(1+t)\,dt = 5/6$

$\cos\theta = \frac{\langle u,v\rangle}{\|u\|\times\|v\|} = \frac{5/6}{\frac{1}{\sqrt{3}}\times\sqrt{\frac{7}{3}}} = \frac{3}{\sqrt{7}}\times\frac{5}{6} = \frac{5}{2\sqrt{7}}$

⑤ $\text{Proj}_v u = \frac{\langle u,v\rangle}{\|v\|^2} v = \frac{5/6}{(7/3)}(1+t) = \cancel{\left(\frac{3}{7}+\frac{3t}{7}\right)} = \frac{5}{14}(1+t)$

$\text{Proj}_u v = \frac{\langle u,v\rangle}{\|u\|^2} u = \frac{5/6}{(1/3)}(t) = \cancel{3t}\ \frac{5t}{2}$

⑥ Unit vector:

$\frac{u}{\|u\|} = \frac{t}{1/\sqrt{3}} = \sqrt{3}t$ and $\frac{v}{\|v\|} = \frac{1+t}{\sqrt{7/3}} = \frac{\sqrt{3}(1+t)}{\sqrt{7}}$

⑦ for orthogonal: $\langle u,v\rangle = 0$ but $\langle u,v\rangle = 1 \neq 0$

for orthonormal: $\langle u,v\rangle = 0$, $\|u\| = 1$, $\|v\| = 1$

but $\langle u,v\rangle = \cancel{3}\,5/6 \neq 0$, $\|u\| \neq 1$ & $\|v\| \neq 1$

$\therefore$ It is neither orthogonal nor orthonormal.

Q. $V = P_2(t)$, $u = t$, $v = 1+t$.

⇒ IP is not given then use std IP on P_2

$\langle a_0 + a_1 t + a_2 t^2, b_0 + b_1 t + b_2 t^2 \rangle = a_0 b_0 + a_1 b_1 + a_2 b_2$

⇒ $\langle 0 + t + 0t^2, 1 + t + 0t^2 \rangle = 0 + 1 + 0$

① $\|u\| = \sqrt{\langle u, u \rangle} = \sqrt{\langle 0+t+0t^2, 1+t+0t^2 \rangle} = \sqrt{0+1+0} = 1$

$\|v\| = \sqrt{\langle v, v \rangle} = \sqrt{\langle 1+t+0t^2, 1+t+0t^2 \rangle} = \sqrt{1+1+0} = \sqrt{2}$

② $d(u,v) = \|u - v\| = \|t - (1+t)\| = \|-1\| \rightarrow w = \sqrt{\langle w, w \rangle}$

$= \sqrt{\langle -1 + 0t + 0t^2, -1 + 0t + 0t^2 \rangle} = \sqrt{(-1)(-1) + 0 + 0} = 1$

③ $\langle u, v \rangle = \langle 0 + t + 0t^2, 1 + t + 0t^2 \rangle = 0 + 1 + 0 = 1.$

$\cos\theta = \dfrac{\langle u, v \rangle}{\|u\| \times \|v\|} = \dfrac{1}{1 \times \sqrt{2}} = \dfrac{1}{\sqrt{2}}$

④ $\text{Proj}_v u = \dfrac{\langle u, v \rangle}{\|v\|^2} v = \dfrac{1}{2}(1+t)$

$\text{Proj}_u v = \dfrac{\langle u, v \rangle}{\|u\|^2} u = \dfrac{1}{1}(t) = t$

⑤ unit vector, $\dfrac{u}{\|u\|} = \dfrac{t}{1} = t$, $\dfrac{v}{\|v\|} = \dfrac{1+t}{\sqrt{2}}$

⑥ for orthogonal, $\langle u, v \rangle = 0$ but $\langle u, v \rangle = 1 \neq 0$.

for orthonormal, $\langle u, v \rangle = 0$, $\|u\| = 1$, $\|v\| = 1$

but $\langle u, v \rangle = 1 \neq 0$, $\|u\| = 1$, $\|v\| = \sqrt{2} \neq 1$

∴ It is neither orthogonal nor orthonormal.

Q. $V = C[-1,1]$, $u = x$, $v = x^2$, $\langle u,v\rangle = \int_{-1}^{1} uv\, dx$

$\Rightarrow$ ① $\|u\| = \sqrt{\langle u,u\rangle} = \sqrt{\int_{-1}^{1} u.u\, dx} = \sqrt{\int_{-1}^{1} x.x\, dx} = \sqrt{\int_{-1}^{1} x^2\, dx}$

$= \sqrt{\left[\frac{x^3}{3}\right]_{-1}^{1}} = \sqrt{\frac{1}{3}+\frac{1}{3}} = \sqrt{\frac{2}{3}}$

$\|v\| = \sqrt{\langle v,v\rangle} = \sqrt{\int_{-1}^{1} v.v\, dx} = \sqrt{\int_{-1}^{1} x^4\, dx} = \sqrt{\left[\frac{x^5}{5}\right]_{-1}^{1}} = \sqrt{\frac{1}{5}+\frac{1}{5}} =$

② $d(u,v) = \|u-v\| = \|x-x^2\| = \sqrt{\langle x-x^2, x-x^2\rangle}$

$= \sqrt{\int_{-1}^{1} (x-x^2)^2\, dx} = \sqrt{\int_{-1}^{1} (x^2+x^4-2x^3)\, dx} = \sqrt{\left[\frac{x^3}{3}+\frac{x^5}{5}-\frac{2x^4}{4}\right]_{-1}^{1}}$

$= \sqrt{\left(\frac{1}{3}+\frac{1}{5}-\frac{1}{2}\right)-\left(\frac{-1}{3}-\frac{1}{5}-\frac{1}{2}\right)} = \sqrt{\frac{16}{15}}$

③ $\langle u,v\rangle = \int_{-1}^{1} u.v\, dx = \int_{-1}^{1} x^3\, dx = \left[\frac{x^4}{4}\right]_{-1}^{1} = \frac{1}{4}-\frac{1}{4} = 0$

$\cos\theta = \frac{\langle u,v\rangle}{\|u\| \times \|v\|} = 0$

④ $\text{Proj}_v u = \frac{\langle u,v\rangle}{\|v\|^2} v = 0$, $\text{Proj}_u v = \frac{\langle u,v\rangle}{\|u\|^2} u = 0$

⑤ unit vector, $\frac{u}{\|u\|} = \frac{x}{\sqrt{2/3}} = \frac{\sqrt{3}\,x}{\sqrt{2}}$

$\frac{v}{\|v\|} = \frac{x^2}{\sqrt{2/5}} = \frac{\sqrt{5}\,x^2}{\sqrt{2}}$

⑥ for orthogonal, $\langle u,v\rangle = 0$

for orthonormal, $\langle u,v\rangle = 0$, $\|u\| = 1$ but $\|u\| = \sqrt{2/3} \neq 1$

$\|v\| = 1$ but $\|v\| = \sqrt{2/5} \neq 1$

$\therefore$ It is orthogonal but <u>not</u> orthonormal.

Orthogonal Set :

Let V be vector space $\langle \rangle$ IP on V, $A = \{v_1, v_2 \dots v_n\}$ be subset of V, Then 'A' is orthogonal set if
$\langle v_i, v_j \rangle = 0$ for all i, j
i.e all elements in 'A' are mutually orthogonal.

Note: ① If 'A' is orthogonal set then 'A' is LI

Gram - Schmidt Process to convert (find) Basis into orthonormal Basis.

let V be vector space, $A = \{v_1, v_2, \dots v_n\}$ be $\boxed{\text{Basis}}$ of V, $\langle \rangle$ IP on V.

Step I: $w_1 = v_1$

Step II: $w_2 = v_2 - \frac{\langle v_2, w_1 \rangle}{\|w_1\|^2} w_1$

Step 3: $w_3 = v_3 - \frac{\langle v_3, w_2 \rangle}{\|w_2\|^2} w_2 - \frac{\langle v_3, w_1 \rangle}{\|w_1\|^2} w_1$

⋮

n^{th} step: $w_n = v_n - \frac{\langle v_n, w_{n-1} \rangle}{\|w_{n-1}\|^2} w_{n-1} - \dots - \frac{\langle v_n, w_1 \rangle}{\|w_1\|^2} w_1$

Orthogonal Basis is $\{w_1, w_2, w_3 \dots w_n\}$

Orthonormal Basis is $\left\{ \frac{w_1}{\|w_1\|}, \frac{w_2}{\|w_2\|} \dots \frac{w_n}{\|w_n\|} \right\}$

Q. Which of the following are orthogonal and orthonormal set.

$V = \mathbb{R}^2$ $\quad A = \{(1,2), (0,1), (1,0)\}$
$\qquad\qquad\qquad\quad v_1 \quad\; v_2 \quad\; v_3$

$\Rightarrow$ IP not given, therefore use std IP on $\mathbb{R}^2$.

$\langle (x_1, x_2)(y_1, y_2) \rangle = x_1 y_1 + x_2 y_2$

① $\langle v_1, v_2 \rangle = \langle (1,2)(0,1) \rangle = 0+2 = 2$

$\langle v_1, v_3 \rangle = \langle (1,2)(1,0) \rangle = 1$

$\langle v_2, v_3 \rangle = \langle (0,1), (1,0) \rangle = 0$

$\therefore$ 'A' is not orthogonal set

$\therefore$ 'A' is not orthonormal set.

Q. Convert given Basis into Orthonormal Basis using Gram's-Schmidt Process.

OR find orthonormal Basis.

$V = \mathbb{R}^2$ $\quad A = \{(1,2)\ (0,1)\}$
$\qquad\qquad\qquad\quad v_1 \qquad v_2$

$\Rightarrow$ IP is not given $\therefore$ use std IP on $\mathbb{R}^2$.

$\langle (x_1, x_2)(y_1, y_2) \rangle = x_1 y_1 + x_2 y_2$

$w_1 = v_1 \qquad \therefore w_1 = (1,2)$

$$w_2 = v_2 - \frac{\langle v_2, w_1 \rangle}{\|w_1\|^2} w_1$$

$\langle v_2, w_1 \rangle = \langle (0,1)(1,2) \rangle = 2$

$\|w_1\| = \sqrt{\langle w_1, w_1 \rangle} = \sqrt{\langle (1,2)(1,2) \rangle} = \sqrt{5}$

$\therefore \|w_1\|^2 = 5$.

$$w_2 = (0,1) - \frac{2}{(\sqrt{5})^2}(1,2) = (0,1) - \frac{2}{5}(1,2)$$

$$= \left(0 - \frac{2}{5}, 1 - \frac{4}{5}\right) = \left(\frac{-2}{5}, \frac{1}{5}\right) = w_2$$

$$\|w_2\| = \sqrt{\langle w_2, w_2 \rangle} = \sqrt{\left\langle \left(\frac{-2}{5}, \frac{1}{5}\right), \left(\frac{-2}{5}, \frac{1}{5}\right) \right\rangle} = \sqrt{\frac{4}{25} + \frac{1}{25}} = \sqrt{\frac{5}{25}}$$

$$= 1/\sqrt{5}$$

$\therefore$ Orthogonal Basis $= \{w_1, w_2\} = \{(1,2), (\frac{-2}{5}, \frac{1}{5})\}$

Orthonormal Basis $= \left\{ \frac{w_1}{\|w_1\|}, \frac{w_2}{\|w_2\|} \right\}$

$= \left\{ \frac{(1,2)}{\sqrt{5}}, \frac{(-2/5, 1/5)}{1/\sqrt{5}} \right\} = \left\{ \left(\frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}}\right), \left(\frac{-2}{\sqrt{5}}, \frac{1}{\sqrt{5}}\right) \right\}$

$V = \mathbb{R}^3$, $A = \{\underset{v_1}{(1,0,0)}, \underset{v_2}{(0,7,0)}, \underset{v_3}{(0,0,10)}\}$

$\langle (x_1, x_2, x_3), (y_1, y_2, y_3) \rangle = x_1 y_1 + x_2 y_2, x_3 y_3$.

$W_1 = v_1 = (1,0,0)$

$W_2 = v_2 - \frac{\langle v_2, w_1 \rangle}{\|w_1\|^2} w_1$

$\langle v_2, w_1 \rangle = \langle (0,7,0), (1,0,0) \rangle = 0$

$\|w_1\| = \sqrt{\langle w_1, w_1 \rangle} =$ ★ First check for orthogonal

$\langle v_1, v_2 \rangle = \langle (1,0,0)(0,7,0) \rangle = 0$

$\langle v_2, v_3 \rangle = \langle (0,7,0)\ (0,0,10) \rangle = 0$

$\langle v_1, v_3 \rangle = \langle (1,0,0), (0,0,10) \rangle = 0$

$\therefore W_1 = v_1, W_2 = v_2, W_3 = v_3$.

$\therefore$ Orthogonal Basis $= \{(1,0,0), (0,7,0)\ (0,0,10)\}$

Orthonormal Basis $= \left\{ \frac{(1,0,0)}{1}, \frac{(0,7,0)}{7}, \frac{(0,0,10)}{10} \right\}$

$V = \mathbb{R}^3$ $A = \{(1,0,0), (0,7,0), (1,7,1)\}$

$\langle v_1, v_2 \rangle = 0$

$\langle v_1, v_3 \rangle = 1$

$\langle v_2, v_3 \rangle = 49$

] common is v_3 i.e problem in v_3.

$\therefore$ only calculate W_3

$\because W_1 = v_1$, $W_2 = v_2$ and calculate W_3.

Orthogonal complement of set : $AX=0$ solⁿ
or, null space of 'A'
solⁿ

Least square Approximation.

let 'A' be $M_{m \times n}$ matrix, $b \in R^m$ then,
Least square solution of $AX=B$ is $\hat{x}$ ($\hat{x} \in R^n$)
such that $d(b, A\hat{x}) \leq d(b, Ax)$ for all $x \in R^n$
i.e $\|b - Ax\|$ minimised.

NOTE:

Least square solution of $AX=B$ is nothing but solution of $A^TAX = A^TB$

Orthogonality of fundamental subspace

left Null (A) = Null (A^T)

NOTE:

1) left Null (A) is subspace of R^n
2) dim (left Null (A)) = $n - \rho(A)$
3) Basis (left Null (A)) = Basis (Null (A^T))
4) Row (A) $\perp$ Null (A)
   i.e every element in Row (A) are perpendicular to every element of Null (A)
5) Col (A) $\perp$ Null (A^T)
   i.e every element in Col(A) are perpendicular to every element of Null (A^T)

Orthogonal complement of set.

let V be vector space, $A = \{v_1, v_2, \dots v_n\}$ be subset of V, $\langle \rangle$ IP on V, then orthogonal complement of 'A' is
$A^\perp = \{u \mid \langle u, v_i \rangle = 0$, for all $i\}$ $AX=0$ system.

NOTE:
1) $A^\perp$ is subspace of R^n
2) $A^\perp$ = solution set of $AX=0$ = Null (A)
3) dim ($A^\perp$) = $n - \rho(A)$

Q Find least square solution of $AX=B$ where

$$A=\begin{bmatrix}0 & 1\\ 1 & 1\\ 2 & 1\end{bmatrix} \quad B=\begin{bmatrix}6\\0\\0\end{bmatrix}$$

$\Rightarrow$ least square solⁿ of $AX=B$ is solⁿ of $A^TAX = A^TB$

$$\begin{bmatrix}0 & 1 & 2\\ 1 & 1 & 1\end{bmatrix}\begin{bmatrix}0 & 1\\ 1 & 1\\ 2 & 1\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix} = \begin{bmatrix}0 & 1 & 2\\ 1 & 1 & 1\end{bmatrix}\begin{bmatrix}6\\0\\0\end{bmatrix}$$

$$\Rightarrow \begin{bmatrix}5 & 3\\ 3 & 3\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix} = \begin{bmatrix}0\\6\end{bmatrix} \rightarrow AX=B \text{ system.}$$

$$\therefore [A|B] = \left[\begin{array}{cc|c}5 & 3 & 0\\ 3 & 3 & 6\end{array}\right]$$

$5R_2 - 3R_1$, $$[A|B] = \begin{array}{c} \begin{array}{cc} x & y \end{array} \\ \left[\begin{array}{cc|c}5 & 3 & 0\\ 0 & 6 & 30\end{array}\right] \end{array} \Rightarrow \text{REF}$$

$\rho(A)=2$, $\rho(A|B)=2$, $n=2$

$AX=B$ have unique solⁿ.

From REF, $5x+3y=0$

$6y=30 \Rightarrow y=5$ and $x=-3$.

$\therefore$ least square solⁿ, $\hat{x} = \begin{bmatrix}x\\y\end{bmatrix} = \begin{bmatrix}-3\\5\end{bmatrix}$

★ This solⁿ minimised the distance betⁿ $A\hat{x}=B$ ★

i.e $\|B - A\hat{X}\| = \left\| \begin{bmatrix}6\\0\\0\end{bmatrix} - \begin{bmatrix}0 & 1\\ 1 & 1\\ 2 & 1\end{bmatrix}\begin{bmatrix}-3\\5\end{bmatrix} \right\|$

$$= \left\| \begin{bmatrix}6\\0\\0\end{bmatrix} - \begin{bmatrix}5\\2\\-1\end{bmatrix} \right\| = \left\| \begin{bmatrix}1\\-2\\1\end{bmatrix} \right\|$$

use standard IP on R^3,

$= \sqrt{6}$

Q. find dimension and Basis of orthogonal complement of given subset.

$V = R^2 \quad A = \{\underset{v_1}{(1,2)}\ \underset{v_2}{(2,4)}\}$

$\Rightarrow A^{\perp} = \left\{ v \in V \;\middle|\; \begin{matrix} \langle v, v_1 \rangle = 0 \\ \langle v, v_2 \rangle = 0 \end{matrix} \right\}$

$= \left\{ v = (x,y) \in V = R^2 \;\middle|\; \begin{matrix} \langle (x,y)\ (1,2) \rangle = 0 \\ \langle (x,y)\ (2,4) \rangle = 0 \end{matrix} \right\}$

$= \left\{ (x,y) \;\middle|\; \begin{matrix} x + 2y = 0 \\ 2x + 4y = 0 \end{matrix} \right\} \qquad AX = 0.$

$[A|B] = \left[\begin{array}{cc|c} 1 & 2 & 0 \\ 2 & 4 & 0 \end{array}\right]$

$R_2 - 2R_1 \Rightarrow [A|B] = \left[\begin{array}{cc|c} 1 & 2 & 0 \\ 0 & 0 & 0 \end{array}\right]$

$\rho(A) = 1, \ \rho(A|B) = 1, \ n = 2.$

Infinite soln.

'x' is Leading entry and 'y' is free

Put, $y = t$.

from REF, $x + 2y = 0$

$x = -2t$

$A^{\perp} = \{(-2t, t) \mid t \in R\}$ *

$A^{\perp}$ is subspace of $V = R^2$

$\dim(A^{\perp}) = \dim(\text{Null}(A))$

or

no. of free variable or $n - \rho(A)$

$\dim(A^{\perp}) = 1$

Basis $(A^{\perp}) = \{(-2,1)\}$

$\because \begin{bmatrix} -2t \\ t \end{bmatrix} = t \begin{bmatrix} -2 \\ 1 \end{bmatrix}$