Date 13/12/24

UNIT-4. MULTIPLE INTEGRAL.

Let $y = f(x)$ then integration is $\int_a^b f(x)\,dx$.
i.e area of $f(x)$ over interval (a,b).

* Now, Double Integration,
let $f(x,y)$ be given funⁿ and R is given region in XY-plane
∴ Area of $f(x,y)$ over given region R is

$$\boxed{\iint_R f(x,y)\,dA} \quad \text{i.e} \quad \boxed{\iint_R f(x,y)\,dx\,dy}$$

Area → Double integration

* Triple Integration.
let $f(x,y,z)$ be given funⁿ, R be region in XYZ plane.
Then volumn of $f(x,y,z)$ over region R is.

$$\boxed{\iiint_R f(x,y,z)\,dA} \quad \text{i.e} \quad \boxed{\iiint_R f(x,y,z)\,dx\,dy\,dz}$$

volume — Tripple integration

* Evaluation of Multiple Integral:
A] If all limits given.

1) $$\int_a^b\int_c^d f(x,y)\,dx\,dy = \int_a^b\left(\int_c^d f(x,y)\,dx\right)dy$$

OR

$$\int_c^d\left(\int_a^b f(x,y)\,dy\right)dx$$

i.e if all limit are constant then we have two choice
① 1st integrate w.r.t 'x' and then w.r.t 'y'.
OR
② 1st integrate w.r.t 'y' and then w.r.t 'x'.

$$\int_a^b\int_c^d\int_e^f f(x,y,z)\,dx\,dy\,dz =$$ ① 1^{st} integrate w.r.t 'x' and then w.r.t 'y' and finally w.r.t 'z'.

OR

② 1^{st} w.r.t y
2^{nd} w.r.t z
3^{rd} wrt x.

OR ③ 1^{st} wrt z
2^{nd} wrt x
3^{rd} wrt y.

2) $$\int_a^b\int_{y=\phi_1(x)}^{y=\phi_2(x)} f(x,y)\,dx\,dy = \int_a^b\left(\int_{y=\phi_1(x)}^{y=\phi_2(x)} f(x,y)\,dy\right)dx$$

i.e 1^{st} integrate wrt 'y' and then wrt 'x'.

3) $$\int_a^b\int_{x=\phi_1(y)}^{x=\phi_2(y)} f(x,y)\,dx\,dy = \int_a^b\left(\int_{x=\phi_1(y)}^{x=\phi_2(y)} f(x,y)\,dx\right)dy.$$

i.e 1^{st} integrate wrt 'x' and then wrt 'y'.

4) $$\int_a^b\int_{y=\phi_1(x)}^{y=\phi_2(x)}\int_{z=\psi_1(x,y)}^{z=\psi_2(x,y)} f(x,y,z)\,dx\,dy\,dz =$$

$$\int_a^b\left[\int_{y=\phi_1(x)}^{y=\phi_2(x)}\left(\int_{z=\psi_1(xy)}^{z=\psi_2(xy)} f(x,y,z)\,dz\right)dy\right]dx$$

i.e 1^{st} integrate wrt 'z', then wrt 'y' and finally wrt 'x'.

5) $$\int_a^b\int_{z=\phi_1(y)}^{z=\phi_2(y)}\int_{x=\psi_1(y,z)}^{x=\psi_2(y,z)} f(x,y,z)\,dx\,dy\,dz =$$

i.e 1^{st} integrate wrt 'x' then wrt 'z' and finally wrt 'y'.

6) $\int_{a}^{b}\int_{x=\phi_1(z)}^{x=\phi_2(z)}\int_{y=\psi_1(x,z)}^{y=\psi_1(x,z)} f(x,y,z)\, dx\, dy\, dz =$

first integrate wrt 'y' then w.r.t 'x' and finally w.r.t 'z'

26/12/24 # When Limit is Given

Q.1. Evaluate $\int_0^1\int_1^2 xy\, dx\, dy$ OR $\int_{y=0}^{y=1}\int_{x=1}^{x=2} xy\, dx\, dy$

OR If $0 \leq y \leq 1$ and $1 \leq x \leq 2$ then Evaluate $\iint xy\, dx\, dy$.

⇒ When limit is constant then we have choice: 1st integrate w.r.t x and then y.

OR 1st w.r.t 'y' and then 'x'.

⇒ first integrate w.r.t 'x' by keeping 'y' constant and finally w.r.t 'y'.

$$\int_0^1\left[\int_1^2 xy\, dx\right] dy = \int_0^1\left[\frac{x^2y}{2}\right]_1^2 dy$$

$$= \int_0^1\left[\frac{4y}{2} - \frac{y}{2}\right] dy = \int_0^1 \frac{3y}{2}\, dy$$

$$= \left[\frac{3y^2}{4}\right]_0^1 = \frac{3}{4}$$

OR first integrate wrt 'y' by keeping 'x' constant and finally wrt 'x'.

$$\int_0^1\left[\int_1^2 xy\, dy\right] dx = \int_0^1\left[\frac{xy^2}{2}\right]_1^2 dx$$

$$= \int_0^1\left[\frac{4x}{2} - \frac{x}{2}\right] dx$$

$$= \int_0^1 \frac{3x}{2} \, dx = \left[\frac{3x^2}{4}\right]_0^1 = \frac{3}{4}$$

$\therefore$ Both are same.

so, we integrate 1st wrt 'x' or wrt 'y' ans will be same

Q.2 $\int_0^1 \int_0^{x^2} xy^2 \, dx \, dy$ → here, only 'x' term is present

$\Rightarrow$ i.e $\int_{x=0}^{x=1} \int_{y=0}^{y=x^2} xy^2 \, dx \, dy$ ∴ it is fun^n of y'.

here, we have no choice, as limit is not constant. so first we have to integrate wrt 'y' and then wrt 'x'

$$\int_0^1 \left[\int_0^{x^2} xy^2 \, dy\right] dx = \int_0^1 \left[\frac{xy^3}{3}\right]_0^{x^2} dx$$

$$= \int_0^1 \left[\frac{x^7}{3} - 0\right] dx = \left[\frac{x^8}{24}\right]_0^1 = \frac{1}{24}$$

Q.3. $\int_0^1 \int_y^{y^2} xy^2 \, dx \, dy$

$\Rightarrow$ i.e $\int_{y=0}^{y=1} \int_{x=y}^{x=y^2} xy^2 \, dx \, dy$

1st integrate wrt 'x' and then wrt 'y'

$$\int_0^1 \left[\int_y^{y^2} xy^2 \, dx\right] dy = \int_0^1 \left[\frac{x^2y^2}{2}\right]_y^{y^2} dy$$

$$= \int_0^1 \left[\frac{y^6}{2} - \frac{y^4}{2}\right] dy = \int_0^1 \frac{y^6}{2} dy - \int_0^1 \frac{y^4}{2} dy$$

$$= \int_0^1 \frac{y^6}{2}\,dy - \int_0^1 \frac{y^4}{2}\,dy = \left[\frac{y^7}{14}\right]_0^1 - \left[\frac{y^5}{10}\right]_0^1$$

$$= \frac{1}{14} - \frac{1}{10} = \frac{10-14}{140} = \frac{-4}{140} = \frac{-2}{70}$$

$$= \frac{-1}{35}$$ → Do not use mod here. (area will be -ve)

#8] When limit is not Given i.e only Region given:

Step 1: We have choice 1st integrate wrt 'x' or 'y' then wrt 'y' or 'x'

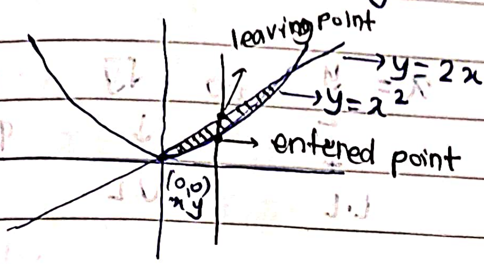
Suppose, 1st integrate wrt 'x' and then wrt 'y'. Then draw a strip (line) parallel to x-axis through given region the point at which strip entered in region is lower limit of 'x' and the point at which strip leave the region is upper limit of 'x' and for limit of 'y' is given by region bounded by points.

Suppose, 1st integrate wrt 'y' and then wrt 'x'. Then draw a strip (line) parallel to y-axis through given region the point at which strip entered in region is lower limit of 'y' and the point at which strip leave the region is upper limit of 'y' and for limit of 'x' is given by region bounded by points.

Q. Evaluate $\iint_R (4x+2)\,dx\,dy$

where 'R' is given Region bounded by the parabola $y = x^2$ and line $y = 2x$.

⇒ Here, limits are not given so first find the limit.



Decide, first integrate wrt 'y' and then wrt 'x'.

$y = x^2$ to $2x$
$\downarrow$ L.L $\quad$ $\downarrow$ U.L

$x = 0$ to 2 by solving
$\downarrow$ L.L $\quad$ $\downarrow$ U.L

$y = 2x$ and $y = x^2$
we get $(0, 2)$

$$\int_0^2 \int_{x^2}^{2x} (4x+2)\, dx\, dy.$$

here 1st integrate wrt 'y and then wrt 'x'.

$$\int_0^2 \left[\int_{x^2}^{2x} 4x + 2 \, dy \right] dx = \cancel{\int_0^2 \frac{4x^2}{2}}$$

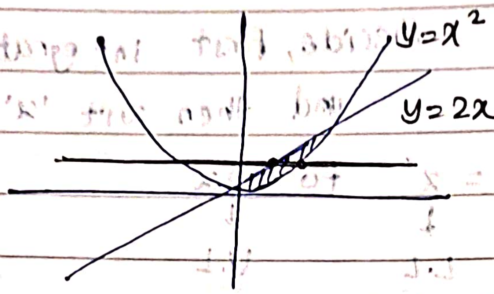
$$= \int_0^2 \left[\frac{4xy}{1} + 2y \right]_{x^2}^{2x} dx = \int_0^2 \left[4x(2x) + 2(2x) - 4x^3 - 2x^2 \right] dx$$

$$= \int_0^2 (8x^2 + 4x - 4x^3 - 2x^2)\, dx = \int_0^2 (6x^2 + 4x - 4x^3)\, dx$$

$$= \left[\frac{6x^3}{3} + \frac{4x^2}{2} - \frac{4x^4}{4} \right]_0^2 = \left[2x^3 + 2x^2 - x^4 \right]_0^2$$

$$= (16 + 8 - 16) = \underline{\underline{8}}$$

OR Decide 1st integrate wrt 'x' and then wrt 'y'.



$x = \frac{y}{2}$ to $\sqrt{y}$ ($\because$ in 1st quadrant x is +ve)

L.L U.L

$y = 0$ to 4

L.L U.L

solve. $\frac{y}{2} = \sqrt{y}$
$\Rightarrow y = 4$

$$\int_0^4 \int_{y/2}^{\sqrt{y}} (4x+2)\, dx\, dy = \int_0^4 \left[\int_{y/2}^{\sqrt{y}} (4x+2)\, dx \right] dy$$

$$= \int_0^4 \left[\frac{4x^2}{2} + 2x \right]_{y/2}^{\sqrt{y}} dy = \int_0^4 \left(\frac{4y}{2} + 2\sqrt{y} - \frac{4y^2}{8} - y \right) dy$$

$$= \int_0^4 \left(2\sqrt{y} - \frac{y^2}{2} + \frac{2y}{2} \right) dy = \left[\frac{2y^{3/2}}{3/2} - \frac{y^3}{6} + \frac{y^2}{2} \right]_0^4$$

$$= \left[\frac{4y^{3/2}}{3} - \frac{y^3}{6} + \frac{y^2}{2} \right]_0^4 = \frac{32}{3} - \frac{64}{6} + \frac{16}{2} = 8$$

27/12/24

Evaluation of double integration using change of variable (Polar co-ordinates)

let $\iint_R f(x,y)$ be given using polar co-ordinate

Put $x = r\cos\theta$, $y = r\sin\theta$

$$\therefore \iint_R f(x,y)\, dx\, dy = \iint_{r\,\theta} f(r\cos\theta, r\sin\theta)\, J\, dr\, d\theta$$

$$= \iint_{r\ \theta} f(r\cos\theta,\ r\sin\theta)\ r\,dr\,d\theta$$

$$J = \frac{\partial(x,y)}{\partial(r,\theta)} = r \quad \text{(for polar co-ordinates)}$$

∵ old limit in term of 'x' and 'y'.
Now, limit are in term of 'r' and 'θ'.

→ for limit of 'r'.

1. Draw a radial strip (radius vector) from starting of region.
2. starting end (inner end) of strip given lower limit of 'r' and outer end of strip gives upper limit of 'r'.

→ for limit of 'θ'.

rotate strip through $\boxed{\text{entire}}$ region.

<u>NOTE:</u> ① for $\iint$ integration use polar co-ordinate if given region is circle or ellipse or $f(x,y)$ contain term like x^2+y^2 or some part of circle and ellipse.

② If region is circle then put $x = r\cos\theta$, $y = r\sin\theta$ ⇒ $\boxed{J = r}$

③ If region is ellips $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

put, $x = ar\cos\theta$
$y = br\sin\theta$

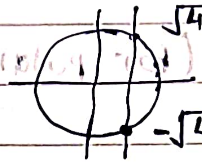
$$\Rightarrow \boxed{J = abr}$$

→ $\int e^{-f(x)} \cdot f'(x)\, dx = -e^{-f(x)}$

→ annulus = region betⁿ two circles

Q. Evaluate. $\iint e^{-x^2-y^2}\, dx\, dy$ over circle $x^2+y^2=4$

⇒ ① By regular method



$$\int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} e^{-x^2-y^2}\, dx\, dy$$

by regular method it is very difficult to solve.

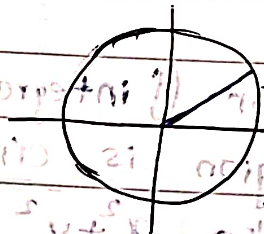
② since region is circle ∴ use polar co-ordinate.

$x = r\cos\theta, \quad y = r\sin\theta$

$$J = \frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} x_r & x_\theta \\ y_r & y_\theta \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r$$

$$\iint e^{-x^2-y^2}\, dx\, dy = \iint e^{-x^2-y^2}\, r\, dr\, d\theta = \iint e^{-r^2\cos^2\theta - r^2\sin^2\theta}\, r\, dr\, d\theta$$

$$= \iint e^{-r^2}\, r\, dr\, d\theta.$$



$\to x^2+y^2=4$.

$r^2=4$

ie $r=2$

limit of 'r' and 'θ'

r : 0 to 2

θ : 0 to 2π

$$\int_{r=0}^{r=2}\int_{\theta=0}^{\theta=2\pi} e^{-r^2}\, r\, dr\, d\theta$$

∴ Here all limit are constant.

$$\int e^{-f(x)} f'(x)\, dx = -e^{-f(x)}$$

$$= \frac{1}{2}\int_{r=0}^{r=2}\int_{\theta=0}^{\theta=2\pi} e^{-r^2} \cdot 2r\, dr\, d\theta$$

$$= \frac{1}{2}\int_{\theta=0}^{\theta=2\pi}\left[\int_{r=0}^{r=2} e^{-r^2} \cdot 2r\, dr\right] d\theta$$

$$= \frac{1}{2}\int_{\theta=0}^{\theta=2\pi} \left[-e^{-r^2}\right]_0^2 d\theta = \frac{1}{2}\int_0^{2\pi} \left[-e^{-4}+1\right] d\theta$$

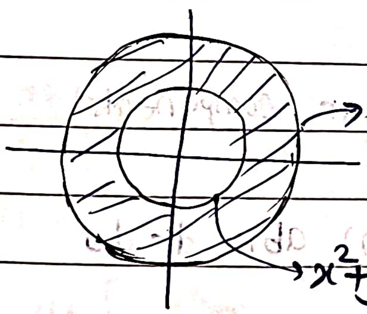
$$= \frac{1}{2}\left[-e\theta^{-4}+\theta\right]_0^{2\pi} = \left[\theta - e\theta^{-4}\right]_0^{2\pi} \cdot \frac{1}{2}$$

$$= \cancel{0-e^{-4}-2\pi+e^{-4}} = \frac{2\pi - 2\pi e^{-4}}{2}$$

$$= \pi(1-e^{-4})$$

Q. $\iint x^2+y^2 \, dx\, dy$ where R is annulus (Region bet^n two circles) bet^n $x^2+y^2=4$ and $x^2+y^2=9$

$\Rightarrow$



$x = r\cos\theta$, $y = r\sin\theta$

$J = r$.

$\to x^2+y^2=9$ i.e. $r=3$

$\to x^2+y^2=4$ i.e. $r=2$

$$\iint x^2+y^2 \, dx dy = \iint (r^2\cos^2\theta + r^2\sin^2\theta) \, r\, dr\, d\theta$$

$$= \iint r^3 \, dr\, d\theta$$

limit of 'r' and 'θ'.

r: 2 to 3

θ: 0 to 2π

$$\int_{r=2}^{r=3}\int_{\theta=0}^{\theta=2\pi} r^3 \, dr\, d\theta$$ $\therefore$ Here all limit constant.

$$= \int_{\theta=0}^{\theta=2\pi}\left[\int_{r=2}^{r=3} r^3 dr\right] d\theta = \int_0^{2\pi}\left[\frac{r^4}{4}\right]_2^3 d\theta$$

$$\int_0^{2\pi} \left[\frac{r^4}{4}\right]_2^3 d\theta = \int_0^{2\pi} \left[\frac{81}{4} - \frac{16}{4}\right] d\theta$$

$$= \left[\frac{65}{4}\theta\right]_0^{2\pi} = \frac{65(2\pi)}{4} = \frac{65\pi}{2}$$

HW

① $\iint x^2+y^2 \, dx\, dy$ over $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

② $\int_0^2 \int_0^{\sqrt{4-y^2}} \frac{y}{\sqrt{(4-x^2)(x^2+y^2)}} \, dx\, dy$

⇒ ① $\iint x^2+y^2 \, dx\, dy$ $\quad \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

here region is ellipse.

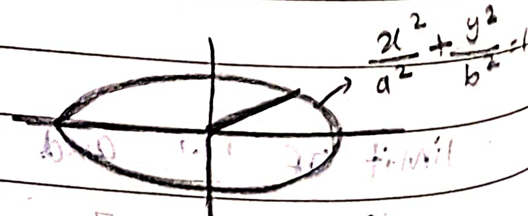
$\therefore$ $x = ar\cos\theta$, $y = br\sin\theta$ (by polar components)

$\because$ $J = abr$

$$\iint x^2+y^2 \, dx\, dy = \iint (a^2r^2\cos^2\theta + b^2r^2\sin^2\theta)\, abr \, dr\, d\theta$$

$$\Rightarrow \iint (a^2\cos^2\theta + b^2\sin^2\theta)\, ab\, r^3 \, dr\, d\theta$$

limits of 'r' and 'θ',



r: 0 to 1.

$$\hookrightarrow \frac{a^2r^2\cos^2\theta}{a^2} + \frac{b^2r^2\sin^2\theta}{b^2} = 1$$

$$\Rightarrow r^2 = 1 \Rightarrow r = 1.$$

θ: 0 to 2π.

$$\therefore \int_{r=0}^{r=1} \int_{\theta=0}^{\theta=2\pi} (a^2\cos^2\theta + b^2\sin^2\theta)\, ab\, r^3 \, dr\, d\theta$$

$$\Rightarrow \int_{\theta=0}^{\theta=2\pi} ab\,(a^2\cos^2\theta + b^2\sin^2\theta)\left[\int_{r=0}^{r=1} r^3\,dr\right] d\theta.$$

$$= \int_{\theta=0}^{\theta=2\pi} ab\,(a^2\cos^2\theta + b^2\sin^2\theta)\left[\frac{r^4}{4}\right]_0^1 d\theta$$

$$= \int_{\theta=0}^{\theta=2\pi} ab\,(a^2\cos^2\theta + b^2\sin^2\theta)\cdot\frac{1}{4}\,d\theta.$$

$$= \frac{ab}{4}\int_0^{2\pi} (a^2\cos^2\theta + b^2\sin^2\theta)\,d\theta.$$

$\rightarrow \cos 2\theta = 2\cos^2\theta - 1$

$\cos^2\theta = \dfrac{\cos 2\theta + 1}{2}$

$\int \cos^2\theta\, d\theta = \dfrac{1}{2}\left(\dfrac{\sin 2\theta}{2} + \theta\right)$

$$= \frac{ab}{4}\left[a^2\left(\frac{\sin 2\theta}{4} + \frac{\theta}{2}\right) + b^2\left(\frac{\theta}{2} - \frac{\sin 2\theta}{4}\right)\right]_0^{2\pi}$$

$\rightarrow \cos 2\theta = 1 - 2\sin^2\theta$

$\sin^2\theta = \dfrac{1 - \cos 2\theta}{2}$

$$= \frac{ab}{4}\left[a^2\left(\frac{\sin 4\theta}{4} + \frac{2\pi}{2}\right) + b^2\left(\frac{2\pi}{2} - \frac{\sin 4\pi}{4}\right)\right]$$

$\int \sin^2\theta\, d\theta = \dfrac{1}{2}\left(\theta - \dfrac{\sin 2\theta}{2}\right)$

$$= \frac{ab}{4}\left[a^2\pi + b^2\pi\right]$$

$$\Rightarrow \frac{ab\pi}{4}(a^2+b^2)$$

$$\int_0^2\int_0^{\sqrt{4-y^2}} \frac{y}{\sqrt{(4-x^2)(x^2+y^2)}}\,dx\,dy.$$

here, limit of 'x' is 0 to $\sqrt{4-y^2}$.

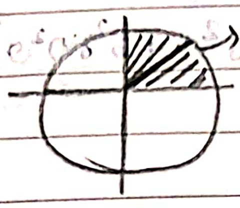
$\sqrt{4-y^2} = x. \Rightarrow x^2 = 4 - y^2.$

$x^2 + y^2 = 4. \longrightarrow$ region is circle.

i.e $x = r\cos\theta$

$y = r\sin\theta$ by polar coordinates

$$\int_0^2 \int_0^{\sqrt{4-y^2}} \frac{y}{\sqrt{4-x^2}\sqrt{(x^2+y^2)}}\, dx\, dy$$



$x^2+y^2=4$

$r=2.$

limit of 'r' : 0 to 2

'θ' : 0 to $\pi/2$

$$\int_{r=0}^{r=2} \int_{\theta=0}^{\theta=\pi/2} \frac{r\sin\theta}{\sqrt{(4-r^2\cos^2\theta)(r^2\sin^2\theta+r^2\cos^2\theta)}}\, r\,dr\,d\theta$$

$$= \int_0^2 \int_0^{\pi/2} \frac{r^{\cancel{2}}\sin\theta\, dr\,d\theta}{\sqrt{4-r^2\cos^2\theta}\cdot \cancel{r}} = \int_0^2 \int_0^{\pi/2} \frac{r\sin\theta\, dr\,d\theta}{\sqrt{4-r^2\cos^2\theta}}$$

$x^2+y^2=4$

$r^2\cos^2\theta + r^2\sin^2\theta = 4$

$\therefore 4-r^2\cos^2\theta = r^2\sin^2\theta$

$$= \int_0^2 \int_0^{\pi/2} \frac{r\sin\theta\, dr\,d\theta}{\sqrt{r^2\sin^2\theta}} = \int_0^2 \int_0^{\pi/2} \frac{r\sin\theta\, dr\,d\theta}{r\sin\theta}$$

$$= \int_0^2 \int_0^{\pi/2} dr\,d\theta = \int_0^{\pi/2} \left[\int_0^2 dr\right] d\theta.$$

$$= \int_0^{\pi/2} [r]_0^2\, d\theta = \int_0^{\pi/2} 2\, d\theta \quad [2\theta]_0^{\pi/2}$$

$$= 2\frac{\pi}{2} - 0 = \underline{\underline{\pi}}$$

Change of order of Integration.

Apply when $\iint_R f(x,y)\, dx\, dy$ is difficult to solve using mentioned limit

eg. $\int_0^1 \int_x^1 \frac{\sin y}{y}\, dx\, dy$

Here, we have to 1st integrate w.r.t 'y' and then w.r.t 'x' but integration is difficult to solve.

→ Now how to change the order of Integration?

* Case I. limits of inner integral are functions of x and integrand is difficult to integrate w.r.t to y first.

Step 1. sketch the region of integration from given limits.
Step 2. Draw a strip parallel to x-axis in ROI.
Step 3. find limits of 'x' from left and right ends of the strip.
Step 4. find limits of 'y' by moving strip from bottom to top in ROI.
Step 5. Using these limits, integrate wrt 'x' first and then resulting integrand wrt 'y'.

NOTE: In such a integral, strip is considered as parallel to y-axis. Hence we change / reverse the strip as per given in step 2.

* Case II. Limits of inner integral are functions of 'y' and integrand is difficult to integrate wrt 'x' first.

Step I. sketch the region of integration from given limits.
Step II. Draw a strip parallel to y-axis in ROI.
Step III. find limits of 'y' from lower and upper ends of the strip.
Step IV. find limits of 'x' by moving strip from left to right in ROI.

Step 5. Using these limits, integrate wrt 'y' first and then resulting integrand wrt 'x'.

NOTE: In such a integral, strip is considered as parallel to x-axis. Hence we change / reverse the strip as given in step 2.

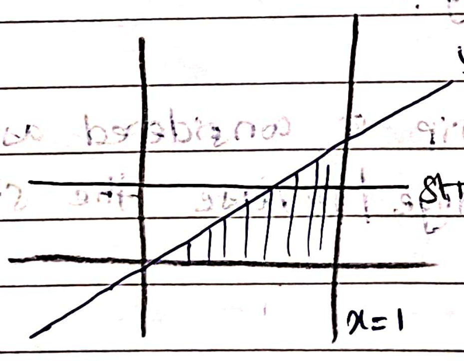
Q. $\int_0^1 \int_y^1 \frac{\sin x}{x} \, dx \, dy$

⇒ Here, 1st integrate wrt 'x' and then wrt "y". but $\int \frac{\sin x}{x} dx$ difficult to solve.

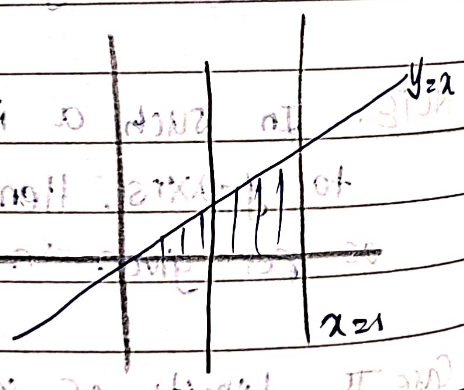
$y = 0$ to 1 and $x = y$ to 1.

∴ change the order of integration.

i.e. 1st integrate wrt 'y' and then wrt 'x'.



orginal limit



change of order

old limit (given limit)

$y = 0$ to 1 and $x = y$ to 1

New limit, $y : 0$ to x

$x : 0$ to 1

∴ $\int_0^1 \int_y^1 \frac{\sin x}{x} \, dx \, dy = \int_0^1 \int_0^x \frac{\sin x}{x} \, dx \, dy$

↓ old

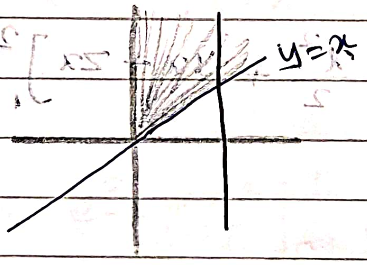
↓ New (order change)

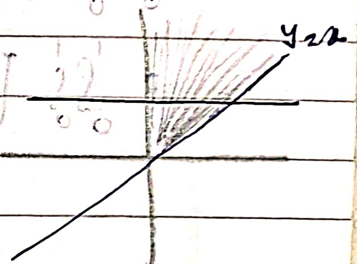
$$\int_0^1\int_0^x \frac{\sin x}{x}\,dx\,dy = \int_0^1\left[\int_0^x \frac{\sin x}{x}\,dy\right]dx$$

$$= \int_0^1\left[\frac{y\sin x}{x}\right]_0^x dx = \int_0^1 \sin x\,dx = \left[-\cos x\right]_0^1$$

$$= -\cos 1 + \cos 0 = \underline{\underline{1-\cos 1}}$$

Q. $\displaystyle\int_0^\infty\int_x^\infty \frac{e^{-y}}{y}\,dx\,dy$





⇒ old limit :- $x = 0$ to ∞

$y = x$ to ∞

New limit : $x = 0$ to ~~∞~~ y

$y = 0$ to ∞

$$\int_0^\infty\int_x^\infty \frac{e^{-y}}{y}\,dx\,dy = \int_0^\infty\int_0^y \frac{e^{-y}}{y}\,dx\,dy$$

↓ old Integration ↓ Integration after order change.

$$= \int_0^\infty\left[\int_0^y \frac{e^{-y}}{y}\,dx\right]dy = \int_0^\infty \left[\frac{x e^{-y}}{y}\right]_0^y dy$$

$$= \int_0^\infty e^{-y}\,dy = \left[-e^{-y}\right]_0^\infty$$

$$= -e^{-\infty} - (-e^{-0}) = \underline{\underline{1}}$$

here, all limits are constant ∴ we have choice but if limits like these $\int_0^1\int_0^1\int_{z=1}^{z=2} f(x,y,z)$ then it is mandatory to integrate 1st wrt z and so

Triple Integration.

Q.1. $\int_0^1\int_0^1\int_1^2 (x+y+z)\,dx\,dy\,dz$

⇒ all limits are constant. ∴ we have choice.
let, 1st integrate wrt 'x' then 'y' and then with 'z'.

$$\int_0^1\int_0^1\left[\int_1^2 (x+y+z)\,dx\right]dy\,dz$$

$$= \int_0^1\int_0^1\left[\frac{x^2}{2}+yx+zx\right]_1^2 dy\,dz = \int_0^1\int_0^1\left[\left(\frac{4}{2}+2y+2z\right)-\left(\frac{1}{2}+y+z\right)\right]dy\,dz$$

$$= \int_0^1\int_0^1\left(2+2y+2z-\frac{1}{2}-y-z\right)dy\,dz.$$

$$= \int_0^1\int_0^1\left(\frac{3}{2}+y+z\right)dy\,dz = \int_0^1\left[\int_0^1\left(\frac{3}{2}+y+z\right)dy\right]dz.$$

$$= \int_0^1\left[\frac{3}{2}y+\frac{y^2}{2}+zy\right]_0^1 dz = \int_0^1\left[\frac{3}{2}+\frac{1}{2}+z\right]dz$$

$$= \int_0^1\left[\frac{4}{2}+z\right]dz = \int_0^1 (2+z)\,dz$$

$$= \left[2z+\frac{z^2}{2}\right]_0^1 = 2+\frac{1}{2} = \frac{5}{2}$$

Q.2. $\int_0^1\int_0^{1-x}\int_0^{x+y} e^z\,dx\,dy\,dz.$

⇒ limit of $z = 0$ to $x+y$
limit of $y = 0$ to $1-x$
limit of $x = 0$ to 1

first integrate wrt z.

$$\int_0^{x+y} e^z dz = \int_0^{x+y} e^z dz = [e^z]_0^{x+y} = e^{x+y} - e^0$$
$$= e^x . e^y - 1.$$

second integrate wrt y.

$$\int_0^{1-x} (e^x . e^y - 1) dy = [e^{x+y} - y]_0^{1-x} = [e^{x+(1-x)} - (1-x)] - [e^x - 0]$$
$$= e - 1 + x - e^x = e - e^x - 1 + x.$$

third integrate wrt x.

$$\int_0^1 [e - e^x - 1 + x] dx = \left[ex - e^x - x + \frac{x^2}{2}\right]_0^1$$

$$= [e - e - 1 + \frac{1}{2}] - [0 - 1 - 0 + 0] = -\frac{1}{2} + 1 = \frac{1}{2}$$

Q.3. $\int_0^1 \int_{y^2}^1 \int_0^{1-x} x \, dz\, dy\, dx$

⇒ limit of z = 0 to $1-x$
limit of y = 0 to 1
limit of x = y^2 to 1.

first integrate wrt z.

$$\int_0^{1-x} x\, dz = [xz]_0^{1-x} = x(1-x) = x - x^2$$

second integrate wrt x.

$$\int_{y^2}^1 (x - x^2) dx = \left[\frac{x^2}{2} - \frac{x^3}{3}\right]_{y^2}^1$$

$$= \frac{1}{2} - \frac{1}{3} - \left(\frac{y^4}{2} - \frac{y^6}{3}\right) = \frac{1}{6} - \frac{y^4}{2} + \frac{y^6}{3}$$

$x, y, z \rightarrow z$ should be function of x and y.
$x, y \rightarrow y$ should be funⁿ of x
$x \rightarrow$ constant limit

Third integrate wrt y.

$$\int_0^1 \left(\frac{1}{6} - \frac{y^4}{2} + \frac{y^6}{3}\right) dy = \left[\frac{y}{6} - \frac{y^5}{10} + \frac{y^7}{21}\right]_0^1$$

$$= \left[\frac{1}{6} - \frac{1}{10} + \frac{1}{21}\right] = \frac{35 - 21 + 10}{210} = \frac{24}{210} = \frac{12}{105}$$

→ $x \quad y \quad z$
$3 \quad 2 \quad 1$

'z' is function of x and y
'y' is function of x
x is limit constant.

In first case limit should (except z)
Second case limit should (except x)
third case limit should be constant.

$$\int_0^1 \int_{y^2}^1 \int_{x+y}^1 x \, dx \, dy \, dz \qquad (x, y, z)$$

Q. $$\int_0^{\pi/2} \int_0^{\cos\theta} \int_0^{\sqrt{1-r^2}} r \, dz \, d\theta \, dr. \qquad (z, r, \theta)$$

⇒ first integrate wrt z.

$$\int_0^{\sqrt{1-r^2}} r \, dz = [rz]_0^{\sqrt{1-r^2}} = r\sqrt{1-r^2}$$

Second integrate wrt r.

$$\int_0^{\cos\theta} r\sqrt{1-r^2} \, dr = \int_0^{\cos\theta} r^2 \sqrt{\frac{1}{r^2} - 1} \, dr$$

When there is not given a limit in triple integration then first integrate wrt z, then y and then x. $(x\ y\ z)$

$z = \sqrt{1-r^2} \Rightarrow z^2 = 1-r^2$

we know, $\boxed{\int [f(x)]^n \cdot f'(x)\, dx = \frac{[f(x)]^{n+1}}{(n+1)} + c}$

$$\int_0^{\pi/2} \frac{-1}{2}\left[\int_0^{\infty} -2r\,(1-r^2)^{1/2}\, dr\right] d\theta = \frac{-1}{2}\int_0^{\pi/2} \frac{(1-r^2)^{3/2}}{3/2}$$

$$= \frac{-1}{3}\int_0^{0}\left[\int_{\pi/2}^{\pi/2}\left[(1-\cos^2\theta)^{3/2} - 1\right] d\theta\right.$$

$$= \frac{-1}{3}\int_0^{\pi/2}(\sin^3\theta - 1)\, d\theta = \frac{\pi}{6} - \frac{2}{9}$$

$\iiint dx\, dy\, dz$ over sphere $x^2+y^2+z^2=1$

In given question limits are not given. therefore, for triple integration compulsory first integrate wrt 'z' then 'y' and then 'x'. (x, y, z)

Now for the limit of 'z' draw a parallel strip to 'z' axis, the point at which strip enter the given region will be the lower limit of 'z' and at the point at which given strip exit given region will be the upper limit.

Now, for the limit of 'x' and 'y' take the projection of given region on x, y plane. (put $z=0$ in eq^n of region).

Now, we know how to find the limit of x, y in double integration

Sphere: $x^2+y^2+z^2=r^2$
tetrahedron: $ax+by+cz+d=0$

$z = \pm\sqrt{1-(x^2+y^2)}$

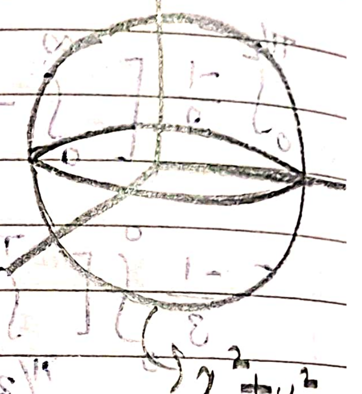
limit of 'z' is $-\sqrt{1-(x^2+y^2)}$ to $+\sqrt{1-(x^2+y^2)}$

limit of 'y' is at $z=0$

$x^2+y^2=1$

$y = \pm\sqrt{1-x^2}$

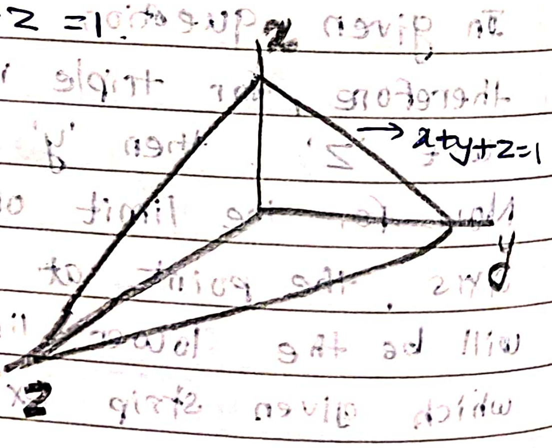
limit of 'y' is $-\sqrt{1-x^2}$ to $+\sqrt{1-x^2}$



limit of 'x' is '-1' to '1'

Q. Evaluate $\iiint_V z\,dv$ or $\iiint_V z\,dx\,dy\,dz$

⇒ Where 'V' is solid tetrahedron bounded by 4 plane.
$x=0,\ y=0,\ z=0$ and $x+y+z=1$



limit:
$z=0$ to $1-x-y$
$y=0$ to $1-x$
$x=0$ to 1

first integrate wrt 'z'.

$$\int_0^{1-x-y} z\,dz = \left[\frac{z^2}{2}\right]_0^{1-x-y} = \frac{(1-x-y)^2}{2}$$

second integrate wrt 'y'

$$\int_0^{1-x} \frac{(1-x-y)^2}{2}\,dy = \left[\frac{(1-x-y)^3}{6}\right]_0^{1-x}$$

$$= \frac{1}{6}\left[(1-x-y)^3\right]_0^{1-x} = \frac{1}{6}\left[(1-x-1+x)-(1-x)^3\right]$$

$$= \frac{1}{6}(1-x)^3$$

Now, integrate wrt 'x'.

$$= \frac{1}{6}\int_0^1 (1-x)^3\,dx = \left[\frac{(1-x)^4}{24}\right]_0^1 = \frac{1}{24}$$

Triple Integration using spherical co-ordinates.

Use these when given volume (region) is sphere or cone or ellipse solid.

r = radius

θ = angle with +ve 'z'

ϕ = angle with +ve 'x'

let, $x = r\sin\theta\cos\phi$

$y = r\sin\theta\sin\phi$

$z = r\cos\theta$

$$\iiint f(x,y,z)\,dx\,dy\,dz = \iiint_{r\,\theta\,\phi} f(r\sin\theta\cos\phi,\ r\sin\theta\sin\phi,\ r\cos\theta)\, r^2\sin\theta\, dr\,d\theta\,d\phi.$$

$$J = \frac{\partial(x,y,z)}{\partial(r,\theta,\phi)} = r^2\sin\theta$$

$\theta = 0$ to π $\quad$ $\phi = 0$ to 2π

$\theta = 0$ to $\pi/2$ $\quad$ $\phi = 0$ to π

$\theta = 0$ to $\pi/2$ $\quad$ $\phi = 0$ to $\pi/2$

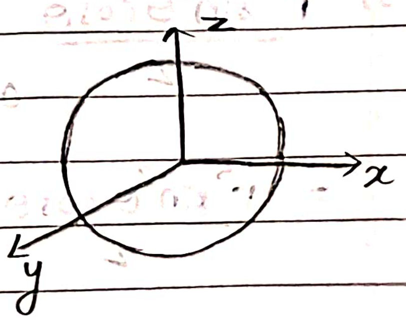
Total sphere $x^2+y^2+z^2=1$

co-ordinates of sphere

$r = 0$ to 1

$\theta = 0$ to π

$\phi = 0$ to 2π



Hemisphere,
$r = 0$ to 1
$\theta = 0$ to $\pi/2$
$\phi = 0$ to π

Sphere in +ve octate
$r = 0$ to 1
$\theta = 0$ to $\pi/2$
$\phi = 0$ to $\pi/2$

Q. $\iiint xyz \, dx \, dy \, dz$ over $x^2+y^2+z^2=4$
$r = 0$ to 2 $\phi = 0$ to 2π
$\theta = 0$ to π

$\Rightarrow$ first integrate wrt 'z'.

$$\int_0^{2\pi} xyz \, dz = xy\left[\frac{z^2}{2}\right]_0^{2\pi} = xy$$

$x = r\sin\theta\cos\phi$
$y = r\sin\theta\sin\phi$
$z = r\cos\theta$

replace x, y, z with

$$\int_{r=0}^{2}\int_{\theta=0}^{\pi}\int_{\phi=0}^{2\pi} r\sin\theta\cos\phi \cdot r\sin\theta\sin\phi \cdot r\cos\theta \cdot r^2\sin\theta \, dr \, d\theta \, d\phi$$

$$\int_0^2\int_0^{\pi}\int_0^{2\pi} r^5\sin^3\theta\cos\theta \cdot \sin\phi\cos\phi \, dr \, d\theta \, d\phi$$

first integrate wrt ϕ

$$\frac{1}{2}\int_0^2\int_0^{\pi}\int_0^{2\pi} r^5\sin^3\theta\cos\theta \sin 2\phi \, dr \, d\theta \, d\phi$$

$$\Rightarrow \frac{r^5\sin^3\theta\cos\theta}{2}\int_0^{2\pi}\sin 2\phi \, d\phi = \frac{r^5\sin^3\theta\cos\theta}{2}\left[\frac{-\cos 2\phi}{2}\right]_0^{2\pi}$$

$$= \frac{r^5\sin^3\theta\cos\theta}{2}\left[-\frac{1}{2}+\frac{1}{2}\right] = 0.$$

eqn: $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=r^2$ $\quad dx\,dy\,dz = abc\; r^2 \sin\theta\, dr\, d\theta\, d\phi$

$x = ar\sin\theta\cos\phi \qquad y = br\sin\theta\sin\phi \qquad z = cr\cos\theta$

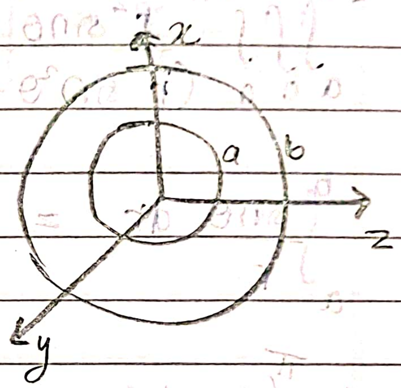
Integration w.r.t 'θ'.

$\Rightarrow 0$//

(4π) ans

$$\iiint_V \frac{dx\,dy\,dz}{(x^2+y^2+z^2)^{3/2}}$$ where 'V' is annulus betn $x^2+y^2+z^2 = a^2$ and $x^2+y^2+z^2=b$

$a<b$



$r = (b-a)$

$r = 0$ to $(b-a)$ = a to b

$\theta = 0$ to π

$\phi = 0$ to 2π

$$\int_0^{(b-a)}\int_0^{\pi}\int_0^{2\pi} \frac{dx\,dy\,dz}{(x^2+y^2+z^2)^{3/2}}$$

$$\int_a^b\int_0^{\pi}\int_0^{2\pi} (x^2+y^2+z^2)^{-3/2}\, dx\, dy\, dz$$

$x = r\sin\theta\cos\phi \quad , \quad z = r\cos\theta$

$y = r\sin\theta\sin\phi$

$$\int^b\int_0^{\pi}\int_0^{2\pi}\left(r^2\sin^2\theta\cos^2\phi + r^2\sin^2\theta\sin^2\phi + r^2\cos^2\theta\right)^{-3/2} r^2\sin\theta\, dr\, d\theta\, d\phi$$

$$\cancel{\int^b\int_0^{\pi}\int_0^{2\pi}\left(r^4\sin^3\theta\cos^2\phi + r^4\sin^3\theta\sin^2\phi + r^4\sin\theta\cos^2\theta\right)^{-3/2}}$$

~~Integrate first wrt z,~~

$$\cancel{\int_a^b\int_0^{\pi}\int_0^{2\pi}}$$

$$\int_a^b\int_0^{\pi}\int_0^{2\pi} \frac{r^2\sin\theta \; dr\,d\theta\,d\phi}{(r^2\sin^2\theta\cos^2\phi + r^2\sin^2\theta\sin^2\phi + r^2\cos^2\theta)^{3/2}}$$

$$\Rightarrow \int_a^b\int_0^{\pi}\int_0^{2\pi} \frac{r^2\sin\theta \; dr\,d\theta\,d\phi}{(r^2\sin^2\theta(\cos^2\phi + \sin^2\phi) + r^2\cos^2\theta)^{3/2}}$$

$$= \int_a^b\int_0^{\pi}\int_0^{2\pi} \frac{r^2\sin\theta \; dr\,d\theta\,d\phi}{(r^2\sin^2\theta + r^2\cos^2\theta)^{3/2}} = \iiint \frac{r^2\sin\theta \; dr\,d\theta\,d\phi}{r^3}$$

$$\int_a^b \frac{\sin\theta}{r}\,dr = \sin\theta\,[\ln(r)]_a^b = \sin\theta \ln\left(\frac{b}{a}\right)$$

$$\ln\left(\frac{b}{a}\right)\int_0^{\pi}\sin\theta\,d\theta = -[\cos\theta]_0^{\pi} = -(-1-1) = 2\ln\left(\frac{b}{a}\right)$$

$$\int_0^{2\pi} 2\ln\left(\frac{b}{a}\right)d\phi = 2\ln\left(\frac{b}{a}\right)[\phi]_0^{2\pi} = 2\ln\left(\frac{b}{a}\right)(2\pi)$$

$$= 4\pi\ln\left(\frac{b}{a}\right)$$

Q. Evaluate $\int_0^1\int_y^{1+y^2} x^2y \; dx\,dy$

$\Rightarrow$ limit of $x = y$ to $1+y^2$

limit of $y = 0$ to 1

first integrate wrt 'x'

$$= \int_0^1\left[\int_y^{1+y^2} x^2y\,dx\right]dy = \int_0^1 y\left[\frac{x^3}{3}\right]_y^{1+y^2} dy$$

$$= \int_0^1 y\left[\frac{(1+y^2)^3}{3} - \frac{y^3}{3}\right]dy$$

$$= \int_0^1 y\left[\frac{1+y^6+3y^2+3y^4-y^3}{3}\right]dy$$

$$= \int_0^1 \frac{(y+y^7+3y^3+3y^5-y^4)}{3}\,dy$$

$$= \frac{1}{3}\left[\frac{y^2}{2}+\frac{y^8}{8}+\frac{3y^4}{4}+\frac{3y^6}{6}-\frac{y^5}{5}\right]_0^1$$

$$= \frac{1}{3}\left[\frac{1}{2}+\frac{1}{8}+\frac{3}{4}+\frac{3}{6}-\frac{1}{5}\right] = \frac{1}{3}\left[1+\frac{7}{8}-\frac{1}{5}\right]$$

$$= \frac{1}{3}\left[\frac{40+35-8}{40}\right] = \frac{67}{120}$$

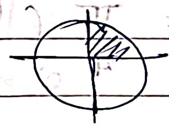
Q. Evaluate $\displaystyle\int_0^1\int_0^{\sqrt{1+x^2}} \frac{dx\,dy}{1+x^2+y^2}$

⇒ limit of 'x' is 0 to 1
limit of 'y' is 0 to $\sqrt{1+x^2}$
~~first integrate wrt 'y'~~.

$$\int_0^1\int_0^{\sqrt{1+x^2}} \frac{dx\,dy}{(\sqrt{1+x^2})^2+y^2}$$

We know that $\displaystyle\int \frac{1}{\sqrt{a^2+x^2}} = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right)$

$$\cancel{= \int_0^1\int_0^{\sqrt{1+x^2}}\left[\frac{dy}{(\sqrt{1+x^2})^2+y^2}\right]dx}$$



$x^2+y^2=1$

⇒ by polar coordinates, put $x = r\cos\theta$, $y = r\sin\theta$.
limit of r: 0 to 1
limit of θ: 0 to $\pi/2$

$$\int_0^1\int_0^{\pi/2} \frac{r\,dr\,d\theta}{1+r^2\cos^2\theta+r^2\sin^2\theta} = \int_0^1\int_0^{\pi/2}\frac{r\,dr\,d\theta}{1+r^2}$$

Put, $1+r^2=u$
$2r\cdot dr = du$
when, $r=0$, $u=1$
$r=1$, $u=2$

$$= \int_0^{\pi/2}\left[\frac{1}{2}\int_1^2 \frac{du}{u}\right]d\theta = \int_0^{\pi/2}\frac{1}{2}\ln(2)\,d\theta$$

$$= \frac{1}{2}\ln(2)\left(\frac{\pi}{2}\right) = \frac{\pi}{4}\ln(2)$$

Alternatively, $x=0$, $x^2 = a^2 - y^2$. further, use substitution

Let, $x = r\cos\theta$, $y = r\sin\theta$. ans $\Rightarrow \frac{\pi a^3}{6}$

limits of r : 0 to a

θ : 0 to $\pi/2$

also

Q. Evaluate $\int_0^a \int_0^{\sqrt{a^2-y^2}} \sqrt{a^2-x^2-y^2}\, dx\, dy$

$\Rightarrow I = \int_0^a \int_0^{\sqrt{a^2-y^2}} \sqrt{a^2-x^2-y^2}\, dx\, dy$

limit of x = 0 to $\sqrt{a^2-y^2}$

limit of y = 0 to a

first integrate w.r.t x,

$I = \int_0^a \left[\int_0^{\sqrt{a^2-y^2}} \sqrt{a^2-x^2-y^2}\, dx \right] dy$ $\quad \because \sqrt{a^2-x^2-y^2} = \sqrt{(a^2-y^2)...}$

We know, $\int \sqrt{a^2-x^2}\, dx = \frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right)$

$I = \int_0^a \left[\frac{x}{2}\sqrt{(a^2-y^2)-x^2} + \frac{(a^2-y^2)}{2}\sin^{-1}\left(\frac{x}{\sqrt{a^2-y^2}}\right)\right]_0^{\sqrt{a^2-y^2}} dy$

$= \int_0^a \left[\frac{\sqrt{a^2-y^2}}{2}\sqrt{(a^2-y^2)-(a^2-y^2)} + \frac{(a^2-y^2)}{2}\sin^{-1}\left(\frac{\sqrt{a^2-y^2}}{\sqrt{a^2-y^2}}\right)\right] - 0$

$= \int_0^a \frac{(a^2-y^2)}{2}\sin^{-1}(1)\, dy = \int_0^a \frac{(a^2-y^2)}{2} \times \frac{\pi}{2}\, dy$

$= \frac{\pi}{4}\int_0^a (a^2-y^2)\, dy = \frac{\pi}{4}\left[a^2 y - \frac{y^3}{3}\right]_0^a$

$= \frac{\pi}{4}\left[a^3 - \frac{a^3}{3}\right] = \frac{\pi}{4}\left(\frac{2a^3}{3}\right) = \frac{\pi a^3}{6}$

Q. Evaluate $\int_0^{\pi/2} \int_{\pi/2}^{\pi} \cos(x+y)\, dx\, dy$

Limit of x = $\pi/2$ to π

limit of y = 0 to $\pi/2$

first integrate wrt x.

$$I = \int_0^{\pi/2}\left[\int_{\pi/2}^{\pi} \cos(x+y)\,dx\right]dy = \int_0^{\pi/2}\left[\sin(x+y)\right]_{\pi/2}^{\pi} dy$$

$$= \int_0^{\pi/2}\left[\sin(\pi+y) - \sin\left(\frac{\pi}{2}+y\right)\right]dy$$

$$= \int_0^{\pi/2}\left[-\sin y - \cos y\right]dy = \cancel{-\cos y}$$

$$= -\int_0^{\pi/2}(\sin y + \cos y)\,dy = -\left(\left[-\cos y\right]_0^{\pi/2} + \left[\sin y\right]_0^{\pi/2}\right)$$

$$= -\left[(0+1)+1\right] = \underline{\underline{-2}}$$

Evaluate $\int_3^4\int_1^2 \frac{dx\,dy}{(x+y)^2}$, limit of $x = 1$ to 2, limit of $y = 3$ to 4

$$I = \int_3^4\int_1^2 \frac{dx\,dy}{(x+y)^2}$$

$$= \int_3^4\left[\int_1^2 \frac{dx}{(x+y)^2}\right]dy = \int_3^4\left[\left[\frac{-1}{(x+y)}\right]_1^2\right]dy$$

$$= \int_3^4\left[\frac{-1}{(2+y)} + \frac{1}{(1+y)}\right]dy = \int_3^4 \frac{1}{1+y}\,dy - \int_3^4 \frac{1}{2+y}\,dy$$

$$= \left[\log(1+y) - \log(2+y)\right]_3^4$$

$$= \left[\log 5 - \log 6\right] - \left[\log(4) - \log 5\right]$$

$$= \log 5 - \log 6 - \log 4 + \log 5 = \log 5 + \log 5 - (\log 6 + \log 4)$$

$$= \log\left(\frac{25}{24}\right)$$

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Q. $\int_0^1 \int_{x^2}^{2-x} xy \, dx \, dy$

$\Rightarrow$ limit of $y = x^2$ to $2-x$
limit of $x = 0$ to 1

$$I = \int_0^1 \left[\int_{x^2}^{2-x} xy \, dy \right] dx = \int_0^1 \left[\frac{xy^2}{2} \right]_{x^2}^{2-x} dx$$

$$= \int_0^1 \left[\frac{x(2-x)^2}{2} - \frac{x(x^2)^2}{2} \right] dx$$

$$= \int_0^1 \frac{x(4+x^2-4x) - x^5}{2} dx = \frac{1}{2} \int_0^1 (4x + x^3 - 4x^2 - x^5)$$

$$= \frac{1}{2} \left[\frac{4x^2}{2} + \frac{x^4}{4} - \frac{4x^3}{3} - \frac{x^6}{6} \right]_0^1$$

$$= \frac{1}{2} \left[\frac{4}{2} + \frac{1}{4} - \frac{4}{3} - \frac{1}{6} \right] = \frac{1}{2} \left[\frac{24+3-16-2}{12} \right] = \frac{9}{24} = \frac{3}{8}$$

Q. $\int_0^1 \int_{0}^{x^2} (x^2+y^2) \, dx \, dy$

$\Rightarrow$ limit of $y = 0$ to x^2
limit of $x = 0$ to 1
first integrate wrt y.

$$= \int_0^1 \left[\int_0^{x^2} (x^2+y^2) \, dy \right] dx = \int_0^1 \left[x^2 y + \frac{y^3}{3} \right]_0^{x^2} dx$$

$$= \int_0^1 \left[x^4 + \frac{x^6}{3} \right] dx = \left[\frac{x^5}{5} + \frac{x^7}{21} \right]_0^1$$

$$= \frac{1}{5} + \frac{1}{21} = \frac{26}{105}$$

classmate

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Q. $\int_0^a \int_0^{\sqrt{ay}} xy \, dx \, dy$

$\Rightarrow$ limit of $x = 0$ to $\sqrt{ay}$

limit of $y = 0$ to a. first integrate wrt x

$$I = \int_0^a \int_0^{\sqrt{ay}} xy \, dx \, dy = \int_0^a \left[\int_0^{\sqrt{ay}} xy \, dx \right] dy$$

$$= \int_0^a \left[\frac{x^2 y}{2} \right]_0^{\sqrt{ay}} dy = \int_0^a \frac{ay^2}{2} dy = \left[\frac{ay^3}{6} \right]_0^a$$

$$= \frac{a^4}{6}$$

Q. $\int_0^1 \int_{y^2}^{y} (1 + xy^2) \, dx \, dy$

limit of $x = y^2$ to y

limit of $y = 0$ to 1.

$\Rightarrow$ first integrate wrt x.

$$\int_0^1 \left[\int_{y^2}^{y} (1+xy^2) dx \right] dy = \int_0^1 \left[x + \frac{x^2 y^2}{2} \right]_{y^2}^{y} dy$$

$$= \int_0^1 \left[y + \frac{y^4}{2} - y^2 - \frac{y^6}{2} \right] dy = \left[\frac{y^2}{2} + \frac{y^5}{10} - \frac{y^3}{3} - \frac{y^7}{14} \right]_0^1$$

$$= \frac{1}{2} + \frac{1}{10} - \frac{1}{3} - \frac{1}{14} = \frac{41}{210}$$

Q. $\int_0^a \int_0^{\sqrt{a^2-y^2}} dx \, dy =$ limit of $x = 0$ to $\sqrt{a^2-y^2}$

limit of $y = 0$ to a

first integrate wrt x.

$$\int_0^a \left[\int_0^{\sqrt{a^2-y^2}} dx \right] dy = \int_0^a [x]_0^{\sqrt{a^2-y^2}} dy$$

$$= \int_0^a \sqrt{a^2-y^2}\, dy = \left[\frac{y}{2}\sqrt{a^2-y^2} + \frac{a^2}{2}\sin^{-1}\left(\frac{y}{a}\right) \right]_0^a$$

$$= \frac{a}{2}(0) + \frac{a^2}{2}\sin^{-1}(1) = \frac{a^2}{2} \times \frac{\pi}{2} = \frac{a^2\pi}{4}$$

Q. $$\int_0^\infty \int_0^\infty e^{-x^2(1+y^2)} \cdot x\, dx\, dy$$

limit of $x = 0$ to ∞

limit of $y = 0$ to ∞

$$\Rightarrow \int_0^\infty \int_0^\infty e^{-x^2-x^2y^2} \cdot x\, dx\, dy$$

$$\Rightarrow \int_0^\infty \int_0^\infty e^{-x^2} e^{-x^2y^2} x\, dx\, dy$$

$$\int_1^a\int_1^b \frac{dx\,dy}{xy}$$

first integrate wrt x limit of $x = 1$ to b

limit of $y = 1$ to a

$$= \int_1^a \left[\int_1^b \frac{dx\,\cancel{dy}}{xy}\right] dy$$

$$= \int_1^a \frac{1}{y}\left[\log x\right]_1^b dy = \int_1^a \frac{1}{y}(\log b - \log 1)\,dy$$

$$= \int_1^a \frac{1}{y}\log b\,dy = \log b\left[\log y\right]_1^a$$

$$= \log b \log a = \log a \log b$$

$$\int_0^1\int_0^1 \frac{dx\,dy}{\sqrt{(1-x^2)(1-y^2)}}$$

limit of $x = 0$ to 1 & limit of $y = 0$ to 1.

first integrate wrt 'x'.

$$\int_0^1\left[\int_0^1 \frac{dx}{\sqrt{(1-x^2)(1-y^2)}}\right] dy = \int_0^1\left[\frac{1}{\sqrt{1-y^2}}\int_0^1 \frac{dx}{\sqrt{1-x^2}}\right] dy$$

$$= \int_0^1 \frac{1}{\sqrt{1-y^2}}\left[\sin^{-1}(x)\right]_0^1 dy = \int_0^1 \frac{1}{\sqrt{1-y^2}}\,\frac{\pi}{2}\,dy$$

$$= \frac{\pi}{2}\int_0^1 \frac{1}{\sqrt{1-y^2}}\,dy = \frac{\pi}{2}\left[\sin^{-1}(y)\right]_0^1$$

$$= \frac{\pi^2}{4}$$

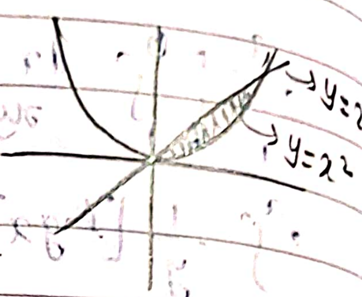
Q. $\iint_R (4x+2)\,dx\,dy$ where 'R' is region bounded by the parabola $y=x^2$ and $y=2x$

$\Rightarrow$ limit of $y = x^2$ to $2x$

$x^2 = 2x$

$x^2 - 2x = 0 \Rightarrow x(x-2) = 0.$

limit of $x = 0$ to 2



first integrate wrt 'y'.

$$\int_0^2 \left[\int_{x^2}^{2x} (4x+2)\,dy \right] dx = \int_0^{x} \cancel{\frac{4x^2}{2}+}$$

$$\Rightarrow \int_0^2 \left[4xy + 2y\right]_{x^2}^{2x} dx = \int_0^2 (8x^2 + 4x - 4x^3 - 2x^2)\,dx$$

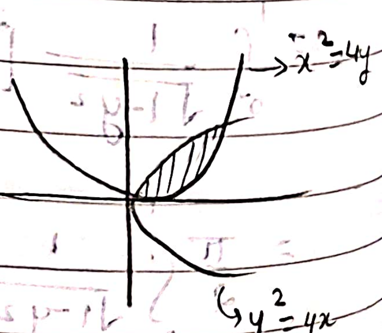
$$= \int_0^2 (6x^2 + 4x - 4x^3)\,dx = \left[\frac{6x^3}{3} + \frac{4x^2}{2} - \frac{4x^4}{4}\right]_0^2$$

$$= (2x^3 + 2x^2 - x^4)_0^2 = 16 + 8 - 16 = \underline{\underline{8}}$$

Q. $\iint_R y\,dx\,dy$ R is region bounded by the parabola $x^2 = 4y$ and $y^2 = 4x$

$\Rightarrow$ first integrate wrt y.

lower limit of y is $\frac{x^2}{4}$



Upper limit of y is $2\sqrt{x}$

$4y - x^2 = 0 \quad \Rightarrow y(4y - x^2) = 0$

$y^2 - 4x = 0 \quad\quad x(y^2 - 4x) = 0.$

$\Rightarrow \; 4y^2 - yx^2 = 0 \quad\quad = 4y^2 - 4x^2$

$+ \; xy^2 - 4x^2 = 0 \quad\quad = 4(y^2 - x^2) = 0.$

limit of $x = 0$ to 4.

first integrate wrt y.

$$= \int_0^4 \left[\int_{x^2/4}^{2\sqrt{x}} y \, dy \right] dx = \int_0^4 \left[\frac{y^2}{2} \right]_{x^2/4}^{2\sqrt{x}} dx = \int_0^4 \left[\frac{4x}{2} - \frac{x^4}{32} \right] dx$$

$$= \left[\frac{2x^2}{2} - \frac{x^5}{160} \right]_0^4 = \left[x^2 - \frac{x^5}{160} \right]_0^4 = \cancel{\left(16 - \frac{4 \times 16 \times 16}{40} \right)}$$

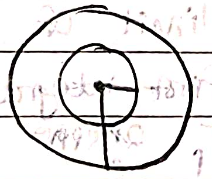
$$= \cancel{16 - \frac{128}{5}} = \cancel{\frac{80 - 128}{5}}$$

$$= \cancel{\frac{-48}{5}} = \left[16 - \frac{1024}{160} \right] = \frac{48}{5}$$

Q. $\iint_R \frac{x^2 y^2}{x^2 + y^2} \, dx \, dy$ where R is annulus betn two circles $x^2 + y^2 = 4$ and $x^2 + y^2 = 9$.

⇒ Put, $x = r\cos\theta$, $y = r\sin\theta$.

limit of r: 2 to 3, θ: 0 to 2π.



$$\int_2^3 \int_0^{2\pi} \frac{r^2\cos^2\theta \; r^2\sin^2\theta}{r^2\cos^2\theta + r^2\sin^2\theta} \, r \, dr \, d\theta = \int_2^3 \int_0^{2\pi} r^3 \cos^2\theta \sin^2\theta \, dr \, d\theta.$$

$\because \cos 2\theta = 1 - 2\sin^2\theta$.

$\sin^2\theta = \frac{1 - \cos 2\theta}{2}$.

$\because \cos 2\theta = 2\cos^2\theta - 1$

$\cos^2\theta = \frac{1 + \cos 2\theta}{2}$

$$\int_2^3 \int_0^{2\pi} r^3 \left(\frac{1 + \cos 2\theta}{2} \right) \left(\frac{1 - \cos 2\theta}{2} \right) dr \, d\theta = \iint \frac{r^3}{4} \left(1 - \cos^2(2\theta) \right) dr \, d\theta$$

$$= \iint \frac{r^3}{4} \sin^2(2\theta) \, dr \, d\theta$$

$\because \cos 2\theta = 1 - 2\sin^2\theta$.

$\cos 4\theta = 1 - 2\sin^2(2\theta)$

$\Rightarrow \sin^2(2\theta) = \frac{1 - \cos 4\theta}{2}$

Triple integration using cylindrical co-ordinate using cylindrical co-ordinates when given volume is cylindrical (or it's parts) or cone.

$$\iiint f(x,y,z)\,dx\,dy\,dz = \iiint_{r\,\theta\,z} f(r\cos\theta,\ r\sin\theta,\ z)\,r\,dr\,d\theta\,dz$$

NOTE:
1. compulsory first integrate wrt 'z' then we have choice
2. for limit of 'z' draw strip parallel to z-axis

$x = r\cos\theta$, $y = r\sin\theta$, $z = z$

$dx\,dy\,dz = r\,dr\,d\theta\,dz$.

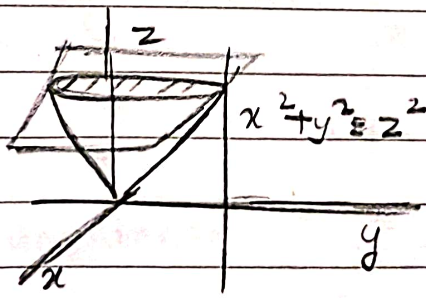
Q. $\iiint \sqrt{x^2+y^2}\,dx\,dy\,dz$, where V is $\underbrace{x^2+y^2+z^2=0}_{\text{cone}}$, $z=0$ & $z>0$ $z=1$.

⇒ limit of $z = r$ to 1.

$x = r\cos\theta$, $y = r\sin\theta$

limit of $r = 0$ to 1

limit of $\theta = 0$ to 2π



$$\iiint \sqrt{x^2+y^2}\,dx\,dy\,dz$$

$$= \iiint \sqrt{r^2\cos^2\theta + r^2\sin^2\theta}\ .\ r\,dr\,d\theta\,dz$$

$$= \iiint \sqrt{r^2}\,.\,r\,dr\,d\theta\,dz = \int_0^1\int_0^{2\pi}\int_r^1 r^2\,dr\,d\theta\,dz$$

* Area $= \iint_R dx\, dy$ where R is region.

volume $= \iiint_V dx\, dy\, dz$ where V is volume

Ex. find Area bounded by $x^2+y^2=1$ in +ve octant in XY plane.

$\Rightarrow$ $r: 0$ to 1

$\theta: 0$ to $\pi/2$

$x = r\cos\theta$

$y = r\sin\theta$

$\to x^2+y^2=1$ i.e. $r=1$

$$\iint_{\substack{x^2+y^2=1 \\ \text{in +ve octant}}} dx\,dy = \int_0^{\pi/2}\int_0^1 r\, dr\, d\theta = \int_0^1 r\, dr \times \int_0^{\pi/2} d\theta = \frac{1}{2}\cdot\frac{\pi}{2} = \frac{\pi}{4}.$$

Ex. find volume bounded by $x^2+y^2+z^2=1$

$\Rightarrow$ volume $= \iiint dx\, dy\, dz$

By spherical co-ordinate

$r: 0$ to 1

$\theta: 0$ to π

$\phi: 0$ to 2π.

$$\text{volume} = \int_{r=0}^{1}\int_{\theta=0}^{\pi}\int_{\phi=0}^{2\pi} r^2\sin\theta\, dr\, d\theta\, d\phi$$

$$= \int_0^1 r^2\, dr \times \int_{\theta=0}^{\pi}\sin\theta\, d\theta \times \int_{\phi=0}^{2\pi} d\phi = \frac{1}{3}(2)(2\pi) = \frac{4\pi}{3}$$

* R: over area outside $\underset{\hookrightarrow 1^{st}}{x^2+y^2=ax}$ and inside $\underset{\hookrightarrow 2^{nd}}{x^2+y^2=2ax}$

$x^2+y^2-ax=0$

$\left(x-\frac{a}{2}\right)^2+y^2=\left(\frac{a}{2}\right)^2$

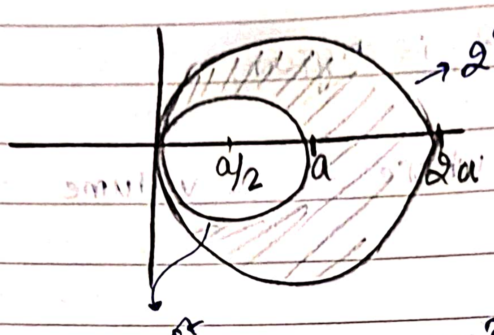
center. $\left(\frac{a}{2},0\right)$ $\quad r=\frac{a}{2}$

$x^2+y^2-2ax=0$

$(x-a)^2+y^2=a^2$

$c(a,0)$, $r=a$

eqn of line by $(y-y_1) = \frac{y_2-y_1}{x_2-x_1}(x-x_1)$



$x^2+y^2 = ax$
$x = r\cos\theta$, $y = r\sin\theta$
$r^2 = ar\cos\theta$.
$r = a\cos\theta$

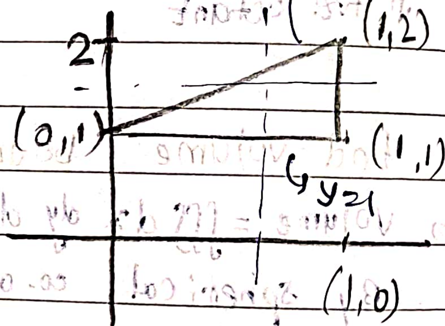
$x^2+y^2 = 2ax$

limit of $\qquad r = 2a\cos\theta$.

$r: a\cos\theta$ to $2a\cos\theta$

$\theta: -\pi/2$ to $\pi/2$ $\qquad$ (θ: angle with +ve x-axis)

* R: vertex $(0,1)$ $(1,1)$ $(1,2)$



① Parallel to y (draw strip)

x: 0 to 1

y: 1 to $1+x$

or

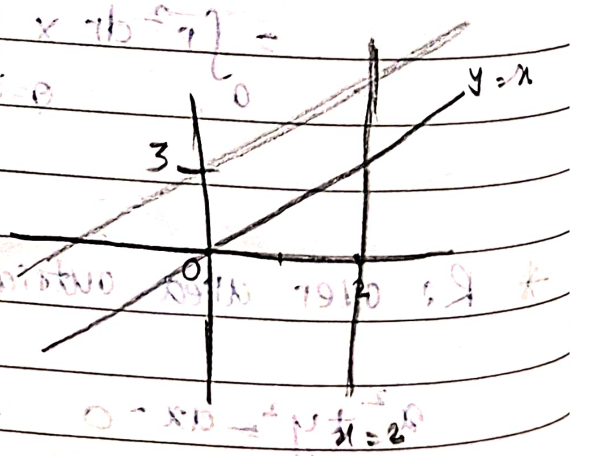
② Parallel to x (draw strip)

x: $y-1$ to 1

y: 1 to 2

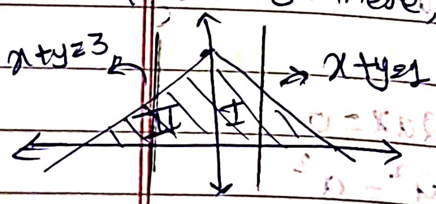
$y-1 = (y-y_1) = m(x-x_1)$
$y-1 = \frac{2}{1}(x-2)$
$y-1 = 2x-4$

* R: $x=0$, $x=2$, $y=x$, $y=x+2$



Note.

If Ques like these, suppose



We have to calculate the area by drawing strip parallel to y-axis. If they are not same then do like these

for I region, limits will be y: 0 to $1-x$

for II region, limits will be y: 0 to $3-x$

area $= \iint_0^{1-x} dx\,dy + \iint_0^{3-x} dx\,dy$ $\qquad$ like these

whenever ...