Date 17/10/24.

Unit. 1

FUNCTION OF SINGLE VARIABLE.

$f : R \to R$

such that $f(x) = y$. called as function of single variable.

x is independent eg, $f(x) = x^2$

y is dependent $f(x) = \sin x$

\# Sequence : sequence is a function $\{\} : N \to R$
and sequence denoted by $\{a_n\}_{n=1}^{\infty}$ or $(a_n)_{n=1}^{\infty}$ or $a_{n\,n=1}^{\infty}$

eg, $\{a_n\}_{n=1}^{\infty} = \frac{1}{n} = \frac{1}{1}, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots$ convergent sequence

$\{a_n\}_{n=1}^{\infty} = n = 1, 2, 3, 4, 5, \dots$ divergent sequence.

$\{a_n\}_{n=1}^{\infty} = 3 = 3, 3, 3, 3, \dots$ constant sequence.

$\{a_n\}_{n=1}^{\infty} = (-1)^n = -1, +1, -1, +1, -1, \dots$
called as oscillatory sequence.

* Convergence and Divergence of sequence.
let 'a_n' be given sequence then

① a_n is convergent iff $\lim_{n\to\infty} a_n$ = exist (finite and unique)

② a_n is divergent iff $\lim_{n\to\infty} a_n = \infty$ or $-\infty$

③ otherwise oscillatory.

eg, ① If $a_n = \frac{1}{n}$ be given sequence then

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1}{n} = 0 \text{ (exist)}$$

$\therefore$ $a_n = \frac{1}{n}$ is convergent and converges to '0'.

② $a_n = n$

$\lim_{n\to\infty} a_n = \lim_{n\to\infty} n = \infty$

$\therefore a_n = n$ is divergent.

③ $a_n = (-1)^n$

$\lim_{n\to\infty} (-1)^n = 1$, if n = even

$= -1$, if n = odd

limit exist but not unique.

$\therefore$ Oscillatory.

* Standard Result of sequence.

① If $a_n = \frac{1}{n^k}$, $k>0$ then $\lim_{n\to\infty} \frac{1}{n^k} = 0$ (Convergent)

② If $a_n = n^k$, $k>0$ then $\lim_{n\to\infty} n^k = \infty$ (divergent)

③ If $a_n = r^n$, $k>0$ then $\lim_{n\to\infty} r^n = 0$ if $r \in (-1,1)$

④ If $a_n = \left(1+\frac{x}{n}\right)^n$ then $\lim_{n\to\infty} \left(1+\frac{x}{n}\right)^n = e^x$, for all $x \in \mathbb{R}$

⑤ If $a_n = \left(1+\frac{1}{n}\right)^n$ then $\lim_{n\to\infty} \left(1+\frac{1}{n}\right)^n = e$

⑥ If $a_n = \left(1-\frac{1}{n}\right)^n$ then $\lim_{n\to\infty} \left(1-\frac{1}{n}\right)^n = \frac{1}{e}$

SERIES:

let $\{a_n\}$ be given sequence of real value then the expression

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \dots + a_n + \dots$$ called series.

eg, $\sum_{n=1}^{\infty} \frac{1}{n} = \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \dots + \frac{1}{n} + \frac{1}{n+1} + \dots$

* Sequence of partial sum:

let $\sum_{n=1}^{\infty} a_n$ be given series then sequence of partial sum (SPS) is $S_n = a_1 + a_2 + a_3 + a_4 + \dots + a_n$

i.e,

$S_1 = a_1$ | if $S_n = \sum_{n=0}^{\infty} ar^n$

$S_2 = a_1 + a_2$

$S_3 = a_1 + a_2 + a_3$ | $S_n = \frac{a(1-r^n)}{(1-r)}$

$S_n = a_1 + a_2 + a_3 + \dots + a_n$

18/10/24.

* Convergence And Divergence of series

Let $\sum_{n=1}^{\infty} u_n$ be given series then (Test for divergence)

① If $\sum_{n=1}^{\infty} u_n$ is convergent then $\lim_{n\to\infty} u_n = 0$ But not conversly.

i.e. If $\lim_{n\to\infty} u_n = 0$ then $\sum_{n=1}^{\infty} u_n$ may or may not convergent.

② If $\lim_{n\to\infty} u_n \neq 0$ then $\sum_{n=1}^{\infty} u_n$ divergent.

(for MCQs) Note: By adding or deleting some term of series does not affect the nature of series.

Date ____ Page ____

Q. Write down, find n^{th} term and check convergence of series.

① $\frac{1}{2} + \frac{2}{3} + \frac{3}{4} + \frac{4}{5} + \dots$

$\Rightarrow \sum_{n=1}^{\infty} \frac{n}{n+1}$ $\quad \therefore$ n^{th} term is $U_n = \frac{n}{n+1}$

Now, $\lim_{n\to\infty} U_n = \lim_{n\to\infty} \frac{n}{n+1} = \lim_{n\to\infty} \frac{n}{n}\left(\frac{1}{1+\frac{1}{n}}\right)$

$\lim_{n\to\infty} \frac{1}{1+\frac{1}{n}} = \frac{1}{1+\frac{1}{\infty}} = \frac{1}{1+0} = 1 \neq 0.$

$\lim_{n\to\infty} U_n \neq 0.$ $\quad \therefore \sum \frac{n}{n+1}$ is divergent.

② $\frac{1}{\sqrt{2}+1} + \frac{1}{\sqrt{3}+\sqrt{2}} + \frac{1}{\sqrt{4}+\sqrt{3}} + \dots$

$\Rightarrow \sum_{n=1}^{\infty} \frac{1}{\sqrt{n+1}+\sqrt{n}}$ $\quad \therefore$ n^{th} term is $U_n = \frac{1}{\sqrt{n+1}+\sqrt{n}}$

Now, $\lim_{n\to\infty} U_n = \lim_{n\to\infty} \frac{1}{\sqrt{n+1}+\sqrt{n}} = \frac{1}{\sqrt{n}\left(\sqrt{1+\frac{1}{n}}+1\right)}$

$= \lim_{n\to\infty} \frac{1}{\sqrt{n}} \cdot \lim_{n\to\infty} \frac{1}{\left(\sqrt{1+\frac{1}{n}}+1\right)}$

$= 0 \times \frac{1}{2} = 0.$

$\therefore \lim_{n\to\infty} U_n = 0$

$\therefore \sum \frac{1}{\sqrt{n+1}+\sqrt{n}}$ may or may not convergent.

③ $\sum \cos\left(\frac{1}{n}\right)$

$\Rightarrow$ n^{th} term is $\cos(1/n)$.

$$\lim_{n\to\infty} u_n = \lim_{n\to\infty} \cos\left(\frac{1}{n}\right) = \cos\left(\lim_{n\to\infty}\frac{1}{n}\right) = \cos 0 = 1 \neq 0.$$

$$\lim_{n\to\infty} \cos\left(\frac{1}{n}\right) \neq 0.$$

$\therefore$ $\sum \cos(1/n)$ is divergent.

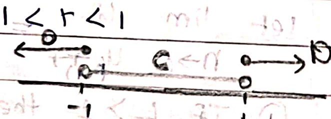
* Geometric series.

$$\sum r^n = 1 + r + r^2 + r^3 + r^4 + \dots = \frac{1}{1-r}$$

$\sum r^n$ convergent if $|r|<1$ i.e. $-1 < r < 1$

$\sum r^n$ divergent if $r \geq 1$

$\sum r^n$ oscillatory if $r \leq -1$



* Test for convergente of series.

→ Use geometric series, when given series in the form of (Number)'

1) **P Test :** The series $\sum \frac{1}{n^p}$ is called 'p' series

① If $p > 1$ then $\sum \frac{1}{n^p}$ is convergent ('p' is any number → real +ve no.*)

② If $p \leq 1$ then $\sum \frac{1}{n^p}$ is divergent.

NOTE: We can use P Test if given series in the form of $\frac{1}{n^p}$ only.

eg. ① $\sum \frac{1}{n^2}$ By P Test $p = 2 > 1$

$\therefore \sum \frac{1}{n^2}$ convergent

② $\sum \frac{1}{n}$ By p Test $p = 1 \leq 1$

$\therefore \sum \frac{1}{n}$ is divergent

③ $\sum \frac{1}{\sqrt{n}}$ $p = \frac{1}{2} \leq 1$ $\therefore \sum \frac{1}{\sqrt{n}}$ divergent

④ $\sum \frac{1}{n^2+1}$ We can't use p Test

$\because$ Not in the form of $\frac{1}{n^p}$

II) D'Alembert ratio Test :

let $\sum_{n=1}^{\infty} U_n$ be series of positive Number.

let $\lim_{n\to\infty} \frac{U_n}{U_{(n+1)}} = L$, then

eg, $U_n = \frac{1}{n}$

$U_{(n+1)} = \frac{1}{n+1}$

① If $L > 1$ then series convergent

② If $L < 1$ then series divergent.

③ If $L = 1$ then Test fail.

OR

let, $\lim_{n\to\infty} \frac{U_{(n+1)}}{U_n} = L.$

① If $L < 1$ then series convergent

② If $L > 1$ then series divergent.

③ ~~If~~ $L = 1$ then test fail

NOTE : Apply above test if given series contain e^n or n^n or $(\text{Number})^n$ or ~~factorial~~ or combination of n^n, $(\text{Number})^n$, ~~n!~~ Factorial

odd : $(2n-1)$
even : $2n$
$\sqrt{\ } \pm \sqrt{\ }$: rationalization.

Q. Write down the n^{th} term of the given series and check the converge

① $\sum_{n=1}^{\infty} \frac{2^n n!}{n^n}$ $\Rightarrow$ By D'Alembert ratio Test.

$u_n = \frac{2^n n!}{n^n}$ $\therefore$ $u_{n+1} = \frac{2^{n+1}(n+1)!}{(n+1)^{n+1}}$

$$\lim_{n\to\infty} \frac{u_n}{u_{(n+1)}} = \frac{2^n n!}{n^n} \times \frac{(n+1)^{n+1}}{2^{n+1}(n+1)!}$$

$$\lim_{n\to\infty} \frac{\cancel{2^n}\cancel{n!}}{n^n} \times \frac{(n+1)^{n+1}}{\cancel{2^n}.2\ (n+1)\cancel{(n)!}} = \lim_{n\to\infty} \frac{(n+1)^n}{n^n} \times \frac{1}{2}$$

$$\lim_{n\to\infty} \left(1+\frac{1}{n}\right)^n \cdot \frac{1}{2} = \frac{1}{2}(e)$$

$\because \lim_{n\to\infty} \left(1+\frac{1}{n}\right)^n = e$. $\quad l = \frac{e}{2} > 1$.

$\therefore$ By D'Alembert ratio test
$l > 1$,
$\sum \frac{2^n n!}{n^n}$ is convergent.

② $\frac{1}{2} + \frac{3!}{2.4} + \frac{5!}{2.4.6} + \dots$

$\Rightarrow$ n^{th} term, $u_n = \frac{(2n-1)!}{2.4.6\dots 2n}$, $u_{n+1} = \frac{(2(n+1)-1)!}{2.4.6\dots 2(n+1)}$

$$u_{n+1} = \frac{(2n+1)!}{2.4.6\dots 2(n+1)}$$

$$\lim_{n\to\infty} \frac{u_n}{u_{n+1}} = \frac{(2n-1)!}{2.4.6\dots 2n} \times \frac{2.4.6\dots 2(n+1)}{(2n+1)!}$$

$$\lim_{n\to\infty} \frac{\cancel{(2n-1)!}}{2.4.6\dots 2n} \times \frac{2.4.6\dots (2n+2)}{(2n+1)\ 2n\ \cancel{(2n-1)!}} = \frac{\cancel{2.4.6}\dots \cancel{2n}(2n+2)}{2.4.\cancel{6}\dots 2n \times (2n+1)2n}$$

$$\lim_{n\to\infty} \frac{2n+2}{2n(2n+1)} = \lim_{n\to\infty} \frac{n\left(2+\frac{2}{n}\right)}{2n^2\left(2+\frac{1}{n}\right)}$$

$$\lim_{n\to\infty} \frac{1}{2n} = 0 \qquad \because l = 0 < 1$$

given series is divergent.

③ $\frac{1}{2} + \frac{1.3}{2.4} + \frac{1.3.5}{2.4.6} + \ldots$

$\Rightarrow u_n = \frac{1.3.5\ldots(2n-1)}{2.4.6\ldots 2n}$, $u_{n+1} = \frac{1.3.5\ldots(2(n+1)-1)}{2.4.6\ldots 2(n+1)} = \frac{1.3.5\ldots(2n+1)}{2.4.6\ldots 2(n+1)}$

$= \frac{2n+2}{2n+1}$

$$\lim_{n\to\infty}\frac{u_n}{u_{n+1}} = \frac{1.3.5\ldots(2n-1)}{2.4.6\ldots 2n} \times \frac{2.4.6\ldots 2n.(2n+2)}{1.3.5\ldots(2n+1-2)(2n+1)}$$

$= 2n-1$

$$\lim_{n\to\infty}\frac{u_n}{u_{n+1}} = \frac{1.3.5\ldots(2n-1)}{2.4.6\ldots 2n} \times \frac{2.4.6\ldots 2(n+1)}{1.3.5\ldots(2n+1)}$$

$$\lim_{n\to\infty}\frac{2n+2}{2n+1} = \lim_{n\to\infty}\frac{n(2+\frac{2}{n})}{n(2+\frac{1}{n})} = 1 \quad \text{Test fails}$$

$$\lim_{n\to\infty} \frac{1.3.5\ldots(2n-1)}{2.4.6\ldots 2n} \times \frac{2.4.6\ldots 2n\,\cancel{(2n+1)}(2n+2)}{1.3.5\ldots(2n)(2n+1)}$$

$$\lim_{n\to\infty} \frac{\cancel{(2n+1)}(2n+2)}{(2n)\cancel{(2n+1)}} = \frac{2n+2}{2n} = \frac{n+1}{n} = 1+\frac{1}{n}$$

$$\lim_{n\to\infty} 1+\frac{1}{n} = 1 \quad \text{Test fails as } l = 1.$$

④ $\sum \frac{n^3}{(n-1)!}$

$\Rightarrow u_n = \frac{n^3}{(n-1)!}$, $u_{n+1} = \frac{(n+1)^3}{(n+1-1)!} = \frac{(n+1)^3}{n!}$

$$\lim_{n\to\infty}\frac{u_n}{u_{n+1}} \Rightarrow \lim_{n\to\infty}\frac{n^3}{(n-1)!}\times\frac{n!}{(n+1)^3} = \lim_{n\to\infty}\frac{n^3 \cdot n\cancel{(n-1)!}}{\cancel{(n-1)!}\,(n+1)^3}$$

$$\lim_{n\to\infty} \frac{n^4}{(n+1)^3} = \infty \qquad \underline{\text{convergent}}$$

$$\lim_{n\to\infty}\frac{n^4}{(n+1)^3} = \frac{n^4}{n^3+1+3n+3n^2} = \infty$$

5 $\sum \frac{n^2}{3^n}$

$\Rightarrow$ $u_n = \frac{n^2}{3^n}$, $u_{n+1} = \frac{(n+1)^2}{3^{n+1}}$

$\lim_{n\to\infty} \frac{u_n}{u_{n+1}} = \frac{n^2}{3^n} \times \frac{3^{n+1}}{(n+1)^2} = \frac{n^2}{\cancel{3^n}} \times \frac{\cancel{3^n}.3}{(n+1)^2}$

$\lim_{n\to\infty} \frac{3\ n^2}{(n+1)^2} \Rightarrow \lim_{n\to\infty} 3 \times \lim_{n\to\infty} \frac{n^2}{(n+1)^2}$

$\lim_{n\to\infty} 3 \times \lim_{n\to\infty} \frac{n^2}{n^2+1+2n} = \lim_{n\to\infty} 3 \frac{\cancel{n^2}}{\cancel{n^2}(1+\frac{1}{n^2}+\frac{2}{n})} = 3. > 1.$

$\therefore$ $\sum \frac{n^2}{3^n}$ is convergent.

6 $\sum \frac{x^n}{(2n)!} \Rightarrow u_n = \frac{x^n}{(2n)!}$, $u_{n+1} = \frac{x^{n+1}}{(2n+2)!}$

$\lim_{n\to\infty} \frac{u_n}{u_{n+1}} = \lim_{n\to\infty} \frac{x^n}{(2n)!} \times \frac{(2n+2)!}{x^n . x} = \lim_{n\to\infty} \frac{\cancel{x^n}}{\cancel{(2n)!}} \times \frac{(2n+2)(2n+1)\cancel{(2n)!}}{\cancel{x^n}.x}$

$\lim_{n\to\infty} \frac{(2n+2)(2n+1)}{x} = \infty > 1$, for all values of x

$\therefore$ It is convergent series.

7 $\sum \frac{n^2}{2^n} + \frac{1}{n^2}$

$\Rightarrow$ solve individually

$u_n = \frac{n^2}{2^n}$, $u_{n+1} = \frac{(n+1)^2}{2^{n+1}}$

$\lim_{n\to\infty} \frac{u_n}{u_{n+1}} = \frac{n^2}{2^n} \times \frac{2^{n+1}}{n^2+1+2n}$

$= \lim_{n\to\infty} \frac{n^2 \times \cancel{2^n} \times 2}{\cancel{2^n}(n^2+1+2n)} = \frac{2\cancel{n^2}}{\cancel{n^2}(1+\frac{1}{n^2}+\frac{2}{n})}$

$= 2. > 1 \therefore$ convergent.

$u_n = \frac{1}{n^2}$ by P-series

$P = 2 > 1$

$\therefore$ It is convergent series.

sum of it also convergent.

$\therefore$ $\sum \frac{n^2}{2^n} + \frac{1}{n^2}$ is convergent series.

ex. $\sum_{n=1}^{\infty} \sin\left(\frac{1}{n}\right)$, $\sum_{n=1}^{\infty} \sin^2\left(\frac{1}{n}\right)$, $\sum_{n=1}^{\infty} \sin^2\left(\frac{1}{n+1}\right)$

V_n $\quad V_n = \frac{1}{n^2}$ $\quad V_n = \frac{1}{(n+1)^2}$

Power of trigo. fun goes to bracket and these is the V_n

$\lim_{n\to\infty} \frac{u_n}{V_n} = \lim_{n\to\infty} \frac{\sin(\frac{1}{n})}{(\frac{1}{n})} = 1$ conv.

* COMPARISION TEST

let $\sum u_n$ & $\sum v_n$ are two series of positive number such that $u_n \le v_n$ for all n.

Then ① If $\sum v_n$ convergent then $\sum u_n$ convergent.

② If $\sum u_n$ divergent then $\sum v_n$ divergent.

* Limit comparison Test:

let $\sum u_n$ and $\sum v_n$ are two series of positive number and let $\lim_{n\to\infty} \frac{u_n}{v_n} = L$

NOTE: $v_n = \frac{1}{n^{q-p}}$ where p is Highest power of 'n' in Numerator.

q is Highest power of 'n' in denominator.

① If $L \ne 0$ and finite then $\sum u_n$ and $\sum v_n$ both convergent or both divergent.

② If $L=0$ then If $\sum v_n$ convergent then $\sum u_n$ convergent.

③ If $L=\infty$ then If $\sum v_n$ divergent then $\sum u_n$ divergent.

NOTE:

Use limit comparison test if given series contain polynomial in 'n' or polynomial under square root.

Q. Check the convergent.

① $\sum \sqrt{n^2+1} - n$ If square root is present, then always do rationalisation first.

$\Rightarrow u_n = \sqrt{n^2+1} - n = \frac{\sqrt{n^2+1} - n}{\sqrt{n^2+1}+n} \times \sqrt{n^2+1} + n$

$= \frac{n^2+1-n^2}{\sqrt{n^2+1}+n} = \frac{1}{\sqrt{n^2+1}+n}$

classmate
Date ______
Page ______

$u_n = \dfrac{1}{\sqrt{n^2+1}+n}$ $\quad \because \vartheta_n = \dfrac{1}{n^{1-0}} = \dfrac{1}{n}.$

$\sum \vartheta_n = \sum \frac{1}{n}.$

$$\lim_{n\to\infty} \frac{u_n}{\vartheta_n} = \lim_{n\to\infty} \frac{n}{\sqrt{n^2+1}+n} = \lim_{n\to\infty} \frac{n}{n}\left(\frac{1}{\sqrt{1+\frac{1}{n^2}}+1}\right)$$

$$= \lim_{n\to\infty} \frac{1}{\sqrt{1+\frac{1}{n^2}}+1} = \frac{1}{2} = L.$$

$L = \frac{1}{2} \neq 0$, finite $\quad \therefore$ Both series behave same

but by p test $\sum \frac{1}{n}$ divergent.

$\because p=1$

$\therefore \sum \sqrt{n^2+1} - n$ is divergent.

② $\sum \dfrac{2n^3+5}{4n^5+1}$

$\Rightarrow$ $u_n = \dfrac{2n^3+5}{4n^5+1}$, $\vartheta_n = \dfrac{1}{n^{5-3}} = \dfrac{1}{n^2}$

$$\lim_{n\to\infty} \frac{u_n}{\vartheta_n} = \lim_{n\to\infty} \left(\frac{2n^3+5}{4n^5+1}\right) \times \frac{n^2}{1} = \lim_{n\to\infty} \frac{2n^5+5n^2}{4n^5+1}.$$

$$\lim_{n\to\infty} \frac{\cancel{n^5}\left(2+\frac{5}{n^3}\right)}{\cancel{n^5}\left(4+\frac{1}{n^5}\right)} = \frac{2}{4} = \frac{1}{2} \neq 0, \text{ finite.}$$

$\because$ Both series behave same.

but by p. test. $\sum \frac{1}{n^2}$ is convergent.

$p=2 > 1.$

$\therefore \sum \dfrac{2n^3+5}{4n^5+1}$ is convergent.

③ $\sum \frac{\sqrt{n}}{n^2+1}$

$\Rightarrow$ $u_n = \frac{\sqrt{n}}{n^2+1}$, $v_n = \frac{1}{n^{2-\frac{1}{2}}} = \frac{1}{n^{3/2}}$

$\lim_{n\to\infty} \frac{u_n}{v_n} = \frac{\sqrt{n}}{n^2+1} \times \frac{n\sqrt{n}}{1} = \frac{n^2}{n^2+1}.$

$\lim_{n\to\infty} \frac{n^2}{n^2+1} = \lim_{n\to\infty} \frac{\cancel{n^2}}{\cancel{n^2}\left(1+\frac{1}{n^2}\right)} = 1 \neq 0$, finite.

$\because$ Both series behave same.

But by p-Test. $\sum \frac{1}{n^{3/2}}$ is convergent $\because \frac{3}{2} > 1$.

$\therefore$ $\sum \frac{\sqrt{n}}{n^2+1}$ is convergent.

④ $\sum \sqrt{n^3+1} - \sqrt{n^3}$

$\Rightarrow$ $u_n = \sqrt{n^3+1} - \sqrt{n^3} = \frac{\left(\sqrt{n^3+1} - \sqrt{n^3}\right)\left(\sqrt{n^3+1} + \sqrt{n^3}\right)}{\left(\sqrt{n^3+1} + \sqrt{n^3}\right)} = \frac{\cancel{n^3}+1-\cancel{n^3}}{\sqrt{n^3+1} + \sqrt{n^3}}$

$u_n = \frac{1}{\sqrt{n^3+1} + \sqrt{n^3}}$, $v_n = \frac{1}{n^{3/2-0}} = \frac{1}{n^{3/2}}$

$\lim_{n\to\infty} \frac{u_n}{v_n} = \frac{1}{\sqrt{n^3+1} + \sqrt{n^3}} \times \frac{n^{3/2}}{1} = \lim_{n\to\infty} \frac{n^{3/2}}{n^{3/2}\left(\sqrt{1+\frac{1}{n^3}} + 1\right)}$

$\lim_{n\to\infty} \frac{1}{\sqrt{1+\frac{1}{n^3}} + 1} = \frac{1}{2} \neq 0$, finite.

$\because$ Both series are same (behave)

But by p-Test $\sum \frac{1}{n^{3/2}}$, $3/2 > 1$ $\rightarrow$ convergent.

$\therefore$ $\sum \sqrt{n^3+1} - \sqrt{n^3}$ is convergent.

⑤ $1+\frac{1}{2^2}+\frac{2^2}{3^3}+\frac{3^3}{4^4}+\frac{4^4}{5^5}+\dots$

$\Rightarrow u_n = \frac{n^n}{(n+1)^{n+1}}$, $u_n = \frac{n^n}{(n+1)^n\,(n+1)}$ $\qquad (n+1)^n \approx n^n$

$u_n = \frac{n^n}{n^n\,(n+1)}$ $\Rightarrow u_n = \frac{1}{n+1}$.

By limit comparison Test, $v_n = \frac{1}{n^{1-0}} = \frac{1}{n}$.

$\lim\limits_{n\to\infty} \frac{u_n}{v_n} = \lim\limits_{n\to\infty} \frac{1}{n+1} \times \frac{n}{1} \Rightarrow \lim\limits_{n\to\infty} \frac{n}{n\left(1+\frac{1}{n}\right)} = 1 \neq 0$, finite.

$\therefore$ Behave of both series is same

$v_n = \frac{1}{n}$ $\sum \frac{1}{n}$ by p-test. $p=1$. it is divergent.

$\therefore 1+\frac{1}{2^2}+\frac{2^2}{3^3}+\frac{3^3}{4^4}+\frac{4^4}{5^5}+\dots$ is divergent.

⑥ $\sum \frac{1}{(n^2+n)^p}$

$\Rightarrow u_n = \frac{1}{(n^2+n)^p} = \frac{1}{[n(n+1)]^p}$ $\qquad n+1 \approx n$.

$u_n = \frac{1}{(n^2)^p} = \frac{1}{n^{2p}}$.

from p-Test.

$\sum \frac{1}{(n^2+n)^p}$ converges if $2p > 1$.
$p > \frac{1}{2}$.

and divergences if $2p \leq 1$.
$p \leq \frac{1}{2}$.

$\therefore \sum \frac{1}{(n^2+n)^p}$ convergent if $p > \frac{1}{2}$.

$\therefore \sum \frac{1}{(n^2+n)^p}$ divergent if $p \leq \frac{1}{2}$.

⑦ $\sum \sqrt[3]{n^3+1} - n$

In Limit comparison Test, if $L=0$ and $\sum V_n$ is divergent then we cannot conclude anything about the convergence or divergence of $\sum U_n$.

$$\sum \sqrt[3]{n^3+1} - n$$

$\Rightarrow$ $U_n = (n^3+1)^{1/3} - n.$

$$V_n = \frac{1}{n^{0-1}} = \frac{1}{n^{-1}} = n.$$

by limit comparison Test, $\lim\limits_{n\to\infty} \frac{U_n}{V_n}$.

$$\lim_{n\to\infty} \frac{U_n}{V_n} = \lim_{n\to\infty} \frac{(n^3+1)^{1/3} - n}{n} = \cancel{\lim_{n\to\infty}} \cancel{n(n^2}$$

$$\lim_{n\to\infty} \frac{n\left(1+\frac{1}{n^3}\right)^{1/3} - n}{n} = \lim_{n\to\infty} \frac{\cancel{n}\left[\left(1+\frac{1}{n^3}\right)^{1/3} - 1\right]}{\cancel{n}} = 0.$$

$\sum \sqrt[3]{n^3+1} - n = 0.$

by p-Test. $V_n = \frac{1}{n^{-1}}$, $P = -1 \leq 1.$ so, it is divergent.

__Examples:__

Check the convergence of the given sequence.

1) $$\sum \frac{\sqrt{n}}{n^2+1}$$

$\Rightarrow$ $U_n = \frac{\sqrt{n}}{n^2+1}$, $V_n = \frac{1}{n^{2-1/2}} = \frac{1}{n^{3/2}}$. $\therefore$ By limit comparison Test.

$$\lim_{n\to\infty} \frac{U_n}{V_n} = \lim_{n\to\infty} \frac{\sqrt{n}}{n^2+1} \times n^{3/2} = \lim_{n\to\infty} \frac{n^2}{n^2+1}$$

$$\lim_{n\to\infty} \frac{\cancel{n^2}}{\cancel{n^2}\left(1+\frac{1}{n^2}\right)} = 1. \neq 0 \text{ and finite.}$$

By p-Test $\sum V_n = \sum \frac{1}{n^{3/2}}$

$P = \frac{3}{2} > 1.$ $\because \sum V_n =$ convergent

$\therefore \sum \frac{\sqrt{n}}{n^2+1}$ is convergent.

2) $\frac{\sqrt{2}-\sqrt{1}}{1} + \frac{\sqrt{3}-\sqrt{2}}{2} + \frac{\sqrt{4}-\sqrt{3}}{3} + \dots$

$\Rightarrow U_n = \frac{\sqrt{n+1}-\sqrt{n}}{n}$, By limit comparision Test.

~~$V_n = \frac{1}{n^{\frac{1}{2}}}$~~ $\quad U_n = \frac{\sqrt{n+1}-\sqrt{n}}{n} \times \frac{\sqrt{n+1}+\sqrt{n}}{\sqrt{n+1}+\sqrt{n}}$.

$U_n = \frac{n+1-n}{n(\sqrt{n+1}+\sqrt{n})} = \frac{1}{n(\sqrt{n+1}+\sqrt{n})} = \frac{1}{n\sqrt{n}\left(\sqrt{1+\frac{1}{n}}+1\right)}$.

$V_n = \frac{1}{n^{3/2-0}} = \frac{1}{n^{3/2}}$

$\lim_{n\to\infty} \frac{U_n}{V_n} = \lim_{n\to\infty} \frac{1}{n\sqrt{n}\left(\sqrt{1+\frac{1}{n}}+1\right)} \times \frac{\cancel{n^{3/2}}}{1} = \frac{1}{2} \neq 0$, finite

Behave of both sequenece is same.

$V_n = \frac{1}{n^{3/2}}$, by P-Test, $P = 3/2 > 1$.

$\because \Sigma V_n =$ convergent.

$\therefore \frac{\sqrt{n+1}-\sqrt{n}}{n}$ is convergent.

3) $\sum \frac{1}{3^n+n}$

$\Rightarrow U_n = \frac{1}{3^n+n}$, $U_{n+1} = \frac{1}{3^{n+1}+n+1}$. by radio Test (fail)

$\lim_{n\to\infty} \frac{U_n}{U_{n+1}} = \frac{1}{3^n+n} \times \frac{3^{n+1}+n+1}{1} = \lim_{n\to\infty} \frac{3^{n+1}+n+1}{3^n+n} = \frac{\infty}{\infty}$ form.

Indetermined form.

By comparision Test;

finding U_n and V_n.

$3^n \leq 3^n$

$3^n \leq 3^n+n$ $\qquad \underbrace{\frac{1}{3^n+n}}_{U_n} \leq \underbrace{\frac{1}{3^n}}_{V_n}$ $\qquad U_n \leq V_n$

$\frac{1}{3^n} \geq \frac{1}{3^n+n}$

$$\sum U_n = \sum \frac{1}{3^n} = \sum \left(\frac{1}{3}\right)^n$$

By Geometric series, $\sum r^n$ convergent if $r \in (-1,1)$

$\therefore \sum \left(\frac{1}{3}\right)^n$ convergent.

$\therefore \sum \frac{1}{3^n+n}$ is convergent.

Power Series.

$$\sum_{n=0}^{\infty} a_n (x-c)^n = a_0 + a_1(x-c) + a_2(x-c)^2 + \ldots \quad (x>0)$$

if $c=0$ then $\sum a_n x^n = a_0 + a_1 x + a_2 x^2 + \ldots$

* Radius of convergence (ROC):

ROC is interval within which power series is convergent and divergent for value of x. outside interval called ROC or interval of convergence.

To find ROC

By ratio Test. $\lim_{n\to\infty} \left|\frac{u_n}{u_{n+1}}\right| = L = \frac{M}{|x-c|}$

If $L>1$, Then By ratio Test convergent $\therefore$ But $L = \frac{M}{|x-c|}$

$\therefore$ ROC $L>1 \Rightarrow \frac{M}{|x-c|} > 1$

$\Rightarrow |x-c| < M \longrightarrow$ ROC (series convergent in this interval)

If $L<1$, Then divergent $\Rightarrow |x-c| > M \rightarrow$ series divergent in this interval

If $L=1$, series may or may not convergent.

→ 'M' is called the radius of convergence

'C' is called centre of convergence

Q.1) find radius of convergence of given series.

1) $\sum_{n=0}^{\infty} \frac{n(x+1)^n(-1)^n}{2n}$, $x>0$

$\Rightarrow \lim_{n\to\infty}\left|\frac{u_n}{u_{n+1}}\right| = \lim_{n\to\infty}\left|\frac{(-1)^n n(x+1)^n}{2n} \times \frac{2(n+1)}{(n+1)(x+1)^{n+1}(-1)^{n+1}}\right|$

$\lim_{n\to\infty}\left|\frac{(2n+2)\cancel{(-1)^n}\cdot\cancel{n}\cancel{(x+1)^n}}{2\cancel{n}\cancel{(n+1)}\cancel{(x+1)^n}(x+1)\cancel{(-1)^n}(-1)}\right| = \lim_{n\to\infty}\left|\frac{1}{(-1)(x+1)}\right|$

$\lim_{n\to\infty}\frac{1}{|x+1|}$ $\Rightarrow L=1.$ $\lim_{n\to 0}\frac{1}{|x+1)}$

IF $L\cancel{>}1 \Rightarrow \frac{1}{|x+1|} > 1$ $\Rightarrow |x+1| \cancel{<} 1$

$-1 < x+1 < 1$

$\boxed{-2 < x < 0}$

2) $\sum_{n=0}^{\infty} \frac{n(x+1)^n(-1)^n}{2^n}$, $x>0$.

$\Rightarrow \lim_{n\to\infty}\left|\frac{u_n}{u_{n+1}}\right| = \lim_{n\to\infty}\left|\frac{(-1)^n n(x+1)^n}{2^n} \times \frac{2^{n+1}}{(-1)^{n+1}(n+1)(x+1)^{n+1}}\right|$

$\lim_{n\to\infty}\left|\frac{\cancel{(-1)^n} n\cancel{(x+1)^n} \times \cancel{2^n}\cdot 2}{\cancel{2^n}\cancel{(-1)^n}(-1)(n+1)\cancel{(x+1)^n}(x+1)}\right|$

$\lim_{n\to\infty}\left|\frac{2n}{(n+1)(x+1)}\right| = \lim_{n\to\infty}\frac{2n}{n+1}\cdot\frac{1}{x+1}.$

$= \lim_{n\to\infty}\frac{2\cancel{n}}{\cancel{n}(1+\frac{1}{n})}\cdot\frac{1}{x+1} = \frac{2}{|x+1|}.$

$\Leftrightarrow$ $L = \frac{2}{|x+1|} > 1$

$L = \frac{2}{|x+1|} > 1$

$\frac{2}{|x+1|} > 1 \Rightarrow |x+1| < 2.$

$-2 < x+1 < 2$

$-3 < x < 1$, but $x > 0$.

$\Rightarrow (0, 1)$

convergent in $(0,1)$

Power series divergent if $L < 1$ — Test fail if $L = 1$

$\frac{2}{|x+1|} < 1 \Rightarrow |x+1| > 2.$

$\Rightarrow \frac{2}{|x+1|} = 1$

$\Rightarrow 2 = |x+1|$

$(2)^2 = |x+1|^2$

$4 = x^2 + 2x + 1$

$x^2 + 2x = 3.$

$x^2 + 2x - 3 = 0.$

$x^2 + 3x - x - 3 = 0$

$x(x+3) - 1(x+3) = 0$

$x = -3, 1.$

$|x+1| > 2$ means:

$x+1 > 2$ or $x+1 < -2$

$x > 1$ or $x < -3$

$x \in (-\infty, -3) \cup (1, \infty)$

25/10/24

Taylor Theorem

Let $f(x)$ defined on $[a, a+h]$

If ① $f(x), f'(x), \dots f^{(n-1)}(x)$ continuous on $[a, a+h]$

② $f^n(x)$ exist for all $x \in (a, a+h)$ then there exist atleast one number θ. $(0 < \theta < 1)$ such that

$$f(a+h) = f(a) + \frac{h}{1!} f'(a) + \frac{h^2}{2!} f''(a) + \dots + \frac{h^{n-1}}{(n-1)!} f^{(n-1)}(a) + \dots + R_n$$

where, $R_n = \frac{h^n}{n!} f^n(a + \theta h)$ is remainder after n^{th} term

Note: $f(x)$ is given function, 'a' is point, 'h' is increament in 'a'

* Taylor series.

$$f(a+h) = f(a) + \frac{h}{1!} f'(a) + \frac{h^2}{2!} f''(a) + \dots + \frac{h^n}{n!} f^n(a) + \dots \quad \text{—①}$$

Use above series when we have to find approximate value $f(x)$.

* another form of Taylor series.

In ① put $h = x-a$

$$f(x) = f(a) + \frac{(x-a)}{1!} f'(a) + \frac{(x-a)^2}{2!} f''(a) + \dots$$

Use above series when we have to find expansion (series) of $f(x)$ at point 'a'.

Q.1) Express $f(x) = 2x^3+3x^2-8x+7$ in power of $(x-2)$

OR

~~Express~~ Expand $f(x) = 2x^3+3x^2-8x+7$ at $a=2$.

OR

Expand $f(x) = 2x^3+3x^2-8x+7$ in ascending power of $x-2$.

$\Rightarrow$ $f(x) = 2x^3+3x^2-8x+7$, $a=2$ $(\because x-a = x-2 \Rightarrow a=2)$

We have

$$f(x) = f(a) + \frac{(x-a)}{1!} f'(a) + \frac{(x-a)^2}{2!} f''(a) + \dots$$

$f(x) = 2x^3+3x^2-8x+7$

$\therefore$ $f(a) = f(2) = 2(2)^3+3(2)^2-8(2)+7 = 16+12-16+7 = 19$.

$f'(x) = 6x^2+6x-8$

$f'(a) = f'(2) = 6(2)^2+6(2)-8 = 24+12-8 = 28$.

$f''(x) = 12x+6$

$f''(a) = f''(2) = 12(2)+6 = 30$.

$f'''(x) = 12$, $f'''(a) = 12$.

$f^{IV}(x) \dots f^{(n)}x = 0$, $f^{IV}(a) = f^{V}(a) = \dots = f^n(a) = 0$

$$2x^3+3x^2-8x+7 = 19 + \frac{x-2}{1}(28) + \frac{(x-2)^2}{2}(30) + \frac{(x-2)^2}{6}(12) + 0.$$

$$\Rightarrow 2x^3+3x^2-8x+7 = 19+28(x-2)+15(x-2)^2+2(x-2)^3$$

2) Expand $f(x) = \log(\cos x)$ at $\pi/4$

$\Rightarrow f(x) = \log(\cos x)$, $a = \pi/4$.

We have,

$$f(x) = f(a) + \frac{x-a}{1!}f'(a) + \frac{(x-a)^2}{2!}f''(a) + \frac{(x-a)^3}{3!}f'''(a) + \dots$$

$f(x) = \log(\cos x)$.

$f(a) = \log\left(\cos\frac{\pi}{4}\right) = \log\frac{1}{\sqrt{2}}$.

$f'(x) = \frac{-\sin x}{\cos x} = -\tan x$, $\quad f'(a) = -\tan\frac{\pi}{4} = -1$.

$f''(x) = -\sec^2 x$, $\quad f''(a) = -\sec^2\frac{\pi}{4} = -2$

$f'''(x) = -2\sec x \sec x \tan x$, $\quad f'''(a) = -2\sec^2\left(\frac{\pi}{4}\right)\tan\left(\frac{\pi}{4}\right)$

$= -2\sec^2 x \tan x.$ $\qquad = -2(2)(1) = -4.$

$$\log(\cos x) = \log\frac{1}{\sqrt{2}} + \frac{x-\frac{\pi}{4}}{1}(-1) + \frac{\left(x-\frac{\pi}{4}\right)^2}{2}(-2) + \frac{\left(x-\frac{\pi}{4}\right)^3}{6}(-4) + \dots$$

$$\Rightarrow \log(\cos x) = \log\frac{1}{\sqrt{2}} - \left(x-\frac{\pi}{4}\right) - \left(x-\frac{\pi}{4}\right)^2 - \frac{2}{3}\left(x-\frac{\pi}{4}\right)^3 + \dots$$

3) Prove that $\frac{1}{1-x} = \frac{1}{3} + \frac{x+2}{3^2} + \frac{(x+2)^2}{3^3} + \dots$

$\Rightarrow f(x) = \frac{1}{1-x}$, $x-a = x+2$

$\Rightarrow a = -2$.

We have, $f(x) = f(a) + \frac{(x-a)}{1} f'(a) + \frac{(x-a)^2}{2!} f''(a) + \dots$

$f(x) = \frac{1}{1-x}$, $f(a) = f(-2) = \frac{1}{3}$

$f'(x) = \frac{1}{(1-x)^2}$, $f'(a) = f'(-2) = \frac{1}{3^2}$

$f''(x) = \frac{2}{(1-x)^3}$, $f''(a) = f''(-2) = \frac{2!}{3^3}$

$f'''(x) = \frac{6}{(1-x)^4}$, $f'''(a) = f'''(-2) = \frac{3!}{3^4}$

$\Rightarrow \frac{1}{1-x} = \frac{1}{3} + \frac{(x+2)}{1!} \frac{1}{3^2} + \frac{(x+2)^2}{2!} \frac{2!}{3^3} + \frac{(x+2)^3}{3!} \cdot \frac{3!}{3^4} + \dots$

Hence proved.

Macluarian Series.

Put $a=0$ in Taylor's series.

$f(x) = f(0) + \frac{x}{1!} f'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \dots$

Q.1) Expand $f(x) = \sin x$ at $a=0$

or

expand $f(x) = \sin x$ or Express $f(x) = \sin x$ in ascending pow[ers] of x.

$\Rightarrow f(x) = \sin x$, $a=0$.

We have, $f(x) = f(0) + \frac{x}{1!} f'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \dots$

$f(x) = \sin x$, $f(0) = \sin 0 = 0$

$f'(x) = \cos x$, $f'(0) = \cos 0 = 1$

$f''(x) = -\sin x$, $f''(0) = -\sin 0 = 0$

$f'''(x) = -\cos x$, $f'''(0) = -\cos x = -1$

$$\sin x = 0 + \frac{x}{1!}(1) + \frac{x^2}{2}(0) + \frac{x^3}{6}(-1) + \dots$$

$$\Rightarrow \sin x = x - \frac{x^3}{6} + \dots$$

* Standard Maclaurian Series.

1) $e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$

2) $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \dots$

3) $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots$

4) $\tan x = x + \frac{x^3}{3} + \frac{2}{15}x^5 + \frac{17}{315}x^7 + \dots$

5) $\sinh x = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \frac{x^9}{9!} + \dots$

6) $\cosh x = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \dots$

7) $\tanh x = x - \frac{x^3}{3} + \frac{2}{15}x^5 - \frac{17}{315}x^7 + \dots$

8) $\sin^{-1}x = \left(x + \frac{x^3}{6} + \frac{3x^5}{40} + \dots\right)$

9) $\cos^{-1}x = \frac{\pi}{2} - \left(x + \frac{x^3}{6} + \frac{3x^5}{40} + \dots\right)$

10) $\tan^{-1}x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \dots$

11) $\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots$

12) $\log(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \dots$

13) $\frac{1}{1-x} = 1 + x + x^2 + x^3 + \dots$

14) $\frac{1}{1+x} = 1 - x + x^2 - x^3 + \dots$

15) $(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \dots$

→ Use regular method (find $f, f', f'', f''' \dots$) when derivatives are easy otherwise use standard series.

Q.2) Use macluarian series, show that

$(1+x)^x = 1 + x - \frac{x^3}{2} + \frac{5}{6}x^4 - \frac{3}{4}x^5 + \dots$

⇒ Let $y = (1+x)^x$

$\therefore \log y = x \log(1+x)$ —— (1)

$\log y = x \log(1+x)$

$= x\left(x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \dots\right)$

$= x^2 - \frac{x^3}{2} + \frac{x^4}{3} - \frac{x^5}{4} + \frac{x^6}{5} - \dots = z$ (say) — ②

By ①, $y = e^{x\log(1+x)} = e^z$.

But $y = e^z$.

Using Maclaurin's series of e^z

$y = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \dots$

Replacing z from ②.

$y = 1 + \left(x^2 - \frac{x^3}{2} + \frac{x^4}{3} - \frac{x^5}{4} + \frac{x^6}{5} - \dots\right)$

$+ \frac{1}{2}\left(x^2 - \frac{x^3}{2} + \frac{x^4}{3} - \frac{x^5}{4} + \frac{x^6}{5} - \dots\right)^2$

$= 1 + x^2 - \frac{x^3}{2} + \frac{x^4}{3} - \frac{x^5}{4} + \frac{1}{2}x^4 - \frac{1}{2}x^5 + \dots$

Highest power of x.

$= 1 + x^2 - \frac{x^3}{2} + \frac{5}{6}x^4 - \frac{3}{4}x^5 + \dots$

Q. $e^{x\cos x} = 1 + x + \frac{x^2}{2} - \frac{x^3}{3} - \frac{11}{24}x^4 + \dots$

$\Rightarrow$ let, $y = e^{x\cos x}$

$\log y = x\cos x \log e = x\cos x$

$\log y = x - \frac{x^3}{2!} + \frac{x^5}{4!} - \dots = z$ (say)

$y = e^z = e^{x - \frac{x^3}{2!} + \frac{x^5}{4!}}$

$$y = 1 + \left(x - \frac{x^3}{2} + \frac{x^5}{24} + \dots\right) + \frac{1}{2!}\left(x - \frac{x^3}{2} + \frac{x^5}{24} + \dots\right)^2$$

$$= 1 + x + \frac{x^2}{2} + x^3\left(\frac{-1}{2} + \frac{1}{6}\right) + x^4\left(\frac{-1}{2} + \frac{1}{4!}\right)$$

$$y = 1 + x + \frac{x^2}{2} + x^3\left(\frac{-1}{3}\right) + \frac{11}{24}x^4 + \dots$$

Q. Using Taylor's Theorem calculate $\sqrt{25.15}$ upto 4 decimals.

$\Rightarrow$ let, $f(x) = \sqrt{x}$. $\Rightarrow f(x+h) = \sqrt{x+h}$.

By Taylor's Theorem,

$$f(x+h) = f(x) + hf'(x) + \frac{h^2}{2!}f''(x) + \dots + \frac{h^n}{n!}f^n(x).$$

Put, $x = 25$ and $h = 0.15$

$\therefore f(x+h) = \sqrt{x+h}$

$= \sqrt{25+0.15}$

$$\sqrt{25.15} = f(25) + (0.15)\ f'(25) + \frac{(0.15)^2}{2!}f''(25) + \dots \quad \text{—(1)}$$

$f(x) = \sqrt{x}$, $f(25) = 5$.

$f'(x) = \frac{1}{2\sqrt{x}}$, $f'(25) = \frac{1}{10} = 0.1$

$f''(x) = \frac{1}{2}\left(\frac{-1}{2}\right)\frac{1}{x^{3/2}}$, $f''(25) = \frac{-1}{500} = -0.002$

Putting values in eq^n (1) and considering only first three values.

$$\sqrt{25.15} = 5 + (0.15)(0.1) + \frac{(0.15)^2}{2!}(-0.002) + \dots$$

$= 5.0150$ (approx).