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Unit I: System of Linear Equation

Rank of Matrix

Let 'A' be any m x n matrix.

I] Row Echelon form (REF) for finding Rank of Matrix.

NOTE:

Use only Row operation on Matrix.

Rule I: Make all entry (if present) zero below first non-zero entry (if present) in first row.

Rule II: Apply Rule I to remaining row of matrix.

Rule III: If given matrix have zero row or during process of REF if we get zero row then shift that Zero row to bottom of the matrix.

Rule IV: Next leading (Pivot) entry must be placed at RHS of previous leading entry.

* Leading Entry: An entry from REF is said to be leading if it is first non-zero entry in a row below that all entries are zero (if present)

* Rank: Number of non-zero Row's in REF called Rank of Matrix and denoted by $\rho(A)$

OR

Rank: Rank of a matrix = No. of leading Entries

Ex. $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$

$R_2 - 3R_1 \Rightarrow \begin{bmatrix} 1 & 2 \\ 0 & -2 \end{bmatrix} \rightarrow$ Ref

$\rho(A) = 2$

$LE = 1, -2$

$R_{12} = \begin{bmatrix} 3 & 4 \\ 1 & 2 \end{bmatrix}$

$3R_2 - R_1 = \begin{bmatrix} 3 & 4 \\ 0 & 2 \end{bmatrix} \rightarrow$ Ref

$\rho(A) = 2, \quad LE = 3, 2$

Q. find the rank of the given matrix.

① $A = \begin{bmatrix} 9 & 3 \\ 0 & 9 \\ 5 & 9 \\ 6 & 8 \\ 6 & 0 \end{bmatrix}$ ⇒ first non-zero entry in first row is '9'.

so, make all the entries zero below 9.

$R_3 = 9R_3 - 5R_1$

$R_4 = 3R_4 - 2R_1$

$R_5 = 3R_5 - 2R_1$

⇒ $A = \begin{bmatrix} 9 & 3 \\ 0 & 9 \\ 0 & 165 \\ 0 & 18 \\ 0 & -6 \end{bmatrix}$ (same for second row.)

first non-zero entry in second row is '9'.

$R_3 = 9R_3 - 165R_2$

$R_4 = R_4 - 2R_2$

$R_5 = 3R_5 + 2R_2$

⇒ $A = \begin{bmatrix} 9 & 3 \\ 0 & 9 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$ Leading entries are 9 and 9.

Rank of matrix A is 2

$\rho(A) = 2$.

② $B = \begin{bmatrix} 6 & 0 & 0 \\ 0 & 0 & 6 \\ 6 & 1 & 4 \\ 3 & 1 & 3 \end{bmatrix}$ ⇒ first non-zero entry in first row is '6'

so, make all entries zero below 6.

$R_2 = 6R_2 - R_1$

$R_3 = R_3 - R_1$

$R_4 = 2R_4 - R_1$

⇒ $B = \begin{bmatrix} 6 & 0 & 0 \\ 0 & 0 & 36 \\ 0 & 1 & 4 \\ 0 & -2 & 6 \end{bmatrix}$ same for second row.

first non-zero entry in second row is '36'.

$R_3 = 9R_3 - R_2$,

$R_4 = 6R_4 - R_2$

$$B = \begin{bmatrix} 6 & 0 & 0 \\ 0 & 0 & 36 \\ 0 & 9 & 0 \\ 0 & 12 & 0 \end{bmatrix}$$

for third row,
first non-zero entry is '9'.
$R_4 = 3R_4 - 4R_3$.

$$B = \begin{bmatrix} 6 & 0 & 0 \\ 0 & 0 & 36 \\ 0 & 9 & 0 \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow{R_2 \leftrightarrow R_3} B = \begin{bmatrix} 6 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 36 \\ 0 & 0 & 0 \end{bmatrix}$$

$\rho(B) = 3$.
LE $= 6, 9, 36$

Q.1. find rank of the matrix, find Pivot element.

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$$

$\Rightarrow$ $$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$$

$R_2 = R_2 - 4R_1$
$R_3 = R_3 - 7R_1$

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & -3 & -6 \\ 0 & -6 & -12 \end{bmatrix}$$

$R_3 = R_3 - 2R_2$

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & -3 & -6 \\ 0 & 0 & 0 \end{bmatrix} \longrightarrow \text{REF}$$

$\therefore$ $\rho(A) = 2$
pivot element $= 1, -3$

Q. $A = \begin{bmatrix} 1 & 2 & -1 & -2 \\ 2 & 2 & -4 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & -2 & 2 \end{bmatrix}$ $\quad R_2 = R_2 - 2R_1$

$A = \begin{bmatrix} 1 & 2 & -1 & -2 \\ 0 & -2 & -2 & 4 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & -2 & 2 \end{bmatrix}$ $\quad R_3 = 2R_3 + R_2$

$A = \begin{bmatrix} 1 & 2 & -1 & -2 \\ 0 & -2 & -2 & 4 \\ 0 & 0 & -4 & 4 \\ 0 & 0 & -2 & 2 \end{bmatrix}$ $\quad R_4 = 2R_4 - R_3$

$A = \begin{bmatrix} 1 & 2 & -1 & -2 \\ 0 & -2 & -2 & 4 \\ 0 & 0 & -4 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix} \longrightarrow \text{REF}$

$\rho(A) = 3$

Leading entries = 1, −2, −4

Q. $B = \begin{bmatrix} 1 & 2 \\ 3 & 1 \\ -1 & 1 \\ 0 & 0 \\ 1 & 1 \end{bmatrix} \Rightarrow \begin{matrix} R_2 = R_2 - 3R_1 \\ R_3 = R_3 + R_1 \\ R_5 = R_5 - R_1 \end{matrix} \Rightarrow \begin{bmatrix} 1 & 2 \\ 0 & -5 \\ 0 & 3 \\ 0 & 0 \\ 0 & -1 \end{bmatrix}$

$R_3 = 5R_3 + 3R_2$
$R_5 = 5R_5 - R_2$ $\rightarrow \begin{bmatrix} 1 & 2 \\ 0 & -5 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$

$\therefore \rho(B) = 2$

leading entries = 1, −5

If order is odd and matrix is skew-symmetric then it's determinant is zero

Q. $C = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 2 & 3 \\ 0 & 4 & 5 \end{bmatrix}$

$\Rightarrow R_2 = R_2 - 3R_1$
$R_3 = R_3 - 5R_1$
$$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 2 & 0 \\ 0 & 4 & 0 \end{bmatrix}$$

$R_3 = R_3 - 2R_2$
$$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$R_1 \leftrightarrow R_2$
$$\begin{bmatrix} 0 & \underset{L_2R}{2} & 0 \\ 0 & 0 & \underset{L_1R}{1} \\ 0 & 0 & 0 \end{bmatrix}$$

$\rho(C) = 2$
leading entries are 2, 1

Q. $D = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ $\rho(D) = 0$

Q. $E = [1 \;\; 2 \;\; 3]$ $\rho(E) = 1$, Leading entry = 1.

NOTE / Shortcut:

1. Let 'A' be $m \times n$ matrix then $\rho(A) \leq$ minimum $\{m, n\}$
2. $\rho(A + B) \leq \rho(A) + \rho(B)$
3. $\rho(AB) \leq \min\{\rho(A), \rho(B)\}$
4. let 'A' be any $m \times n$ matrix then $\rho(A) = \rho(A^T)$
5. let 'A' be any $n \times n$ matrix then

   (a) $\rho(A) = n$, iff $|A| \neq 0$

   (b) $\rho(A) < n$, iff $|A| = 0$

* **Row Reduced echelon form (RREF)**

first convert given matrix into REF then make all leading entry '1' and entry above and below of all Leading entry must be 'zero' (if Present) is called as RREF

Q. Convert given matrix into RREF and find LE

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$$

$\Rightarrow$ $R_2 \rightarrow R_2 - 4R_1$
$R_3 \rightarrow R_3 - 7R_1$

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & -3 & -6 \\ 0 & -6 & -12 \end{bmatrix}$$

$R_3 \rightarrow R_3 - 2R_2$

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & -3 & -6 \\ 0 & 0 & 0 \end{bmatrix} \rightarrow \text{REF}$$

$\rho(A) = 2$, LE $= 1, -3$

$\frac{R_2}{-3}$

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

$R_1 - 2R_2$

$$A = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix} \rightarrow \text{RREF}$$

$\rho(A) = 2$ and LE $= 1, 1$

H.W

Q. find 'k' such that $\rho(A) = 3$ and $\rho(A) < 3$

$$A = \begin{bmatrix} k & 1 & 2 \\ 0 & k & 1 \\ 0 & 0 & k \end{bmatrix}$$

$k = 0$

$$A = \begin{bmatrix} 0 & 1 & 2 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$\rho(A) = 2$

m= no. of eq^n
n= no. of unknown/ variable

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Solution of System of Linear equation (SLE). AX=B of order m×n

Ex. $x + y = 0$ —①
$x - y = 0$ —②

$$\underset{A}{\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}} \underset{X}{\begin{bmatrix} x \\ y \end{bmatrix}} = \underset{B}{\begin{bmatrix} 0 \\ 0 \end{bmatrix}}$$

Where, 'A' is matrix of coeff.
'X' is matrix of unknown
'B' is matrix of constant.

* Homogeneous SLE
If RHS of every eq^n is zero then it is called as Homogeneous SLE and denoted by AX=0

* Non-Homogeneous SLE
If RHS of atleast one eq^n is non-zero then it is called as Non-Homogeneous SLE and denoted by AX=B $(B \neq 0)$

* Augmented Matrix of SLE
Ex. $2x + 8y = 3$ m=2, n=2
$x + 9y = 2$ AX=B

$$[A|B] = \left[\begin{array}{cc|c} 2 & 8 & 3 \\ 1 & 9 & 2 \end{array}\right]$$
A   B

$AX = B$ SLE of order $m \times n$, m = no. of eq^n, n = no. of unknown

$AX = 0$ (HOMO)

① If $\rho(A) = \rho(A|B) = n$

Then $AX = 0$ is consistent (sol^n exist) and $AX = 0$ have trivial sol^n (zero sol^n)

NOTE: 1. Trivial sol^n: value of all unknown are zero.

2. In this case, No free Variable.

② If $\rho(A) = \rho(A|B) < n$

Then $AX = 0$ is consistent and $AX = 0$ have infinitely many non-zero sol^n

(Trivial sol^n + infinitely many non-zero sol^n)

NOTE: In this case we get atleast one free Variable.

$AX = B$, $B \neq 0$ (Non-HOMO)

① If $\rho(A) = \rho(A|B) = n$

Then $AX = B$ is consistent and $AX = B$ have unique sol^n (unique non-zero sol^n)

NOTE:

1. In this case No free Variables

② If $\rho(A) = \rho(A|B) < n$

Then $AX = B$ is consistent and $AX = B$ have infinitely many non-zero sol^n (only infinite non-zero sol^n)

NOTE:

1. In this case we get atleast one free variables.

③ If $\rho(A) \neq \rho(A|B)$

Then $AX = B$ is inconsistent

i.e. No sol^n

NOTE:

In this case, No free Variable

Homo. have always sol^ns that's why it cannont inconsistent

Q. Solve the system completely if it consistent.

$2x + 3y = 0$

$4x + 5y = 0$

$\Rightarrow$ $\begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ $m = 2$ (no. of eq^{n}s)

$n = 2$ (no. of unknowns)

$A \quad X = 0$ $AX = 0$ system.

corresponding Augmented matrix.

$[A|B] = \left[\begin{array}{cc|c} 2 & 3 & 0 \\ 4 & 5 & 0 \end{array}\right]$

Convert $[A|B]$ into REF

$R_2 - 2R_1$, $[A|B] = \left[\begin{array}{cc|c} 2 & 3 & 0 \\ 0 & -1 & 0 \end{array}\right] \rightarrow$ REF

$\rho(A) = 2$, $\rho(A|B) = 2$, $n = 2$

$\because \rho(A) = \rho(A|B) = n$.

$\therefore$ It has trivial sol^n. $(x = 0, y = 0)$

Q. solve the system completely if it consistent. $2x + 3y = 0$

$4x + 6y = 0$

$\Rightarrow$ $m = 2$, $n = 2$, $AX = 0$ system

Corresponding Augmented matrix,

$[A|B] = \left[\begin{array}{cc|c} 2 & 3 & 0 \\ 4 & 6 & 0 \end{array}\right]$

Convert $[A|B]$ into REF

$R_2 - 2R_1$, $[A|B] = \begin{array}{c} \begin{array}{cc} x & y \end{array} \\ \left[\begin{array}{cc|c} 2 & 3 & 0 \\ 0 & 0 & 0 \end{array}\right] \\ A \quad\quad B \end{array} \rightarrow$ REF

$\rho(A) = 1$, $\rho(A|B) = 1$, $n = 2$

$\therefore AX = 0$ is consistent

$\rho(A) = \rho(A|B) < n$

$AX = 0$ have infinitely many sol^{n}s.

from REF 2 is LE and 2 is coeff of 'x'

$\therefore x$ is LE and y is free

Put, $y = t$, $t \in R$

from REF , $2x + 3y = 0$

$x = -\frac{3y}{2} = -\frac{3t}{2}$

$\Rightarrow$ $\boxed{x = -\frac{3t}{2}}$ and $\boxed{y = t}$

but as we say it has infinitely many sol^ns that means, $t \in R$ and there are infinite values of t i.e of y. and therefore infinite values of 'x'.

$\therefore$ It has infinitely many sol^ns.

Q. $2x + 3y = 1$ $AX = B$.

$4x + 6y = 2$

$\rightarrow$ $m = 2$, $n = 2$, $AX = B$ system

Corresponding augmented matrix.

$$[A \mid B] = \left[\begin{array}{cc|c} 2 & 3 & 1 \\ 4 & 6 & 2 \end{array}\right]$$

convert $[A \mid B]$ into REF

$R_2 - 2R_1$, $[A \mid B] = \left[\begin{array}{cc|c} 2 & 3 & 1 \\ 0 & 0 & 0 \end{array}\right] \longrightarrow$ REF

$\rho(A) = 1$, $\rho(A \mid B) = 1$, $n = 2$.

$\therefore$ $AX = B$ is consistent

$\rho(A) = \rho(A \mid B) < n$

Q. $2x + 3y = 1$

$4x + 6y = 35$

$\Rightarrow$ $m=2$, $n=2$, $AX=B$ system

corresponding augmented matrix.

$$[A|B] = \left[\begin{array}{cc|c} 2 & 3 & 1 \\ 4 & 6 & 35 \end{array}\right]$$

Convert $[A|B]$ into REF

$$R_2 - 2R_1, \quad [A|B] = \begin{array}{c} \begin{array}{cc} x & y \end{array} \\ \left[\begin{array}{cc|c} 2 & 3 & 1 \\ 0 & 0 & 33 \end{array}\right] \end{array} \rightarrow \text{REF}$$

A B

$\rho(A) = 1$, $\rho(A|B) = 2$

$\therefore$ $AX=B$ is inconsistent

$\rho(A) \neq \rho(A|B)$

i.e. No solution

Q. $2x + 3y = 1$

$x + 6y = 35$

$\Rightarrow$ $m=2$, $n=2$, $AX=B$ system

corresponding augmented matrix.

$$[A|B] = \left[\begin{array}{cc|c} 2 & 3 & 1 \\ 1 & 6 & 35 \end{array}\right]$$

convert $[A|B]$ into REF

$$2R_2 - R_1, \quad [A|B] = \begin{array}{c} \begin{array}{cc} x & y \end{array} \\ \left[\begin{array}{cc|c} 2 & 3 & 1 \\ 0 & 9 & 69 \end{array}\right] \end{array} \rightarrow \text{REF}$$

A B

$\rho(A) = 2$, $\rho(A|B) = 2$, $n = 2$

$\therefore$ $AX=B$ is consistent

$AX=B$ have unique solution.

from REF, $2x + 3y = 1$

$9y = 69$

$y = \frac{69}{9} = \frac{23}{3}$ and $x = -11$

Q. $x + 2y + 2z + 2w = 0$
$x + 2y + 2z + 14w = 0$

$\Rightarrow$ $m=2$, $n=4$, $AX=0$ system

Corresponding augmented matrix.

$$[A|B] = \left[\begin{array}{cccc|c} 1 & 2 & 2 & 2 & 0 \\ 1 & 2 & 2 & 14 & 0 \end{array}\right]$$

Convert $[A|B]$ into REF.

$$R_2 - R_1, \quad \begin{array}{c} \begin{array}{cccc} x & y & z & w \end{array} \\ \left[\begin{array}{cccc|c} 1 & 2 & 2 & 2 & 0 \\ 0 & 0 & 0 & 12 & 0 \end{array}\right] \\ A \qquad\qquad B \end{array} \rightarrow \text{REF}$$

$\rho(A) = 2$, $\rho(A|B) = 2$, $n=4$

$\rho(A) = \rho(A|B) \neq n$

$\rho(A) = \rho(A|B) < n$

$\therefore$ $AX=0$ is consistent and have infinitely many non-zero soln.

from REF, 1 and 12 are LE and 1 is coeff of x,
12 is coeff of w.

$\therefore$ x and w are LE

$\therefore$ y and z are free

Put, $y = t_1$ and $z = t_2$

from REF, $x + 2y + 2z + 2w = 0$

$12w = 0$

$\boxed{w=0}$

$x + 2t_1 + 2t_2 + 0 = 0$

$\Rightarrow$ $x = -2t_1 - 2t_2$

$y = t_1$

$z = t_2$

$w = 0$.

Q. Find the value of α, β such that given system have ① unique sol^n ② Infinite sol^n ③ No sol^n.

$$x + 2y + 3z = 1$$
$$4x + 5y + 6z = 2$$
$$7x + 8y + \alpha z = \beta$$

$\Rightarrow$ ① Unique sol^n.

$m = 3$, $n = 3$, $AX = B$ system.

Corresponding augmented matrix

$$[A \mid B] = \left[\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 4 & 5 & 6 & 2 \\ 7 & 8 & \alpha & \beta \end{array}\right]$$

convert $[A \mid B]$ into REF

$R_2 \to R_2 - 4R_1$
$R_3 \to R_3 - 7R_1$

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 0 & -3 & -6 & -2 \\ 0 & -6 & \alpha - 21 & \beta - 7 \end{array}\right]$$

$R_3 \to R_3 - 2R_2$

$$\begin{array}{c} \begin{array}{ccc} x & y & z \end{array} \\ \left[\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 0 & -3 & -6 & -2 \\ 0 & 0 & \alpha - 9 & \beta - 3 \end{array}\right] \\ \begin{array}{cc} A & B \end{array} \end{array}$$

for ① unique sol^n. $\rho(A) = \rho(A \mid B) = n$.

i.e. $\rho(A) = \rho(A \mid B) = 3$

$\rho(A) = 3$ if $\alpha - 9 \neq 0$ and $\beta \in R$.

② Infinite sol^n $\rho(A) = \rho(A \mid B) < n$

$\rho(A) = \rho(A \mid B) < 3$

$\rho(A) < 2$ if $\alpha - 9 = 0$, $\beta - 3 = 0$

③ No sol^n $\rho(A) \neq \rho(A \mid B)$

if $\alpha - 9 = 0$ and $\beta - 3 \neq 0$

HW Q.① Find value of 'k' such that given system have ① trivial soln ② Infinite soln

$$x + 2y + 3z = 0$$
$$4x + 5y + 6z = 0$$
$$7x + 8y + kz = 0$$

② find k such that
$$kx + 2y + 3z = 0$$
$$2x + ky + 6z = 0$$
$$x + y + kz = 0$$

③ $x + y + z + w = 0$

④ $x + 2y + 3z + 4w = 0$
$y - z + x = 0$
$3x + 4y - z = 0$

⑤ $2x + 3y = 0$
$4x + 3y = 0$
$3x - y = 0$
$y + x = 0$

⇒ ①
$$x + 2y + 3z = 0$$
$$4x + 5y + 6z = 0$$
$$7x + 8y + kz = 0$$

$m = 3$, $n = 3$, $AX = 0$ system.

corresponding augmented matrix is,

$$[A \mid B] = \left[\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 4 & 5 & 6 & 0 \\ 7 & 8 & k & 0 \end{array}\right]$$

convert $[A \mid B]$ into REF

$R_2 \to R_2 - 4R_1$
$R_3 \to R_3 - 7R_1$

$$\begin{array}{cccc} x & y & z & \\ \end{array}$$
$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 0 & -3 & -6 & 0 \\ 0 & -6 & k-21 & 0 \end{array}\right]$$
A B

~~I. trivial soln: $\rho(A) = \rho(A \mid B) = n$~~

$\rho(A) = 3$, ~~$\rho(A \mid B)$~~

$R_3 \to R_3 - 2R_2$

$$\begin{array}{ccc} x & y & z \end{array}$$
$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 0 & -3 & -6 & 0 \\ 0 & 0 & k-9 & 0 \end{array}\right]$$
A B

$\rho(A) = \rho(A|B) = n$ for trivial soln.

$n = 3$. $\rho(A) = 3$ then $k-9 \neq 0$ $\therefore k \neq 9$

$\rho(A|B) = 3$ then $k-9 \neq 0$

① $\therefore k \neq 9$ (for trivial soln)

② for infinite many non-zero soln. $\Rightarrow \rho(A) = \rho(A|B) < n$

$n = 3$ $\rho(A)$ should be less than 3.

$\rho(A|B)$ should be less than 3.

$k - 9 = 0$

$\therefore k = 9$ (for infinite soln)

② $kx + 2y + 3z = 0$ $m = 3$, $n = 3$, $AX = 0$ system.

$2x + ky + 6z = 0$ corresponding Augmented matrix is.

$x + y + kz = 0$.

$$[A|B] = \left[\begin{array}{ccc|c} k & 2 & 3 & 0 \\ 2 & k & 6 & 0 \\ 1 & 1 & k & 0 \end{array}\right]$$

Convert $[A|B]$ into REF.

$R_3 \leftrightarrow R_1$ $\left[\begin{array}{ccc|c} 1 & 1 & k & 0 \\ 2 & k & 6 & 0 \\ k & 2 & 3 & 0 \end{array}\right]$

$R_2 \to R_2 - 2R_1$

$R_3 \to \frac{1}{k}R_3 - R_1$ $\left[\begin{array}{ccc|c} 1 & 1 & k & 0 \\ 0 & k-2 & 6-2k & 0 \\ 0 & \frac{2}{k}-1 & \frac{3-k^2}{k} & 0 \end{array}\right]$

$R_3 \to R_3 + \frac{R_2}{k}$ $\left[\begin{array}{ccc|c} 1 & 1 & k & 0 \\ 0 & k-2 & 6-2k & 0 \\ 0 & 0 & \frac{-k^2-2k+9}{k} & 0 \end{array}\right]$

for trivial soln, $\rho(A) = \rho(A|B) = n$

$n = 3$ $\rho(A) = 3$.

$\frac{-k^2-2k+9}{k} \neq 0$ $\therefore k \neq \frac{-3 \pm 3\sqrt{5}}{2}$

zero row and zero column does not affect the Rank.

for infinitely many non-zero solⁿ.

$\rho(A) = \rho(A|B) < n$

$\rho(A) < 3$.

$$\frac{-k^2-2k+9}{k} = 0 \Rightarrow k = \frac{-3 \pm 3\sqrt{5}}{2}$$

OR

$$[A|B] = \left[\begin{array}{ccc|c} k & 2 & 3 & 0 \\ 2 & k & 6 & 0 \\ 1 & 1 & k & 0 \end{array}\right]$$

But in $AX=0$ $\rho(A) = \rho(A|B)$

zero row or column does not affect the rank of matrix.

so, $A = \begin{bmatrix} k & 2 & 3 \\ 2 & k & 6 \\ 1 & 1 & k \end{bmatrix}_{3\times3}$ $n=3$

we have $\therefore \rho(A) = n$, iff $|A| \neq 0$

$\rho(A) < n$, iff $|A| = 0$

$|A| = k(k^2-6) - 2(2k-6) + 3(2-k) = k^3 - 13k + 18$

① $AX=0$ have trivial solⁿ.

$\rho(A) = \rho(A|B) = n = 3$

$\therefore \rho(A) = 3$ if $|A| \neq 0 \Rightarrow k^3 - 13k + 18 \neq 0$.

② $AX=0$ have infinite non-zero solⁿ.

$\rho(A) = \rho(A|B) < n = 3$

$\rho(A) < n$ iff $|A| = 0$. $\Rightarrow k^3 - 13k + 18 = 0$

④ $m=3$, $n=4$, $AX=0$ system

corresponding augmented matrix,

$$[A|B] = \left[\begin{array}{cccc|c} 1 & 2 & 3 & 4 & 0 \\ 1 & 1 & -1 & 0 & 0 \\ 3 & 4 & -1 & 0 & 0 \end{array}\right]$$

convert $[A|B]$ into REF.

$R_2 \to R_2 - R_1$
$R_3 \to R_3 - 3R_1$

$$[A|B] = \left[\begin{array}{cccc|c} 1 & 2 & 3 & 4 & 0 \\ 0 & -1 & -4 & -4 & 0 \\ 0 & -2 & -10 & -12 & 0 \end{array}\right]$$

$R_3 \to R_3 - 2R_2$

$$[A|B] = \left[\begin{array}{cccc|c} 1 & 2 & 3 & 4 & 0 \\ 0 & -1 & -4 & -4 & 0 \\ 0 & 0 & -2 & -4 & 0 \end{array}\right]$$

A B

$\rho(A) = 3$, $\rho(A|B) = 3$, $n = 3$.

$\therefore \rho(A) = \rho(A|B) = n = 3$. $\therefore$ trivial soln.

(5) $m = 4$, $n = 2$ $AX = 0$ system.

Corresponding augmented matrix,

$$[A|B] = \left[\begin{array}{cc|c} 2 & 3 & 0 \\ 4 & 3 & 0 \\ 3 & -1 & 0 \\ 1 & 1 & 0 \end{array}\right]$$

$R_4 \leftrightarrow R_1$

$$[A|B] = \left[\begin{array}{cc|c} 1 & 1 & 0 \\ 4 & 3 & 0 \\ 3 & -1 & 0 \\ 2 & 3 & 0 \end{array}\right]$$

$R_2 \to R_2 - 4R_1$
$R_3 \to R_3 - 3R_1$
$R_4 \to R_4 - 2R_1$

$$[A|B] = \left[\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & -1 & 0 \\ 0 & -4 & 0 \\ 0 & 1 & 0 \end{array}\right]$$

$\Rightarrow R_3 \to R_3 - 4R_2$
$R_4 \to R_4 + R_2$

$$[A|B] = \left[\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

A B

$\rho(A) = 2$, $\rho(A|B) = 2$

$n = 2$.

$\therefore \rho(A) = \rho(A|B) = n = 2$. $\therefore$ trivial soln.

(3) $m = 1$, $n = 4$, $AX = 0$ system. (Homo)

$[A|B] = \left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 0 \end{array}\right] \to$ REF.

$\rho(A) = 1$, $\rho(A|B) = 1$

$\because \rho(A) = \rho(A|B) < 4 = n$. $\therefore$ Infinitely many non-zero soln.

LE $= 1$, $\therefore$ LE $= x$

and y, z, w are free.

Put $y = t_1$, $z = t_2$, $w = t_3$ $\Rightarrow$ from REF

$x + y + z + w = 0$

$\Rightarrow x = -(t_1 + t_2 + t_3)$, $y = t_1$, $z = t_2$, $w = t_3$

Gauss Jordan Elimination Method. to find sol^n of system of linear eq^n.

Step I: System is given

Step II: write [A | B] matrix

Step III: convert [A|B] into $\boxed{\text{RREF}}$

Step IV: Apply condition from sol^n of SLE and find value of unknown.

Using Gauss Jordan elimination method find the sol^n of given system.

$$x + 2y + 3z = 1$$
$$4x + 5y + 6z = 2$$
$$7x + 8y + 9z = 3$$

$m = 3$, $n = 3$, $AX = B$ (system is non-Homo)

$$[A|B] = \left[\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 4 & 5 & 6 & 2 \\ 7 & 8 & 9 & 3 \end{array}\right]$$

convert [A|B] into RREF. firstly convert into REF

$R_2 \to R_2 - 4R_1$
$R_3 \to R_3 - 7R_1$

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 0 & -3 & -6 & -2 \\ 0 & -6 & -12 & -4 \end{array}\right] = [A|B]$$

$R_3 \to R_3 - 2R_2$

$$[A|B] = \left[\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 0 & -3 & -6 & -2 \\ 0 & 0 & 0 & 0 \end{array}\right] \to \text{REF}$$

convert it into RREF,

$R_1 \to 3R_1 + 2R_2$

$$[A|B] = \left[\begin{array}{ccc|c} 3 & 0 & -3 & -1 \\ 0 & -3 & -6 & -2 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$R_1 \to \frac{R_1}{3}$, $R_2 \to \frac{R_2}{-3}$ $\quad [A|B] = \begin{array}{cccc} x & y & z & \\ \end{array}$

$$[A|B] = \left[\begin{array}{ccc|c} 1 & 0 & -1 & -1/3 \\ 0 & 1 & 2 & 2/3 \\ 0 & 0 & 0 & 0 \end{array}\right] \to RREF$$

$\rho(A) = 2$ and $\rho(A|B) = 2$, $n = 3$

$AX = B$ is consistent and infinitely many non-zero soln.

x and y are LE.

z is free

Put $z = t$, $t \in R$ from RREF

$x - z = -1/3 \Rightarrow x - z = \frac{-1}{3}$

$y + 2z = 2/3$

$$x = \frac{-1}{3} + z = \frac{-1}{3} + t = \frac{-1+3t}{3}.$$

$$y = \frac{2}{3} - 2z = \frac{2}{3} - 2t = \frac{2-6t}{3}$$

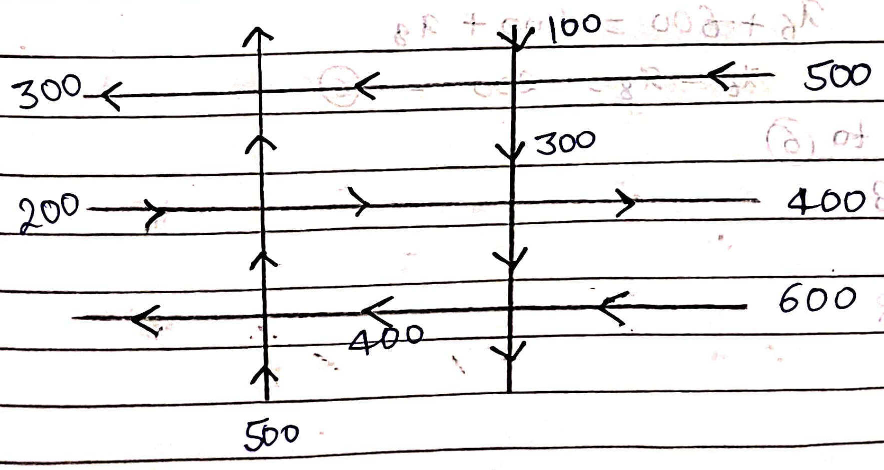
$\therefore x = \frac{-1+3t}{3}$, $y = \frac{2-6t}{3}$, $z = t$. $\quad (t \in R)$

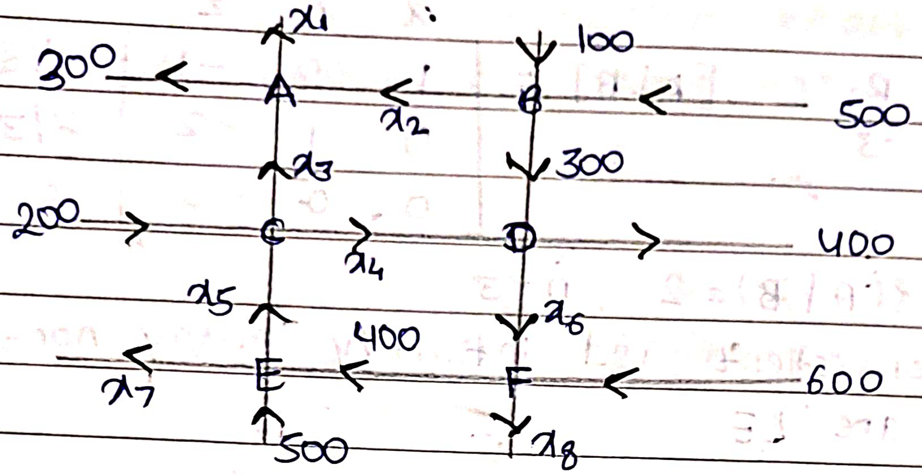
* Application of SLE.

I] Network flow

Total in flow = Total out flow

ex. solve the Network.





at point A: $x_2 + x_3 = x_1 + 300$

$x_1 - x_2 - x_3 = -300$ — ①

at point B: $100 + 500 = x_2 + 300$

$x_2 = 300$ — ②

at point C: $200 + x_5 = x_3 + x_4$.

$x_3 + x_4 - x_5 = 200$ — ③

at point D: $300 + x_4 = 400 + x_6$

$x_4 - x_6 = 100$ — ④

at point E: $500 + 400 = x_5 + x_7$

$x_5 + x_7 = 900$ — ⑤

at point F: $x_6 + 600 = 400 + x_8$

$x_6 - x_8 = -200$ — ⑥

From eqⁿ ① to ⑥

$AX = B$

$m = 6$

$n = 8$

$$[A \mid B] = \begin{array}{c} \begin{array}{cccccccc} x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & x_7 & x_8 \end{array} \\ \left[\begin{array}{cccccccc|c} 1 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & -300 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 300 \\ 0 & 0 & 1 & 1 & -1 & 0 & 0 & 0 & 200 \\ 0 & 0 & 0 & 1 & 0 & -1 & 0 & 0 & 100 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 900 \\ 0 & 0 & 0 & 0 & 0 & \cancel{0}1 & 0 & -1 & -200 \end{array}\right] \end{array} \rightarrow \text{REF.}$$

$\qquad\qquad\qquad\qquad A \qquad\qquad\qquad\qquad\qquad B.$

$\rho(A) = 6$, $\rho(A|B) = 6$, $n = 8$ (system is non-Homo)

$\therefore \rho(A) = \rho(A|B) < n$. (Infinite many non-zero sol^n).

LE = $x_1, x_2, x_3, x_4, x_5, x_6$

free = x_7, x_8

Put, $x_7 = t_1$ and $x_8 = t_2$ $(t_1 \,\&\, t_2 \in R)$

from 5 and 6

$x_5 + x_7 = 900$, $x_6 - x_8 = -200$

$\Rightarrow x_5 + t_1 = 900$ $\Rightarrow x_6 - t_2 = -200$

$\Rightarrow x_5 = 900 - t_1$ $\Rightarrow x_6 = t_2 - 200$

like these find all the values of x_1 to x_8.