Vectors and The Geometry of Space

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1 Vectors

1.1 Introduction

In all the previous years of our high school mathematics, we have dealt with scalar quantities. These are quantities that only have a *magnitude*. For example, the time it takes for me to walk from home to school is 15min. 15min is the magnitude of this quantity. The number of moles in a molecule is 6.02×10^{23}, this number is the magnitude of the quantity.

Most of our mathematics so far have been with scalar quantities, now, we introduce *vectors*. A vector quantity is one that has both a magnitude, and a *direction*. For example, the velocity of a car is $60km/h$ due North. 60 km/h is the magnitude and North is the direction of the velocity vector. The wind is blowing at a rate of 12km/h due Southwest. 12km/h is the magnitude and Southwest is the direction of the velocity vector. The weight of a boxer is 200lb. 200lb is the magnitude and downwards is the direction of the weight vector.

Another key thing about vectors is that they are straight lines drawn from one point to another. When provided with 2 points, we can draw a vector and determine the *components* of that vector.

There are various representations of vectors. Matrix notation:

$$\vec{u} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

Component notation:

$$\vec{u} = [x, y, z] = \langle x, y, z \rangle = x\hat{i} + y\hat{j} + z\hat{k} = x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k}$$

Magnitude and direction notation:

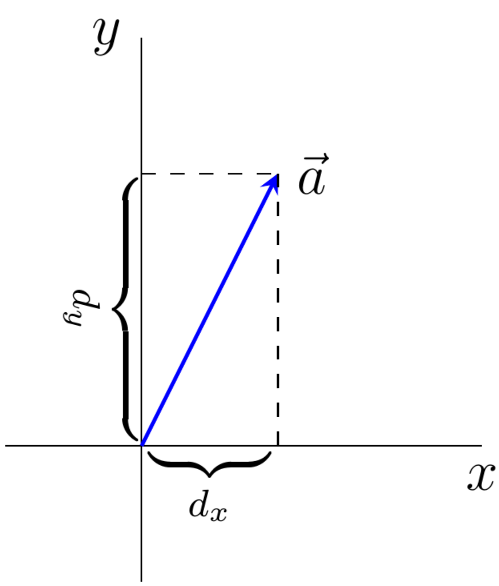
$$\vec{u} = a[A\,\theta\,B]$$

Here, a is the magnitude, and the expression $[A\ \theta\ B]$ would normally be replaced with something like $[N30^oE]$ which is read as "North, 30 degrees East". This is more commonly used in high school physics. Despite the difference in notation, all these notations are equally valid and you may use whichever one you wish to use. However, in your first course of multivariable calculus, you will see mostly the use of component or matrix notation. Throughout this handout, I will be using component notation to denote a vector. We will explore the meaning of this notation in the next section.

1.2 2 Dimensional Space: Vector Components

Since direction in 1D space can be determined using only a sign convention, (for example, if the sign of the magnitude is negative, then direction is left, if sign of the magnitude is positive, direction is right) we will discuss this topic on the xy-plane.

When we graph a vector on the xy-plane, it will look like the figure below:



As you can see, the vector $\vec{a}$ is a straight line. Vectors will not be curved! (This is only in Cartesian Coordinates, when working with curvilinear coordinates, you will see that vectors that are straight with respect to the coordinate axes are curved) Since $\vec{a}$ is a straight line, we can see that by drawing a straight line from a to the x axis, we can create a right triangle. This triangle will have the vector $\vec{a}$ as its hypotenuse. Now, using Pythagoras' theorem, we can determine the length of the other two sides. We know that:

$$|\vec{a}|^2 = d_x^2 + d_y^2$$

$$\boxed{\therefore |\vec{a}| = \sqrt{(d_x)^2 + (d_y)^2}}$$

Note $|\vec{a}|$ is used to denote the magnitude of $\vec{a}$. As you can guess, we cannot square a direction. So we only use the magnitude of the vector. Note that when we change the length of a vector, we will not change its direction. We are just increasing its magnitude. (This is NOT the case when we mutliply by a negative scalar, this also reverses the direction and changes the magnitude)

Now, using some simple trigonometry, we can determine a relationship between the magnitude of $\vec{a}$ and its ***components***; d_x and d_y. Here, d_x is the x component of the vector and d_y is the y component.

Let us call the angle between the x axis and $\vec{a}$, θ. Assuming that we know this angle, then:

$$\sin(\theta) = \frac{d_y}{|\vec{a}|} \quad \cos(\theta) = \frac{d_x}{|\vec{a}|}$$

Hence we get the following relationships:

$$\boxed{d_y = |\vec{a}|\sin(\theta) \qquad d_x = |\vec{a}|\cos(\theta)}$$

And we can see that if we use these expressions in the formula we derived for the magnitude of a vector, the relationship holds. Hence both the formula and the relationships are valid.
Proof:

$$|\vec{a}| = \sqrt{(d_x)^2 + (d_y)^2} = \sqrt{(|\vec{a}|\cos(\theta))^2 + (|\vec{a}|\sin(\theta))^2}$$

$$= \sqrt{|\vec{a}|^2 \cdot (\cos^2(\theta) + \sin^2(\theta))} = \sqrt{|\vec{a}|^2} = |\vec{a}|$$

When dealing with things such as forces, dynamics problems, projectile motion problems and addition and subtraction of vectors, we use the vector components to calculate the result.

1.3 Basic Operations with Vectors

Here are a few ground rules for performing operations with vectors.
If $\vec{u}$ is a vector and k is a scalar quantity, then:

$$k \cdot \vec{u} = \vec{u} \cdot k$$

$$k \pm \vec{u} = undefined$$

$$k \cdot \vec{u} = \langle k \cdot x_u, k \cdot y_u \rangle$$

If $\vec{u} = \langle x_u, y_u \rangle$ and $\vec{v} = \langle x_v, y_v \rangle$ then:

$$\vec{u} \pm \vec{v} = \langle x_u \pm x_v, y_u \pm y_v \rangle$$

Here are a few properties of vector operations:

$$\vec{u} + \vec{v} = \vec{v} + \vec{u}$$

$$(\vec{u} + \vec{v}) + \vec{w} = \vec{u} + (\vec{v} + \vec{w})$$

$$0\vec{u} = \vec{0}$$

$$\vec{u} + \vec{0} = \vec{u}$$

$$\vec{u} + (-\vec{u}) = \vec{0}$$

$$1\vec{u} = \vec{u}$$

$$a(b\vec{u}) = (ab)\vec{u}$$

$$a(\vec{u} + \vec{v}) = a\vec{u} + a\vec{v}$$

$$(a + b)\vec{u} = a\vec{u} = b\vec{u}$$

You may have noticed that the zero in the third expression is denoted as a vector. This is purely a conceptual issue; it is not possible to multiple a scalar value by a vector and get a scalar as the product. Therefore, we denote the product as the *zero vector*. The zero vector, denoted by $\vec{0}$, is a vector with a magnitude equal to 0. Since it has no magnitude, there is no need to define a direction for it as it will not extend towards any coordinate axis. Every time we subtract two vectors and both the components of the result are zero, or when we multiply a vector by a scalar factor of 0, the result is always the zero vector, $\vec{0}$ not just 0.

1.4 Unit Vectors

As we discussed in section 1.1, a vector has both a direction and a magnitude. In different applications of the concept of vectors, we sometimes come across situations where we are only interested in the direction of a vector not its magnitude. If we do not wish for the magnitude of the direction to effect our vector at hand, we must turn it into 1 since if the magnitude is 1, only the direction of the result will change not the magnitude. This is called a *unit vector*. It is a vector with a direction but magnitude of 1.

To make a unit vector, we first take any vector we want the direction of. Let us take some vector $\vec{u} = \langle 2, 5\rangle$ as an example. Then we divide the vector by its magnitude in order to only keep its direction. Note that when you multiply a vector by a scalar, the direction does not change, only the magnitude will.

$$\vec{u} = \frac{1}{\sqrt{2^2 + 5^2}}\langle 2, 5\rangle = \frac{1}{\sqrt{29}}\langle 2, 5\rangle = \left\langle \frac{2}{\sqrt{29}}, \frac{5}{\sqrt{29}} \right\rangle$$

To denote a unit vector and distinguish it from the initial vector, we remove the arrow on the top and replace it with a hat. The unit vector of vector $\vec{u}$ is denoted by $\hat{u}$. Hence:

$$\hat{u} = \left\langle \frac{2}{\sqrt{29}}, \frac{5}{\sqrt{29}} \right\rangle$$

Making things general, for some vector $\vec{v} = \langle v_1, v_2 \rangle$,

$$\boxed{\hat{v} = \frac{\vec{v}}{|\vec{v}|}}$$

You may ask how is this of any use? Let us take a look at an example to further understand the importance of a unit vector.

Example Gary the snail is applying a force of 48N to a weight that is on an inclined surface, keeping it from falling down. The inclination of the surface is made such that it is always parallel to the water that comes out of the Trevi fountain in Rome whose trajectory can be described by the vector $\vec{u} = \langle \sqrt{23}, -11 \rangle$. Express the vector that describes the force Gary is applying to the weight in component notation.

Solution We know that the magnitude of the force being applied is 48N. Hence, all we need is finding the direction of the force vector and multiply it by the magnitude to get the force vector.
Knowing that the trajectory of the water and the inclined surface are parallel implies that they have the same direction. As a result, if we find the direction of the water's trajectory, we have found the direction of the inclined surface which is the same as the direction the force is applying applied towards.
Since we are not interested in the magnitude of the water's trajectory, we only take the direction which is the unit vector $\hat{u}$.

$$\hat{u} = \frac{\vec{u}}{|\vec{u}|} = \frac{\langle \sqrt{23}, -11 \rangle}{\sqrt{23 + (-11)^2}} = \frac{\langle \sqrt{23}, -11 \rangle}{\sqrt{144}} = \frac{\langle \sqrt{23}, -11 \rangle}{12} = \left\langle \frac{\sqrt{23}}{12}, \frac{-11}{12} \right\rangle$$

Now all we have to do is multiply the unit vector by the magnitude which will give us the force vector.

$$\vec{F} = 48 \cdot \hat{u} = 48 \cdot \left\langle \frac{\sqrt{23}}{12}, \frac{-11}{12} \right\rangle = \langle 4\sqrt{23}, -44 \rangle$$

Therefore, the force being applied to the weight is represented by the vector $\vec{F} = \langle 4\sqrt{23}, -44\rangle$.
You may have noticed that in section 1.1, one of the component notations is:

$$\vec{u} = u_1\hat{\imath} + u_2\hat{\jmath} + u_3\hat{k}$$

Here, $\hat{\imath}$, $\hat{\jmath}$, and $\hat{k}$ are unit vectors. $\hat{\imath} = \langle 1, 0, 0\rangle$ is 1 step in the x direction. Similarly, $\hat{k} = \langle 0, 1, 0\rangle$. It is 1 step in the y direction. And later in 3D space, $\hat{k} = \langle 0, 0, 1\rangle$ is 1 step in the z direction.

1.5 The Dot Product

1.5.1 Introduction to The Dot Product

The dot product is an extremely important and useful operation one can perform with vectors. It is used to calculate quantities such as work, projection, etc. In this section we will explore the dot product and will later apply it to concepts in the geometry of space.
When we take the dot product of two vectors, it gives us the projection of the first vector onto the second vector. It is denoted by:

$$\vec{u} \cdot \vec{v}$$

This expression is basically asking, how much of vector $\vec{u}$ is in $\vec{v}$? If you recall components, you can think about the dot product as a way fro us to get a vector's component in another vector. Unlike how trigonometry limited us to components in the x and y directions, the dot products allows us to define components in any direction we want. Note that with vectors, it matters what multiplication sign you use. The meaning of $\vec{u}\cdot\vec{v}$ is completely different than $\vec{u}\times\vec{v}$ and we will get to this more in detail.

The dot product of two vectors is only meaningful when the vectors are tail-to-tail meaning that the arrows are not touching the other vector. When the ends are connected, we say the vectors are tail-to-tail. The definition of the cross product is:

$$\boxed{\vec{u} \cdot \vec{v} = |\vec{u}||\vec{v}|\cos\theta}$$

Where $|\vec{u}|$ and $|\vec{v}|$ are the magnitudes of vectors $\vec{u}$ and $\vec{v}$ respectively and θ is the angle between two vectors. Another method to calculate the dot product when not given the magnitudes or the angle between the two vectors is through components. Another definition of the dot product is:

$$\boxed{\vec{u} \cdot \vec{v} = u_1v_1 + u_2v_2 = \sum_{k=1}^{n} u_kv_k = |\vec{u}||\vec{v}|\cos\theta}$$

Notice that the dot product of two vectors does NOT give a vector, but a scalar value. Using this definition, given the component of the two vectors, we can calculate the angle between them. Therefore, the angle between two vectors $\vec{u}$ and $\vec{v}$ with components $\vec{u} = \langle u_1, u_2 \rangle$ and $\vec{v} = \langle v_1, v_2 \rangle$ is:

$$\theta = \cos^{-1}\left(\frac{u_1 v_1 + u_2 v_2}{|\vec{u}||\vec{v}|}\right) = \arccos\left(\frac{\vec{u} \cdot \vec{v}}{|\vec{u}||\vec{v}|}\right)$$

Now, the question becomes how can we apply these concepts ? Or even better, how can we work our way backwards from having the full vector to finding the individual components of it ?

1.5.2 Applications of The Dot Product: Work and Projection

One of the most important quantities we learn about in physics class is Work. Most of us know that the work done by mechanical energy is, by definition, the amount of force applied per unit distance. We know that the formula for work done, given the force and the displacement, is:

$$W = F\Delta d$$

However, now that we are working with vectors and since Work is a scalar quantity, we are only interested in the magnitude of the Force and Displacement vectors. Hence:

$$\boxed{W = |\vec{F}||\vec{\Delta d}|}$$

We also know that force is not always applied in the direction of the displacement. There may be an angle between the two vectors and if we recall from physics class, the formula in this situation becomes:

$$W = |\vec{F}||\vec{\Delta d}|\cos(\theta)$$

Now that we are familiar with what the dot product is, we can see that this formula is extremely similar to that of the dot product. Hence we can write the work done as:

$$W = \vec{F} \cdot \vec{\Delta d}$$

Or if you reviewed the "Anti-Derivatives" section in the previous handout, you can see that the Work done is defined as:

$$W = \int_{s_1}^{s_2} F\, ds$$

Once we do vector calculus in an advanced section, you can see that the work done by a force vector field on a curve C is defined by the line integral:

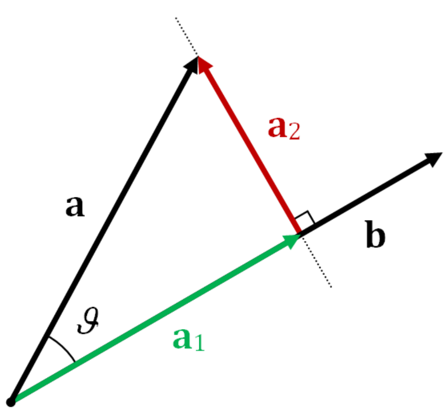
$$W = \int_C \vec{F} \cdot d\vec{s}$$

Note that if it is on a closed, simple curve, then:

$$W = \oint_C \vec{F} \cdot d\vec{s} = 0$$

This follows from the definition of work. If the work is done over a closed path, then there is no displacement and hence the work done is equal to 0. We will look at this in detail in a future handout on Multivariable Calculus. For now however, we are dealing with calculus and geometry with respect to one variable only and we will keep it this way.

Another application of the dot product is vector projection. We are now familiar with how to represent vectors in different ways and we know that given the components of a vector, we can represent it in any way we want. However, we can ask ourselves that given the magnitude of the vector, how can we determine the components ? Is there a way to work the system backwards ? The answer is, surprisingly, yes!



Let us imagine two vectors $\vec{a}$ and $\vec{b}$ such that the angle between the two vectors is ϑ (theta) and that $\vec{a}$ and $\vec{b}$ are tail to tail as shown in the figure on the left. In this case, given that $a_2 \perp \vec{b}$, we can use some simple trigonometry to determine the length of a_1:

$$\cos\vartheta = \frac{a_1}{|\vec{a}|} \implies a_1 = |\vec{a}|\cos\vartheta$$

If we consider $\vec{b}$ a direction, then a_1 is a component of $\vec{a}$ in the direction of $\vec{b}$. It is very important to understand that components, are not necessarily always in the x and y direction. Sometimes, we express the vector components in terms of other vectors' direction. In this case, given that a_2 is a component of $\vec{a}$ in the direction of some other vector, $\vec{c}$, then:

$$\vec{a} = a_1\hat{b} + a_2\hat{c}$$

Therefore, from now on you will see me use unit vectors in component notation more often.

Here, a_1, is the *projection* of $\vec{a}$ onto $\vec{b}$. Or, a_2 is the *projection* of $\vec{a}$ onto $\vec{c}$.

The general formula for finding the projection of a vector onto another is:

$$proj_{\vec{a}}\vec{b} = \frac{\vec{a} \cdot \vec{b}}{|\vec{a}|^2}$$

Note that here we have a scalar value divided by another scalar. Hence, projection is a scalar. However, there is also a vector projection which is the same thing with one small difference: once we have the scalar value, we multiply it by the unit vector to have a vector with the magnitude of the projection. In other words, the vector projection of $\vec{a}$ onto $\vec{b}$ is:

$$\boxed{proj_{\vec{a}}\vec{b} = \frac{\vec{a} \cdot \vec{b}}{|\vec{a}|^2}\hat{a}}$$

The proof of this will be left as an exercise to the reader. I encourage you to try and prove this as a good geometry exercise. A hint would be to consider the implications of the trigonometric ratios.

Before moving on to the Cross Product, it is important to do a quick interlude on *orthogonality* and *orthonormality*. Of course, this looks like very fancy word but it is not at all difficult to understand. *Orthogonal* means that two geometric objects are perpendicular to each other. When two geometric objects such as two vectors are perpendicular to each other, we state that the two vectors are orthogonal.

The reason I mentioned this term is due to its great use in the rest of this handout, especially in the lines and planes section. For now, we consider orthogonality a special case of the dot product. When we are asked whether two vectors are orthogonal or not, we can answer the question given the components of the vectors.

Let us consider two vectors $\mathbf{u} = \langle u_1, u_2 \rangle$ and $\mathbf{v} = \langle v_1, v_2 \rangle$ with magnitudes $||\mathbf{u}||$ and $||\mathbf{v}||$ respectively. (Note that vectors can also be denoted by bold letters too. This is because in older textbooks, since LaTeX was not invented, they used bold characters to distinguish between a vector and a scalar instead of the arrow) Orthogonality of these two vectors, implies that the angle between them is $\pi/2$. Hence:

$$\mathbf{u} \cdot \mathbf{v} = ||\mathbf{u}||\,||\mathbf{v}|| \cos\left(\frac{\pi}{2}\right)$$

$$\implies \mathbf{u} \cdot \mathbf{v} = 0 \ \left(\cos\left(\frac{\pi}{2}\right) = 0\right)$$

$$\boxed{\therefore \mathbf{u} \perp \mathbf{v} \iff \mathbf{u} \cdot \mathbf{v} = 0}$$

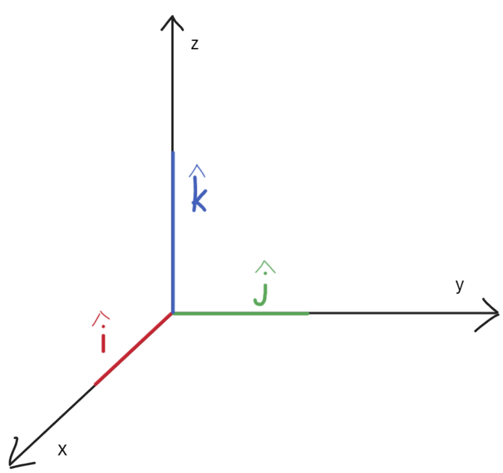
Note that if you have not looked into it already, the symbol $\implies$ means "implies that" and the symbol $\perp$ means "is perpendicular to". You have also seen $\iff$ which means "if and only if" and can also be written as "iff" instead of the symbol. This notation is extremely important to learn and get comfortable with. The statement above can also be written as:

$$\therefore \mathbf{u} \perp \mathbf{v} \text{ iff } \mathbf{u} \cdot \mathbf{v} = 0$$

It is helpful to learn this notation and get comfortable with it as it comes in very handy both as a shortcut in note taking and as a proper and more formal mathematical notation. Now we know that given the components of two vectors, we can determine whether they are orthogonal or not. We will come back to the idea of orthogonality when discussing lines and planes later on.

1.6 3 Dimensional Space

Up to this point, we have established the ground work for discussing vectors in our good old 2 dimensional *plane*. This *plane*, included the x and the y axes. Now, I ask you to look at a literal corner of your room and label the x and y axes. You may notice that there are not just 2 axes and that our life has not been on a piece of paper for the past years of our existence. We live in a world with not 2, but 3 spatial dimensions. I know that you are probably thinking "and 1 time dimension" but since we are not discussing Special or General Relativity or Quantum Mechanics, I ask that you please let go of that idea and only consider spatial dimensions.



In order to properly discuss and capture the geometry of our world, we introduce a z axis which is *orthogonal* to both x and y. Considering the room corner scenario, the x and y axes are the right and left edges on the floor, and z is the axis that goes up to the ceiling. You can also consider a box. The length and width of this box will be the x and y axes whereas the height would be an example of the z axis. We can embed this, of course using some good drawing skills, on a page. Consider the diagram on the left. This a typical representation of 3D space on paper. As you can see, the z axis goes up and down, the y axis goes right and left, and the x axis goes in and out of the page. Note that you may see a different choice of labeling the axes in physics. Remember that since we are in 3 dimensional space, we also have to introduce unit vector $\hat{k} = \langle 0, 0, 1 \rangle$ and our vectors will no longer have 2, but 3 component as it must correspond to the number of dimensions we are working with. Hence:

$$\mathbf{u} = \langle u_1, u_2 \rangle \rightarrow \mathbf{u} = \langle u_1, u_2, u_3 \rangle$$

From now on, we will continue to apply the concepts we have developed and those that we will develop soon, in 3d space. As a review, let us restate some ground vector operations for 3d. However, know that these are true for as many dimensions as one may need:

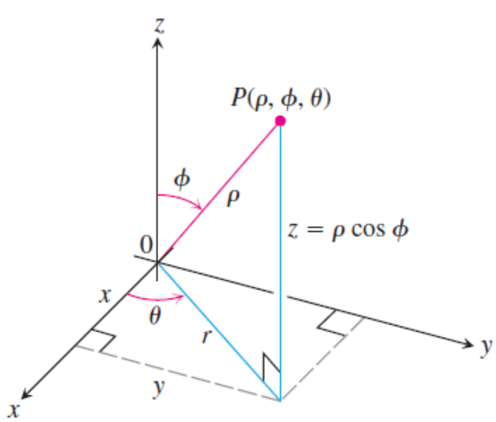
$$\mathbf{u} = \langle u_1, u_2, u_3 \rangle \quad \mathbf{v} = \langle v_1, v_2, v_3 \rangle$$

$$\mathbf{u} \pm \mathbf{v} = \langle u_1 + v_1, u_2 + v_2, u_3 + v_3 \rangle$$

$$\mathbf{u} \cdot \mathbf{v} = \langle u_1 \cdot v_1, u_2 \cdot v_2, u_3 \cdot v_3 \rangle = ||\mathbf{u}||\,||\mathbf{v}|| \cos(\theta)$$

You may have noticed, that I did not show that the formula for the magnitude of a vector is also the same. This because in order to do so, we need proof. The previous statements were simply properties and rules that are established and not necessarily need to be proven. The magnitude formula, however, can be derived and hence, we will derive this using some simple trigonometry in 3 dimensional space.

It should be obvious, up to this point, that having a fundamentally strong understanding of the algebraic and geometric interpretations and implications of vectors demands an exceptionally deep and thorough understanding of trigonometry. Hence, I advise you to keep practicing trigonometry and remembering the formulas and identities as these will be extremely helpful throughout Calculus and Vectors.



Now, as I promised, we will derive the formula for the magnitude of a vector in 3D space. In order to help us better understand this matter, let us use the diagram on the left to better visualize the problem. As it is evident, we are trying to find the vector that connects point P to point O with O standing for the origin. Let us call $||\mathbf{u}|| = \overline{OP}$. Now we use Pythagoras' theorem:

$$r = \sqrt{x^2 + y^2}$$

$$\mathbf{u} = \langle u_1, u_2, u_3 \rangle \mid u_1 = x,\ u_2 = y,\ u_3 = z$$

$$\implies ||\mathbf{u}|| = \sqrt{r^2 + z^2}$$

However, we know the value of r. Hence we can make the following substitution:

$$||\mathbf{u}|| = \sqrt{(\sqrt{x^2+y^2})^2 + z^2} = \sqrt{x^2+y^2+z^2}$$

Since we know that $x, y,$ and z correspond to $u_1, u_2,$ and u_3, then we know that:

$$\boxed{||\mathbf{u}|| = \sqrt{(u_1)^2 + (u_2)^2 + (u_3)^2}}$$

Which is very similar to the formula for 2D vectors but with a third term. The reason I included the angles in the diagram is because you can also do this entire process with trigonometric functions and $||\mathbf{u}||$ in order to prove the formula for yourself. We will not, however, go through this proof although I encourage you to try and prove it. It is very similar to the proof we used for the 2D formula but with slightly more manipulation. It is left as an exercise for the reader to prove that for n dimensional vectors, the distance formula for a euclidean vector (a vector that is a straight line) is:

$$||\mathbf{u}|| = \sqrt{(u_1)^2 + (u_2)^2 + (u_3)^2 + \cdots + (u_n)^2} = \sqrt{\sum_{k=1}^{n} u_k^2}$$

The summation sign $\sum$ is not meant to be intimidating, although it may be, but rather an introduction through practice to this type of notation. All this means, is that you sum over the indices. We will go through more detail on this in the Notation section of this handout.

1.7 The Cross Product

Now that we have established our framework with vectors in 3D space, we are ready to learn the cross product. However, before doing so, we need a short interlude on Linear Algebra, matrices and determinants of them specifically. Note that this is not too advanced and you should learn this in order to learn the cross product. Hence, you cannot skip this section.

1.7.1 Interlude on Linear Algebra: Matrices and Determinants

The term Linear Algebra is pretty self explanatory; It is the algebra of linear systems. When dealing with a system of linear equations, analyzing behavior of these equations and applications of them and even in the study of differential equations, Linear Algebra proves itself to be one of the most important and useful branches of mathematics. Let us start by introducing the single, most fundamental, yet most basic idea of linear algebra: a matrix. One way to think of a matrix, is to think of it simply as a collection of numbers, or if you are into computer science, it is an array of numbers. Here is an example of a matrix **A**. Note that to represent matrices, we use boldface.

$$\mathbf{A} = \begin{pmatrix} a_{00} & a_{01} & a_{02} & a_{03} & a_{04} & \cdots & a_{0n} \\ a_{10} & a_{11} & a_{12} & a_{13} & \cdots & \cdots & a_{1n} \\ a_{20} & \vdots & \vdots & \ddots & & \vdots & a_{2n} \\ \vdots & \vdots & \vdots & & \ddots & \vdots & \vdots \\ a_{m0} & a_{m1} & a_{m2} & a_{m3} & a_{m4} & \cdots & a_{mn} \end{pmatrix}$$

The matrix **A** is simply a $n \times m$ size matrix. The indexing system is such that the first digit on the subscript corresponds to the row and the second corresponds to the column number. You can start either from 0 or 1, there is no advantage to either one. You may ask, how can we use this ? Well, here is how matrices help us represent a system of linear equations. Let us say we have a system of linear equations:

$$\begin{aligned} w + 2x - 3y + 5z &= 0 \\ 2w + 3x + y - 2z &= 0 \\ -w - 7x - 11y + z &= 0 \\ 81w - \pi x + \phi y - ez &= 0 \end{aligned}$$

Then we can represents this system of linear equations in a 4×4 matrix. Such that:

$$\mathbf{S} = \begin{pmatrix} 1 & 2 & -3 & 5 \\ 2 & 3 & 1 & -2 \\ -1 & -7 & -11 & 1 \\ 81 & -\pi & \phi & -e \end{pmatrix}$$

All we did was bring the coefficients of the variables in the appropriate row. In other words, get rid of all the variables and simply write the coefficients. We then know that the 1st column corresponds to variable w, the second column to variable x, third column to variable y, and the last one to variable z. Or, we can think that the first row corresponds to the first equation, second row to the second equation, third row to the third equation and last row to the fourth equation.

Another way to think of matrices, is a collection of not just numbers, but vectors! We can have a matrix that is a collections of as many vectors as we want. Recall, the matrix notation for a vector **u** with components u_1, u_2 and u_3, is a 3×1 matrix.

$$\mathbf{u} = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}$$

We can see that the first row, corresponds to the x component, second row to the y component, and third row to the z component. Using this rule, we can represents multiple vectors in 1 matrix.
Let us apply this to three vectors and write these in a 3×3 matrix. Given the vectors:

$$\mathbf{u} = u_1\hat{\imath} + u_2\hat{\jmath} + u_3\hat{k}$$

$$\mathbf{v} = v_1\hat{\imath} + v_2\hat{\jmath} + v_3\hat{k}$$

$$\mathbf{w} = w_1\hat{\imath} + w_2\hat{\jmath} + w_3\hat{k}$$

We can write a matrix $\mathbf{V}$ - there is no preference on what you call this matrix - that holds all these vectors for us:

$$\mathbf{V} = \begin{pmatrix} u_1 & v_1 & w_1 \\ u_2 & v_2 & w_2 \\ u_3 & v_3 & w_3 \end{pmatrix}$$

I will not go into more details about matrices as that is the subject for another handout. However, now that we know a little about matrices, we can learn about the determinant of a matrix.

I will not fully explain what the determinant is. That will make us go off the topic, we will only learn enough to be able to calculate the determinant and then apply it to the cross product.

To calculate the determinant of a matrix, we first start by learning to do it for a 2×2 matrix. Then we expand the idea to larger matrices. Let us say we have a matrix $\mathbf{A}$:

$$\mathbf{A} = \begin{pmatrix} a_{00} & a_{01} \\ a_{10} & a_{11} \end{pmatrix}$$

Now, to denote that we are taking the determinant of the matrix, we denote it as $\det(\mathbf{A})$. Once we do so, we change the curved parenthesis, or square brackets, to straight lines:

$$\det(\mathbf{A}) = \begin{vmatrix} a_{00} & a_{01} \\ a_{10} & a_{11} \end{vmatrix}$$

This is equal to the following statement:

$$\det(\mathbf{A}) = a_{00}a_{11} - a_{01}a_{10}$$

For example, if I have a matrix $\mathbf{A}$:

$$\mathbf{A} = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}$$

Then the determinant of A;

$$\det(\mathbf{A}) = \begin{vmatrix} 1 & 3 \\ 2 & 4 \end{vmatrix} = (1 \cdot 4) - (2 \cdot 3) = 4 - 6 = -2$$

Now, here is a question: How can we take the determinant of a 3×3 matrix such as matrix $\mathbf{V}$ which we mentioned earlier ?

The first step is very obvious, it is simply a matter of notation:

$$\det(\mathbf{V}) = \begin{vmatrix} u_1 & v_1 & w_1 \\ u_2 & v_2 & w_2 \\ u_3 & v_3 & w_3 \end{vmatrix}$$

Now, here is the process: In order to get the determinant of this matrix, we must turn it into a combination of determinants of 2×2 matrices. To do so follow this process:

1. Choose component at index 00
2. Cover the row and column with your hand
3. Write down the component you covered
4. Multiply it by the 2×2 matrix that is not covered by your hand
5. Move on to the next component in the same row
6. Repeat the process

This process is known as Cramer's Rule. Here is an example of how to do this with a 3×3 matrix. Let us say we have a matrix $\mathbf{A}$ such that:

$$\mathbf{A} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$$

Now we apply Cramer's Rule:

$$\det(\mathbf{A}) = \begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix} = 1 \cdot \begin{vmatrix} 5 & 6 \\ 8 & 9 \end{vmatrix} - 2 \cdot \begin{vmatrix} 4 & 6 \\ 7 & 9 \end{vmatrix} + 3 \cdot \begin{vmatrix} 4 & 5 \\ 7 & 8 \end{vmatrix}$$

$$= 1 \cdot [(5 \cdot 9) - (6 \cdot 8)] - 2[(4 \cdot 9) - (6 \cdot 7)] + 3 \cdot [(4 \cdot 8) - (5 \cdot 7)] = -3 + 12 - 9 = 0$$

You may ask, where did the negative sign for the -2 came from ? This is something that we have to do in order to guarantee that we get the correct result. The second term, always has a negative multiplied to it. Note that the determinant of a matrix is not always a scalar value!

Let us now generalize Cramer's Rule. For a 3×3 matrix $\mathbf{A}$:

$$\mathbf{A} = \begin{bmatrix} a_{00} & a_{01} & a_{02} \\ a_{10} & a_{11} & a_{12} \\ a_{20} & a_{21} & a_{22} \end{bmatrix}$$

The determinant is calculated through following Cramer's Rule:

$$\det(\mathbf{A}) = \begin{vmatrix} a_{00} & a_{01} & a_{02} \\ a_{10} & a_{11} & a_{12} \\ a_{20} & a_{21} & a_{22} \end{vmatrix} = a_{00} \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} - a_{01} \begin{vmatrix} a_{10} & a_{12} \\ a_{20} & a_{22} \end{vmatrix} + a_{02} \begin{vmatrix} a_{10} & a_{11} \\ a_{20} & a_{21} \end{vmatrix}$$

$$= a_{00}[(a_{11} \cdot a_{22}) - (a_{12} \cdot a_{21})] - a_{01}[(a_{10} \cdot a_{22}) - (a_{12} \cdot a_{20})] + a_{02}[(a_{10} \cdot a_{21}) - (a_{11} \cdot a_{21})]$$

As you can see, the calculations may get very tedious and so it is important to write every single calculation to reduce the chance of making a mistake. Now that we have learned a little bit about determinants and matrices, we can get back on track to learning the cross product. I, however, highly encourage you to learn more linear algebra. A great resource for the topic is Gilbert Strang's Introduction to Linear Algebra.

Before we move on, I would like to note, for those of you who are interested, the general case of the determinant for some $n \times n$ matrix $\mathbf{A}$, where S_n is the symmetric group, is given by:

$$\det(\mathbf{A}) = \sum_{\sigma \in S_n} \left(\operatorname{sgn}(\sigma) \prod_{k \to n} a_{k,\sigma(k)} \right)$$

Where $\sigma(k)$ is the bijective function from the set $\{1, 2, \ldots, n\}$ to itself and $k, \sigma(k)$ is an index in $\mathbf{A}$.

1.7.2 Taking the Cross Product

The dot product of two vectors, is also called the scalar product. This is, as suggested by the name, because the dot product of two vectors always returns a scalar value. The cross product, on the other hand, returns a vector value.

This vector is always orthogonal to the two vectors that were multiplied. As you may have already imagined such as situation, you may have noticed that it is not possible to have a vector orthogonal to two other distinct vectors at the same time in 2D space. Hence, we can only define this quantity in 3 or higher dimensions of space.

Given two vectors $\mathbf{u} = \langle u_1, u_2, u_3 \rangle$ and $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$ the cross product of the two vectors is denoted by the following expression:

$$\mathbf{u} \times \mathbf{v}$$

Hence, it is important to know which multiplication symbol we use when dealing with vectors as they carry completely different meanings. Now as you may recall, we stated that the dot product of two vectors, is equal to the product of the vectors' magnitudes and the cosine of the angle between them. In the case of the cross product, only the *magnitude* of the resultant vector is given by:

$$\boxed{||\mathbf{u} \times \mathbf{v}|| = ||\mathbf{u}||\, ||\mathbf{v}|| \sin \theta}$$

I encourage you to try and prove this formula using the information I will provide you with shortly. It is a slightly difficult proof and hence I will not go over it. You may ask, why does one need to know about linear algebra to calculate the cross product. Well, as you may recall, there was also a way to calculate the dot product using the components of the vectors. For the cross product, the method provides us with the resultant vector's components. So how do we do this ?

The first step, is to write the two vectors, and the unit vectors of the Cartesian coordinate system, in a 3 × 3 matrix like this:

$$\mathbf{A} = \begin{pmatrix} \hat{i} & \hat{j} & \hat{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{pmatrix}$$

After you have done this, taking the determinant of this 3×3 matrix will give you the resultant vector of the cross product which, according to Cramer's rule, is:

$$\mathbf{u} \times \mathbf{v} = \det(\mathbf{A}) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} = \hat{i}\begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix} - \hat{j}\begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix} + \hat{k}\begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix}$$

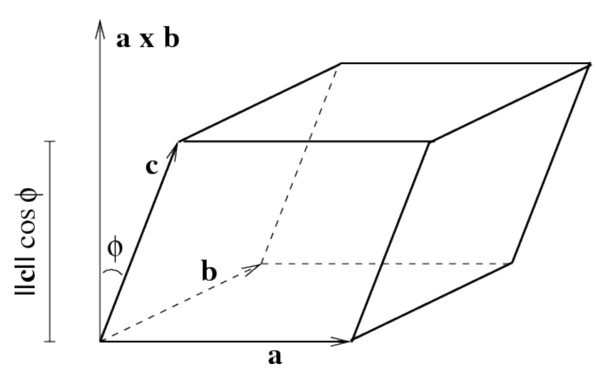
$$\boxed{\mathbf{u} \times \mathbf{v} = \hat{i}(u_2v_3 - u_3v_2) - \hat{j}(u_1v_3 - u_3v_1) + \hat{k}(u_1v_2 - u_2v_1)}$$

While I am aware that some teachers may choose a different approach to teach the cross product, in higher levels of math, this is how the cross product is calculated and all shortcuts come from this method. Any other method that you have learned is applying the same process but with other approaches.

1.7.3 Applications of The Cross Product: Magnetism, Torque, Parallelepipeds

Many applications of the cross product that you will come across will be in physics and even chemistry! However, before we get into those, let us explore the more mathematical application (although it is still applied in physics as well) which is finding the volume of a parallelepiped.

The first question that comes to mind is "What even is a parallelepiped ?" and I do not blame you for asking this question. Unless you have had a chance to take a course on Geometry or done some contest level mathematics, you probably have never encountered a parallelepiped.



The figure on the left is an example of a parallelepiped. It is a parallelogram base with sides that are parallelograms themselves. The formula for the surface area is fairly obvious. It is given by:

$$SA = 2||\mathbf{b}||\,(||\mathbf{a}|| + ||\mathbf{c}||) + 2\,(||\mathbf{a}||\,||\mathbf{c}||\cos\phi)$$

Where ϕ is the angle between vector $\mathbf{c}$ and vector $\mathbf{a} \times \mathbf{b}$ which is the vertical axis and hence we can use the dot product to define this angle:

$$\boxed{\phi = \arccos\left(\frac{\mathbf{c} \cdot (\mathbf{a} \times \mathbf{b})}{||\mathbf{c}||\,||\mathbf{a} \times \mathbf{b}||}\right)}$$

Note that $\arccos\theta$ is the same thing as $\cos^{-1}\theta$. Which one to choose is simply a matter of taste, however, in integration and differentiation, it is better to use arccos, arcsin and arctan as you will not mistake them for cosine to the power of negative 1 for example. The argument is that if you write $\cos^{-1}$ how should the reader know if you are discussing $\frac{1}{\cos\theta}$ or the inverse of the cosine function. Hence, from now on we will no longer use the $\cos^{-1}$ notation and will apply the more convenient notation. We will discuss this in slightly more detail in the notations section.

The volume of a parallelepiped is only slightly more complicated. We know that the idea behind volume formulae is finding the area of the base and multiplying it by the height. This is a very powerful interpretation. However, In order to apply it, we can use the property of a cross product. The volume of a parallelepiped is given by the following expression:

$$V = (\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}$$

This is also called the *triple scalar product* of the three vectors. At first, it is fairly obvious to solve to this; first determine the cross product of **a** and **b** and then dot product it with vector **c**. However, there is a quicker way which gives us the answer right away and, as you may have guessed already, it is through determinants!

Recall from the interlude that we can write a collection of vectors in matrices either vertically or horizontally. In this case, we can write all three vectors by components horizontally in a matrix like this:

$$\mathbf{V} = \begin{bmatrix} c_1 & c_2 & c_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{bmatrix}$$

Now, all we have to do to find the volume of the parallelepiped, is taking the absolute value of the determinant of this 3×3 matrix:

$$V = |\det(\mathbf{V})| = \begin{vmatrix} c_1 & c_2 & c_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

Which as you may recall is calculated through Cramer's rule:

$$V = c_1 \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} - c_2 \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} + c_3 \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}$$

$$\therefore V = |c_1(a_2b_3 - a_3b_2) - c_2(a_1b_3 - a_3b_1) + c_3(a_1b_2 - a_2b_1)|$$

When applying this formula, you can simply write the numbers in the 3×3 matrix and go from there.

Another application of the cross product is in Electromagnetic Theory. We can apply the cross product in order to determine the vector that describes the magnetic force on a moving charge. In grade 12 physics, we learn that given a magnetic field strength $\vec{B}$ and a charge q with velocity $\vec{v}$, the applied magnetic force is given by:

$$|\vec{F}| = q|\vec{v}|\,|\vec{B}|\sin\Theta$$

Or written in bold notation:

$$||\mathbf{F}|| = q||\mathbf{v}||\,||\mathbf{B}||\sin\Theta$$

However, now that we know about the cross product, we can replace $||\mathbf{v}||\,||\mathbf{B}||\sin\Theta$ with $\mathbf{v}\times\mathbf{B}$ which will return a vector. This allows for us to have both the magnitude of a vector and its direction. Hence the formula becomes:

$$\mathbf{F} = q\mathbf{v}\times\mathbf{B}$$

Or if we use the arrow notation for vectors:

$$\boxed{\vec{F} = q\vec{v}\times\vec{B}}$$

Finally, the torque of a dynamic system is another area in physics where the cross product comes in handy. In physics, we learn that torque is similar to linear force but in a rotating body. This quantity is represented by the Greek letter $\vec{\tau}$ (tau) and is calculated through the formula:

$$|\vec{\tau}| = |\vec{F}||\vec{r}|\sin\theta$$

Again, we can replace the expression with a simpler expression:

$$\boxed{\vec{\tau} = \vec{r}\times\vec{F}}$$

As you can see there are many many more applications of the cross product in all sorts of different areas. We will not go into those just know that it is very applicable.

This concludes the chapter on vectors. Now, that we have learned about vectors and are fairly confident in our ability to work with vectors, we will start working with slightly more complex geometrical objects such as lines and planes. These objects will outline the framework for an introduction to multivariable calculus and vector calculus. Near the end of the handout, I will do a brief introduction to tensors which can be described as a parent geometric object to vectors and scalars. However, we will first start by introducing lines and planes.

2 Lines and Planes

The entirety of this section is discussed in $\mathbb{R}^3$ (3D space) as a line can be simply represented by a linear function in $\mathbb{R}^2$ (2D space) and a plane is just a rectangle in a lower dimension. We will start by discussing lines, then taking the idea and applying to the concept of planes.

2.1 Lines

Let us approach this concept carefully. It is very simple but can get complicated as we are no longer in $\mathbb{R}^2$. But to better understand the topic, we will start by understanding a vector representation of lines and relating it to a linear function, then moving our way up to $\mathbb{R}^3$.

2.1.1 Representing lines in $\mathbb{R}^2$

Let us start by going all the way back to grade 9. The very first type of function that we learn about, and perhaps the simplest one, is a linear function of the form: $f(x) = mx + b$. The quantity that interests us the most about this function is m which is the slope. In 1 dimension, of course, a slope has no meaning. However, when we expand to 2 dimensions, we learn that there are different slopes. Any line that makes a non-zero angle with the x axis has a non-zero slope. We can prove this by recalling the slope is given by:

$$m = \tan(\theta) = \frac{f(x_1) - f(x_0)}{x_1 - x_0}$$

Where θ is the angle between the line and the x axis. Hence, for a non-zero theta, the slope will be a non-zero value. There is, however, another way to think of lines in $\mathbb{R}^2$ which is through vectors. We can think of a line as a vector whose magnitude is equal to the length of the line. So we can represent a line like this:

$$f(x) = x \langle m_1, m_2 \rangle + b$$

There is one small problem, however; multiplying the slope vector by a scalar value x will give us a vector, but we are adding a scalar to this vector which is not possible. Hence, we need a vector to be added to the equation instead of a y-intercept. Hence, the equation becomes a vector equation and takes the form:

$$\vec{f}(x) = x \langle m_1, m_2 \rangle + \langle b_1, b_2 \rangle$$

However, we commonly use different variables when it comes to lines in order to differentiate them from scalar functions or vector valued curves. Hence, we use the form:

$$\boxed{\vec{r}(t) = t \langle m_1, m_2 \rangle + \langle b_1, b_2 \rangle \quad \text{or} \quad \mathbf{r}(t) = t\, [m_1, m_2] + [b_1, b_2]}$$

2.1.2 Representing lines in $\mathbb{R}^3$

Lines in $\mathbb{R}^3$ are not so different than what we discussed in the previous section. As a matter of fact, all we need to do in order to represent a straight line in space is adding a z component to our vectors. Hence we get the general form of a line in space:

$$\vec{r}(t) = t\,\langle m_1, m_2, m_3\rangle + \langle b_1, b_2, b_3\rangle$$

Now, the question is whether we can come up with other representations of lines that would be useful in solving problems given only certain information about the line.

If we think about one of the earliest things we learned in this unit, components of vectors, we can define such a quantity for a line as well. If we consider the affect of each of these quantities incorporated into this equation for a line, we can see that, for example, the x component of $\vec{r}$ is never effected by the m_2 or m_3 components of the slope vector. Neither is it by b_2 or b_3. If we write $\vec{r}$ by its components, we get a set of equations that give us the results we want with less information:

$$\boxed{\begin{aligned} r_x &= tm_1 + b_1 \\ r_y &= tm_2 + b_2 \\ r_z &= tm_3 + b_3 \end{aligned}}$$

Which is of the exact same form of a scalar linear equation of a line in Cartesian coordinates: $y = mx + b$. This is called the *parametric* equation of a line which we can see is titled this way since the parameters of m and b are now used instead of the vectors.

Another representation is simply given by a manipulation of the parametric form. If we solve for t in all three equations, we get 3 different representations of t that all have the same form:

$$t = \frac{r_x - b_1}{m_1}$$
$$t = \frac{r_y - b_2}{m_2}$$
$$t = \frac{r_z - b_3}{m_3}$$

Since all of these quantities are equal, we can set them equal to each other which gives the *symmetric* equation of a line:

$$\boxed{\frac{r_x - b_1}{m_1} = \frac{r_y - b_2}{m_2} = \frac{r_z - b_3}{m_3}}$$

An important note is that there is a restriction when writing the symmetric form for the equation of a line. This restriction can be instantaneously seen since the slope vector's components are in the denominator of a fraction and we know that we cannot divide by 0. Hence, we get a restriction:

$$m_1, m_2, m_3 \neq 0$$

Again, this is only the case for a symmetric equation. Now, let us say that we face such a situation for an x component of the slope vector, as an example. This may occur with any component. How would we re-write our symmetric equation to account for this ? Here is how; we ask ourselves, what does it mean for a slope component to be zero ? It means that there is no change in the x component of our line. Hence, it is equal to a constant which happens to be b_1 in this case. Hence we can write our equation like this:

$$r_x = b_1, \ \frac{r_y - b_2}{m_2} = \frac{r_z - b_3}{m_3}$$

You may also be given a standard form of a line in $\mathbb{R}^2$ which is of the form $Ax + By + C = 0$ and be asked to state the generalization of this form for $\mathbb{R}^3$ in which case you must remember that a scalar function is of no use in $\mathbb{R}^3$ and that it has to be able to describe some type of vector. Hence there is no standard form of a line in space.

Many of the characteristics of a line are in its slope vector. For example, we can use this vector to find the angle between two lines using the following formula:

$$\vec{m_1} \cdot \vec{m_2} = |\vec{m_1}|\,|\vec{m_2}| \cos\theta$$
$$\implies \theta = \arccos\left(\frac{\vec{m_1} \cdot \vec{m_2}}{|\vec{m_1}|\,|\vec{m_2}|}\right)$$

Or we can use them to find a perpendicular vector to 2 lines using their cross product:

$$\vec{m_3} = \vec{m_1} \times \vec{m_2} = |\vec{m_1}|\,|\vec{m_2}| \sin\theta$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ m_{11} & m_{12} & m_{13} \\ m_{21} & m_{22} & m_{23} \end{vmatrix}$$

$$= (m_{12}m_{23} - m_{13}m_{22})\hat{i} - (m_{11}m_{23} - m_{13}m_{21})\hat{j} + (m_{11}m_{22} - m_{12}m_{21})\hat{k}$$

Now that we have explored the world of lines, let us think about what happens when we put a collection of lines together ? What geometric object is found through this thought experiment ?

2.2 Planes

To imagine a plane, we can simply think of a piece of paper. For the geometry that we will be working with, which is Euclidean geometry meaning that our coordinates and the space we work with is flat, we can think of planes as flat surfaces embedded in $\mathbb{R}^3$. There reason why we cannot picture a plane in $\mathbb{R}^2$ is because the entirety of 2D space is already expressed inside a plane. In order to picture an object that is intrinsically 2D, we must embed it into a higher dimensional space. We will discuss this in more detail in the section on tensors. Note that we cannot define a plane completely since this definition will be a recursive definition with use of other undefined objects in Euclid's Elements.

2.2.1 Standard and Vector Equation of a Plane

In order to define a plane mathematically, we will need the plane's normal vector which we will call $\vec{n} = \langle n_x, n_y, n_z \rangle$ and some point that we know is on the plane. Let us give this point coordinates (x_0, y_0, z_0). We know that if we pick any other point on the plane with coordinates (x, y, z), then the dot product of $\vec{n}$ and the vector connecting the first and the second point will have to be zero (due to their orthogonality). Hence we get:

$$\vec{n} \cdot \langle x - x_0, y - y_0, z - z_0 \rangle = 0$$

$$\boxed{\implies n_x(x - x_0) + n_y(y - y_0) + n_z(z - z_0) = 0}$$

This is the vector equation of a plane. Now by a little bit of re-arranging we get:

$$n_x x + n_y y + n_z z - (n_x x_0 + n_y y_0 + n_z z_0) = 0$$

We can call the third term D, and call $\vec{n}$'s components $\vec{n} = \langle A, B, C \rangle$ which then gives us the scalar equation of a plane:

$$\boxed{Ax + By + Cz - D = 0 \quad \text{or} \quad Ax + By + Cz = D}$$

We can see that given either one of these equations, we can immediately read off the components of the normal vector to the plane.

In order to come up with the equation of a plane, we may use 2 intersecting lines, 3 points, 1 point and a line, or 2 parallel lines. Given 2 intersecting lines, we can simply take their cross product to get the normal vector. This is also the same process with 3 points. However, this time we have to take a step before hand and get 2 lines using the given points. To do that we simply perform a subtraction to find the vector connecting the two points. Then we repeat this process but changing one of the point. This will give us 2 vectors on the plane which we can then take the cross product of to get the normal vector to the plane. It should be fairly obvious that to get D we simply substitute in the coordinates of 1 point on the plane and solve for D.

3 Advanced: Introduction to Vector Calculus

In this section, it is assumed that you are familiar with differentiation and basic integration.

3.1 Vector Valued Functions

Every since being introduced to functions, we have been dealing with scalar values. We have always defined a type of function which takes in a scalar value and maps it to a set of scalars, meaning, it returns a scalar value. However, who is to say that a function can only do that ? Why can we not have a function that returns a vector instead of a scalar ? This brings up the great subject of *vector calculus*. It is the study of the behaviors of vector valued functions. A vector valued function f is denoted by $\vec{f}(x)$. The general form of a vector valued function is:

$$\vec{r}(t) = f(t)\hat{i} + g(t)\hat{j} + h(t)\hat{k}$$

Where $g(x), p(x)$ and $q(x)$ are scalar valued functions. An example would be this:

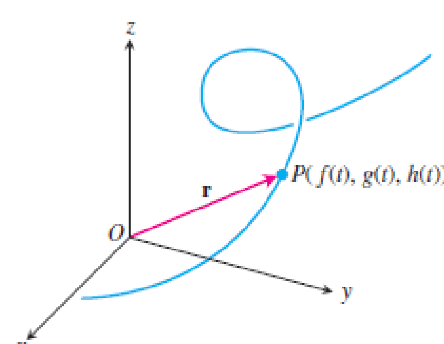
$$\vec{r}(t) = t^2\hat{i} - \cos(t)\hat{j} + e^t\hat{k}$$

As you may have guessed, if we sub in some point like $t = 2$, we get a vector with components:

$$\vec{r}(2) = 4\hat{i} + 0.42\hat{j} + 9\hat{k}$$

under the assumption that $e \approx 3$.

In an intuitive interpretation, a vector valued function takes a point and assigns a vector to it over its domain. A geometric visualization would be some curve in a 3 dimensional space like the figure on the left. Here, some point on the function will have the form $P(f(t), g(t), h(t))$ as shown in the figure. What is really happening is that when we sub in a value to $\vec{r}$ which gives us a vector whose tail is at the origin. Then we take the point at the end of this



vector and put it together with the end of other values of
the function to get a curve.

3.2 Differentiating and Integration Of Vector Valued Functions

Differentiating our new type of functions is very simple. First let us start by learning how we evaluate limits for vector functions. The actual calculation is quit simple. It follows this formula:

$$\boxed{\lim_{t\to a}\vec{r}(t) = \lim_{t\to a} f(t)\hat{i} + \lim_{t\to a} g(t)\hat{j} + \lim_{t\to a} h(t)\hat{k}}$$

But conceptually, what is happening is we are observing a vector function, or a curve, get infinitesimally close to becoming some vectors. Following the theorem stating that if two vectors are equal then their components must be equal, as the curve approaches the limit, the components must get infinitesimally close to the targeted vector's components. Hence, intuitively our formula makes sense. The same idea applies to differentiation. When we differentiate a curve, what we are doing is seeking the slope of its tangent vectors. If we have the components of that tangent vector, then we can use Pythagoras' theorem to calculate its slope. Hence, we most come up with a way to calculate the components of tangent vector. Similar to limits, we break the curve down into its components, and looking the rate of change of those components individually and then put them together. Hence, the rule for differentiation is:

$$\boxed{\frac{d}{dt}\vec{r}(t) = \frac{d}{dt}f(t)\hat{i} + \frac{d}{dt}g(t)\hat{j} + \frac{d}{dt}h(t)\hat{k}}$$

NOTE! Another notation for differentiation with respect to t ONLY, is to put a dot on top of the function of differentiation. Again, this is only valid when the variable of differentiation is t. An example would be:

$$\frac{d}{dt}x(t) = \dot{x}$$

This is known as Newton's dot notation. It appears very commonly in Classical Mechanics such as the Euler-Lagrange equation. We may also apply this to our function $\vec{r}(t)$:

$$\frac{d}{dt}\vec{r}(t) = \dot{\vec{r}} = \dot{f}\,\hat{i} + \dot{g}\,\hat{j} + \dot{h}\,\hat{k}$$

However, you will not see this notation very commonly in mathematics textbooks and will see it appearing more in physics. Now, that we have learned about differentiating a vector-valued function, let us explore integration of these functions. As we know, the integral means taking the anti-derivative. With ordinary scalar functions, all we had to do was think back wards; "What function's derivative gives me the function of integration ?" Similarly, we must think backwards again but apply it to each component separately. If differentiation of each components separately gives us the derivative, then integration of each component separately gives us the anti-derivative, a.k.a. the integral. Therefore:

$$\boxed{\int dt\,\vec{r}(t) = \left(\int dt\,f(t)\right)\hat{\imath} + \left(\int dt\,g(t)\right)\hat{\jmath} + \left(\int dt\,h(t)\right)\hat{k}}$$

Note, another notation we must get used to hear is seeing the differential right after the integral sign and before the function of derivative. They are basically the same thing! It is just a matter of preference. In mathematics, you will almost never see the differential in the beginning and see it right in the end. However, in physics, there are many who prefer to write the differential before the function and right after the integral sign. Again, it is just a matter of taste and preference. Just like using square brackets or round parenthesis for matrices, use which ever one you like!

$$\int \vec{r}(t)\,dt = \int dt\,\vec{r}(t) = \left(\int f(t)\,dt\right)\hat{\imath} + \left(\int g(t)\,dt\right)\hat{\jmath} + \left(\int h(t)\,dt\right)\hat{k}$$

3.3 Application of Vector Calculus

In this section, we will apply calculus of vectors to a typical physics problem; projectile motion. If you have ever taken physics, you know that there are five kinematic equations that govern the motion of particles in classical mechanics;

$$\Delta s = v_0\Delta t + \frac{1}{2}a\Delta t^2$$

$$\Delta s = v_1\Delta t - \frac{1}{2}a\Delta t^2$$

$$\Delta s = \frac{1}{2}\Delta t\,(v_0 + v_1)$$

$$v_1 = v_0 + a\Delta t$$

$$(v_1)^2 = (v_0)^2 + 2a\Delta s$$

However, we also know that quantities like velocity and acceleration are vectors! Let us use the first equation, but break it off into its components:

$$\Delta s_x = v_0\cos\theta\Delta t$$

$$\Delta s_y = v_0\sin\theta\Delta t - \frac{1}{2}g\Delta t^2$$

Note that this is for an idea projectile with no other acceleration or force acting on it. We will get to that too. Now that we have a vector in its components, we can use 1 function to represent it! A vector valued function $\vec{s}(t)$ can be used to model the motion of any ideal projectile in space:

$$\boxed{\vec{s}(t) = (v_0 \cos\theta)t\ \hat{\imath} + \left[(v_0 \sin\theta)t - \frac{1}{2}gt^2\right]\hat{\jmath}}$$

Now, let us say there is wind that is coming in some direction. This wind is represented by the vector $\vec{w} = \langle w_x, w_y \rangle$. To take this into account, we simply subtract the components from each other! So if there is wind, then the position function becomes:

$$\vec{s}(t) = [(v_0 \cos\theta)t - w_x]\,\hat{\imath} + \left[(v_0 \cos\theta)t - \frac{1}{2}gt^2 - w_y\right]\hat{\jmath}$$

There are many many more applications of vector calculus. A great source for furthering studying of the subject is *Calculus: Early Transcendentals* by James Stewart.