A Journey Through Differential Equations

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A Note From The Author

Good day, this handout was started as a guide to University of Toronto's MAT234: Differential Equations. The reader is assumed to be familiar with concepts from applied linear algebra along with single variable and multivariable calculus. Special thanks to $\mathcal{J}$ake for motivating me to make this handout and helping me with grammar so we can actually use it, thanks to $\mathcal{J}$ulian for reminding me what I got in high school calculus, and thanks to $\mathcal{J}$asmine for tolerating my stupid jokes. Props to the $\mathcal{J}$s!

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1 Introduction to Differential Equations

We start our journey with the question; "What is a differential equation ?" To answer this, we must first understand properly, what is a derivative and its significance.

As we recall from our ordinary first year calculus, a derivative quantifies the slope of the tangent line to a curve with respect to some variable. In the case of 3 or higher dimensions, the partial derivatives quantify this slope with respect to only 1 of the variables.

We also know, both by intuition and definition, that the slope of this tangent line quantifies the rate of change of our function. Thus, derivatives allow us to model change in a quantity of interest. What, then, is the power of differential equations if we can use derivatives ? The issue is that we do not always know what the function looks like. Unless you are in an exam scenario for calculus, you will almost never know the function, but you will know some characteristics of that function. All of our empirical observations guide our scientific discovery, and some of the easiest empirical observations is change of a quantity. Almost anyone can measure how temperature changes in a room, or how fluid velocity changes from one point to another, but pretty much no one knows what exactly is the function whose derivatives perfectly describe this quantity.

Thus, to understand differential equations, is to understand how quantities change, and how we can discover the nature of such quantity, through simple empirical observations, and some very hard core mathematical gymnastics.

This is the calculus picture of differential equations. There is another less thought of way to think about differential equations. This is through the language of linear algebra. If we recall the idea of

linear transformations, we would know that they are simply "maps" which obey linearity conditions:

$$T : \mathbb{R}^n \to \mathbb{R}^m \text{ is linear } \iff T(c \cdot x) = c \cdot T(x) \quad \text{and} \quad T(x \pm y) = T(x) \pm T(y)$$

for some $c \in \mathbb{R}$ and $x, y \in \mathbb{R}^n$. Now, let us observe our ordinary derivative rules. Suppose we have two functions, f and g both dependent on some $x \in \mathbb{R}$. We know that the following derivative rules are true:

$$\frac{d}{dx}(c \cdot f) = c\frac{df}{dx} \qquad \frac{d}{dx}(f \pm g) = \frac{df}{dx} + \frac{dg}{dx}$$

Then we can say by inspection that the derivative is indeed a linear transformation. It is the simplest form of a differential operator (a matrix which involves the derivative operator in some of its elements). One can indeed say, that in general a differential equation can be expressed as the following format:

$$D\mathbf{x} = \mathbf{y}$$

Where D is the differential operator, $\mathbf{x}$ is the function which satisfies this differential equation and $\mathbf{y}$ is the image (from linear algebra) of the differential equation. Later on, we will also see that differential equations have eigenvalues, eigenvectors and eigenbasis. It is easy to get lost in the dust with the beauty (and difficulty) of the topic of differential equations, and thus, do not rush with the topics. DO NOT! Take your time, read through everything carefully and make sure you properly understand everything for yourself and not just nodding along. Try to really show that every theorem, claim or definition makes sense. This topic is the alphabet of engineering and the physical sciences. Understanding differential equations is to understand the universe and pretty much all engineering phenomena at their core. We will begin this pursuit with the simplest type, and keep building on it. So buckle up! (By the way, buckling also has a differential equation!)

2 Separable Equations

Differential equations share many characteristics to algebraic equations. Similar to how we characterize algebraic equations, we can categorize differential equations. Identifying the category of a DE will indeed help us skip over the trouble of figuring what exactly is the correct approach and take us directly to grinding the mechanics of the solution.

2.1 The Gymnastics

We will start our journey with Ordinary Differential Equations (ODEs). Hopefully we remember, that an ordinary differential equation is an algebraic equation relating the derivatives of a function to itself. The equation must involve at least one derivative of the function. There are many types of differential equations. We will introduce them by categorizing them and seeing that there are families of equations which allow us to use the same algorithm to solve them. The first of these are separable equations that can be written in the form:

$$g(y)y' = f(x)$$

Where $y(x)$ is the function we want to solve for, y' is its first derivative, and $f(x)$ is what is called a forcing term. The term $g(y)$ is any coefficient of y' which only depends on y and not directly on x. These equations are rather simple to solve. We simply apply integration such that:

$$\int g(y)\,dy = \int f(x)\,dx$$

Let us explore some examples. Take the equation $y' - y = 0$. In this example, we have that $f(t) = 1$ and $g(y) = 1/y$. We can rearrange this equation to:

$$\frac{dy}{dt} = y \implies \frac{1}{y}\frac{dy}{dt} = 1 \implies \ln y = t + c_0 \implies y(t) = Ae^t;\ A \in \mathbb{R}^+$$

Of course the step of having dt isolated on one side is slightly non-rigorous but it helps with understanding. Another example is:

$$\frac{1}{1+y^2}y' = \frac{t\sin(t)}{y}$$

We can re write this equation as:

$$\frac{y}{1+y^2}dy = t\sin(t)\,dt \implies \int \frac{y}{1+y^2}\,dy = \int t\sin(t)\,dt \implies y = Ae^{\frac{1}{4}(-t\cos t+\cos t)} - 1; A \in \mathbb{R}$$

2.2 Initial and Boundary Value Problems

In the last part, we saw how we can obtain the form of the solution to a separable differential equation. But there was always an annoying A that never left. This is because we never specified any initial values. These are conditions we put on our solution, in order to go from a family of solutions to a single solution. Take the first differential equation as an example, had we put the condition $y(0) = 2\pi$, then we must have that $Ae^0 = 2\pi \implies A = 2\pi$. And thus, instead of a family of exponentials, we would only have $y(t) = 2\pi e^t$ as the only solution. We may even have conditions on the function's derivatives. For example, we could have had $y'''(0) = 3$ and that would give us $y(t) = 3e^t$.

2.3 Implicit Solutions

Consider the following differential equation:

$$\frac{dy}{dt} = \frac{t^2 - 1}{y^2 + 1}$$

We can see immediately that this is a separable equation. So let us solve for the solution:

$$\int y^2 + 1\, dy = \int t^2 - 1\, dt \implies \frac{1}{3}y^3 + y = \frac{1}{3}t^3 - t + C$$

Wait a second, we applied our method, we said it is guaranteed to work, but it did not! Why is that ? In fact, the solution did work. It is simply the fact that the algebra of our solution is too messy. Thus, what we do is leave it as such. If we recall from implicit differentiation, we often had situations where we could differentiate an equation's dependent variable with respect to its independent variable, without actually isolating for it. This is where it comes to use. We can see the behavior of $y(t)$ without actually solving for $y(t)$. One use case of this to give it to numerical solvers. This is now only an algebraic solution which is significantly easier for the solver to deal with than a differential equation. So if you ever see implicit form, leave it at that and call it a day.

3 Exact ODEs and The Integrating Factor Method

We have now pretty much handled the simplest type of differential equation. So without further ado, let us dive into a more interesting type. One where the solution is not as straight forward.

3.1 Homogeneous First Order Ordinary Differential Equations

I know, the title is quite a mouthful. So let us examine it step by step. What does homogeneous mean ? Consider the matrix form of a differential equation:

$$D\mathbf{x} = \mathbf{y}$$

Consider the scenario where $\mathbf{y} = 0$. This gives us an ODE of the form $D\mathbf{x} = 0$. This is the definition of the homogeneous equation. When there are absolutely no terms independent of our function or its derivatives. An example is:

$$\frac{dy}{dt} - 3ty + y^2 = 0$$

This ODE does not consist of any terms that only contain t. They all have some dependence of y. A simple way to think about it, is if we put all the y terms on the left and all the t terms on the right, the right hand will be 0. So that is the definition of homogeneous. In linear algebra, it would be all equations where we are solving for the kernel of a differential operator D.
What about first order ? A first order ODE is one where the highest order derivative of y involved is 1. So the equation would only contain a first derivative of y or its first derivative along with itself. An example of a first order equation is:

$$\frac{dy}{dt} - y = 5t \sin t$$

Lastly, we know that the term *ordinary* means that we are strictly dealing with single variable derivatives and there are no partial derivatives involved. So let us combine these together. Some examples of homogeneous first order ODEs are Newton's cooling law and Kirchhoff's voltage law for transient first order linear circuits.

3.2 Bernoulli's Equation and Its Solution

In the last section, we looked at differential equations of the form:

$$D\mathbf{x} = 0$$

the solution of such a differential equation will lie in the kernel of our differential operator. So $\mathbf{x} \in \ker(D)$. We will come back to the power of this picture when we discuss Laplace transforms and generalized Integral transforms. For now, let us expand on our current knowledge. Now, consider all differential equations, where D only contains 0th and 1st derivative operators. But now, we consider non-homogeneous equations. Consider the following differential equation:

$$\frac{dy}{dt} + p(t)y = 0$$

One can immediately tell that the solution is of the form:

$$y(t) = C_0 e^{-\int P(t)\,dt} + C_1$$

But now, let us consider a non-homogeneous equation of the form:

$$\frac{dy}{dt} + p(t)y = q(t)$$

How would we approach this ? This is a lot less intuitive. one cannot really isolate for functions of t and y. In fact, this is not a separable equation. So how would we approach it ?
Let us recall the product rule from differentiation:

$$\frac{d}{dt}[v(t)w(t)] = v'(t)w(t) + v(t)w'(t)$$

What does this tell us ? Normally, we would say that the derivative of two functions' product is the sum of each function's first derivatives times the other function. But now, let us look at it differently. This is telling us that if we ever see the combination $v'w + vw'$, we can turn it into $(vw)'$. How is this of any use, you may ask ? Let us go back to our differential equation. Suppose find some function $v(t)$ and multiply to both sides of our differential equation. What is special

about this $v(t)$? We define $v(t)$ as any function that has the following property:

$$v(t)p(t) = v'(t)$$

For example, if $p(t) = \sin(t)$, then we choose $v(t) = e^{\cos t}$ which means $v'(t) = e^{\cos t} \cdot \sin t = v(t)p(t)$. And this is precisely how we use the magic of this function. We multiply both sides of our differential equation with this mysterious function:

$$v(t)y'(t) + v(t)p(t)y = v(t)q(t)$$

Now, we defined v such that $vp = v'$. Then we can replace the coefficient of y with v':

$$\implies v(t)y'(t) + v'(t)y(t) = v(t)q(t)$$

Do you see the trick ? Now, the left hand of our differential equation is the combination of one function's derivative multiplied by the other function! This is indeed the definition of the product rule!

$$\frac{d}{dt}[v(t)y(t)] = v'(t)y(t) + v(t)y'(t) \implies [v(t)y(t)]' = v(t)q(t)$$

This powerful method allows us to isolate for $y(t)$ by integrating and then dividing by $v(t)$ (yes there is a subtlety here that we will get to).

$$\implies y(t) = \frac{1}{v(t)} \int v(t)q(t)\,dt$$

Now, let us replace $v(t)$ with a function of $p(t)$. We know from how we defined v that:

$$p(t) = \frac{v'(t)}{v(t)} \implies v(t) = e^{-\int p(t)\,dt}$$

We already knew this from separable differential equations! Then, we may replace $v(t)$ with the above expression to get:

$$y(t) = \frac{1}{e^{-\int p(t)\,dt}} \int e^{-\int p(t)\,dt} q(t)\,dt$$

I know that this looks scary but let us do a few examples and we'll get used to it. The ODE we just solved is known as Bernoulli's equation. There are many Bernoulli-s. This is one of them.

Consider the following differential equation:

$$4y' - 2\,y = \frac{1}{2}\sin t$$

To solve it, and make our life easy, let us first find $v(t)$. This function is called the integrating factor. To find it, we use the equation

$$v(t) = e^{-\int p(t)\,dt}$$

But there is a little issue. The coefficient of y' is not 1. It is 4. So we must divide the whole equation by 4 in order to account for this and fid the correct $p(t)$ and $q(t)$. So we re-write our equation as $y' - \frac{1}{2}y = \frac{1}{8}\sin t$. Now, we may use the formula:

$$v(t) = e^{-\int \frac{-1}{2}\,dt} = e^{\frac{1}{2}t}$$

Next, we jump straight into the solution:

$$y(t) = \frac{1}{v(t)}\int v(t)q(t)\,dt = \frac{1}{8e^{t/2}}\int e^{t/2}\sin t\,dt$$

We know from our elementary calculus knowledge that this integral can be solved by parts. I'll leave that as an exercise for you. The answer would be

$$y(t) = \frac{1}{8e^{t/2}}\cdot\frac{-4}{5}e^{t/2}\cos t + \frac{1}{8e^{t/2}}\frac{2}{5}e^{t/2}\sin t = \frac{1}{20}\sin t - \frac{1}{10}\cos t$$

But be careful. These integrals can get very complicated, very fast. just imagine if we had $p(t) = 2t$. Something as simple as that would lead to a gaussian integral involving sines. This would require far more advanced techniques of integration.

Before we move on, let us make a quick observation. We already know that our function satisfies the differential equation. But what about its individual components ? If we plug those individual components in, we actually find that they both satisfy the differential equation! So we have a theorem: If $y(t) = y_1(t) + y_2(t) + \ldots y_n(t)$ is a solution, then each $y_i(t)$ on its own is also a solution to the differential equation. Keep this in mind as it will come back later.

Using this, is there a different approach to Bernoulli's equation? Consider the previous differential

equation:

$$y' - \frac{1}{2}y = \frac{1}{8}\sin t$$

Let us ignore the forcing term on the right hand. For this new differential equation, $y' = y/2$, we can easily find that $y = e^{t/2}$. Now, obviously this solution by itself is not a solution to the original ODE. But, what if we add this solution to our original solution with sinusoids ? If we say:

$$y(t) = Ae^{t/2} + C_1 \sin t + C_2 \cos t$$

The exponential component will always cancel out, and the sinusoidal part will always satisfy the non-homogeneous equation. Thus, we can say that this function is the full solution of our differential equation. The solution of the homogeneous equation is known as the particular solution, $y_p(t)$ and the solution of the non-homogeneous equation is known as the complementary solution $y_c(t)$. In general, for the solution of a differential equation $D\mathbf{x} = \mathbf{y}$, we must have that:

$$\mathbf{x} = \mathbf{x}_c + \mathbf{x}_p \quad s.t. \quad \mathbf{x}_p \in \ker(D), \;\; D\mathbf{x}_c = \mathbf{y}$$

Note that this is true for every ordinary differential equation, not just the example we did! This result is known as Abel's Theorem.

3.3 Solving Exact Ordinary Differential Equations

In this section, we will learn how to solve almost all ODEs that are homogeneous and first order. These are called, exact ODEs. What does an "exact" equation mean ? It is any type of differential equation that can be written in the following form:

$$M(x, y)\, dx + N(x, y)\, dy = 0$$

By this definition, any (potential) function that can satisfy the following identity is the solution to the ODE:

$$dF = M(x, y)\, dx + N(x, y)\, dy$$

Following this definition, we can say an ODE is exact if it satisfies that:

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Why is that ? Because then, by definition, we must satisfy that the second partial derivatives of any function are equal to each other. How would we approach solving these equations ? We know that if we have an ordinary (first order) equation of the form:

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0$$

then we have an exact ODE. We also know, that the functions M and N must satisfy that: $M_y = N_x$. Suppose we find a mysterious function $\varphi(x,y) : \mathbb{R}^2 \to \mathbb{R}$ which satisfies:

$$\begin{cases} \frac{\partial \varphi}{\partial x} = M(x,y) \\ \frac{\partial \varphi}{\partial y} = N(x,y) \end{cases}$$

Then we can re-write the form of our ordinary differential equation as the following:

$$\frac{\partial \varphi}{\partial x} + \frac{\partial \varphi}{\partial y}\frac{dy}{dx} = 0$$

Now, it may look like we made things worse by introducing partial derivatives. But, if we look just a little closer, we can see a pattern. Remember the multivariable chain rule ? We have that:

$$\frac{d}{dx}[f(x,y)] = \frac{\partial f}{\partial x}\frac{dx}{dx} + \frac{\partial f}{\partial y}\frac{dy}{dx}$$

If we observe, just a little closely, we can see that we have the exact equation here! Because we have that:

$$\frac{d}{dx}[\varphi(x,y(x))] = \frac{\partial \varphi}{\partial x}\frac{dx}{dx} + \frac{\partial \varphi}{\partial y}\frac{dy}{dx}$$

The derivative of x with respect to itself is just 1, so this reduces to our ordinary differential equation! Thus, we can re-write our ODE as:

$$\frac{d}{dx}\varphi(x,y(x)) = 0 \implies \varphi(x,y(x)) = c$$

If we recall from multivariable calculus, given some vector field $\mathbf{F} : \mathbb{R}^2 \to \mathbb{R}^2$, one of the classic exercises was, given that this vector field is a gradient field (meaning it is conservative), find a potential function, where had to solve the equation:

$$\mathbf{F} = \nabla f \implies f_x = F_1 \qquad f_y = F_2$$

We then had that $f = \int F_1 \, dx = g(x, y) + h(y)$ and $f = \int F_2 \, dy = j(x, y) + k(x)$. After this, we looked for the common terms and gathered it all together to find f. This is a similar problem. Once we find the potential function, $\varphi(x, y(x))$, we set it equal to a constant (like finding the general form of the isosurfaces) and then solve for $y(x)$. Consider the following ODE:

$$-\alpha x e^{-\alpha t} + e^{-\alpha t} \frac{dx}{dt} = 0$$

First, we must check whether this is exact or not. One can see that $M(t, x) = -\alpha x e^{-\alpha t}$ and $N(t, x) = e^{-\alpha t}$. We can easily see that the identity $\partial_t M = \partial_x N$ is satisfied. So this is an exact ODE. So, let us find the potential function. We know that:

$$\begin{cases} \partial_t \varphi = -\alpha x e^{-\alpha t} & \implies \varphi(t, x) = \int -\alpha x e^{-\alpha t} \, dt = x e^{-\alpha t} + f(x) \\ \partial_x \varphi = e^{-\alpha t} & \implies e^{-\alpha t} + f'(x) = e^{-\alpha t} \end{cases}$$

Using the second equation, we have an expression for $f(x)$:

$$e^{-\alpha t} + f'(x) = e^{-\alpha t} \implies f(x) = C_0$$

This gives us the final form for the potential function of our equation:

$$\varphi(t, x) = x e^{-\alpha t} + C_0$$

Now, all we have to do is find the isoline of our potential function and it gives us the solution. We will combine the constant of the curve with C_0 so that we have $\varphi = C_1$ and $C = C_1 - C_0$. Then:

$$x e^{-\alpha t} = C \implies x(t) = C e^{\alpha t}$$

And that is the solution to our original differential equation! Now, after learning to solve Bernoulli's equation, this may seem like a bit of an over kill. However, this method is an extremely powerful tool both for more complicated equations, and for when we want to solve an equation numerically. You may think that we are done, but no! Sometimes, we can turn ODEs into exact form. For example, consider the following differential equation:

$$(k+T)e^x + xe^x \frac{dT}{dx} = 0 \quad s.t. \quad T(1) = T_0$$

We can consider it in the form:

$$M(x,T) = (k+T)e^x \quad N(x,T) = xe^x$$

But these two do not satisfy the exactness condition since $M_T \neq N_x$. So what can we do ? This gives rise to the idea of the integrating factor. This is where it really comes from. The integrating factor transforms a non-exact equation into an exact one. The idea is that there exists some function, $\mu(x,y)$ such that it satisfies:

$$\frac{\partial(\mu M)}{\partial y} = \frac{\partial(\mu N)}{\partial x}$$

and this definition is exactly how we calculate it! So how do we go about it in this case ? We have:

$$\frac{\partial}{\partial T}\mu M = \frac{\partial \mu}{\partial T}M + \mu\frac{\partial M}{\partial T} \qquad \frac{\partial}{\partial x}\mu N = \frac{\partial \mu}{\partial x}N + \mu\frac{\partial N}{\partial x}$$

Setting these equal to each other gives the following equation:

$$\frac{\partial \mu}{\partial T}M - \frac{\partial \mu}{\partial x}N = (N_x - M_T)\mu$$

Now, this is a *partial differential equation*. These types of differential equations are even more complicated than ordinary equations. So the trick here is to make life simple. First, let us consider the case where μ is only dependent on x. Then, this PDE simplifies to the following ODE:

$$\frac{d\mu}{dx} = \frac{M_T - N_x}{N}\mu \implies \mu(x) = \exp\int \frac{M_T - N_x}{N}\,dx = \exp\int \frac{e^x - e^x - xe^x}{xe^x}\,dx = e^{-x}$$

Now, let us repeat the process but this time, assume μ is only T-dependent:

$$\frac{d\mu}{dT} = \frac{N_x - M_T}{M}\mu \implies \mu(T) = \exp\int \frac{e^x + xe^x - e^x}{(k+T)e^x}\,dT = (k+T)e^x$$

Since this assumption led to a result that is still dependent on T, it is not a valid integrating factor. So the correct integrating factor is indeed $\mu(x,T) = e^{-x}$. That is our integrating factor. Now that we actually found this factor, we will go back and readjust our differential equation with the following changes:

$$\begin{cases} M \to \tilde{M} = \mu M \\ N \to \tilde{N} = \mu N \end{cases} \implies \tilde{M} + \tilde{N}\frac{dT}{dx} = 0$$

and the potential function we get is:

$$\begin{cases} \frac{\partial\varphi}{\partial T} = k + T \implies \varphi(x,T) = kT + \frac{1}{2}T^2 + f(x) \\ \frac{\partial\varphi}{\partial x} = x \implies f'(x) = x \implies f(x) = \frac{1}{2}x^2 \end{cases}$$

And so our potential function is:

$$\varphi(x,T) = kT + kT^2/2 + x^2/2$$

Now, the solution of our differential equation is the isolines of this surface:

$$\varphi(x,T) = C_0 \implies T = \sqrt{C + k^2 - x^2} - k$$

3.4 Non-linear Equations: Reduction to Linear Form

So far, we have been dealing with Ordinary Differential Equations of first degree. This means that the highest power of y in our ordinary differential equations has been 1. In this chapter we will explore first order non-linear equations. Suppose have a non-linear differential equation of the form:

$$y' + g(x)y = p(x)y^\alpha$$

How can we deal with this ? The trick, as it turns out, is to make a substitution. We can define a new function, that will change the form of the ODE to a linear equation. This is called reduction to linear form. We will then solve for that function and use that solution to find the true expression of the original function. The function substitution for ODEs of the above form is $u(x) = [y(x)]^{1-\alpha}$. Why ? Well, let us see what happens when we differentiate this:

$$u' = (1-\alpha)y^{-\alpha}y' = (1-\alpha)y^{-\alpha}(py^{\alpha} - gy)$$

Some simplifying gives us:

$$u' = (1-\alpha)(p - gy^{1-\alpha}) = (1-\alpha)(p - gu)$$

This gives the following differential equation for u:

$$u' + (1-\alpha)g(x)u = (1-\alpha)p(x)$$

We already know how to take care of this! Our integrating factor is simply:

$$v(x) = \exp\left((\alpha - 1)\int g(x)\,dx\right)$$

and our solution is just:

$$u(x) = u_p(x) + \frac{1}{v(x)}\int v(x)(1-\alpha)p(x)\,dx$$

We then use this to solve for the original function, $y(x) = [u(x)]^{1/1-\alpha}$.

4 Picard-Lindelöf Theorem

Before we start, just a note that this may also be called the Existence and Uniqueness Theorem, Picard's Existence Theorem or the Cauchy-Lipschitz Theorem. Consider a differential equation for the function $y(t)$. Suppose we have some rectangular region, D, around a point (t_0, y_0) bounded

by $[t_0 - \varepsilon, t_0 + \varepsilon]$ and $[y_0 - b, y_0 + b]$. Consider the following initial value problem:

$$\frac{dy}{dt} = f(t, y(t)) \quad y(t_0) = y_0$$

where dy/dt and $\partial f/\partial y$ are continuous on D. Consider a new region around our point, R which is bounded by $[t_0 - a, t_0 + a]$ where $a < \varepsilon$. So essentially R is a region with a smaller width than D, and they are both centered around (t_0, y_0). What the existence and uniqueness theorem says is the following; There is always a smaller region R for which the IVP has a unique solution.
Think about this theorem like the Intermediate Value Theorem. It allows us to show that there exists a single unique solution to an IVP without actually finding one. The key here is showing that $\partial f/\partial y$ exist on the interval and is continuous. Consider the following IVP:

$$y' = t + y \quad y(0) = 1$$

We observe that $f(t, y(t)) = t + y$. This is continuous everywhere. We have that $\partial f/\partial y = 1$. Which is also continuous. So we have by existence and uniqueness theorem, that for all regions, D for which $(0, 1) \in D$, there exists a unique solution to the IVP.
Let us try this for a more interesting IVP such as the following:

$$y' = y^{2/3} + t^2 y \quad y(0) = 0$$

We can see that $f(t, y) = t^2 y + y^{2/3}$. We can see that $y^{2/3}$ is continuous for all real numbers. But we find that

$$\frac{\partial f}{\partial y} = t^2 + \frac{2}{3} y^{-1/3}$$

This is not continuous everywhere. The second term has a discontinuity at $y = 0$ which is the initial value. What does this mean ? We know that $f(t, y)$ exists so the solution does exist. However, this discontinuity means that the solution is not unique. In fact there are families of solutions to this IVP. You will see that any family of functions of the form:

$$y(t) = \begin{cases} 0, & x \leq 0 \\ \left(\frac{x-c}{3}\right)^3 & x > 0 \end{cases}$$

are all solutions to our IVP. So, the solution is not unique.

5 Higher Order Equations

So far, we have had a long discussion on ordinary differential equations involving some function $y : \mathbb{R} \to \mathbb{R}$ and its first derivative, dy/dx. These are all differential equations of the first order. Now, we will extend the ideas we have had so far in order to accommodate differential equations of the second order, where the second derivative of our function is also present. For example, the following equation is a second-order equation:

$$y'' + 3ty' - y = 0$$

It involves, at least, the second derivative of y. Solving second order ODEs is a little more heavily dependent on using an *ansatz*. This words means a guess. When we make an educated guess (based on our elementary knowledge of calculus) we call it an ansatz, which is the German word for guessing. I know you're excited, so I won't keep you waiting any longer!

5.1 Homogeneous Second Order Equations

First, we will approach solving second order homogeneous equations, similar to how we did with first order equations, and then we will define a methodology for generalized second-order equations. Remember, once again, that a differential equation, is just a differential operator, D, acting on the solution and returning an expression:

$$D\mathbf{x} = \mathbf{y}$$

And solving a homogeneous equation is equivalent to solving for the Kernel or Null Space of our differential operator. Consider a second order homogeneous equation of the form:

$$y''(t) + ay'(t) + by(t) = 0$$

Here is how we approach solving it; this equation tells us that we have a function, whose derivatives add and subtract to 0. Thus, we must have that the derivatives of the function look very similar, if not identical, to the original function. What is a function we know that looks like that ? Yes, it

is the exponential. So as an ansatz, we will substitute in the guess $y = e^{rt}$. Let us substitute this into our equation:

$$r^2 e^{rt} + are^{rt} + be^{rt} = 0$$

Getting rid of the exponential terms leaves us with a polynomial equation:

$$r^2 + ar + b = 0$$

This is called the characteristic polynomial or characteristic equation of our differential equation. The values of r that satisfy this polynomial equation, are those that will allow our ansatz to satisfy the differential equation. We can see that the solutions to r are:

$$r = \{a/2 + \sqrt{a^2 - 4b}/2,\ a/2 - \sqrt{a^2 - 4b}/2\}$$

So the solution to our differential equation is:

$$y(t) = e^{a/2} \cdot \left(e^{\frac{\sqrt{a^2-4b}}{2}} + e^{-\frac{\sqrt{a^2-4b}}{2}} \right)$$

now, as we may re-call, sometimes solutions to polynomial equations are not strictly real numbers. In fact, r may turn out to be a complex number! What would that mean for our solution ? Currently, we have 2 exponential functions. Now, suppose that our r values are $r = r_1 \in \mathbb{R}$ and $r = r_2 \in \mathbb{C}$. Then, what would r_2 look like ? It would look something like this: $r_2 = a + ib$. Then, what would the solution look like? It would be:

$$y(t) = e^{r_2 t} + e^{(a+ib)t}$$

Now, the second term looks rather interesting. We can break it up into: $e^{at}e^{ibt}$. However, remember that we have Euler's formula for complex exponentials:

$$e^{i\theta} = \cos\theta + i\sin\theta$$

This is a very important formula. It is the underlying foundation of Fourier series and the Fourier transform which we will visit later. So we simplify the term with the imaginary number in the

exponent to a sum of sines and cosines!

$$y(t) = e^{r_2 t} + e^{a}(\cos(bt) + i\sin(bt))$$

Do not let the i bother you. Later on, I will introduce a much more intuitive and easier way to think about i than just the square root of negative one. All this means is that there is a linear combination of sines and cosines with exponential amplitudes and an exponentially increasing mean value. This is how we generally approach homogeneous equations. However, remember that there is also the trivial solution of 0, so you must always include constant factors.

5.2 Non-Homogeneous Equations

We have so far explored differential equations that are homogeneous. However, second-order equations are not always so nice. For example, in lots of vibration problems, we see systems that have an external forcing term acting on the system, making the equations of motion non-homogeneous. We learned earlier, from Abel's Theorem, that the solutions of a non-homogeneous still involve the solution of the corresponding homogeneous equation, so we will still need to solve that. But now there is a second linearly independent solution due to the forcing term. These are equations of the form:

$$ay'' + by' + cy = f(t)$$

Where $f : \mathbb{R} \to \mathbb{R}$ is assumed to be smooth and $(a, b, c) \in \mathbb{R}^3$ such that $a \neq 0$ (to ensure that our equation is second-order). So how do we solve these ? We will learn 2 approaches: Method of Undetermined Coefficients and Variation of Parameters. For now we will stick to constant coefficients, for which we will use method of undetermined coefficients. This method heavily relies on a good ansatz. Suppose we are trying to solve the above differential equation. For each type of $f(t)$ our approach to an ansatz will be slightly different. The simplest is when $f(t)$ is an exponential. For exponentials, we will guess that $y = f(t)$. Then, substitute this into our equation to find the coefficients, a, b, c. For example,

$$y'' - 3y' - 2y = e^t$$

Our ansatz for the particular solution (the one for the non-homogeneous equation) is $y_P = Ae^t$. Then, we substitute this into our equation and find the value of A for which the equation is satisfied:

$$Ae^t - 3Ae^t - 2Ae^t = e^t \implies A = \frac{-1}{4}$$

You can check to see that the complementary solution (for the homogeneous equation) is

$$y_C = e^{3t/2}\left(e^{-\sqrt{17}t/2} + e^{\sqrt{17}t/2}\right)$$

So the final solution is:

$$y = y_C + y_P = C_1 e^{3t/2}\left(C_2 e^{-\sqrt{17}t/2} + C_3 e^{\sqrt{17}t/2}\right) - \frac{1}{4}e^t$$

Now, suppose we were solving the following equation:

$$y'' - 3y' + 2y = e^t$$

Here, when we make our ansatz, we get that $A = 0$. What does this mean ? The math is telling us that this is not a linearly independent solution form the complementary solution, it is already contained in the solution to the homogeneous equation. In this case there are 2 things you can do; either decide that it makes sense and move on to finding the complementary solution or make a different ansatz to see if it works. In the end, the way to make this decision is purely intuition based, which comes from lots and lots of practice with ODEs. If you were to make a different ansatz, you want to make sure that it is linearly independent, so you can try something like $y = Ate^t$, which for this case you will quickly realize would not work and then move on.

For the case where $f(t)$ is a some sinusoid or a variant of a sinusoid, we would make a similar guess, along with the shifted sinusoid. For example, if $f(t) = \sin(3t)$ then our guess would be $y_P = A\sin(3t) + B\cos(3t)$. If it were a variant, a more complicated one like $f(t) = t\sin(2t)$, then we adjust accordingly; $y_P = At\sin(2t) + Bt\cos(3t)$. This one is similar to the previous case so I will only leave a few exercises on it at the end and won't do an example.

The case where $f(t)$ is a polynomial is an interesting case. Consider the following equation:

$$y'' + y' - 6y = 2t^3 - t + 3$$

Here, f is a third degree polynomial. This means, our ansatz for y_P is a general cubic polynomials. So we must guess that $y_P = At^3 + Bt^2 + Ct + D$ and then we must solve for all of these coefficients. It is not as bad as it looks, it will actually simplify quite a lot when we substitute it in:

$$6At + 2B + 3At^2 + 2Bt + C - 6At^3 - 6Bt^2 - 6Ct - 6D = 2t^3 - t + 3$$

$$\implies (-6A)t^3 + (3A - 6B)t^2 + (6A + 2B - 6C)t + (-6D + C) = 2t^3 - t + 3$$

Once we match the coefficients we get the following linear system:

$$\left(\begin{array}{cccc|c} -6 & 0 & 0 & 0 & 2 \\ 3 & -6 & 0 & 0 & 0 \\ 6 & 2 & -6 & 0 & -1 \\ 0 & 0 & 1 & -6 & 3 \end{array}\right) \rightarrow \left(\begin{array}{cccc|c} 1 & 0 & 0 & 0 & -1/3 \\ 0 & 1 & 0 & 0 & -1/6 \\ 0 & 0 & 1 & 0 & -2/9 \\ 0 & 0 & 0 & 1 & -29/54 \end{array}\right)$$

You will see overtime that this matrix setup will always be lower triangular so it is a bit of an overkill to do it with Gauss-Jordan elimination, but it does not hurt. To put this whole ansatz algorithm in summary:

$$\begin{aligned} \text{if } f(t) = ae^{\alpha t} &\implies \text{guess } y_P = Ae^{\alpha t} \\ \text{if } f(t) = a\sin(\omega t) \text{ or } f(t) = b\cos(\omega t) &\implies \text{guess } y_p = A\sin(\omega t) + B\cos(\omega t) \\ \text{if } f(t) = a\sin(\omega t) + b\cos(\omega t) &\implies \text{guess } y_p = A\sin(\omega t) + B\cos(\omega t) \\ \text{if } f(t) = a_n t^n + a_{n-1}t^{n-1} + \cdots + a_1 t + a_0 &\implies \text{guess } y_p = A_n t^n + A_{n-1}t^{n-1} + \ldots A_1 t + A_0 \end{aligned}$$

Now what if we had a more complicated differential equation ? Something like:

$$ay'' + by' + cy = e^{-\alpha t}\sin(\omega t)$$

Based on our previous experience, the only logical approach for an ansatz is to guess the solution

$y_p = Ae^{-\alpha t}\sin(\omega t) + Be^{-\alpha t}\cos(\omega t)$ and find the coefficients. Now remember, by Abel's theorem, if we have that y_{P_1} and y_{P_2} are both solutions to a differential equations, then their sum will also be a solution! So if we have that $f(t)$ is a sum of a bunch of complicated functions, we will solve for each of those cases individually using the method of undetermined coefficients and then add up the solutions. I know that all of this feels like it came out of thin air, but the issue is that the reason for guessing comes from the method of variation of parameters which is not part of the course, hence, I will leave it as an extra section in the last chapter if you are interested.

5.3 Euler-Cauchy Equations

So far, we have solved many differential equations. Mostly, our equations have had constant coefficients. These are pretty straightforward, special cases of differential equations. But what about equations with variable coefficients ? A set of ODEs that are of this type are Euler-Cauchy equations. They have the form:

$$t^2y'' + aty' + by = 0$$

So how do we solve these equations ? Our ansatz for this equation is polynomials. Why ? If we have simple polynomial, then after a differentiation, we lost 1 degree but the multiple of t to y' will make up for it, and twice differentiating takes 2 degrees off which will then be fixed by the t^2 coefficient of y''. So we guess that

$$y(t) = t^m \implies y'(t) = mt^{m-1},\ y''(t) = m(m-1)t^{m-2}$$

Substituting these into the Euler-Cauchy equation gives us that

$$m(m-1)t^m + amt^m + bt^m = 0 \implies m(m-1) + am + b = 0$$

Simplifying this expression, the final form for our exponent is:

$$m^2 + (a-1)m + b = 0$$

and from this the solution for m is:

$$m = \frac{1 - a \pm \sqrt{(a-1)^2 - 4b}}{2} = \{m_1, m_2\}$$

Thus, the final form of the Euler-Cauchy equations is:

$$y(t) = C_1 t^{m_1} + C_2 t^{m_2}$$

5.4 Variation of Parameters

Before we start our discussion of variation of parameters, we must first discuss a tool called the Wronskian. For now, we define the Wronskian for two functions f, g as:

$$W(f, g) = fg' - f'g$$

This tool will come in rather handy and we will discuss more deeply in the first section of our Laplace transforms chapter. Let us go back to the general non-homogeneous equation

$$y'' + p(t)y' + q(t)y = r(t)$$

The method of variation of parameters assumes that p, q and r are continuous. So what is this mysterious method ? Suppose we have a homogeneous solution to our equation:

$$y_C = c_1 y_1(t) + c_2 y_2(t)$$

Lagrange's idea was to consider this solution but now "vary its parameters". Essentially, we let them be functions of t and then solve for them in terms of y_1, y_2, p, q and r. So our guess is:

$$y_P = u(t)y_1 + v(t)y_2$$

Now we substitute this into the equation:

$$y_P' = u'y_1 + uy_1' + v'y_2 + vy_2' \qquad y_P'' = u'y_1' + v'y_2' + uy_1'' + vy_2''$$

$$\implies u'y_1' + v'y_2' + uy_1'' + vy_2'' + pu'y_1 + puy_1' + pv'y_2 + pvy_2' + quy_1 + qvy_2 = r$$

Collecting like terms gives us the following equations

$$u(y_1'' + py_1' + qy_1) + v(y_2'' + py_2' + qy_2) + u'y_1' + v'y_2' = r$$

Since the terms in the parentheses are simply the homogeneous equation - as we assumed y_1 and y_2 are homogeneous solutions, we have the following relation:

$$u'y_1' + v'y_2' = r$$

We also have from the earlier solution that

$$u'y_1 + v'y_2 = 0$$

Multiplying the first equation by $-y_2$ and second one by y_2', we get

$$u'(y_1y_2' - y_1'y_2) = -y_2r \implies u'W = -y_2r \qquad v'(y_1y_2' - y_1'y_2) = -y_1r \implies v'W = y_1r$$

where W is the Wronskian of the two homogeneous solutions. Thus, the method of variation of parameters gave us a solution based on the Wronskian of the homogeneous solutions and the solutions themselves:

$$y_P(t) = \left(\int \frac{-y_2r}{W}\,dt\right) y_1(t) + \left(\int \frac{y_1r}{W}\,dt\right) y_2(t)$$

The reason why these integrals exists is due to the assumption that p, q and r are continuous. We can easily generalize this to n-th order Linear ODEs. First, a bit of notation about the Wronskian. The notation W_k means we take the Wronskian matrix (which we will see later) and make all its k-th column's elements 0 except the last one which we make a 1. Then the variation parameters for an n-th ODE says:

$$y_P(t) = \sum_{i=1}^{n} \left(\int \frac{W_i}{W} r(t)\,dt\right) y_i(t)$$

6 The Laplace Transform

As we have see so far, differential equations - despite their beauty - can get rather tedious and difficult to solve. In this section, we will learn a tool which feels like magic and is abused (which is an understatement) quite a lot by engineers. This tool allows us to take a differential equation and transform it into an algebraic equation! Have you guessed what this tool is called ? (it really is anyone's guess)

6.1 Variable Spaces, Function Spaces, Transforms and The Wronskian

In our earlier first year linear algebra, we explored the concept of vector spaces. We generally (unless you transferred in from Engineering Science) look at Euclidean vectors spaces such as $\mathbb{R}^2, \mathbb{R}^3$ and in general $\mathbb{R}^n$. We look at vectors in these Euclidean vectors spaces, and linear transformations from one Euclidean vector space to another. A vector space, however, is only 1 kind of space. It is a set of vectors, and an element of this space or set is called a vector. In this brief section, we are going to extend the idea of a space to functions which will provide us with a much more profound understanding of Laplace, and later on, Fourier transforms. You do not really need to know the content from this section, although I highly recommend it.

So first, let us get rid of this fancy work *space*. When we say a something space, it means the set of all of those things. For example, a variable space is a set of numbers, a vector space is a set of vectors and thus, a function space is just a set of functions which satisfy a certain condition. For example, we can take the function space of all functions $f : \mathbb{R}^2 \to \mathbb{R}$ which satisfy injectivity. We can take the set of functions that are both injective and surjective and call it the space of bijections. Now, what is the point ? If you recall, we had a set of tools when we worked with vector-spaces such as dot products and projections. These tools are what makes this idea so powerful.

Consider a function space X and two functions, $f(x), g(x) \in X$. The addition and subtraction of functions is similar to vectors and we are already familiar with that. But what about the dot product ? The dot product is a way for us to know how much of one vector is in another. Similarly, we have a method to see how much of one function is in another. This is called the inner product, denoted by $\langle f, g \rangle$. Tow calculate this, what we do is take the area under the graph of the two functions' product:

$$\langle f, g \rangle = \int_{-\infty}^{\infty} f(x)g(x)\,dx$$

Consequently, two functions are said to be orthogonal, if and only if:

$$\langle f, g \rangle = 0$$

So there's the analog of our dot product. But what about transformations ? This is where things get interesting. We are looking for a mapping φ such that $\varphi(f) \notin X$. Rather, it takes our function into a new function space, Y. This means, instead of our function being in terms of variable x, they become in terms of variable y. In mathematical notation:

$$f \xrightarrow{\varphi} \varphi\{f(x)\}(y)$$

Our new function is now $\varphi\{f\}$. To make things less confusing, we can just put a tilde or a hat on top of our function:

$$f(x) \xrightarrow{\varphi} \tilde{f}(y)$$

Why would we need this ? Similar to how sometimes vectors in different basis are easier to deal with for finding things like eigenvalues, eigenvectors, orthogonalization algorithms and basis. This is indeed what we mean when we said earlier that an ODE has linearly independent solutions; it meant that their inner product is 0.

So now we understand function transforms. But what do these transforms typically look like ? We discussed what φ does, but we did not state how. One way these transformations are done is through integration. These are called integral transforms. Here is how it works; we have a function $f(x) \in X$ which we wish to transform into a new domain, Y. We define a transfer funtion, $K(x, y)$ which takes inputs from both variable spaces. Then, we take the inner product of this function with our function, $f(x)$:

$$\tilde{f}(y) = \varphi\{f(x)\}(y) = \langle f(x), K(x, y) \rangle = \int_{-\infty}^{\infty} f(x) K(x, y)\, dx$$

These are generalized integral transforms which allow us to take a function in one space and redefine it for another space, sort of like what change of basis did for a vector. Before we move on, there is an easier way to tell whether two functions are linearly independent. Suppose $f(t)$ and $g(t)$ are two functions. If we wish to see if these two functions are linearly independent, we use a tool called the

Wronskian. This is a matrix defined based on functions. It is denoted by $W[f,g]$ and it is defined as:

$$W[f,g] = \det \begin{bmatrix} f & g \\ f' & g' \end{bmatrix}$$

More generally, for $y_1, \ldots, y_n$ we have

$$W[y_1, \ldots y_n] = \det \begin{bmatrix} y_1 & y_2 & \cdots & y_n \\ y_1' & y_2' & \cdots & y_n' \\ \vdots & \vdots & \ddots & \vdots \\ y_1^{(n-1)} & y_2^{(n-1)} & \cdots & y_n^{(n-1)} \end{bmatrix}$$

Why is this helpful ? As it turns out, two functions are said to be linearly independent, if there exists some point $t = t_0$ such that:

$$W[f,g](t_0) \neq 0 \implies f \text{ and } g \text{ are linearly independent}$$

Note that this does not go the other way! Just because f, g are linearly independent does not mean that their Wronskian at some point is non-zero!
In fact, we can see this from a general homogeneous second order ODE's Wronskian. Suppose f, g are both solutions to:

$$y'' + p(t)y' + q(t) = 0$$

Then the Wronskian would be:

$$W(t) = W(t_0) \exp\left(- \int_{t_0}^{t} p(t)\, dt \right)$$

We see that if $W(t_0) = 0$ then we have $W(t) = 0$ so f, g must be independent.

6.2 Exploring Laplace Transforms

Consider the following differential equation:

$$\frac{dy}{dt} - 3y = 0$$

Let us ignore the fact that this equation is separable. One approach to solving this equation is to transform it to a different domain. With problems that we cannot solve in the given domain or initial domain of the problem, integral transforms allow us to take that problem in a different domain or space, solve it there, and then transform it back in order to arrive at the answer in the desired domain. This is done by defining a linear, integral transformation. An example of this is the famous Laplace transform. Let us define $f : \mathbb{R} \to \mathbb{R}$ such that $f = f(t)$. What the Laplace transform allows us to do is take our function, from a time space t into a frequency domain s so that $f(t) \xrightarrow{\mathcal{L}} F(s)$. It is defined as:

$$F(s) = \mathcal{L}\left\{f(t)\right\}(s) = \int_0^{+\infty} f(t)e^{-st}\,dt$$

So in this case we have that $K(s,t) = e^{-st}$. Here s is a complex number. The reason for this choice of $K(s,t)$ is rather deep and I highly encourage you to watch 3blue1brown's series on the Laplace transform and if you have further questions come talk to me in person and I'll be happy to tell you about it.

Let us explore a few *curious* properties of the Laplace transform. Let us compute the transform of some common functions like e^{at}, $\sin(\omega t)$ and $\cos(\omega t)$:

$$\mathcal{L}\left\{e^t\right\} = \int_0^{\infty} e^{at}e^{-st}\,dt = \int_0^{\infty} e^{(a-s)t}\,dt = \frac{1}{a-s}e^{-(s-a)t}\Big|_0^{\infty} = \frac{1}{s-a}$$

for sine and cosine, we will use a trick. We know that $e^{i\omega t} = \cos(\omega t) + i\sin(\omega t)$. So if we take the Laplace transform of $e^{i\omega t}$, we will have the sum of sine and cosine's Laplace transforms:

$$\mathcal{L}\left\{e^{i\omega t}\right\}(s) = \int_0^{\infty} e^{i\omega t}e^{-st}\,dt = \int_0^{\infty} e^{(i\omega - s)t}\,dt = \frac{1}{s-i\omega} = \frac{s+i\omega}{s^2+\omega^2} = \frac{s}{s^2+\omega^2} + i\frac{\omega}{s^2+\omega^2}$$

So we have that

$$\mathcal{L}\left\{\sin(\omega t)\right\} = \Im(\mathcal{L}\left\{e^{i\omega t}\right\}) = \frac{\omega}{s^2+\omega^2} \qquad \mathcal{L}\left\{\cos(\omega t\right\}) = \Re(\mathcal{L}\left\{e^{i\omega t}\right\}) = \frac{s}{s^2+\omega^2}$$

Notice that for both, at $s = i\omega$, the function blows up. Despite the fact that we are in the frequency domain, we are still representing the same function, just in terms of a different variable.

For example, if $f(s) = 1/s^2 + \omega^2$ represents the magnitude of a vibrational amplitude, then we can interpret this value of $s = i\omega$ as the resonant frequency.

6.3 Applying The Laplace Transform

We have spent some time on understanding the Laplace transform. But why are we learning about it in a differential equations course ? This goes back to an important property of the Laplace transform; the transform of a function's derivative! Let us compute $\mathcal{L}\{f'(t)\}$:

$$\mathcal{L}\{f'(t)\} = \int_0^\infty f'(t)e^{-st}\,dt = f(t)e^{-st}\Big|_0^\infty + s\int_0^\infty f(t)e^{-st}\,dt = s\mathcal{L}\{f\}(s) - f(0) = sF(s) - f(0)$$

So the Laplace transform of a function's derivative is $\mathcal{L}\{f\} = sF(s) - f(0)$. And for the second derivative, you can check to see that $\mathcal{L}\{f''(t)\} = s^2F(s) - sf(0) - f'(0)$. You may see a pattern. Indeed we have that:

$$\mathcal{L}\left\{f^{(n)}(t)\right\} = s^nF(s) - \sum_{i=0}^{n-1} s^i f^{(n-1-i)}(0)$$

Every time we take the Laplace transform, we turn a n-th order derivative in the time domain into an n-th order polynomial in the s domain. So how can we use this to solve differential equations ? Well, let us consider the differential equation we saw in 6.2:

$$\frac{dy}{dt} - 3y = 0$$

Now, we take the Laplace transform of both sides:

$$\mathcal{L}\left\{\frac{dy}{dt} - 3y\right\} = \mathcal{L}\{0\} \implies \mathcal{L}\{y'\} - 3\mathcal{L}\{y\} = 0$$

Now, we represent $y(t)$'s Laplace transform as $Y(s)$ and solve for $Y(s)$:

$$sY(s) - y(0) - 3Y(s) = 0 \implies Y(s) = \frac{y(0)}{s-3}$$

And there is our solution in the s domain. But we are not done yet, we must have a way back to the time domain (although this domain is perfectly sufficient to analyze the solution). So how ? This is achieved through the inverse Laplace transform. Once we transform the function to our

frequency domain and solve the problem, we can transform it back to the time domain using the inverse Laplace transform, defined as

$$f(t) = \mathcal{L}^{-1}\{F(s)\}(t) = \frac{1}{2\pi i}\lim_{T\to\infty}\int_{\gamma-iT}^{\gamma+iT} F(s)e^{st}\,ds$$

Now, for the sake of saving pages of calculations and many hours of our precious time, we will opt to use a table of Laplace transforms. Similarly, we have more important things to do than to waste our time on calculating this complicated integral, so we will opt to use a table of Inverse Laplace transforms (you can simply google "table of Laplace transforms"). We know that $\mathcal{L}\left\{e^{3t}\right\} = 1/s-3$. So naturally:

$$Y(s) = y(0)\frac{1}{s-3} \implies y(t) = y(0)e^{3t}$$

This also illustrates the role of initial values in Laplace transforms. As you can see, the Laplace transform is an extremely powerful tool in solving differential equations. We see later that you can also solve PDEs with the Laplace transform (if you read the further supplements).

Before we move on, let us explore some interesting properties of the Laplace transform with integrals. We showed that the Laplace transform allows us to turn derivatives into polynomial expressions. This property allowed us to solve differential equations very easily. Now, although we will not cover that here (you may consult 11.6), similar to how we can have equations involving functions and their derivatives, we can have equations involving functions and their integrals. These are called *Integral Equations*. Believe it or not, we can use Laplace transforms for those as well, but for that, we need to know what the Laplace transform of an integral is.

$$\mathcal{L}\left\{\int_0^t f(\tau)\,d\tau\right\} = \int_0^\infty e^{-st}\int_0^t f(\tau)\,d\tau\,dt = \int_0^\infty f(\tau)\int_\tau^\infty e^{-st}\,dt\,d\tau = \frac{1}{s}\int_0^\infty f(\tau)e^{-s\tau}\,d\tau = \frac{1}{s}F(s)$$

As you can see we similar to how we had $\mathcal{L}\{f'\} = sF - f(0)$, we have that $\mathcal{L}\left\{\int f\right\} = F/s$. This intuitively does make sense as we know that integrals and derivatives are inverse of each other by the fundamental theorem of calculus.

6.4 Existence, Uniqueness and Shifting Theorems

We can use the properties of exponentials to arrive at a shifting theorem for Laplace transforms. Essentially, we want to explore what a shift in the frequency domain looks like in the time domain. Essentially, if $f(t) \xrightarrow{\mathcal{L}} F(s)$ then how is $F(s-a)$ related to $f(t)$? Well let us see what it looks like when we look at the definition for the shifted Laplace transform:

$$F(s-a) = \int_0^\infty e^{-(s-a)t} f(t)\, dt = \int_0^\infty e^{at} f(t) e^{-st}\, dt$$

You can observe that the function $e^{at}f(t)$ is only dependent on t. So we can say that:

$$\mathcal{L}\left\{e^{at}f(t)\right\} = F(s-a) \iff \mathcal{L}^{-1}\{F(s-a)\} = e^{at}f(t)$$

What does this physically mean ? Suppose $f(t)$ represents the amplitude of a wave. Then, if we amplify the wave by an exponential factor e^{at} it would result in a shift of the resonant frequency of wave by a.

Now a small question is how would we know if a function even has a Laplace transform ? Some functions do not! Although this barely has any practicality in engineering, it is important to be aware of it. A function $f : \mathbb{R} \to \mathbb{R}$ must not be divergent, essentially meaning it should not grow "too fast". To mathematically express this, we say that there must exist at least two constants, $M, k \in \mathbb{R}$ such that $f(t)$ satisfies:

$$|f(t)| \leq Me^{kt}$$

This is called satisfying growth of exponential order. Although $f(t)$ does not need to be continuous, it does need to be *piecewise continuous*. This means that it is allowed to have finite jump discontinuities, but other types of discontinuities are not allowed. So it is continuous over intervals it is defined on, but not anything else. The existence theorem for Laplace transforms states the following:

If $f(t)$ is defined and piecewise continuous on every finite interval on the semi-axis $t \geq 0$ and satisfies exponential growth order with M, k for all $t \geq 0$, then $\mathcal{L}\{f(t)\}$ is defined for all $s > k$.

The uniqueness theorem for Laplace transforms says if two functions have the sample Laplace transforms, they must be completely identical (although they may differ at isolated points).

6.5 The Heaviside Function and Dirac Delta Distribution

Consider a unit step function, also known as the Heaviside function $H(t)$ is used to represent a jump. It is defined as:

$$H(t-a) = \begin{cases} 0 & t < a \\ 1 & t > a \end{cases}$$

It is extremely useful in signal analysis and piecewise forcing. So if we multiply a function $f(t-a)$ to it, we can have a shifted signal that starts at $t = a$:

$$f(t-a)H(t-a) = \begin{cases} 0 & t < a \\ f(t-a) & t > a \end{cases}$$

Lots of forcing terms look like piecewise signals. Thus, it is helpful to know its Laplace transform:

$$\mathcal{L}\{f(t-a)H(t-a)\} = \int_0^\infty e^{-st} f(t-a)H(t-a)\,dt = \int_a^\infty e^{-st} f(t-a)\,dt$$

Now to get rid of the shifting, we do a u-substitution: $u = t - a \implies t = u + a$

$$\int_a^\infty e^{-st} f(t-a)\,dt = \int_a^\infty e^{-s(u+a)} f(u)\,du = e^{-as}F(s) \implies \mathcal{L}\{f(t-a)H(t-a)\} = e^{-as}F(s)$$

So there is the Heaviside function's Laplace transform. This will be of great use in Control Theory, Signal Analysis and PDEs. Another important step function in engineering and physics is the Dirac Delta function. It is defined as:

$$\delta(t-a) = \begin{cases} 0 & t \neq a \\ \infty & t = a \end{cases}$$

It has the powerful property that:

$$\int_{-\infty}^\infty \delta(t-a)\,dt = 1 \qquad \int_{-\infty}^\infty f(t)\delta(t-a)\,dt = f(a)$$

In fact, most of the times, the second integral is how we define the Dirac delta in the first place! Unlike the Heaviside function which shows a continuous signal shift, the Dirac delta is used to

model a sudden impulse to the system. Its Laplace transform is then:

$$\mathcal{L}\{\delta(t-a)\} = \int_0^\infty e^{-st}\delta(t-a)\,dt = e^{-as}$$

This intuitively makes sense as we know from transient response in subjects like circuit theory, that when we have a sudden current impulse, the transient response follows exponential decay, which is precisely what the Laplace transform is telling us!

7 Systems of Differential Equations

Similar to how we learn algebraic equations, we first approach differential equations by learning how to solve 1 individual equation. Now, similar to systems of algebraic equations, we can also have systems of differential equations. These equations are slightly more interesting as solving one individually would not work because the solution of 1 would immediately change the other. So we need new techniques to deal with systems of equations. An example of such a system is a coupled mass-spring-damper system:

$$\begin{cases} m_1\ddot{x}_1 + \mu(\dot{x}_1 - \dot{x}_2) + k(x_1 - x_2) + k_1 x = 0 \\ m_2\ddot{x}_2 + \mu(\dot{x}_2 - \dot{x}_1) + k(x_2 - x_1) = F_0\sin(\omega t) \end{cases}$$

As you can see, we cannot solve one equation and ignore the other. It just doesn't work. So how can we deal with such systems ?

7.1 Representing Systems of ODEs

To represent such systems, we use (similar to algebraic equations) matrices. Suppose we have the following system:

$$\begin{cases} x' = 4x + 3y \\ y' = -3x + 4y \end{cases}$$

One way to think about this system is that we have some vector $\mathbf{x} = (x \quad y)$. Then, the system above represents how its components evolve. So we represent the above system with matrices:

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 4 & 3 \\ -3 & 4 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

If we let the number matrix be represented by M, then we have that:

$$\dot{\mathbf{x}} = M\mathbf{x}$$

And now, similar to functions, we actually treat this like a separable equation:

$$\mathbf{x} = \mathbf{x}(0)e^{Mt}$$

But I'm getting ahead of myself. For now, just understand that we use matrices and vectors to represents systems of differential equations. We can represent the system we looked at first as:

$$M\ddot{\mathbf{x}} + C\dot{\mathbf{x}} + K\mathbf{x} = \begin{pmatrix} 0 \\ F_0 \sin(\omega t) \end{pmatrix}$$

Where we have that

$$M = \begin{pmatrix} m_1 & 0 \\ 0 & m_2 \end{pmatrix} \quad C = \begin{pmatrix} \mu & -\mu \\ -\mu & \mu \end{pmatrix} \quad K = \begin{pmatrix} k + k_1 & -k \\ -k & k \end{pmatrix}$$

If you look at the above equation, it looks just like 1 ordinary differential equation, and believe it or not, it is actually solved in a similar manner.

7.2 Matrices and Exponentials

In order to solve systems of differential equations, we must first understand the notion of what it means to raise a number to the power of a matrix. Specifically, what it means to raise e to the power of some matrix M like we saw earlier. When we have matrix exponents, we are not looking at the exponentiation definition as iterative multiplication of the base. What we really mean is

utilizing the Taylor series definition of the exponential function:

$$e^t = \sum_{k=0}^{n} \frac{t^k}{k!}$$

Thus, to raise to the power of a matrix means to use the Taylor series definition on the matrix:

$$e^M = \sum_{k=0}^{n} \frac{1}{k!} M^k = M + \frac{1}{2}M^2 + \frac{1}{6}M^3 + \dots$$

And we (hopefully) know from linear algebra how to calculate powers of matrices with iterative multiplication. This is where diagonalization becomes so important, because solving systems of ODEs requires lots of matrix exponentiation which in turn requires lots of matrix multiplication. Since diagonalization algorithms significantly reduce the computation time for matrix multiplication, they are crucial to ODE solvers and engineering simulation software.
Let us try this on a few matrices. Suppose

$$M = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

As you may have noticed, this matrix corresponds to rotation by $\pi/2$. So what is e^M ? Let us calculate the first four powers:

$$M^1 = M \quad M^2 = M \cdot M = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} = -I$$

Continuing this curious pattern for powers of 3 and 4 gives:

$$M^3 = M^2 \cdot M = -I \cdot M = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = -M \quad M^4 = M^2 \cdot M^2 = I \cdot I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I$$

Geometrically, we calculated 4 rotations by $\pi/2$. We know that since a circle is 2π, $M^5 = M$, $M^6 = M^2, \dots$ which means we group every fourth matrix together:

$$e^M = M + \frac{1}{2}M^2 + \frac{1}{6}M^3 + \frac{1}{24}M^4 + \frac{1}{108}M^5 + \frac{1}{648}M^6 + \dots$$

$$= I\left(\sum_{k=0}^{\infty}\frac{(-1)^k}{(2k)!}\right) + M\left(\sum_{k=0}^{\infty}\frac{(-1)^k}{(2k+1)!}\right) = I\cos 1 + M\sin 1$$

$$\implies e^M = e^{\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}} = \begin{pmatrix}\cos 1 & -\sin 1\\ \sin 1 & \cos 1\end{pmatrix}$$

So there is the result of our matrix exponentiation. What would have happened if we took e^{tM} ? If we had this, then M would have had t in its components and then the resulting exponentiated matrix would correspond to rotation by t radians:

$$e^{tM} = \begin{pmatrix}\cos t & -\sin t\\ \sin t & \cos t\end{pmatrix}$$

Why does this matter though ? Before we get to that, let us first try to build a visual understanding of what matrix exponentiation looks like. We remember from linear algebra that matrices represent linear transformation. Then, matrix exponentiation is essentially continuously applying a linear transformation. Rather than immediately changing the basis vectors to the transformed vectors, it will gradually take them there. It allows us to control the rate at which a linear transformation is being applied. This is how we geometrically think about linear transformations and their exponentials.

7.3 Solving Separable Systems

Suppose we have a system of the following form:

$$\begin{cases}\dot{x} = f(x)\\ \dot{y} = g(y)\end{cases}$$

We can easily solve this system since they are both separable equation and they are not coupled, meaning there is not x in the y equation and no y in the x equation. So we get:

$$\begin{cases}\int \dot{x}(t)\,dx = \int f(x(t))\,dt\\ \int \dot{y}(t)\,dy = \int g(y(t))\,dt\end{cases} \implies x(t) = \int f(x(t))\,dt, \quad y(t) = \int g(y(t))\,dt$$

But usually things are not so simple. What if our equations are coupled ? Meaning, what if our equations involved both x and its derivatives, and y and its derivatives ? Then it gets a little more complicated. Indeed, we need to solve the two equations simultaneously. One approach is to find the level curves of the solution, essentially finding y as a function of x. Consider the following system:

$$\begin{cases} x' = f(x, y) \\ y' = g(x, y) \end{cases}$$

This is a coupled equation. So, how can we solve this ? First let us formulate the general approach. We can see that

$$\frac{y'}{x'} = \frac{dy/dt}{dx/dt} = \frac{dy}{dx} = \frac{f(x, y)}{g(x, y)}$$

We then isolate for the terms involving y and x on each side and integrate. Consider the following example:

$$\begin{cases} \dot{x} = x(1 - y) \\ \dot{y} = y(1 - x) \end{cases}$$

If we take the ratio of $\dot{y}$ to $\dot{x}$, we get:

$$\frac{\dot{y}}{\dot{x}} = \frac{dy/dt}{dx/dt} = \frac{dy}{dx} = \frac{y(1 - x)}{x(1 - y)}$$

Now algebraically separating y and x gives:

$$\frac{1 - y}{y}\, dy = \frac{1 - x}{x}\, dx \implies \ln y - y = \ln x - x + C$$

This is an implicit relation for the solution of our differential equation!

7.4 First-Order Linear Systems

Now, we will learn a more explicit approach to solving first-order linear systems. Here is where matrix exponents come to use. To start, we already discussed that we can represent some first order systems with matrices:

$$\dot{\mathbf{x}} = M\mathbf{x}$$

These are homogeneous systems, which we turned into "one" ODE for a vector. The scalar version: $x' = \lambda x$ had the natural solution $x = Ce^{\lambda t}$. The natural approach for homogeneous systems is to then guess an exponential, but with a vector coefficient instead!

$$\text{guess } \mathbf{x}(t) = \mathbf{v}e^{\lambda t} \implies \lambda \mathbf{v}e^{\lambda t} = M\mathbf{v}e^{\lambda t} \implies (M - \lambda I) = 0$$

So the general solution of homogeneous systems is the exponential of the eigenvalue of our matrix M times t:

$$\mathbf{x} = \mathbf{x}(0)e^{At}$$

This shows us how eigenvalues naturally arise when we deal with systems of equations. An $n \times n$ matrix would correspond to n eigenvalues. If the eigenvalue is real, then the solution would be an exponential with that phase factor. If it is a complex eigenvalue, it corresponds to a sinusoidal function with frequency $\Im(\lambda)$. The function that satisfies our system, is then called an *eigenfunction* of the system. But what about non-homogeneous equations ? Once again similar to scalar equations, we take an ansatz. The general first order linear system is:

$$\dot{\mathbf{x}} = M\mathbf{x} + \mathbf{b}$$

Recall from Abel's theorem, if $\mathbf{x}_1$ and $\mathbf{x}_2$ are solutions, then $\mathbf{x}_1 \pm \mathbf{x}_2$ is a solution. The solution to the homogeneous equation must also be included, so the particular solution, $\mathbf{x}_h$ is one solution that is only an eigenfunction to $\dot{\mathbf{x}} = M\mathbf{x}$. The particular solution is now what we are interested in. The undetermined coefficients approach is rather similar to scalar ODEs. We guess the particular solution based on the form of the $\mathbf{b}$ term, similar to how we did in 5.2. For example, if we have the following system:

$$\dot{\mathbf{x}} = \begin{pmatrix} 1 & 2 \\ 3 & 2 \end{pmatrix} \mathbf{x} + t \begin{pmatrix} 2 \\ -4 \end{pmatrix}$$

The extra term involves a polynomial function, so our guess is that our particular solution involves a polynomial of degree 1 along with vector coefficients!

$$\mathbf{x}_P = t\mathbf{c}_0 + \mathbf{c}_1$$

Substituting it in gives us that:

$$\mathbf{c}_0 = M\mathbf{c}_0 t + M\mathbf{c}_1 + t\mathbf{b} \implies t(M\mathbf{c}_0 + \mathbf{b}) + (M\mathbf{c}_0 - \mathbf{c}_1) = 0$$

Now a simple approach to solving this system is to set each term equal to 0:

$$M\mathbf{c}_0 + \mathbf{b} = 0 \implies \mathbf{c}_0 = -M^{-1}\mathbf{b}$$
$$M\mathbf{c}_0 - \mathbf{c}_1 = 0 \implies \mathbf{c}_1 = M\mathbf{c}_0$$

After completing these matrix vector products we find that:

$$\mathbf{c}_0 = \begin{pmatrix} 3 \\ -5/2 \end{pmatrix} \qquad \mathbf{c}_1 = \begin{pmatrix} -11/4 \\ 23/8 \end{pmatrix}$$

And then our final solution becomes:

$$\mathbf{x} = \mathbf{x}(0)e^{Mt} + t\begin{pmatrix} 3 \\ -5/2 \end{pmatrix} + \begin{pmatrix} -11/4 \\ 23/8 \end{pmatrix}$$

This is one way to solve first order linear systems. But how about our approach with variable coefficients ? This requires the use of variation of parameters which we do not explore here.

8 Series Solutions

We have explored quite a lot in terms of Ordinary Differential Equations. This section will be the last which we will talk about them for now. Even though we have so far learned that the method of variation of parameters allows us to solve ODEs with variable coefficient, the standard method of approaching these problems is to use power series.

8.1 The Power Series Method

Recall from our single variable calculus course that we can express functions as power series. For some famous functions, we have some well known series such as their Taylor expansion! In general,

we have that:

$$y(t) = \sum_{n=0}^{\infty} a_n t^n$$

Then, the simplest thing we can do to solve differential equations is to guess that y has the above form, and then solve for the coefficients! For example, consider the following differential equation:

$$y' - y = 0$$

Now we already know that the solution to this is the exponential function. But now, let us see if we can arrive t this solution through the power series of y:

$$\text{let } y(t) = \sum_{n=0}^{\infty} a_n t^n \implies \sum_{n=0}^{\infty} n a_n t^{n-1} - a_n t^n = 0$$

This gives us the following summation:

$$(a_1 + 2a_2 x + 3a_3 x^2 + \dots) - (a_0 + a_1 x + a_2 x^2 + \dots) = 0$$

Now let us isolate the coefficients. We have that $a_1 - a_0 = 0 \implies a_1 = a_0$ which is trivial, it depends on the initial conditions. What about the rest ? We have that

$$\begin{aligned} 2a_2 - a_1 = 0 &\implies a_2 = \frac{a_1}{2} = \frac{a_0}{2} \\ 3a_3 - a_2 = 0 &\implies a_3 = \frac{a_2}{3} = \frac{a_0}{6} \\ 4a_4 - a_3 = 0 &\implies a_4 = \frac{a_3}{4} = \frac{a_0}{24} \end{aligned}$$

And so on. Do you see a pattern ? Yes, indeed we have that:

$$a_n = \frac{a_0}{n!}$$

Thus, the solution to our differential equation is the power series

$$y(t) = \sum_{n=0}^{\infty} \frac{a_0}{n!} t = a_0 e^t$$

And this is the magic of series solutions! Even though we assumed that y is a polynomial power series, the equation imposed all the necessary conditions needed for us to see that it is exponential in nature. This is a very handy method when we have variable coefficient! How do we know that series solutions exist to a differential equation ? For some differential equation:

$$y'' + p(x)y' + q(x) = r(x) \quad y(x_0) = y_0$$

A unique series solution of the equation exists if and only if p, q and r are analytic functions (meaning they have a derivative) and that the sequence of partial sums of the coefficients centered at $x = x_0$ converges with a radius of convergence $R > 0$.
Do not worry about what it means for a function to be analytic. This requires knowledge of complex analysis which we will not deal with here. For the purposes of MAT234, just think of it as saying the function has a defined derivative over its domain.

8.2 The Legendre Equation

In this section, we are going to explore a special differential equation, whose solution turns out to be very important in physics and engineering. This is the Legendre associated polynomial.
The following is called the Legendre differential equation:

$$(1 - x^2)y'' - 2xy' + 2y = 0$$

So let us try and solve this using power series! We have that

$$y = \sum_{n=0}^{\infty} a_n x^n \implies y' = \sum_{n=0}^{\infty} n a_n x^{n-1}, \quad y'' = \sum_{n=0}^{\infty} n(n-1) a_n x^{n-2}$$

Now remember, what we are interested in, are the coefficients a_n which make the differential equation true. So let us explore what happens:

$$\implies \sum_{n=0}^{\infty} n(n-1) a_n x^{n-2} - \sum_{n=0}^{\infty} n(n-1) a_n x^n - \sum_{n=0}^{\infty} 2n a_n x^n + \sum_{n=0}^{\infty} 2 a_n x^n = 0$$

I will leave it to you as an exercise to go through the hand calculations. Write out the sum, collect the like terms and solve for your coefficients. Then, re-write the final form of the solution and you should get something that looks like this:

$$y = a_1 x + a_0(1 - x - \frac{1}{3}x^2 - \frac{1}{5}x^3 - \dots) = a_1 x + a_0 P_1(x)$$

The minus sign is a convention here. The polynomial in the bracket is sometimes denoted by $P_1(x)$. This is the Legendre associated polynomial. The subscript of 1 is there because we did not solve all families of the Legendre equation. You may look at section 10.2 for more details. This is the first degree Legendre polynomial and it is the solution to the first degree Legendre differential equation.

8.3 Frobenius' Method

In the end of 8.1, we said that a second-order linear differential equation has a unique solution if and only if the coefficient and image functions are analytic functions. We then said that for our purposes, we can say that analytic means the function has a derivative at every point. Now as it turns out, there is an extension to the power series method known as Frobenius' method. This method allows us to deal with certain forms of p, q, r which are not analytic, but are "not too bad". Specifically, we are interested in equations of the form

$$y'' + \frac{b(x)}{x}y' + \frac{c(x)}{x^2}y = 0$$

Where $b(x)$ and $c(x)$ are functions that are analytic at 0. Frobenius' method says that this equation has at least 1 solution which has the form

$$y = x^r \sum_{n=0}^{\infty} a_n x^n$$

To see where Frobenius' method comes from, we get rid of the denominators:

$$y''x^2 + b(x)xy' + c(x)y = 0$$

Now, we apply a power series expansion to $b(x)$ and $c(x)$, or if they are polynomials we leave them as they are:

$$b(x) = b_0 + b_1 x + b_2 x^2 + \dots \qquad c(x) = c_0 + c_1 x + c_2 x^2 + \dots$$

If we take derivatives of our ansatz, we see that

$$y' = \sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1} \qquad y'' = \sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2}$$

We then input these into the re-arranged ODE which becomes nasty, and then we equate each coefficient of each x^{r+k} to 0. This results in a system of equations for the n-th coefficient, a_n. What Frobenius method does is considering the smallest degree, x^r which has the coefficient

$$[r(r-1) + b_0 r + c_0] a_0 = 0$$

Since a_0 cannot be 0, we must have that the quadratic equation in the brackets be 0. This results in what is called the *indicial equation*:

$$r(r-1) + b_0 r + c_0 = 0$$

So r must satisfy this equation. Now, we know that there are 3 possible cases for the zeroes of this quadratic, either we have a double root, roots differing by integers, or roots that do not differ by an integer amount. Now based on the Euler-Cauchy equation back in 5.3, we do not really expect the last two but let us consider them anyways. For the double root case, we have $r = \frac{1}{2}(1 - b_0)$ and

$$y_1(x) = x^r (a_0 + a_1 x + a_2 x^2 + \dots) \qquad y_2(x) = y_1(x) \ln x + x^r (A_1 x + A_2 x^2 + \dots)$$

If the roots of the indicial equation differ by an integral amount, we have r_1 and r_2

$$y_1(x) = x^{r_1} (a_0 + a_1 x + a_2 x^2 + \dots) \qquad y_2(x) = k y_1(x) \ln x + x^{r_2} (A_0 + A_1 x + A_2 x^2 + \dots)$$

Lastly, when r_1 and r_2 differ by a non-integer amount, we have

$$y_1(x) = x^{r_1} (a_0 + a_1 x + a_2 x^2 + \dots) \qquad y_2(x) = x^{r_2} (A_0 + A_1 x + A_2 x^2 + \dots)$$

8.4 The Bessel Equation

One of the most important equations that will show up time over time is the Bessel equation:

$$x^2y'' + xy' + (x^2 - \nu^2)y = 0 \qquad \nu \geq 0$$

This ODE turns out to be a fundamental characteristic of systems which obey a cylindrical symmetry, similar to how Laplace's equation shows up with spherical symmetries. We can solve this equation now using Frobenius' method! We know that it must have a solution of the form:

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

So let us try it in the equation

$$\sum_{n=0}^{\infty}(n+r)(n+r-1)a_n x^{n+r} + \sum_{n=0}^{\infty}(n+r)a_n x^{n+r} + \sum_{n=0}^{\infty} a_n x^{n+r+2} - \nu^2 \sum_{n=0}^{\infty} a_n x^{n+r} = 0$$

I will leave this tedious work for you to enjoy. But the indicial equation for the first term of the solution becomes

$$(r+\nu)(r-\nu) = 0$$

The roots of this equation are $r_1 = \nu$ and $r_2 = -\nu$. For the second and $n = s$ term, they are:

$$r(r+1)a_1 + (r+1)a_1 - \nu^2 a_1 = 0$$
$$(s+r)(s+r-1)a_s + (s+r)a_s + a_{s-2} - \nu^2 a_s = 0$$

If we take the root $r = \nu$, the first equation simplifies to $(2\nu+1)a_1 = 0 \implies a_1 = 0$. If we combine the terms in the second equation, we find

$$(s+2\nu)sa_s + a_{s-2} = 0$$

This means all odd coefficients will be 0. And the even coefficients for $s = 2n$ leads to the following formula:

$$a_{2n} = \frac{(-1)^n a_0}{2^{2n} n!(\nu+1)(\nu+2)\dots(\nu+n)}$$

Now consider these coefficients for the special case of $\nu = m$. Then, by inserting these into the solution that we guessed for Bessel's equation, we get the following function:

$$J_n(x) = x^m \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{2^{2n+m} n!(m+n)!}$$

This is the n-th order Bessel function of the first kind. You can use the ratio test to show that this series converges for all x and it is defined for all x. Now what about the cases where $\nu \neq m$? For this case we apply the extension of the factorial function which is the Gamma function. We know from MIE231 that the Gamma function has the following property:

$$\Gamma(\nu+1) = \int_0^{\infty} t^{\nu} e^{-t}\, dt = \nu\Gamma(\nu) = \nu!$$

So the Bessel function of order ν of the first kind becomes

$$J_\nu(x) = x^\nu \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{2^{2n+\nu} n!\Gamma(\nu+n+1)}$$

This is the solution to the Bessel equation and is very frequently used in physics and engineering, especially when it comes to mechanical vibrations and signal analysis.

9 Partial Differential Equations

Up until now, we have spent a great deal of time studying Ordinary Differential Equations. These were equations where our variable or function of interest was only dependent on 1 variable. However, the real world is different. In reality, functions rarely ever depend on only one variable. Rather, they depend on several variables. And similar to ODEs, we can write equations which may involve partial derivatives of a function. These are known as Partial Differential Equations or simply PDEs. They are exponentially harder to solve than ODEs. In fact, they are so difficult that we create whole categories for a single equation! We will start our endeavor by learning about Fourier series and the Fourier transform which serves as extremely powerful tools for solving PDEs.

9.1 Fourier Series and The Fourier Transform

We start with building our toolbox, the most important tool of which are the Fourier transform and the Fourier series. Let's dive in. First, what is the Fourier Series ? The Fourier Series is a tool which allows us to break down a function into an infinite sum of sine and cosine waves. We may have some functions that are very difficult to deal with, however, we already know that dealing with sines and cosines are easy so transforming our function to a sum of sines and cosines allows us to break free of the trouble! The formula for the Fourier series of a function $f : \mathbb{R} \to \mathbb{R}$ is:

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \sin(nx) + \sum_{m=1}^{\infty} b_m \cos(mx)$$

Let us make sense of everything here. First of all, what do m and n in the sine and cosine represent ? If we are trying to express a function as a sum of sines and cosines, we never know how much of each we need. Sometimes, we may need different frequencies of sines and cosines! We may even need only sines but a combination of different frequencies of sine. Hence the n and m. Now what about a_n and b_m ? What do those mean ? Well, for each different frequency of sine and cosine, we may need a certain amplitude of each. Those coefficients - called the Fourier coefficients - are where this fact comes into play. They allow us to fine tune the amplitudes depending on the frequency of the sine and cosine wave. What about a_0 ? Why is there a division by 2? This comes from the average value of the sinusoid. After all, the following is the general form of a sinusoid:

$$g(t) = \frac{g_{max} + g_{min}}{2} + \frac{g_{max} - g_{min}}{2} \sin(2\pi f t + \varphi)$$

The coefficient of the sinusoid is the amplitude which is given by half the difference between the maximum and the minimum value of the function. The additional term at the beginning is the average value of the function. That is exactly the role $a_0/2$ plays here. However, now we need a new formula for those coefficients as with changing functions like $f(x)$ that is simply not going to cut it! we know the average value of a function over a domain $[x_0, x_1]$, so we already know that the first term of the Fourier series is given by:

$$\frac{a_0}{2} = \frac{1}{L} \int_{x_0}^{x_1} f(x)\, dx$$

Where $L = |x_1 - x_0|$. Now how can we find the Fourier coefficients ? Intuitively, those coefficients tell us how much of each frequency of that sinusoid is in our function. The way to do this is the following: first, take the product of our function with that sinusoid to amplify the behavior of our function on the sinusoid. Then, we take the area under this new curve, and normalize by the length of the interval to give us the overlap of our functions:

$$a_n = \frac{2}{L}\int_{x_0}^{x_1} f(x)\sin(\omega n x)\,dx \qquad b_m = \frac{2}{L}\int_{x_0}^{x_1} f(x)\cos(\omega m x)\,dx$$

As an exercise let us compute the Fourier series for the following $f : \mathbb{R} \to \mathbb{R}$ such that:

$$f(t) = \begin{cases} 1 & t > 0 \\ -1 & t < 0 \end{cases}$$

over the interval $[-\pi, \pi]$. Our job is to essentially just calculate the Fourier coefficients. First, let us find the average value:

$$\frac{a_0}{2} = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)\,dt = \frac{1}{2\pi}\left(\int_{-\pi}^{0} -1\,dt + \int_{0}^{\pi} 1\,dt\right) = 0$$

And this intuitively does make sense. If a function is -1 for half of its domain and 1 for the other half of the domain, then intuitively, we could have just said:

$$\frac{a_0}{2} = \frac{1 + (-1)}{2} = 0$$

What about the Fourier coefficients ? Let us see! For the case of sine we have:

$$b_m = \frac{1}{\pi}\left(\int_{-\pi}^{0} -\cos(m\omega t)\,dt + \int_{0}^{\pi} \cos(m\omega t)\,dt\right) = \frac{1}{\pi}\left(\frac{1}{m\omega}\sin(m\omega t)\Big|_0^{\pi}\right) = 0$$

So we know that our function has no pure sine wave. What about sine ? For that we calculate the first coefficient:

$$a_n = \frac{2}{\pi}\left(\int_{-\pi}^{0} -\sin(n\omega t)\,dt + \int_{0}^{\pi} \sin(n\omega t)\,dt\right) = \frac{2}{\pi}\left(\frac{-1}{n\omega}\cos(n\omega t)\Big|_0^{\pi}\right) = \frac{2}{n\pi\omega}(1 - (-1)^{\omega n})$$

So we have no pure cosine wave but no sine waves. Thus, the Fourier series of our signal is:

$$f(t) = \frac{2}{\pi\omega} \sum_{m=1}^{\infty} \frac{1 - (-1)^{n\omega}}{n} \sin(n\omega t)$$

So that is how we calculate Fourier series for functions. As you saw, it was a lot of work. Suppose we only wanted to know how much a certain frequency was present in our function, then would it make sense to derive the entire Fourier series just to find out how much sine and cosine are present for that frequency ? No, it does not! It is way too much work and computation and includes lots of unnecessary steps. So how do we get around this ? We use the Fourier transform! The Fourier transform returns the frequency domain equivalent of our time-domain function. We can then use this to analyze our function in the frequency domain, similar to Laplace transforms! Here is how we define the Fourier transform of a function:

$$\mathcal{F}\{f(t)\} = \hat{f}(\xi) = \int_{-\infty}^{\infty} e^{-2\pi i \xi t} f(t)\, dt$$

Where $\xi \in \mathbb{R}$ is the frequency meaning $\omega = 2\pi\xi$. To go back from the Fourier domain to the time domain, we use the inverse Fourier transform:

$$\mathcal{F}^{-1}\left\{\hat{f}(\xi)\right\} = f(t) = \int_{-\infty}^{\infty} e^{2\pi i \xi t} \hat{f}(\xi)\, d\xi$$

Where are the sine and cosine you may ask ? They are inside the exponential function! Remember Euler's golden identity:

$$e^{i\theta} = \cos\theta + i\sin\theta$$

So instead of considering each function separately - which also makes the integration more complicated usually - we just use an exponential with a complex input! For example, the Fourier domain - or frequency - domain equivalent of a simple damped oscillator $f(t) = e^{-\alpha t}\cos(t),\ t \geq 0$ is:

$$\mathcal{F}\{f(t)\} = \hat{f}(\omega) = \int_{-\infty}^{\infty} e^{-i\omega t} e^{-\alpha t} \cos(t)\, dt = \int_{0}^{\infty} e^{-i\omega t - \alpha t} \cos(t)\, dt$$

Now we use the fact that $\cos(t) = \frac{1}{2}(e^{it} + e^{-it})$ to simplify the integral:

$$\hat{f}(\omega) = \frac{1}{2}\int_0^\infty e^{-\alpha t}(e^{i(1-\omega)t} + e^{-i(1+\omega)t})\,dt = \frac{1}{2}\left(\frac{1}{\alpha + i(\omega-1)} + \frac{1}{\alpha + i(\omega+1)}\right)$$

So there is our function's Fourier transform. This allows us to analyze the frequency behavior of our sinusoid without needing to work with the difficult time domain, furthermore, it allows us to solve differential equations easier, in a domain that makes more sense. Before we move on, let us note a curious fact about the Fourier transform. It can be re-written as:

$$\mathcal{F}\{f(t)\} = \int_{-\infty}^{\infty} e^{-i\omega t} f(t)\,dt = \int_{-\infty}^{\infty} \cos(\omega t) f(t)\,dt - i\int_{-\infty}^{\infty} \sin(\omega t) f(t)\,dt$$

We define the real part of this, $\Re(\mathcal{F}\{f(t)\})$ as the Fourier cosine transform of f and the imaginary part, $\Im(\mathcal{F}\{f(t)\}$ as the Fourier sine transform of f. Fourier analysis is an extremely powerful approach to differential equations and the foundation of digital signals theory.

9.2 PDE Problems

Unlike ODEs, where even if we did not know an initial or boundary condition we could just let it be some constant, PDEs are heavily dependent on their initial and boundary conditions. In fact, they will completely change the form of the solution! There is a much more deep reason why boundary conditions and initial conditions play such an important role for PDEs but we will not get into that; after all, this is for engineering not mathematics. To demonstrate different kinds of PDE problems, we will use the 1 dimensional transport equation as an example:

$$\frac{\partial u}{\partial t} + b\frac{\partial u}{\partial x} = 0$$

We typically like to think of one parameter as a time parameter and the others like spatial parameters, even if they're not really time and space parameters. This just helps with intuition.
We can think of initial values as conditions which says at time $t = 0$, what is the form of our solution. In order to know how our function behaves, we must know the state it is in before it evolves with respect to the PDE. So typically initial conditions have the form:

$$u(x, 0) = f(x)$$

PDE problems with a given initial condition are called initial value problems. But once again, this is usually enough only for ODEs. For PDEs, we also need constraints on our spatial behavior. For example, if we are modeling a standing wave on a string of length L, then we must have that at the end points of the standing wave, the amplitude of the wave be 0 since it is bounded! So we would have a condition like $u(0,t) = 0,\ u(L,t) = 0$. Or in general, we would have:

$$u(0,t) = g(t) \qquad u(L,t) = h(t)$$

Suppose it was a 2 dimensional equation. Then we would need similar conditions on the second spatial variable, y. These set of constraints on our PDE are called Boundary conditions. Often times, problems which have both boundary and initial conditions are known as Initial-Boundary Value Problems (IVBP). These are also known as Dirichlet problems (the "t" is silent).
There is some more terminology that we need to get familiar with. For any differential equation, we said that we have a set of functions which are invariant under the differential operator. Let us go back to the linear algebra picture:

$$Lx = f$$

where L is the differential operator and x is the function we are solving for. For the transport equation we have that

$$\left(\frac{\partial}{\partial t} + b\frac{\partial}{\partial x}\right) u = 0$$

We can identify the terms as $L = \frac{\partial}{\partial t} + b\frac{\partial}{\partial x}$, $x = u$ and $f = 0$. There are a set of functions which satisfy this differential equation with some scaling coefficients. What this means is we have

$$Lx = \lambda x \implies (L - \lambda)x = 0$$

This is indeed the same as the eigenvector equation. However, since we are now talking about functions instead of vectors, these would be called the *eigenfunctions* of the differential equation with a corresponding eigenvalue. These eigenfunctions behave according to their eigenvalues and their coefficients are determined by the Dirichlet problem.

9.3 Separation of Variables

In this section, we are going to learn perhaps one of the best methods to solve PDEs, known as separation of variables. As the name suggests, we are literally separating variables. What does this mean ? Suppose we have a PDE involving the function $u(x,t)$. For example, consider Fick's second law for non-steady state diffusion:

$$\frac{\partial u}{\partial t} = -D\frac{\partial^2 u}{\partial x^2}$$

This is hard to solve by inspection. In fact, it is very hard. So how can we deal with this ? Well, we have spent a good deal of time on ODEs. So what if there was a method to transform this PDE into an ODE ? This is exactly what separation of variables allows us to do. There are different ways of performing separation of variables. The following is a multiplicative separation of variable. Here is what we do; We make a bold assumption that our multivariable function is a product of 2 or more single variable functions. Then, we substitute this assumption into our PDE. Consider Fick's second law. What if we assume $u(x,t) = f(x)\cdot g(t)$? Then, our PDE becomes:

$$f\cdot g' = -D(f''\cdot g) \implies -D\frac{f''}{f} = \frac{g'}{g}$$

It may look like we did not achieve much. But, pay attention. On the right hand side, we have a fraction that only depends on t. On the left hand side, we have a fraction that only depends on x, and our PDE told us that this ration must be equal! The only way two expressions depending on different variables can be equal, is for both of them to be constants! This means we can set all these fractions equal to some constant λ:

$$-D\frac{f''}{f} = \frac{g'}{g} = \lambda \implies \frac{f''}{f} = \frac{-\lambda}{D}, \qquad \frac{g'}{g} = \lambda$$

Now look at what just happened! We just turned our PDE into the following two ODEs!

$$\begin{cases} f''(x) + \frac{\lambda}{D}f(x) = 0 \\ g'(t) - \lambda g(t) = 0 \end{cases}$$

And good thing we have a huge toolbox for solving these! In fact we get that:

$$f(x) = A_0 \cos\left(\sqrt{\frac{\lambda}{D}}x\right) \qquad g(t) = e^{-\lambda t}$$

Now, we go back to where we assumed $u(x,t) = f(x) \cdot g(t)$. Thus, to find u we must multiply these two functions together.

$$u(x,t) = A_0 e^{-\lambda t} \cos\left(\sqrt{\frac{\lambda}{D}}x\right)$$

All of this was very beautiful and elegant. However, there are some subtleties we skipped over. For example, what is A_0 ? What is the meaning of λ ? Is this true for all BVPs of the diffusion equation ? Are there other angular frequencies of cosine that satisfy this equation ? In fact the expression we found is only 1 solution to the diffusion equation. The big problem we have right now is λ. What exactly is it ? There are only a certain set of λs which satisfy our equation, λ_n. Each of these are eigenvalues for our differential equation. However, we know from linear algebra that every eigenvalue must have a corresponding eigenvector, which in our case is an *eigenfunction*. So we must find all eigenfunctions and their corresponding values in order to completely solve the diffusion equation. On the other hand, we know that by linearity, The following is also a solution:

$$u(x,t) = \sum_{n=0}^{\infty} A_n f_n(x) g_n(x)$$

In fact, this shows that separation of variables naturally leads to a series solution, not a single one. Let us go back and analyze our solutions in further detail. The spatial solution $f(x)$ yields the following result:

$$f_n(x) = \cos\left(\sqrt{\frac{\lambda}{D}}\frac{n\pi x}{L}\right)$$

These form an orthogonal basis on the boundary condition interval, $[0, L]$. Thus, any linear combination is also a solution:

$$f(x) = \sum_{n=1}^{\infty} A_n \cos\left(\sqrt{\frac{\lambda}{D}}\frac{n\pi x}{L}\right)$$

This is indeed the Fourier series! We have, then, that

$$A_n = \frac{2}{L}\int_0^L u(x,t)\cos\left(\sqrt{\frac{\lambda}{D}}\frac{n\pi x}{L}\right)\,dx$$

Thus, we see that the presence of eigenvalues and linearity leads naturally to a series solution of the diffusion equation which takes the following form:

$$u(x,t) = \sum_{n=1}^{\infty} A_n e^{-\lambda_n t}\cos\left(\sqrt{\frac{\lambda}{D}}\frac{n\pi x}{L}\right)$$

This shows the importance of the Fourier series in interpreting solutions of PDEs. We must be very careful when we apply separation of variables and be mindful of the natural emergence of Fourier series in this method.

9.4 The Heat Equation

Now we have pretty much built the necessary tool box to approach some famous PDEs. The first of these will be the famous heat conduction equation:

$$\frac{\partial u}{\partial t} = \alpha^2\frac{\partial^2 u}{\partial x^2}$$

How would we solve this ? Well, we use our new found tool, the Fourier transform. But first, we must add some boundary conditions to this equation. We consider the following:

$$u(x,0) = f(x), \quad u(0,t) = u(L,t) = 0$$

for a solid of length L. First, let us express the Fourier transform for the right hand side derivative:

$$\mathcal{F}\left\{\frac{\partial^2 u}{\partial x^2}\right\} = \int_{-\infty}^{\infty} e^{-i\omega x}\frac{\partial^2 u}{\partial x^2}\,dx$$

Now, we can use 2 iterated integration by parts to turn the position second derivative into just $u(x,t)$. The boundary terms that arise will be 0 because our position boundary conditions are 0

for all times, t. Thus we get:

$$\mathcal{F}\left\{\frac{\partial^2 u}{\partial x^2}\right\} = -\omega^2 \int_{-\infty}^{\infty} e^{-i\omega x} u(x,t)\, dx = -\omega^2 \hat{u}(\omega, t)$$

The same trick, however, does not work so well for the time derivative. Remember, we are transforming x to ω domain. So we are not dealing with t. Thus, we can just take out the derivative operator from the Fourier transform:

$$\mathcal{F}\left\{\frac{\partial u}{\partial t}\right\} = \int_{-\infty}^{\infty} e^{-i\omega x} \frac{\partial u}{\partial t}\, dx = \frac{\partial}{\partial t} \int_{-\infty}^{\infty} e^{-i\omega x} u(x,t)\, dx = \frac{\partial \hat{u}}{\partial t}$$

So the heat equation in the Fourier domain becomes:

$$\frac{\partial \hat{u}}{\partial t} = -\omega^2 \alpha^2 \hat{u}$$

This is a separable equation! We know that:

$$\hat{u}(\omega, t) = \hat{u}(\omega, 0) e^{-\alpha^2 \omega^2 t}$$

Now, we must transform this back to the position domain, x. To do so, we calculate the inverse Fourier transform:

$$\mathcal{F}^{-1}\left\{\hat{u}(\omega, t)\right\} = \int_{-\infty}^{\infty} e^{i\omega x} \hat{u}(\omega, 0) e^{-\alpha^2 \omega^2 t}\, d\omega$$

This integral is very complicated and involves convolution theory, so we will not explicitly calculate it, however, you can check for yourself (or later when we do Green's functions) that this yields the following integral:

$$u(x,t) = \int_{-\infty}^{\infty} \frac{1}{\sqrt{4\omega t}} \exp\left(-\frac{(x-y)^2}{4\omega t}\right) f(y)\, dy$$

The surprising thing about this solution is that it looks awfully similar to Gauss' Normal distribution formula! Thus, we see that heat distribution follows a normal distribution in a solid which intuitively would also make sense. If we have a heat source on a solid, we expect heat to be at a maximum near the source and gradually radiate out on the surface and then the bulk of the material. As you may have noticed, the heat equation and the diffusion equation have the same form. In fact, they are identical equations, mathematically. So what is the difference between the Fourier series approach

and the Fourier transform approach ? For finite domains, the Fourier series is the approach that is used, because as you may expect, an infinite sum with increasing amplitudes will diverge. There is plenty of arguments to be made about whether the Fourier series converges or diverges. This is why typically it is good to use the Fourier transform for infinite domains.

9.5 The Wave Equation

The wave equation is another type of PDE. This equation is seen quite often in mechanical vibrations, electromagnetic theory and even mechanical buckling. Most importantly, it describes the propagation of waves through a medium. It is vary similar to the heat equation, however, now instead of a first order time derivative, we have a second order time derivative:

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$$

This is the one dimensional equation. In higher dimensions, the wave equation is of the form:

$$\frac{\partial^2 u}{\partial t^2} = c^2 \nabla^2 u$$

Here, c is the speed of the wave and $u(x,t)$ describes the amplitude of the wave. We can solve these equations using separation of variables and Fourier transform. We will not explore the Fourier transform approach as that would require defining multi-dimensional Fourier transforms and introduces many complexities. Instead, we will continue with separation of variables! Let us assume 3 spatial dimensions and 1 time dimension (boy it's about to get very messy, very fast!). Then the wave equation is:

$$\frac{\partial^2 u}{\partial t^2} = c^2 \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \right) \qquad (x,y,z) \in [0, L_x] \times [0, L_y] \times [0, L_z]$$

Where we say $u = 0$ for all boundary points. This is called a Dirichlet problem. We also introduce the following set of initial conditions:

$$u(x,y,z,0) = f(x,y,z) \qquad \left. \frac{\partial u}{\partial t} \right|_{t=0} = g(x,y,z)$$

Now we introduce our separation of variables where $\mathbf{x} = (x, y, z)$:

$$u(\mathbf{x}, t) = X(x)Y(y)Z(z)T(t)$$

This yields that

$$\frac{1}{c^2}T''XYZ = X''YZ + XY''Z + XYZ''$$

After we separate, we get this messy expression:

$$\frac{1}{c^2}\frac{T''}{T} = \frac{X''}{X} + \frac{Y''}{Y} + \frac{Z''}{Z} = -\lambda$$

We choose $-\lambda$ because it will make our ODEs simpler. Now we have a system of equations on the right hand side. So what do we do ? We do the following; let $\lambda = \lambda_x + \lambda_y + \lambda_z$ such that each one satisfies:

$$\frac{X''}{X} = -\lambda_x \qquad \frac{Y''}{Y} = -\lambda_y \qquad \frac{Z''}{Z} = -\lambda_z$$

The rest is pretty much the same procedure as the 1 dimensional example! The only difference between this wave equation and heat equation is that we have a second order time derivative which turns our time function into a sinusoid instead of an exponential since its ODE will be a second order one without a first order term. Thus, our solutions will be:

$$X_n(x) = \sin\left(\frac{n\pi x}{L_x}\right) \qquad Y_m(y) = \sin\left(\frac{m\pi y}{L_y}\right) \qquad Z_p(z) = \sin\left(\frac{p\pi z}{L_z}\right)$$
$$\lambda_x = \left(\frac{n\pi}{L_x}\right)^2 \qquad \lambda_y = \left(\frac{m\pi}{L_y}\right)^2 \qquad \lambda_z = \left(\frac{p\pi}{L_z}\right)^2$$

Now what about the time term ? Right now, we have the values for all the Lambda terms. The time term depends on all three of them, so it will have all three parameters in it! In fact, the solution for $T(t)$ is the solution to

$$T'' = -c^2(\lambda_x + \lambda_y + \lambda_z)T$$

Let ω_{mnp} represent the net angular frequency of the time function. Then

$$T(t) = A_{nmp}\sin(\omega_{nmp}t) + B_{nmp}\cos(\omega_{nmp}t); \quad \omega_{mnp} = c\sqrt{\left(\frac{n\pi}{L_x}\right)^2 + \left(\frac{m\pi}{L_y}\right)^2 + \left(\frac{p\pi}{L_z}\right)^2}$$

Since these are all singular solutions of the wave equations, a full solution is a Fourier series of the individual sinusoids. This also gives us - albeit very very nasty - expressions for A_{nmp} and B_{nmp}. The final, complete solution, to the 3 dimensional wave equation is:

$$u(x,y,z,t) = \sum_{p=1}^{\infty}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\left[A_{nmp}\sin(\omega_{nmp}t) + B_{nmp}\cos(\omega_{nmp}t)\right]\sin\left(\frac{n\pi x}{L_x}\right)\sin\left(\frac{m\pi y}{L_y}\right)\sin\left(\frac{p\pi z}{L_z}\right)$$

Where the Fourier coefficients will be calculated by the following triple integrals:

$$A_{nmp} = \frac{2^3}{L_xL_yL_z}\iiint_L f(x,y,z)\sin\left(\frac{n\pi x}{L_x}\right)\sin\left(\frac{m\pi y}{L_y}\right)\sin\left(\frac{p\pi z}{L_z}\right)\,dV$$
$$B_{nmp} = \frac{2^3}{\omega_{nmp}L_xL_yL_z}\iiint_L g(x,y,z)\sin\left(\frac{n\pi x}{L_x}\right)\sin\left(\frac{m\pi y}{L_y}\right)\sin\left(\frac{p\pi z}{L_z}\right)\,dV$$

Where $L = [0, L_x] \times [0, L_y] \times [0, L_z]$. Yes, this is ugly, but it works. If you know an easier way, please let me know - we will actually learn an easier way called Green's functions.

9.6 Laplace's Equation

Laplace's equation is a very important equation. It was first introduced to us when we discussed potential functions and conservative vector fields. We know that for some vector field, f is a potential function if and only if it satisfies Laplace's equation:

$$\nabla^2 f = 0$$

So how can we solve this equation ? Consider the 2 dimensional Laplace equation:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

We also apply the following boundary conditions:

$$u(0,y) = g(y) \qquad u(L,y) = 0$$
$$u(x,0) = 0 \qquad u(x,H) = 0$$

The solutions to this equation are known as *harmonic functions*, these play an important role in describing complicated wave functions and . So how does one solve this equation ? Once again, we perform a separation of variables! (yes, once again) We will also later explore the Green's function for this equation. So once again, we go ahead and perform a multiplicative separation:

$$u(x,y) = X(x)Y(y) \implies X''Y + XY'' = 0 \implies \frac{X''}{X} = -\frac{Y''}{Y} = \lambda$$

$$\implies \frac{d^2X}{dx^2} - \lambda X = 0; \;\; X(L) = 0 \qquad \frac{d^2Y}{dy^2} - \lambda Y = 0; \;\; Y(0) = Y(H) = 0$$

Now the second equation we can solve since we have sufficient boundary conditions, however, the first equation we do not have enough conditions. Without knowing the value of λ we cannot do anything. So we apply the Fourier series:

$$\lambda_n = \left(\frac{n\pi}{H}\right)^2 \qquad Y_n(y) = \sin\left(\frac{n\pi y}{H}\right)$$

We then use this to solve the first equation, with the addition of $X(L) = 0$ boundary condition we get:

$$X_n(x) = B_n \sinh\left(\frac{n\pi}{H}(x-L)\right)$$

Now, we have enough information to form the product solution:

$$u_n(x,y) = \sum_{n=0}^{\infty} B_n \sinh\left(\frac{n\pi}{H}(x-L)\right)\sin\left(\frac{n\pi y}{H}\right)$$

To find the appropriate solution, we simply employ the condition $u(0,y) = g(y)$:

$$u(0,y) = \sum_{n=0}^{\infty} B_n \sinh\left(\frac{n\pi}{H}(-L)\right)\sin\left(\frac{n\pi y}{H}\right) = g(y)$$

Now we could just use orthogonality of the sines, but there is an easier approach. We can just observe that this is a Fourier sine series. Except, instead of the coefficient just being B_n, the coefficient of the sine is

$$B_n \sinh\left(\frac{-n\pi L}{H}\right)$$

So the expression for the Fourier coefficient, B_n is a little messy this time, we have that

$$B_n = \frac{2}{H \sinh\left(\frac{-n\pi L}{H}\right)} \int_0^H g(y) \sin\left(\frac{n\pi y}{H}\right)\, dy$$

This is the final differential equation!

10 Further Supplements for Deeper Understanding

10.1 Generalized Integral Transforms

So far, we have discussed Fourier and Laplace transforms. We can sort of see a pattern between the original transform and its inverse. In fact, let us take a moment to explore the general concept of an integral transform. The idea is the same; we are taking a function and expressing it in a different function space. We do this because sometimes there are certain properties of functions in different function spaces that work out nicer than the original function space. This transformation or mapping is achieved via integrating the function of interest. In general, an integral transform has the form:

$$\tilde{f}(u) = \int_S f(v) K(v, u)\, dv$$

Where S is the space of transformation we are interested in. The function $K(v, u)$ is our transform function. Sometimes, this function has a non-empty kernel in our function spaces. If this is true, we can write an inverse transform of this form, using the inverse mapping of K. So the inverse mapping, $K^{-1}(u, v)$ is our inverse transform function and the inverse transform is:

$$f(v) = \int_{S^*} \tilde{f}(u) K^{-1}(u, v)\, du$$

Where S^* is our new function space's equivalent domain interval. Understanding integral transforms is an *integral* (ha ha) part of understanding the mathematical foundations of signal analysis and

differential equations.

10.2 Legendre's Equation

Okay, take a deep breath! This one is going to be one of the most mathematically demanding sections of this entire handout. This shebang starts with the famous Legendre's differential equations;

$$(1-x^2)y'' - 2xy' + \left(\alpha(\alpha+1) - \frac{\beta^2}{1-x^2}\right)y = 0 \quad \alpha, \beta \in \mathbb{C}$$

This equation is linear and second order. Thus, it has 2 linearly independent solutions. These solutions are expressed in terms of the gamma function, $\Gamma(n)$, and the Gaussian hypergeometric function, ${}_2F_1(a, b; c, z)$. These functions are defined, respectively as:

$$\Gamma(n) = \int_0^\infty x^{n-1}e^{-x}\,dx$$

$${}_2F_1(a,b;c,z) = \sum_{k=0}^{\infty} \frac{(a)_k(b)_k}{(c)_k}\frac{(z)_k}{k!}, \quad s.t.\,(x)_k = \prod_{j=0}^{k-1}(x-j)$$

the terms of the form $(x)_n$ are known as Pochhammer Symbols which is analogous to a falling factorial function. Using these two functions, we can formulate the solutions of Legendre's differential equation as:

$$P_\alpha^\beta(z) = \frac{1}{\Gamma(1-\beta)}\left[\frac{z+1}{z-1}\right]^{\beta/2} {}_2F_1\left(-\alpha, \alpha+1; 1-\beta; \frac{1-z}{2}\right) \quad s.t. |1-z| < 2$$

and

$$Q_\alpha^\beta(z) = \frac{\sqrt{\pi}\,\Gamma(\alpha+\beta+1)}{2^{\alpha+1}\Gamma(\alpha+3/2)}\frac{e^{i\beta\pi}(z^2-1)^{\beta/2}}{z^{\alpha+\beta+1}}\,{}_2F_1\left(\frac{\alpha+\beta+1}{2}, \frac{\alpha+\beta+2}{2}; \alpha+\frac{3}{2}; \frac{1}{z^2}\right), \quad s.t. |z| > 1$$

and yes, these are very very long expressions, for a very complicated differential equation. We will not cover how these solutions are derived as that is a very heavy subject. We will accept that $P_\alpha^\beta(z)$ and $Q_\alpha^\beta(z)$ are linearly independent solutions to Legendre's differential equation. These are also called the associated Legendre polynomials.

10.3 Laplace's Equation and Spherical Harmonics

Consider a scalar field, $f : \mathbb{R}^3 \to \mathbb{C}$ that is twice differentiable on all its variables, which satisfies Laplace's equation, meaning:

$$\nabla^2 f = 0$$

If we express this in spherical coordinates:

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial f}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial f}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 f}{\partial\phi} = 0$$

To solve this differential equation, we apply a separation of variables such that $f(r,\theta,\phi) = R(r)Y(\theta,\phi)$ to obtain:

$$\frac{1}{R}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) = \lambda, \quad \frac{1}{Y}\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial Y}{\partial\theta}\right) + \frac{1}{Y}\frac{1}{\sin^2\theta}\frac{\partial^2 Y}{\partial\phi} = -\lambda$$

Applying a separation of variables to the second equation again, with $Y(\theta,\phi) = \Theta(\theta)\Phi(\phi)$ we get firstly:

$$\frac{1}{\Phi}\frac{d^2\Phi}{d\phi^2} = -m^2, \quad s.t.\, m \in \mathbb{C},\ Re(m) \in \mathbb{Z}$$

Since Φ is period over intervals of 2π, m must be integer and thus, Φ is of the form:

$$\Phi(\phi) = Ne^{im\phi}$$

We also find, for $\Theta(\theta)$ the equation:

$$\lambda\sin^2\theta + \frac{\sin\theta}{\Theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right) = m^2$$

This equation is of the form of a Sturm-Liouville differential equation. Thus, we can infer that $\lambda = l(l+1)$ for some $l \in \mathbb{Z}\ s.t.\ l \geq |m|,\ l \geq 0$. Moreover, if we apply a change of variables such that $t = \cos\theta$, we see that the equation turns into Legendre's equation! Thus, the solution will be the associated Legendre polynomials. Hence:

$$Y_l^m(\theta,\phi) = Ne^{im\phi}P_l^m(\cos\theta)$$

Here, $Y_l^m : S^2 \to \mathbb{C}$ is called a *spherical harmonic function of degree l and order m* and the polynomial $P_l^m : [-1, 1] \to \mathbb{R}$ is its associated Legendre Polynomial. The domain of Y is S^2 since (θ, ϕ) are coordinates for the 2-sphere manifold. This plays a fundamental role in deriving the eigenvalues of the radial equation for different atoms.

10.4 Laguerre's Equation

Here we are with another interlude about differential equations! Fasten your seat-belts because this one ain't getting any easier! The goal of this interlude is for you to understand Laguerre's polynomials. Similar to Legendre polynomials, Laguerre polynomial are also non-trivial solutions to a differential equation. This differential equation, obviously, is known as Laguerre's differential equation and it is the following equation:

$$xy'' + (\alpha + 1 - x)y' + ny = 0, \quad s.t.\, y = y(x),\, n \in \mathbb{Z}^+ \cup \{0\}$$

The solution of this general equation are called generalized Laguerre polynomials or associated Laguerre polynomials. Similar to Legendre's polynomials, we will not go into how these solutions are derived as that would require an entire handout on differential equations (which is hopefully done by the time you are reading this). The associated Laguerre polynomial that solves this differential equation is the following:

$$L_n^\alpha(x) = \frac{x^{-\alpha}}{n!}\left(\frac{d}{dx} - 1\right)^n x^{n+\alpha}$$

This operator form is derived through L's generating form:

$$\sum_{n=0}^{\infty} t^n L_n^\alpha(x) = \frac{1}{(1-t)^{\alpha+1}} e^{\frac{-tx}{1-t}}$$

The key thing that makes this polynomial important is that it also appears in one of the representations of the spherical harmonic function.

Another helpful representation of this function is through integration over a complex circle, C in the complex plane counterclockwise such that:

$$L_n^\alpha(x) = \frac{1}{2\pi i} \oint_C \frac{e^{-xt/1-t}}{(1-t)^{\alpha+1} t^{n+1}}\, dt$$

10.5 Green's Functions

At last, we are almost done! This last topic is perhaps the most powerful method, ever invented, to solve differential equations. It is Green's functions. So what is this method ? Suppose we have a differential equation

$$Lx = f$$

L is a linear differential operator, x is the function we are trying to solve for and f is the independent terms. Then the Green's function, denoted by G must satisfy the following identity:

$$LG = -\delta$$

where δ is the Dirac delta distribution. For example, let us consider the Helmholtz equation:

$$\nabla^2 u + k^2 u = f$$

Here, $L = \nabla^2 + k^2$, and the function we want to solve for is $x = u$. So the Green's function must satisfy

$$(\nabla^2 + k^2)G(\mathbf{r}, \mathbf{r}_0) = [\delta(\mathbf{r} - \mathbf{r}_0)]^n$$

The reason we raised δ to the n-th power is because we did not specify the number of dimensions considered in the Laplacian operator. For 3D, the Helmholtz equation's Green's function will satisfy

$$\nabla^2 G + k^2 G = \delta^3 = \delta(x - x_0)\delta(y - y_0)\delta(z - z_0)$$

So what is the power of this method ? The "juice" is in the following; if $G(\mathbf{r} - \mathbf{r}_0)$ is a Green's function, then

$$u(\mathbf{r}) = \int_S f(\mathbf{r}_0)G(\mathbf{r} - \mathbf{r}_0)\, dV(\mathbf{r}_0)$$

So all we need is the initial condition and the Green's function to find the solution. Here, S is the region for which we defined the problem. Remember that with PDEs it is very important to have a well defined set of initial and boundary conditions, otherwise, it could lead to contradictions in our solution.

So how do we compute the Green's function for the Helmholtz equation ? We know that based

on what the Helmholtz equation represents (forced 3 dimensional wave propagation) we know that the solution must obey translational invariance. This means, if we shift the origin by $\mathbf{r}_0$, it will not change the solution's form. So we express a change of variables

$$r = |\mathbf{r} - \mathbf{r}_0| \implies G(\mathbf{r} - \mathbf{r}_0) \to G(r)$$

We see that based on this, spherical coordinates could be a more natural choice. So the Helmholtz equation becomes

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{dG}{dr}\right) + k^2 G = 0$$

We can easily solve this ODE:

$$G(r) = \frac{1}{r}\left(Ae^{ikr} + Be^{-ikr}\right)$$

Let us consider a wave that is propagating outwards, meaning $B = 0$ as we would not consider the e^{-ikr} term. Then

$$G(r) = A\frac{e^{ikr}}{r}$$

But what about A ? By the definition of the Green's function, we have that

$$\nabla^2 G + k^2 G = -\delta(r)$$

If we integrate both sides over a small ball B_ε of radius ε.

$$\oint_{B_\varepsilon} (\nabla^2 + k^2) G \, dV = \oint_{B_\varepsilon} -\delta(r) \, dV = -1$$

We know the second integral is -1 based on the property of the Dirac Delta distribution which we saw earlier in 6.5. Now in the limit as the radius of the ball approaches 0,

$$\lim_{r\to 0} e^{ikr} = 1 \implies G(r) \overset{\varepsilon\to 0}{\sim} \frac{A}{r}$$

So the following happens to the $k^2 G$ term:

$$\oint_{B_\varepsilon} k^2 G \, dV \sim k^2 \int_0^\varepsilon G(r) r^2 \, dr = Ak^2 \int_0^\varepsilon r \, dr \overset{\varepsilon\to 0}{=} 0$$

This means we can completely get rid of it for a small sphere boundary condition. Thus, we only have that

$$\oint_{B_\varepsilon} \nabla^2 G \, dV = -1 \overset{\text{Divergence Theorem}}{\Longrightarrow} \oint_{C_\varepsilon} \nabla G \cdot dA = -1$$

where C_ε is just the contour of the ball. Now, let usu compute the integral

$$\oint_{C_\varepsilon} \nabla G \cdot dA = \int_0^\varepsilon \frac{dG}{dr} \cdot dA$$

On the sphere, $r = \varepsilon$ which means, $dA = \mathbf{r}\varepsilon^2 d\Omega$. This simply a change of variables! So now, we can simplify the integral to:

$$\int_{C_\varepsilon} \frac{dG}{dr} \varepsilon^2 \, d\Omega$$

In order to further simplify our calculations, we take a Taylor expansion for the exponential function in $G(r)$:

$$G(r) = A\frac{e^{ikr}}{r} \overset{r\to\varepsilon}{\sim} \frac{A}{r}(1 + ik\varepsilon + O(\varepsilon^2)) \implies \frac{dG}{dr} \sim A\frac{(ikr-1)e^{ikr}}{r} = A\left(-\frac{1}{\varepsilon^2} - O\left(\frac{1}{\varepsilon^2}\right)\right)$$

Substituting this into our integral, we find that

$$\oint_{C_\varepsilon} A\left(-\frac{1}{\varepsilon^2} - O\left(\frac{1}{\varepsilon^2}\right)\right)\varepsilon^2 \, d\Omega = \oint_{C_\varepsilon} -A \, d\Omega = -4\pi A$$

Thus, we have that $-4\pi A = -1 \implies A = 1/4\pi$. Therefore, the final form of our Green's function is:

$$G(r) = G(\mathbf{r}, \mathbf{r}_0) = \frac{e^{ik|\mathbf{r}-\mathbf{r}_0|}}{4\pi|\mathbf{r}-\mathbf{r}_0|}$$

Now, we can easily find the solution:

$$u(\mathbf{r}) = \int_{\mathbb{R}^3} \frac{e^{ik|\mathbf{r}-\mathbf{r}_0|}}{4\pi|\mathbf{r}-\mathbf{r}_0|} f(\mathbf{r}_0) \, d^3\mathbf{r}_0$$

Note that the notation

$$\int_{\mathbb{R}^3} d^3\mathbf{r} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} dx \, dy \, dz$$

So there is our solution! But we also need an intuition for why it is that we did all this work. So what is the Green's function ? Physically, the Green's function represents the response of a

system to a point source. What does this mean ? Consider the Helmholtz equation. This equation represents some sort of wave propagation. It is a type of time-independent wave equation. We considered this equation with a closed spherical boundary condition. So the Green's function of the Helmholtz equation represents the response of this system to a point source at the center of the sphere which generates propagating waves. Recall that earlier we said that the Dirac delta represents an impulse. So if we want to simulate an impulse to a system, we add a Dirac delta term. But this is only half of the puzzle! The Green's function of the equation will then tell us how our system reacts to this impulse! It is the other half of the story. Suppose we take Maxwell's equations for a spherical domain. A Dirac delta tells us that there is a point source in the center of this sphere which emits electromagnetic radiation. The Green's function of those set of Maxwell's equations will then tell us the behavior of these propagating waves. For the heat equation, the Green's function will tell us how heat is distributed throughout the system if there is an impulse of heat at a point.

So why does the following hold true ?

$$u(\mathbf{r}) = \int_S G(\mathbf{r}, \mathbf{r}_0) f(\mathbf{r}_0)\, dV(\mathbf{r}_0)$$

We said that the Green's function shows how our system reacts to an impulse. So by integrating over the product of the Green's function and the initial condition, we are adding up every bit of response to infinitely many impulses. We are putting all these infinitesimal behaviors together in a continuous matter through the integral which allows us to see a continuous (not necessarily in the rigorous sense of the word continuous) function as the solution. The integral essentially states that the behavior of the function at each infinitesimal impulse must obey the Green's function, thus, integrating it over every possible point defined by the Dirichlet problem gives us the full solution to the PDE.