# Limits and Continuity

Barsam Rahimi

November 2023

## 1 Behaviour of Functions

Note: This handout is simply a guide through the concept of a limit. There are topics here labeled as "Advanced". This means that they are not taught in Ontario's High School and will be taught in University. Feel free to skip these topics if you struggle to understand the previous topics. In the field of mathematics, functions are used to establish a relationship between 2 or more variables (we will explore functions of several variables in later chapters). A function is formally defined as a set of rules which connects the elements of one set to another. The function f(x) connects the elements of a set D (which is all the possible values of x) to the elements of another set R (all the possible values of f(x)). The details of what these sets can include depends on the function we are dealing with. Take the function $f(x) = ln(x)$ where ln denotes the natural logarithm (the logarithm with base e). For this function, 0 is not an element of set D as the logarithm is not defined for the value of 0. For further information on this topic, you may refer to Thomas' Calculus: Early Transcendentals or Stewart's Calculus.

## 2 End Behavior: Asymptotes and Functions at infinity

When we face some rational function f(x), we see that there are vertical and horizontal asymptotes involved. The way f(x) behaves near these lines or curves is known as end behavior. Take the function $f(x) = \frac{(x-1)}{(3x-2)(x+5)}$ as an example. When analyzing the end behavior of the function, we state the following:
as $x \to \infty$, $f(x) \to \frac{1}{3}$
as $x \to -\infty$, $f(x) \to \frac{1}{3}$
as $x \to -5^{+}$, $f(x) \to \infty$
as $x \to \frac{2}{3}^{-}$, $f(x) \to \infty$
The idea of limits is in nature similar to that of end behavior. However, with limits we use a more formal and efficient notation which can also be used to prove certain properties of functions and explore concepts such as continuity and discontinuity, an infinite series or sequence and even the area

under a curve which will turn out to be one of the most important tools ever invented by humanity. However, that is the subject of another chapter.

## 3 Limit Notation: Turning End Behavior to Limits

Let us use the same function in the previous section as an example. Let us define f(x) to be:

$$f(x) = \frac{(x-1)}{(3x-2)(x+5)}$$

Here are some examples of limit notation:

$$\lim_{x \to -5^+} f(x) = \infty$$

$$\lim_{x \to \infty} f(x) = \frac{1}{3}$$

The first statement above reads as such: "The limit as x approaches negative five from the right of f of x is equal to infinity". The second one reads: "The limit as x approaches to infinity of f of x is equal to one third". Note that the following 2 statements have the exact same meaning:

$$\lim_{x \to \infty} f(x) = \frac{1}{3}$$

$$as\ x \to \infty, f(x) \to \frac{1}{3}$$

All we have done is bring the first part of the statement which starts with "as" and placed it under "lim". Then we state what is the function we are applying the limit to and write the value approached by the function after the equal sign. Here are some more examples of turning end behavior to limit notation.

$$as\ x \to \frac{1}{2}, f(x) \to \infty \qquad \lim_{x \to \frac{1}{2}} f(x) = \infty$$

$$as\ x \to \infty, f(x) \to 2 \qquad \lim_{x \to \infty} f(x) = 2$$

$$as\ x \to \frac{3}{5}, f(x) \to -\infty \qquad \lim_{x \to \frac{3}{5}} f(x) = -\infty$$

$$as\ x \to -\infty, f(x) \to \infty \qquad \lim_{x \to -\infty} f(x) = \infty$$

# 4 Evaluating Limits

The above examples all are simply about notation. However, when presented with an actual limit problem, will not be able to achieve any result. The concept of a limit is not simply related to the asymptotic behavior or end behavior of a function. It can be defined for all values within the domain of the function. So, let us take a deeper look at limits and how to evaluate them.

**Limit Theorems** let us define 2 functions f(x) and g(x)

let L represent the limit as $x$ approaches $x_0$ of f(x) and let M represent the limit as $x$ approaches $x_0$ of g(x) then the following rules will hold for limits as x approaches some constant $x_0$, c also represents some constant:

$$\lim_{x\to x_0} c = c$$

$$\lim_{x\to x_0} f(x) \pm g(x) = L \pm M$$

$$\lim_{x\to x_0} f(x) \times g(x) = L \times M$$

$$\lim_{x\to x_0} \frac{f(x)}{g(x)} = \frac{L}{M}, M \neq 0$$

$$\lim_{x\to x_0} k \cdot f(x) = k \cdot L$$

$$\lim_{x\to x_0} [f(x)]^n = L^n, n > 0$$

$$\lim_{x\to x_0} \sqrt[n]{f(x)} = L^{\frac{1}{n}}, n \neq 0$$

Let us look at some examples to better understand how to evaluate limits. DNE is for Does Not Exist;

$$\lim_{x\to 4} x^3 - 3x^2 + 6x - 2 = \lim_{x\to 4} x^3 - 3 \cdot \lim_{x\to 4} x^2 + 6 \cdot \lim_{x\to 4} x - 2$$

$$= (4)^3 - 3 \cdot (4)^2 + 6 \cdot (4) - 2 = 64 - 48 + 24 - 2 = 38$$

$$\lim_{x\to e^{ix}} \ln x = \ln e^{ix} = ix \cdot \ln e = ix, \quad i = \sqrt{-1}$$

$$\lim_{x\to\infty} \frac{x-2}{x^2-4} = \lim_{x\to\infty} \frac{x-2}{(x-2)(x+2)}$$

$$= \lim_{x\to\infty} \frac{1}{x+2} = \frac{1}{\lim_{x\to\infty} x + \lim_{x\to\infty} 2} = \frac{1}{\infty} = 0$$

$$\lim_{x\to\infty} sin(x) = D.N.E.$$

# 5 Continuity

In the previous grades, we have learned about what a continuous function is. A simple - but rather trivial, informal and improper - definition of a continuous function is a function which can be drawn on the coordinate system without the pencil/pen being lifted off the page. For example, for drawing the function $f(x) = x$, we can draw a straight line on the coordinate system without ever lifting our pen and re-putting it on the page, therefore, $f(x)$ is continuous. This definition, however, cannot be accepted as a general definition for continuity. As a matter a fact, the idea of a limit turns out to be the very concept we need in order to define what a continuous function is.

According to the proper definition of continuity, there are 4 statements that must be true about a function for it to be continuous: (c is some arbitrary constant within the domain of $f(x)$)

Let $f(x)$ be some function, then $f(x)$ is continuous at an interior point $x = c$ of its domain if and only if the following conditions are true for the function:

$$f(c)\ exists \mid c \in \Re$$

$$\lim_{x \to c} f(x)\ exists$$

$$\lim_{x \to c} f(x) = f(c)$$

$$\lim_{x \to c^+} f(x) = \lim_{x \to c^-} f(x)$$

The last condition may not be applicable to some functions such as $f(x) = sin(x)$ Let us apply this to a 2 examples in order to test for continuity of these function.

Example 1: (e denotes Euler's number, $\varphi$ denotes the golden ratio)

$$f(x) = \begin{cases} 0 & x \leq -4 \\ e^{xlog_\pi x} & -4 < x \leq 1 \\ sin(xcos^{-1}(x)) & 1 < x < 5 \\ \varphi^2 & 5 \leq x \end{cases}$$

Let us examine this function using the tests for continuity, c = 1

$$f(1) = 1$$

$$\lim_{x \to 1} f(x) = 1 = f(1)$$

$$\lim_{x \to 1^+} f(x) \neq \lim_{x \to c^-} f(x)$$

Since the very last condition of continuity is not met, the function $f(x)$ is not a continuous function. It has a jump discontinuity at $x = 1$.

Example 2:

$$f(x) = e^{ix}, i = \sqrt{-1}$$

Let us examine this function using $x = \frac{\ln \varphi}{i}$

$$f(\frac{\ln \varphi}{i}) = \varphi$$

$$\lim_{x \to \frac{\ln \varphi}{i}} f(x) = \varphi = f(\frac{\ln \varphi}{i})$$

As the final condition is not applicable, we can conclude that the function $f(x) = e^{ix}$ is continuous.

Note:

Euler's number, e, is defined as the following expression which I encourage you to try to prove:

$$e = \lim_{x \to \infty} \left(1 + \frac{1}{x}\right)^x$$

The Instantaneous Rate Of Change (IROC), also known as the slope of the tangent line or simply the derivative of a function, can be defined by the following expression:

$$IROC_f \ or \ \frac{df}{dx} \ or \ f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

We will discuss derivatives in greater detail in the next chapter.

# 6 Advanced: Proper Definition of a Limit

In the previous chapters, we introduced limits as a simple notation rather than a true concept of Calculus which is used to prove theorems and conjectures. We have, in fact, very poorly defined the concept of a limit. In order to use this concept in further proofs of mathematical tools and theorems, we must have a precise, formal definition of the limit. Therefore, let us take a look at a rather simple "thought experiment". Suppose we are watching the values of a function f(x) as x approaches $x_0$ (without taking on the value of $x_0$ itself ). Certainly we want to be able to say that f(x) stays within one tenth or one hundredth or a thousandth of a unit from L, where L is the limit of f(x) as $x$ approaches some number $x_0$ within the domain of f. Now, to define the limit as a number in some arbitrary range would not be an accurate definition as this range may be different and does not describe the infinitesimal approaching of a number to another. To define limit, we need to think about the difference between the limit and the actual value of the function at $x_0$. If the limit is an infinitesimal approaching of a function's value, then the difference between the function's value at $x_0$ and its limit must be infinitely small. In simpler words, if I subtract a number and another number which is infinitely close to the first one, the result must be infinitely small, but not zero as then we will be talking about the function itself and not its limit.

Therefore, let us define two, arbitrary, infinitely small constants, $\epsilon$ and $\delta$. Both of these constants are infinitely small, but greater than zero. They do not have a true value and are a representation of a concept of a number rather than an actual number.

A limit, we have learned, will always be infinitely close to the value of the function at all points assuming the function is continuous. If such definition is true, then we can define the limit, in words, as a value which is infinitely close to the value of the function such that it is even smaller than $\epsilon$. And if this statement is correct, it implies that the difference between the value of $x$ and $x_0$ must also be so small that it is smaller than some infinitely small constant like $\delta$.

Thus, formally, the limit can be defined through the following:

If a function has a limit, L, at $x = x_0$, then:

$$0 < |x - x_0| < \delta \implies |f(x) - L| < \epsilon$$

Any limit that satisfies these two conditions can be proven to exist. If we can show that there exists a $\delta$ for every epsilon or vice-versa, then we have proven that the limit exists.

Example: Prove that

$$\lim_{x \to 2} 5x - 3 = 7$$

*Proof:*

$$0 < |x - 2| < \delta \implies |5x - 3 - 7| < \epsilon$$

$$\implies |5x - 10| < \epsilon \implies 5|x - 2| < \epsilon \implies \frac{\epsilon}{5} > |x - 2|$$

$$\implies \frac{\epsilon}{5} = \delta$$

Since there exists an $\epsilon$ for every $\delta$, we have proven that the limit stays within an infinitely small distance of 7. Therefore, we have proven that the limit exists. $\square$

Note: the $\square$ symbol denotes the end of a proof. You may also use "*QED*" or simply "End of Proof".

Now that we have defined a formal definition for limits, let us use the definition to prove the limit theorems introduced in section 4. Here we will explore the second theorem, however, the rest may be done as an exercise in order to properly understand the way limit proofs work.

Example: Given that $\lim_{x \to c} f(x) = L$ and $\lim_{x \to c} g(x) = M$, prove that

$$\lim_{x \to c} f(x) + g(x) = L + M$$

*Proof:*

Let $\epsilon > 0$ be given. We want to find a positive number $\delta$ such that for all x

$$0 < |x - c| < \delta \implies |\,(f(x) + g(x)) - (L + M)\,| < \epsilon$$

$$\implies |(f(x) - L) + (g(x) - M)| \leq |f(x) - L| + |g(x) - M| < \epsilon$$

Since $\lim_{x \to c} f(x) = L$, there exists some number, $\delta_1 > 0$ such that for all x

$$0 < |x - c| < \delta_1 \implies |f(x) - L| < \epsilon/2$$

Similarly, Since $\lim_{x \to c} g(x) = M$, there exists some number, $\delta_2 > 0$ such that for all x

$$0 < |x - c| < \delta_2 \implies |g(x) - M| < \epsilon/2$$

let $\delta = \min\{\delta_1, \delta_2\}$, if $0 < |x - c| < \delta$ then $|x - c| < \delta_1$ and so $|f(x) - L| < \epsilon/2$. Using the same process for g(x) we can show that:

$$|\,(f(x) + g(x)) - (L + M)\,| < \epsilon/2 + \epsilon/2 = \epsilon$$

Since there exists an $\epsilon/2$ for every $delta_n$, the limit for each component individual exists and so their sum must also exists. Therefore, we have proven that $lim_{x \to c} f(x) + g(x) = L + M$ $\square$