Applications of Derivatives

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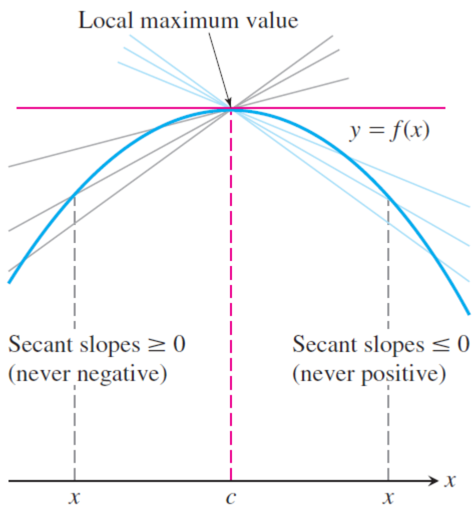
Note: The pre-requisites to this handout are the Limits and Continuity and the Differentiation handout.
Reminder: The topics that are labelled as advanced are not necessarily hard to understand, but they are simply not taught in the Ontario High School curriculum. Feel free to skip these if you wish to focus on the curriculum only.

1 Extreme Value Theorem

As we have learned in Advanced Functions, many functions have local and absolute extreme values. In this chapter, we explore how we can find the exact coordinates of these points using derivatives. This will lead us into our next topics which are optimization and curve sketching.
In order to properly explore the extreme value theorem, let us first define, what a local and global maximum and minimum are.
If a function $f(x)$ at some point $x = c$ is greater than all the other values of f within its domain, then $f(c)$ is a **global or absolute maximum**. In other words, it is the greatest value the function ever takes. Similarly, if a function $f(x)$ at some point $x = c$ is smaller than all the other values of f within its domain, then $f(c)$ is a **global or absolute minimum**. Meaning it is the smallest value the function ever takes in its domain. Maximum and minimum values are called extreme values of the function f. Absolute maxima or minima are also referred to as global maxima or minima. Now how can we define a local minimum or maximum ? Informally, it can be described as a point where there are no other extreme values *nearby*. Formally, however, such definition is not enough and we need proper mathematical terms. A function f has a **local maximum** value at a point c within its domain D if $f(x) \leq f(c)$ for all $x \in D$ lying in some open interval containing c. Similarly, a function f has a **local minimum** value at a point c within its domain D if $f(c) \leq f(x)$ for all $x \in D$ lying in some open interval containing c.
Now that we have established a proper definition for these terms, the question becomes how can we find these values ? The answer lies within the use of derivatives and their geometric interpreta-

tion. As we discussed in the previous handout, the derivative of a function at some point x gives the slope of the tangent line to that function at that point. However, when we look at the pattern in the sloep of the tangent near local or global maxima or minima, we see a pattern. Let us look at this figure and analyze what is happening to the derivative near the local maximum:



When looking at this figure, you may notice a pattern; when the function gets close to the local max from the left its slope becomes less steep and as it approaches the local max from the right the same thing happens. This pattern continues until we reach the local max itself where we see that the slope is exactly equal to zero and we have a horizontal line. The case is the same for local minimums and is true for all functions. Hence, we can find the exact coordinates of the local or global extreme of any function by determining where its derivative is equal to zero. This can be expressed also using continuity. If $f(x)$ has a local extreme at $x = c$ and is continuous over its domain, then:

$$\lim_{x \to c^+} \frac{df}{dx} = \lim_{x \to c^-} \frac{df}{dx} = 0$$

Therefore, we can conclude that $f(x)$ ***may*** have a local extreme value at $x = c$. This theorem says that a function's first derivative is always zero at an interior point where the function has a local extreme value and the derivative is defined. Hence the only places where a function f can possibly have an extreme value (local or global) are:

1. interior points where $f'(x) = 0$
2. interior points where $f'(x)$ is undefined
3. endpoints of the domain of $f(x)$

Note: when we set the derivative of a function equal to zero, all values that make it zero and are within the domain of the function are called *critical numbers*.

Example: Find the global maximum and minimum values of the function $f(x) = x^4 - 2x^3 + 4x^2 - 3x - 2$ over the interval $[-7, 10]$.

Solution: In order to determine the points of local and global extrema, according to the extreme value theorem, we must determine the zeroes of the derivative and test each one to determine the global max and min.

$$f'(x) = 4x^3 - 6x^2 + 8x - 3 = (2x-1)(2x^2 - 2x + 3)$$

From this we see that the roots of the derivative are:

$$x = \left\{ \frac{1}{2}, \frac{1 \pm i\sqrt{5}}{2} \right\}$$

Hence we can only test $\frac{1}{2}$ and the endpoints which are -7 and 10.

$$f(\frac{1}{2}) = \frac{5}{16},\ f(-7) = 1888,\ f(10) = 8368$$

Therefore, the function $f(x)$ has a global maximum at $f(10)$ and a global minimum at $f(\frac{1}{2})$.

2 Advanced: The Mean Value Theorem

In order to understand the mean value theorem, we first explore Rolle's theorem and then use that result to prove the mean value theorem.

We know that derivative of any constant function is zero. However, is there a more complicated function whose derivative is also zero ? and if such function exists, how can we related this to others with the same derivative ? Rolle's theorem states that if a differentiable function crosses a horizontal line at two different points, there is at least one point between them where the tangent to the graph is horizontal and the derivative is zero. If some function $f(x)$ over the interval $[a, b]$ is continuous and differentiable, if f has a derivative of 0 at both a and b, then there exists some value, c within the interval [a,b] such that $f'(c) = 0$ or in other words, if the derivative of the function at two points is zero, it implies that there is a local maximum or minimum between those two points where the derivative is also zero.

The mean value theorem, takes a more broad, useful take on Rolle's theorem with the small change that says the derivatives do not need to be zero, but they will be equal! Take a function $f(x)$, if we draw a secant line from $f(a)$ to $f(b)$, then according to the mean value theorem, the slope of this secant line is equal to the slope of the tangent line at c. In mathematical notation:

$$AROC_{a\ to\ b} = \frac{f(b) - f(a)}{b - a} = f'(c)$$

Note that $f'(a)$ does not need to be equal to $f'(b)$ for this theorem to be true! In fact, it does not even require f to be differentiable at a or b, it only needs to be continuous. This implies that any function whose derivative is zero, must be a constant function. However, this also implies that if the derivatives of two functions are equal, $f'(x) = g'(x)$, then there exists some constant C such that $f(x) = g(x)+C$ which explains how the derivatives of the functions can show the relationship between the two functions.

Challenge: Let $f(x) = px^2 + qx + r$ be a quadratic function defined on a closed interval [a, b]. Show that there is exactly one point c in (a, b) at which f satisfies the conclusion of the Mean Value Theorem. Then, determine the coordinate of the absolute maximum value of the function given that the second derivative of the function $f'(c) = f(b^2 - a^3)$, $b = \ln(a) + c^e$ and $a^2x^2 + bx - c = 0$ has 1 real solution.

3 Curve Sketching

For most of this section, we are applying the extreme value theorem to find the coordinates of extreme values, points of inflection, vertical tangents, cusps and concavity in order to properly be able to sketch the graph of a function. Let us first start by taking a greater dive into the concept of concavity and a point of inflection. The first question we must ask is what is concavity ?

3.1 Concavity

You may recall from grade 10 when writing an equation for quadratics, if the a value for some quadratic function $f(x) = ax^2 + bx + c$ was positive, then the quadratic is facing up and has a maximum. The "facing up" part in this statement, however, is not a properly defined property of a function. Hence we introduce "concavity", a property which states whether a function is facing up or down. You may ask why is this an official definition if it is defined the same way. The key factor that makes the difference is that concavity can be determined through differentiating a function.

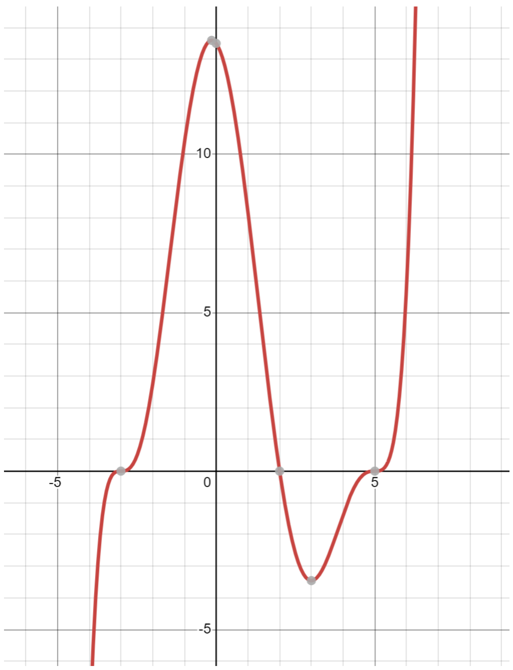
The theorem is that if the second derivative of a function is positive, then it implies that the function is facing upwards. If it is negative, then the function is facing downwards. Or in other words, if we examine some function $f(x)$ over some interval $x \in [a, b]$, then $f(x)$ is:

$$C.U \;\; if \;\; \frac{d^2f}{dx^2} > 0, \;\; C.D \;\; if \;\; \frac{d^2f}{dx^2} < 0$$

This is known as "The Second Derivative Test For Concavity". Keep in mind, this is only allowed if the function is twice-differentiable.

3.2 Points of Inflection

As we recall from advanced functions, a point of inflection is when a polynomial in its factored form has a cubic, quintic or any power of the order $2n + 1$. However, the problem is that there are many other types of functions who also have points of inflection and those cannot be determined simply by looking at a "factored" form of the equation. Therefore, we need a new approach to determining the coordinates of a point of inflection or simply P.O.I. and as we examine this concept closely, we see that the key is in derivatives. Let us examine the graph of the function $f(x) = \frac{1}{500}(x-5)^3(x+3)^3(x-2)$:



We can see that the function has a POI at the points (-3,0) and (5,0). However, what makes this different than a typical zero of the function ? If we examine closely, we can see that the slope of the tangent line to the function at these points, is equal to zero! Therefore, a POI can be determined by using the first derivative test! (Setting the derivative equal to zero and solving for all x values).

This claim brings up one quick problem, however; If the first derivative test can be used to determine both POI and local/global extreme values, then how can we distinguish a POI from an extreme value of the function ? The answer is concavity. Let us examine concavity at a local minimum value of the function at approximately $x = 3$. We can see that the concavity of the function does not change. In other words, both on the right side and on the left side of the extreme value, the function is facing up and therefore concave up. So there is no change in concavity. However, when looking at the points of inflection at the cubic roots, we see that the function changes its concavity. Take the point (5,0) for example. To the left of the point, the function is C.U. (Concave Up) but to the right it is C.D. (Concave Down). The same can be observed at the point (-3,0). A similar pattern can be observed when considering other polynomial functions. Hence we can define a POI to be where the derivative is zero and the concavity of the function changes. Unlike an extreme value where the concavity stays the same.

Although this is the main takeaway of the theorem, we can also deduce that if the concavity of the function changes, that implies that the second derivative of the function changes its sign across a POI. Hence, it must become zero at the coordinates of the POI. Which gives us the second part of the theorem; at a POI of some function:

$$\frac{d^2f}{dx^2} = 0$$

3.3 The Second Derivative Tests For Extreme Values

Now that we have established proper definitions of the point of inflection and local extreme values, we need to explore another aspect of the second derivative test. In the previous section, we used it to determine whether a point is a local extreme or a POI. Nonetheless, we find that there is one small problem with finding local extremes. Let's say that we calculate a function's derivative, we determine that one of the critical numbers is not a POI but a local extreme value. Now what ? Well, we want to be able to say that at some point $(x, f(x))$, there is either a local max or a local min. But how can find which one ? How do we know if the point is a maximum or a minimum ? The approach that we will take is using the second derivative. If we go back to the figure at page 2, we can see that before a local maximum, the slope is increasing, and after that, it starts to decrease. In other words, *the rate of change of the slope is changing.* When we examine a local maximum, we see that the derivative decreases, and for a local minimum the derivative increases. Hence we can deduce the theorem, for a local max:

$$\frac{d^2f}{dx^2} < 0$$

And for a local min, the theorem is:

$$\frac{d^2f}{dx^2} > 0$$

If the second derivative is equal to zero, the we deduce that the test is inconclusive.

3.4 The Algorithm for Curve Sketching

1. Identify the domain of the function and any symmetries the curve may have
2. Find the derivatives f' and f''
3. Find the critical points of f, if any, and identify the function's behavior at each one
4. Find where the curve is increasing and where it is decreasing
5. Find the points of inflection, if any occur, and determine the concavity of the curve
6. Identify any asymptotes that may exist
7. Plot key points, such as the intercepts and the points found in Steps 3–5, and sketch the curve together with any asymptotes that exist.

4 Optimization

What are the dimensions of a rectangle with fixed perimeter having maximum area? What are the dimensions for the least expensive cylindrical can of a given volume? How many items should be produced for the most profitable production run? Each of these questions asks for the best, or *optimal*, value of a given function. In *optimization*, our primary goal is to determine that optimal value and apply to solve many real world problems. This section is a demonstration of just how key of a role Calculus plays in our everyday life and the industrialization of our world.
***Note: The best way to be able to understand and learn to solve almost all optimization problems is through practice, practice and practice!!!
Before we start, I will give you a general guidline of how to appraoch and solve an optimization problem.

1. *Read the problem.* Read the problem until you understand it. What is given? What is the unknown quantity to be optimized?

2. *Draw a picture.* Label any part that may be important to the problem.

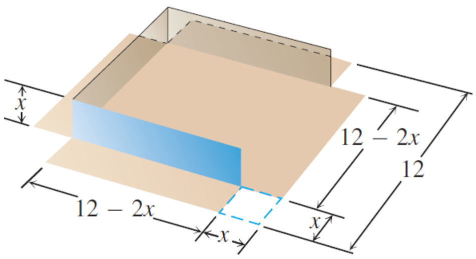
3. *Introduce variables.* List every relation in the picture and in the problem as an equation or algebraic expression, and identify the unknown variable.

4. *Write an equation for the unknown quantity.* If you can, express the unknown as a function of a single variable or in two equations in two unknowns. This may require considerable manipulation.

5. *Test the critical points and endpoints in the domain of the unknown.* Use what you know about the shape of the function's graph. Use the first and second derivatives to identify and classify the function's critical points.

In order to understand this topic, I will only include examples for this topic. Again, the only way to learn this is to practice, there is no description or definition I can give for a better understanding of the topic.
Example 1 An open-top box is to be made by cutting small congruent squares from the corners of a 12cm by 12cm sheet of tin and bending up the sides. How large should the squares cut from the corners be to make the box hold as much as possible?

Solution: The first step is to visualize this box to get a better understanding of the problem.



This figure represents the box discussed in the context of the problem. The phrase *hold as much as possible* implies that we are looking for the maximum volume the shape can hold. In order to do so, we define a function for the volume of the shape with respect to x:

$$V(x) = l \cdot w \cdot h = (12 - 2x)(12 - 2x)(x)$$

Now in order to make the differentiation process easier, instead of keeping it in a factored form, we turn it into standard form so that we can apply power rule:

$$V(x) = 4x(x^2 - 12x + 36) = 4x^3 - 48x^2 + 144x$$

Now, regarding the domain, we see that since sides of the full shape have to be 12cm, our independent variable can only cut it up to half. If it is at half, then our shape has a width/length of $x \leq 6$ and we known that some amount is cut so that means x has to start from 0. So we get the full domain: $0 \leq x \leq 6$. Now we apply the extreme value theorem in order to determine the value that maximizes the volume:

$$\frac{dV}{dx} = 12x^2 - 96x + 144$$

$$\frac{dV}{dx} = 0 \implies 12(x^2 - 8x + 12) = 0$$

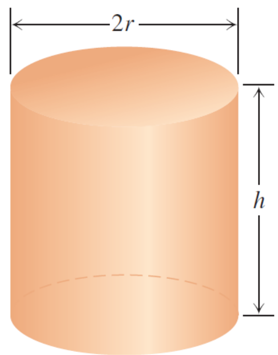
$$\implies x^2 - 8x + 12 = 0 \implies x = \{6, 2\}$$

Of the two zeros, $x = 2$ and $x = 6$, only 2 lies in the interior of the function's domain and makes the critical-point list. The values of $V(x)$ at this one critical point and two endpoints are:

$$V(0) = 0, \ V(6) = 0, \ V(2) = 128cm^3$$

Hence we get our final answer: For the solution $x = 2cm$, the box will have a maximum volume.

Example 2 You have been asked to design a one-liter can shaped like a right circular cylinder. What dimensions will use the least material?



Solution: In order to better visualize the problem we draw the diagram on the left. Then we establish our domain; $r > 0,\ h > 0$. By examining the question carefully, we learn that the phrase *the least material* implies that we are looking for the minimum surface area in order to use less material. Now, we define the function that models the surface area with respect to the r and h:

$$A(r,h) = 2\pi r^2 + 2\pi rh = 2\pi r(r+h)$$

In order to make this a single variable function, we use the information given to us in the question about the volume. We are told that the volume of the cylinder is 1L or 1000 cubic meters. Hence:

$$V = \pi r^2 h = 1000m^3$$

From this we obtain a relationship between r and h:

$$h = \frac{1000}{\pi r^2}$$

Now substituting this into $A(r,h)$, we obtain a single variable function, $A(r)$:

$$A(r) = 2\pi r^2 + 2\pi r\left(\frac{1000}{\pi r^2}\right) = 2\pi r^2 + \frac{2000}{r}$$

Now we use the extreme value theorem to determine the r value that minimizes the surface area:

$$A'(r) = 4\pi r - \frac{2000}{r^2} = 0 \implies \frac{2000}{r^2} = 4\pi r$$

$$\implies 4\pi r^3 = 2000 \implies r = \sqrt[3]{\frac{500}{\pi}}$$

After obtaining the value of r, we see that the value of h (after some algebra) is:

$$h = 2\sqrt[3]{\frac{500}{\pi}}$$

In order to make sure that our response actually minimizes the area, we check the endpoints in $A(r,h)$:

$$A(0,0) = 0, \quad A\left(\sqrt[3]{\frac{500}{\pi}}, 2\sqrt[3]{\frac{500}{\pi}}\right) \approx 553.58 units^2$$

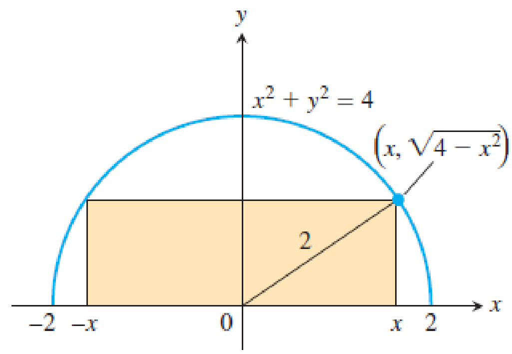
Since $r = 0$ and $h = 0$ are not in the domain, 0 is not a valid solution. Hence, our answer is correct.

Now that we applied our knowledge of optimization in more practical examples, let us apply it to some more theoretical questions in mathematics and physics.

Example 3 A rectangle is to be inscribed in a semicircle of radius 2. What is the largest area the rectangle can have, and what are its dimensions ?

Solution: The first step, as usual, is to visualize the problem using a diagram. In this case, the shape that we are trying to optimize is bounded by another function:

$$f(x) = \sqrt{4 - x^2}$$



From this relationship, we obtain the length of the rectangle to be:

$$l = f(x) = \sqrt{4 - x^2}$$

And the width of the rectangle is simply equal to $2x$. Hence we get the equation for the area:

$$A(x) = l \cdot w = 2x\sqrt{4 - x^2}$$

Now, we repeat the same steps as every other optimization problem; applying the extreme value theorem and determining the maximum area.

$$\frac{dA}{dx} = 2\sqrt{4 - x^2} - \frac{2x^2}{\sqrt{4 - x^2}}$$

$$\frac{dA}{dx} = 0 \implies \sqrt{4 - x^2} = \frac{x^2}{\sqrt{4 - x^2}}$$

$$\implies 4 - x^2 = x^2 \implies x = \sqrt{2}$$

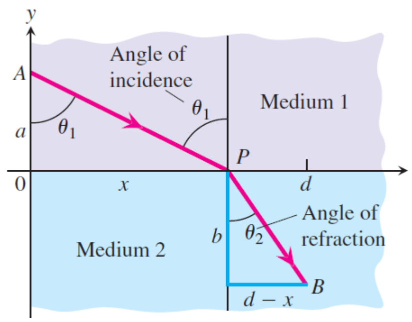
Note that the answer $x = -\sqrt{2}$ was eliminated due to the restriction on x being greater than 0.

$$A(0) = 0,\ A(2) = 0,\ A(\sqrt{2}) = 4$$

Thus, the maximum area of the rectangle bounded by the x axis and the function $f(x) = \sqrt{4 - x^2}$ is 4 units squared.

At this point you can see that once the equations are set up, the steps taken to solve the problem are almost exactly the same. Now, let us take a look at a physical example of applying optimization.

Example 4 The speed of light depends on the medium through which it travels, and is generally slower in denser media. Fermat's principle in optics states that light travels from one point to another along a path for which the time of travel is a minimum. Describe the path that a ray of light will follow in going from a point A in a medium where the speed of light is c_1 to a point B in a second medium where its speed is c_2.



Solution: Since light traveling from A to B follows the quickest route, we look for a path that will minimize the travel time. We assume that A and B lie in the xy-plane and that the line separating the two media is the x-axis.In a uniform medium, where the speed of light remains constant, "shortest time" means "shortest path," and the ray of light will follow a straight line. Thus the path from A to B will consist of a line segment from A to a boundary point P, followed by another line segment from P to B. We know that the time passed is:

$$Time = \frac{Distance}{Speed}$$

For the light to travel from point A to point P, the time is:

$$t_1 = \frac{AP}{c_1} = \frac{\sqrt{a^2 + x^2}}{c_1}$$

For the light to travel from P to B:

$$t_2 = \frac{PB}{c_2} = \frac{\sqrt{b^2 + (d-x)^2}}{c_2}$$

The total time then is the sum of these expressions:

$$\sum t = t_1 + t_2 = \frac{\sqrt{a^2 + x^2}}{c_1} + \frac{\sqrt{b^2 + (d-x)^2}}{c_2}$$

When we think about this equation, it may seem intimidating at first, however, the only variable that we are concerned with is x. Hence, we have expressed the total time as a differentiable function of x over the interval [0,d].

Now we want to determine the minimum time taken in order to find the shortest distance which means we must apply the extreme value theorem to this function.

$$\frac{dt}{dx} = \frac{x}{c_1\sqrt{a^2+x^2}} - \frac{d-x}{c_2\sqrt{b^2+(d-x)^2}}$$

Now if we wish to determine this relationship in terms of θ_1 and θ_2, we use the trigonometric ratios:

$$\frac{dt}{dx} = \frac{\sin\theta_1}{c_1} - \frac{\sin\theta_2}{c_2}$$

since using the Pythagorean theorem, we get:

$$\frac{x}{\sqrt{a^2+x^2}} = \sin\theta_1$$

$$\frac{d-x}{\sqrt{b^2+(d-x)^2}} = \sin\theta_2$$

Setting the position derivative of $t(x)$ equal to 0, we get:

$$\frac{\sin\theta_1}{c_1} = \frac{\sin\theta_2}{c_2}$$

This equation is Snell's Law or the Law of Refraction, and is an important principle in the theory of optics. It describes the path the ray of light follows.

5 Advanced: Indeterminate Forms and L'Hôpital's Rule

When evaluating limits, you may recall that we sometimes faced the form $\frac{0}{0}$ as the answer to the limit which meant that we had to do some simplification in order to evaluate the proper limit. The form $\frac{0}{0}$ is known as *Indeterminate Form*. This is the only indeterminate form that we are introduced to in the Ontario Curriculum. However, there are a total of 7 indeterminate forms in which the limit must be simplified. We will explore the 7 indeterminate forms first. Let us evaluate the following limit:

$$\lim_{x\to 0}\frac{x-\sin x}{x^3} = \frac{0}{0}??$$

We cannot apply the Quotient Rule since the limit of the denominator is 0. Moreover, in this case, both the numerator and denominator approach 0, and $\frac{0}{0}$ is undefined. Such limits may or may not exist in general, but the limit does exist for expression under discussion by applying l'Hôpital's Rule.

5.1 L'Hôpital's Rule

This rule is an extremely important and useful theorem in mathematics (and yes, it is called the hospital rule accrediting the French mathematician Guillaume François Antoine de l'Hôpital). This rule is true for all indeterminate forms of $\frac{0}{0}$ and $\frac{\infty}{\infty}$.

When we have two functions $f(x)$ and $g(x)$, if both functions at some point $x = a$ are equal to zero, then

$$\lim_{x \to a} \frac{f(x)}{g(x)}$$

cannot be evaluated using the substitution method as it results in the indeterminate form $\frac{0}{0}$ which is meaningless. The other indeterminate forms that may occur when evaluating limits are:

$$\frac{\infty}{\infty},\ \infty \cdot 0,\ \infty - \infty,\ 0^0,\ 1^\infty,\ \infty^0$$

These are also called indeterminate forms. Sometimes, but not always, limits that lead to indeterminate forms may be found by cancellation, rearrangement of terms, or other algebraic manipulations which is how we did it in the first handout on Limits and Continuity. Now we can draw on our knowledge of derivatives. Since both $f(a)$ and $g(a)$ are equal to 0, we can say that

$$f'(x) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

and the same can be true for $g(x)$. L'Hôpital's Rule enables us to draw on our success with derivatives to evaluate limits that otherwise lead to indeterminate forms

This rule states that if both f and g are differentiable over their domain and at point $x = a$, and if $x \neq a$, then

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$$

***NOTE: You can only apply L'Hôpital's Rule if the limit results in the indeterminate forms $\frac{0}{0}$ or $\frac{\infty}{\infty}$. In other situations or indeterminate forms, you are NOT allowed to apply this formula unless you can change the form to something that results in the above two indeterminate forms. Let us explore a few examples.

5.2 Indeterminate forms $\frac{\infty}{\infty}$, $\infty \cdot 0$ and $\infty - \infty$

Example 1: $\frac{\infty}{\infty}$ Indeterminate form Evaluate the following limit:

$$\lim_{x \to \infty} \frac{\ln x}{2\sqrt{x}}$$

Since substituting infinity into the expression will give us the indeterminate form, we are able to use L'Hôpital's Rule. Hence we get:

$$\lim_{x\to\infty} \frac{\ln x}{2\sqrt{x}} = \lim_{x\to\infty} \frac{1/x}{1/\sqrt{x}} = \lim_{x\to\infty} \frac{1}{\sqrt{x}} = 0$$

Example 2: $\infty \cdot 0$ Indeterminate form Evaluate the following limit:

$$\lim_{x\to\infty} \left(x \sin \frac{1}{x} \right)$$

This limit by itself is not one that we can use L'Hôpital's Rule on. So we must perform some manipulation. Let us introduce a variable substitution such that $h = \frac{1}{x}$. Now substituting this in, we get:

$$\lim_{x\to\infty} \left(x \sin \frac{1}{x} \right) = \lim_{h\to 0} \left(\frac{1}{h} \sin h \right) = \lim_{h\to 0} \frac{\sin h}{h} = \lim_{h\to 0} \frac{\cos h}{1} = 1$$

Example 3: $\infty - \infty$ Indeterminate form Evaluate the following limit:

$$\lim_{x\to 0} \left(\frac{1}{\sin x} - \frac{1}{x} \right)$$

This shows an indeterminate form that is not solvable through the direct use of L'Hôpital's Rule. So we change the problem by taking the common denominator. Then we apply L'Hôpital's Rule:

$$\lim_{x\to 0} \left(\frac{1}{\sin x} - \frac{1}{x} \right) = \lim_{x\to 0} \frac{x - \sin x}{x \sin x} = \lim_{x\to 0} \frac{1 - \cos x}{\sin x + x \cos x} = \lim_{x\to 0} \frac{\sin x}{2\cos x - x \sin x} = \frac{0}{2} = 0$$

5.3 Indeterminate Powers and Logarithms

When presented with the forms $1^\infty, 0^0$and ∞^0, we cannot use L'Hôpital's Rule. In most of these cases, a simple algebraic manipulation would not be helpful in turning the limit into an indeterminate form where L'Hôpital's Rule is applicable. Hence we must employ a new technique. For indeterminate powers, a method that works is first, taking the logarithm of the expression and then applying L'Hôpital's Rule.

If $\lim_{x\to a} f(x) = L$ then:

$$\lim_{x\to a} f(x) = \lim_{x\to a} e^{\ln f(x)} = e^L$$

Using this, we can evaluate almost all limits.

Example 4: 1^∞ Indeterminate form Evaluate the following limit:

$$\lim_{x\to 0} (1+x)^{\frac{1}{x}}$$

If we call this expression $f(x)$, then we the key to solving this is to take the natural logarithm of $f(x)$:

$$\ln f(x) = \frac{1}{x} \cdot \ln(1+x)$$

$$\lim_{x \to 0} \ln f(x) = \lim_{x \to 0} \frac{\ln(1+x)}{x} = \lim_{x \to 0} \frac{\frac{1}{1+x}}{1} = \frac{1}{1} = 1$$

$$\therefore \lim_{x \to 0} (1+x)^{\frac{1}{x}} = e^1 = e$$

Example 5: ∞^0 Indeterminate form Evaluate the following limit:

$$\lim_{x \to \infty} x^{\frac{1}{x}}$$

We apply the same approach as the previous example to this example:

$$\ln f(x) = \frac{1}{x} \cdot \ln x$$

$$\lim_{x \to \infty} \ln f(x) = \lim_{x \to \infty} \frac{\ln x}{x} = \lim_{x \to \infty} \frac{1/x}{1} = 0$$

Here are some challenges for you to try on your own. Thank you to Dr.Moshtagh for providing these questions.

Challenge Evaluate the following limits if possible.

$$a)\ \lim_{x \to \infty} \frac{e^x}{\ln x}$$

$$b)\ \lim_{x \to 0^+} \frac{\ln(x^2+x) - \ln x - x}{x^2}$$

$$c)\ \lim_{x \to \infty} \left(\sqrt{x^2+4} - \sqrt{x^2-1}\right)$$

$$d)\ \lim_{x \to 0} (1+x)^{\cot x}$$

6 *Recommended, Advanced: The Anti-Derivative

We have studied how to find the derivative of a function. However, many problems require that we recover a function from its known derivative (from its known rate of change). For instance, we may know the velocity function of an object falling from an initial height and need to know its height at any time. In other words, we want to determine a function $F(x)$ from its derivative, $f(x)$. In this case, $F(x)$ is known as the *anti-derivative* of $f(x)$. We will see in the next handouts that anti-derivatives

are the link connecting the two major elements of calculus: derivatives and definite integrals.
To express the idea in algebraic notation, if $F'(x) = f(x)$, then $F(x)$ is the anti-derivative of $f(x)$. When applying this idea to functions, the key question to ask ourselves is "What function's derivative would be equal to the function at hand?" The answer to this question is the anti-derivative. The process of recovering a function $F(x)$ from its derivative $f(x)$ is called *anti-differentiation.* We use capital letters such as F to represent an anti-derivative of a function$f$, G to represent an anti-derivative of g, and so forth. Let's take a look at a few examples:

$$f(x) = 2x \implies F(x) = x^2$$

$$g(x) = \cos x \implies G(x) = \sin x$$

$$h(x) = \frac{1}{x} + 2e^{2x} \implies H(x) = \ln|x| + e^{2x}$$

We are simply working our way backwards from the derivative to the original function. Note that if we differentiate $F(x), G(x)$ or $H(x)$ we will get the lowercase equivalent of the function. There is, however, one small problem with our solutions; $F(x) = x^2$ is not the only function whose derivative is equal to $f(x) = 2x$. Recall that the derivative of any constant is equal to 0. So if I add or subtract any constant term to my anti-derivative, without knowing what the initial function is, those solutions would be equally valid. To demonstrate with an example, if $f(x) = 2x$, then $F(x) = 2x$ is one valid solution. But another set of valid solutions could be $F(x) = 2x+3$, $F(x) = 2x+5$, $F(x) = 2x+23412$, etc. because if we differentiate the anti-derivative, we would still get $f(x) = x^2$ since the constant term that we add will be cancelled out to a 0. Therefore, every time we determine an anti-derivative without knowing more information about the function but its equation, we must consider a potentially existing constant term that got cancelled out in the differentiation process. Hence, the anti-derivative of any function $f(x)$ is not just $F(x)$ but $F(x) + C$ were C denotes the constant term that may or may not have been part of the original function. The value of this constant term will be determined through the context provided by the problem. If there is not enough information, we simply leave the term as C. Here is an example:
Find an anti-derivative of $f(x) = 4x^3 - e^x$ that satisfies the condition $F(0) = 3$.
Solution We first start but anti-differentiating the function, this time with the C term:

$$f(x) = 4x^3 - e^x \implies F(x) = x^4 - e^x + C$$

Now we use the initial conditions given to us by the problem to determine the value of C.

$$F(0) = 0 - 1 + C = 3 \implies C = 4$$

$$\therefore F(x) = x^4 - e^x + 4$$

The use of initial values is extremely important in the topic of differential equations.

6.1 Application to Physics

We have seen that the derivative of the position function of an object gives its velocity, and the derivative of its velocity function gives its acceleration. If we know an object's acceleration, then by finding an anti-derivative we can recover the velocity, and from an anti-derivative of the velocity we can recover its position function. Let us apply this to a physics problem:

Example The velocity of an object is calculated from the function $v(t) = 3t - 4t^2 + t^3$ where $v(t)$ is in meters per second. Determine the displacement of the object from the time $t = 1$ to $t = 3$.
Solution Since we know that the position is the anti-derivative of velocity with respect to time, we simply anti-differentiate $v(t)$ to get the position function:

$$v(t) = 3t - 4t^2 + t^3 \implies s(t) = \frac{3}{2}t^2 - \frac{4}{3}t^3 + \frac{1}{4}t^4$$

$$\Delta s = s(3) - s(1) = \left(\frac{27}{2} - 36 + \frac{81}{4}\right) - \left(\frac{3}{2} - \frac{4}{3} + \frac{1}{4}\right) \approx -2.67m$$

Note that the reason we got a negative quantity is that displacement is a *vector* quantity not a scalar one, meaning it has a direction. The negative sign indicates that the direction is towards negative t.

6.2 The Indefinite Integral

A special symbol is used to denote the collection of all anti-derivatives of a function f.
The collection of all anti-derivatives of a function $f(x)$ with respect to x is called the *Indefinite Integral* of f with respect to x. It is denoted by the following:

$$F(x) = \int f(x)dx$$

The $\int$ symbol is an *Integral Sign.* The function $f(x)$ is the *Integrand* and x is the *variable of integration.* After the integral sign in the notation we just defined, the integrand function is always followed by a differential to indicate the variable of integration. We will have more to say about why this is important in the handout on Integration. Using this new notation that we just introduced, let us restate the anti-derivatives or indefinite integrals in the first example to get more comfortable with the notation:

$$f(x) = 2x \implies F(x) = \int 2x dx = x^2 + C$$

$$g(x) = \cos x \implies G(x) = \int \cos x dx = \sin x + C$$

$$h(x) = \frac{1}{x} + 2e^{2x} \implies H(x) = \int \frac{1}{x} + 2e^{2x} dx = \ln|x| + e^{2x} + C$$

Here, I will include a few challenge questions for you to practice basic integration. We will explore the concept of integration in much greater detail in the integration handout.

Challenge Determine the following integrals:

$$\int \cos\theta(\tan\theta + \sec\theta)\, d\theta$$

$$\int \frac{t\sqrt{t} + \sqrt{t}}{t^2}\, dt$$

$$\int e^{3x} + 5e^{-x}\, dx$$

$$\int e^{-\pi x^2} \arctan\left(\frac{3\sec(x) - \ln|x^{\cos(x)}|}{\arcsin(\cos(3x))}\right)\, dt$$

Extra Bonus Challenge 1: Given $f(x) = ax^2 + 2bx + c$ with $a > 0$. By considering the minimum, prove that $f(x) \geq 0$ for all real x if and only if $b^2 - ac \leq 0$.

Extra Bonus Challenge 2: **Schwarz's Inequality** is an extremely useful discovery in the analysis of vector innerproducts of an innerproduct vector space. It is considered one of the most important and widely used inequalities in mathematics. In the previous exercise, let

$$f(x) = (a_1x + b_1)^2 + (a_2x + b_2)^2 + (a_3x + b_3)^2 + \cdots + (a_nx + b_n)^2 = \sum_{k=1}^{n}(a_kx + b_k)^2$$

and deduce Schwarz's inequality which states that

$$(a_1b_1 + a_2b_2 + a_3b_3 + \cdots + a_nb_n)^2 \leq (a_1^2 + a_2^2 + a_3^2 + \cdots + a_n^2)(b_1^2 + b_2^2 + b_3^2 + \cdots + b_n^2)$$

Or in Sigma Notation which we will discuss in the Integration handout:

$$\sum_{k=1}^{n}(a_kb_k)^2 \leq \sum_{k=1}^{n}(a_k^2)(b_k^2)$$