Integration

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1 Introduction

Integration is perhaps one of the most important tools ever invented in mathematics. There are 2 main branches when it comes to Calculus; Differential and Integral Calculus. We have explored, albeit briefly, the beautiful world of Differential Calculus and some of its applications. Now, it is time to learn about the second part, Integral Calculus. This handout is by no means a breeze page turner, however, if you are willing to sit down, take the time and actually try to understand what is going on, I guarantee that you will not regret it!

2 Sigma Notation and Summation Formulas

2.1 Sigma Notation

In order to learn integrals in great detail, we must first familiarize ourselves with summation. After all, once we learn integration we realize that it is a summation after all. To start, we must learn sigma notation. Let us say we want to evaluate the following sum:

$$1 + 2 + 3 + 4 + 5 + 6$$

Hopefully we know how to add up these numbers. If not, I suggest going back to kindergarten and using your fingers to add them up! There is another method to represent this sum, obviously it is very tedious and dumb to write out the entire summation when we are adding a billion numbers. So, we use *sigma notation.* If you have ever done computer science or programming, sigma notation is just like a for-loop. Here is how you would express the sum above in sigma notation:

$$\sum_{n=1}^{6} n = 1 + 2 + 3 + 4 + 5 + 6$$

Let us analyse what is happening here. The variable n is known as a *dummy variable* or *dummy index*. What this means is that it does not really matter what letter you use to represent the index. You may use whichever index you like! Some other common indices are k, j, i and m. What you are basically doing is putting the value of the index and then increasing the number by one and add it to the next number and so on. Let us say we want to add numbers going up in increments of 2. To represent this sum, we use

$$2 + 4 + 6 + 8 + 10 + 12 = \sum_{i=1}^{6} 2i$$

Similarly, if they were to increase in increments of 3 it would be:

$$3 + 6 + 9 + 12 + 15 + 18 = \sum_{j=1}^{6} 3j$$

To generalize it, if numbers are going up by increments of μ, then the summation would be:

$$\mu + 2\mu + 3\mu + 4\mu + 5\mu = \sum_{k=1}^{5} \mu k$$

Note that if you have five terms, the indexing goes up to 5. So the top number would change. Similarly, you can start the indexing from 0! Consider the following sum:

$$\sum_{n=1}^{5} n = 1 + 2 + 3 + 4 + 5$$

but this can also be written like this:

$$\sum_{n=0}^{4} n + 1 = 1 + 2 + 3 + 4 + 5$$

We can even start at a higher or a negative index!

$$\sum_{n=2}^{7} n-2 = \sum_{n=5}^{10} n-5 = \sum_{n=-3}^{2} n+4 = \sum_{n=1}^{5} n = \sum_{n=0}^{4} n+1$$

2.2 Evaluating Summations

Now comes a question, is there a faster way to calculate summations ? Take for example, the summation from 1 to 1000 going up by increments of one. The summation would be

$$\sum_{n=1}^{1000} n = 1+2+3+\cdots+1000$$

How can we calculate this sum ? To do so we use the following formula:

$$\boxed{\sum_{n=1}^{k} n = \frac{k(k+1)}{2}} \tag{1}$$

It is very helpful especially when numbers get too big and we can no longer do the whole pairing up trick. Proving this formula is done through a technique known as *proof by induction.*

Proof:

$$\text{let} \quad P(k) = \frac{k(k+1)}{2}$$

for $n=1,$ the statement is true. Now, we assume that the statement is true for $n=k$

$$\implies \sum_{n=1}^{k+1} n = \frac{k(k+1)}{2} + k + 1$$

$$= \frac{k(k+1)+2k+2}{2} = \frac{(k+1)(k+2)}{2}$$

Since $P(k)$ holds true for $n=k+1,$ it also holds for $0 \leq n \leq k$ $\blacksquare$

A similar approach may be taken to prove the following formulas:

$$\boxed{\begin{aligned}\sum_{n=1}^{k} n^2 &= \frac{k(k+1)(2k+1)}{6} \qquad (2)\\ \sum_{n=1}^{k} n^3 &= \left[\frac{k(k+1)}{2}\right]^2 \qquad (3)\end{aligned}}$$

Deriving the other formulas for higher powers is left as an exercise to the reader. Using these formulas we can easily calculate summations extremely fast! Before we look at an example, we must know that constant rule. It states that when there is a constant multiplied by the summation expression, we can factor it out as such:

$$\sum_{n=1}^{k} c \cdot f(n) = c \cdot \left(\sum_{n=1}^{k} f(n)\right) \qquad (4)$$

$$\text{there is also a rule when there is nothing: } \sum_{n=1}^{k} c = k \cdot c \qquad (5)$$

3 The Integral & The Fundamental Theorem

One of the handiest tools in all of mathematics, the sciences, engineering and even areas like business, economics, the social sciences and even health/life sciences is integration. In this section, we will take a deep dive into what the integral is and we will explore how to evaluate them with better methods and apply them to problems in economics, engineering, physics and statistics.

3.1 The Riemann Sum

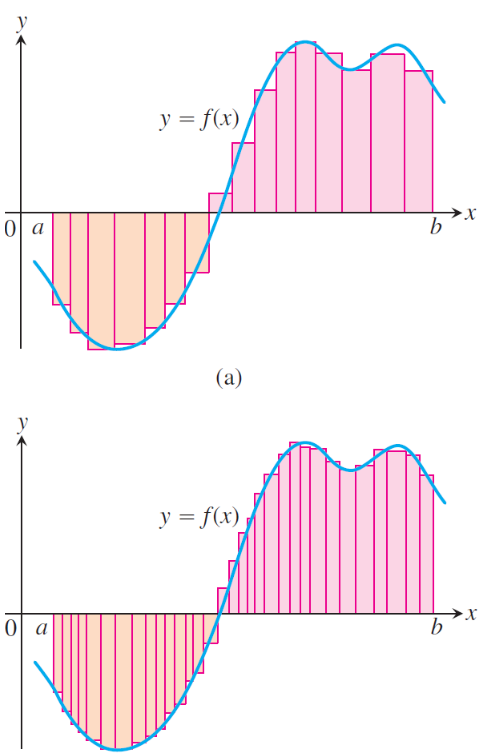
In order to define the integral, we must first understand what it is! To do so let us start with a little problem; How can we find the area under some curve $f(x)$ given the equation of the curve ?

To understand how this is possible, let us consider the simplest type of function: a horizontal line. Take some horizontal function $f(x) = 2$. Obviously, since this is a straight line, calculating the area under the graph from some number $x = a$ to $x = b$ is just taking the area of a rectangle with length $b - a$ and height 2.

Next, let us try this with a linear function $f(x) = x$ in the same interval. Obviously this time we cannot take the area using 1 rectangle. While it would give us some sort of area, it would not be accurate. We will come back to this idea in a second. The most accurate way is to consider the area to be the area of a trapezoid and go one from there. So this gives us the area underneath the graph to be $\frac{1}{2}[f(a) + f(b)](b - a)$. Now, let us go back to the rectangle method. Is there a way to calculate the area under the graph with rectangles only ?

The answer, fortunately, is yes. Let us go back to our good old linear function. If we take one rectangle under the graph, it gives us some of the area but not all of it. So, let us replace the giant rectangle with 2 rectangles of equal width that are smaller than the previous rectangle. We have covered some more area to cover but we do not have all of it.

However, we have a more accurate result than the previous rectangle. So, let us add another rectangle. This same idea can be applied to all curves. As we add rectangles, the width of all of them get smaller and smaller in order to fit more number of rectangles in the same area as shown in the figure on the next page.



Now, how can we reach the highest accuracy ? How can we have an exact value for this ? And more importantly, how can we calculate this ? Obviously one method is to sit down and try to calculate the area of each individual rectangle. I do not know about you but I personally do not have that patience and would prefer a shortcut. Let us formulate the process we went through. The area of one rectangle is its length multiplied by its width. The width is some interval Δx and the length will be the y value or $f(x)$. So:

$$A = f(x)\Delta x$$

Now, what about multiple rectangles ? When we add these rectangles, we can write the sum of their areas as such (with n being the number of rectangles we have):

$$A = f(x_1)\Delta x_1 + f(x_2)\Delta x_2 + f(x_3)\Delta x_3 + \cdots + f(x_n)\Delta x_n$$

We can also write this expression using sigma notation:

$$A = \sum_{k=1}^{n} f(x_k)\Delta x_k$$

This is so that we are adding the area of n number of rectangles together. Now, let us go back to the initial problem, how can we make this an exact value ? Well, we know that the more rectangles we have the more accurate the area is going to get. So how about when we have infinitely many rectangles ? Then, we reach maximum accuracy which is an exact number. Obviously, we do not just let the sum go up to infinity as it will be much more difficult to compute, but instead use a limit such that

$$\boxed{A = \lim_{n\to\infty} \sum_{k=1}^{n} f(x_k)\Delta x_k} \tag{6}$$

This is known as the *Riemann Sum* or the definition of the integral. Now, let us say that we are only interested in the area of a graph in a specific interval $x \in [a, b]$ where both are within the domain of the function. To find the area, we will have to make some substitutions. If we divide the area into n subintervals (or rectangles), the width of each rectangle will be $\Delta x = (b-a)/n$ and the height would be f at a plus the number of widths we have accounted for so far which is $f(x_k) = f(a + k(b-a)/n)$. This gives us the expression for calculating the area of any curve over some interval:

$$\boxed{A = \lim_{n\to\infty} \sum_{k=1}^{n} f\left(a + k \cdot \frac{b-a}{n}\right)\left(\frac{b-a}{n}\right)} \tag{7}$$

Before we use integrals, let us do an example of using Riemann sums.

Example Determine the area under the curve $f(x) = x^2 - 1$ over the interval $x \in [0, 2]$ (take a few minutes and try to solve this problem without looking at the answer)

Solution We begin by writing expression (6) for $f(x) = x^2 - 1$:

$$A = \lim_{n\to\infty} \sum_{k=1}^{n} f\left(a + k \cdot \frac{b-a}{n}\right)\left(\frac{b-a}{n}\right)$$

$$= \lim_{n\to\infty} \sum_{k=1}^{n} \left[\left(a + k \cdot \frac{b-a}{n}\right)^2 - 1\right]\left(\frac{b-a}{n}\right)$$

Now we substitute in the values and calculate!

$$A = \lim_{n\to\infty} \sum_{k=1}^{n} \left[\left(0 + k \cdot \frac{2-0}{n}\right)^2 - 1\right]\left(\frac{2-0}{n}\right)$$

$$= \lim_{n\to\infty} \sum_{k=1}^{n} \left[k^2 \frac{4}{n^2} - 1\right]\left(\frac{2}{n}\right) = \lim_{n\to\infty} \sum_{k=1}^{n} k^2 \frac{8}{n^3} - \frac{2}{n}$$

$$= \lim_{n\to\infty} \frac{8}{n^3} \cdot \frac{n(n+1)(2n+1)}{6} - 2 = \lim_{n\to\infty} \frac{8}{n^3} \cdot \frac{2n^3 + 3n^2 + n}{6} - 2$$

$$= \lim_{n\to\infty} \frac{8}{3} + \frac{4}{n} + \frac{4}{3n^2} - 2 = \frac{8}{3} + 0 + 0 - 2 = \frac{2}{3}$$

You may have noticed that finding the area under the graph of even the simplest functions using the Riemann sum is accurate but also very long and tedious. There is plenty of room to make calculation errors and it takes way too long. How can we further simplify this calculation and make it faster ?

3.2 The Indefinite Integral

We have studied how to find the derivative of a function. However, many problems require that we recover a function from its known derivative (from its known rate of change). For instance, we may know the velocity function of an object falling from an initial height and need to know its height at any time. In other words, we want to determine a function $F(x)$ from its derivative, $f(x)$. In this case, $F(x)$ is known as the *anti-derivative* of $f(x)$. We will see in the next handouts that anti-derivatives are the link connecting the two major elements of calculus: derivatives and definite integrals.

To express the idea in algebraic notation, if $F'(x) = f(x)$, then $F(x)$ is the anti-derivative of $f(x)$.

When applying this idea to functions, the key question to ask ourselves is "What function's derivative would be equal to the function at hand?" The answer to this question is the anti-derivative. The process of recovering a function $F(x)$ from its derivative $f(x)$ is called *anti-differentiation*. We use capital letters such as F to represent an anti-derivative of a function$f$, G to represent an anti-derivative of g, and so forth. Let's take a look at a few examples:

$$f(x) = 2x \implies F(x) = x^2$$

$$g(x) = \cos x \implies G(x) = \sin x$$

$$h(x) = \frac{1}{x} + 2e^{2x} \implies H(x) = \ln|x| + e^{2x}$$

We are simply working our way backwards from the derivative to the original function. Note that if we differentiate $F(x), G(x)$ or $H(x)$ we will get the lowercase equivalent of the function. There is, however, one small problem with our solutions; $F(x) = x^2$ is not the only function whose derivative is equal to $f(x) = 2x$. Recall that the derivative of any constant is equal to 0. So if I add or subtract any constant term to my anti-derivative, without knowing what the initial function is, those solutions would be equally valid. To demonstrate with an example, if $f(x) = 2x$, then $F(x) = 2x$ is one valid solution. But another set of valid solutions could be $F(x) = 2x + 3$, $F(x) = 2x + 5$, $F(x) = 2x + 23412$, etc. because if we differentiate the anti-derivative, we would still get $f(x) = x^2$ since the constant term that we add will be cancelled out to a 0. Therefore, every time we determine an anti-derivative without knowing more information about the function but its equation, we must consider a potentially existing constant term that got cancelled out in the differentiation process. Hence, the anti-derivative of any function $f(x)$ is not just $F(x)$ but $F(x) + C$ were C denotes the constant term that may or may not have been part of the original function. The value of this constant term will be determined through the context provided by the problem. If there is not enough information, we simply leave the term as C. Here is an example:
Find an anti-derivative of $f(x) = 4x^3 - e^x$ that satisfies the condition $F(0) = 3$.
Solution We first start but anti-differentiating the function, this time with the C term:

$$f(x) = 4x^3 - e^x \implies F(x) = x^4 - e^x + C$$

Now we use the initial conditions given to us by the problem to determine the value of C.

$$F(0) = 0 - 1 + C = 3 \implies C = 4$$

$$\therefore F(x) = x^4 - e^x + 4$$

The use of initial values is extremely important in the topic of differential equations. A special symbol is used to denote the collection of all anti-derivatives of a function f.

The collection of all anti-derivatives of a function $f(x)$ with respect to x is called the *Indefinite Integral* of f with respect to x. It is denoted by the following:

$$F(x) = \int f(x)\,dx$$

The $\int$ symbol is an *Integral Sign.* The function $f(x)$ is the *Integrand* and x is the *variable of integration.* After the integral sign in the notation we just defined, the integrand function is always followed by a differential to indicate the variable of integration. Using this new notation that we just introduced, let us restate the anti-derivatives or indefinite integrals in the first example to get more comfortable with the notation:

$$f(x) = 2x \implies F(x) = \int 2x\,dx = x^2 + C$$

$$g(x) = \cos x \implies G(x) = \int \cos x\,dx = \sin x + C$$

$$h(x) = \frac{1}{x} + 2e^{2x} \implies H(x) = \int \frac{1}{x} + 2e^{2x}\,dx = \ln|x| + e^{2x} + C$$

Now that we know the basic process of calculating integrals we must learn what they mean and what is their purpose.

3.3 The Definite Integral

In this part, we will develop a geometric interpretation of what the integral is. In simple words, we use the integral to calculate the area under a curve. Let us recall expression (6). Now, let us analyze this formula one more time. It is a summation over the areas of rectangles. The area is given by the value of the function multiplied by some interval Δx. But, what happens when we divide the area under the curve to infinitely many rectangles ? There is not change to the height, but the width gets infinitely small. In calculus, when intervals get infinitely small, instead of using Δ we use d so that an infinitesimally small change in x would no longer be Δx but dx instead. This is known as a *differential of x.* This gives us the expression:

$$A = \lim_{n\to\infty} \sum_{k=1}^{n} f(x_k)\, dx$$

Now, we make the final change; instead of writing the limit and summation over and over again we introduce the symbol $\int$ which is just a stretched out S that stands for sum. This sign is known as the *integral sign.* This gives us a final expression we can use to calculate the area under any curve $f(x)$:

$$A = \int f(x)\, dx \tag{8}$$

Which will be $F(x)$ with anti-derivative notation. What if we are interested in the area of the function from $x = a$ to $x = b$ only ? Then, we use the fundamental theorem of calculus. But before we learn what that is, let us learn the notation for it. To denote that we are evaluating the area under a curve from a to b, we use the notation:

$$A = \int\limits_{a}^{b} f(x)\, dx \quad \text{or} \quad A = \int_{a}^{b} f(x)\, dx \tag{9}$$

This allows us to compute the exact area under a curve. But how do we use it ? Once we have the anti-derivative of the function, we use the formula:

$$A = \int\limits_{a}^{b} f(x)\, dx = [F(x) + C]\Big|_{a}^{b} = F(b) - F(a)$$

This is a part of the fundamental theorem of calculus. We are going to explore this theorem in the next section. But before that, we should learn a little bit more about the definite integral. Here are some properties of the indefinite integral:

$$\textit{Order of Integration:} \quad \int_a^b f(x)\,dx = -\int_b^a f(x)\,dx \tag{10}$$

$$\textit{Zero Width Interval:} \quad \int_a^a f(x)\,dx = 0 \tag{11}$$

$$\textit{Constant Multiple:} \quad \int_a^b k \cdot f(x)\,dx = k \cdot \int_a^b f(x)\,dx \tag{12}$$

$$\textit{Sum and Difference Rule:} \quad \int_a^b (f(x) \pm g(x))\,dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx \tag{13}$$

$$\textit{Additivity:} \quad \int_a^b f(x)\,dx + \int_b^c f(x)\,dx = \int_a^c f(x)\,dx \tag{14}$$

$$\textit{Domination:} \quad f(x) \geq g(x) \text{ on } [a,b] \implies \int_a^b f(x)\,dx \geq \int_a^b g(x)\,dx \tag{15}$$

We can also use the definite integral to find the average value of an integrable function over some interval $[a, b]$ using the expression

$$\boxed{\operatorname{avg}(f(x)) = \bar{f}(x) = \frac{1}{b-a}\int_a^b f(x)\,dx} \tag{16}$$

The proof of this theorem is very simple, hence, it is left as an exercise to the reader, however, try it after learning the fundamental theorem. Now, we are ready to move on to the fundamental theorem of calculus.

3.4 The Fundamental Theorem

3.4.1 Part 1

Consider the following integral function

$$F(x) = \int_a^x f(t)\,dt$$

Here we have defined was is called an elementary version of an *integral function.* These are functions that are defined as the integral of another function. We will explore them further later on. This equation gives us a way to define new functions, but its importance now is the connection it makes between integrals and derivatives. If f is any continuous function, then the Fundamental Theorem asserts that F is a differentiable function of x whose derivative is f itself. At every value of x, it asserts that

$$\frac{d}{dx}F(x) = f(x)$$

To analyze why this is true, let us look at the derivative of the function. From the definition, the derivative would be the difference quotient as h approaches 0;

$$F'(x) = \frac{F(x+h) - F(x)}{h}$$

Now, if $h > 0$, then the expression is just subtracting the areas under the graph from x to $x + h$ and if h is infinitely small, then we can take the are to just be a rectangle of length h and height $f(x)$ which gives us:

$$F(x+h) - F(x) \approx hf(x)$$

And from this, by dividing by the approximation of h, we get

$$F'(x) = \lim_{h\to 0} \frac{F(x+h) - F(x)}{h} = f(x) \tag{17}$$

And therefore, we get the first part of the fundamental theorem of calculus:

$$\boxed{F'(x) = \frac{d}{dx}\int_a^x f(t)\,dt = f(x)} \tag{18}$$

Equation 18 is essentially saying that the derivative is the opposite operation to the integral. However, it is also very useful when computing integrals. For example, take the following integral:

$$\frac{dy}{dx} = \frac{d}{dt} \int_{x}^{5} 3t \sin(t) \, dt$$

Obviously we do not yet know how to do this integral in order to sub in the integral limits yet. However, the theorem allows us to compute this integral without even integrating!

$$\frac{dy}{dx} = -\frac{d}{dt} \int_{5}^{x} 3t \sin(t) \, dt = -3x \sin(x)$$

3.4.2 Part 2

The second part of the fundamental theorem is the evaluation theorem. It is very simple, for a function $f(x)$ with anti-derivative $F(x)$

$$\boxed{\int_{a}^{b} f(x) \, dx = F(b) - F(a)} \tag{19}$$

Another notation for the right hand side is $F(x)\Big|_a^b$, you may also see $F(x)\Big]_a^b$. This tells us the general algorithm for evaluating a definite integral: first find the anti-derivative of the function, then substitute in the values to find the area. Note, if the area of the graph goes under the x-axis, you may need to divide the area to sub-intervals and taking the absolute value of the areas then add them all up.

3.4.3 Part 3

If we gather up expressions (19) and (18), we get the final part of the fundamental theorem

$$\boxed{\int_{a}^{b} F'(x) \, dx = F(x)\Big|_a^b = F(b) - F(a)} \tag{20}$$

This is known as the net change theorem. The conclusions of the Fundamental Theorem tell us several things. Equation (18) tells us that if you first differentiate the function F and then integrate the result, you get the function F back (adjusted by an integration constant). In a sense, the processes of integration and differentiation are "inverses" of each other. The Fundamental Theorem also says that every continuous function f has an anti-derivative F. It shows the importance of finding anti-derivatives in order to evaluate definite integrals easily. Furthermore, it says that the differential equation $dy/dx = f(x)$ has a solution for every continuous function f.

3.4.4 Interpreting Indefinite Integrals

This is a very short paragraph. Now that we have learned about the geometric interpretation of definite integrals, it is very easy to imagine what the indefinite integral is. It is what we call an *area function*. Basically, when a function is integrated, it gives us a new function that can be used to calculate the area under the initial function's curve. If F is the anti-derivative (or the integral) of some curve $f(x)$, then the area of the curve f over some interval $[a, b]$ can be easily calculated by first computing the area function and then substituting in the points. So The anti-derivative or the integral of a function, gives us a new function allows us to compute the area under that initial function over any interval the function is defined in.

4 Techniques of Integration

In this chapter, we are going to take a deep dive into some common techniques of computing integrals. Then we will use these techniques to expand our understanding of the behavior of functions. Before we move on to learning these techniques, remember to open your eyes when doing integrals. Not every integral is as complicated as it may seem. Many are easily solved by algebraically manipulating the integrand in some way. With that in mind, let's get into it!

4.1 The Reverse Chain Rule

Consider the following equation:

$$\frac{d}{dx}\left(\frac{u^{n+1}}{n+1}\right) = u^n \frac{du}{dx}$$

We know that this is true as long as $n \neq -1$. But what can we learn from this ? Looking at this expression from a different perspective tells us that the anti-derivative of any function of the form $f(x) = u^n$ (where u is a function of x which is denoted by $u = u(x)$) is $u^{n+1}/(n+1) + C$. Therefore

$$\int u \frac{du}{dx}\, dx = \frac{u^{n+1}}{n+1} + C$$

Now, instead of writing the expression $du/dx \cdot dx$ we can simply write du as they are equal. Note, you are NOT allowed to cancel out the differentials of x in the denominator or numerator. We will get to why this is true in a moment. Using this substitution, our integral takes the form:

$$\int u^n\, du = \frac{u^{n+1}}{n+1} + C \tag{21}$$

The reason we cannot cancel out the differentials is because of the property of functions. Note that the derivative of any function at some point may be equal to zero. Hence, if we cancel out the derivatives, it is the equivalent of dividing 0 by 0 which is nonsense and not allowed! You can prove very easily that the differentials simplify to give the formula that Gottfried Leibniz discovered:

$$du = \frac{du}{dx} dx \tag{22}$$

A quick review that a differential is another way of saying the derivative but without specifying with respect to what variable. However, it can also be used as a way to denote an infinitesimally small change in some quantity. In this case, du of equation (21) has the geometrical interpretation of an infinitesimal change in u but the algebraic interpretation of being a differential. We can use this to come up with a method of integration known as the reverse chain rule or simply *u-substitution*. To use *u-substitution*, we must take the approach of looking at du as a derivative. We can write an integral like this:

$$\int f(u)\, du$$

However, we have also learned about composite functions such as $f(g(x))$ before. The question that is answered through the reverse chain rule is how we integrate such functions. Given an integral of the form

$$\int f(g(x))g'(x)\, dx$$

We can make a substitution. If we take the derivative of some function $F(g(x))$, we get that

$$\frac{d}{dx}F(g(x)) = f(g(x))g'(x)$$

Hence, the anti-derivative of the right hand side would be the expression at the left. Therefore we get:

$$\int f(g(x))g'(x)\,dx = F(g(x)) + C \tag{23}$$

But where is the substitution ? To substitute, we let $u = g(x)$, then

$$\frac{du}{dx} = g'(x)$$

And therefore

$$\int f(g(x))g'(x)\,dx = \int f(u)\frac{du}{dx}\,dx$$

Now we can use our formula in equation (22) to simplify the integral to

$$\boxed{\int f(g(x))g'(x)\,dx = \int f(u)\,du} \tag{24}$$

This is *u-substitution*. To better understand how this works, let us try an example.

Example Compute the following integral:

$$I = \int \sqrt{3x+5}\,dx$$

Solution Obviously this integral cannot be computed by inspection or using equation (21). Hence, we make a substitution:

$$\text{let } u = 3x+5$$

Then

$$du = 3\,dx \quad \implies \quad dx = \frac{du}{3}$$

And now the integral takes the form:

$$\int \frac{1}{3}\sqrt{u}\,du = \int \frac{1}{3}u^{1/2}\,du = \frac{1}{3}\int u^{1/2}\,du$$

This allows us to use (21) to calculate the integral:

$$\frac{1}{3}\int u^{1/2}\,du = \frac{1}{3}\frac{2u^{3/2}}{3} + C = \frac{2}{9}\sqrt{u^3} + C$$

And finally, we substitute the expression for u back into our answer to get the final answer

$$\therefore I = \frac{2}{9}\sqrt{(3x+5)^3} + C$$

Example Compute the integral

$$I = \int x^2 e^{x^3}\,dx$$

To compute this integral, we make the substitution

$$u = x^3 \quad \Longrightarrow \quad du = 3x^2\,dx$$

And the integral simplifies to

$$\int x^2 e^{x^3}\,dx = \int 3e^u\,du = 3e^u + C = 3e^{x^3} + C$$

Before we move on to the next section, we must learn how to use the substitution method with definite integrals.

Example Determine the area under the curve $f(x) = 2x\sqrt{x^2+2}$ over the interval $[2, 4]$

Solution First we must write the expression representing this area:

$$A = \int_2^4 2x\sqrt{x^2+2}\,dx$$

Now we make the substitution

$$u = x^2 + 2 \qquad \Longrightarrow \qquad du = 2x\,dx$$

But before we substitute in this value, we must also readjust the integral boundaries. We must see what is the value of u when x is 2 and 4 and replace those with the current integral boundaries:

$$u(2) = 2^2 + 2 = 6, \quad u(4) = 4^2 + 2 = 18$$

Now we are ready to re-write the integral in terms of our substitution

$$A = \int\limits_{6}^{18} \sqrt{u}\,du = \left[\frac{2u^{3/2}}{3} + C\right]_6^{18} = 36\sqrt{2} - 4\sqrt{6} \approx 41.11$$

It is important to always remember to change the integral boundaries when using $u-substitution$

4.2 Integration By Parts

This technique is perhaps one of the most useful integration techniques when it comes to doing integrals in physics problems where functions do not necessarily have a defined expression and may vary from scenario to scenario. It is also extremely useful with evaluating other types of integrals that would not normally be solvable by other techniques. When it comes to integration by parts, it is usually about integrals of the form

$$\int f(x)g(x)\,dx$$

Let us consider the product rule for this multiplication of functions:

$$\frac{d}{dx}[f(x)g(x)] = f'(x)g(x) + f(x)g'(x)$$

and in terms of indefinite integrals

$$\int \frac{d}{dx}[f(x)g(x)]\,dx = \int f'(x)g(x)\,dx + \int f(x)g'(x)\,dx$$

And now with a bit of rearranging, we get that

$$\int f(x)g'(x)\,dx = \int \frac{d}{dx}[f(x)g(x)]\,dx - \int f'(x)g(x)\,dx$$

We also know, from equation (18) of the fundamental theorem of calculus, that the first term on the right hand side will simply be equal to $f(x)g(x)$. This gives us the integration by parts formula:

$$\boxed{\int f(x)g'(x)\,dx = f(x)g(x) - \int f'(x)g(x)\,dx} \tag{25}$$

Perhaps this formula would be easier to apply if we write it using u and v with u representing $f(x)$ and $v = g(x)$, then

$$\boxed{\int u\,dv = uv - \int v\,du}$$

The hope is that with a proper choice of what u and v will be representing, the second integral on the right will be easier to evaluate than the left hand side. This allows us to compute much more complex integrals. Let us try a few examples.

Example Compute the following integral:

$$\int x\cos(x)\,dx$$

To compute this, we make the following substitution

$$u = x \qquad dv = \cos(x)\,dx$$

Here is why we make this choice. We know that in the second term, u will be differentiated. So if we set u equal to something that will disappear or become simpler after it is differentiated, we make the second integral much easier to solve. In this case,

$$du = dx \qquad v = \int \cos(x)\,dx = \sin(x)$$

And therefore, we get that our initial integral is equal to:

$$\int x\cos(x)\,dx = x\sin(x) - \int \sin(x)\,dx$$

We know by inspection that the second integral is equal to $-\cos(x)$ and hence our integral is:

$$\therefore I = x\sin(x) + \cos(x)$$

Now, let us look at a more interesting example. Something with a little twist.

Example Solve the following integral

$$\int e^x \cos(x)\,dx$$

Solution In order to compute this integral, we will use integration by parts. We make the substitution

$$u = e^x \quad dv = \cos(x)\,dx$$

$$\implies du = e^x\,dx \quad v = \sin(x)$$

And we get:

$$I = e^x \sin(x) - \int e^x \sin(x)$$

But now we have to solve the second integral with integration by parts again!

$$u = e^x \quad dv = \sin(x)\,dx$$

$$du = e^x\,dx \quad v = -\cos(x)$$

And therefore, the integral becomes:

$$I = e^x \sin(x) + e^x \cos(x) - \int e^x \cos(x)\,dx$$

Now, it may look like we are stuck in a loop of integration by parts, but open your eyes! The integral on the right hand side is equivalent to the integral we are solving for in the first place!

$$\int e^x \cos(x)\,dx = e^x \sin(x) + e^x \cos(x) - \int e^x \cos(x)\,dx$$

$$\implies 2\int e^x \cos(x)\,dx = e^x \sin(x) + e^x \cos(x) + C$$

$$\therefore \int e^x \cos(x)\,dx = \frac{e^x \sin(x) + e^x \cos(x)}{2} + C$$

There is a shortcut that is quite popular for solving integrals involving integration by parts. It is called the tabular method and it works something like this:

Once you pick your u and dv, instead of going ahead and using the integration by parts formula, list out all derivatives of u until you reach 0 and all integrals of dv until you have the same number as the number of derivatives. For example, for the integral

$$\int x^2 e^x\,dx$$

We let $f(x) = x^2$ and $g(x) = e^x$ and then construct a table as such:

$f(x)$ and its derivatives		**$g(x)$ and its integrals**
x^2	$(+)$	e^x
$2x$	$(-)$	e^x
2	$(+)$	e^x
0		e^x

Now, The result is the multiple of the first term on the left and second term on the right. Then we add/subtract these terms with alternating signs. So the answer is:

$$I = x^2 e^x - 2xe^x + 2e^x + C \qquad \text{(you are encouraged to check this using the formula)}$$

When using the integration by parts formula for definite integrals, the formula is:

$$\boxed{\int_a^b u\,dv = uv\Big|_a^b - \int_a^b v\,du} \tag{26}$$

4.3 Trigonometric Substitution

Trigonometric substitution is a technique that is extremely useful when we are dealing with integrals involving the forms

$$\int \sqrt{a^2+x^2}\,dx \qquad \int \sqrt{a^2-x^2}\,dx \qquad \int \sqrt{x^2-a^2}\,dx$$

or their reciprocals. Let us start with the concept of what we are actually doing. Then, we will work our way into the computation.
Let us recall, that if we have a right triangle with sides x and a, the hypotenuse of this right triangle is given by Pythagoras' theorem to be $\sqrt{x^2+a^2}$. Hence, we can build a triangle out of this form. Then, we introduce an angle θ such that:

$$x = a\tan\theta \qquad dx = a\sec^2\theta\,d\theta$$

And hence we get that:

$$\int \sqrt{a^2+x^2}\,dx = \int \sqrt{a^2+a^2\tan^2\theta}\,a\sec^2\theta\,d\theta$$

Now with a bit of simplification, we get

$$= \int \sqrt{a^2(1+\tan^2\theta)}\,a\sec^2\theta\,d\theta = a^2\int \sec^3\theta\,d\theta$$

In order to solve this integral, we use integration by parts, considering $u = \sec\theta$ and $dv = \sec^2\theta\,d\theta$. After computing this integral, we arrive at:

$$a^2\frac{1}{2}(\sec\theta + \tan\theta + \ln|\sec\theta+\tan\theta|) + C$$

Now it is time to work our way back from the substitution. We know that $\tan\theta = x/a$. we also know that $\sec\theta = \sqrt{1+\tan^2\theta}$ hence we have our final answer:

$$\int \sqrt{x^2+a^2}\,dx = \frac{1}{2}a^2\sqrt{1+\left(\frac{x}{a}\right)^2} + \frac{xa}{2} + \frac{1}{2}a^2\ln\left|\sqrt{1+\left(\frac{x}{a}\right)^2} + \frac{x}{a}\right| + C$$

There are many other alternate forms of writing this. But the idea is to use trigonometric substitution. Let us try one of the second form:

$$\int \frac{dx}{\sqrt{a^2-x^2}}$$

Note that here, the integrand is the reciprocal of the integrand of the second form. The interesting fact is that it does not matter! Either way, the substitution will be the same. For this integral, we make the substitution:

$$x = a\sin\theta \qquad dx = a\cos\theta\,d\theta$$

$$\int \frac{dx}{\sqrt{a^2-x^2}} = \int \frac{a\cos\theta}{\sqrt{a^2(1-\sin^2\theta)}}\,d\theta = \int \frac{a\cos\theta}{a\cos\theta}\,d\theta = \int d\theta = \theta + C$$

Now we go back and isolate for theta from our substitution to get the final answer:

$$\theta = \arcsin\left(\frac{x}{a}\right) \implies \int \frac{dx}{\sqrt{a^2-x^2}} = \arcsin\left(\frac{x}{a}\right) + C$$

The idea is the same, but just to further solidify our understanding, let us try an example from the third form:

$$\int \sqrt{x^2-a^2}\,dx$$

For this form, we will use the substitution

$$x = a\sec\theta \qquad dx = a\sec\theta\tan\theta\,d\theta$$

$$\int \sqrt{x^2-a^2}\,dx = a^2\int \tan^2\theta\sec\theta\,d\theta$$

Now, we will use a trick and re-write $\tan^2\theta$ as $1-\sec^2\theta$. This allows us to break up the integral to:

$$= a^2 \int \sec^3\theta \, d\theta - a^2 \int \sec\theta \, d\theta$$

We already know the answer to the first integral. But what about the second one? To evaluate this integral, we multiply the whole thing by the expression

$$\frac{\sec\theta + \tan\theta}{\sec\theta + \tan\theta}$$

which is equal to 1. Hence it takes the form:

$$a^2 \int \sec\theta \, d\theta = a^2 \int \frac{\sec^2\theta + \sec\theta\tan\theta}{\sec\theta + \tan\theta} \, d\theta$$

At first sight it may look like we have made a big mistake. But open your eyes, the numerator is just the derivative of the denominator! So we can make a u substitution:

$$u = \sec\theta + \tan\theta \qquad du = \sec^2\theta + \sec\theta\tan\theta$$

Hence

$$a^2 \int \frac{\sec^2\theta + \sec\theta\tan\theta}{\sec\theta + \tan\theta} \, d\theta = a^2 \int \frac{1}{u} \, du$$

Now, we must remember that the derivative of the natural logarithm, is

$$\frac{d}{du} \ln u = \frac{1}{u} \, du$$

This allows us to define the logarithm as an integral function!

$$\boxed{\ln(x) = \int_0^x \frac{dt}{t}} \tag{27}$$

So, we can use this to say that the integral above, is simply equal to

$$a^2 \int \frac{1}{u} \, du = a^2 \ln|u| + C$$

And now we must substitute everything back:

$$I_2 = a^2 \ln|\sec\theta + \tan\theta| + C$$

$$I_1 = \frac{1}{2}a^2(\sec\theta + \tan\theta + \ln|\sec\theta + \tan\theta|) + C$$

$$I = I_1 - I_2$$

$$\sec\theta = \frac{x}{a} \qquad \tan\theta = \sqrt{1 + \left(\frac{x}{a}\right)^2}$$

Therefore, we get the final answer,

$$\int \sqrt{x^2 - a^2}\,dx = \frac{1}{2}a^2 \ln\left|\frac{x}{a} + \sqrt{1 + \left(\frac{x}{a}\right)^2}\right| + \frac{xa}{2} + \frac{1}{2}a^2\sqrt{1 + \left(\frac{x}{a}\right)^2} + C$$

Notice that throughout this section, we have also used trigonometric substitution when the square root is in the denominator. There is one last thing we can use trigonometric substitution for. That, is when we have a square root of a quadratic in either denominator or numerator of the integrand. For example,

$$\int \frac{dx}{\sqrt{x^2 - 3x + 5}}$$

Obviously, the quadratic in the question is not factor-able. However, we can use a technique we learned back when we were first learning about quadratics: completing the square! If we complete the square in this integrand, we get

$$\int \frac{dx}{\sqrt{x^2 - 3x + 5}} = \int \frac{dx}{\sqrt{\left(x - \frac{3}{2}\right)^2 + \frac{11}{4}}}$$

Now, instead of going through all the trouble of breaking the quadratic into different fractions (which is a method we will also learn soon) or making multiple rigorous substitutions, we will simply make the substitution:

$$x = \sqrt{\frac{11}{4}}\tan\theta + \frac{3}{2} \qquad dx = \frac{11}{4}\sec^2\theta\,d\theta$$

And this transforms the integral to:

$$I = \int \frac{\frac{11}{4}\sec^2\theta\, d\theta}{\sqrt{\frac{11}{4}}\sec\theta} = \sqrt{\frac{11}{4}}\int \sec\theta\, d\theta = \sqrt{\frac{11}{4}}\ln|\sec\theta + \tan\theta| + C$$

Finally, we replace theta with x in order to get the final expression:

$$\int \frac{dx}{\sqrt{x^2 - 3x + 5}} = \sqrt{\frac{11}{4}}\ln\left|\sqrt{1 + \left(\frac{2x+3}{\sqrt{11}}\right)^2} + \frac{2x+3}{\sqrt{11}}\right| + C$$

We can also do this when we have a quadratic that is not under a square root! This technique should give you the inverse of some sort of a trigonometric function as the anti-derivative plus the constant of integration or any additional terms that may be included. Before we finish up this section, if you recall the multiple forms of the Pythagorean identity with trigonometric functions, you will realise that this whole time, we were making use of this identity to simplify our integrals. It is important to keep in mind that not every integrand should be integrated right away! You still may need to perform some sort of algebraic manipulation before integrating. So keep your eyes open and do not be scared to manipulate the integral. This is the biggest mistake that calculus students make. To summarize this whole section;

$$\boxed{\begin{array}{l}
\text{if} \quad \int \sqrt{x^2 - a^2}\, dx \quad \text{or} \quad \int \frac{dx}{\sqrt{x^2 - a^2}} \implies \text{substitute} \quad x = a\sec\theta \\
\text{if} \quad \int \sqrt{x^2 + a^2}\, dx \quad \text{or} \quad \int \frac{dx}{\sqrt{x^2 + a^2}} \implies \text{substitute} \quad x = a\tan\theta \\
\text{if} \quad \int \sqrt{a^2 - x^2}\, dx \quad \text{or} \quad \int \frac{dx}{\sqrt{a^2 - x^2}} \implies \text{substitute} \quad x = a\sin\theta \\
\text{if} \quad \int \sqrt{ax^2 + bx + c}\, dx \quad \text{or} \quad \int \frac{dx}{\sqrt{ax^2 + bx + c}} \implies \text{substitute} \quad x = \frac{\sqrt{4ac - b^2}}{2a}\tan\theta - \frac{b}{2a}
\end{array}}$$

We will dive in a little bit deeper into integrals involving trigonometric functions and methods we can use to solve them in the section "Trigonometric Integrals". To continue, we have to learn about integrals involving rational functions.

4.4 Partial Fractions Method

This section shows how to express a rational function (a quotient of polynomials) as a sum of simpler fractions, called partial fractions, which are easily integrated. For instance, the rational function $(5x-3)/(x^2-2x-3)$ can be rewritten as:

$$\frac{5x-3}{x^2-2x-3} = \frac{2}{x+1} + \frac{3}{x-3}$$

This makes the integration of this function a lot easier as we can simply rewrite the integral:

$$\int \frac{5x-3}{x^2-2x-3}\,dx = \int \frac{2}{x+1}\,dx + \int \frac{3}{x-3}\,dx$$

Which as we know can be evaluated to logarithmic anti-derivatives plus the constant of integration. The method involved in writing fractions as a sum of simpler fractions is known as the *partial fractions method*.

To be able to apply this technique, we must learn the theorem which originates from rational functions and algebra. The theorem states that rational functions whose numerator's degree is lower than their denominator's can be written as a sum of multiple other rational functions. This is known as the *Heaviside method*. Given that

$$\frac{f(x)}{g(x)} = \frac{f(x)}{(x-r_1)(x-r_2)\dots(x-r_n)}$$

Such that each r is a root of the function $g(x)$, we can find a set of numbers, A_i such that:

$$\begin{aligned}
A_1 &= \frac{f(r_1)}{(r_1-r_2)\dots(r_1-r_n)} \\
A_2 &= \frac{f(r_2)}{(r_2-r_1)(r_2-r_3)\dots(r_2-r_n)} \\
&\vdots \qquad\qquad \vdots \\
A_n &= \frac{f(r_n)}{(r_n-r_1)(r_n-r_2)\dots(r_n-r_{n-1})}
\end{aligned}$$

And now, we can re-write the initial fraction as a sum of other fractions:

$$\frac{f(x)}{g(x)} = \sum_{i=1}^{n} \frac{A_i}{(x - r_i)}$$

This expression, in contrast to the previous function, can be easily integrated through the logarithmic integral formula. There may also be situations where you have a quadratic or even a cubic in the denominator. In that case, you would still break it to partial fractions but you must follow one extra rule; the degree of the numerator of the partial fraction must be one less than the degree of its denominator. These rules will be easier to follow once we go over a few examples. The main idea is that if the fraction:

$$\frac{f(x)}{g(x)} = \frac{f(x)}{(x - r)(ax^2 + bx + c)}$$

You would NOT break it off like this:

$$\frac{f(x)}{g(x)} = \frac{A}{x - r} + \frac{B}{ax^2 + bx + c}$$

But instead, you would have to include a linear function $Bx + C$ at the numerator of the second fraction. So the correct expansion is:

$$\frac{f(x)}{g(x)} = \frac{A}{x - r} + \frac{Bx + C}{ax^2 + bx + c}$$

This is also true if you have a cubic term. If the denominator is cubic, then the numerator must be $Bx^2 + Cx + D$ and so on. Let us go through a few examples.

Compute the following integral:

$$\int \frac{x^2 + 4x + 1}{(x - 1)(x + 1)(x + 3)}\, dx$$

To solve this integral, we first break off the fraction to:

$$\frac{x^2 + 4x + 1}{(x - 1)(x + 1)(x + 3)} = \frac{A}{x - 1} + \frac{B}{x + 1} + \frac{C}{x + 3}$$

Now, we must take the common denominator.

$$\frac{x^2+4x+1}{(x-1)(x+1)(x+3)} = \frac{(A+B+C)x^2+(4A+2B)x+3A-3B-C}{(x-1)(x+1)(x+3)}$$

If we recall the function identity, we know that we can set the coefficients of the numerator of the initial fraction, equal to the coefficients of the numerator of the new fraction. This gives us 3 equations for our 3 unknowns:

$$\begin{aligned} A+B+C &= 1 \\ 4A+2B &= 4 \\ 3A-3B-C &= 1 \end{aligned}$$

These equations yield the solution $A = 3/4, B = 1/2, C = -1/4$. Hence our integral becomes:

$$I = \frac{3}{4}\int \frac{1}{x-1}\,dx + \frac{1}{2}\int \frac{1}{x+1}\,dx - \frac{1}{4}\int \frac{1}{x+3}\,dx$$

Which tells us that the integral is equal to:

$$\int \frac{x^2+4x+1}{(x-1)(x+1)(x+3)}\,dx = \frac{3}{4}\ln|x-1| + \frac{1}{2}\ln|x+1| - \frac{1}{4}\ln|x+3| + C$$

Let us try another one. This one, will have some twists to it!

Compute the following integral:

$$\int \frac{1}{x^3+1}\,dx$$

The first step is to break off the integrand to its partial fractions:

$$\frac{1}{(x+1)(x^2-x+1)} = \frac{A}{x+1} + \frac{Bx+C}{x^2-x+1}$$

After taking the common denominator we get the identity

$$(A+B)x^2 + (B+C-A)x + A + C = 1$$

which gives us the three equations

$$
\begin{aligned}
A + B &= 0 \\
B + C - A &= 0 \\
A + C &= 1
\end{aligned}
$$

and they yield the solutions $A = 1/3,\ B = -1/3,\ C = 2/3$. And now, we can break off our integral to 3 integrals;

$$\int \frac{1}{x^3+1}\,dx = \int \frac{1}{3} \cdot \frac{1}{x+1} + \frac{1}{3} \cdot \frac{2-x}{x^2-x+1}\,dx = \frac{1}{3}\int \frac{1}{x+1} + \frac{2}{x^2-x+1} - \frac{x}{x^2-x+1}\,dx$$

$$I_1 = \frac{1}{3}\ln|x+1| + C$$

For the second term, we must complete the square and then we use trigonometric substitution:

$$I_2 = \frac{1}{3}\int \frac{2}{(x-\frac{1}{2})^2 + \frac{3}{4}}\,dx$$

$$x = \sqrt{\frac{3}{4}}\left(\tan\theta - \frac{1}{2}\right)$$

$$dx = \sqrt{\frac{3}{4}}\sec^2\theta\,d\theta$$

$$\implies I_2 = \frac{2}{3}\sqrt{\frac{3}{4}}\int \frac{4}{3}\frac{\sec^2\theta}{\sec^2\theta}\,d\theta = \frac{4\sqrt{3}}{9}\theta + C = -\frac{4}{\sqrt{3}}\arctan\left(\frac{2x-1}{\sqrt{3}}\right) + C$$

And finally, we reach the third term,

$$I_3 = \frac{1}{3}\int \frac{x}{x^2-x+1}\,dx$$

To compute this one, we make a u-substitution:

$$u = x - \frac{1}{2} \qquad dx = dx$$

$$\implies I_3 = \frac{1}{3}\int \frac{u+\frac{1}{2}}{u^2+\frac{3}{4}}\,du = \frac{1}{3}\left(\int \frac{u}{u^2+\frac{3}{4}}\,du + \frac{1}{2}\int \frac{1}{u^2+\frac{3}{4}}\,du\right) = \frac{1}{\sqrt{3}}\arctan\left(\frac{2u}{\sqrt{3}}\right) + \frac{1}{2}\ln\left|u^2 + \frac{3}{4}\right| + C$$

After substituting x back in we get:

$$I_3 = \frac{1}{\sqrt{3}} \arctan\left(\frac{2x-1}{\sqrt{3}}\right) + \frac{1}{2} \ln|x^2 - x + 1| + C$$

And finally, we get that the integral is equal to:

$$\int \frac{1}{x^3+1}\, dx = \frac{1}{3} \ln|x+1| + \frac{1}{2} \ln|x^2 - x + 1| - \sqrt{3} \arctan\left(\frac{2x-1}{\sqrt{3}}\right) + C$$

Before we end off this section, it is worth noting that whenever the degree of numerator is higher than denominator, perform long division and use partial fractions (only if needed) on the remainder only. Now, we have learned pretty much the general techniques of integration. There are a few that we will talk about in the end of this handout and few that we will go over right now. I highly recommend you to practice at least 3 hard integrals a day to get comfortable with integrals. You have to practice a lot in order to reach a point where you can tell what type of substitution you should make or which method you should use to integrate.

4.5 Trigonometric Integrals

Trigonometric integrals involve algebraic combinations of the six basic trigonometric functions. In principle, we can always express such integrals in terms of sines and cosines, but it is often simpler to work with other functions, such as the integral

$$\int \sec^2\theta\, d\theta = \tan\theta + C$$

The general idea is to use identities to transform the integrals we have to find into integrals that are easier to work with.

4.5.1 Products of Powers of Sines and Cosines

Let us begin with integrals of the form:

$$\int \sin^m x \cos^n x\, dx$$

where m and n are non-negative integers. For the case of zero it is quite simple. So we will not get into that. For all positive cases, we divide the appropriate substitutions into 3 cases based on whether the powers are odd or even.

If **m is odd** and n is even: write $m = 2k+1$ and then use the identity: $\sin^2 x = 1 - \cos^2 x$

to obtain: $\sin^m x = \sin^{2k+1} x = (\sin^2 x)^k \sin x = (1 - \cos^x)^k \sin x$

then combine the single $\sin x$ with dx and set it equal to $-d(\cos x)$

If **n is odd** and m is even: write $n = 2k+1$ and then use the identity: $\cos^2 x = 1 - \sin^2 x$

to obtain: $\cos^n x = \cos^{2k+1} x = (\cos^2 x)^k \cos x = (1 - \sin^x)^k \cos x$

then combine the single $\cos x$ with dx and set it equal to $d(\sin x)$

If **both m and n are even**, substitute:

$$\sin^2 x = \frac{1 - \cos 2x}{2} \qquad \cos^2 x = \frac{1 + \cos 2x}{2}$$

to reduce the integrand to one in lower powers of $\cos 2x$

Let us try a few examples to get an understanding of the idea. Compute the following integral:

$$\int \sin^3 x \cos^2 x \, dx$$

This is an example of case one. Hence, we follow the procedure

$$I = \int \sin^2 x \cos^2 x \sin x \, dx = \int (1 - \cos^2 x) \cos^2 x \sin x \, dx$$

Now, we make our substitution:

$$u = \cos x \quad du = -\sin x \, dx$$

$$\implies I = -\int (1 - u^2) u^2 \, du = \int u^4 - u^2 \, du = \frac{u^5}{5} - \frac{u^3}{3} + C$$

After re-substituting u as cosine, we get the final answer:

$$I = \frac{\cos^5 x}{5} - \frac{\cos^3 x}{3} + C$$

Compute the following integral

$$\int \cos^5 x \, dx$$

This integral is an example of the second case, meaning $m = 0$ counts as m is even. Hence we follow the procedure for case 2;

$$\int \cos^5 x \, dx = \int \cos^4 x \cos x \, dx = \int (1 - \sin^2 x)^2 \cos x \, dx$$

$$\text{let } u = \sin x \qquad du = \cos x$$

$$I = \int (1 - u^2)^2 \, du = \int 1 - 2u^2 + u^4 \, du = u - \frac{2}{3}u^3 + \frac{1}{5}u^5 + C$$

And now we substitute the trigonometric function back in to get the final result:

$$I = \sin x - \frac{2}{3}\sin^3 x + \frac{1}{5}\sin^5 x + C$$

For the final example, compute the following integral:

$$\int \sin^2 x \cos^4 x \, dx$$

This is an example of the 3rd case, hence, we turn both functions to $\cos 2x$ using the double angle formulae:

$$\int \sin^2 x \cos^4 x \, dx = \int \left(\frac{1 - \cos 2x}{2}\right)\left(\frac{1 + \cos 2x}{2}\right)^2 \, dx$$

$$= \frac{1}{8}\int (1 - \cos 2x)(1 + 2\cos 2x + \cos^2 2x) \, dx = \frac{1}{8}\int 1 + \cos 2x - \cos^2 2x - \cos^3 2x \, dx$$

$$= \frac{1}{8}\left[x + \frac{1}{2}\sin 2x - \int (\cos^2 2x + \cos^3 2x \, dx)\right]$$

To solve the first integral, we use:

$$\int \cos^2 2x \, dx = \frac{1}{2}\int 1 + \cos 4x \, dx = \frac{1}{2}x + \frac{1}{8}\sin 4x + C$$

and for the second one we use:

$$\int \cos^3 2x \, dx = \int (1 - \sin^2 2x)\cos 2x \, dx = \frac{1}{2}\int (1 - u^2) \, du = \frac{1}{2}\sin 2x - \frac{1}{6}\sin^3 2x + C$$

Combining these results gives us the final answer

$$\int \sin^2 x \cos^4 x \, dx = \frac{1}{16}x - \frac{1}{64}\sin 4x + \frac{1}{48}\sin^3 2x + C$$

4.5.2 Eliminating Square Roots

In this part we will only do 2 examples that contains the general cases. as the rest is the same pattern. We will use the identities $\sin^2(nx/2) = (1 - \cos(nx))/2$ and $\cos^2(nx/2) = (1 - \cos(nx))/2$ to eliminate square roots of trigonometric functions in the integrand.
Compute the following the integral

$$\int \sqrt{1 - \cos(nx)} \, dx$$

To compute this integral we use the first identity introduced earlier to turn it into the form

$$\int \sqrt{2\sin^2\left(\frac{nx}{2}\right)} \, dx$$

And from here, we eliminate the square root and integrate to get:

$$= \int \sqrt{2}\sin\left(\frac{nx}{2}\right), dx = \frac{2\sqrt{2}}{n}\cos\left(\frac{nx}{2}\right) + C$$

$$\therefore \int \sqrt{1 - \cos(nx)} \, dx = \frac{2\sqrt{2}}{n}\cos\left(\frac{nx}{2}\right) + C$$

We take a similar approach with integrals of the form:

$$\int \sqrt{1 + \cos(nx)} \, dx$$

after using the second identity we introduced, you should get:

$$I = \frac{2\sqrt{2}}{n}\sin\left(\frac{nx}{2}\right) + C$$

This whole section should really outline the great importance of being confident in trigonometric identities and being able to see where to use which identity in order to simplify the computation of the integral at hand.

4.5.3 Powers of Tangents and Secants

To approach integrals of the power of tangent or secant of a variable, we use the identities $\tan^2 x = \sec^2 x - 1$ and $\sec^2 x = \tan^2 x + 1$ in order to simplify the integral. The steps are mostly the same as the ones discussed in the first subsection of this section. Let us try a few examples.

Compute the following integral:

$$\int \tan^4 x \, dx$$

To solve this integral, we first break off the power to 2 terms of tangent squared:

$$= \int \tan^2 x \cdot \tan^2 x \, dx = \int \tan^2 x \cdot (\sec^2 x - 1) \, dx = \int \tan^2 x \sec^2 x - \tan^2 x \, dx$$

$$= \frac{1}{3} \tan^3 x - \int \sec^2 x - 1 \, dx = \frac{1}{3} \tan^3 x - \int \sec^2 x \, dx + \int dx$$

$$I = \frac{1}{3} \tan^3 x - \tan x + x + C$$

Compute the following integral:

$$\int \sec^3 x \, dx$$

This integral is not easy to solve. To solve this one (although we have solved it before) we use integration by parts to get:

$$= \sec x \tan x - \int \tan^2 x \sec x \, d = \sec x \tan x - \int \sec^3 x - \sec x \, dx$$

$$\int \sec^3 x \, dx = \sec x \tan x - \int \sec^3 x \, dx + \ln|\sec x + \tan x| + C$$

And now, after re-arranging and then simplifying, we get the final result:

$$\therefore \int \sec^3 x \, dx = \frac{1}{2} \sec x \tan x + \frac{1}{2} \ln|\sec x + \tan x| + C$$

You may ask, why have we not divided the C term by 2 ? The reason is this; since we are not given the value of C, it could be any arbitrary constant. Hence, we will automatically consider any algebraic simplification to be incorporated into the constant term so that there is no need to readjust it.

4.5.4 Products of Sines and Cosines

In this part, we will learn how to approach integrals of the forms:

$$\int \sin mx\, \sin nx\, dx \qquad \int \sin mx\, \cos nx\, dx \qquad \int \cos mx\, \cos nx\, dx$$

These integrals come up quite frequently in harmonic analysis (specifically, they are used to calculate Fourier coefficients used to write the Fourier series of periodic functions) and so they are quite useful to know how to calculate. To do so, we use the following identities which can be easily derived from the compound angle formulae:

$$\boxed{\begin{aligned} \sin mx\, \sin nx &= \frac{1}{2}[\cos[(m-n)x] - \cos[(m+n)x]] \\ \sin mx\, \cos nx &= \frac{1}{2}[\sin[(m-n)x] + \sin[(m+n)x]] \\ \cos mx\, \cos nx &= \frac{1}{2}[\cos[(m-n)x] + \cos[(m+n)x]] \end{aligned}} \quad \begin{aligned} (28) \\ (29) \\ (30) \end{aligned}$$

We will only do one example as the rest is following the same procedure. Compute the following integral:

$$\int \sin(\alpha x) \cos(\beta x)\, dx$$

We will use equation (29) to compute this integral:

$$I = \frac{1}{2} \int \sin(\alpha - \beta)x + \sin(\alpha + \beta)x\, dx = -\frac{1}{2(\alpha - \beta)} \cos([\alpha - \beta]x) - \frac{1}{2(\alpha + \beta)} \cos([\alpha + \beta]x) + C$$

4.6 Improper Integrals

In this short section, we will discuss how to solve improper definite integrals. But what are improper integrals ? These are integrals that involve infinities in their integral boundaries. The following integrals are all examples of improper integrals:

$$\int_{-\infty}^{+\infty} e^{-x^2}\, dx \qquad \int_{0}^{\infty} e^{-x^3} \cos(5x)\, dx \qquad \int_{-\infty}^{+\infty} \frac{\ln(x^2+1)}{x^4+1}\, dx$$

The approach to solving these types of integrals is actually quite simple; just replace the infinity in the integral boundary with some made up variable, integrate and then take the limit as the variable approaches the initial boundary. Let us do a few examples of this.

Example Is the area under the curve $f(x) = \ln(x)/x^2$ finite ? If so, what is its value ?

To do this problem, we first write out the expression for the area under the curve $f(x)$ knowing that its domain is the interval [1,∞]:

$$A = \int_0^\infty \frac{\ln x}{x^2}\, dx$$

Now, we replace the integral's upper boundary with some variable β to get:

$$A = \lim_{\beta \to \infty} \int_0^\beta \frac{\ln x}{x^2}\, dx$$

Then, we compute the anti-derivative using integration by parts to get:

$$A = \lim_{\beta \to \infty} \frac{-\ln x - 1}{x}\bigg|_0^\beta + 1 = \lim_{\beta \to \infty} \frac{-\ln \beta - 1}{\beta} + 1 = \lim_{\beta \to \infty} \frac{-\ln \beta}{\beta} - 0 + 1$$

To compute this limit, we will use L'Hôpital's rule to obtain:

$$A = \lim_{\beta \to \infty} \frac{-1/\beta}{1} + 1 = 1$$

and thus we get that:

$$\int_0^\infty \frac{\ln x}{x^2}\, dx = 1$$

Aside from computing such integrals, there is a long and deep discussion we must do about convergence and divergence. However, we will postpone this discussion to a future handout so that we can discuss it in greater detail. To better understand the two concepts we need a deep understanding of series and sequences which deserves its own handout.

5 Final Point

I hope that this handout was a good starting point for understanding integrals. We will discuss further on applications of integration, series and sequences and multivariable calculus in future handouts. Before we finish off, I have left a few challenge problems for you to try:

Challenge 1 Prove Equation (10) (*hint: use the properties of Riemann Sums*)

Challenge 2 Compute the following integral:

$$\int (\ln x)^{\ln x} \left[\frac{1}{x} + \frac{\ln(\ln x)}{x}\right] dx$$

Challenge 3 Compute the following integral:

$$\int_0^\infty \frac{\ln x}{x^3}\, dx$$

Challenge 4 Prove the following property of integrals, also known as King's Rule:

$$\int_a^b f(x)\, dx = \int_a^b f(a+b-x)\, dx$$

Challenge 5 Compute the following integral:

$$\int \frac{\sin x}{2+\sin x}\, dx$$

Challenge 6 Compute the following integral:

$$\int \sqrt{\tan x}\, dx$$

Challenge 7 Derive an expression which gives the exact result for the gamma function:

$$\Gamma(n) = \int_0^\infty x^{n-1} e^{-x}\, dx$$

Which has the property: $\Gamma(n) = (n-1)!$ for all $n \in \mathbb{Z}^+$.