Introduction to Waves

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December 2023

1 What Is Harmony ?

Everyday, when we walk around, when we breath, when we are sleeping, when we are studying physics, etc. we hear all sorts of different sounds from the environment around us as the waves travel through a medium and reach our ears. Some of these sounds are more pleasant to our ears than others. Take for example, the sound of someone screaming versus the sound that a very good pianist makes with the piano. How pleasing this sound is depends not just on what is used to make the sound but what is the sound that is being produced, or in other words, what pitch is being heard by us. Now, any pitch when we hear it sounds good by itself. It is the combination of these pitches which make things complicated and really determine whether we will get an even better sound or worst.

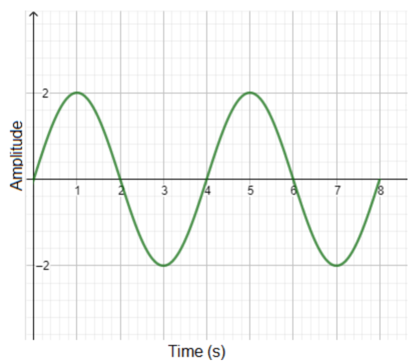
How one learns which pitches to play together in order to make a good sound is known as harmony. It is a way of determining whether notes are dissonant together or not. For example, playing a C and a D together will sound very dissonant but a C and an E will sound nice. But the real question is why do these work ? How do we know it works ? and that is when physics comes in!

2 The Basics of Waves

Sound as we now it, is a wave. When we hear a sound, it is the result of the waves reaching our ears. These waves travel through a medium and hit our ear drums which helps us ”hear” the sound. Every wave, just like sound, has some properties that are used to describe it. In this section, we will explore the properties of waves and then we will apply them to understand harmony and musical chords.

2.1 Properties of Waves

The first property of a wave we will learn about is the amplitude of a wave. Most of the time, we will work with waves that are sinusoidal functions. Here is an example of a wave's graph:



If this wave is represented by some function, $\psi(t)$, then the amplitude, ψ, is the y value. In the context of sound waves, the greater the amplitude is, the louder the sound will be. When searching for a function that describes a waves behavior, we are looking for a relationship between the position or time and the amplitude of a wave. The unit used to measure the amplitude is usually m (meters). However, this may vary depending on the context. Before introducing the period, we need to know what a crest and a trough is. When looking at the graph of a wave, we see that the graph has "ups" and "downs". The points where the function's value is at a maximum, are known as crests and where the function's value is at a minimum, are known as troughs. The next fundamental property of a wave, is its period. The period of a wave is represented using the Greek letter τ (tau). A quick note on this letter, in Einstein's Theory of Relativity, τ is used for time elapsed and in Engineering Physics, it is used to represent a quantity called torque. In waves, however, we use this to represent the period. This is when the amplitude is expressed as a function of time. The period is defined as the *time* between two crests or trough. t is the time that it takes for the wave to do one cycle (which means for the wave to go from one crest to another crest or one trough to another trough). The reciprocal of this quantity is known as the frequency of the wave, in mathematical terms:

$$f = \frac{1}{\tau}$$

The frequency will tell us how many cycles a wave can do in 1 second. Hence, the unit for period is s (seconds) and for frequency, it is 1/s. In the context of a sound wave, the frequency will tell us how high pitched the wave will be. The greater the frequency, the higher the pitch. However, using the inverse proportionality of f and τ, we can conclude that the greater the period of a wave is, the lower the frequency and hence the lower the pitch. Therefore, when we hear the sound of a high pitched instrument like a Piccolo or Flute, we know that the waves generated have a much smaller period than the waves generated by a Double Bass or Tuba. Now, there is only one more property of a wave that we need to be familiar with which is the wavelength. In the figure above, we express the wave's amplitude as a function of time. However, what would happen if we express the amplitude as a function of position ? Remember that we can have a wave's equation both as a function of time and as a function of position. When we express the amplitude, ψ, as a function of position, x, then the period is meaningless. Remember that the period, is a temporal quantity not a spatial one. Since we cannot use τ, we introduce a new quantity, λ (lambda) which represents the *wavelength* of a wave. This quantity, tells us the *distance* between two crests or trough and hence, its unit is in meters as well.

The next property is the speed of a wave. The speed of any object is defined as the displacement over some time interval or:

$$\vec{v} = \frac{\vec{\Delta d}}{\Delta t}$$

In the case of waves, what quantities can we use as displacement and time ? If we take a look back at the properties of a wave, we learned that the period, is a temporal quantity. So we can guess that if we have a formula for the velocity of a wave, the denominator is going to have something to do with the period. so we get:

$$v = \frac{?}{\tau}$$

Now, if we can figure out what is the distance that the wave travels over some period, then we can find Δd. Fortunately, we also know about the wavelength of a wave which is the distance a wave travels over the time interval, τ. Hence, we can replace Δd with λ and we get that the velocity of a wave is determined by the following expression:

$$v = \frac{\lambda}{\tau}$$

Since we know that the period is the reciprocal of the frequency, we can say the following:

$$f = \frac{1}{\tau} \implies v = \frac{1}{\tau} \cdot \lambda = f\lambda$$

$$\therefore v = \frac{\lambda}{\tau} = f\lambda$$

This wave equation is true for all waves of non-changing period, frequency and wavelength.

2.2 Principle of Superposition

Now that we have learned about the basic properties of waves, we need to know how waves interact with each other. What happens when two waves meet with each other ? The answer is actually very simple, whenever two waves meet, in order to determine the resultant wave's amplitude, we add the amplitude of all the waves together. Or in mathematical terms, when n number of waves meet, the resultant wave's amplitude is determined by:

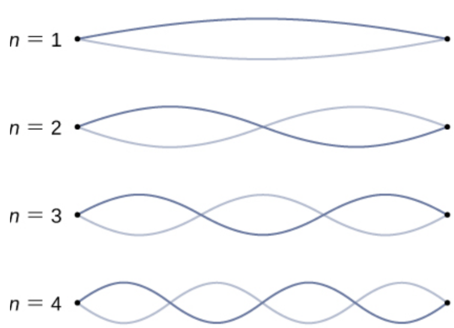
$$\psi_{result} = \psi_1 + \psi_2 + \psi_3 + \cdots + \psi_n = \sum_n \psi_n$$

When waves meet and the sum of their amplitudes create a greater amplitude, then the interaction is called *constructive interference.* If this sum creates a smaller amplitude (if the amplitude of a wave's sign is the opposite of the other) then the resultant interaction is called *destructive interference.*

3 Standing Waves

Normally when we think about waves, we think of moving sinusoidal functions through a medium. A standing wave, however, as the name suggests is a wave that is bounded at its limits. Essentially, they are a result of interference coupled with boundary conditions.

Boundary conditions, are the restrictions that we place on a standing wave. Let us imagine a rope. Let us say that an endpoint of this rope is tied to a wall and the other endpoint to another wall. Hence, the rope is *bounded* by the two walls. The endpoints of the rope are its boundary conditions.



The figure on the right shows multiple harmonics of a standing wave. The points in the end of the rope where the boundary conditions are, or where the amplitude of the wave is equal to zero are called *nodes.* On the contrary, the peaks of the wave, or the crests and troughs, are called the *anti-nodes.* The variable n, denotes the number of anti-nodes present in the standing wave. We usually use the variable L to denote the distance between the two endpoints of the wave, or in the case of a rope, the length of the rope is L. Each of these waves is called a *harmonic*. For example, $n = 3$ denotes the third harmonic. The first harmonic, where $n = 1$, is called the fundamental frequency. The reason for that is because the frequency of all the other harmonics, are an integer multiple of the fundamental frequency. Or in mathematical form:

$$f_n = nf_1$$

Hence, given the frequency of the first harmonic, and the value of n, we can calculate the frequency of the nth harmonic using the above formula. The other harmonics of the wave are for a vibrating string are called the *overtones* which are integral multiples of the fundamental frequency.
Now, let us express the wave length for the different harmonics. we know that the wave length must be one full cycle. Hence, for the second harmonic, $\lambda = L$. Now for the case of the first harmonic, $\lambda = L/2$. Let us continue this pattern and see what we obtain:

$$n = 1: \quad \lambda = 2L$$

$$n = 2: \quad \lambda = L$$

$$n = 3: \quad \lambda = \frac{2L}{3}$$

$$n = 4: \quad \lambda = \frac{1}{2}L$$

$$n: \quad \lambda = \frac{2L}{n}$$

Hence we have now found a relationship between the wavelength and the length of the wave and the harmonic. But, since we know that $v = f\lambda$, we can now relate this to the frequency and speed as well. Hence we get a very important equation:

$$f_n = \frac{nv}{2L} = nf_1$$

4 Resonance

Resonance is a very common word in households. However, in the context of physics, not many people know about what it really is. Let us get right into it. Resonance, is caused by forced vibration. As we saw in the previous section, every string has its *natural frequency*. Now, an interesting thing happens when we force a vibration and the oscillation creates a frequency that is approximately equal to the natural frequency. Due to the principle of superposition and the equivalence of the two frequencies, the amplitude of the vibration suddenly increases! This phenomenon is called resonance.

As we discussed in section 2.1, when the amplitude increases, the volume increases. Hence, when resonance occurs, the volume (and intensity) of a sound wave increases. Musicians apply this knowledge when composing pieces, harmonizing, and even playing to better project in concert halls and over an ensemble when needed without the need to press or attack their instrument.

Note, for future reference, the intensity of a sound wave is usually calculated through the power it transmits. Power of a sound wave is given by:

$$P = \frac{1}{2}\mu 4\pi^2 f^2 \psi^2 v$$

Now, here we must introduce a new quantity called the *angular velocity* or *angular frequency* denoted by the Greek letter lower case omega, ω. In the context of a mass rotating around a point in circular motion, ω is defined as:

$$\omega = \frac{\Delta\theta}{\Delta t} = \frac{d\theta}{dt}$$

For the case of waves, we can also say that it is defined as:

$$\omega = 2\pi f = \frac{2\pi}{\tau}$$

For simplicity, we now express power in terms of ω:

$$P = \frac{1}{2}\mu\omega^2\psi^2 v$$

The letter μ (mu) is another Greek letter which is a constant representing the mass per unit length of the vibrating string. Now, we apply the definition of intensity which is, the distribution of power per unit surface area. In other words, if σ (sigma) represents the surface area, then:

$$I = \frac{P}{\sigma}$$

And if you know calculus, then:

$$P = \int \vec{I} \cdot d\vec{\sigma}$$

The surface area, or σ depends on the type of surface we are considering. For a sphere, the surface area is given by $\sigma = 4\pi r^2$. Hence:

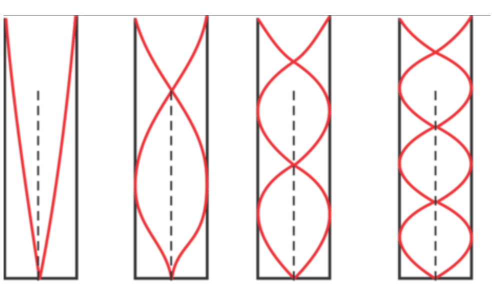
$$I = \frac{P}{4\pi r^2} = \frac{\mu\omega^2\psi^2 v}{8\pi r^2}$$

From this we see that intensity is directly proportional to the square of the amplitude and the square of the frequency. Hence, when resonance occurs, since both the amplitude and the frequency increase, the intensity of the wave increases significantly, making for an even better tool for projection in a concert hall for a musician.

5 Air Columns

In the last two sections, we discussed standing waves in detail. We can use that knowledge to explain the way instruments like a piano, violin, cello, double bass, harp, etc. work. But what about woodwind and brass instruments ? Unlike the previous category, these instruments do not make a sound through the vibration of a string. The way they create their beautiful sound is through air columns.

An air column is essentially a visualization of a tube through which an air wave travels. There are 2 types of air columns: open and closed. Open means that both ends are open while closed means that one end is closed and the other is open. Note that the laws of physics for all of these types of air columns are the same.



Now, as we did we standing waves, we will apply the same analysis to the air columns. First we apply the analysis to a closed end air column with a fixed length. Let us call the length of the tube L. If L is fixed, this means that the wavelength must be changing to accommodate the length. How can we determine a relationship between λ and L ? We know that lambda is supposed to be the distance between two crests or troughs. Hence, we get:

$$L = \frac{\lambda_1}{4}$$

$$L = \frac{3\lambda_2}{4}$$

$$L = \frac{5\lambda_3}{4}$$

And so on. Examining this relationship, we see that the common factor is always $\lambda_n/4$ multiplied by an odd number. Therefore:

$$L = \frac{(2n-1)\lambda_n}{4}, \quad n \in \mathbb{N}$$

$$\implies \lambda_n = \frac{4L}{(2n-1)} = \frac{v}{f_n}$$

$$\therefore f_n = \frac{(2n-1)v}{4L}$$

Now we apply the same analysis but given that λ is fixed and L is changing. Then:

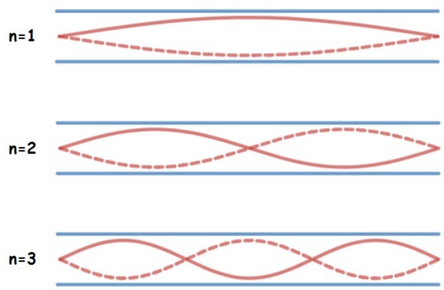
$$L_1 = \frac{\lambda}{4}$$

$$L_2 = \frac{3\lambda}{4}$$

$$L_3 = \frac{5\lambda}{4}$$

$$L_n = \frac{(2n-1)\lambda}{4}$$

Now let us do this analysis on an open ended air column with a fixed f and one time with a fixed L. Once we gather these formulas, we can describe the motion of waves in different air columns either with varying frequency or varying length of tube.



For an open air column with a fixed L:

$$L = \frac{\lambda_1}{2}$$

$$L = \lambda_2$$

$$L = \frac{3\lambda_3}{2}$$

Continuing this pattern, we can see that the length is always a natural multiple of $\lambda/2$. Therefore:

$$L = \frac{n\lambda_n}{2}$$

$$\implies \lambda_n = \frac{2L}{n}$$

Now, since we know that $v = f_n\lambda_n$, we can replace lambda with v/f_n to get:

$$\frac{v}{f_n} = \frac{2L}{n} \implies f_n = \frac{nv}{2L}$$

Which is the same equation we got for standing waves! Now let us examine, for the last time, an open air column, with a fixed f.

$$L_1 = \frac{\lambda}{2}$$

$$L_2 = \lambda$$

$$L_3 = \frac{3\lambda}{2}$$

Continuing the pattern, we get:

$$L_n = \frac{n\lambda}{2}$$

$$\Delta L = L_{n+1} - L_n = \frac{\lambda}{2}$$

6 Advanced: The Wave Equation

When describing a wave mathematically, we seek to describe the amplitude of the wave as a function of position, time or both. The equation relating these is known as the wave equation which states that the "behavior" of a standing wave with some amplitude $\psi(x_0, x_1, x_2, \ldots, t)$(pronounced as "psi of x not, x one, x two, ..., t) and speed c is described using the equation:

$$\nabla^2\psi = \frac{1}{c^2}\frac{\partial^2\psi}{\partial t^2} \ \ or \ \ \sum_n \partial^2_{x_n}\psi = \frac{1}{c^2}\frac{\partial^2\psi}{\partial t^2}$$

For the case of 2 dimensions, 1 spatial dimension and 1 temporal dimension, the wave equation simplifies to:

$$\frac{\partial^2\psi}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2\psi}{\partial t^2}$$

And for n spatial dimensions, where x_k represents the kth spatial dimension, and a temporal dimension, the wave equation is:

$$\sum_{k\to n}\frac{\partial^2\psi}{\partial x_k^2} = \frac{1}{c^2}\frac{\partial^2\psi}{\partial t^2}$$

6.1 The Solution to the Wave Equation: 1 Spatial Dimension

In this section, we will explore how to solve the wave equation, a partial differential equation, in the context of only 1 space dimension, x, and 1 time dimension, t. In order to do so, we employ an extremely powerful technique in solving partial differential equations called "separation of variables". In this technique, we assume that our solution, $\psi(x, t)$ is a product of two single variable functions, 1 of space and 1 of time, such that $\psi(x, t) = f(x) \cdot g(t)$. Substituting this back into our PDE (partial differential equation), we get:

$$f''(x)g(t) = \frac{1}{c^2}g''(t)f(x)$$

At first glance, it may seem like we made the problem worse, but through a simple manipulation, we get the expression:

$$\frac{f''(x)}{f(x)} = \frac{g''(t)}{c^2 g(t)}$$

Now you may ask, how is this better ? The key is that we can call this whole expression equal to some constant, k, which is an eigenvalue and thus $f(x)$ and $g(t)$ are eigenfunctions. Hence, we turn the expression into:

$$\frac{f''(x)}{f(x)} = k, \quad \frac{g''(t)}{c^2 g(t)} = k$$

And now, we can turn our extremely difficult partial differential equation, into 2 ordinary differential equations, which we already know how to solve using various methods!

$$f''(x) - kf(x) = 0, \quad g''(x) - c^2 kg(x) = 0$$

Now we must apply the boundary conditions to determine the solutions. Since this equation represents vibration of standing waves, we know that:

$$\psi(0,t) = f(0)g(t) = 0 \qquad \psi(L,t) = f(L)g(t) = 0$$

We show that k must be negative. For $k = 0$, the general solution of the first ODE is $f(x) = ax + b$ and to satisfy the boundary condition, $a = b = 0$ which is of no interest. For positive $k = \mu^2$, the only possible solution is to set f always equal to zero which is of no interest. Hence we are left with $k = -p^2$. Then, the solution to the ODE is:

$$f(x) = A\cos px + B\sin px$$

Since $f(L) = 0$, it means that for $B \neq 0$:

$$\sin pL = 0 \implies pL = n\pi \implies p = \frac{n\pi}{L}$$

Setting $B = 1$, we thus obtain infinitely many solutions. Hence, $f(x) = f_n(x)$ where:

$$f_n(x) = \sin\frac{n\pi}{L}x, \quad n \in \mathbb{N}$$

Now, let us solve the second ODE with $k = -p^2 = -(n\pi/L)^2$. Hence:

$$g''(t) + \lambda_n^2 g(t) = 0, \qquad \lambda_n = \frac{cn\pi}{L}$$

Hence we obtain that a general solution satisfying the boundary conditions is:

$$g_n(t) = B_n \cos \lambda_n t + B_n^* \sin \lambda_n t$$

Therefore, the solution to our wave equation is:

$$\psi_n(x,t) = (B_n \cos \lambda_n t + B_n^* \sin \lambda_n t) \sin \frac{n\pi}{L} x$$

These functions are called the eigenfunctions, or characteristic functions, and the values $\lambda_n = cn\pi/L$ are the eigenvalues or characteristic values of the vibrating string. The set $\{\lambda_1, \lambda_2, \ldots, \lambda_n\}$ is called the spectrum.

We see that each ψ_n represents a harmonic motion having the frequency $\lambda_n/2\pi = cn/2L$ cycles per unit time. This motion is called the nth normal mode of the string. The first normal mode is known as the fundamental mode where $n = 1$ and the others are known as overtones; musically they give the octave, octave plus fifth, etc.

Tuning is done by changing the tension T on the string. Our formula for the frequency $\lambda_n/2\pi = cn/2L$ of ψ_n with $c = \sqrt{\tau/\rho}$ confirms that effect because it shows that the frequency is proportional to the tension. T cannot be increased indefinitely, but can you see what to do to get a string with a high fundamental mode? (Think of both L and ρ) Why is a violin smaller than a double-bass?

Now, there is one more problem with our solution. The eigenfunctions that we got satisfy the wave equation and the boundary conditions (string fixed at the ends). A single ψ_n, however, will generally not satisfy the initial conditions. But since the wave equation is linear and homogeneous, it follows from Fundamental Theorem that the sum of finitely many solutions is a solution itself. To obtain a solution that also satisfies the initial conditions, we consider the infinite series (with $\lambda_n/2\pi = cn/2L$ as before):

$$\psi(x,t) = \sum_{n=1}^{\infty} (B_n \cos \lambda_n t + B_n^* \sin \lambda_n t) \sin \frac{n\pi}{L} x$$

Satisfying the first initial condition, we get:

$$\psi(x,0) = \sum_{n=1}^{\infty} B_n \sin \frac{n\pi}{L} x = f(x)$$

Hence we must choose the B_n's so that $\psi(x,0)$ becomes the Fourier sine series of $f(x)$. Thus:

$$B_n = \frac{2}{L} \int_0^L f(x) \sin \frac{n\pi}{L} x \; dx$$

Also satisfying the second initial condition, for $t = 0$, we get:

$$\frac{\partial \psi}{\partial t} = \sum_{n=1}^{\infty} B_n^* \lambda_n \sin \frac{n\pi}{L} x = g(x)$$

Similarly, we must choose B_n^*'s so that $\partial\psi/\partial t$ becomes the Fourier sine series. Thus:

$$B_n^* \lambda_n = \frac{2}{L} \int_0^L g(x) \sin \frac{n\pi}{L} x \; dx$$

Setting $\lambda_n = cn\pi/L$, we obtain:

$$B_n^* = \frac{2}{cn\pi} \int_0^L g(x) \sin \frac{n\pi}{L} x \; dx$$

Finally, we have a general solution that satisfies the boundary conditions and the initial conditions of the wave equation provided that the series converges and so do the series obtained by differentiating (12) twice term-wise with respect to x and t and have the sums $\partial^2\psi/\partial x^2$ and $\partial^2\psi/\partial t^2$ respectively, which are continuous.

According to our derivation, the solution $\psi(x,t)$ is at first a purely formal expression, but we shall now establish it. For the sake of simplicity we consider only the case when the initial velocity, $g(x)$ is zero. In that case, the B_n^*'s equal zero and the solution reduces to:

$$\psi(x,t) = \sum_{n=1}^{\infty} B_n \cos \lambda_n t \sin \frac{n\pi}{L} x$$

Now we evaluate the summation formula for this expression:

$$\psi(x,t) = \frac{1}{2}\sum_{n=1}^{\infty} B_n \sin\left\{\frac{n\pi}{L}(x-ct)\right\} + \frac{1}{2}\sum_{n=1}^{\infty} B_n \sin\left\{\frac{n\pi}{L}(x+ct)\right\}$$

These two series are those obtained by substituting $x - ct$ and $x + ct$ respectively, for the variable x in the Fourier sine series for $f(x)$. Thus:

$$\psi(x,t) = \frac{1}{2}[f^*(x-ct) + f^*(x+ct)]$$

where $f^*(x)$ is the odd periodic extension of f with the period 2L (Fig. 289). Since the initial deflection $f(x)$ is continuous on the interval $0 \leq x \leq L$ and zero at the endpoints, it follows from $\psi(x,t)$ that is a continuous function of both variables x and t for all values of the variables. By differentiating $\psi(x,t)$ we see that it is a solution of the wave equation, provided $f(x)$ is twice differentiable on the interval $0 < x < L$, and has one-sided second derivatives at $x = 0$ and $x = L$ which are zero. Under these conditions $\psi(x,t)$ is established as a solution of the wave equation, satisfying the initial conditions with $g(x) = 0$.