Electricity and Magnetism SPH4U1

Barsam Rahimi

May 2024

1 Derivation of Coulomb's Law

Consider a spherical surface with a test charge q at its center. The electrical flux of the surface of this sphere is given by one of Maxwell's equations of electromagnetism known as Gauss' law:

$$\oint \vec{E} \cdot d\vec{A} = \frac{q}{\epsilon_0}$$

Now if we consider some perpendicular electric field line:

$$= \oint E \, dA \cos(0) = \frac{q}{\epsilon_0}$$

If the electric field is varying over time or pint, then we must integrate with respect to that variable using green's theorem. Otherwise, the electric field will just be a constant we can factor out:

$$E \oint dA = \frac{q}{\epsilon_0}$$

And after integrating we get:

$$E \cdot A = E \cdot 4\pi r^2 = \frac{q}{\epsilon_0} \implies E = \frac{q}{4\pi r^2 \epsilon_0}$$

We now use k to denote the constant:

$$k = \frac{1}{4\pi\epsilon_0}$$

Note that ϵ_0 is the permittivity of free space which is also a constant. This gives the form:

$$\boxed{E = k\frac{q}{r^2}} \tag{1}$$

We must now recall the definition of the electric field. The electric field is defined as *the force per unit charge in an electric field* or in mathematical notation:

$$E = \frac{F}{Q} \implies F = EQ$$

Substituting this into the electric field equation gives us Coulomb's law which quantifies the electric force between two charges q and Q for a constant electric field where r is the distance between the two charges:

$$\boxed{F = k\frac{qQ}{r^2}} \tag{2}$$

2 Introduction to Field Theory

Imagine an empty space with a single charge in it. What does the charge do to announce its presence in the space ? It sends out signals to its surroundings which is what we call a *field*.
To test the strength of a field, we introduce what is called a test charge. A test charge is assumed to be positive and arbitrarily small so that it does not have its own electric field. Remember that if there is a charge, then there is a field and vice versa.
We send the test charge into the field which interacts with it and causes a force. That is how we measure the strength of the field;

$$\boxed{\vec{F} = q\vec{E}} \tag{3}$$

Or in a calculus definition, you can say the field is a derivative of the force:

$$\boxed{\lim_{q \to 0^+} \vec{E} = \frac{\vec{F}}{q} \implies \vec{E} = \frac{\partial \vec{F}}{\partial q}} \tag{4}$$

Which after differentiating you get the same definition:

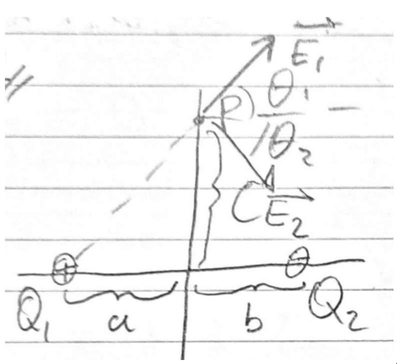
$$\boxed{\vec{E} = k\frac{Q}{r^2}} \tag{5}$$

If there are multiple electric fields present, then:

$$\boxed{\sum \vec{E} = \vec{E}_1 + \vec{E}_2 + \cdots = \sum_{i=1}^{n} \vec{E}_i} \tag{6}$$

2 Electric field lines can never cross each as the net force between them can only point in 1 direction!

3 Sample Problem 1



Consider the situation described by the diagram on the left. Given Q_1, Q_2, a, b, and c, determine the magnitude of the electric field at point P.

To do so we first write out the angles in terms of sides in order to reduce number of unknowns:

$$\theta_1 = \arctan\left(\frac{c}{a}\right) \quad \theta_2 = \arctan\left(\frac{c}{b}\right)$$

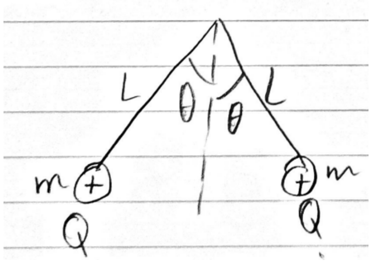
Then the electric fields due to Q_1 and Q_2 are respectively given by

$$|\vec{E}_1| = k\frac{Q_1}{a^2 + c^2} \quad |\vec{E}_2| = k\frac{Q_2}{b^2 + c^2}$$

$$\therefore |\vec{E}| = \sqrt{(|\vec{E}_1|\cos(\theta_1) + |\vec{E}_1|\cos(360 - \theta_1)^2 + (|\vec{E}_2|\sin(\theta_2) + |\vec{E}_2|\sin(360 - \theta_2))^2}$$

Let us try another one.

4 Sample Problem 2



Consider the situation described by the diagram on the left. Given the values for θ, L, and m, determine Q.

If we assume that one charge is a field charge and the other one is a source charge and draw their FBDs, we get:

$$|\vec{F}| = T\sin\theta \qquad T\cos\theta = mg$$

$$\implies |\vec{F}| = mg\tan\theta = k\frac{Q^2}{4L^2\sin^2\theta}$$

$$\therefore Q = \sqrt{\frac{4L^2\sin^2\theta mg\tan\theta}{k}} = 2L\sin\theta\sqrt{\frac{mg\tan\theta}{k}}$$

5 Electrical Energy and Its Application

Let us recall the definition of the work done by potential energy:

$$PE = \vec{F}\cdot\vec{\Delta d}$$

Now, what happens when we consider this case for an electric force:

$$\implies PE = |\vec{F}_{\text{elec}}|\cdot|\vec{\Delta d}|$$

What happens when we divide both sides by some charge quantity q:

$$\frac{PE}{q} = \frac{|\vec{F}_{\text{elec}}|}{q}|\vec{\Delta d}|$$

What is it that we see from this formula ? The left hand side is by definition *the work done per unit charge*. This is also known as *Voltage* represented by the letter V. This gives us:

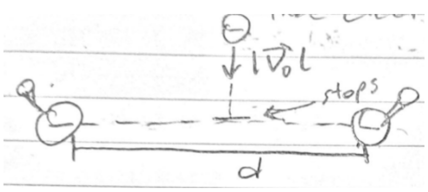
$$V = \frac{|\vec{F}_{\text{elec}}|}{q}|\vec{\Delta d}|$$

Now, look at the first term we have on the right hand side. This is the same as a rearrangment of equation (3)! Hence we can replace this term with $|\vec{E}|$. This gives us the formula for voltage across some electric field:

$$\boxed{V = |\vec{E}|\,|\vec{\Delta d}| = \frac{PE_{\text{elec}}}{q}} \tag{7}$$

And from this we get an expression that we can use in the conservation of energy. We know that the potential energy is given by qV from expression (7) and that is what we are going to use in order to solve for quantities using the law of conservation of energy. There are also other forms of this expression such as kqQ/r which is fairly easy to derive, just replace F with coulomb's law and $|\Delta d|$ with r. Let us try a problem with this formula to see it in action before moving on to magneto-statics.

6 Sample Problem 3



Consider the situation described by the diagram on the left. The particles all have a charge equal to the charge of an electron which is denoted by e. The particle will move down in a straight line but once it reaches the point in between the two other charges, it stops moving. Given the distance d, mass of the electron m_e, the fact that the two charges on the left and right are pinned down, and that the electron initially is coming from an infinitely far away distance, determine the initial velocity of the free particle, $|\vec{v_0}|$.

To solve this problem we use the law of conservation of energy:

$$\frac{1}{2}m|\vec{v_0}|^2 + qV_0 = \frac{1}{2}m|\vec{v_1}|^2 + qV_1$$

Since the particle stops moving, its final velocity $|\vec{v_1}| = 0$. The final voltage of the particle is given by

$$V_1 = \sum k\frac{qQ}{r} = \frac{-2ke}{d} + \frac{-2ke}{d} = \frac{-4ke}{d}$$

Note that the reason why there is a negative sign is because the charge of an electron is negative not positive. Hence, e by itself is not sufficient. A minus sign fixes the sign convention. The reason why we multiplied by 2 is because there are 2 charges, each with their own electric field which we know from equation (6) implies that we must add them together. Now what about the initial voltage ? The problem tells us that the electron is coming from an infinitely far distance. Hence, the voltage initially is given by:

$$V_0 = \frac{-ke}{\infty} + \frac{-ke}{\infty} = 0$$

Hence, the second term on the left hand side goes to 0. This leaves us with:

$$\frac{1}{2}m_e|\vec{v}_0|^2 = -e \cdot \frac{-4ke}{d}$$

$$\implies |\vec{v}_0| = \sqrt{\frac{8ke^2}{m_e d}}$$

and this is our final answer.

7 Magneto-statics: Derivation of Lorentz Force

To derive the first equation for a magnetic force on a charged particle, we once again find our way to Maxwell's equations; though this time, the laws we will be using are Faraday's law of induction and Gauss' law for magnetism. Faraday's law states that:

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \implies \oint_{\partial\Sigma} d\vec{\ell} \cdot \vec{E} = -\int_{\Sigma} d\vec{A} \cdot \frac{\partial \vec{B}}{\partial t}$$

Where $\vec{B}$ is the magnetic field and Σ is some simple closed path. Now, we use expression (3) and the fundamental theorem of line integrals to re-write Faraday's law as

$$\oint_{\partial\Sigma} d\vec{\ell} \cdot \frac{\vec{F}}{q} = -\frac{\partial}{\partial t} \int_{\Sigma} d\vec{A} \cdot \vec{B}$$

Now let us assume that the wire in this context is stationary. In order to further proceed, we use the Leibniz Integration theorem and Guass' law for magnetism:

$$\nabla \cdot \vec{B} = 0$$

$$\implies \oint_{\partial\Sigma} d\vec{\ell} \cdot \frac{\vec{F}}{q} = -\int_{\Sigma} d\vec{A} \cdot \frac{\partial \vec{B}}{\partial t} + \oint_{\partial\Sigma} \vec{v} \times \vec{B}\, d\vec{\ell}$$

And now, we use the differential form of Faraday's law and the contour integral theorem to replace the second integral

$$\oint_{\partial\Sigma} d\vec{\ell} \cdot \frac{\vec{F}}{q} = \oint_{\partial\Sigma} d\vec{\ell} \cdot \vec{E} + \oint_{\partial\Sigma} \vec{v} \times \vec{B}\, d\vec{\ell}$$

And this simply gives us the following equation known as the *Lorentz force* equation:

$$\boxed{\vec{F} = q\vec{E} + q\vec{v} \times \vec{B}} \tag{8}$$

Where $\times$ denotes the cross product. The first part of this equation we are familiar with, it is the electric force. What about the second part ? The second part represents the magnetic force on the particle. Hence the equation you will use in your high school years:

$$\boxed{|\vec{F}_{\text{magnetic}}| = q|\vec{v}||\vec{B}| \sin\theta} \tag{9}$$

Where θ is the angle between the velocity of the particle and the magnetic field lines. Now let us take this to a circuit scenario. Maxwell's equations tell us that electrical and magnetic fields are connected. So if there is an electric circuit, there is also a magnetic field. How do we take this into consideration for circuits ? To do so, we consider a simple fact, that velocity can be written as

$$\vec{v} = \frac{\vec{\ell}}{t}$$

and the definition of electrical current

$$i = \frac{q}{t}$$

And now we substitute the first definition to get:

$$\vec{F} = q\frac{\vec{\ell}}{t} \times \vec{B}$$

After substituting the definition of current, we get a form appropriate for circuits:

$$\boxed{\vec{F} = i\vec{\ell} \times \vec{B}} \tag{10}$$

or as a scalar equation

$$\boxed{|\vec{F}_{\text{magnetic}}| = i|\vec{\ell}||\vec{B}|\sin\theta} \tag{11}$$

You will hardly ever use Equation (8) in high school. Equations (9) and (11) are the ones you really need to know and will use in your test questions. Before we do some examples, we need to consider a special case. Suppose the charge is travelling in a circular path. What are the implications of this ? We remember from circular motion that the centripetal force is perpendicular to the velocity! This makes $\theta = 1$ in equation (9). We also know that the net force is also equal to $\vec{F} = m\vec{v}^2/r$. So we can set these equal to each other and get a new equation for magneto-static problems in the case of circular motion:

$$q|\vec{v}||\vec{B}| = \frac{m|\vec{v}|^2}{r}$$

$$\boxed{\implies \frac{q}{m} = \frac{|\vec{v}|}{r|\vec{B}|}} \tag{12}$$

The term on the left is often referred to as the *charge to mass ratio* of the field charge. Now, we are ready to try a few examples.