A Brief Introduction to General Relativity

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April 2024

1 What To Take For Your Own Scientific Paper

The most important part of the paper is your model, your equation, the mathematical expression that supports your finding. In physics, all claims must be supported by proper mathematical derivations and the interpretations of the theories arise from analyzing those mathematical models. Unless your work is extremely theoretical and cannot be tested through any type of experimentation (which is a very rare case), you need some sort of experimental data to support your theory. If the data that you find from an experiment approximately matches up with your model, then your theory is both mathematically and experimentally consistent. However, in the case where it does not, there are 3 possibilities; an error in your experimentation such as not taking into account all there is to be thought of, improper lab procedure, incorrect data processing , etc. Another possibility is having an inconsistent mathematical model. If you are sure that your experimentation has been performed correctly and you have considered everything, then your model is probably inconsistent with the previously discovered laws and must be adjusted or re-derived. The third case is when you have both experimental errors and an inconsistent mathematical model. Hopefully this will not happen! Another thing to keep in mind when writing a physics paper is knowing when to use what. If you are talking about particles at a subatomic level such as electrons or particle accelerators, you need to use quantum mechanics and classical mechanics will not be consistent with the experimental findings! In the case of using astronomical data and massive objects such as stars, planets and black holes, you need to use the theory of general relativity and again, cannot use classical mechanics. This is due to the fact that these minor adjustments make all the difference in these two fields. A very small decimal place could make the difference between a particle existing or not existing in the quantum world. Good luck!

2 Classical Mechanics

To understand relativity we must first start with the idea of classical mechanics. This is the elementary mechanics developed by many physicists but initially started by Sir Isaac Newton and his 3 famous laws of motion. Classical mechanics is very useful in describing most everyday objects that most of us encounter on a daily basis. To know how much fuel is needed to fly a rocket, predict the motion of a car based on the force on it, calculate the velocity of wind fields, etc. we can use classical mechanics. The 4 main equations in classical mechanics governing kinematics, the study and analyzing of motion, are:

$$\Delta s = v_0 \Delta t + \frac{1}{2} a \Delta t^2$$

$$\Delta s = \frac{1}{2} \Delta t \left(v_0 + v_1 \right)$$

$$v_1 = v_0 + a \Delta t$$

$$(v_1)^2 = (v_0)^2 + 2a\Delta s$$

Where Δs is a displacement, v_1 and v_0 are final and initial velocities respectively, a is the acceleration of the object and t is a measure of the time interval.

In the end of the day, all of these equations originate from Newton's 2nd law, stating that the net force on an object is proportional by its mass to the object's acceleration! Or in other words, the famous equation:

$$\sum \vec{F} = m\vec{a}$$

Here, the Greek letter $\sum$ means the sum of all the forces, or the net force. The reason why $\vec{F}$ and $\vec{a}$ have little arrows on top of them is because they are vectors. This means that they have not just a numerical value associated with them, but also a direction in space. There are 2 things that are great about this equation of Newton: it's precise meaning that it gives you an accurate prediction of a net force and allows you to make predictions based on this equation, and it is everywhere! This equation does not just say "that one time someone pushed their car with this force, it was proportional to its acceleration". It tells us that every time in the history of the universe, whenever anyone applied a force to any object, its value was proportional to the object's acceleration. These are what separates a good equation or model from a less sophisticated one: room for predictions and everywhereness!

Another important law in all of classical mechanics is conservation of mechanical energy. This law states that energy in any closed system free of dissipative forces such as friction, air resistance, drag, etc. is constant. In other words, if we add all the energies in the system such as its potential energy, kinetic energy, electrical potential energy, heat energy, etc. in the initial state of the closed system, it will be equal to the sum of all the energies in the final state of the system:

$$KE_1 + PE_1 + EE_1 + HE_1 + \dots = KE_2 + PE_2 + EE_2 + HE_2 + \dots$$

Here is the same equation but this time, the mathematical formulas for calculating the energies are included if you are interested in that:

$$\frac{1}{2}m|\vec{v_0}|^2 + mgh_0 + qV_0 + mc\Delta T + \dots = \frac{1}{2}m|\vec{v_1}|^2 + mgh_1 + qV_1 + mc\Delta T + \dots$$

This law will come in handy in pretty much all of your high school and first year physics problems.

Aside from Conservation of Mechanical Energy, everything we discussed here is about Newtonian mechanics. There is another branch of classical mechanics called Hamiltonian mechanics which is mathematically more sophisticated so we will leave it for now. However, it does come in handy in general relativity so if you know calculus, do not hesitate to learn about it. It is just a different perspective on classical mechanics. Hamiltonian mechanics uses the Hamiltonian and the Euler-Lagrange Equation(s) of motion. In the end, all these different views are connected by the law of conservation of mechanical energy.

3 Gravity in Newtonian Mechanics

The theory of General Relativity, is a theory on the geometry of space-time. But before we go there, we must first understand Newtonian Gravity. The question is how did Newton define gravity and why did Einstein disagree with his formulation of gravity ?

In classical mechanics, gravity is regarded as a force of nature. It is the pull that stops us from jumping more than a certain distance, keeping our cars on earth and making apples fall from trees. In Newtonian mechanics, the gravitational force between two bodies with masses M and m whose distance is r is formulated such that:

$$\vec{F}_{\text{gravity}} = \frac{GMm}{r^2}\vec{e}$$

Where G is known as Newton's gravitational constant, as measured by Henry Cavendish in the famous Cavendish experiment, M and m are the masses of the two objects, r is the distance between them and $\vec{e}$ is a unit directionality vector.

The story of Newton and the apple falling from the tree was probably not true and made up by Newton himself. Even before this formulation we knew that gravity existed! What is so special and important about Newton's discovery is that realization about the *everywhereness* of gravity. What made him so famous was the realization that the same gravitational force causing the apple to fall from the tree was also responsible for keeping the planets in their orbits and effecting the path that stars take in galaxies and so much more.

Another thing that is truly amazing about this equation is that it allows one to instantaneously make a prediction and learn a very counterintuitive fact about the universe; that all objects fall at the same rate. Here is how we do it: Since the force on any object is directly proportional to its acceleration, we can set the two force equations equal to each other:

$$\frac{GMm}{r^2} = ma$$

And now with a little simplification which is through cancelling the masses, we learn that the acceleration of an object does not depend on its mass:

$$a = \frac{GM}{r^2}$$

What this tells us, which was proven in the famous Apollo experiment, is that the only reason a hammer and a feather fall at different rates on earth is due to dissipative factors such as air resistance and turbulence. For centuries the laws of classical mechanics were shown to be incredibly successful and scientists thought that now it was just a matter of figuring out all the different forces and fitting them within Newton's equations. Then comes to the stage Albert Einstein who changes lots of known principles in physics.

4 Special Relativity

The adjective special means that the theory deals only with inertial reference frames, which are frames in which Newton's laws are valid. (Einstein's general theory of relativity treats the more challenging situation in which reference frames can undergo gravitational acceleration; in this chapter the term relativity implies only inertial reference frames.) Starting with two deceivingly simple postulates, Einstein stunned the scientific world by showing that the old ideas about relativity were wrong, even though everyone was so accustomed to them that they seemed to be unquestionable common sense. This supposed common sense, however, was derived only from experience with things that move rather slowly. Einstein's relativity, which turns out to be correct for all physically possible speeds, predicted many effects that were, at first study, bizarre because no one had ever experienced them. In particular, Einstein demonstrated that space and time are entangled; that is, the time between two events depends on how far apart they occur, and vice versa. Also, the entanglement is different for observers who move relative to each other. One result is that time does not pass at a fixed rate, as if it were ticked off with mechanical regularity on some master grandfather clock that controls the universe. Rather, that rate is adjustable: Relative motion can change the rate at which time passes. Prior to 1905, no one but a few daydreamers would have thought that. Now, engineers and scientists take it for granted because their experience with special relativity has reshaped their common sense.

4.1 Postulates of Relativity

In order to start a theory of relativity, we first need 2 postulates that always hold true. We will use these postulates throughout our proofs and derivations. At first, they may seem irrelevant or simple but we will see that they are in fact quite deep assumptions.
The first postulate of Einstein's relativity is very simple:

The laws of physics have the same form, meaning they hold true, in all inertial reference frames.

Here, an inertial reference frame refers to the reference frame of an observer who is not accelerating, meaning the net force on them is equal to 0. We will come back to this definition as it is not compatible with general relativity but for now we can accept it. This one is not very difficult to understand. It is almost assumed by most people who start with relativity for the first time. The second postulate is what makes Einstein's relativity different from Galileo's:

The speed of light, c, is the same and is independent of the relative motion of the observer.

The postulate, if you have studied Electromagnetic Theory you know, is derived from James Clerk Maxwell's famous equations. What this is telling us is that no matter how fast an observer is moving, the speed of light from everyone's point of view is always c. Even if an observer is moving at a speed of $0.9c$, they will still experience the speed of light to be c, it will not slow down or speed up. It is very important to understand this as the rest of our derivations will depend on this fact.

4.2 Simultaneity, Time Dilation and Spacetime

Imagine yourself standing beside a railroad where a train that is going at an extremely fast speed passes by. Now imagine that two lightning strikes the ground when this train is passing by with a distance of 100m. If you are observing the train from outside, then you see the lightning hit the ground at the same time. They are simultaneous. Now, imagine yourself inside the train. Now, what do you experience if two lightning strikes simultaneously for an observer on the ground outside the train ? The answer is that you will see one of the lights earlier than the other one. You are moving towards one lightning and away from the other one. Since the speed of light is the same, then the distance one light ray travels is going to be greater than the distance traveled by the other one as the train moves towards it.

Hence, an *event* or experience that may be simultaneous for one observer, or reference frame, may not be simultaneous for another.

This implies that the time experienced by the two observers is not the same! This means that the idea of two clocks ticking at the same rate all the time is not true. In fact, relativity shows us that if a clock is moving, no matter what type of clock it is, it will tick slightly slower than a clock of a ground observer. This difference is given by the formula:

$$t = \frac{\tau}{\sqrt{1 - \frac{v^2}{c^2}}} = \gamma\tau$$

Where γ (gamma) is:

$$\gamma = \frac{1}{\sqrt{1 - v^2/c^2}}$$

Or if we make a substitution of $\beta = v^2/c^2$ we get:

$$\gamma = \frac{1}{\sqrt{1-\beta^2}}$$

The time at the right hand side of the equation is called the "proper time" and t is the coordinate time. t is the time that the observer in the train experiences and τ is the proper time which is what the ground observer sees.

We can also see that when the speed of the observer is above the speed of light, we have a square root of a negative which is imaginary and that is bad. This explains why objects cannot move faster than the speed of light.

This also allows us to derive the length contraction formula:

$$L = \frac{L_0}{\gamma} = L_0\sqrt{1-\beta^2}$$

This connection between length contraction and time dilation shows us that there is a connection between spatial coordinates and time coordinates. As Hermann Minkowski, Einstein's math teacher, put it: "Henceforth, space by itself, and time by itself, are doomed to fade away into mere shadows, and only a kind of union of the two will preserve an independent reality".
The reason this theory is called "special" relativity is because it only deals with inertial reference frames that are moving at constant velocities. The theory of General Relativity is mostly focused on the geometry of space-time and all types of reference frames including accelerated ones. The mathematics of Special Relativity are quite complicated, but it is nowhere near the difficulty of General Relativity. Hence, let us approach the topic cautiously.

5 Spacetime: Minkowskian Geometry

To understand Minkowskian Geometry, let us first recall the most basic theorem in Euclidean geometry, the geometry where your surface is flat. That is, Pythagoras' theorem: A distance square is equivalent to the square of the sum of its components.

$$S^2 = x^2 + y^2$$

In this geometry, what we are doing is calculating the distance between two things. In Minkowskian geometry, however, we are calculating the *elapsed time* between 2 *events*. Then, Pythagoras' theorem takes the following form:

$$\tau^2 = t^2 - x^2$$

Here, the distance between the two events is replaced with the proper time and the spatial dimension has a minus sign beside it. This is Minkowskian geometry; where the coordinates we are working with are the proper time, the coordinate time and our spatial coordinates.
You can see from this that what we are really interested in is describing spacetime. However, this came with a big problem, the concept of gravity did not fit into the special theory of relativity. As it turns out, the key was to understand space time. To do so we have to go back a long time.

5.1 Euclid's Parallel Postulate

Many years ago, Euclid was trying to write a book called "Euclid's Elements" to establish all known mathematics of his time. To do so, he established 5 axioms, well-known facts that he would base his theories off of. The 5th axiom, also called the parallel postulate, is the most famous of them.
This postulate states that one two lines are intersected by another line, if the interior angles made by these intersections are 90 degrees, then the lines will stay parallel to each other. While this may sound logical to high school students, mathematicians were highly skeptical of this and tried to prove its falseness for centuries just to realize that the point was never to prove whether this is true or false.

It turns out that this postulate opens up doors to new geometries. In Euclidean geometry, the geometry where our surface is intrinsically flat, which is what we learn in high school and elementary school, the parallel postulate holds true. However, if geometries were given a curved surface, the 5th axiom will not hold true and we have to change it.

If the surface we are working with is shaped like a saddle, then we call it Hyperbolic geometry, or negatively curved geometry. You may also have a surface that is shaped like a sphere which gives us Spherical geometry or positively curved geometry. In these geometries, our 5th axiom will change.

In Hyperbolic geometry, since the surface is negatively curved, the two lines will diverge (move away from each other) and in Spherical geometry they converge (get closer and closer to each other until they intersect). Then, Carl Friedrich Gauss comes along and realizes you may have a surface that is, in some places, negatively curved and, positively curved in others and flat in some regions as well. Suddenly, mathematicians realized that we can have an infinite number of geometries each with their unique properties where in some places, Euclid's parallel lines converge and in others they diverge.

5.2 Pythagoras' Theorem In Different Geometries

In Euclidean geometry, we know that Pythagoras' theorem says that the square of the x, y and z components gives us the distance between two points. It has the form:

$$S^2 = x^2 + y^2 + z^2$$

Note that both x, y and z have coefficients of $+1$. This will come in handy later. Before we take this further, we know that this formula is also true for a change in position (which is the same thing as displacement or distance). Hence, we will use ΔS to denote a change in distance. Then, our theorem takes the form:

$$\Delta S^2 = \Delta x^2 + \Delta y^2 + \Delta z^2$$

In spherical geometry, a system were we measure a point based on its distance from the origin, r, and the angle this line's x-component makes with the x axis, θ, and the angle its z-component makes with the z axis, ϕ, this theorem takes the form:

$$\Delta S^2 = \Delta r^2 + r^2 \Delta\theta^2 + r^2 \sin^2\theta \Delta\phi^2$$

Do not try to prove this as to derive this you would need to calculate the metric of this space. We will learn what that means in a second. In calculus, you learn that if we are observing an infinitesimally small change, we replace Δ with a d which turns the formulas into:

$$dS^2 = dx^2 + dy^2 + dz^2$$

$$dS^2 = dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta \, d\phi^2$$

Now let us look at where we got these from.

5.3 Generalization of Pythagoras' Theorem: The Metric Tensor

Usually in math and physics, when wanting to generalize a formula, what we do is look at a series of cases and examine the pattern. Then we come up with a way to mathematically describe these patterns.
What do we see here ? This is the question that Bernard Riemann, Gauss' student and one of the greatest mathematicians of all time also known for the famous Riemann Hypothesis, also asked himself and he came up with a brilliant way to describe geometries.

Riemann realized that if we know the length of every single infinitesimal curve in the surface, then that tells us what type of surface we are working with, including its curvature. However, since curved lines and coordinates are hard to work with, we zoom in on them. In calculus, we learn that when we zoom in on curvy lines enough, they are a collection of infinitely small straight lines. That is why we turned ΔS into dS. Now, all we need to find the geometry is the version of Pythagoras' theorem in that specific geometry.

To do that, we examine the pattern between the two distance formulas we introduced in the previous section. We see that on the left hand side, we have some small *interval* squared and it is always equal to the sum of the change in each coordinate squared times some coefficient. We also introduce a theory where we say that really, what is going on is we are multiplying 2 of our dimensions such as xy or z^2 which is the same thing as zz and then multiplying that coefficient. It just so happens that in Euclidean geometry, the coefficients are zero when two non-identical dimensions are multiplied to each other. So we can rewrite Pythagoras' theorem like this:

$$dS^2 = Adx\,dx + Bdx\,dy + Cdx\,dz + Ddy\,dx + Edy\,dy + Fdy\,dz + Gdz\,dx + Hdz\,dy + Idz\,dz$$

$$= Adx^2 + Bdx\,dy + Cdx\,dz + Ddy\,dx + Edy^2 + Fdy\,dz + Gdz\,dx + Hdz\,dy + Idz^2$$

Setting $B, C, D, F, G,$ and $H = 0$, and then setting $A, E,$ and $I = 1$ gives us Pythagoras' theorem. We can see a similar thing happening in the spherical geometry case.

What Riemann proposed was to put all these coefficients in a collection, called a matrix, and we call this matrix g_{mn}. This matrix is often referred to as the *metric tensor.*

$$g_{mn} = \begin{pmatrix} g_{00} & g_{01} & g_{02} \\ g_{10} & g_{11} & g_{12} \\ g_{20} & g_{21} & g_{22} \end{pmatrix} = \begin{pmatrix} A & B & C \\ D & E & F \\ G & H & I \end{pmatrix}$$

This is solely to make our notation more compact and efficient. To make things even more compact, we are going to reintroduce some notation. Instead of writing all our coordinates over and over again, we will introduce a notation such that $\{x, y, z\}$ is now all represented by $\{X^1, X^2, X^3\} = X^m$. So instead of writing:

$$dS^2 = dx^2 + dy^2 + dz^2$$

We will simply write:

$$dS^2 = \sum_{m,n} g_{mn}\, dX^m\, dX^n$$

The sigma, $\sum_{m,n}$ means that we sum over all the possible indices of m and n. When Einstein was learning this geometry, he was very tired of writing all these summation signs everywhere. So he invented something called the "Einstein Summation Convention" where every time you have an index repeating downstairs and upstairs, it is assumed that you sum over it and there is no need to write the summation sign for it. So the expression becomes even more compact! It ultimately takes the form:

$$dS^2 = g_{mn}\, dX^m\, dX^n$$

You can think of the metric as a factor that helps scale your theorem with respect to your coordinate system of choice. Now, how are we going to use this to find an equation in Minkowskian Geometry?

5.4 Characterizing Curvature: The Riemann Tensor

So far, we have solved the issue of changing coordinate systems by introducing the metric, $g_{\mu\nu}$. However, we have not yet discussed the curvature of the surface itself. To discuss the curvature, we use a tool called the Riemann Curvature tensor denoted by $\mathcal{R}^{\lambda}_{\rho\mu\nu}$ and as you may have noticed, it has 4 indices. What this means is that it is a 4×4 matrix, where each term corresponds to a 4×4 matrix itself! So it is technically a 16×16 matrix, but not really.

The Riemann curvature tensor allows us to figure out the curvature of the geometry of the surface we are working with. So rather than just taking into account the curvature of our coordinate systems and measuring their variations, we can use the Riemann tensor to calculate the measure of the curvature at each point on our surface. This opens the door to having infinitely many geometries each with their own individual curvatures. We will come back to $\mathcal{R}^{\lambda}_{\rho\mu\nu}$ and expand on it to come up with the famous Einstein Field Equations.

5.5 Pythagoras' Theorem in Minkowskian Geometry

Now that we have all the tools we need, it is time to finally properly define spacetime. In spacetime, there are 4 dimensions: 3 spatial dimensions, x, y, z and 1 time dimension t. This system is represented by:

$$\{t, x, y, z\} = \{X^0, X^1, X^2, X^3\} = X^\mu \; or \; X^\nu$$

The indices μ (pronounced "mu") and ν (pronounced "nu") are dummy indices. This means it does not matter what letter you use. There is however 1 convention. You may have noticed that this time, we started numbering the coordinates with 0 and worked our way up. This is because physicists like to use multiple dimensions of space. There may be 1, 4 or even 9 (if you are a string theorist) spatial dimensions. However, the time dimension never changes. There is always only 1 time dimension. Hence, we give it the index 0, to only start the real numbering with spatial dimensions. This brings up another small convention in General Relativity; when we use a Latin or English letter as an index, it means we start the index from 1, however, if we use Greek letters like μ and ν, it means we are starting from 0. Let us recall, Riemann's generalization of Pythagoras' theorem in m or n dimensions was formulated as:

$$dS^2 = g_{mn} \, dX^m \, dX^n$$

This is, of course, using the summation convention. Now, we can write this with Greek indices and we have ourselves an equation in Minkowskian Geometry, or, in spacetime. However, this time, as we recall, in Minkowskian Geometry, by the length, we mean the proper time τ. Hence, the equation takes the form:

$$d\tau^2 = g_{\mu\nu}\, dX^\mu\, dX^\nu$$

This equation is perhaps the most important equation in General Relativity. It allows you to analyze and predict behavior of objects near black holes, massive bodies in spacetime, trajectories and even geodesics which you will find an analysis of in the advanced version of this handout. Now, all that there is left is modeling gravity. How does gravity help us put all these pieces together to come up with a general theory ?

6 Modeling Gravity in General Relativity

The starting idea of General Relativity was to figure out the true nature of gravity. Einstein learned this when he was on an elevator in a conference in Prussia to give a lecture on Special Relativity. The answer to gravity lies in the equivalence principle, also described by Einstein as the happiest thought of his life!

6.1 The Equivalence Principle

The equivalence principle is perhaps the most important concept that one must understand in the theory of Relativity. Let us dive into it!

When Einstein came up with this principle in 1907, most people found it difficult to understand how this can be helpful. His happiest thought was imagining a man falling off a rooftop! The reason this thought made him so happy was that he realized there is absolutely no difference between gravity and accelerating. When the man is falling off the roof, there is no experiment he can do to determine whether he is falling off at constant acceleration or if he is in deep space, traveling upwards at the same constant acceleration. In other words, gravity and acceleration are the same! This is the fundamental idea around this entire theory.

6.2 Trajectories and Geodesics in Spacetime

Another important thing to know in order to understand General Relativity is how objects follow a trajectory in spacetime. The rule is all objects follow a straight trajectory in spacetime. They always travel on a straight line. Whether it is a soccer ball, an astronaut or a beam of light, they all follow the same trajectories, just at different velocities.

What matters is when we have mass in our spacetime. When there is mass, it causes the fabric of spacetime to bend towards that mass so that it pulls in all the straight lines towards itself. Hence, the existence of mass causes the *curvature* of space time. "But how does that help with defining gravity ?" you may ask. Well, the answer is in a beautiful thought experiment.

Let us recall that in Newtonian mechanics, gravity is known as a force of attraction that pulls two objects towards each other. Now, imagine two people, Alice and Bob, on earth. Alice is from India and Bob is from Texas. Imagine that both of these people now start walking towards the north pole on a straight line on the surface of earth. Because earth is a sphere, as these people walk towards the north pole, they also start to get closer to each other and will eventually meet at the north pole. It *feels as if there was a force that pulled them to each other*. However, that is not the reality of it! The reality is that there is no force between the Alice and Bob, they simply followed a straight line path and it just so happened that these paths got closer to each other to create the *effect of what seems like a force but is really is just a consequence of the geometry of the surface!* It was not a force of attraction that pulled the two people towards each other, it was just due to the fact that the geometry was curved which caused straight line paths between the two people's position to get closer to each other and look curved. This is the same case in general relativity; when there is a mass in spacetime, it bends the geometry and creates curvature. This causes the shortest path, a straight line, to seem as if it is bent. But really, it IS the shortest path, just on a curved geometry. If you look at the paths that airplanes follow, on a sphere it seems like a straight line but when you flatten out the surface, it looks curved. They are following a *geodesic* on earth. The shortest path from one point to another in a specific geometry is called a *Geodesic*.

This brings up the concept of gravity!

6.3 What Is Gravity ?

The concept of gravity in General Relativity is no longer looked at as a ”force of attraction between two or more bodies”. When we go back to the north pole scenario, we see that the only reason it seems like there is a force of attraction between the two people is because the geometry is curved. This is the same case with gravity. The only reason a man falls from a roof top to the ground is because he is following a straight line path through spacetime which is curved due to the presence of the mass of earth, bending the geometry as a consequence.

Hence, we come up with a better definition of gravity; it is not a force of attraction, but an effect of *the curvature of space time due to the presence of mass.*

Let us review, the presence of mass causes spacetime to bend. Since objects always follow geodesics in spacetime, they will keep traveling in straight lines. When spacetime is bent, those geodesics look like curved lines that are bent towards the mass. This causes objects near a massive body to fall towards it which creates an effect called gravity. If there is no mass in the spacetime, then the spacetime is flat and objects will still follow straight line paths.

Now that we have redefined gravity in a way that it is consistent with Special Relativity, we must come up with an equation! How do we model gravity ?

6.4 Einstein’s Field Equations

Since we want to define gravity, we are looking for some equation that replaces:

$$a = \frac{GM}{r^2}$$

Obviously looking at this equation, on the right hand side we need something that can tell us about the mass present in spacetime. On the left hand, since we have defined acceleration as objects following geodesics in spacetime near a curved section, we must replace acceleration with a measure of the curvature of spacetime at that point.

Recall that we can find the curvature of the geometry we are in using the Riemann tensor, $\mathcal{R}^{\lambda}_{\rho\mu\nu}$ but what do we do about the mass ?

To do this, Einstein came up with the *energy-momentum tensor*. We know from the famous equation:

$$E = \gamma mc^2$$

or its extended form involving momentum, $\vec{p}$:

$$E = \sqrt{(mc^2)^2 + (\vec{p}\,c)^2}$$

that energy and momentum are related to mass. Hence, if we can relate the presence of energy or momentum to the curvature of spacetime, we have our equation. To do so, we use the energy momentum tensor, $T_{\mu\nu}$ which tells us about things like stress, heat flow, pressure, etc. in spacetime. Now, we make our first guess! If curvature is characterized by the Riemann tensor and energy-momentum by $T_{\mu\nu}$ then the obvious guess would be to set those equal to each other:

$$\mathcal{R}^{\lambda}_{\rho\mu\nu} = T_{\mu\nu}$$

And unfortunately this does not work! This is because the Riemann tensor is a 4 by 4 matrix of 4 by 4 matrices whereas the energy-momentum tensor is a 4 by 4 matrix of numbers. So it does not work. To deal with this, some industrious Italians such as Professor Ricci used a technique called *tensor contraction* which can be used to decrease the number of indices of your tensor. I will not go over how it works, but it works! Hence Professor Ricci could distill the Riemann tensor to something called the Ricci tensor, $\mathcal{R}_{\mu\nu}$ and he could distill that down to something called the curvature scalar, $\mathcal{R}$. Given these tools, Einstein's second guess was:

$$\mathcal{R}_{\mu\nu} =?\, T_{\mu\nu}$$

And it did not work! This is because energy is not conserved in certain areas in spacetime and other factors I will not go too deep into. After many many guesses and simplifications, Einstein finally came up with his equation:

$$\mathcal{R}_{\mu\nu} - \frac{1}{2}\mathcal{R}g_{\mu\nu} = 8\pi G\, T_{\mu\nu}$$

This equation is the correct connection between the curvature of spacetime and the presence of energy and momentum. There are many other forms of this equation which use different tensors and different derivation techniques, all of which are valid. I have only introduced one form and all the other forms imply the same results as this form. This is why the subsection is called Einstein's Field Equations and not Equation. Another form is:

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = \kappa T_{\mu\nu}$$

Where $G_{\mu\nu}$ is the Einstein tensor, Λ is the cosmological constant and $\kappa = 8\pi G/c^4$. This brings up another form:

$$\mathcal{R}_{\mu\nu} - \frac{1}{2}\mathcal{R}g_{\mu\nu} + \Lambda g_{\mu\nu} = \kappa T_{\mu\nu}$$

Since the definition of the Einstein tensor is:

$$G_{\mu\nu} \equiv \mathcal{R}_{\mu\nu} - \frac{1}{2}\mathcal{R}g_{\mu\nu}$$

You can see now that these equations are not just symbols for the sake of inscrutability but they have real meanings and allow us to make tons of predictions about the universe. Objects such as black holes, wormholes and concepts such as the expansion and shape of the universe, spinning black holes, interference of two dying stars, etc. all arise from analyzing the field equations of general relativity. This is what makes it such a powerful and successful theory; It allows scientists to make predictions about the universe and they can come up with experiments to constantly check the consistency of the theory. General Relativity was proposed in 1915, yet more than half the things we know about the universe through this theory were discovered after its proposition and formulation by Albert Einstein. To this day, the theory is still being tested and so far has been consistent with our experimental data making it one of the most powerful theories of physics. Two good sources for further studying the topic are online lectures by Prof. Leonard Susskind at Stanford University and Prof. Scott Hughes at MIT, and books; *An Introduction to General Relativity, Spacetime and Geometry* by Prof. Sean Carroll at Johns Hopkins University and formerly California Institute of Technology, and *General Relativity: The Theoretical Minimum* by Prof. Leonard Susskind and Andre Cabannes at MIT. I also suggest studying the entire *The Theoretical Minimum Series* by Prof. Leonard Susskind if you are interested in a thorough study of physics at a high level.