Chapter 9 - Laplace Transform

Fourier Transform

$\int_{-\infty}^{\infty} f(t)e^{-j\omega t}dt$

What is Laplace Transform

The Laplace Transform (LT) is the **generalization** of the Fourier Transform to **include general complex exponential signals** of the form

$$e^{st} = e^{(\sigma+j\omega)t} = e^{\sigma t}e^{j\omega t}$$

compare $\longleftrightarrow$ $e^{j\omega t}$

We can generalize because e^{st} also follows:

$$e^{st} \xrightarrow{T} H(s)e^{st} = e^{st}\int_{-\infty}^{\infty} h(\tau)e^{-s\tau}d\tau$$

compare $\longleftrightarrow$ $e^{j\omega t} \to H(j\omega)\, e^{j\omega t}$

😍For **intuitive understanding of Laplace Transform**, you can refer to the following video, which has a lot of 3D plot visualizations to help you better understand!😍

https://www.youtube.com/watch?v=n2y7n6jw5d0

Bilateral Laplace Transform

In this course, we only concern **Bilateral Laplace Transform**.

Bilateral Laplace Transform means that the integral starts from negative infinity to positive infinity.

The formula is

$$X(s) = \int_{-\infty}^{\infty} x(t)e^{-st}dt$$

$$x(t) = \frac{1}{2\pi j}\int_{\sigma-j\infty}^{\sigma+j\infty} X(s)e^{st}ds$$

$$s = \sigma + j\omega$$

LT pairs can be denoted as:

$$x(t) \leftrightarrow X(s)$$ & ROC   always specify ROC!

Typically, you should **RARELY** use the above 2 equations during the exam. Instead, please use common LT pairs on the exam paper to simplify the calculation.

Relationship between Laplace Transform & Fourier Transform

We can derive the following relationship:

$$X(s)|_{s=j\omega} = \mathcal{F}\{x(t)\}$$

$$X(s) = \mathcal{F}\{x(t)e^{-\sigma t}\}, \text{where } s = \sigma + j\omega$$

Summary:

Fourier Transform is a **slice** of Laplace transform.

Region of Convergence (ROC)

Concept

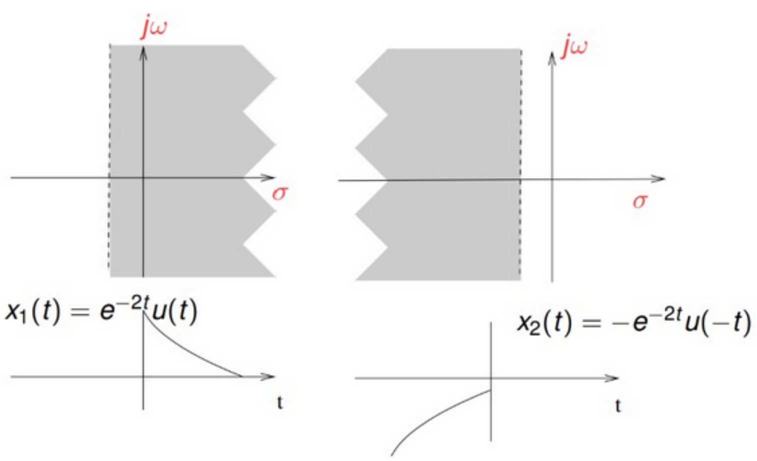
ROC is **the set of values of s** on the complex plane for which the **bilateral Laplace transform is guaranteed to exist/converge**. That is,

$$ROC = \{s \in C : \int_{-\infty}^{\infty} |x(t)|e^{-real\{s\}t}dt < \infty\}$$

To define **a pair of Laplace Transform**, you **have to** specify its **ROC**!

Plot

Often we display the ROC of a signal using the **complex s-plane** as shown in the following figures.



$X(s) = \frac{1}{s+2}$

- The horizontal axis is called σ axis, and the vertical axis is called $j\omega$ axis.
- The shaded region indicates ROC.

- Dotted lines is used to indicate boundaries if ROC doesn't include its edges.
- **If the shaded region includes the $j\omega$ axis, then the FT of the signal exists.**

Poles and Zeros for Rational Laplace Transform

If the Laplace transform of a signal $x(t)$, $X(s)$, has the form:

$$X(s) = \frac{N(s)}{D(s)} = \frac{\sum_{m=0}^{M} b_m s^m}{\sum_{n=0}^{N} a_n s^n} = G\frac{\prod_{k=1}^{M}(s - z_k)}{\prod_{k=1}^{N}(s - p_k)}$$

- **Zeros**: (z_1,z_2, . . ., z_M), sometimes infinity
- **Poles**: (p_1, p_2, . . ., p_N), sometimes infinity
- **Gain**: G = $\frac{b_M}{a_N}$

Questions:

1. **What do zeros & poles mean?**

   Let z_1 be zero, and p_1 be pole, we can obtain

$$X(z_1) = 0$$
$$X(p_1) = \infty$$

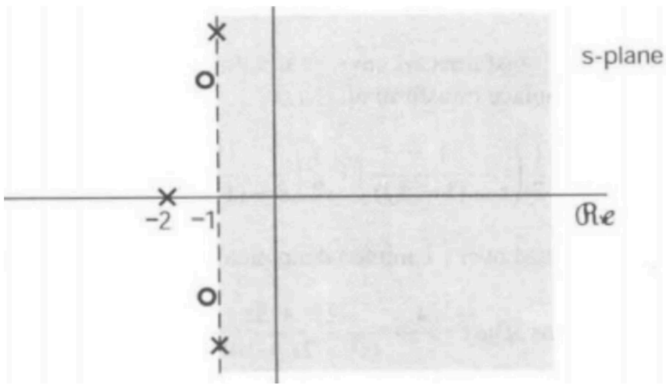
2. **How to judge zeros/poles at ∞?**

   If $lim_{s\to\infty}H(s) = 0$, there is a zero at infinity.

   If $lim_{s\to\infty}H(s) \to \infty$, there is a pole at infinity.

Pole-zero plot

Using 'o' to represent zeros, using 'X' to represent poles



Property of ROC

This part is used to judge the ROC of Laplace Transform.

- The ROC of $X(s)$ consists of **stripe-shaped** regions parallel to the $j\omega$ axis in the s plane.
- The ROC of rational LT does not contain any poles.
- If $x(t)$ has a finite duration and is absolutely integrable, the ROC is the whole s plane.
- If $x(t)$ is **right sided** (remaining zeros when $t < t_0$), and if the line $Re\{s\} = \sigma_0$ is in the ROC, then all values of s whose $Re\{s\} > \sigma_0$ will also be in the ROC.

- If $x(t)$ is **left sided**, and if the line $Re\{s\} = \sigma_0$ is in the ROC, then all values of s whose $Re\{s\} < \sigma_0$ will also be in the ROC.
- If $x(t)$ is **two sided**, and if the line $Re\{s\} = \sigma_0$ is in the ROC, then the ROC is composed of a stripe-shaped region including the line $Re\{s\} = \sigma_0$ on the s plane.
- If $X(s)$ is rational, its ROC is constrained by the poles or extents to infinity.
- If $X(s)$ is rational, and if $x(t)$ is right sided, the ROC is on the right of the rightest pole.
- If $X(s)$ is rational and left sided, its ROC is on the left of the leftest pole.

You can refer to this video by Prof. Oppenheim for detailed explanation, which derive from scratch: https://www.bilibili.com/video/BV1SB4y1C7yx?p=20&vd_source=0be2819c805c6da41df6967754921b05

ROC and System Properties

The Laplace transform $H(s)$ of a system's impulse response $h(t)$ is called its system function.

- Stability ⇔ $j\omega$ axis in ROC ⇔ $h(t)$ absolutely integrable
- Causality ⇔ $h(t)$ is a right-sided signal ⇔ ROC of the system function H(s) is the right-half plane (RHP) to the right of the rightmost pole.
- Stability and Causality ⇔ All of its poles lie strictly within the LHP (left-half of the s-plane).
- ***CAUTION***: Consider poles at ±∞! e.g. $H(s) = s$ has a pole at s = ∞.
- All improper systems are unstable!
- **A diffeq system is stable** iff all roots of its characteristic polynomial are in the left half plane.

Geometric Properties of FT from pole-zero plot

Given the pole-zero plot corresponding to the transfer function $H(s)$ of an LTI system, we can sketch the magnitude response $|H(\omega)|$ and phase response $\angle H(\omega)$:

$$H(\omega) = G\frac{(j\omega - z_1)\cdot \ldots \cdot (j\omega - z_m)}{(j\omega - p_1)\cdot \ldots \cdot (j\omega - p_n)}$$

Magnitude response:

$$|H(\omega)| = G\frac{|j\omega - z_1|\cdot \ldots \cdot |j\omega - z_m|}{|j\omega - p_1|\cdot \ldots \cdot |j\omega - p_n|}$$

Phase response:

$$\angle H(\omega) = \angle G + \angle(j\omega - z_1) + \ldots + \angle(j\omega - z_m) - \angle(j\omega - p_1) - \ldots - \angle(j\omega - p_n)$$

See video by Oppenhium if you don't understand

Common Laplace Transform & Inverse Laplace Transform

Common Laplace Transform

Table of Laplace transform pairs

$f(t)$	$F(s)$	ROC
$\delta(t)$	1	$\forall s$
$u(t)$	$\frac{1}{s}$	$\text{real}\{s\} > 0$
$t^n u(t)$	$\frac{n!}{s^{n+1}}$	$\text{real}\{s\} > 0$
$e^{-at}u(t)$	$\frac{1}{s+a}$	$\text{real}\{s\} > \text{real}\{-a\}$
$-e^{-at}u(-t)$	$\frac{1}{s+a}$	$\text{real}\{s\} < \text{real}\{-a\}$
$t^n e^{-at}u(t)$	$\frac{n!}{(s+a)^{n+1}}$	$\text{real}\{s\} > \text{real}\{-a\}$

$f(t)$	$F(s)$	notes
$\sin(\omega_0 t)\, u(t)$	$\frac{\omega_0}{s^2+\omega_0^2}$	$\text{real}\{s\} > 0$
$\cos(\omega_0 t)\, u(t)$	$\frac{s}{s^2+\omega_0^2}$	$\text{real}\{s\} > 0$
$e^{-at}\cos(\omega_0 t)\, u(t)$	$\frac{s+a}{(s+a)^2+\omega_0^2}$	$\text{real}\{s\} > \text{real}\{-a\}$
$e^{-at}\sin(\omega_0 t)\, u(t)$	$\frac{\omega_0}{(s+a)^2+\omega_0^2}$	$\text{real}\{s\} > \text{real}\{-a\}$
$u_n(t) = \frac{d^n}{dt^n}\delta(t)$	s^n	$\forall s$
$u_{-n}(t) = \underbrace{u(t) * \ldots * u(t)}_{\text{n times}}$	$\frac{1}{s^n}$	$\text{real}\{s\} > 0$

Property of Laplace Transform

Properties of the Laplace Transform

	Time	Laplace	ROC (of result)
Linearity	$a_1 f_1(t) + a_2 f_2(t)$	$a_1 F_1(s) + a_2 F_2(s)$	contains $\text{ROC}_1 \cap \text{ROC}_2$
Time shift	$f(t-\tau)$	$e^{-s\tau}F(s)$	same
Time-scaling	$f(at),\ a \neq 0$	$\frac{1}{\lvert a\rvert}F\left(\frac{s}{a}\right)$	ROC/a
Time reversal	$f(-t)$	$F(-s)$	$-\text{ROC}$
Convolution	$f_1(t) * f_2(t)$	$F_1(s) \cdot F_2(s)$	contains $\text{ROC}_1 \cap \text{ROC}_2$
Frequency shift	$f(t)e^{j\omega_0 t}$	$F(s - j\omega_0)$	same
Frequency shift	$f(t)e^{s_0 t}$	$F(s - s_0)$	$\text{ROC} + \text{real}\{s_0\}$
Time Differentiation	$\frac{d^n}{dt^n}f(t)$	$s^n F(\omega)$	contains ROC
s-domain Differentiation	$(-t)^n f(t)$	$\frac{d^n}{ds^n}F(\omega)$	same
Integration	$\int_{-\infty}^{t} f(\tau)\, d\tau = f(t) * u(t)$	$\frac{1}{s}F(s)$	contains $\text{ROC} \cap \{\text{real}\{s\} > 0\}$
DC Value	$\int_{-\infty}^{\infty} f(t)\, dt = F(0)$		must contain $s = 0$

✨Inverse Laplace Transform

Requirement: Given rational $X(s)$ and **ROC**, we are required to invert it to find signal $x(t)$.

Steps:

- Decompose $X(s)$ into **the sum of atomic expressions** by doing **PFE**.
- For each expression, find ROC. Select the one including the ROC of LT.
- For each expression, use LT table and LT properties to get the result. Then sum them up.
- ***CAUTION***: If the ROC is not given explicitly, discuss all the situations!!!

There are cases that given the pole-zero plot and DC value, and ask you to find x(t). In this case, you need to find the gain G with certain points.

Exercise

The system function of a causal LTI system is

$$H(s) = \frac{s+1}{s^2+2s+2}$$

Determine and sketch the response ($y(t)$) when the input is

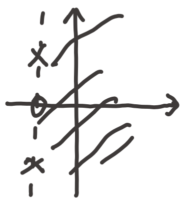
$$e^{-|t|}, \quad -\infty < t < \infty$$

Hint: You can use $e^{-|t|} = e^{-t}u(t) + e^{t}u(-t)$

Answer:

$$x(t) = e^{-|t|} = e^{-t}u(t) + e^{t}u(-t)$$

$$X(s) = \frac{1}{s+1} - \frac{1}{s-1} = \frac{-2}{(s+1)(s-1)}, \quad -1 < \mathcal{R}\{s\} < 1$$



$$H(s) = \frac{s+1}{s^2+2s+2} = \frac{s+1}{(s+1)^2+1} \qquad \mathcal{R}\{s\} > -1$$

$$Y(s) = H(s)X(s) = \frac{-2}{(s-1)(s^2+2s+2)} \qquad -1 < \mathcal{R}\{s\} < 1$$

Do PFE

$$Y(s) = \frac{-2/5}{s-1} + \frac{2}{5}\left[\frac{s+1}{(s+1)^2+1}\right] + \frac{4}{5}\left[\frac{1}{(s+1)^2+1}\right]$$

$$y(t) = \frac{2}{5}e^{t}u(-t) + \frac{2}{5}e^{-t}\cos(t)u(t) + \frac{4}{5}e^{-t}\sin(t)u(t)$$

✨Differential Equation

Solve **algebraic expression** by Linear Constant Coefficient Differential Equation.

Solve **ROC** by stability & causality & etc. conditions.

Exercise

The system function of a causal LTI system is

$$H(s) = \frac{s+1}{s^2+2s+2}$$

Express $H(s)$ as a differential equation.

$$H(s) = \frac{Y(s)}{X(s)} = \frac{s+1}{s^2+2s+2}$$

$$(s^2+2s+2)Y(s) = (s+1)X(s)$$

$$\frac{d^2y(t)}{dt^2} + 2\frac{dy(t)}{dt} + 2y(t) = \frac{d}{dt}x(t) + x(t)$$

Block Diagram

In this part, you should be capable of **both**:

1. Writing $H(s)$ according to the block diagram. Then analyze the system with $H(s)$. e.g. How to solve the Impulse Response $h(t)$? etc.
2. Draw the block diagram according to $H(s)$.

Parallel Interconnection

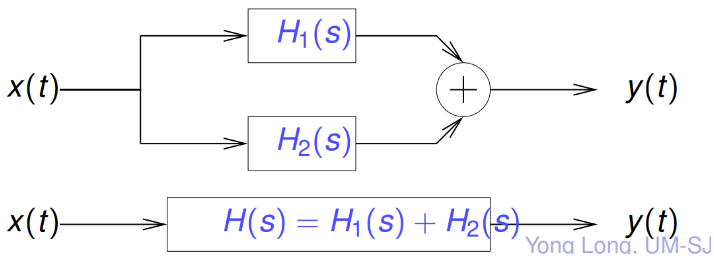
Thus the overall frequency response of two LTI systems connected in parallel is given by the sum of the frequency responses of the individual systems:

$$H(\omega) = H_1(\omega) + H_2(\omega).$$

The overall system (transfer) function is

$$H(s) = H_1(s) + H_2(s).$$

Distributive



Series Interconnection

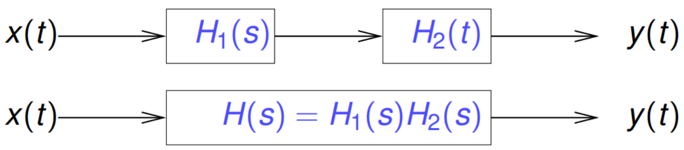
Thus the overall frequency response of two LTI systems connected in series is given by the *product* of the frequency responses of the individual systems:

$$H(\omega) = H_1(\omega)H_2(\omega).$$

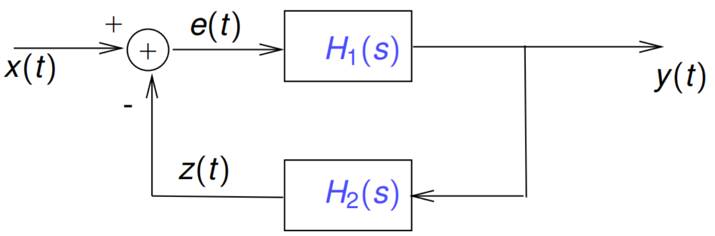
The overall system (transfer) function is

$$H(s) = H_1(s)H_2(s).$$

Associative



Feedback Interconnection



$$\begin{cases} y(t) = e(t) * h_1(t), \\ e(t) = x(t) - z(t), \\ z(t) = y(t) * h_2(t) \end{cases} \implies y(t) = [x(t) - y(t) * h_2(t)] * h_1(t)$$

$$H(\omega) = \frac{Y(\omega)}{X(\omega)} = \frac{H_1(\omega)}{1 + H_1(\omega)H_2(\omega)}$$

$$H(s) = \frac{Y(s)}{X(s)} = \frac{H_1(s)}{1 + H_1(s)H_2(s)}$$

If you are not sure whether you can write the $H(s)$ according to the block diagram, take the above as an example and practice.

Exercise

Exercise 1

Draw the block diagram of the causal LTI system with system function:

$$H(s) = \frac{1}{s+3}$$

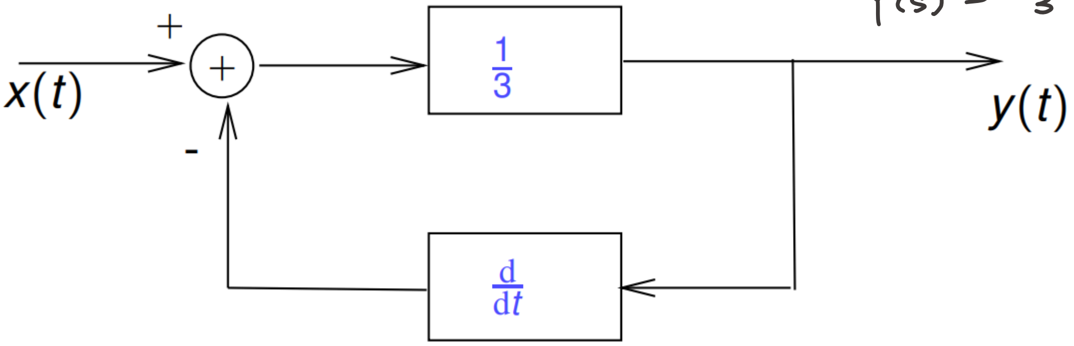
$$\frac{Y(s)}{X(s)} = \frac{1}{s+3}$$

1. Use differentiator to implement. Mention, the differentiator is both difficult to implement and extremely sensitive to noise.

$$sY(s) + 3Y(s) = X(s)$$

$$Y(s) = \frac{1}{3}X(s) - \frac{s}{3}X(s)$$

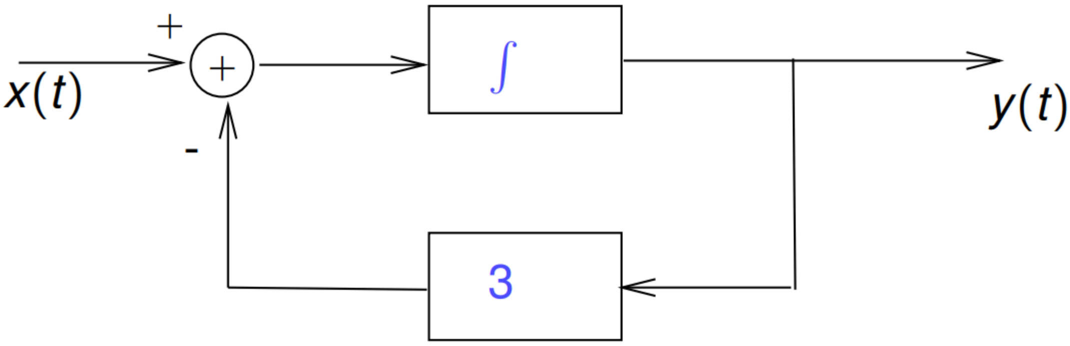
Answer:



2. Use integrator to implement.

Answer:

$$sY(s) = X(s) - 3Y(s)$$



For normal cases, please use integrator ($\frac{1}{s}$) to draw the diagram

Exercise 2:

Draw the block diagram for the causal LTI system with system function:

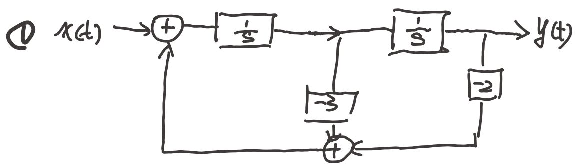
$$H(s) = \frac{1}{(s+1)(s+2)}$$

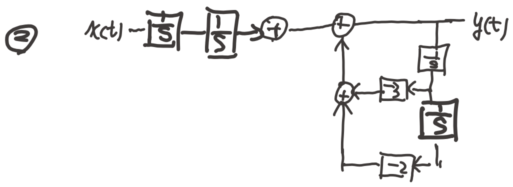
Use **Direct Form**, **Cascade Form**, and **Parallel From**

Answer:

use general block diagram

1. Direct Form





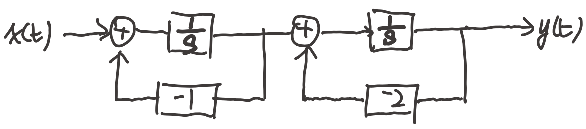
$$H(s) = \frac{Y(s)}{X(s)} = \frac{1}{s^2+3s+2}$$

$$(s^2+3s+2)Y(s) = X(s)$$

$$Y(s) = \frac{1}{s^2}X(s) - \frac{3}{s}Y(s) - \frac{2}{s^2}Y(s)$$

2. Cascade Form

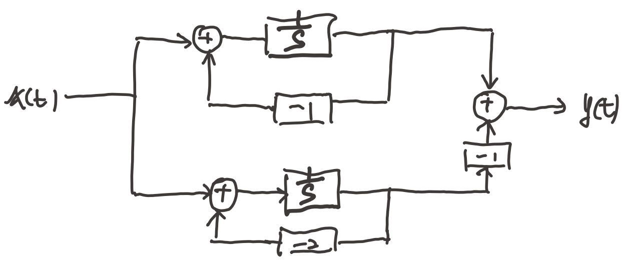
$$H(s) = \frac{1}{s+1} \cdot \frac{1}{s+2}$$



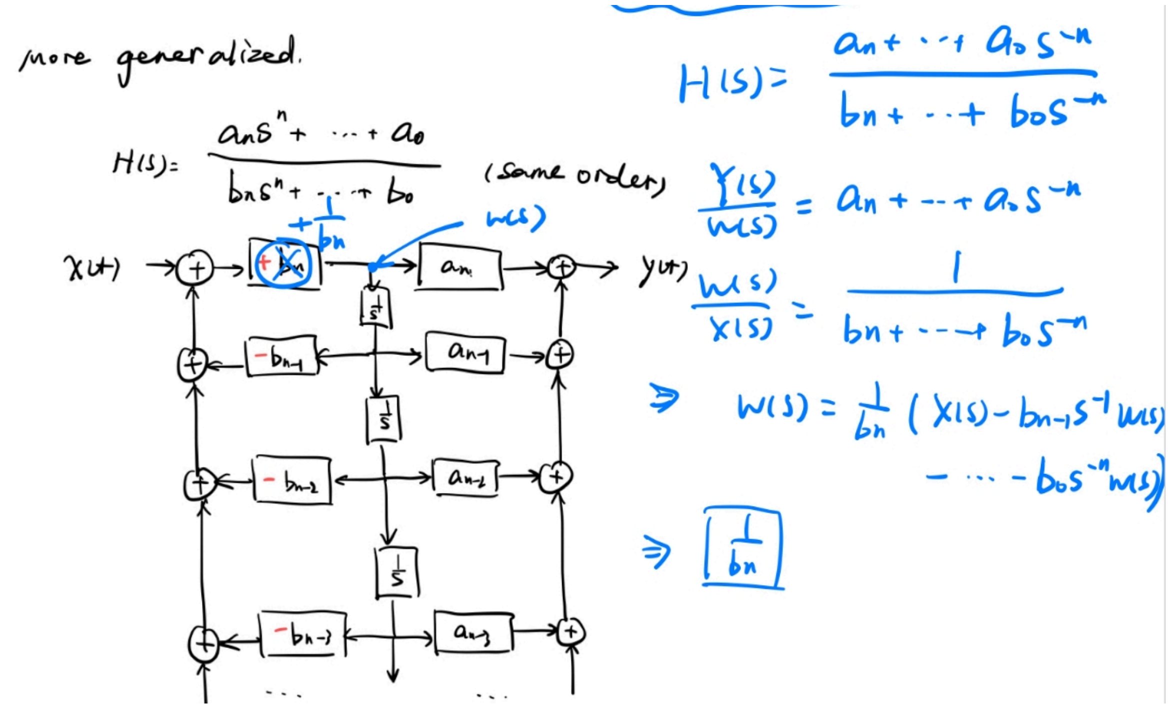
3. Parallel From

PFE!

$$H(s) = \frac{1}{s+1} - \frac{1}{s+2}$$



General block diagram for rational systems



- $m = n$: Just apply the method.
- $m < n$: e.g. $H(s) = \frac{1}{s^2+3s+2}$, we just first fill all the item in the upper to make the upper and lower part are at the same order. Finally **remove the branch** with zero coefficient.
- $m > n$: e.g. $H(s) = s$. It has poles at $s = \infty$, which makes the system improper (All rational but improper systems are unstable!!), thus **won't appear in real cases**.

Notes

Before the exam:

1. **Review all the quizzes!**
2. Review homework answers.
3. Gain proficiency in calculation (PFE, Common Laplace Transform, differential equation, etc.).

During the exam:

1. For solving differential equation problem, consider Laplace Transform first. Do NOT use time domain methods to solve it!
2. **Always judge and specify the ROC.**

Reference

1. Hu fan. 216 final recitation class.
2. Long yong. 216 slides.