Chapter 2 - Convolution

Linear and Time-Invariant (LTI) Systems

What is LTI Property?

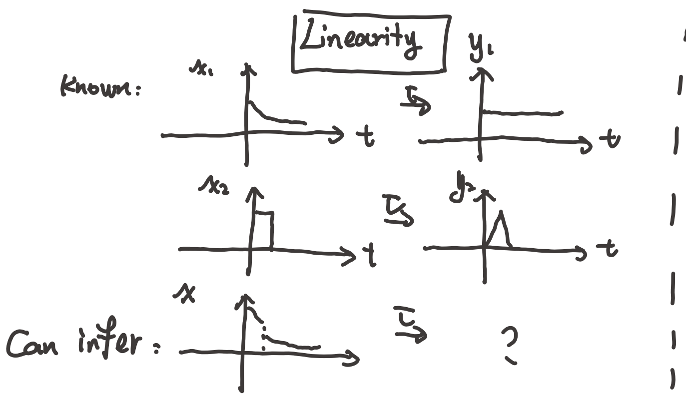
Linearity:

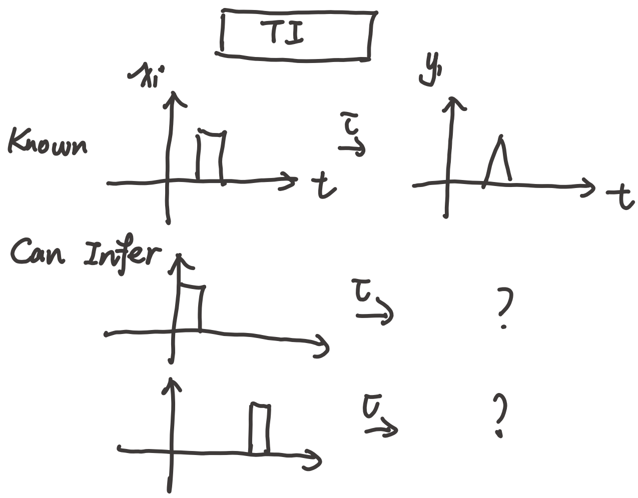
For a linear system, if $x_1(t) \xrightarrow{T} y_1(t)$ and $x_2(t) \xrightarrow{T} y_2(t)$, then $ax_1(t) + bx_2(t) \xrightarrow{T} ay_1(t) + by_2(t)$.

Time-Invariance:

For a time-invariant system, if $x(t) \xrightarrow{T} y(t)$, then $x(t - t_0) \xrightarrow{T} y(t - t_0)$.

Graphical Understanding of LTI Property:

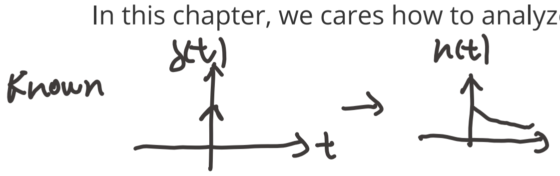




Why LTI Property Matters?

LTI property **simplify** the analysis of a system. Specifically, it makes the system design and response analysis easier. To design an LTI system, we only need to care about its impulse response $h(t)$, since an LTI system is **completely described** by its impulse response $h(t)$. Great Simplification!

In this chapter, we cares how to analyze LTI system using **convolution** methods.



Can Infer: All Inputs $\xrightarrow{T}$ All Outputs!

Convolution

What is Convolution?

$x(t)$ can be decomposed into delayed impulses

$$x(t) = \int_{-\infty}^{\infty} x(\tau)\delta(t-\tau)d\tau.$$

Denoted

$$\delta(t) \xrightarrow{\mathcal{T}} h(t),$$

For LTI system, the corresponding response $y(t)$ can be derived using LTI property, which is

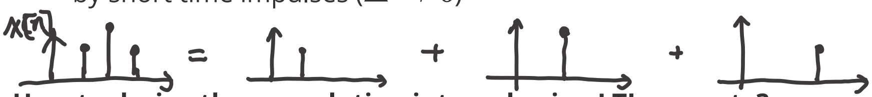
$$y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau = x(t) * h(t).$$

Concept Question:

1. **Why we can decompose $x(t)$ into delayed impulses?**
   - Hint: Start from discrete case $x[n]$ to understand the idea, and build an approximation by short time impulses ($\Delta \to 0$)

Continuous Case

Discrete case



Calculus!

2. **How to derive the convolution integral using LTI property?**
   - Hint 1: $x(\tau)$ is a constant so that we can use linearity property.
   - Hint 2: $\delta(t-k) \xrightarrow{\mathcal{T}} h_k(t)$, since the system is TI, $\delta(t-k) \xrightarrow{\mathcal{T}} h(t-k)$

omplicated case : $x(t) \rightarrow \boxed{\text{system}} \rightarrow y(t) = \int_{-\infty}^{\infty} x(\tau) h_k(t) d\tau$ (Linearity)

$= \int_{-\infty}^{\infty} x(\tau) h(t-\tau) d\tau$ (TI)

Decompose $\downarrow$

Simplified case : $\delta(t) \rightarrow \boxed{\text{system}} \rightarrow h(t)$

(Use integral definition) $x(t) = \int_{-\infty}^{\infty} x(\tau)\delta(t-\tau) d\tau$

Why Convolution?

An LTI system is completely described by its impulse response $h(t)$.

Any system whose input-output relationship can be expressed in $y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau = x(t) * h(t)$ is time-invariant.

If you can write the representation into this form, you do not need to prove TI first!

Pratical Question:

1. Known the representation of a LTI system, $y(t) = F(x(t))$, how to solve $h(t)$?
   - By definition:

     $h(t)$ is the output signal when input signal $x(t)$ is $\delta(t)$.

     $h(t) = F(\delta(t))$
   - By convolution:

     $y(t) = h(t) * x(t) = F(x(t))$

     solve $h(t)$ by transforming the $F(x(t))$ to convolution form

   You should handle both!

Exercise:

1. Find the impulse response of the LTI systems.

$$y(t)=\int_{-3}^{3}\tau^2x(t-\tau)d\tau+\int_{-\infty}^{t+1}(t-\tau+3)^{-2}x(\tau)d\tau$$

① $-3<\tau<3 \Rightarrow rect(\frac{\tau}{6})$ ② $\tau<t+1 \Rightarrow t-\tau+1>0 \Rightarrow u(t-\tau+1)$

$y(t)=\int_{-\infty}^{\infty} rect(\frac{\tau}{6})\tau^2 x(\underline{t-\tau})d\tau+\int_{-\infty}^{\infty} u(\underline{t-\tau}+1)(\underline{t-\tau}+3)^{-2}x(\tau)d\tau$

$=[rect(\frac{t}{6})t^2]*x(t)+[u(t+1)(t+3)^{-2}]*x(t)$

$h(t)=rect(\frac{t}{6})t^2+u(t+1)(t+3)^{-2}$

Summary: How to get turn $y(t)$ to $h(t)*x(t)$?

1. Use $u(t)$ to represent range of integration
2. Turn to convolution form (first find $t-\tau$ and τ , then substitute them with t).

Graphical Method for Convolution Calculation

For time-limited signals, it's more convenient and intuitive to calculate the convolution with the help of graphics.

Steps: (Exchange the role of h and x is fine)

- Fold: fold $h(\tau)$ about $\tau=0$ to get $h(-\tau)$
- Shift: shift $h(-\tau)$ by t to get $h(t-\tau)$
- Multiply: multiply $x(\tau)$ by $h(t-\tau)$ for every τ
- Integrate: $y(t)=\int_{-\infty}^{\infty}x(\tau)h(t-\tau)d\tau$

If you don't understand, please refer to this video (from 32:40):

https://www.bilibili.com/video/BV1SB4y1C7yx?p=4&vd_source=0be2819c805c6da41df6967754921b05

!!!

If you do not have 100% confidence, use algebra methods than graphical methods in the exam, since algebra methods enable you to gain more partial points.

Properties of Convolution

- $x(t)*h(t)=h(t)*x(t)$
- $[x(t)*h_1(t)]*h_2(t)=x(t)*[h_1(t)*h_2(t)]$
- $x(t)*\delta(t)=x(t)$
- $x(t)*\delta(t-t_0)=x(t-t_0)$
- $\delta(t-t0)*\delta(t-t1)=\delta(t-t0-t1)$
- $\underline{x(t-t0)*h(t-t1)=y(t-t0-t1)}$ ☆ Important! Used frequently!
- $\frac{d}{dt}y(t)=[\frac{d}{dt}x(t)]*h(t)=[\frac{d}{dt}h(t)]*x(t)$

- $\int_{-\infty}^{\infty} y(t) = [\int_{-\infty}^{\infty} x(t)dt][\int_{-\infty}^{\infty} h(t)dt]$
- $y(at) \neq x(at) * h(at)$

For properties which are not included, please derive from integral form!

More in https://zhuanlan.zhihu.com/p/150737244

LTI Systems properties via $h(t)$

1. **Casual**:

   An LTI System is casual iff its impulse response $h(t) = 0$ for all $t < 0$.

2. **Memoryless:**

   An LTI System is memoryless iff its impulse response $h(t) = a\delta(t)$.

   Otherwise, the system is **dynamic**.

   (Note: A finite impulse response or FIR system means $h(t) \neq 0, t_1 < t < t_2.$)

3. **BIBO stable**:

   An LTI system is BIBO stable iff its impulse response is absolutely integrable, i.e. $\int_{-\infty}^{\infty} |h(t)|dt < \infty$.

4. **Invertible**:

   An LTI system is invertible iff there exist an inverse system whose impulse response $h_i(t)$ satisfies

$$h(t) * h_i(t) = \delta(t)$$

Exercise:

1. Determine whether the system is causal, stable and static via $h(t)$.

$$y(t) = \int_{-\infty}^{t} (t-\tau)e^{-(t-\tau)}x(\tau)d\tau$$

$$y(t) = \int_{-\infty}^{\infty} u(t-\tau)(t-\tau)\, e^{-(t-\tau)} x(\tau)\, d\tau.$$

$$h(t) = u(t)\, t\, e^{-t}$$

causal : ✓

stable : $\int_{-\infty}^{\infty} |u(t)\, t e^{-t}| = \int_0^{\infty} |t e^{-t}| dt = 1 < \infty$

Dynamic : $h(t) \neq 0$ for $t > 0$.

Step Response

Step response of an LTI system is defined as the response of the system to an input signal $s(t)$ that is unit step.

$$\delta(t) \xrightarrow{\tau} h(t)$$

$$u(t) \xrightarrow{\tau} s(t) = \int_{-\infty}^{\infty} u(t-\tau)h(\tau)d\tau$$

$$s(t) = u(t) * h(t) = \int_{-\infty}^{t} h(\tau)d\tau$$

$$h(t) = \frac{d}{dt}s(t)$$

CT systems described by differential equation models

Tips:

Solving differential equation in the time domain is NOT covered in the Mid1 !!!!!

Also, it is not the focus of VG216. The course emphasizes more on solving differential equations on the frequency domain, which will be covered in Chapter 3&9

Linear constant-coefficient differential equation:

$$\sum_{k=0}^{N} a_k \frac{d^k}{dt^k} y(t) = \sum_{k=0}^{N} b_k \frac{d^k}{dt^k} x(t)$$

Will not appear in the exam !

Approach:

$$y(t) = y_h(t) + y_p(t)$$

$y_h(t)$: homogeneous solution, or zero-input response, or natural response.

$y_p(t)$: particular solution, or zero-state response, or forced response.

Initial rest condition

Initial rest states that if $x(t) = 0$ for $t < t_0$, then $y(t) = 0$ for $t < t_0$.

It means that the output must be zero up until the time when the input becomes zero.

Causal and LTI system ⇔ initial rest

Convolution of Common Signals

Copy them on your CTPP! Copy them on your CTPP! Copy them on your CTPP!

- $k * f(t) = k\int_{-\infty}^{\infty} f(t)dt = k\times$(area of $f(t)$ over R)
- $f(t) * u(t) = \int_{-\infty}^{t} f(t)dt$
- $u(t) * u(t) = tu(t)$
- $u(t) * e^{-at}u(t) = \frac{1}{a}(1-e^{-at})u(t)$
- $e^{-at}u(t) * e^{-at}u(t) = te^{-at}u(t)$
- $e^{-a_1t}u(t) * e^{-a_2t}u(t) = \frac{e^{-a_1t}-e^{-a_2t}}{a_2-a_1}u(t)$
- $f(t) * \sum_{m=-\infty}^{\infty} \delta(t-mT) = \sum_{m=-\infty}^{\infty} f(t-mT)\backslash$

- $\frac{1}{a}rect(\frac{t}{a}) * rect(\frac{t}{a}) = tri(\frac{t}{a})$
- $\frac{1}{a}rect(\frac{t-t_0}{a}) * rect(\frac{t-t_1}{a}) = tri(\frac{t-t_0-t_1}{a})$

Be familiar with all of them, and derive them by yourself once.

You can use them directly in your exam!

Useful Tips:

1. **Write all the $Arect(at)$ signal to the form of $Au(t-t_0) - Au(t-t_1)$ first when calculating the convolution of signals.**
2. If you can find the common convolution results, use them directly in the exam.

Let's Practice!

1. Find the response of the input $x(t) = u(t)$, when $h(t) = e^{-at}u(t)$

   Method 1: Using convolution definition

   Method 2: Using the results above

Q: why here is u(t) ?

A: when t<0.
we need 0
instead of
$\int_0^t e^{-a(t-\tau)}d\tau$!

$$y(t) = x(t) * h(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)\,d\tau$$

$$= \int_{-\infty}^{\infty} u(\tau)e^{-a(t-\tau)}u(t-\tau)\,d\tau$$

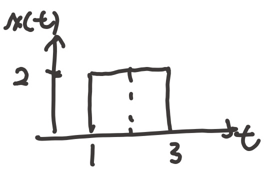
$$= u(t)\int_0^t e^{-a(t-\tau)}\,d\tau \quad (\tau \ge 0, t-\tau \ge 0 \Longrightarrow 0 \le \tau \le t)$$

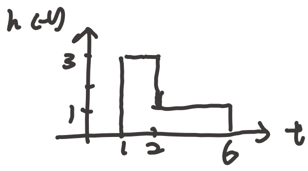
$$= e^{-at}\left(\frac{1}{a}e^{a\tau}\Big|_0^t\right)u(t)$$

$$= \frac{1}{a}(1-e^{-at})u(t)$$

Take care of the range of integral and use the results above in the exam if possible.

2. Given





Solve: y(t) expression

Step 1: Write x(t) & h(t) expression

Tips: write u(t) so that you can use $u(t) * u(t) = tu(t)$

$x(t) = 2u(t-1) - 2u(t-3)$

$h(t) = 3u(t-1) - 2u(t-2) - u(t-6)$

Step 2: Solve y(t) using convolution

$y(t) = x(t) * h(t) = [2u(t-1) - 2u(t-3)] * [3u(t-1) - 2u(t-2) - u(t-6)]$

$= 6u(t-1) * u(t-1) - 4u(t-1) * u(t-2) - 2u(t-1) * u(t-6) - 6u(t-3) * u(t-1)$

$+ 4u(t-3) * u(t-2) + 2u(t-3) * u(t-6)$

$= 6(t-2)u(t-2) - 4(t-3)u(t-3) - 2(t-7)u(t-7) - 6(t-4)u(t-4) + 4(t-5)u(t-5)$

$+ 2(t-9)u(t-9)$

*Step 3: If Graph is needed, Plug in value and connect them.

Tips

If you found it's really hard to understand some chapters of this course, you can:

1. Go to corresponding chapter of MIT open course: https://www.bilibili.com/video/BV1SB4y1C7yx/?spm_id_from=333.999.0.0&vd_source=0be2819c805c6da41df6967754921b05
2. Read the slides/textbook page by page

Before the exam:

1. **Review all the quizzes!**
2. Review homework answers.
3. Gain proficiency in convolution calculation.
4. Check whether all the contents on summary sheet have been on your ctpp.

During the exam:

1. Use convolution of common signals if possible.
2. Pay attention to the range of integral.
3. For the problem with a "!", it means that only zero points or full points. Take care of them!
4. Pay attention to "Time-invariant" or "Time-variant" in the problem

Reference

1. Yuan Jiaming. *VE216 Mid 1 RC part 2.*
2. Long Yong. *VE216 slides.*

5. Always Remember to use Sifting & Sampling property to simplify the reprensentation!

a constant!

Sifting property: $\int_{-\infty}^{\infty} x(t)\delta(t-t_0)\,dt = \underline{x(t_0)}$ if x continuous at t_0

Sampling property: $x(t)\delta(t-t_0) = \underline{x(t_0)}\delta(t-t_0)$

a constant

e.g. HW2 T3 & T5