Chapter 9 - Laplace Transform

First of All

~40%

The Laplace Transform typically accounts for **30%** of the final exam.

The main topics include:

- Laplace Transform and Inverse Laplace Transform
- Solving Differential Equations
- Block Diagrams

What is Laplace Transform

The Laplace Transform (LT) is the **generalization** of the Fourier Transform to **include general complex exponential signals** of the form

$$e^{st} = e^{(\sigma + j\omega)t} = e^{\sigma t} e^{j\omega t}$$

We can generalize because e^{st} also follows:

$$e^{st} \xrightarrow{T} H(s)e^{st} = e^{st} \int_{-\infty}^{\infty} h(\tau) e^{-s\tau} d\tau$$

😍For **intuitive understanding of Laplace Transform**, you can refer to the following video, which has a lot of 3D plot visualizations to help you better understand!😍

https://www.youtube.com/watch?v=n2y7n6jw5d0

Bilateral Laplace Transform

In this course, we only concern **Bilateral Laplace Transform**.

Bilateral Laplace Transform means that the integral starts from negative infinity to positive infinity.

The formula is

$$X(s) = \int_{-\infty}^{\infty} x(t)e^{-st}dt$$

$$x(t) = \frac{1}{2\pi j}\int_{\sigma-j\infty}^{\sigma+j\infty} X(s)e^{st}ds$$

$$s = \sigma + j\omega$$

Typically, you should **RARELY** use the above 2 equations during the exam. Instead, please **use common LT pairs** on the exam paper to simplify the calculation.

LT pairs can be denoted as:

$$x(t) \leftrightarrow X(s), ROC$$

Always identify ROC !!

Common Laplace Transform

Table of Laplace transform pairs

$f(t)$	$F(s)$	ROC
$\delta(t)$	1	$\forall s$
$u(t)$	$\frac{1}{s}$	$\text{real}\{s\} > 0$
$t^n u(t)$	$\frac{n!}{s^{n+1}}$	$\text{real}\{s\} > 0$
$e^{-at}u(t)$	$\frac{1}{s+a}$	$\text{real}\{s\} > \text{real}\{-a\}$
$-e^{-at}u(-t)$	$\frac{1}{s+a}$	$\text{real}\{s\} < \text{real}\{-a\}$
$t^n e^{-at}u(t)$	$\frac{n!}{(s+a)^{n+1}}$	$\text{real}\{s\} > \text{real}\{-a\}$

$f(t)$	$F(s)$	notes
$\sin(\omega_0 t)\,u(t)$	$\frac{\omega_0}{s^2+\omega_0^2}$	$\text{real}\{s\} > 0$
$\cos(\omega_0 t)\,u(t)$	$\frac{s}{s^2+\omega_0^2}$	$\text{real}\{s\} > 0$
$e^{-at}\cos(\omega_0 t)\,u(t)$	$\frac{s+a}{(s+a)^2+\omega_0^2}$	$\text{real}\{s\} > \text{real}\{-a\}$
$e^{-at}\sin(\omega_0 t)\,u(t)$	$\frac{\omega_0}{(s+a)^2+\omega_0^2}$	$\text{real}\{s\} > \text{real}\{-a\}$
$u_n(t) = \frac{d^n}{dt^n}\delta(t)$	s^n	$\forall s$
$u_{-n}(t) = \underbrace{u(t) * \ldots * u(t)}_{\text{n times}}$	$\frac{1}{s^n}$	$\text{real}\{s\} > 0$

Property of Laplace Transform

Properties of the Laplace Transform

	Time	Laplace	ROC (of result)
Linearity	$a_1 f_1(t) + a_2 f_2(t)$	$a_1 F_1(s) + a_2 F_2(s)$	contains $\text{ROC}_1 \cap \text{ROC}_2$
Time shift	$f(t-\tau)$	$e^{-s\tau}F(s)$	same
Time-scaling	$f(at),\ a \neq 0$	$\frac{1}{\lvert a\rvert}F\left(\frac{s}{a}\right)$	ROC/a
Time reversal	$f(-t)$	$F(-s)$	$-\text{ROC}$
Convolution	$f_1(t) * f_2(t)$	$F_1(s) \cdot F_2(s)$	contains $\text{ROC}_1 \cap \text{ROC}_2$
Frequency shift	$f(t)e^{j\omega_0 t}$	$F(s - j\omega_0)$	same
Frequency shift	$f(t)e^{s_0 t}$	$F(s - s_0)$	$\text{ROC} + \text{real}\{s_0\}$
Time Differentiation	$\frac{d^n}{dt^n}f(t)$	$s^n F(\omega)$	contains ROC
s-domain Differentiation	$(-t)^n f(t)$	$\frac{d^n}{ds^n}F(\omega)$	same
Integration	$\int_{-\infty}^{t} f(\tau)\,d\tau = f(t) * u(t)$	$\frac{1}{s}F(s)$	contains $\text{ROC} \cap \{\text{real}\{s\} > 0\}$
DC Value	$\int_{-\infty}^{\infty} f(t)\,dt = F(0)$		must contain $s = 0$

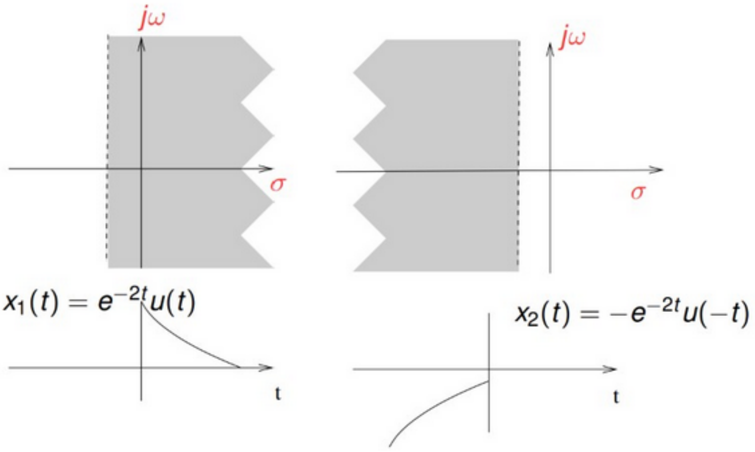
Region of Convergence (ROC)

ROC is **the set of values of s** on the complex plane for which the **bilateral Laplace transform is guaranteed to exist/converge**. That is,

$$ROC = \{s \in C : \int_{-\infty}^{\infty} |x(t)|e^{-real\{s\}t} dt < \infty\}$$

To define **a pair of Laplace Transform**, you **have to** specify its **ROC**!

Often we display the ROC of a signal using the **complex s-plane** as shown in the following figures.



Poles and Zeros for Rational Laplace Transform

If the Laplace transform of a signal $x(t)$, $X(s)$, has the form:

$$X(s) = \frac{N(s)}{D(s)} = \frac{\sum_{m=0}^{M} b_m s^m}{\sum_{n=0}^{N} a_n s^n} = G\frac{\prod_{k=1}^{M}(s - z_k)}{\prod_{k=1}^{N}(s - p_k)}$$

- **Zeros** ($X(z_1) = 0$): (z_1,z_2, . . ., z_M), sometimes infinity
- **Poles** ($X(p_1) = \infty$): (p_1, p_2, . . ., p_N), sometimes infinity
- **Gain**: G = $\frac{b_M}{a_N}$

Questions:

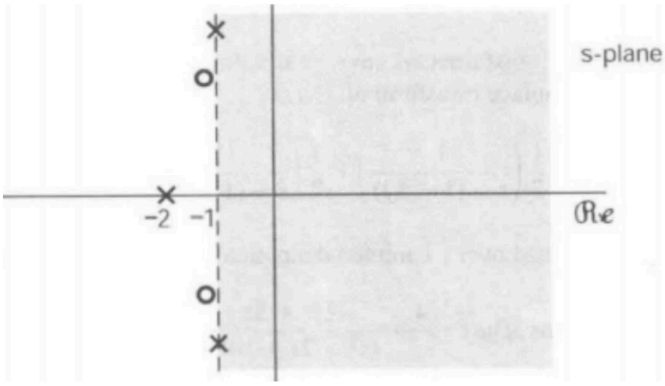
1. **How to judge zeros/poles at** ∞?

   If $lim_{s\to\infty} H(s) = 0$, there is a zero at infinity.

   If $lim_{s\to\infty} H(s) \to \infty$, there is a pole at infinity.

Pole-zero plot

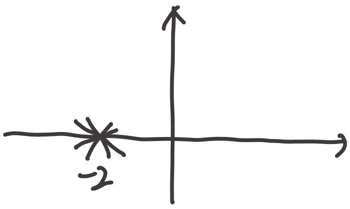
Using 'o' to represent zeros, using 'X' to represent poles



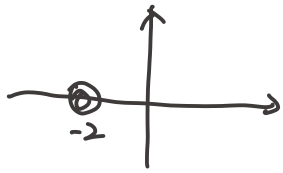
Questions:

How to represent double zeros/poles on the pole-zero plot?

$H(s) = \frac{1}{(s+2)^2}$



$H(s) = (s+2)^2$



Property of ROC

This part is used to **judge the ROC of Laplace Transform**.

- The ROC of $X(s)$ consists of **stripe-shaped** regions parallel to the $j\omega$ axis in the s plane.
- The ROC of rational LT does not contain any poles.
- If $x(t)$ has a finite duration and is absolutely integrable, the ROC is the whole s plane.
- If $x(t)$ is **right sided** (remaining zeros when $t < t_0$), and if the line $Re\{s\} = \sigma_0$ is in the ROC, then all values of s whose $Re\{s\} > \sigma_0$ will also be in the ROC.
- If $x(t)$ is **left sided**, and if the line $Re\{s\} = \sigma_0$ is in the ROC, then all values of s whose $Re\{s\} < \sigma_0$ will also be in the ROC.
- If $x(t)$ is **two sided**, and if the line $Re\{s\} = \sigma_0$ is in the ROC, then the ROC is composed of a stripe-shaped region including the line $Re\{s\} = \sigma_0$ on the s plane.
- If $X(s)$ is rational, its ROC is constrained by the poles or extents to infinity.
- If $X(s)$ is rational, and if $x(t)$ is right sided, the ROC is on the right of the rightest pole.
- If $X(s)$ is rational and left sided, its ROC is on the left of the leftest pole.

You can refer to this video by Prof. Oppenheim for detailed explanation, which derive from scratch: https://www.bilibili.com/video/BV1SB4y1C7yx?p=20&vd_source=0be2819c805c6da41df6967754921b05

Questions:

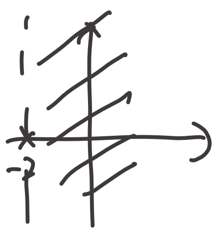
How to judge ROC?

$x(t) = e^{-2t}u(t)$ right-side

$\Updownarrow$

$X(s) = \frac{1}{s+2} \Rightarrow s = -2$

Poles & Right-sided/ left-sided/ finite duration signals



ROC and System Properties

The Laplace transform $H(s)$ of a system's impulse response $h(t)$ is called its system function.

- Stability ⇔ $j\omega$ axis in ROC ⇔ $h(t)$ absolutely integrable
- Causality ⇔ $h(t)$ is a right-sided signal ⇔ ROC of the system function H(s) is the right-half plane (RHP) to the right of the rightmost pole.
- Stability and Causality ⇔ All of its poles lie strictly within the LHP (left-half of the s-plane).
- ***CAUTION***: Consider poles at ±∞! e.g. $H(s) = s$ has a pole at s = ∞.
- **A diffeq system is stable** iff all roots of its characteristic polynomial are in the left half plane.

Geometric Properties of FT from pole-zero plot

Given the pole-zero plot corresponding to the transfer function $H(s)$ of an LTI system, we can sketch the magnitude response $|H(\omega)|$ and phase response $\angle H(\omega)$:

$$H(\omega) = G\frac{(j\omega - z_1)\cdot \ldots \cdot (j\omega - z_m)}{(j\omega - p_1)\cdot \ldots \cdot (j\omega - p_n)}$$

Magnitude response:

$$|H(\omega)| = G\frac{|j\omega - z_1|\cdot \ldots \cdot |j\omega - z_m|}{|j\omega - p_1|\cdot \ldots \cdot |j\omega - p_n|}$$

Phase response:

$$\angle H(\omega) = \angle G + \angle(j\omega - z_1) + \ldots + \angle(j\omega - z_m) - \angle(j\omega - p_1) - \ldots - \angle(j\omega - p_n)$$

✨Inverse Laplace Transform

Requirement: Given rational $X(s)$ and **ROC**, we are required to invert it to find signal $x(t)$.

Steps:

- Decompose $X(s)$ into **the sum of atomic expressions** by doing **PFE**.
- For each expression, find ROC. Select the one including the ROC of LT.
- For each expression, use LT table and LT properties to get the result. Then sum them up.
- ***CAUTION***: If the ROC is not given explicitly, discuss all the situations!!!

There are cases that given the pole-zero plot and DC value, and ask you to find x(t). In this case, you need to find the gain G with certain points.

✨Differential Equation

Solve **algebraic expression** by Linear Constant Coefficient Differential Equation.

Solve **ROC** by stability & causality & etc. conditions.

Block Diagram

In this part, you should be capable of **both**:

1. Writing $H(s)$ according to the block diagram. Then analyze the system with $H(s)$. e.g. How to solve the Impulse Response $h(t)$? etc.
2. Draw the block diagram according to $H(s)$.

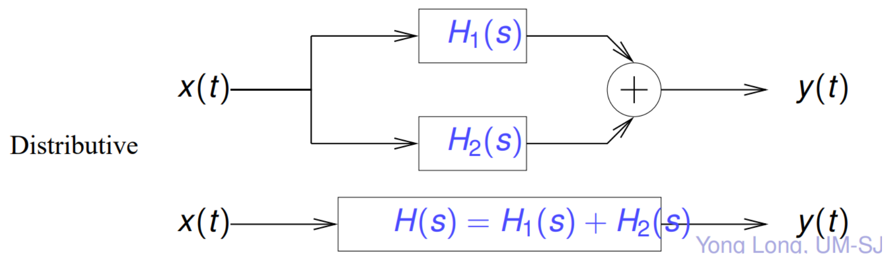
Parallel Interconnection

Thus the overall frequency response of two LTI systems connected in parallel is given by the sum of the frequency responses of the individual systems:

$$H(\omega) = H_1(\omega) + H_2(\omega).$$

The overall system (transfer) function is

$$H(s) = H_1(s) + H_2(s).$$



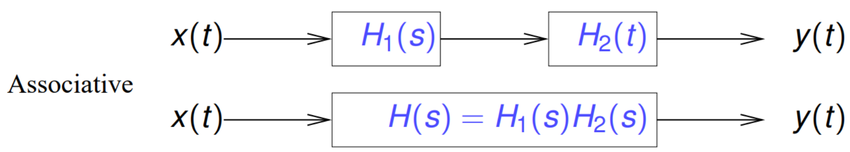
Series Interconnection

Thus the overall frequency response of two LTI systems connected in series is given by the *product* of the frequency responses of the individual systems:

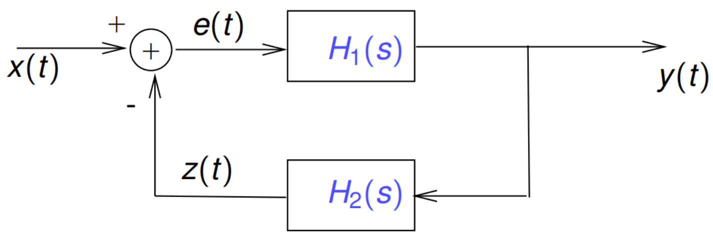
$$H(\omega) = H_1(\omega)H_2(\omega).$$

The overall system (transfer) function is

$$H(s) = H_1(s)H_2(s).$$



Feedback Interconnection



$$\begin{cases} y(t) = e(t) * h_1(t), \\ e(t) = x(t) - z(t), \\ z(t) = y(t) * h_2(t) \end{cases} \Longrightarrow y(t) = [x(t) - y(t) * h_2(t)] * h_1(t)$$

$$H(\omega) = \frac{Y(\omega)}{X(\omega)} = \frac{H_1(\omega)}{1 + H_1(\omega)H_2(\omega)}$$

$$H(s) = \frac{Y(s)}{X(s)} = \frac{H_1(s)}{1 + H_1(s)H_2(s)}$$

If you are not sure whether you can write the $H(s)$ according to the block diagram, take the above as an example and practice.

Exercise

Draw the block diagram for the causal LTI system with system function:

$$H(s) = \frac{1}{(s+1)(s+2)}$$

Use **Direct Form**, **Cascade Form**, and **Parallel From**

Answer:

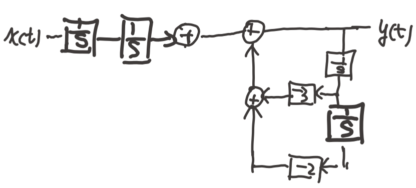
1. Direct Form

or use general form directly

$$H(s) = \frac{Y(s)}{X(s)} = \frac{1}{s^2+3s+2}$$

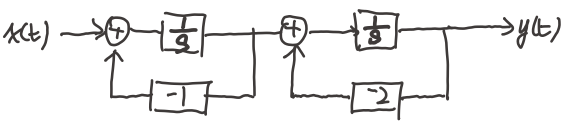
$$(s^2+3s+2)Y(s) = X(s)$$

$$Y(s) = \frac{1}{s^2}X(s) - \frac{3}{s}Y(s) - \frac{2}{s^2}Y(s)$$



2. Cascade Form

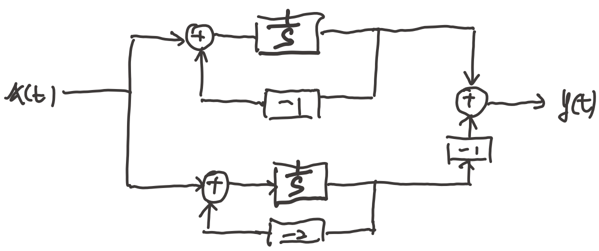
$$H(s) = \frac{1}{s+1} \cdot \frac{1}{s+2}$$



3. Parallel Form

PFE!

$$H(s) = \frac{1}{s+1} - \frac{1}{s+2}$$

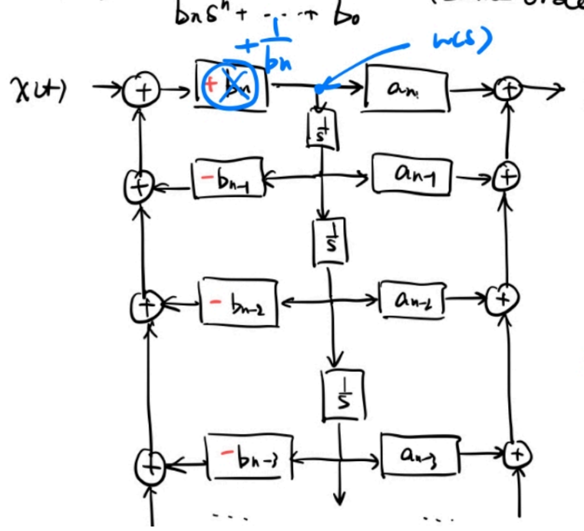


General block diagram for rational systems

Copy it on your CTPP!!!!!!!!!

more generalized.

$$H(s) = \frac{a_n s^n + \cdots + a_0}{b_n s^n + \cdots + b_0} \quad \text{(same order)}$$



$$H(s) = \frac{a_n + \cdots + a_0 s^{-n}}{b_n + \cdots + b_0 s^{-n}}$$

$$\frac{Y(s)}{W(s)} = a_n + \cdots + a_0 s^{-n}$$

$$\frac{W(s)}{X(s)} = \frac{1}{b_n + \cdots + b_0 s^{-n}}$$

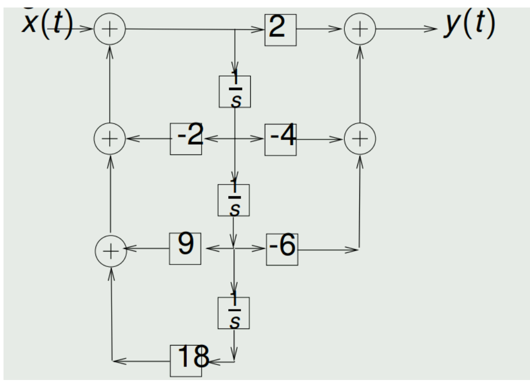
$$\Rightarrow W(s) = \frac{1}{b_n}\left(X(s) - b_{n-1} s^{-1} W(s) - \cdots - b_0 s^{-n} W(s)\right)$$

$$\Rightarrow \boxed{\frac{1}{b_n}}$$

- $m = n$: Just apply the method.
- $m < n$: e.g. $H(s) = \frac{1}{s^2+3s+2}$, we just first fill all the item in the upper to make the upper and lower part are at the same order. Finally **remove the branch** with zero coefficient.
- $m > n$: e.g. $H(s) = s$. It has poles at $s = \infty$, which makes the system nonproper (All rational but non-proper systems are unstable!!), thus **won't appear in real cases**.

Exercise

Consider the causal signal processing system described by the following block diagram.

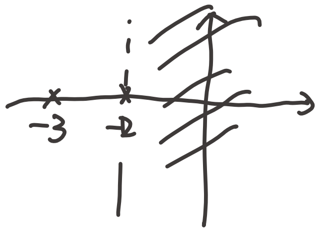


Step 1: Find H(s) according to block diagram

$$H(s) = \frac{2s^3 - 4s^2 - 6s}{s^3 + 2s^2 - 9s - 18}$$

Step 2: find zeros and poles

$$H(s) = \frac{2s(s+1)}{(s+2)(s+3)}$$



1. Determine if this system is BIBO stable. Explain your answer.
2. Find the impulse response of this system.
3. Find the differential equation corresponding to this system.
4. Given the input signal $x(t) = (e^{-t}u(t)) * u(t)$, find $Y(s)$ and its ROC.
5. Find the output signal $y(t)$.

$\frac{1}{s+1}$ $Re\{s\} > -1$ $\quad$ $\frac{1}{s}$ $Re\{s\} > 0$

1. since $-3, -2 < 0$ stable

2. $H(s) = \frac{2s(s+1)}{(s+2)(s+3)}$ ROC: $Re\{s\} > -2$

To solve $h(t)$

$$H(s) = 2 - \frac{8s+12}{(s+2)(s+3)}$$

$$\frac{a}{s+2} + \frac{b}{s+3}$$

$$a = \frac{8s+12}{s+3}\Big|_{s=-2} = \frac{-4}{1} = -4$$

$$b = \frac{8s+12}{s+2}\Big|_{s=-3} = 12$$

$$H(s) = 2 + \frac{4}{s+2} - \frac{12}{s+3} \quad \text{ROC: } Re\{s\} > -2$$

$$\Updownarrow$$

$$h(t) = 2\delta(t) + 4e^{-2t}u(t) - 12e^{-3t}u(t)$$

3. $H(s) = \frac{Y(s)}{X(s)} = \frac{2s^2+2s}{s^2+5s+6}$

$$(s^2+5s+6)Y(s) = (2s^2+2s)X(s)$$

$$\frac{d^2}{dt^2}y(t) + 5\frac{d}{dt}y(t) + 6y(t)$$

$$= 2\frac{d^2}{dt^2}x(t) + 2\frac{d}{dt}x(t)$$

4. $\underline{Y(s) = H(s)X(s)}$

Step 1: $X(s) = \frac{1}{s(s+1)}$ $Re\{s\} > 0$

Step 2: $Y(s) = \frac{2}{(s+2)(s+3)}$

$$= \frac{2}{s+2} - \frac{2}{s+3} \quad Re\{s\} > -2$$

5. $y(t) = 2e^{-2t}u(t) - 2e^{-3t}u(t)$

A causal LTI system with impulse response $h(t)$ has the following properties:

(1) • When the input to the system is $x(t) = e^{2t}$ for all t, the output is $y(t) = \frac{1}{6}e^{2t}$ for all t. $\in$

(2) • The impulse response $h(t)$ satisfies the differential equation
$\frac{dh(t)}{dt} + 2h(t) = (e^{-4t})u(t) + bu(t)$, where b is an unknown constant.

Determine the system function $H(s)$ of the system, consistent with the information above. There should be no unknown constants in your answer; that is, the constant b should not appear in the answer.

$\boxed{\text{Wrong}}$ version

$x(t) = e^{2t}$ $\quad y(t) = \frac{1}{6}e^{2t}$

does not apply for all input !!!

$\cancel{y(t) = \frac{1}{6}x(t)}$ ✗

$\cancel{H(s) = \frac{1}{6}}$

$x(t) \longrightarrow \boxed{H(s)} \longrightarrow y(t)$

(1) $x(t) = e^{2t} \longrightarrow \boxed{H(s)} \longrightarrow \frac{1}{6}e^{2t}$

$$e^{2t} \longrightarrow H(2)e^{2t} = \frac{1}{6}e^{2t}$$

$$H(2) = \frac{1}{6}$$

(2) $sH(s) + 2H(s) = \frac{1}{s+4} + b\frac{1}{s}$ $\quad Re\{s\} > 0$

$$H(s) = \frac{s + b(s+4)}{s(s+4)(s+2)}$$

$$H(2) = \frac{2+6b}{2(2+4)(2+2)} = \frac{1}{6}$$

$$\Rightarrow b = 1$$

$$H(s) = \frac{2}{s(s+4)}$$

Notes

Before the exam:

1. **Review all the quizzes! Review all the quizzes! Review all the quizzes!**
2. Review homework answers.
3. Gain proficiency in calculation (PFE, Common Laplace Transform, differential equation, etc.).

During the exam:

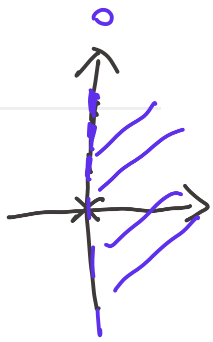
1. For solving differential equation problem, consider Laplace Transform first. Do NOT use time domain methods to solve it!
2. **Always judge and specify the ROC.**

Common Mistakes:

1. Marginal stable

Not stable

since does not include $j\omega$ axis.



Reference

1. Hu fan. 216 final recitation class.
2. Long yong. 216 slides.