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What is Laplace Transform

$$\text{Fourier Transform} \\ \int_{-\infty}^{\infty} f(t)e^{-j\omega t} dt$$

The Laplace Transform (LT) is the **generalization** of the Fourier Transform to **include general complex exponential signals** of the form

$$e^{st} = e^{(\sigma + j\omega)t} = e^{\sigma t} e^{j\omega t} \quad \xleftrightarrow{\text{compare}} \quad e^{j\omega t}$$

We can generalize because e^{st} also follows:

$$e^{st} \xrightarrow{\tau} H(s)e^{st} = e^{st} \int_{-\infty}^{\infty} h(\tau)e^{-s\tau} d\tau \quad \xleftrightarrow{\text{compare}} \quad e^{j\omega t} \rightarrow H(j\omega)e^{j\omega t}$$

👉 For **intuitive understanding of Laplace Transform**, you can refer to the following video, which has a lot of 3D plot visualizations to help you better understand! 🎥

<https://www.youtube.com/watch?v=n2y7n6jw5d0>

Bilateral Laplace Transform

In this course, we only concern **Bilateral Laplace Transform**.

Bilateral Laplace Transform means that the integral starts from negative infinity to positive infinity.

The formula is

$$X(s) = \int_{-\infty}^{\infty} x(t)e^{-st} dt$$

$$x(t) = \frac{1}{2\pi j} \int_{\sigma-j\infty}^{\sigma+j\infty} X(s)e^{st} ds$$

$$s = \sigma + j\omega$$

LT pairs can be denoted as:

$$x(t) \leftrightarrow X(s) \quad \& \quad ROC \quad \text{always specify ROC!}$$

Typically, you should **RARELY** use the above 2 equations during the exam. Instead, please use common LT pairs on the exam paper to simplify the calculation.

Relationship between Laplace Transform & Fourier Transform

We can derive the following relationship:

$$X(s)|_{s=j\omega} = \mathcal{F}\{x(t)\}$$

$$X(s) = \mathcal{F}\{x(t)e^{-\sigma t}\}, \text{ where } s = \sigma + j\omega$$

Summary:

Fourier Transform is a **slice** of Laplace transform.

Region of Convergence (ROC)

Concept

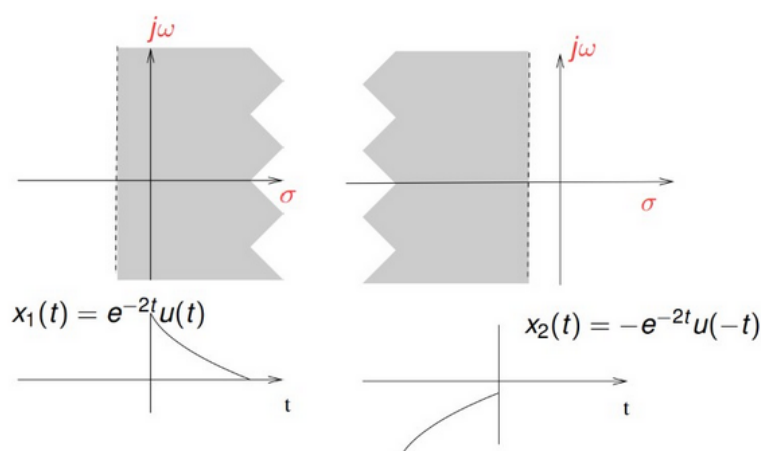
ROC is **the set of values of s** on the complex plane for which the **bilateral Laplace transform is guaranteed to exist/converge**. That is,

$$ROC = \{s \in C : \int_{-\infty}^{\infty} |x(t)|e^{-\text{real}\{s\}t} dt < \infty\}$$

To define a **pair of Laplace Transform**, you **have to** specify its **ROC**!

Plot

Often we display the ROC of a signal using the **complex s-plane** as shown in the following figures.



- The horizontal axis is called σ axis, and the vertical axis is called $j\omega$ axis.
- The shaded region indicates ROC.

- Dotted lines is used to indicate boundaries if ROC doesn't include its edges.
- **If the shaded region includes the $j\omega$ axis, then the FT of the signal exists.**

Poles and Zeros for Rational Laplace Transform

If the Laplace transform of a signal $x(t)$, $X(s)$, has the form:

$$X(s) = \frac{N(s)}{D(s)} = \frac{\sum_{m=0}^M b_m s^m}{\sum_{n=0}^N a_n s^n} = G \frac{\prod_{k=1}^M (s - z_k)}{\prod_{k=1}^N (s - p_k)}$$

- **Zeros:** (z_1, z_2, \dots, z_M) , sometimes infinity
- **Poles:** (p_1, p_2, \dots, p_N) , sometimes infinity
- **Gain:** $G = \frac{b_M}{a_N}$

Questions:

1. What do zeros & poles mean?

Let z_1 be zero, and p_1 be pole, we can obtain

$$X(z_1) = 0$$

$$X(p_1) = \infty$$

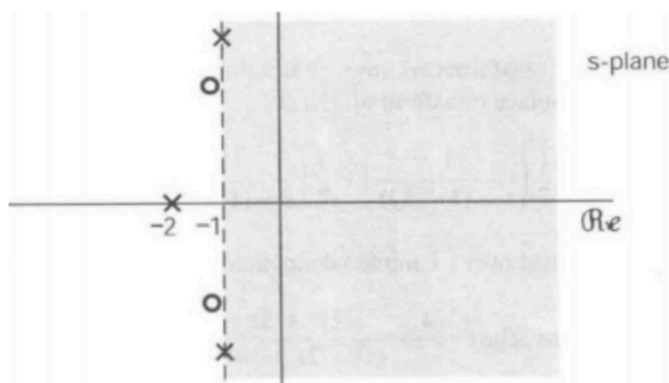
2. How to judge zeros/poles at ∞ ?

If $\lim_{s \rightarrow \infty} H(s) = 0$, there is a zero at infinity.

If $\lim_{s \rightarrow \infty} H(s) \rightarrow \infty$, there is a pole at infinity.

Pole-zero plot

Using 'o' to represent zeros, using 'x' to represent poles



Property of ROC

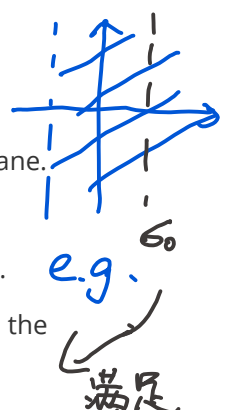
This part is used to judge the ROC of Laplace Transform.

- The ROC of $X(s)$ consists of **stripe-shaped** regions parallel to the $j\omega$ axis in the s plane.
- The ROC of rational LT does not contain any poles.
- If $x(t)$ has a finite duration and is absolutely integrable, the ROC is the whole s plane.
- If $x(t)$ is **right sided** (remaining zeros when $t < t_0$), and if the line $\text{Re}\{s\} = \sigma_0$ is in the ROC, then all values of s whose $\text{Re}\{s\} > \sigma_0$ will also be in the ROC.

$x(t)$ right-sided



ROC



- If $x(t)$ is **left sided**, and if the line $\text{Re}\{s\} = \sigma_0$ is in the ROC, then all values of s whose $\text{Re}\{s\} < \sigma_0$ will also be in the ROC.
- If $x(t)$ is **two sided**, and if the line $\text{Re}\{s\} = \sigma_0$ is in the ROC, then the ROC is composed of a stripe-shaped region including the line $\text{Re}\{s\} = \sigma_0$ on the s plane.
- If $X(s)$ is rational, its ROC is constrained by the poles or extends to infinity.
- If $X(s)$ is rational, and if $x(t)$ is right sided, the ROC is on the right of the rightmost pole.
- If $X(s)$ is rational and left sided, its ROC is on the left of the leftmost pole.

You can refer to this video by Prof. Oppenheim for detailed explanation, which derive from scratch: https://www.bilibili.com/video/BV1SB4y1C7yx?p=20&vd_source=0be2819c805c6da41df6967754921b05

ROC and System Properties

The Laplace transform $H(s)$ of a system's impulse response $h(t)$ is called its system function.

- Stability $\Leftrightarrow j\omega$ axis in ROC $\Leftrightarrow h(t)$ absolutely integrable
- Causality $\Leftrightarrow h(t)$ is a right-sided signal \Leftrightarrow ROC of the system function $H(s)$ is the right-half plane (RHP) to the right of the rightmost pole.
- Stability and Causality \Leftrightarrow All of its poles ^{lie} strictly within the LHP (left-half of the s -plane).
- **CAUTION:** Consider poles at $\pm\infty$! e.g. $H(s) = s$ has a pole at $s = \infty$.
- All ~~improper~~ ^{nonproper} systems are unstable! $(m < n)$
- **A diffeq system is stable** iff all roots of its characteristic polynomial are in the left half plane.

Geometric Properties of FT from pole-zero plot

Given the pole-zero plot corresponding to the transfer function $H(s)$ of an LTI system, we can sketch the magnitude response $|H(\omega)|$ and phase response $\angle H(\omega)$:

$$H(\omega) = G \frac{(j\omega - z_1) \cdot \dots \cdot (j\omega - z_m)}{(j\omega - p_1) \cdot \dots \cdot (j\omega - p_n)}$$

Magnitude response:

$$|H(\omega)| = G \frac{|j\omega - z_1| \cdot \dots \cdot |j\omega - z_m|}{|j\omega - p_1| \cdot \dots \cdot |j\omega - p_n|}$$

Phase response:

$$\angle H(\omega) = \angle G + \angle(j\omega - z_1) + \dots + \angle(j\omega - z_m) - \angle(j\omega - p_1) - \dots - \angle(j\omega - p_n)$$

See video by Oppenheim if you don't understand

Common Laplace Transform & Inverse Laplace Transform

Common Laplace Transform

Table of Laplace transform pairs

$f(t)$	$F(s)$	ROC
$\delta(t)$	1	$\forall s$
$u(t)$	$\frac{1}{s}$	$\text{real}\{s\} > 0$
$t^n u(t)$	$\frac{n!}{s^{n+1}}$	$\text{real}\{s\} > 0$
$e^{-at} u(t)$	$\frac{1}{s+a}$	$\text{real}\{s\} > \text{real}\{-a\}$
$-e^{-at} u(-t)$	$\frac{1}{s+a}$	$\text{real}\{s\} < \text{real}\{-a\}$
$t^n e^{-at} u(t)$	$\frac{n!}{(s+a)^{n+1}}$	$\text{real}\{s\} > \text{real}\{-a\}$

$f(t)$	$F(s)$	notes
$\sin(\omega_0 t) u(t)$	$\frac{\omega_0}{s^2 + \omega_0^2}$	$\text{real}\{s\} > 0$
$\cos(\omega_0 t) u(t)$	$\frac{s}{s^2 + \omega_0^2}$	$\text{real}\{s\} > 0$
$e^{-at} \cos(\omega_0 t) u(t)$	$\frac{s+a}{(s+a)^2 + \omega_0^2}$	$\text{real}\{s\} > \text{real}\{-a\}$
$e^{-at} \sin(\omega_0 t) u(t)$	$\frac{\omega_0}{(s+a)^2 + \omega_0^2}$	$\text{real}\{s\} > \text{real}\{-a\}$
$u_n(t) = \frac{d^n}{dt^n} \delta(t)$	s^n	$\forall s$
$u_{-n}(t) = \underbrace{u(t) * \dots * u(t)}_{n \text{ times}}$	$\frac{1}{s^n}$	$\text{real}\{s\} > 0$

Property of Laplace Transform

Properties of the Laplace Transform

	Time	Laplace	ROC (of result)
Linearity	$a_1 f_1(t) + a_2 f_2(t)$	$a_1 F_1(s) + a_2 F_2(s)$	contains $\text{ROC}_1 \cap \text{ROC}_2$
Time shift	$f(t - \tau)$	$e^{-s\tau} F(s)$	same
Time-scaling	$f(at), a \neq 0$	$\frac{1}{ a } F\left(\frac{s}{a}\right)$	ROC/a
Time reversal	$f(-t)$	$F(-s)$	$-\text{ROC}$
Convolution	$f_1(t) * f_2(t)$	$F_1(s) \cdot F_2(s)$	contains $\text{ROC}_1 \cap \text{ROC}_2$
Frequency shift	$f(t) e^{j\omega_0 t}$	$F(s - j\omega_0)$	same
Frequency shift	$f(t) e^{s_0 t}$	$F(s - s_0)$	$\text{ROC} + \text{real}\{s_0\}$
Time Differentiation	$\frac{d^n}{dt^n} f(t)$	$s^n F(s)$	contains ROC
s-domain Differentiation	$(-t)^n f(t)$	$\frac{d^n}{ds^n} F(s)$	same
Integration	$\int_{-\infty}^t f(\tau) d\tau = f(t) * u(t)$	$\frac{1}{s} F(s)$	contains $\text{ROC} \cap \{\text{real}\{s\} > 0\}$
DC Value	$\int_{-\infty}^{\infty} f(t) dt = F(0)$		must contain $s = 0$

✦ Inverse Laplace Transform

Requirement: Given rational $X(s)$ and **ROC**, we are required to invert it to find signal $x(t)$.

Steps:

- Decompose $X(s)$ into **the sum of atomic expressions** by doing **PFE**.
- For each expression, find ROC. Select the one including the ROC of LT.
- For each expression, use LT table and LT properties to get the result. Then sum them up.
- CAUTION:** If the ROC is not given explicitly, discuss all the situations!!!

There are cases that given the pole-zero plot and DC value, and ask you to find $x(t)$. In this case, you need to find the gain G with certain points.

Exercise

The system function of a causal LTI system is

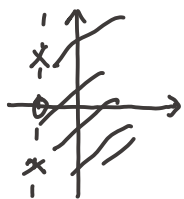
$$H(s) = \frac{s+1}{s^2+2s+2}$$

Determine and sketch the response ($y(t)$) when the input is

$$e^{-|t|}, \quad -\infty < t < \infty$$

Hint: You can use $e^{-|t|} = e^{-t}u(t) + e^t u(-t)$

Answer:



D. PFE

$$\begin{aligned} x(t) &= e^{-|t|} = e^{-t}u(t) + e^t u(-t) \\ X(s) &= \frac{1}{s+1} - \frac{1}{s-1} = \frac{-2}{(s+1)(s-1)}, \quad -1 < \text{Re}\{s\} < 1 \\ H(s) &= \frac{s+1}{s^2+2s+2} = \frac{s+1}{(s+1)^2+1}, \quad \text{Re}\{s\} > -1 \\ Y(s) &= H(s)X(s) = \frac{-2}{(s-1)(s^2+2s+2)}, \quad -1 < \text{Re}\{s\} < 1 \\ Y(s) &= \frac{-2/s}{s-1} + \frac{2}{8} \left[\frac{s+1}{(s+1)^2+1} \right] + \frac{4}{8} \left[\frac{1}{(s+1)^2+1} \right] \\ y(t) &= \frac{2}{5} e^t u(-t) + \frac{2}{5} e^{-t} \cos(t) u(t) + \frac{4}{5} e^{-t} \sin(t) u(t) \end{aligned}$$

✦ Differential Equation

Solve **algebraic expression** by Linear Constant Coefficient Differential Equation.

Solve **ROC** by stability & causality & etc. conditions.

Exercise

The system function of a causal LTI system is

$$H(s) = \frac{s+1}{s^2+2s+2}$$

Express $H(s)$ as a differential equation.

$$\begin{aligned} H(s) &= \frac{Y(s)}{X(s)} = \frac{s+1}{s^2+2s+2} \\ (s^2+2s+2)Y(s) &= (s+1)X(s) \\ \frac{d^2 y(t)}{dt^2} + 2 \frac{dy(t)}{dt} + 2y(t) &= \frac{d}{dt} x(t) + x(t) \end{aligned}$$

Block Diagram

In this part, you should be capable of **both**:

1. Writing $H(s)$ according to the block diagram. Then analyze the system with $H(s)$. e.g. How to solve the Impulse Response $h(t)$? etc.
2. Draw the block diagram according to $H(s)$.

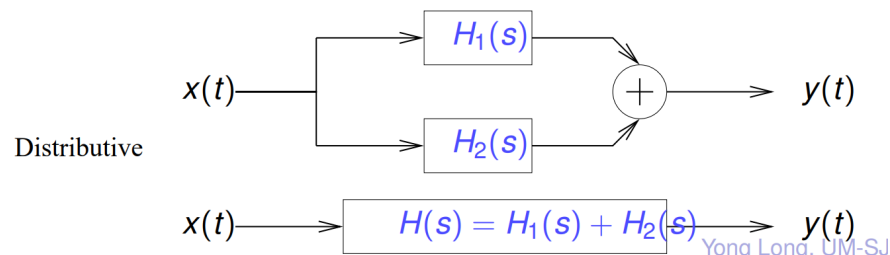
Parallel Interconnection

Thus the overall **frequency response** of two LTI systems connected in parallel is given by the **sum** of the frequency responses of the individual systems:

$$H(\omega) = H_1(\omega) + H_2(\omega).$$

The overall **system (transfer) function** is

$$H(s) = H_1(s) + H_2(s).$$



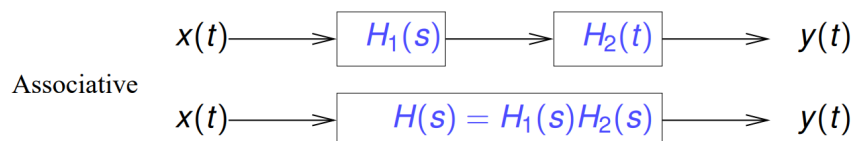
Series Interconnection

Thus the overall **frequency response** of two LTI systems connected in series is given by the **product** of the frequency responses of the individual systems:

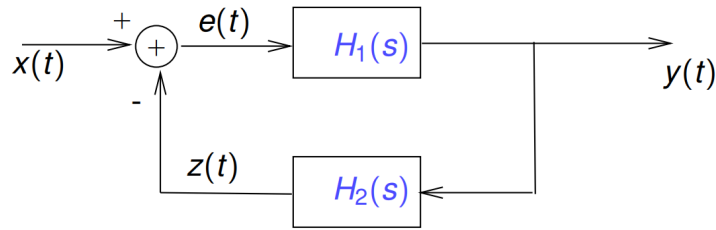
$$H(\omega) = H_1(\omega)H_2(\omega).$$

The overall **system (transfer) function** is

$$H(s) = H_1(s)H_2(s).$$



Feedback Interconnection



$$\begin{cases} y(t) = e(t) * h_1(t), \\ e(t) = x(t) - z(t), \\ z(t) = y(t) * h_2(t) \end{cases} \implies y(t) = [x(t) - y(t) * h_2(t)] * h_1(t)$$

$$H(\omega) = \frac{Y(\omega)}{X(\omega)} = \frac{H_1(\omega)}{1 + H_1(\omega)H_2(\omega)}$$

$$H(s) = \frac{Y(s)}{X(s)} = \frac{H_1(s)}{1 + H_1(s)H_2(s)}$$

If you are not sure whether you can write the $H(s)$ according to the block diagram, take the above as an example and practice.

Exercise

Exercise 1

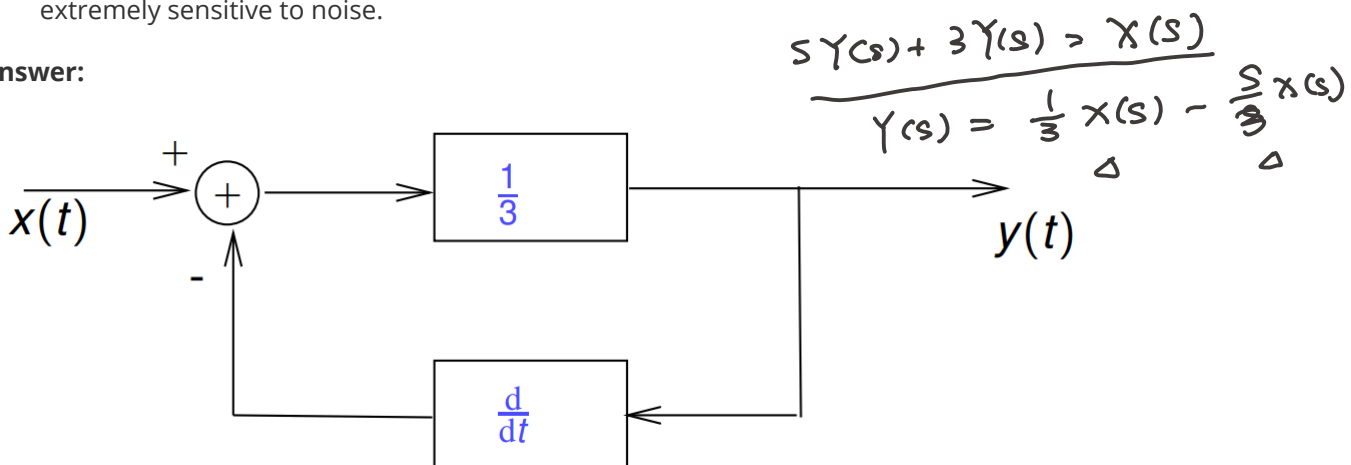
Draw the block diagram of the causal LTI system with system function:

$$H(s) = \frac{1}{s+3}$$

$$\frac{Y(s)}{X(s)} = \frac{1}{s+3}$$

1. Use differentiator to implement. Mention, the differentiator is both difficult to implement and extremely sensitive to noise.

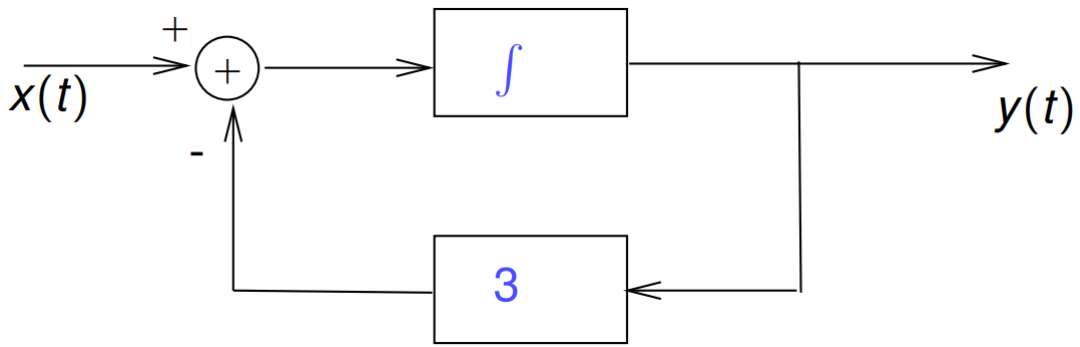
Answer:



2. Use integrator to implement.

Answer:

$$sY(s) = X(s) - 3Y(s)$$



For normal cases, please use integrator ($\frac{1}{s}$) to draw the diagram

Exercise 2:

Draw the block diagram for the causal LTI system with system function:

$$H(s) = \frac{1}{(s+1)(s+2)}$$

Use **Direct Form**, **Cascade Form**, and **Parallel Form**

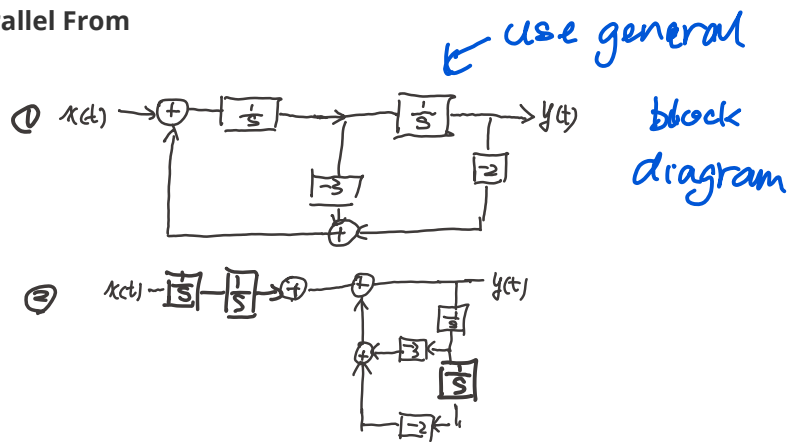
Answer:

1. Direct Form

$$H(s) = \frac{Y(s)}{X(s)} = \frac{1}{s^2 + 3s + 2}$$

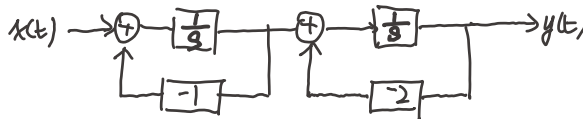
$$(s^2 + 3s + 2)Y(s) = X(s)$$

$$Y(s) = \frac{1}{s^2} X(s) - \frac{3}{s} Y(s) - \frac{2}{s} Y(s)$$



2. Cascade Form

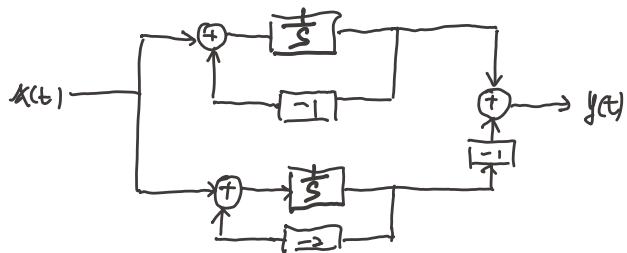
$$H(s) = \frac{1}{s+1} \cdot \frac{1}{s+2}$$



3. Parallel Form

PFE!

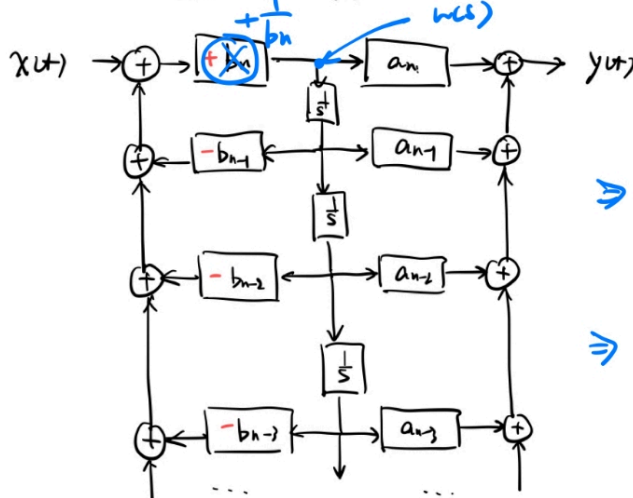
$$H(s) = \frac{1}{s+1} - \frac{1}{s+2}$$



General block diagram for rational systems

more generalized.

$$H(s) = \frac{a_n s^n + \dots + a_0}{b_n s^n + \dots + b_0} \quad (\text{same order})$$



$$H(s) = \frac{a_n + \dots + a_0 s^{-n}}{b_n + \dots + b_0 s^{-n}}$$

$$\frac{Y(s)}{W(s)} = a_n + \dots + a_0 s^{-n}$$

$$\frac{w(s)}{x(s)} = \frac{1}{b_n + \dots + b_0 s^{-n}}$$

$$\Rightarrow W(s) = \frac{1}{b_n} (x(s) - b_{n-1}s^{-1}w(s) - \dots - b_0s^{-n}w(s))$$

$$\Rightarrow \boxed{\frac{1}{bn}}$$

- $m = n$: Just apply the method.
- $m < n$: e.g. $H(s) = \frac{1}{s^2+3s+2}$, we just first fill all the item in the upper to make the upper and lower part are at the same order. Finally **remove the branch** with zero coefficient.
- $m > n$: e.g. $H(s) = s$. It has poles at $s = \infty$, which makes the system improper (All rational but improper systems are unstable!!), thus **won't appear in real cases**.

Notes

Before the exam:

1. **Review all the quizzes!**
2. Review homework answers.
3. Gain proficiency in calculation (PFE, Common Laplace Transform, differential equation, etc.).

During the exam:

1. For solving differential equation problem, consider Laplace Transform first. Do NOT use time domain methods to solve it!
2. **Always judge and specify the ROC.**

Reference

1. Hu fan. 216 final recitation class.
2. Long yong. 216 slides.