Chapter 9 - Laplace Transform

Laplace Transform

The Laplace Transform (LT) is the **generalization** of the Fourier Transform to include general complex exponential signals of the form

$$e^{st} = e^{(\sigma+j\omega)t} = e^{\sigma t}e^{j\omega t}$$

We can generalize because e^{st} also follows:

$$e^{st} \xrightarrow{\tau} H(s)e^{st} = e^{st}\int_{-\infty}^{\infty} h(\tau)e^{-s\tau}d\tau$$

Laplace Transforms come in two flavors:

- Bilateral Laplace Transform (two-sided)
- Unilateral Laplace Transform (one-sided)

In this course, we only concern **Bilateral Laplace Transform**.

Bilateral Laplace Transform

Bilateral Laplace Transform means that the integral starts from negative infinity to positive infinity.

The formula is

$$X(s) = \int_{-\infty}^{\infty} x(t)e^{-st}dt$$
$$x(t) = \frac{1}{2\pi j}\int_{\sigma-j\infty}^{\sigma+j\infty} X(s)e^{st}ds$$
$$s = \sigma + j\omega$$

LT pairs can be denoted as:

$$x(t) \leftrightarrow X(s)$$

Relation between Laplace Transform & Fourier Transform

We can derive the following relationship:

$$X(s)|_{s=j\omega} = \mathcal{F}\{x(t)\}$$
$$X(s) = \mathcal{F}\{x(t)e^{-\sigma t}\}, \text{where } s = \sigma + j\omega$$

These two relationships can help you understand the properties below.

Region of Convergence (ROC)

Concept

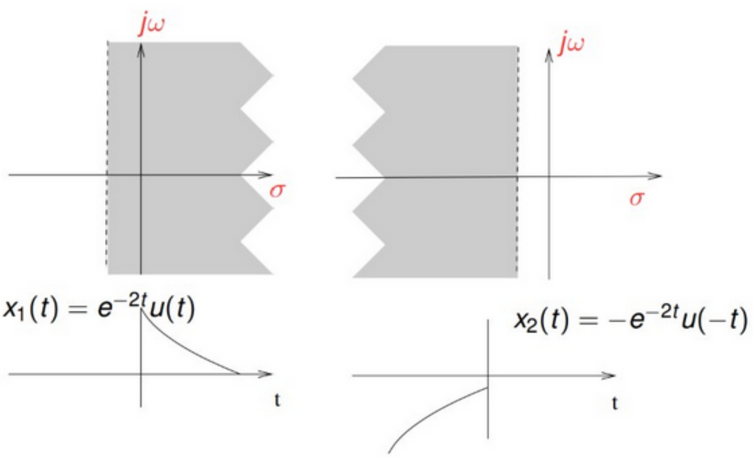
ROC is **the set of values of s** on the complex plane for which the **bilateral Laplace transform is guaranteed to exist/converge**. That is,

$$ROC = \{s \in C : \int_{-\infty}^{\infty} |x(t)|e^{-real\{s\}t}dt < \infty\}$$

To define a Laplace Transform, you **have to** specify its **ROC**!

Display

Often we display the ROC of a signal using the **complex s-plane** as shown in the following figures.



- The horizontal axis is called σ axis, and the vertical axis is called $j\omega$ axis.
- The shaded region indicates ROC.
- Dotted lines is used to indicate boundaries if ROC doesn't include its edges.
- **If the shaded region includes the $j\omega$ axis, then the FT of the signal exists.**

Rational Laplace Transform

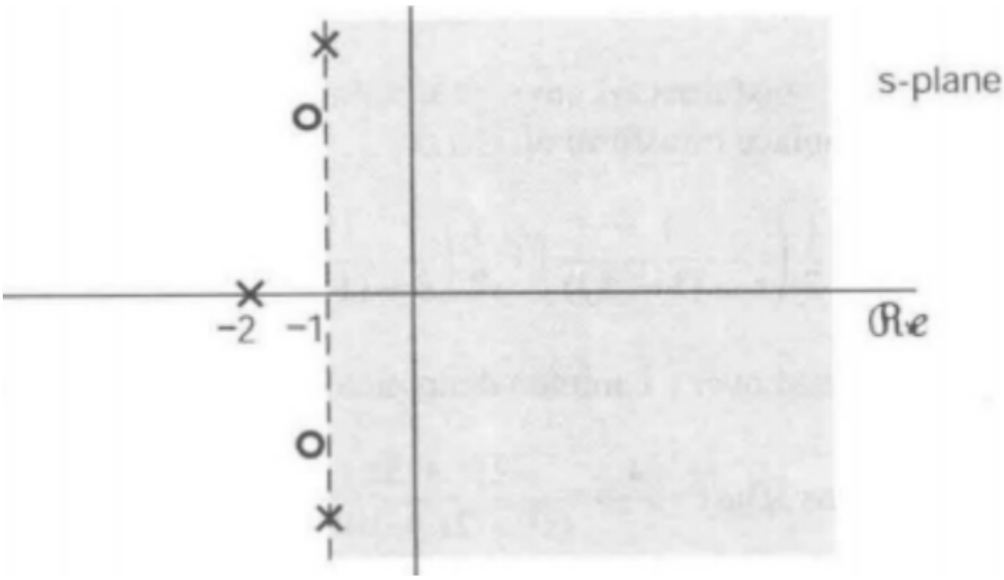
If the Laplace transform of a signal $x(t)$, $X(s)$, has the form:

$$X(s) = \frac{N(s)}{D(s)} = \frac{\sum_{m=0}^{M} b_m s^m}{\sum_{n=0}^{N} a_n s^n} = G\frac{\prod_{k=1}^{M}(s - z_k)}{\prod_{k=1}^{N}(s - p_k)}$$

- **Zeros**: (z_1,z_2, . . ., z_M), sometimes infinity
- **Poles**: (p_1, p_2, . . ., p_N), sometimes infinity
- **Gain**: G = $\frac{b_M}{a_N}$

Pole-zero plot

Using 'o' to represent zeros, using 'X' to represent poles



Property of ROC

- The ROC of $X(s)$ consists of **stripe-shaped** regions parallel to the $j\omega$ axis in the s plane.
- The ROC of rational LT does not contain any poles.
- If $x(t)$ has a finite duration and is absolutely integrable, the ROC is the whole s plane.
- If $x(t)$ is **right sided** (remaining zeros when $t < t_0$), and if the line $Re\{s\} = \sigma_0$ is in the ROC, then all values of s whose $Re\{s\} > \sigma_0$ will also be in the ROC.
- If $x(t)$ is **left sided**, and if the line $Re\{s\} = \sigma_0$ is in the ROC, then all values of s whose $Re\{s\} < \sigma_0$ will also be in the ROC.
- If $x(t)$ is **two sided**, and if the line $Re\{s\} = \sigma_0$ is in the ROC, then the ROC is composed of a stripe-shaped region including the line $Re\{s\} = \sigma_0$ on the s plane.
- If $X(s)$ is rational, its ROC is constrained by the poles or extents to infinity.
- If $X(s)$ is rational, and if $x(t)$ is right sided, the ROC is on the right of the rightest pole.
- If $X(s)$ is rational and left sided, its ROC is on the left of the leftest pole.

Explanation:

https://www.bilibili.com/video/BV1SB4y1C7yx?p=20&vd_source=0be2819c805c6da41df6967754921b05

ROC and System Properties

The Laplace transform $H(s)$ of a system's impulse response $h(t)$ is called its system function.

- Stability ⇔ $j\omega$ axis in ROC ⇔ $h(t)$ absolutely integrable
- Causality ⇔ $h(t)$ is a right-sided signal ⇔ ROC of the system function H(s) is RHP
- Stability and Causality ⇔ All of its poles lie strictly within the LHP.
- ***CAUTION***: Consider poles at ±∞! e.g. $H(s) = s$ has a pole at s = ∞.
- All improper systems are unstable!
- **A diffeq system is stable** iff all roots of its characteristic polynomial are in the left half plane.

Geometric Properties of FT from pole-zero plot

Given the pole-zero plot corresponding to the transfer function $H(s)$ of an LTI system, we can sketch the magnitude response $|H(\omega)|$ and phase response $\angle H(\omega)$:

$$H(\omega) = G\frac{(j\omega - z_1)\cdot \ldots \cdot (j\omega - z_m)}{(j\omega - p_1)\cdot \ldots \cdot (j\omega - p_n)}$$

Magnitude response:

$$|H(\omega)| = G\frac{|j\omega - z_1|\cdot \ldots \cdot |j\omega - z_m|}{|j\omega - p_1|\cdot \ldots \cdot |j\omega - p_n|}$$

Phase response:

$$\angle H(\omega) = \angle G + \angle(j\omega - z_1) + \ldots + \angle(j\omega - z_m) - \angle(j\omega - p_1) - \ldots - \angle(j\omega - p_n)$$

Common Laplace Transform & Inverse Laplace Transform

Common Laplace Transform

Table of Laplace transform pairs

$f(t)$	$F(s)$	ROC
$\delta(t)$	1	$\forall s$
$u(t)$	$\frac{1}{s}$	$\text{real}\{s\} > 0$
$t^n u(t)$	$\frac{n!}{s^{n+1}}$	$\text{real}\{s\} > 0$
$e^{-at}u(t)$	$\frac{1}{s+a}$	$\text{real}\{s\} > \text{real}\{-a\}$
$-e^{-at}u(-t)$	$\frac{1}{s+a}$	$\text{real}\{s\} < \text{real}\{-a\}$
$t^n e^{-at}u(t)$	$\frac{n!}{(s+a)^{n+1}}$	$\text{real}\{s\} > \text{real}\{-a\}$

$f(t)$	$F(s)$	notes
$\sin(\omega_0 t)\,u(t)$	$\frac{\omega_0}{s^2+\omega_0^2}$	$\text{real}\{s\} > 0$
$\cos(\omega_0 t)\,u(t)$	$\frac{s}{s^2+\omega_0^2}$	$\text{real}\{s\} > 0$
$e^{-at}\cos(\omega_0 t)\,u(t)$	$\frac{s+a}{(s+a)^2+\omega_0^2}$	$\text{real}\{s\} > \text{real}\{-a\}$
$e^{-at}\sin(\omega_0 t)\,u(t)$	$\frac{\omega_0}{(s+a)^2+\omega_0^2}$	$\text{real}\{s\} > \text{real}\{-a\}$
$u_n(t) = \frac{d^n}{dt^n}\delta(t)$	s^n	$\forall s$
$u_{-n}(t) = \underbrace{u(t) * \ldots * u(t)}_{\text{n times}}$	$\frac{1}{s^n}$	$\text{real}\{s\} > 0$

Property of Laplace Transform

Properties of the Laplace Transform

	Time	Laplace	ROC (of result)
Linearity	$a_1 f_1(t) + a_2 f_2(t)$	$a_1 F_1(s) + a_2 F_2(s)$	contains $\text{ROC}_1 \cap \text{ROC}_2$
Time shift	$f(t-\tau)$	$e^{-s\tau} F(s)$	same
Time-scaling	$f(at),\ a \neq 0$	$\frac{1}{\lvert a\rvert} F\left(\frac{s}{a}\right)$	ROC/a
Time reversal	$f(-t)$	$F(-s)$	$-\text{ROC}$
Convolution	$f_1(t) * f_2(t)$	$F_1(s) \cdot F_2(s)$	contains $\text{ROC}_1 \cap \text{ROC}_2$
Frequency shift	$f(t) e^{j\omega_0 t}$	$F(s - j\omega_0)$	same
Frequency shift	$f(t) e^{s_0 t}$	$F(s - s_0)$	$\text{ROC} + \text{real}\{s_0\}$
Time Differentiation	$\frac{d^n}{dt^n} f(t)$	$s^n F(\omega)$	contains ROC
s-domain Differentiation	$(-t)^n f(t)$	$\frac{d^n}{ds^n} F(\omega)$	same
Integration	$\int_{-\infty}^{t} f(\tau)\, d\tau = f(t) * u(t)$	$\frac{1}{s} F(s)$	contains $\text{ROC} \cap \{\text{real}\{s\} > 0\}$
DC Value	$\int_{-\infty}^{\infty} f(t)\, dt = F(0)$		must contain $s = 0$

Inverse Laplace Transform

Requirement: Given rational $X(s)$ and **ROC**, we are required to invert it to find signal $x(t)$.

Steps:

- Decompose $X(s)$ into **the sum of atomic expressions** by doing **PFE**.
- For each expression, find ROC. Select the one including the ROC of LT.
- For each expression, use LT table and LT properties to get the result. Then sum them up.
- ***CAUTION***: If the ROC is not given explicitly, discuss all the situations!!!

There are cases that given the pole-zero plot and DC value, and ask you to find x(t). In this case, you need to find the gain G with certain points.

Exercise

The system function of a causal LTI system is

$$H(s) = \frac{s+1}{s^2 + 2s + 2}$$

Determine and sketch the response (y(t)) when the input is

$$e^{-|t|}, \quad -\infty < t < \infty$$

Hint: You can use $e^{-|t|} = e^{-t}u(t) + e^{t}u(-t)$

Answer:

$$x(t) = e^{-|t|} = e^{-t}u(t) + e^{t}u(-t)$$

$$X(s) = \frac{1}{s+1} - \frac{1}{s-1} = \frac{-2}{(s+1)(s-1)}, \quad -1 < \Re\{s\} < 1$$

We also have

$$H(s) = \frac{s+1}{s^2+2s+2}$$

The poles of $H(s)$ are $-1 \pm j$, and since $h(t)$ is causal, we have that the ROC of $H(s)$ is $\Re\{s\} > -1$

$$Y(s) = H(s)X(s) = \frac{-2}{(s^2+2s+2)(s-1)}, \quad \text{ROC} : -1 < \Re\{s\} < 1$$

Rewrite it as

$$Y(s) = -\frac{2/5}{s-1} + \frac{2}{5}\left[\frac{s+1}{(s+1)^2+1}\right] + \frac{4}{5}\left[\frac{1}{(s+1)^2+1}\right]$$

we get,

$$y(t) = \frac{2}{5}e^{t}u(-t) + \frac{2}{5}e^{-t}\cos(t)u(t) + \frac{4}{5}e^{-t}\sin(t)u(t)$$

Differential Equation

How to solve

Solve **algebraic expression** by Linear Constant Coefficient Differential Equation.

Solve **ROC** by stability & causality & etc. conditions.

Block Diagram

In this part, you should be capable of:

1. Writing $H(s)$ according to the block diagram. Then analyze the system with $H(s)$. e.g. How to solve the Impulse Response $h(t)$? etc.
2. Draw the block diagram according to $H(s)$.

Parallel Interconnection

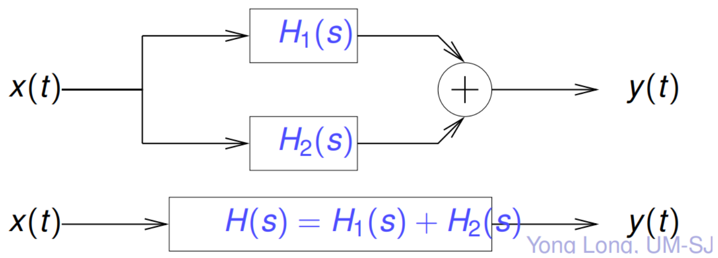
Thus the overall frequency response of two LTI systems connected in parallel is given by the sum of the frequency responses of the individual systems:

$$H(\omega) = H_1(\omega) + H_2(\omega).$$

The overall system (transfer) function is

$$H(s) = H_1(s) + H_2(s).$$

Distributive



Series Interconnection

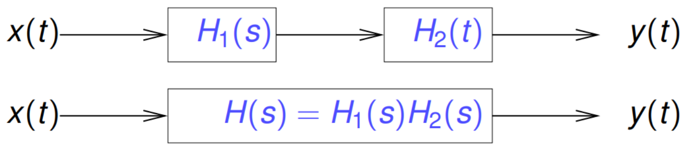
Thus the overall frequency response of two LTI systems connected in series is given by the *product* of the frequency responses of the individual systems:

$$H(\omega) = H_1(\omega)H_2(\omega).$$

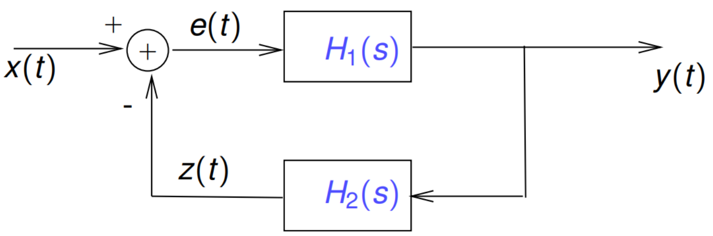
The overall system (transfer) function is

$$H(s) = H_1(s)H_2(s).$$

Associative



Feedback Interconnection



$$\begin{cases} y(t) = e(t) * h_1(t), \\ e(t) = x(t) - z(t), \\ z(t) = y(t) * h_2(t) \end{cases} \implies y(t) = [x(t) - y(t) * h_2(t)] * h_1(t)$$

$$H(\omega) = \frac{Y(\omega)}{X(\omega)} = \frac{H_1(\omega)}{1 + H_1(\omega)H_2(\omega)}$$

$$H(s) = \frac{Y(s)}{X(s)} = \frac{H_1(s)}{1 + H_1(s)H_2(s)}$$

If you are not sure whether you can write the $H(s)$ according to the block diagram, take the above as an example and practice.

Exercise

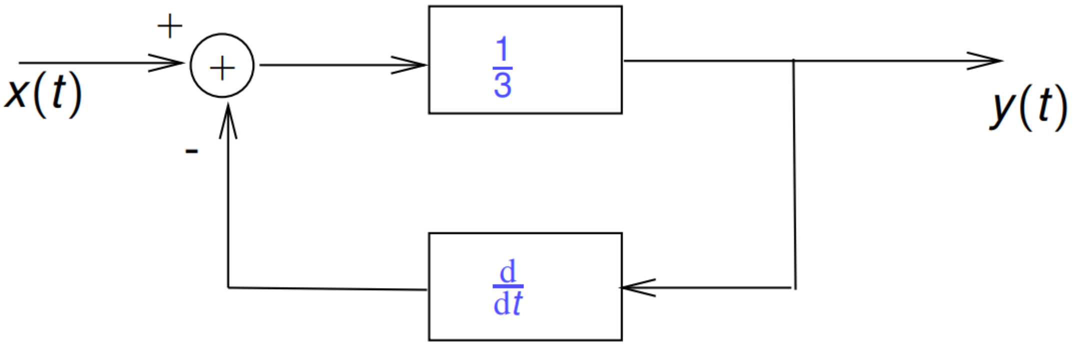
Exercise 1

Draw the block diagram of the causal LTI system with system function:

$$H(s) = \frac{1}{s+3}$$

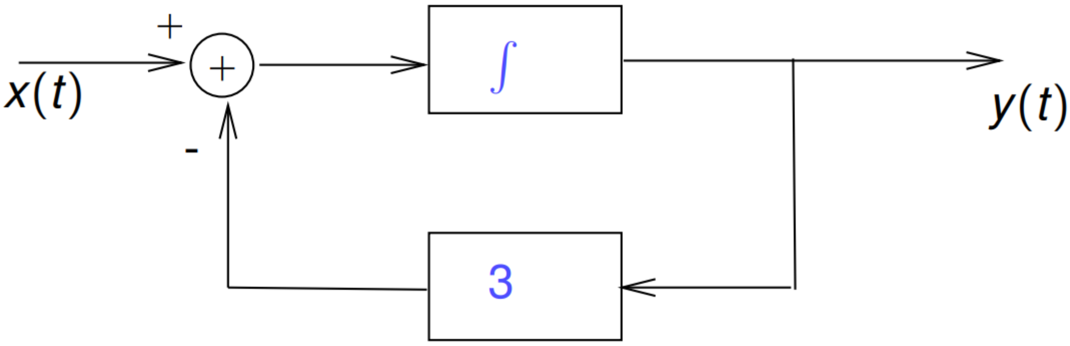
1. Use differentiator to implement. Mention, the differentiator is both difficult to implement and extremely sensitive to noise.

Answer:



2. Use integrator to implement.

Answer:



For normal cases, please use integrator ($\frac{1}{s}$) to draw the diagram

Exercise 2:

Draw the block diagram for the causal LTI system with system function:

$$H(s) = \frac{1}{(s+1)(s+2)}$$

Use **Direct Form**, **Cascade Form**, and **Parallel From**

Answer:

1. Direct Form

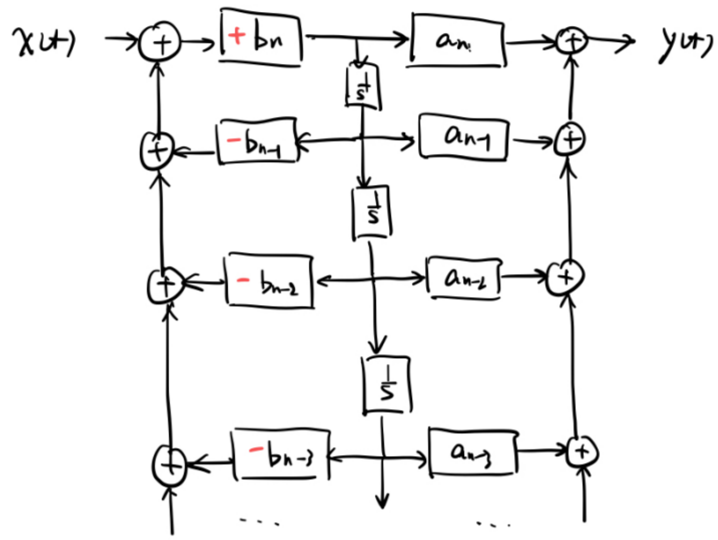
2. Cascade Form

3. Parallel From

General block diagram for rational systems

more generalized.

$$H(s) = \frac{a_n s^n + \cdots + a_0}{b_n s^n + \cdots + b_0} \quad \text{(same order)}$$



- $m = n$: Just apply the method.
- $m < n$: e.g. $H(s) = \frac{1}{s^2+3s+2}$, we just first fill all the item in the upper to make the upper and lower part are at the same order. Finally **remove the branch** with zero coefficient.
- $m > n$: e.g. $H(s) = s$. It has poles at $s = \infty$, which makes the system improper (All rational but improper systems are unstable!!), thus **won't appear in real cases**.

Exercise

Exercise 1:

Suppose we are given the following information about a causal and stable LTI system S with impulse response $h(t)$ and a rational system function $H(s)$:

(1) When the input to the system is $x(t) = e^{-3t}$ for all t, the output is $y(t) = -e^{-3t}$ for all t.

(2) When the input is $u(t)$, the output is absolutely integrable.

(3) When the input is $tu(t)$, the output is not absolutely integrable.

(4) The signal $\frac{d^2}{dt^2}h(t) + 4\frac{d}{dt}h(t) + 1h(t)$ is of finite duration.

(5) $H(s)$ has exactly one zero at infinity.

Determine $H(s)$ and its region of convergence.

How to solve problem: Given several information of $x(t)$ and $y(t)$, find $H(s)$

Steps:

1. eigenfunction e^{kt}

   $x(t) = e^{kt}$, then $y(t) = H(k)e^{kt}$
2. for certain $x(t)$, $y(t)$ is absolutely integrable/ or not.
   1. absolutely integrable => ROC contains $j\omega$ axis
   2. not absolutely integrable => ROC doesn't contain $j\omega$ axis
3. $y(t)$ is of finite duration

   => ROC is the whole plane

   => no poles exist for $Y(s)$
4. $H(s)$ has one zero at infinity

   => $n - m = 1$, where n is the degree of denominator
5. $H(s)$ has one pole at infinity

   => $m - n = 1$, where n is the degree of denominator

Exercise 2:

Suppose that the signal

$$y(t) = e^{-2t}u(t)$$

is the output of a causal system with the system function

$$H(s) = \frac{s-1}{s+1}$$

1. Find and plot two possible input $x(t)$ for the output $y(t)$.

2. If there exists a stable (but not necessarily causal) system, if it takes $y(t)$ as the input, its output will be $x(t)$. Which $x(t)$ is it? Find the impulse response of the system, and prove by directly calculate convolution.

Notes

Before the exam:

1. **Review all the quizzes!**
2. Review homework answers.
3. Gain proficiency in calculation (PFE, Common Laplace Transform, differential equation, etc.).
4. Check whether all the contents on summary sheet have been on your ctpp.

During the exam:

1. For solving differential equation problem, consider Laplace Transform first. Do NOT use time domain methods to solve it!
2. Always judge and specify the ROC.

Reference

1. Hu fan. 216 final recitation class.
2. Long yong. 216 slides.