Chapter 9 - Laplace Transform

Laplace Transform

The Laplace Transform (LT) is the **generalization** of the Fourier Transform to include general complex exponential signals of the form

$$e^{st} = e^{(\sigma+j\omega)t} = e^{\sigma t}e^{j\omega t}$$

We can generalize because e^{st} also follows:

$$e^{st} \xrightarrow{\tau} H(s)e^{st} = e^{st}\int_{-\infty}^{\infty}h(\tau)e^{-s\tau}d\tau$$

Laplace Transforms come in two flavors:

- Bilateral Laplace Transform (two-sided)

- Unilateral Laplace Transform (one-sided)

In this course, we only concern **Bilateral Laplace Transform**.

Bilateral Laplace Transform

Bilateral Laplace Transform means that the integral starts from negative infinity to positive infinity.

The formula is

$$X(s) = \int_{-\infty}^{\infty} x(t)e^{-st}dt$$

$$x(t) = \frac{1}{2\pi j}\int_{\sigma-j\infty}^{\sigma+j\infty} X(s)e^{st}ds$$

$$s = \sigma + j\omega$$

LT pairs can be denoted as:

$$x(t) \leftrightarrow X(s)$$

Relation between Laplace Transform & Fourier Transform

We can derive the following relationship:

$$X(s)|_{s=j\omega} = \mathcal{F}\{x(t)\}$$

$$X(s) = \mathcal{F}\{x(t)e^{-\sigma t}\}, \text{where } s = \sigma + j\omega$$

These two relationships can help you understand the properties below.

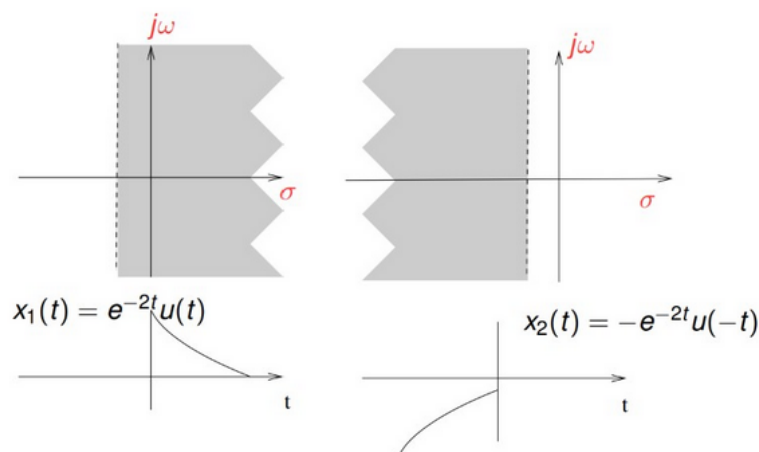# Region of Convergence (ROC)

Concept

ROC is **the set of values of s** on the complex plane for which the **bilateral Laplace transform is guaranteed to exist/converge**. That is,

$$ROC = \{s \in C : \int_{-\infty}^{\infty} |x(t)|e^{-real\{s\}t}dt < \infty\}$$

To define a Laplace Transform, you **have to** specify its **ROC**!

Display

Often we display the ROC of a signal using the **complex s-plane** as shown in the following figures.

- The horizontal axis is called σ axis, and the vertical axis is called $j\omega$ axis.
- The shaded region indicates ROC.
- Dotted lines is used to indicate boundaries if ROC doesn't include its edges.
- **If the shaded region includes the $j\omega$ axis, then the FT of the signal exists.**
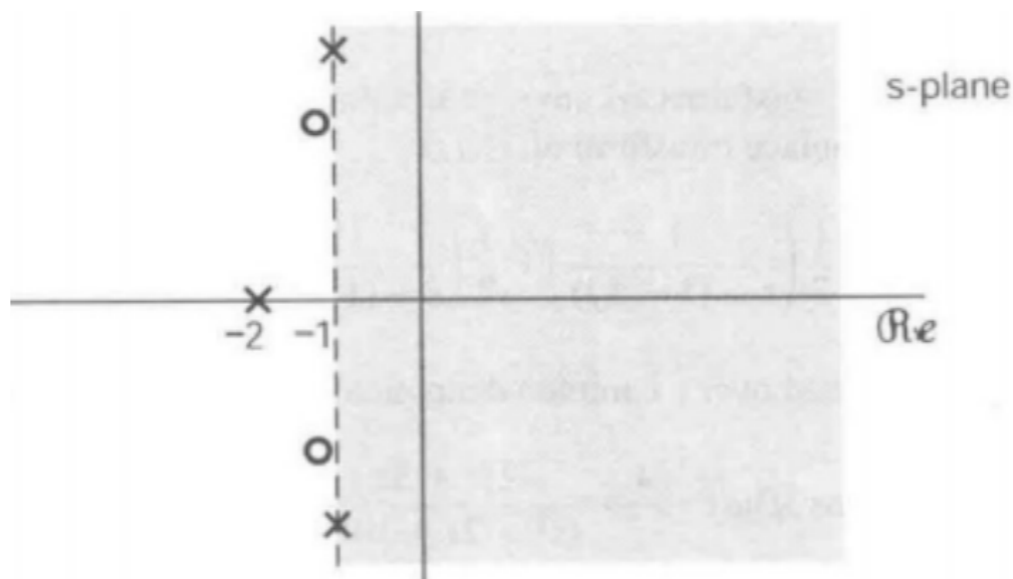
Rational Laplace Transform

If the Laplace transform of a signal $x(t)$, $X(s)$, has the form:

$$X(s) = \frac{N(s)}{D(s)} = \frac{\sum_{m=0}^{M} b_m s^m}{\sum_{n=0}^{N} a_n s^n} = G \frac{\prod_{k=1}^{M}(s - z_k)}{\prod_{k=1}^{N}(s - p_k)}$$

- **Zeros**: $(z_1, z_2, \ldots, z_M)$, sometimes infinity
- **Poles**: $(p_1, p_2, \ldots, p_N)$, sometimes infinity
- **Gain**: G = $\frac{b_M}{a_N}$

Pole-zero plot

Using 'o' to represent zeros, using 'X' to represent poles

Property of ROC

- The ROC of $X(s)$ consists of **stripe-shaped** regions parallel to the $j\omega$ axis in the s plane.
- The ROC of rational LT does not contain any poles.
- If $x(t)$ has a finite duration and is absolutely integrable, the ROC is the whole s plane.
- If $x(t)$ is **right sided** (remaining zeros when $t < t_0$), and if the line $Re\{s\} = \sigma_0$ is in the ROC, then all values of s whose $Re\{s\} > \sigma_0$ will also be in the ROC.
- If $x(t)$ is **left sided**, and if the line $Re\{s\} = \sigma_0$ is in the ROC, then all values of s whose $Re\{s\} < \sigma_0$ will also be in the ROC.
- If $x(t)$ is **two sided**, and if the line $Re\{s\} = \sigma_0$ is in the ROC, then the ROC is composed of a stripe-shaped region including the line $Re\{s\} = \sigma_0$ on the s plane.
- If $X(s)$ is rational, its ROC is constrained by the poles or extents to infinity.
- If $X(s)$ is rational, and if $x(t)$ is right sided, the ROC is on the right of the rightest pole.
- If $X(s)$ is rational and left sided, its ROC is on the left of the leftest pole.

Explanation:

ROC and System Properties

The Laplace transform $H(s)$ of a system's impulse response $h(t)$ is called its system function.

- Stability $\Leftrightarrow j\omega$ axis in ROC $\Leftrightarrow h(t)$ absolutely integrable
- Causality $\Leftrightarrow h(t)$ is a right-sided signal $\Leftrightarrow$ ROC of the system function H(s) is RHP
- Stability and Causality $\Leftrightarrow$ All of its poles lie strictly within the LHP.
- **CAUTION**: Consider poles at $\pm\infty$! e.g. $H(s) = s$ has a pole at s = ∞.
- All improper systems are unstable!
- **A diffeq system is stable** iff all roots of its characteristic polynomial are in the left half plane.

Geometric Properties of FT from pole-zero plot

Given the pole-zero plot corresponding to the transfer function $H(s)$ of an LTI system, we can sketch the magnitude response $|H(\omega)|$ and phase response $\angle H(\omega)$:

$$H(\omega) = G\frac{(j\omega - z_1) \cdot \ldots \cdot (j\omega - z_m)}{(j\omega - p_1) \cdot \ldots \cdot (j\omega - p_n)}$$

Magnitude response:

$$|H(\omega)| = G\frac{|j\omega - z_1| \cdot \ldots \cdot |j\omega - z_m|}{|j\omega - p_1| \cdot \ldots \cdot |j\omega - p_n|}$$

Phase response:

$$\angle H(\omega) = \angle G + \angle(j\omega - z_1) + \ldots + \angle(j\omega - z_m) - \angle(j\omega - p_1) - \ldots - \angle(j\omega - p_n)$$

Common Laplace Transform & Inverse Laplace Transform

Common Laplace Transform

Table of Laplace transform pairs

$f(t)$	$F(s)$	ROC
$\delta(t)$	1	$\forall s$
$u(t)$	$\dfrac{1}{s}$	$\text{real}\{s\} > 0$
$t^n u(t)$	$\dfrac{n!}{s^{n+1}}$	$\text{real}\{s\} > 0$
$e^{-at}u(t)$	$\dfrac{1}{s+a}$	$\text{real}\{s\} > \text{real}\{-a\}$
$-e^{-at}u(-t)$	$\dfrac{1}{s+a}$	$\text{real}\{s\} < \text{real}\{-a\}$
$t^n e^{-at}u(t)$	$\dfrac{n!}{(s+a)^{n+1}}$	$\text{real}\{s\} > \text{real}\{-a\}$

$f(t)$	$F(s)$	notes
$\sin(\omega_0 t)\, u(t)$	$\dfrac{\omega_0}{s^2 + \omega_0^2}$	$\text{real}\{s\} > 0$
$\cos(\omega_0 t)\, u(t)$	$\dfrac{s}{s^2 + \omega_0^2}$	$\text{real}\{s\} > 0$
$e^{-at}\cos(\omega_0 t)\, u(t)$	$\dfrac{s+a}{(s+a)^2 + \omega_0^2}$	$\text{real}\{s\} > \text{real}\{-a\}$
$e^{-at}\sin(\omega_0 t)\, u(t)$	$\dfrac{\omega_0}{(s+a)^2 + \omega_0^2}$	$\text{real}\{s\} > \text{real}\{-a\}$
$u_n(t) = \dfrac{d^n}{dt^n}\delta(t)$	s^n	$\forall s$
$u_{-n}(t) = \underbrace{u(t) * \ldots * u(t)}_{n \text{ times}}$	$\dfrac{1}{s^n}$	$\text{real}\{s\} > 0$

Property of Laplace Transform

Properties of the Laplace Transform

	Time	Laplace	ROC (of result)
Linearity	$a_1 f_1(t) + a_2 f_2(t)$	$a_1 F_1(s) + a_2 F_2(s)$	contains $\mathrm{ROC}_1 \cap \mathrm{ROC}_2$
Time shift	$f(t - \tau)$	$e^{-s\tau} F(s)$	same
Time-scaling	$f(at),\ a \neq 0$	$\dfrac{1}{\|a\|} F\left(\dfrac{s}{a}\right)$	ROC/a
Time reversal	$f(-t)$	$F(-s)$	$-\mathrm{ROC}$
Convolution	$f_1(t) * f_2(t)$	$F_1(s) \cdot F_2(s)$	contains $\mathrm{ROC}_1 \cap \mathrm{ROC}_2$
Frequency shift	$f(t) e^{j\omega_0 t}$	$F(s - j\omega_0)$	same
Frequency shift	$f(t) e^{s_0 t}$	$F(s - s_0)$	$\mathrm{ROC} + \mathrm{real}\{s_0\}$
Time Differentiation	$\dfrac{d^n}{dt^n} f(t)$	$s^n F(\omega)$	contains ROC
s-domain Differentiation	$(-t)^n f(t)$	$\dfrac{d^n}{ds^n} F(\omega)$	same
Integration	$\int_{-\infty}^{t} f(\tau)\, d\tau = f(t) * u(t)$	$\dfrac{1}{s} F(s)$	contains $\mathrm{ROC} \cap \{\mathrm{real}\{s\} > 0\}$
DC Value	$\int_{-\infty}^{\infty} f(t)\, dt = F(0)$		must contain $s = 0$

Inverse Laplace Transform

Requirement: Given rational $X(s)$ and **ROC**, we are required to invert it to find signal $x(t)$.

Steps:

- Decompose $X(s)$ into **the sum of atomic expressions** by doing **PFE**.

- For each expression, find ROC. Select the one including the ROC of LT.

- For each expression, use LT table and LT properties to get the result. Then sum them up.

- *CAUTION*: If the ROC is not given explicitly, discuss all the situations!!!

There are cases that given the pole-zero plot and DC value, and ask you to find x(t). In this case, you need to find the gain G with certain points.

Exercise

The system function of a causal LTI system is

$$H(s) = \frac{s + 1}{s^2 + 2s + 2}$$

Determine and sketch the response (y(t)) when the input is

$$e^{-|t|}, \quad -\infty < t < \infty$$

Hint: You can use $e^{-|t|} = e^{-t} u(t) + e^{t} u(-t)$

Answer:

$$x(t) = e^{-|t|} = e^{-t}u(t) + e^t u(-t)$$

$$X(s) = \frac{1}{s+1} - \frac{1}{s-1} = \frac{-2}{(s+1)(s-1)}, \quad -1 < \Re\{s\} < 1$$

We also have

$$H(s) = \frac{s+1}{s^2 + 2s + 2}$$

The poles of $H(s)$ are $-1 \pm j$, and since $h(t)$ is causal, we have that the ROC of $H(s)$ is $\Re\{s\} > -1$

$$Y(s) = H(s)X(s) = \frac{-2}{(s^2 + 2s + 2)(s-1)}, \quad \text{ROC}: -1 < \Re\{s\} < 1$$

Rewrite it as

$$Y(s) = -\frac{2/5}{s-1} + \frac{2}{5}\left[\frac{s+1}{(s+1)^2 + 1}\right] + \frac{4}{5}\left[\frac{1}{(s+1)^2 + 1}\right]$$

we get,

$$y(t) = \frac{2}{5}e^t u(-t) + \frac{2}{5}e^{-t}\cos(t)u(t) + \frac{4}{5}e^{-t}\sin(t)u(t)$$

Differential Equation

How to solve

Solve **algebraic expression** by Linear Constant Coefficient Differential Equation.

Solve **ROC** by stability & causality & etc. conditions.

Block Diagram

In this part, you should be capable of:

1. Writing $H(s)$ according to the block diagram. Then analyze the system with $H(s)$. e.g. How to solve the Impulse Response $h(t)$? etc.

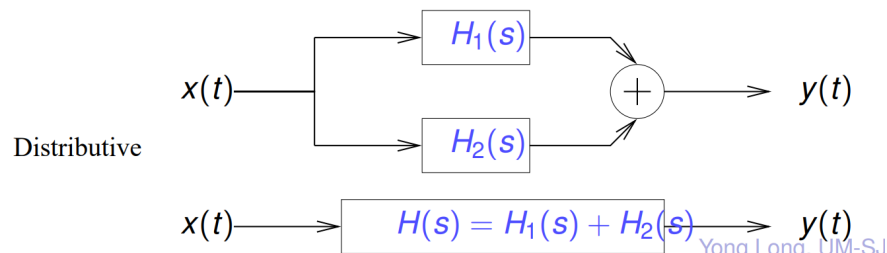2. Draw the block diagram according to $H(s)$.

Parallel Interconnection

Thus the overall frequency response of two LTI systems connected in parallel is given by the sum of the frequency responses of the individual systems:

$$H(\omega) = H_1(\omega) + H_2(\omega).$$

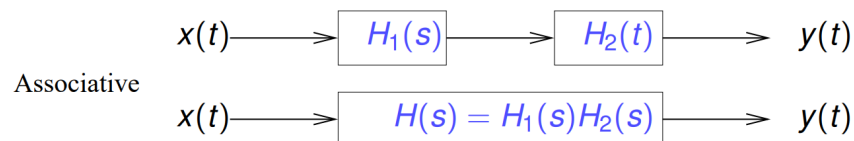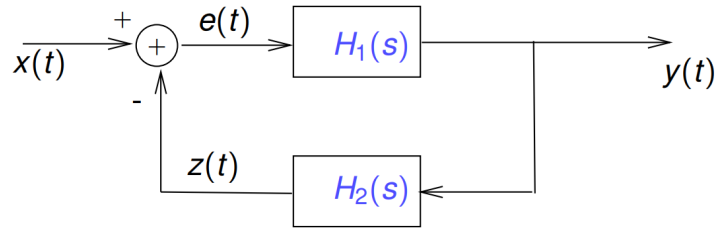The overall system (transfer) function is

$$H(s) = H_1(s) + H_2(s).$$

Distributive

$$x(t) \longrightarrow \boxed{H_1(s)}$$
$$x(t) \longrightarrow \oplus \longrightarrow y(t)$$
$$\boxed{H_2(s)}$$

$$x(t) \longrightarrow \boxed{H(s) = H_1(s) + H_2(s)} \longrightarrow y(t)$$

Series Interconnection

Thus the overall frequency response of two LTI systems connected in series is given by the *product* of the frequency responses of the individual systems:

$$H(\omega) = H_1(\omega)H_2(\omega).$$

The overall system (transfer) function is

$$H(s) = H_1(s)H_2(s).$$

$$x(t) \longrightarrow \boxed{H_1(s)} \longrightarrow \boxed{H_2(t)} \longrightarrow y(t)$$

Associative

$$x(t) \longrightarrow \boxed{H(s) = H_1(s)H_2(s)} \longrightarrow y(t)$$

Feedback Interconnection

$$\begin{cases} y(t) = e(t) * h_1(t), \\ e(t) = x(t) - z(t), \\ z(t) = y(t) * h_2(t) \end{cases} \Longrightarrow y(t) = [x(t) - y(t) * h_2(t)] * h_1(t)$$

$$H(\omega) = \frac{Y(\omega)}{X(\omega)} = \frac{H_1(\omega)}{1 + H_1(\omega)H_2(\omega)}$$

$$H(s) = \frac{Y(s)}{X(s)} = \frac{H_1(s)}{1 + H_1(s)H_2(s)}$$

If you are not sure whether you can write the $H(s)$ according to the block diagram, take the above as an example and practice.
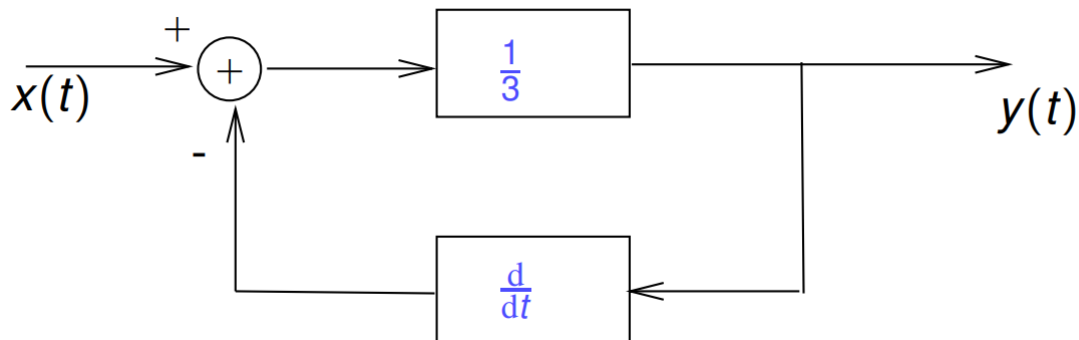
Exercise

Exercise 1

Draw the block diagram of the causal LTI system with system function:

$$H(s) = \frac{1}{s+3}$$

1. Use differentiator to implement. Mention, the differentiator is both difficult to implement and extremely sensitive to noise.

Answer:

2. Use integrator to implement.

Answer:
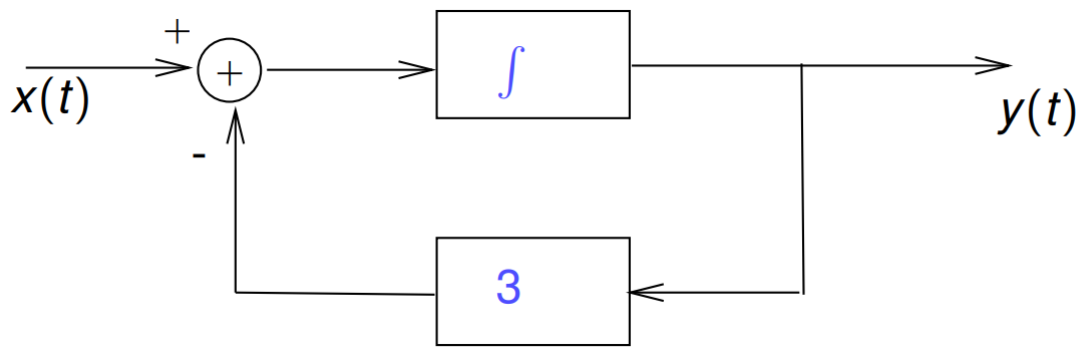
For normal cases, please use integrator ($\frac{1}{s}$) to draw the diagram

Exercise 2:

Draw the block diagram for the causal LTI system with system function:

$$H(s) = \frac{1}{(s+1)(s+2)}$$

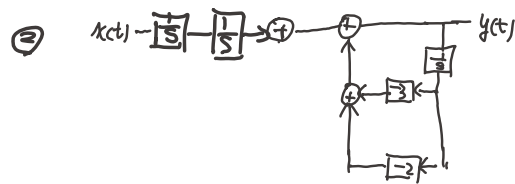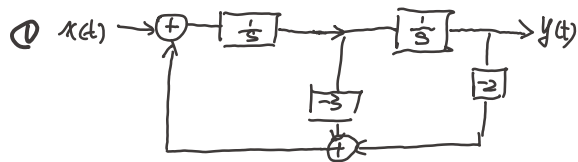Use **Direct Form**, **Cascade Form**, and **Parallel From**

Answer:

1. Direct Form

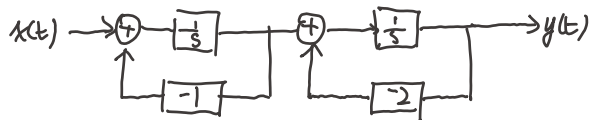$H(S) = \frac{Y(s)}{X(s)} = \frac{1}{s^2+3s+2}$

$(s^2+3s+2)\,Y(s) = X(s)$

$Y(s) = \frac{1}{s^2}X(s) - \frac{3}{s}Y(s) - \frac{2}{s^2}Y(s)$

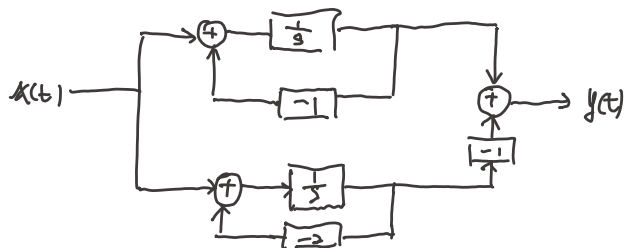$\quad = \frac{1}{s}\left(\frac{1}{s}X(s) - 3\,Y(s) - \frac{2}{s}Y(s)\right)$

2. Cascade Form

$H(s) = \frac{1}{s+1} \cdot \frac{1}{s+2}$
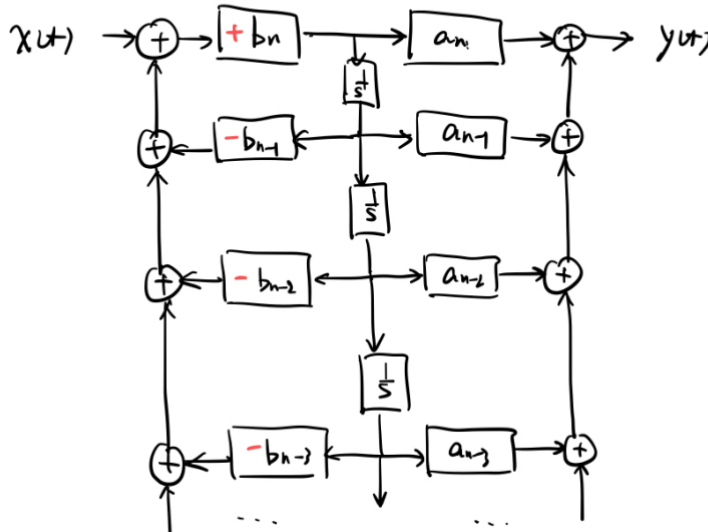

3. Parallel From

$H(s) = \frac{1}{s+1} - \frac{1}{s+2}$

General block diagram for rational systems

More generalized.

$$H(s) = \frac{a_n s^n + \cdots + a_0}{b_n s^n + \cdots + b_0} \quad \text{(same order)}$$

- $m = n$: Just apply the method.

- $m < n$: e.g. $H(s) = \frac{1}{s^2+3s+2}$, we just first fill all the item in the upper to make the upper and lower part are at the same order. Finally **remove the branch** with zero coefficient.

- $m > n$: e.g. $H(s) = s$. It has poles at $s = \infty$, which makes the system improper (All rational but improper systems are unstable!!), thus **won't appear in real cases**.

Exercise

Exercise 1:

Suppose we are given the following information about a causal and stable LTI system S with impulse response $h(t)$ and a rational system function $H(s)$:

(1) When the input to the system is $x(t) = e^{-3t}$ for all t, the output is $y(t) = -e^{-3t}$ for all t.

(2) When the input is $u(t)$, the output is absolutely integrable.

(3) When the input is $tu(t)$, the output is not absolutely integrable.

(4) The signal $\frac{d^2}{dt^2}h(t) + 4\frac{d}{dt}h(t) + 1h(t)$ is of finite duration.

(5) $H(s)$ has exactly one zero at infinity.

Determine $H(s)$ and its region of convergence.

(1) Remind: $e^{st} \xrightarrow{S} H(s) e^{st}$

Since $e^{-3t} \xrightarrow{S} -e^{-3t}$

then $H(-3) = -1$

(2) $x(t) = u(t)$

$X(s) = \frac{1}{s}$, $\text{Re}\{s\} > 0$

$Y(s) = \frac{1}{s}H(s)$ absolutely integrable

$\Rightarrow$ the ROC of $\frac{1}{s}H(s)$ contains $j\omega$ axis

$\Rightarrow$ s=0 not a pole for $\frac{1}{s}H(s)$

$\Rightarrow$ $H(s) = s(\cdots)$ s is a factor of the numerator of $H(s)$

(3) $x(t) = t\,u(t)$

$X(s) = \frac{1}{s^2}$, $\text{Re}\{s\} > 0$

$Y(s) = \frac{1}{s^2} H(s)$ not absolutely integrable

$\Rightarrow$ s^2 is not a factor of the numerator of $H(s)$

(4) $Y(s) = s^2 H(s) + 4s H(s) + H(s)$

$= (s^2 + 4s + 1) H(s)$

$\Rightarrow$ no pole for $(s^2 + 4s + 1) H(s)$

$\Rightarrow$ $H(s) = G\,\dfrac{\cdots}{s^2 + 4s + 1}$

(5) $H(s) = \dfrac{as + b}{cs^2 + ds + e}$

$H(-3) = \dfrac{G-3}{-2} = 1 \Rightarrow G = -\frac{2}{3}$ causal $\Rightarrow$ $\text{Re}\{s\} > -2$

$\Rightarrow$ $\boxed{H(s) = \dfrac{G\,s}{s^2 + 4s + 1} = -\dfrac{2}{3}\dfrac{s}{s^2 + 4s + 1},\ \text{Re}\{s\} > -2}$

How to solve problem: Given several information of $x(t)$ and $y(t)$, find $H(s)$

Steps:

1. eigenfunction e^{kt}

   $x(t) = e^{kt}$, then $y(t) = H(k)e^{kt}$

2. for certain $x(t)$, $y(t)$ is absolutely integrable/ or not.

      1. absolutely integrable => ROC contains $j\omega$ axis

      2. not absolutely integrable => ROC doesn't contain $j\omega$ axis

3. $y(t)$ is of finite duration

   => ROC is the whole plane

   => no poles exist for $Y(s)$

4. $H(s)$ has one zero at infinity

   => $n - m = 1$, where n is the degree of denominator

5. $H(s)$ has one pole at infinity

   => $m - n = 1$, where m is the degree of denominator

Exercise 2:

Suppose that the signal

$$y(t) = e^{-2t}u(t)$$

is the output of a causal system with the system function

$$H(s) = \frac{s-1}{s+1}$$

1. Find and plot two possible input $x(t)$ for the output $y(t)$.

2. If there exists a stable (but not necessarily causal) system, if it takes $y(t)$ as the input, its output will be $x(t)$. Which $x(t)$ is it? Find the impulse response of the system, and prove by directly calculate convolution.

1. $y(t) = e^{-2t}u(t) \Rightarrow Y(s) = \frac{1}{s+2}$, $Re\{s\} > -2$

$Y(s) = X(s) H(s) = \frac{1}{s+2}$

$X(s) = \frac{s+1}{(s-1)(s+2)} = \frac{a}{s-1} + \frac{b}{s+2}$

$a = \frac{s+1}{s+2}\Big|_{s=1} = \frac{2}{3}$

$b = \frac{s+1}{s-1}\Big|_{s=-2} = \frac{1}{3}$

$X(s) = \frac{2}{3(s-1)} + \frac{1}{3(s+2)}$

① $x(t) = -\frac{2}{3} e^{t} u(-t) + \frac{1}{3} e^{-2t} u(t)$, $-2 < Re\{s\} < 1$

② $x(t) = \frac{2}{3} e^{t} u(t) + \frac{1}{3} e^{-2t} u(t)$, $Re\{s\} > 1$

2. Let the system be $H_2(s)$

$\frac{1}{s+2} H_2(s) = \frac{(s+1)}{(s+2)(s-1)}$

$\Rightarrow H_2(s) = \frac{s+1}{s-1}$ ROC: $Re\{s\} < 1$

ROC of $x(t)$: $-2 < Re\{s\} < 1$

$H_2(s) = 1 + \frac{2}{s-1}$, $Re\{s\} < 1$

$h_2(t) = 1 - 2 e^{t} u(-t)$

Prove Omitted

Notes

Reference

1. Hu fan. 216 final recitation class.

2. Long yong. 216 slides.