Chapter 2 - Convolution

Linear and Time-Invariant (LTI) Systems

What is LTI Property?

Linearity:

For a linear system, if $x_1(t) \xrightarrow{\mathcal{T}} y_1(t)$ and $x_2(t) \xrightarrow{\mathcal{T}} y_2(t)$, then $ax_1(t) + bx_2(t) \xrightarrow{\mathcal{T}} ay_1(t) + by_2(t)$.

Time-Invariance:

For a time-invariant system, if $x(t) \xrightarrow{\mathcal{T}} y(t)$, then $x(t - t_0) \xrightarrow{\mathcal{T}} y(t - t_0)$.

Why LTI Property Matters?

LTI property **simplify** the analysis of a system. Specifically, it makes the system design and response analysis easier.

In this chapter, we cares how to analyze LTI system using **convolution** methods.

Questions:

1. **How to design an LTI system?**

   We only need to care about its impulse response $h(t)$, since an LTI system is completely described by its impulse response $h(t)$.

2. **How to determine the response $y(t)$ of LTI systems?**

   Known $h(t)$, we can determine the response $y(t)$ following these steps:

   1. Decompose the input signal $x(t)$ into delayed impulses $\delta(t - \tau)$.
   2. Use convolution to solve: $y(t) = x(t) * h(t)$.

Convolution

What is Convolution?

$x(t)$ can be decomposed into delayed impulses

$$x(t) = \int_{-\infty}^{\infty} x(\tau)\delta(t-\tau)d\tau.$$

Denoted

$$\delta(t) \xrightarrow{\tau} h(t),$$

For LTI system, the corresponding response $y(t)$ can be derived using LTI property, which is

$$y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau = x(t) * h(t).$$

Question:

1. **Why we can decompose $x(t)$ into delayed impulses?**
   - Hint: Start from discrete case $x[n]$ to understand the idea, and build an approximation by short time impulses ($\Delta \to 0$)

2. **How to derive the convolution integral using LTI property?**
   - Hint 1: $x(\tau)$ is a constant so that we can use linearity property.
   - Hint 2: $\delta(t-k) \xrightarrow{\tau} h_k(t)$, since the system is TI, $\delta(t-k) \xrightarrow{\tau} h(t-k)$

Why Convolution?

An LTI system is completely described by its impulse response $h(t)$.

Any system whose input-output relationship can be expressed in $y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau = x(t) * h(t)$ is time-invariant.

THEREFORE, turn $y(t)$ to the form $h(t) * x(t)$, then we can directly get $h(t)$.

In many cases, there's NO need to substitute $x(t)$ with $\delta(t)$ in the algebra representation of $y(t)$ to solve $h(t)$.

Exercise:

1. Find the impulse response of the LTI systems.

$$y(t) = \int_{-3}^{3} \tau^2 x(t-\tau)d\tau + \int_{-\infty}^{t+1} (t-\tau+3)^{-2} x(\tau)d\tau$$

Summary: How to turn $y(t)$ expression to the form $h(t) * x(t)$?

1. Use $u(t)$ to substitute range of integration
2. Turn to convolution form (first find $t - \tau$ and τ , then substitute them with t).

Graphical Method for Convolution Calculation

For time-limited signals, it's more convenient and intuitive to calculate the convolution with the help of graphics.

Steps: (Exchange the role of h and x is fine)

- Fold: fold $h(\tau)$ about $\tau = 0$ to get $h(-\tau)$
- Shift: shift $h(-\tau)$ by t to get $h(t - \tau)$
- Multiply: multiply $x(\tau)$ by $h(t - \tau)$ for every τ
- Integrate: $y(t) = \int_{-\infty}^{\infty} x(\tau)h(t - \tau)d\tau$

If you don't understand, please refer to this video (from 32:40):

https://www.bilibili.com/video/BV1SB4y1C7yx?p=4&vd_source=0be2819c805c6da41df6967754921b05

Properties of Convolution

- $x(t) * h(t) = h(t) * x(t)$
- $[x(t) * h_1(t)] * h_2(t) = x(t) * [h_1(t) * h_2(t)]$
- $x(t) * \delta(t) = x(t)$
- $x(t) * \delta(t - t_0) = x(t - t_0)$
- $\delta(t - t0) * \delta(t - t1) = \delta(t - t0 - t1)$
- $x(t - t0) * h(t - t1) = y(t - t0 - t1)$
- $\frac{d}{dt}y(t) = [\frac{d}{dt}x(t)] * h(t) = [\frac{d}{dt}h(t)] * x(t)$
- $\int_{-\infty}^{\infty} y(t) = [\int_{-\infty}^{\infty} x(t)dt][\int_{-\infty}^{\infty} h(t)dt]$
- $y(at) \neq x(at) * h(at)$

For properties which are not included, please **derive from integral form of convolution**!

Exercise:

1. One of following two statements is correct, and the other is incorrect.

- If $y(t) = h(t) * x(t)$ then $y(t-3) = h(t-3) * x(t-3)$;
- Or if $y(t) = h(t) * x(t)$ then $y(t-3) = h(t) * x(t-3)$.

2. Show that

$$\frac{1}{a} rect(\frac{t}{a}) * rect(\frac{t}{a}) = tri(\frac{t}{a}) \quad \text{where } a > 0$$

Hint: $rect(t) * rect(t) = tri(t)$

LTI Systems properties via $h(t)$

1. **Causal**:

   An LTI System is causal iff its impulse response $h(t) = 0$ for all $t < 0$.

2. **Memoryless:**

   An LTI System is memoryless iff its impulse response $h(t) = a\delta(t)$.

   Otherwise, the system is **dynamic**.

   (Note: A finite impulse response or FIR system means $h(t) \neq 0, t_1 < t < t_2.$)

3. **BIBO stable**:

   An LTI system is BIBO stable iff its impulse response is absolutely integrable, i.e. $\int_{-\infty}^{\infty} |h(t)| dt < \infty$.

4. **Invertible**:

   An LTI system is invertible iff there exist an inverse system whose impulse response $h_i(t)$ satisfies

$$h(t) * h_i(t) = \delta(t)$$

Exercise:

1. Determine whether the system is causal, stable and static via $h(t)$.

$$y(t) = \int_{-\infty}^{t} (t-\tau) e^{-(t-\tau)} x(\tau) d\tau$$

Step Response

Step response of an LTI system is defined as the response of the system to an input signal $s(t)$ that is unit step.

$$\delta(t) \xrightarrow{\tau} h(t)$$

$$u(t) \xrightarrow{\tau} s(t) = \int_{-\infty}^{\infty} u(t-\tau)h(\tau)d\tau$$

$$s(t) = u(t) * h(t) = \int_{-\infty}^{t} h(\tau)d\tau$$

$$h(t) = \frac{d}{dt}s(t)$$

CT systems described by differential equation models

Linear constant-coefficient differential equation:

$$\sum_{k=0}^{N} a_k \frac{d^k}{dt^k} y(t) = \sum_{k=0}^{N} b_k \frac{d^k}{dt^k} x(t)$$

Approach:

$$y(t) = y_h(t) + y_p(t)$$

$y_h(t)$: homogeneous solution, or zero-input response, or natural response.

$y_p(t)$: particular solution, or zero-state response, or forced response.

Initial rest condition

Initial rest states that if $x(t) = 0$ for $t < t_0$, then $y(t) = 0$ for $t < t_0$.

It means that the output must be zero up until the time when the input becomes zero.

Causal and LTI system ⇔ initial rest

Exercise:

1. Find the expression of response of the CT system described by the linear constant-coefficient differential equation.

$$\frac{d}{dt}y(t) + 10y(t) = 2x(t), \qquad y(0) = 1, \qquad x(t) = u(t)$$

Summary - How to find step response or impulse response of diffeq systems?

1. Find general solution $y_h(t)$
2. Set $x(t) = u(t)$, find particular solution $y_p(t)$
3. Solve all the constants according to initial rest condition.
4. $s(t)$ determined, solve impulse response (if required) according to $h(t) = \frac{d}{dt}s(t)$. Remember to add $u(t)$ according to initial rest.

Convolution of Common Signals

Copy them on your CTPP! Copy them on your CTPP!

- $k * f(t) = k\int_{-\infty}^{\infty} f(t)dt = k\times$(area of $f(t)$ over R)
- $f(t) * u(t) = \int_{-\infty}^{t} f(t)dt$
- $u(t) * u(t) = tu(t)$
- $u(t) * e^{-at}u(t) = \frac{1}{a}(1 - e^{-at})u(t)$
- $e^{-at}u(t) * e^{-at}u(t) = te^{-at}u(t)$
- $e^{-a_1 t}u(t) * e^{-a_2 t}u(t) = \frac{e^{-a_1 t} - e^{-a_2 t}}{a_2 - a_1}u(t)$
- $f(t) * \sum_{m=-\infty}^{\infty} \delta(t - mT) = \sum_{m=-\infty}^{\infty} f(t - mT)\backslash$
- $\frac{1}{a}rect(\frac{t}{a}) * rect(\frac{t}{a}) = tri(\frac{t}{a})$
- $\frac{1}{a}rect(\frac{t-t_0}{a}) * rect(\frac{t-t_1}{a}) = tri(\frac{t-t_0-t_1}{a})$

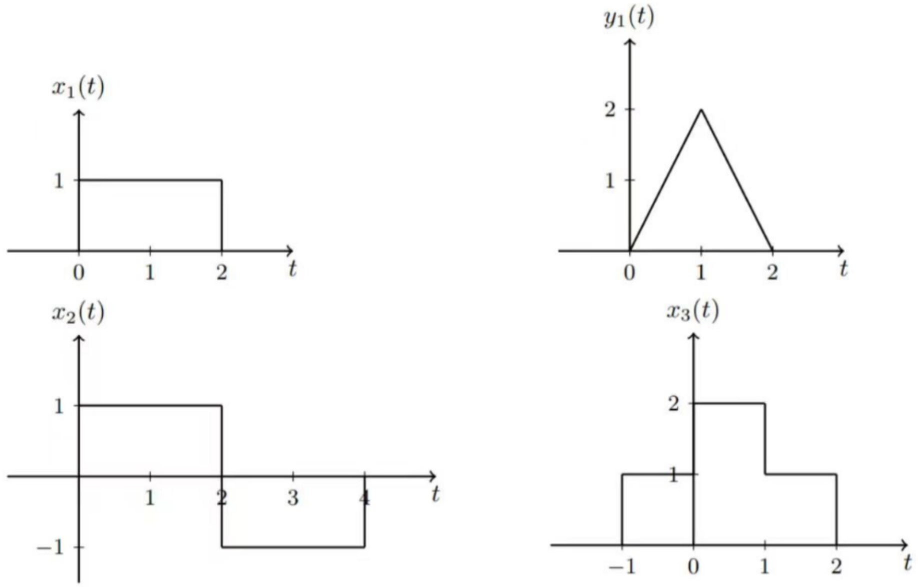
Be familiar with all of them, deduce them by yourself and use them in the exam!

Exercises

1. Compute the following convolution:

$$u(t) * t^2u(t)$$

2. Consider an LTI system whose response to the signal $x_1(t)$ is the signal $y_1(t)$ which are illustrated below.
   (a) Determine the response of the system to the input $x_2(t)$ depicted below.
   (b) Determine the response of the system to the input $x_3(t)$ depicted below.



Notes

Before the exam:

1. **Review all the quizzes!**
2. Review homework answers.
3. Gain proficiency in convolution calculation.
4. Check whether all the contents on summary sheet have been on your ctpp.

During the exam:

1. If algebra doesn't work, go for graphics.
2. Use convolution of common signals if possible.
3. If you use method of substitution, pay attention when substituting the range of integral.

Reference

1. Yuan Jiaming. *VE216 Mid 1 RC part 2.*
2. Long Yong. *VE216 slides.*