Inflation Induced Planck-Size Black Hole Remnants As Dark Matter

Pisin Chen

Stanford Linear Accelerator Center
Stanford University, Stanford, CA 94309, USA

Abstract

While there exist various candidates, the identification of dark matter remains unresolved. Recently it was argued that the generalized uncertainty principle (GUP) may prevent a black hole from evaporating completely, and as a result there should exist a Planck-size BHR at the end of its evaporation. We speculate that the stability of BHR may be further protected by supersymmetry in the form of extremal black hole. If this is indeed the case and if a sufficient amount of small black holes can be produced in the early universe, then the resultant BHRs can be an interesting candidate for DM. We demonstrate that this is the case in the hybrid inflation model. By assuming BHR as DM, our notion imposes a constraint on the hybrid inflation potential. We show that such a constraint is not fine-tuned. Possible observational signatures are briefly discussed.

1 Introduction

It is by now widely accepted that dark matter (DM) constitutes a substantial fraction of the present critical energy density in the universe. However, the nature of DM remains an open problem. There exist many DM candidates, most of them are non-baryonic weakly interacting massive particles (WIMPs), or WIMP-like particles[1]. Figure 1 shows the masses and cross sections of WIMP (or WIMP-like) candidates[2]. By far the DM candidates that have been more intensively studied are the lightest supersymmetric (SUSY) particles such as neutralinos or gravitinos, and the axions (as well as the axinos). There are additional particle physics inspired DM candidates[1]. A candidate which is not as closely related to particle physics is the relics of primordial black holes (PBHs)[3,4]. Earlier it was suggested that PBHs are a natural candidate for

DM[5]. More recent studies[6] based on the PBH production from the "blue spectrum" of inflation demand that the spectral index $n_s \sim 1.3$, but this possibility may be ruled out by recent WMAP experiment[7]. There are also other studies on the idea of PBH as DM[8].
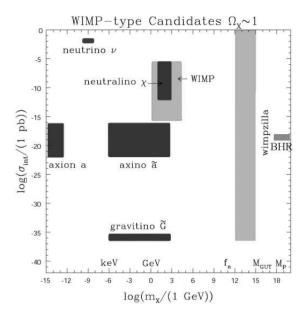

Fig. 1. Cross sections and masses of WIMP dark matter candidates. Black hole remnant (BHR) is included.

One weakness of the notion of PBH as DM is the ambiguity on the final property of small black holes. The standard view of black hole thermodynamics[9,10] does not provide an answer as to whether a small black hole should evaporate entirely, or leave something else behind, which we refer to as a black hole remnant (BHR). Numerous calculations of black hole radiation properties have been made from different points of view[11], and some hint at the existence of remnants, but none appear to give a definitive answer.

In a recent paper[12], a generalized uncertainty principle (GUP)[15,16,17,18] was invoked to argue that the total evaporation of a black hole may be prevented, and as a result there should exist a black hole remnant with Planck mass and size. Here we speculate that the stability of such BHR may be further protected by supersymmetry, in the form of the extremal black hole[13]. Such a BHR is totally inert, with no attributes other than gravitational interaction, and is thus a natural candidate for DM. It remains unclear, however, whether such a notion can be smoothly incorporated into the standard cosmology.

We note that certain inflation models naturally induce a large number of small black holes. As a specific example, we demonstrate that the hybrid inflation[20,21,22] cosmology can in principle yield the necessary abundance of primordial BHRs for them to be the primary source of dark matter. We show that such a construction is not fine-tuned.

2 Generalized Uncertainty Principle and Black Hole Remnant

As a result of string theory[16], or noncommutative spacetime algebra[17,19], or general considerations of quantum mechanics and gravity[15,18], the standard uncertainty principle must be modified when the gravity effect is included. A heuristic derivation may be made on dimensional grounds. Consider a particle such as an electron being observed by means of a photon with momentum p. The usual Heisenberg argument leads to an electron position uncertainty given by $\hbar/\Delta p$. But we should add to this a term due to the gravitational interaction between the electron and the photon, and that term must be proportional to G times the photon energy, or Gpc. Since the electron momentum uncertainty Δp will be of order of p, we see that on dimensional grounds the extra term must be of order $G\Delta p/c^3$. Note that there is no $\hbar$ in the extra term when expressed in this way. The effective position uncertainty is therefore

$$\Delta x \geq \frac{\hbar}{\Delta p} + \xi^2 l_p^2 \frac{\Delta p}{\hbar} \; , \tag{1}$$

where $l_p = (G\hbar/c^3)^{1/2} \approx 1.6 \times 10^{-33}$cm is the Planck length. Here the dimensionless coefficient $\xi \sim \mathcal{O}(1)$ can be considered either as related to the string tension from the string theory's motivation, or simply as a factor to account for the imprecision of our heuristic derivation which can only be fixed by a precise theory of quantum gravity in the future. Note that Eq.(1) has a minimum value of $\Delta x_{min} = 2\xi l_p$, so the Planck length (or ξl_p) plays the role of a fundamental length.

The Hawking temperature for a spherically symmetric black hole may be obtained in a heuristic way with the use of the standard uncertainty principle and general properties of black holes[12]. In the vicinity of the black hole surface there is an intrinsic uncertainty in the position of any vacuum-fluctuating particle of about the Schwarzschild radius, $\Delta x \approx r_s = 2GM_{\mathrm{BH}}/c^2$, due to the behavior of its field lines[23] - as well as on dimensional grounds. This leads to a momentum uncertainty Δp. Identifying Δpc as the characteristic energy of the emitted photon, and thus as a characteristic temperature (with the insertion of a calibration factor $1/4\pi$), one arrives at the celebrated Hawking temperature, $T_{\mathrm{H}} = \hbar c^3/8\pi GM_{\mathrm{BH}}$.

Applying the same argument, but invoking the GUP, one then finds a modified black hole temperature,

$$T_{\mathrm{GUP}} = \frac{m_p c^2}{4\pi} \frac{\mu}{\xi^2} \left[1 \mp \sqrt{1 - \xi^2/\mu^2} \right] \; , \tag{2}$$

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where $\mu \equiv M_{BH}/m_p$ is the BH mass in units of the Planck mass: $m_p = (\hbar c/G)^{1/2} \approx 1.2 \times 10^{19}$GeV. This agrees with the Hawking result for large mass if the negative sign is chosen, whereas the positive sign has no evident physical meaning. Note that the temperature becomes complex and unphysical for mass less than ξm_p and Schwarzschild radius less than $2\xi l_p$, the minimum size allowed by the GUP. At $\mu = \xi$, T_{GUP} is finite but its slope is infinite, which corresponds to zero heat capacity of the black hole. The BH evaporation thus comes to a stop, and what remains is a inert BHR with mass $\mu = \xi$.

If there are g species of relativistic particles, then the BH evaporation rate (assuming the Stefan-Boltzmann law) is

$$\dot{\mu} = -\frac{16g}{t_{ch}}\frac{\mu^6}{\xi^8}\left[1 - \sqrt{1 - \xi^2/\mu^2}\right]^4 , \tag{3}$$

where $t_{ch} = 60(16)^2\pi t_p \approx 4.8 \times 10^4 t_p$ is a characteristic time for BH evaporation, and $t_p = (\hbar G/c^5)^{1/2} \approx 0.54 \times 10^{-43}$sec is the Planck time. Note that the energy output given by Eq.(3) is finite at the end point where $\mu = \xi$, i.e., $d\mu/dt|_{\mu=\xi} = -16g/(\xi^2 t_{ch})$. Thus the hole with an initial mass μ_i evaporates to a remnant in a time given by

$$\begin{aligned}
\tau &= \frac{t_{ch}}{16g}\Big[\frac{8}{3}\mu_i^3 + \frac{8}{3}(\mu_i^2 - \xi^2)^{3/2} - 4\xi^2(\mu_i^2 - \xi^2)^{1/2} \\
&\qquad -8\xi^2\mu_i + 4\xi^3\cos^{-1}\frac{\xi}{\mu_i} + \frac{19}{3}\xi^3 - \frac{\xi^4}{\mu_i}\Big] \\
&\approx \frac{\mu_i^3}{3g}t_{ch}, \qquad \mu_i \gg 1 .
\end{aligned} \tag{4}$$

The evaporation time in the $\mu_i \gg 1$ limit agrees with the standard Hawking picture.

3   The Issue of Black Hole Remnant Stability

Even if the GUP may prevent a small black hole from total evaporation, it remains unclear whether the resultant BHR will be prohibited from decaying into the vacuum. In our previous work[12] we argued that the total collapse of a black hole may be prevented by dynamics (i.e., the GUP) and not by symmetry, just like the prevention of hydrogen atom from collapse by the standard uncertainty principle[24]. In a closer look the hydrogen atom argument may be only fortuitous, as there exist counter-examples such as the finite lifetime of the positronium. Therefore to protect the stability of the BHR, the existence of a symmetry principle in the system appears essential.

In this regard supersymmetry, in particular supergravity, stands a very good chance of providing such a protection to BHR. It is well-known that the no-hair theorem allows for only three attributes of a classical black hole, namely, its mass, charge and angular momentum. For the extreme Kerr-Newman BH, we have the limiting case where

$$M^2 = Q^2 + a^2 \ , \tag{5}$$

where M is the BH mass, Q the BH charge associated with certain gauge symmetry, and $a \equiv S/M$ is the BH angular momentum per unit mass. In the special case where the BH has no angular momentum, it reaches the extremal condition of $M = Q$. It has been shown that in certain specific realizations of supergravity, there exist extremal black hole solutions[13]. As supergravity "charge" is in units of Planck mass, this condition dictates that the BH mass is bounded by the Planck mass. We speculate that the extremal condition should remain intact even when SUSY is broken.

If the primordial black holes were generated in the very early epoch of the universe, such as the one immediately following inflation, it is likely that SUSY was still unbroken. Then the PBHs so generated could be either straight-forwardly the SUSY extremal BHs governed by supergravity, or the small but classical BHs described above. We believe that, governed by the GUP, the latter would soon reduce to Planck-size BHRs whose final state then coincides with the extremal BH condition. In either case we expect the existence of BHRs at Planck size. It is unclear, however, whether this notion can be physically realized. More efforts are evidently needed beyond the simple-minded comments made here. A study in this direction based on string theory is currently underway[14].

4 Hybrid Inflation and Black Hole Production

We now return to the scenario of a semi-classical BH and combine this notion with the hybrid inflation proposed by A. Linde[20,25]. In the hybrid inflation model two inflaton fields, (ϕ, ψ), are invoked. Governed by the inflation potential,

$$V(\phi, \psi) = \left(M^2 - \frac{\sqrt{\lambda}}{2} \psi^2 \right)^2 + \frac{1}{2} m^2 \phi^2 + \frac{1}{2} \gamma \phi^2 \psi^2 \ , \tag{6}$$

ϕ first executes a "slow-roll" down the potential, and is responsible for the more than 60 e-folds expansion while ψ remains zero. When ϕ eventually reduces to a critical value, $\phi_c = (2\sqrt{\lambda} M^2/\gamma)^{1/2}$, it triggers a phase transition

that results in a "rapid-fall" of the energy density of the ψ field, which lasts only for a few e-folds, that ends the inflation.

The equations of motion for the fields are

$$\ddot{\phi} + 3H\dot{\phi} = -(m^2 + \gamma\psi^2)\phi ,$$
$$\ddot{\psi} + 3H\dot{\psi} = (2\sqrt{\lambda}M^2 - \gamma\phi^2 - \lambda\psi^2)\psi , \tag{7}$$

subject to the Friedmann constraint,

$$H^2 = \frac{8\pi}{3m_p^2}\Big[V(\phi,\psi) + \frac{1}{2}\dot{\phi}^2 + \frac{1}{2}\dot{\psi}^2\Big] . \tag{8}$$

The solution for the ψ field in the small ϕ regime, measured backward from the end of inflation, is

$$\psi(N(t)) = \psi_e \exp(-sN(t)) , \tag{9}$$

where $N(t) = H_*(t_e - t)$ is the number of e-folds from t to t_e and $s = -3/2 + (9/4 + 2\sqrt{\lambda}M^2/H_*^2)^{1/2}$ and $H_* \simeq \sqrt{8\pi/3}M^2/m_p$.

We now show how a large number of small black holes can result from the second stage of inflation. Quantum fluctuations of ψ induce variations of the starting time of the second stage inflation, i.e., $\delta t = \delta\psi/\dot{\psi}$. This translates into perturbations on the number of e-folds, $\delta N = H_*\delta\psi/\dot{\psi}$, and therefore the curvature contrasts.

It can be shown that[26] the density contrast at the time when the curvature perturbations re-enter the horizon is related to δN by

$$\delta \equiv \frac{\delta\rho}{\rho} = \frac{2+2w}{5+3w}\delta N , \tag{10}$$

where $p = w\rho$ is the equation of state of the universe at reentry. From Eq.(9), it is easy to see that $\dot{\psi} = sH_*\psi$. At horizon crossing, $\delta\psi \sim H_*/2\pi$. So with the initial condition (at $\phi = \phi_c$) $\psi \sim H_*/2\pi$, we find that $\delta N \sim 1/s$. Thus

$$\delta \sim \frac{2+2w}{5+3w}\frac{1}{s} . \tag{11}$$

As w is always of order unity, we see that the density perturbation can be sizable if s is also of order unity. With an initial density contrast $\delta(m) \equiv \delta\rho/\rho|_m$, the probability that a region of mass m becomes a PBH is[27]

$$P(m) \sim \delta(m)e^{-w^2/2\delta^2} . \tag{12}$$

Let us assume that the universe had inflated e^{N_c} times during the second stage of inflation. From the above discussion we find[25]

$$e^{N_c} \sim \Big(\frac{2m_p}{sH_*}\Big)^{1/s} \ . \tag{13}$$

At the end of inflation the physical scale that left the horizon during the phase transition is $H_*^{-1}e^{N_c}$. If the second stage of inflation is short, i.e., $N_c \sim \mathcal{O}(1)$, then the energy soon after inflation may still be dominated by the oscillations of ψ with $p = 0$, and the scale factor of the universe after inflation would grow as $(tH_*)^{2/3}$. The scale $(tH_*)^{2/3}H_*^{-1}e^{N_c}$ became comparable to the particle horizon $(\sim t)$, or $t \sim (tH_*)^{2/3}H_*^{-1}e^{N_c}$, when

$$t \sim t_h = H_*^{-1}e^{3N_c} \ . \tag{14}$$

At this time if the density contrast was $\delta \sim 1$, then BHs with size $r_s \sim H_*^{-1}e^{3N_c}$ would form with an initial mass

$$\mu_i \sim \frac{m_p}{H_*}e^{3N_c} \equiv \alpha\frac{m_p}{H_*}\Big(\frac{2m_p}{sH_*}\Big)^{3/s} \ . \tag{15}$$

A dimensionless parameter α is introduced to account for the dynamic range of the gravitational collapse. Since H_* depends on M while s on M and λ, the initial BH mass depends only on the mass and the coupling in the ψ-sector of the hybrid inflation.

5   Black Hole Remnants as Dark Matter

By identifying BHRs as DM and assuming hybrid inflation as the progenitor of PBHs, we in effect impose a constraint between H_* and s (or equivalently, M and λ). Though constrained, these parameters are not so fine-tuned, as we will show in the following analysis.

We wish to estimate the present abundance of the BHRs created by hybrid inflation. To do so we should track the evolution of the post-inflation PBHs through different cosmological epochs. The newly introduced "black hole epoch" $(t_h \le t \le \tau)$ would in principle involve evaporation and mergers of PBHs as well as their accretion of radiation; the details of which can be intricate. For the purpose of a rough estimate and as a good approximation, however, one is safe to neglect these detailed dynamics and only keep track of the BH evaporation throughout the BH epoch. We assume that the universe was matter dominant at the time t_h when PBHs were formed. Due to Hawking

evaporation, the universe would later become radiation dominant. Since the rate of BH evaporation rises sharply in its late stage, the crossing time t_x of this transition can be roughly estimated by integrating Eq.(3) from μ_i to $\mu_i/2$. This gives $t_x \sim 7/8\tau$. When the Hubble expansion effect is included, where the radiation density dilutes faster than the matter density by one power of the scale factor, the crossing time would be even closer to τ. For our purpose, we can simply assume that the entire BH epoch was matter dominant.

The radiation to matter density ratio at the end of BH epoch, with Hubble expansion included, can then be estimated as

$$\frac{\Omega_{\gamma,\tau}}{\Omega_{\mathrm{BHR},\tau}} \sim \frac{1}{\xi} \int_{t_h}^{\tau} dt \dot{\mu} \left(\frac{t}{\tau}\right)^{2/3} \equiv \beta \frac{\mu_i}{\xi} \ . \tag{16}$$

Here we introduce another parameter β to account for a range of possible minor variations of the evolution due to the dynamical details in the black hole epoch. Not surprisingly, the density ratio is just roughly the initial BH mass over what remains in its remnant. As we will see below, the typical BH mass in our scenario, while small in astrophysical sense, is nonetheless much larger than the Planck mass, i.e., $\mu_i \gg 1$. Furthermore, the effective reheating temperature through Hawking evaporation in this case would be much higher than the energy scales associated with the standard model of particle physics and baryogenesis. We thus assume that the standard cosmology would resume after the black hole epoch. To conform with the standard cosmology, our assumption that DM is predominantly contributed from BHR demands that, by the time $t \sim t_{eq} \sim 10^{12}$sec the density contributions from radiation and BHR should be about equal, namely,

$$\frac{\Omega_{\gamma,t_{eq}}}{\Omega_{\mathrm{BHR},t_{eq}}} \sim 1 \sim \left(\frac{\tau}{t_{eq}}\right)^{1/2} \frac{\Omega_{\gamma,\tau}}{\Omega_{\mathrm{BHR},\tau}} = \left(\frac{\tau}{t_{eq}}\right)^{1/2} \beta \frac{\mu_i}{\xi} \ . \tag{17}$$

Since τ is uniquely determined by the initial BH mass μ_i (cf. Eq.(4)), the above condition translates into a constraint on H_* and s in hybrid inflation through Eq.(15):

$$\frac{m_p}{H_*} \left(\frac{2m_p}{sH_*}\right)^{3/s} \sim \left(\frac{3g\xi^2}{\alpha^5\beta^2} \frac{t_{eq}}{t_{ch}}\right)^{1/5} \ . \tag{18}$$

Figure 2 shows the region in the (H_*, s) parameter space that satisfies the above condition. We assume $g = 100$ and $\xi = 1$, and let $0.3 \leq \alpha \leq 3$ and $0.3 \leq \beta \leq 3$. The allowed (H_*, s) values are shown in the darkened region. We see that within the constraint that s be of the order unity so that the metric perturbation δ be not exponentially small, there exists a wide range of H_* that could produce the right amount of PBHs.
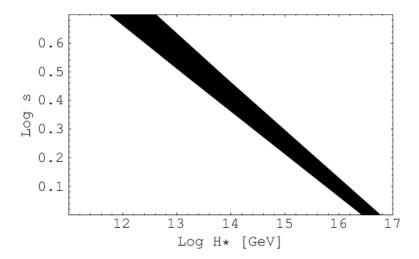
Fig. 2. Region in the hybrid inflation (H_*, s) parameter space where the induced black hole remnants would provide the right abundance for dark matter.

As an example, we take $H_* \sim 5 \times 10^{13}$ GeV and $s \sim 3$. Assume that the universe was matter-dominated when the curvature perturbation reentered the horizon. Then the density contrast is $\delta \sim 1/7$, and the fraction of matter in the BH is $P(m) \sim 10^{-2}$. From Eq.(13), $e^{N_c} \sim 54$. So the total number of e-folds is $N_c \sim 4$. The black holes were produced at the moment $t_h \sim 2 \times 10^{-33}$ sec, and had a typical mass $M_{\mathrm{BH}i} \sim 4 \times 10^{10} m_p$. Let $g \sim 100$. Then the time it took for the BHs to reduce to remnants, according to Eq.(4), is $\tau \sim 5 \times 10^{-10}$ sec. The "black hole epoch" thus ended in time before baryogenesis and other subsequent epochs in the standard cosmology. As suggested in Ref. [25], such a post-inflation PBH evaporation provides an interesting mechanism for reheating. Note that due to the continuous evaporation process and the Hubble expansion, the BH reheating should result in an effective temperature which is sufficiently lower than the Planck scale.

6 Discussion

Our arguments for the existence of BHR based on GUP is heuristic. The search for its deeper theoretical foundation is currently underway[14]. As interactions with BHR are purely gravitational, the cross section is extremely small, and direct observation of BHR seems unlikely. One possible indirect signature may be associated with the cosmic gravitational wave background. Unlike photons, the gravitons radiated during evaporation would be instantly frozen. Since, according to our notion, the BH evaporation would terminate when it reduces to a BHR, the graviton spectrum should have a cutoff at Planck mass. Such a cutoff would have by now been redshifted to $\sim \mathcal{O}(10^4)$ GeV. Another possible GW-related signature may be the GWs released during the gravitational col-

lapse at $t \sim t_h$. The frequencies of such GWs would by now be in the range of $\sim 10^7 - 10^8$ Hz. It would be interesting to investigate whether these signals are in principle observable. Another possible signature may be some imprints on the CMB fluctuations due to the thermodynamics of PBH-CMB interactions. These will be further investigated.

Acknowledgments

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