Upper and Lower Bounds for Capacities of Gaussian-like Channels

Studente: Giuseppe Catalano
Relatori: Prof. Vittorio Giovannetti, Prof. Giacomo De Palma

Introduction

Following the affirmation of quantum mechanics, which represented an epochal turning point in the development of physics, new fields of study have emerged leading to the development of innovative quantum technologies. Among these, quantum communication is gaining more and more importance, with the possibility of creating a quantum internet network [1] and ensuring secure communication through quantum key distribution protocols [2], based on properties such as entanglement, the no-cloning theorem, and the impossibility of obtaining information from a quantum state without disturbing it [3]. For these technologies to evolve, it is necessary to develop coding and decoding protocols that allow one to transcribe, in real physical states, the information one wants to send. At the same time, it is of fundamental importance to study how information, which propagates in communication lines, is degraded by noise. In this way it is possible to adapt the protocols to interactions with the external environment, which is always present in realistic cases.

In quantum information theory, a noise is described by a map from the space of the states of the system to itself that is linear, trace-preserving and completely positive; a map Φ equipped with these properties is called *quantum channel*. All quantum channels have a description in terms of a unitary coupling U with an environment in a quantum pure state; we call this description *Stinespring representation* and, for a quantum channel Φ, we obtain the following expression:

$$\Phi(\rho_S) = \mathrm{Tr}_E \left[U \rho_S \otimes |0\rangle_E \langle 0| U^\dagger \right]. \tag{1}$$

Let's consider the following communication scenario with two characters, Alice and Bob: Alice wants to send the information contained in a classical stochastic variable x to Bob and, to do so, she encodes every instance of the variable x in a state ρ_x of a quantum system called *quantum carrier*. Alice sends the quantum carrier, whose state will then be degraded by the channel of Φ and, as a result, Bob will receive the state $\Phi(\rho)$. Finally, Bob will perform a decoding procedure on the state $\Phi(\rho)$, obtaining part of the information encoded by Alice. Iterating this process it is possible to compute the asymptotic ratio between the information received by Bob and that sent by Alice; this quantity is called *transmission rate*. In other words, it represents the average information transmitted with each use of the channel using specific encoding and decoding protocols.
In the entanglement-assisted case, the communication protocol is slightly different to benefit from the presence of entanglement: Alice and Bob share an entangled pair $|\psi\rangle_{AB}$, Alice performs a certain encoding ε on the quantum system that she controls and then sends it to Bob. In the end, Bob will receive the state $(\Phi_A \circ \varepsilon_A \otimes \mathbb{1}_B)(|\psi\rangle_{AB} \langle\psi|)$ and will perform the decoding procedure on it.

To quantify the efficiency of a quantum channel, we introduce the concept of capacity of a quantum channel. The *capacity* of a quantum channel Φ is defined as the maximum rate at which information can be transmitted faithfully when the quantum carrier is subject to the noise described by the channel Φ. In quantum information theory, however, this definition is not exhaustive. There are, in fact, different types of information that can be transmitted and, for each of them, there is a specific notion of capacity, obtained by maximizing a certain functional on the possible encoding. My thesis work is focused on *entanglement-assisted classical capacity* $C_E(\Phi)$, which is the maximum rate at which classical information can be transmitted when the two parts share an unlimited amount of entanglement. The value of $C_E(\Phi)$ is obtained by maximizing a functional called *mutual quantum information* $I(\Phi, \rho)$ over the possible input states ρ, defined as follows:

$$I(\Phi, \rho) = S(\rho) + S(\Phi(\rho)) - S(\tilde{\Phi}(\rho)). \tag{2}$$

In this definition, $S(\rho)$ is the Von Neumann entropy of the state ρ, which, in this context, takes the name of *input entropy*; $S(\Phi(\rho))$ is the *output entropy*, i.e. the Von Neumann entropy of $\Phi(\rho)$ and $S(\tilde{\Phi}(\rho))$ is the *exchange entropy*, i.e. the output entropy of the complementary channel of Φ. Given a quantum channel Φ and a Stinespring representation $\Phi(\rho_S) = \mathrm{Tr}_E[U_{SE}\rho_S \otimes |0\rangle_E \langle 0| U_{SE}^\dagger]$, the complementary channel $\tilde{\Phi}$ takes the following form: $\tilde{\Phi}(\rho_S) = \mathrm{Tr}_S[U_{SE}\rho_S \otimes |0\rangle_E \langle 0| U_{SE}^\dagger]$. Essentially, the complementary channel gives us the output of the environment after the interaction with the system S.

In the context of quantum communication, the use of continuous-variable quantum systems is particularly relevant as they provide a quantum description of the propagating electromagnetic field [4]. A quantum system is said to be a *continuous variables* system when it is associated with an infinite-dimensional Hilbert space and its principal observables have a continuous spectrum. Bosonic systems exhibit these properties and represent a remarkable example as their formal properties are very well characterized from a mathematical point of view [5, 6] and they can be easily manipulated experimentally [7].
The fundamental tools for investigating the properties of Bosonic systems are Gaussian states and Gaussian channels. They provide, respectively, the description of the ground and thermal states of quadratic hamiltonians in the creation and annihilation operators and the transformations that preserve the structure of the Gaussian states [8].

Thesis project

Although, in ideal cases, quantum systems can be seen as subject to the action of one single noise, in real cases the noise a physical system is subject to is not unique. Therefore, to address this problem, one can take into consideration convex combinations of Gaussian channels, hence considering more realistic scenarios. [9, 10]. Quantum channels with fluctuating parameters are of great importance because this phenomenon is observed in the atmosphere, where it is caused by air turbulence. Therefore any implementation of quantum communication that does not rely on fiber-optical infrastructure has to deal with this kind of channel [11, 12].
However, since Gaussian channels do not form a convex set, such combinations do not have the good mathematical properties that Gaussian channels have. In particular, it has been shown that the states that maximize the quantum mutual information of Gaussian channels are Gaussian [13], while in the case of convex combinations of Gaussian channels it is not known in general what are the states that optimize communication. My thesis is placed in this context, and it aims to study the entanglement-assisted classical capacity of some families of channels built as convex combinations of thermal attenuators. A *thermal attenuator* is a particular type of Gaussian channel that describes the interaction of an input state ρ with a thermal environment σ via a beam-splitter of transmissivity $\lambda \in [0, 1]$ [14]. We can give the following Stinespring representation of a thermal attenuator $\varepsilon_{\lambda,\sigma}$:

$$\varepsilon_{\lambda,\sigma}(\rho_S) = \mathrm{Tr}_E \left[U_\lambda \rho_S \otimes \sigma_E U_\lambda^\dagger \right]. \tag{3}$$

It is worth noting that it is possible to limit the optimization of the quantum mutual information to the set of diagonal states in the Fock basis, exploiting the concavity of the quantum mutual information with respect to the input and the fact that the considered channels are phase covariant channels [15].

In the first part of my thesis, we studied convex combinations of thermal attenuators having different transmissivities and the same environment, considering as environment the vacuum state $|0\rangle_E \langle 0|$. A thermal attenuator with the environment in the void state is named *lossy channel* and it describes a photon loss with probability 1-λ; it is the prototype for optical communication through a lossy fiber, since thermal photons are negligible at room temperature [16]. Basically, a convex combination of lossy channels represents a lossy scenario where the loss parameter fluctuates; this channel takes the following form:

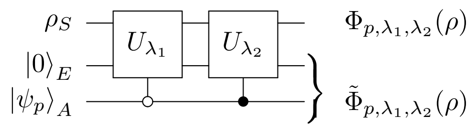
$$\Phi_{p,\lambda_1,\lambda_2} = p\varepsilon_{\lambda_1,|0\rangle\langle 0|} + (1-p)\varepsilon_{\lambda_2,|0\rangle\langle 0|}, \tag{4}$$

that is, in fact, a family of channels parametrized by $p, \lambda_1, \lambda_2 \in [0, 1]$.

The convex combination between two lossy channels has been built as follows: an ancillary qubit A, in the state $|\psi_p\rangle = \sqrt{p}\,|0\rangle + \sqrt{1-p}\,|1\rangle$, has been added to the environment to select, through appropriate controlled-U gates, one or the other thermal attenuator; this has allowed us to construct easily the complementary channel of the convex combination under investigation and to reconstruct the analytical form of quantum mutual information. Given the following Stinespring representation of a lossy channel:

$$\varepsilon_{\lambda,|0\rangle\langle 0|}(\rho) = \mathrm{Tr}_E[U_\lambda \rho_S \otimes |0\rangle_E \langle 0|\, U_\lambda^\dagger], \tag{5}$$

the circuit that represents the channel $\Phi_{p,\lambda_1,\lambda_2}$, and its complementary $\tilde{\Phi}_{p,\lambda_1,\lambda_2}$ is the following:



In my thesis, I analytically proved that, if $\lambda_1 = 1$ and $\lambda_2 = 0$, the quantum state ρ that maximizes the quantum mutual information $I(\Phi_{p,\lambda_1,\lambda_2}, \rho)$, among the states with energy E, has the following form:

$$\rho(E, p_0) = p_0\,|0\rangle\langle 0| + (1-p_0)\sigma_N, \tag{6}$$

where σ_N is a Gibbs state of the hamiltonian $H_1 = \lim\limits_{\Delta\to\infty} a^\dagger a + \Delta\,|0\rangle\langle 0|$ with mean energy equal to N and $N = \frac{E}{1-p_0}$. Moreover, the parameter p_0 depends on the channel itself and the energetic constraint; namely $p_0 = p_0(E, p)$.
Going deeper in the study of this family of channels, we considered the more general cases in which:

- λ_1 is a free parameter and $\lambda_2 = 0$ is fixed,
- λ_2 is a free parameter and $\lambda_1 = 1$ is fixed,
- λ_1 and λ_2 are both free parameters.

In all these cases it emerged that it was possible to find, among the states of the family in eq. (6), some states that perform better than the thermal one with the same energy, therefore giving a better lower bound to the quantum mutual information.
In order to improve the lower bound previously found we used an iterative method, whose steps, once fixed the energy E, are the following:

- *step 0*: evaluate the quantum mutual information in correspondence with the thermal state with energy equal to E,
- *step 1*: the lower bound is the one found with the one-parameter maximization of $I(\Phi, \rho)$ with respect to p_0 among the states of the family $\rho(E, p_0)$ in eq. (6),
- *step k*: at each step, we add one parameter to the family of states, including all the states in the previous family and then we maximize over all the free parameters.

The family of states at the k-th step is:

$$\rho_E^{(k)} = \sum_{n=0}^{k-1} p_n \prod_{i=0}^{n-1}(1-p_i)\,|n\rangle\langle n| + \prod_{n=0}^{k-1}(1-p_n)\sigma_N^{(k)} \tag{7}$$

where $\sigma_N^{(k)}$ is the Gibbs state of the hamiltonian $H_k = \lim\limits_{\Delta\to\infty} a^\dagger a + \Delta\sum_{n=0}^{k-1}|k\rangle\langle k|$ with mean energy equal to N. Using this iterative procedure, we found states that perform better, in terms of quantum mutual information, although it emerges that the performance improvement is smaller and smaller as k increases.
In all cases outlined above, it emerged that there are states which allow one to obtain a value of quantum mutual information greater than the one that we obtain in correspondence with the thermal state with the same energy. This is an interesting result since it would be possible, in principle, to use such states as encoding states, to improve communication performance. Furthermore, in some cases it has been possible to find the optimal state, that is the state that maximizes the value of quantum mutual information, that gives us the explicit value of the capacity $C_E(\Phi)$.

In the second part of my thesis, I studied convex combinations of thermal attenuators with the same transmissivity, while considering the environment at different temperatures. This model has been analyzed in a completely numerical way. The numerical optimization has been done using Python and, in particular, the libraries NumPy and QuTiP [17]; the latter was very useful as it provides tools that allow one to compute, in an optimized way, states and operators necessary for the purposes of this thesis. In this case, taking into account the values of the numerical errors, no state emerged that allows one to obtain values of quantum mutual information significantly higher than those obtained in correspondence with the thermal state with equal energy.

References

[1] Sidhu, J.S., et al., *Advances in space quantum communications*, IET Quant. Comm. **2(4)**, 182–217 (2021).

[2] Ekert A., *Quantum cryptography based on Bell's theorem*, Physical review letters, **67 6**, 661-663, (1991).

[3] Nielsen, M., and Chuang, I., Quantum Computation and Quantum Information: 10th Anniversary Edition., *Cambridge University Press*, Cambridge, (2010).

[4] Petersen V. et al., Gaussian State Description of Squeezed Light, *Phys. Rev. A*, **72**, 5, 053812 (2005).

[5] Caves C. M., Drummond P. D., Quantum limits on bosonic communication rates, *Rev. Mod. Phys.*, **66**, 2, 481 (1994).

[6] Holevo, A. and Giovannetti, V. *Quantum channels and their entropic characteristics.* Reports on progress in physics. Physical Society (Great Britain). **75**, 046001, (2012).

[7] Weedbrook, C., et al. *Gaussian quantum information*, Rev. Mod. Phys. **84**, 621-669 (2012).

[8] Serafini A., *Quantum Continuous Variables: A Primer of Theoretical Methods*, 10.1201/9781315118727, (2017).

[9] László R. et al., Fading channel estimation for free-space continuous-variable secure quantum communication, *New Journal of Physics*, **21**, 123036 (2019).

[10] P. Papanastasiou et al., *Continuous-variable quantum key distribution in uniform fast-fading channels*, Phys. Rev. A **97**, 032311 (2018).

[11] Liao SK., Cai WQ., Liu WY. et al., Satellite-to-ground quantum key distribution, *Nature* **549**, 43–47 (2017).

[12] Bedington R., Arrazola J.M., Ling A., Progress in satellite quantum key distribution, textitnpj Quantum Inf, **3**, 30 (2017).

[13] Holevo, A. and Werner, R. *Evaluating capacities of Bosonic Gaussian channels.* Physical Review A. **63**, (2000).

[14] Holevo A. S., *Quantum Systems, Channels, Information: A Mathematical Introduction*, Berlin, Boston: De Gruyter, (2012).

[15] Mark M. Wilde, *Quantum Information Theory*, Cambridge University Press, (2013).

[16] Eisert J. and Wolf M., Gaussian Quantum Channels, *Quantum Information with Continuous Variables of Atoms and Light* Chapter 2, Imperial College Press, (2007).

[17] Johansson J. R., Nation P. D. and Nori F.: "QuTiP: An open-source Python framework for the dynamics of open quantum systems.", Comp. Phys. Comm. **183**, 1760–1772 (2012)