Lower Bounds for Capacities of Gaussian-like Channels: a Realistic Quantum Communication Scenario

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Introduction

Quantum information exists in two forms: discrete and continuous. The most famous example of quantum information in the discrete form is that contained in a qubit, which is a two-level quantum system. Continuous-variable quantum systems, on the other hand, allow quantum information to be encoded in degrees of freedom with a continuous spectrum, such as those associated with the position and momentum of a particle. A quantum system is said to be a *continuous-variable* system when it is associated with an infinite-dimensional Hilbert space and its relevant observables have continuous spectra. Such systems are useful not only in the field of quantum communication [1, 2] but also in quantum computation [3] and quantum metrology [4], therefore their study assumes particular relevance. Among these fields, quantum communication is gaining more and more importance, with the possibility of creating quantum internet [5–7] and ensuring secure communication through quantum key distribution protocols [8], based on properties of quantum mechanics such as entanglement, no-cloning theorem and the impossibility of obtaining information from a quantum state without perturbing it [9]. In order for these technologies to evolve, it is necessary to develop encoding and decoding protocols that allow one to transcribe the information to be sent in physical states. At the same time, it is essential to investigate how this information propagates in the communication lines employed, in order to better control the presence of noise and the way in which this degrades the information to be transmitted. Therefore, optimizing the possible encoding states and transmission protocols is a crucial point in this field.
Bosonic systems, formally identical to a quantum harmonic oscillator, represent the prototype of continuous-variable quantum systems; this category includes, for instance, the oscillation modes of the electromagnetic field, the vibrational modes of solids and the excitons. Among these, the best candidates to implement quantum communication protocols are the modes of the electromagnetic field, thanks to the know-how accumulated over the years in both theoretical and experimental sense [10, 11], which allowed the development of tools to effectively manipulate quantum information.

The main tools for studying these systems are Gaussian states and Gaussian transformations. They provide, respectively, a description of the ground and thermal states of Hamiltonians quadratic in the creation and annihilation operators and the transformations that preserve the Gaussian structure of the states. In fact, transformations such as those associated with a beam splitter or a phase shifter, which are indeed Gaussian, map Gaussian states to Gaussian states.
A particularly relevant class of Gaussian channels is given by the thermal attenuators. A *thermal attenuator* $\varepsilon_{\lambda,\sigma}$ is a map that describes the interaction between an input state ρ and a bosonic thermal bath in the state σ, mediated by a beam splitter with transmissivity λ; when σ is the vacuum state, the channel is said to be a *lossy* channel. This channel describes a photon loss

with probability 1-λ and it is the prototype for optical communication through a lossy fiber, since thermal photons are negligible at room temperature [12].

Quantum communication scenario

Consider the following communication scenario: Alice wants to send the information contained in a stochastic classical variable x to Bob. In order to do it, she encodes every instance of the variable x in a quantum state ρ_x and sends it to Bob through a communication line, where ρ_x undergoes some noise, represented by a quantum channel Φ. Bob receives the output state $\Phi(\rho_x)$ and, by measuring it, he wants to obtain the maximum possible knowledge of the message that Alice has sent.

To quantify the quality of the communication in the presence of a certain noise, it is necessary to introduce the notion of capacity of a quantum channel. The *capacity* of a quantum channel is defined as the maximum rate at which the information can be transmitted faithfully under the presence of a certain noise [13]. There exist several notions of capacity, depending on what kind of information one is interested in, and all of them can be obtained by optimizing an entropic functional over the possible encodings of the information.

- The *classical capacity* $C(\Phi)$, obtained maximizing the Holevo information $\chi(\Phi, \{p_x, \rho_x\}) = S\left(\sum_x p_x \Phi(\rho_x)\right) - \sum_x p_x S\left(\Phi(\rho_x)\right)$. This is the case of the scenario just described.

- The *entanglement-assisted classical capacity* $C_E(\Phi)$, obtained by maximizing the quantum mutual information $I(\Phi, \rho) = S(\rho) + S(\Phi(\rho)) - S(\tilde{\Phi}(\rho))$. Here the scenario is slightly different, in fact Alice and Bob share a maximally entangled pair and Alice encodes the information in her half of the state and then sends it to Bob through the channel.

- The *quantum capacity* $Q(\Phi)$, obtained by maximizing the coherent information $J(\Phi, \rho) = S(\Phi(\rho)) - S(\tilde{\Phi}(\rho))$. In this case, the difference is that Alice does not send classical information but quantum information, in the form of a quantum state.

In these definitions, $S(\rho)$ is the Von Neumann entropy of the state ρ, while $\tilde{\Phi}$ is the complementary channel of Φ. Given a quantum channel Φ and a Stinespring representation $\Phi(\rho_S) = \mathrm{Tr}_E[U_{SE}\rho_S \otimes |0\rangle_E \langle 0| U_{SE}^\dagger]$, the complementary channel $\tilde{\Phi}$ takes the following form: $\tilde{\Phi}(\rho_S) = \mathrm{Tr}_S[U_{SE}\rho_S \otimes |0\rangle_E \langle 0| U_{SE}^\dagger]$. Essentially, the complementary channel gives us the output of the environment after the interaction with the system S.

Research proposal

All real systems are subjected to noise, which is not necessarily unique. Very often, in fact, the noise parameters can fluctuate and the best way to schematize this situation, on a physical and mathematical level, is to consider a system that evolves under a convex combination of channels, where each channel represents a possible noise acting on the system [7]. In this way, it is possible to construct a more realistic model of noise and the greater the number of channels involved, the better will be the approximation. Quantum channels with fluctuating transmissivity are of great importance because this phenomenon is observed in the atmosphere, where it is caused by air turbolence. Therefore any implementation of quantum communication which does not rely on

fiber-optical infrastructure has to deal with this kind of channels [14, 15].
Gaussian channels have been deeply studied in the last decades: it is known the value of their entanglement-assisted classical capacity and, in some cases, also their classical and quantum capacity is known [16]. However, they do not form a convex set, therefore if we take them in a convex combination, we reach an uncharted territory. My thesis is placed in this framework and aims to find the entanglement-assisted classical capacity of some convex combinations of Gaussian channels.
This means finding the encoding state ρ that, for a certain channel Φ, returns the maximum of the quantum mutual information $I(\Phi,\rho) = S(\rho) + S(\Phi(\rho)) - S(\tilde{\Phi}(\rho))$. In order to maximize $I(\Phi,\rho)$ it is useful to know that, when Φ is a convex combination of thermal attenuators, the maximum of $I(\Phi,\rho)$ is attained on a state that is diagonal in the Fock basis [17]. This is due to the properties of the thermal attenuators and to the concavity of $I(\Phi,\rho)$ with respect to the input state, therefore, it is not possible to use this feature for the maximization of $\chi(\Phi, \{p_x, \rho_x\})$ and $J(\Phi,\rho)$.
In the case of Gaussian channels, it is known that the best possible encoding is given by a Gaussian state [18]; however, in my thesis I found that this is not always the case when we consider convex combinations of Gaussian channels. In fact, I found encoding states that perform better than the thermal state with the same energy and this leads to better performance in terms of quantum mutual information. In my future research I would like to deepen the study carried out during my thesis work; in particular, I would like to find the $C_E(\Phi)$ of those channels for which I found only a lower bound. Moreover, I plan to study also other capacities, like the classical and the quantum capacity.
A more daring idea would be to characterize channels that are integral convex combinations of Gaussian channels:

$$\Phi = \int_0^1 d\lambda p(\lambda) \varepsilon_{\lambda, |0\rangle\langle 0|}, \tag{1}$$

where $\varepsilon_{\lambda,|0\rangle\langle 0|}$ is a lossy channel with transmissivity λ and $p(\lambda)$ is the probability distribution of the transmissivities. This could be a very realistic model of noise and, in particular, the channel Φ will be parametrized by the parameters of the chosen distribution. Some candidates for the distribution $p(\lambda)$ are the following:

- the Gaussian distribution, as a first general approach to model the fluctuations of λ;
- the Weibull distribution, as the fluctuations of the noise parameters may be due to atmospheric interactions and this distribution is particularly suitable for modeling phenomena of this type; indeed, it has already been proposed as a possible distribution of the transmissivities [7, 19, 20].

With this modeling, the quantum channel will no longer be characterized by the transmissivity λ but it will depend on the distribution $p(\lambda)$. Essentially, the new parameters of the noise will be the parameters of the distribution $p(\lambda)$.

How to implement the integral combinations

The first step to perform this calculation is transforming the integral convex combination into a more treatable convex combination

$$\Phi = \int_0^1 d\lambda p(\lambda) \varepsilon_{\lambda,|0\rangle\langle 0|} \simeq \sum_{n=1}^{d} p_n \varepsilon_{\lambda_n,|0\rangle\langle 0|} = \Phi^{(d)}, \tag{2}$$

where the p_n are chosen to approximate the distribution $p(\lambda)$ with a stepwise function. In particular, once fixed the distribution $p(\lambda)$ and the order of the approximation d, that is the number of p_n in the discrete distribution, we can split the interval $[0,1]$ in d equal subintervals and, for each of them, we have to choose a value of p_n and one of λ_n. A possible choice might be taking λ_n as the central point of the subinterval and p_n as the integral of the distribution $p(\lambda)$ over the $n-th$ subinterval. Another possibility for the choice of λ_n could be using the median value of the $n-th$ subinterval, i.e. if we consider an interval $[a_n, b_n]$, the value of λ_n will be the one such that:

$$\int_{a_n}^{\lambda_n} p(\lambda)d\lambda = \int_{\lambda_n}^{b_n} p(\lambda)d\lambda. \tag{3}$$

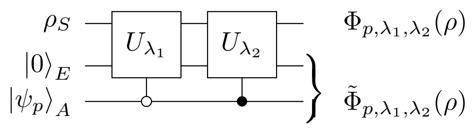
Once fixed the value of the probabilities p_n and the transmissivities λ_n, we have fixed the channel at the order d. The challenging part in the calculation of the quantum mutual information of a quantum channel is computing the exchange entropy $S(\tilde{\Phi}(\rho))$, i.e. the Von Neumann entropy of the output of the complementary channel; a simple way to construct the complementary map of a convex combination of channels is introduced in the following.
Consider a convex combination of lossy channels:

$$\Phi_{p,\lambda_1,\lambda_2} = p\varepsilon_{\lambda_1,|0\rangle\langle 0|} + (1-p)\varepsilon_{\lambda_2,|0\rangle\langle 0|}, \tag{4}$$

where $\varepsilon_{\lambda,|0\rangle\langle 0|}$ has the following Stinespring representation:

$$\varepsilon_{\lambda,|0\rangle\langle 0|}(\rho) = \mathrm{Tr}_E[U_\lambda \rho_S \otimes |0\rangle_E \langle 0| U_\lambda^\dagger]. \tag{5}$$

It is possible to enlarge the environment by introducing an ancillary qubit A in the state $|\psi_p\rangle = \sqrt{p}\,|0\rangle + \sqrt{1-p}\,|1\rangle$ that selects which channel acts on the system S. In this way, it is possible to represent the channel $\Phi_{p,\lambda_1,\lambda_2}$ and its complementary with the following circuit:



This can be easily generalized to the case of the channel $\Phi^{(d)} = \sum_{n=1}^{d} p_n \varepsilon_{\lambda_n,|0\rangle\langle 0|}$ using an ancillary qudit in the state $|\psi_p\rangle = \sum_{n=1}^{d} \sqrt{p_n}\,|n\rangle$ and d *controlled* $-$ U gates.
It is worth observing that this technique can be used with all kinds of channels and all kinds of distributions, therefore this procedure allows generalizations that could describe a huge variety of physical situations.

Ideas to perform the calculation

As already mentioned, for some classes of channels, it is possible to maximize $I(\Phi, \rho)$ considering only diagonal input states, without this being restrictive at all. A good way to approach this maximization could be an iterative method like the one I used in my thesis: the aim is to find better lower bounds to $C_E(\Phi)$ at each step of the procedure, having fixed a certain value E for the energy of the input state. The idea is to start from a one-parameter family of states with energy E and then, at each step, add a parameter to the family on which we optimize, considering a new set of states that includes all the states in the previous step. More generally, the objective is to develop iterative or perturbative methods that start from the solution of a known case and then going to improve the estimate of the capacity.

In order to perform these optimizations, it will be necessary to compute the quantum mutual information several times; this can be computationally hard, especially in the part of the exchange entropy. In my thesis I proved that, when we consider the channel reported in eq. (2) and a Fock-diagonal input, the output state is Fock-diagonal itself. Similarly, if we consider the complementary channel reported in eq. (2) and the same input, we obtain outputs that are block diagonal with $d \times d$ blocks. Starting from these considerations, it is possible to compute the quantum mutual information in a simple way: it could be possible to diagonalize the output of the complementary channel blockwise and, in order to reduce the computation time, one could parallelize the algorithm associating a core (or a CUDA core if we are doing this on a NVIDIA GPU) to each block.

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