247.

ON THE TRANSMISSION OF LIGHT THROUGH AN ATMOSPHERE CONTAINING SMALL PARTICLES IN SUSPENSION, AND ON THE ORIGIN OF THE BLUE OF THE SKY.

[*Philosophical Magazine*, XLVII. pp. 375—384, 1899.]

THIS subject has been treated in papers published many years ago*. I resume it in order to examine more closely than hitherto the attenuation undergone by the primary light on its passage through a medium containing small particles, as dependent upon the number and size of the particles. Closely connected with this is the interesting question whether the light from the sky can be explained by diffraction from the molecules of air themselves, or whether it is necessary to appeal to suspended particles composed of foreign matter, solid or liquid. It will appear, I think, that even in the absence of foreign particles we should still have a blue sky†.

The calculations of the present paper are not needed in order to explain the general character of the effects produced. In the earliest of those above

* *Phil. Mag.* XLI. pp. 107, 274, 447 (1871); XII. p. 81 (1881). [Vol. I. pp. 87, 104, 518.]

† My attention was specially directed to this question a long while ago by Maxwell in a letter which I may be pardoned for reproducing here. Under date Aug. 28, 1873, he wrote :—

"I have left your papers on the light of the sky, &c. at Cambridge, and it would take me, even if I had them, some time to get them assimilated sufficiently to answer the following question, which I think will involve less expense to the energy of the race if you stick the data into your formula and send me the result....

"Suppose that there are N spheres of density ρ and diameter s in unit of volume of the medium. Find the index of refraction of the compound medium and the coefficient of extinction of light passing through it.

"The object of the enquiry is, of course, to obtain data about the size of the molecules of air. Perhaps it may lead also to data involving the density of the æther. The following quantities are known, being combinations of the three unknowns,

M = mass of molecule of hydrogen ;

N = number of molecules of any gas in a cubic centimetre at 0° C. and 760 B.

s = diameter of molecule in any gas :—

referred to I illustrated by curves the gradual reddening of the transmitted light by which we see the sun a little before sunset. The same reasoning proved, of course, that the spectrum of even a vertical sun is modified by the atmosphere in the direction of favouring the waves of greater length.

For such a purpose as the present it makes little difference whether we speak in terms of the electromagnetic theory or of the elastic solid theory of light; but to facilitate comparison with former papers on the light from the sky, it will be convenient to follow the latter course. The small particle of volume T is supposed to be small in all its dimensions in comparison with the wave-length (λ), and to be of optical density D' differing from that (D) of the surrounding medium. Then, if the incident vibration be taken as unity, the expression for the vibration scattered from the particle in a direction making an angle θ with that of *primary vibration* is

$$\frac{D'-D}{D}\frac{\pi T}{r\lambda^2}\sin\theta\cos\frac{2\pi}{\lambda}(bt-r)^*, \qquad \text{..................(1)}$$

r being the distance from T of any point along the secondary ray.

In order to find the whole emission of energy from T we have to integrate the square of (1) over the surface of a sphere of radius r. The element of area being $2\pi r^2 \sin\theta d\theta$, we have

$$\int_0^\pi \frac{\sin^2\theta}{r^2}\,2\pi r^2\sin\theta d\theta = 4\pi\int_0^{\frac{1}{2}\pi}\sin^3\theta d\theta = \frac{8\pi}{3};$$

so that the energy emitted from T is represented by

$$\frac{8\pi^3}{3}\frac{(D'-D)^2}{D^2}\frac{T^2}{\lambda^4}, \qquad \text{............................(2)}$$

Known Combinations.

MN = density.

Ms^2 from diffusion or viscosity.

Conjectural Combination.

$\frac{6M}{\pi s^3}$ = density of molecule.

"If you can give us (i) the quantity of light scattered in a given direction by a stratum of a certain density and thickness; (ii) the quantity cut out of the direct ray; and (iii) the effect of the molecules on the index of refraction, which I think ought to come out easily, we might get a little more information about these little bodies.

"You will see by *Nature*, Aug. 14, 1873, that I make the diameter of molecules about $\frac{1}{1000}$ of a wave-length.

"The enquiry into scattering must begin by accounting for the great observed transparency of air. I suppose we have no numerical data about its absorption.

"But the index of refraction can be numerically determined, though the observation is of a delicate kind, and a comparison of the result with the dynamical theory may lead to some new information."

Subsequently he wrote, "Your letter of Nov. 17 quite accounts for the observed transparency of any gas." So far as I remember, my argument was of a general character only.

* The factor π was inadvertently omitted in the original memoir.

on such a scale that the energy of the primary wave is unity per unit of wave-front area.

The above relates to a single particle. If there be n similar particles per unit volume, the energy emitted from a stratum of thickness dx and of unit area is found from (2) by introduction of the factor ndx. Since there is no waste of energy on the whole, this represents the loss of energy in the primary wave. Accordingly, if E be the energy of the primary wave,

$$\frac{1}{E}\frac{dE}{dx} = -\frac{8\pi^3 n}{3}\frac{(D'-D)^2}{D^2}\frac{T^2}{\lambda^4}; \quad \ldots\ldots\ldots\ldots\ldots\ldots(3)$$

whence

$$E = E_0 e^{-hx}, \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(4)$$

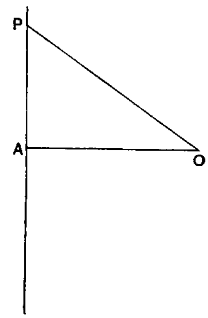
where

$$h = \frac{8\pi^3 n}{3}\frac{(D'-D)^2}{D^2}\frac{T^2}{\lambda^4}. \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(5)$$

If we had a sufficiently complete expression for the scattered light, we might investigate (5) somewhat more directly by considering the resultant of the primary vibration and of the secondary vibrations which travel in the same direction. If, however, we apply this process to (1), we find that it fails to lead us to (5), though it furnishes another result of interest. The combination of the secondary waves which travel in the direction in question has this peculiarity, that the phases are no more distributed at random. The intensity of the secondary light is no longer to be arrived at by addition of individual intensities, but must be calculated with consideration of the particular phases involved. If we consider a number of particles which all lie upon a primary ray, we see that the phases of the secondary vibrations which issue along this line are all the same.

The actual calculation follows a similar course to that by which Huygens' conception of the resolution of a wave into components corresponding to the various parts of the wave-front is usually verified. [See for example Vol. III. p. 74.] Consider the particles which occupy a thin stratum dx perpendicular to the primary ray x. Let AP (Fig. 1) be this stratum and O the point where the vibration is to be estimated. If $AP = \rho$, the element of volume is $dx \,.\, 2\pi\rho\, d\rho$, and the number of particles to be found in it is deduced by introduction of the factor n. Moreover, if $OP = r$, $AO = x$, $r^2 = x^2 + \rho^2$, and $\rho\, d\rho = r\, dr$. The resultant at O of all the secondary vibrations which issue from the stratum dx is by (1), with $\sin\theta$ equal to unity,

Fig. 1.



$$n\,dx \,.\, \int_x^\infty \frac{D'-D}{D}\frac{\pi T}{r\lambda^2}\cos\frac{2\pi}{\lambda}(bt - r)\, 2\pi r\, dr,$$

or

$$n\,dx \,.\, \frac{D'-D}{D}\frac{\pi T}{\lambda}\sin\frac{2\pi}{\lambda}(bt - x). \quad \ldots\ldots\ldots\ldots\ldots\ldots(6)$$

To this is to be added the expression for the primary wave itself, supposed to advance undisturbed, viz., $\cos\frac{2\pi}{\lambda}(bt-x)$, and the resultant will then represent the whole actual disturbance at O as modified by the particles in the stratum dx.

It appears, therefore, that to the order of approximation afforded by (1) the effect of the particles in dx is to modify the phase, *but not the intensity*, of the light which passes them. If this be represented by

$$\cos\frac{2\pi}{\lambda}(bt-x-\delta), \qquad \text{.........................(7)}$$

δ is the *retardation* due to the particles, and we have

$$\delta = nTdx\,(D'-D)/2D. \qquad \text{.......................(8)}$$

If μ be the refractive index of the medium as modified by the particles, that of the original medium being taken as unity, $\delta=(\mu-1)\,dx$, and

$$\mu-1 = nT(D'-D)/2D. \qquad \text{.......................(9)}$$

If μ' denote the refractive index of the material composing the particles regarded as continuous, $D'/D=\mu'^2$, and

$$\mu-1=\tfrac{1}{2}nT(\mu'^2-1), \qquad \text{.......................(10)}$$

reducing to

$$\mu-1=nT(\mu'-1) \qquad \text{.........................(11)}$$

in the case where $\mu'-1$ can be regarded as small.

It is only in the latter case that the formulæ of the elastic-solid theory are applicable to light. In the electric theory, to be preferred on every ground except that of easy intelligibility, the results are more complicated in that when $(\mu'-1)$ is not small, the scattered ray depends upon the shape and not merely upon the volume of the small obstacle. In the case of *spheres* we are to replace $(D'-D)/D$ by $3(K'-K)/(K'+2K)$, where K, K' are the dielectric constants proper to the medium and to the obstacle respectively*; so that instead of (10)

$$\mu-1=\frac{3nT}{2}\,\frac{\mu'^2-1}{\mu'^2+2}. \qquad \text{.......................(12)}$$

On the same suppositions (5) is replaced by

$$h=24\pi^3 n\,\frac{(\mu'^2-1)^2}{(\mu'^2+2)^2}\,\frac{T^2}{\lambda^4}. \qquad \text{....................(13)}$$

On either theory

$$h=\frac{32\pi^3(\mu-1)^2}{3n\lambda^4}, \qquad \text{.........................(14)}$$

* *Phil. Mag.* XII. p. 98 (1881). [Vol. I. p. 533.] For the corresponding theory in the case of an *ellipsoidal* obstacle, see *Phil. Mag.* Vol. XLIV. p. 48 (1897). [Vol. IV. p. 305.]

a formula giving the coefficient of transmission in terms of the refraction, and of the *number of particles per unit volume.*

We have seen that when we attempt to find directly from (1) the effect of the particles upon the transmitted primary wave, we succeed only so far as regards the retardation. In order to determine the attenuation by this process it would be necessary to supplement (1) by a term involving

$$\sin 2\pi (bt - r)/\lambda;$$

but this is of higher order of smallness. We could, however, reverse the process and determine the small term in question *à posteriori* by means of the value of the attenuation obtained indirectly from (1), at least as far as concerns the secondary light emitted in the direction of the primary ray.

The theory of these effects may be illustrated by a completely worked out case, such as that of a small rigid and fixed spherical obstacle (radius c) upon which plane waves of sound impinge*. It would take too much space to give full details here, but a few indications may be of use to a reader desirous of pursuing the matter further.

The expressions for the terms of orders 0 and 1 in spherical harmonics of the velocity-potential of the secondary disturbance are given in equations (16), (17), § 334. With introduction of approximate values of γ_0 and γ_1, viz.

$$\gamma_0 + kc = \tfrac{1}{3}k^3c^3, \qquad \gamma_1 + kc = \tfrac{1}{2}\pi + \tfrac{1}{6}k^3c^3,$$

we get

$$[\psi_0] + [\psi_1] = -\frac{k^2c^3}{3r}\left(1 + \frac{3\mu}{2}\right)\cos k(at - r) + \frac{k^5c^6}{9r}\left(1 - \frac{3\mu}{4}\right)\sin k(at - r), \quad\text{...(15)}\dagger$$

in which c is the radius of the sphere, and $k = 2\pi/\lambda$. This corresponds to the primary wave

$$[\phi] = \cos k(at + x), \quad\text{..........................(16)}$$

and includes the most important terms from all sources in the multipliers of $\cos k(at - r)$, $\sin k(at - r)$. Along the course of the primary ray ($\mu = -1$) it reduces to

$$[\psi_0] + [\psi_1] = \frac{k^2c^3}{6r}\cos k(at - r) + \frac{7k^5c^6}{36r}\sin k(at - r). \quad\text{......(17)}$$

We have now to calculate by the method of Fresnel's zones the effect of a distribution of n spheres per unit volume. We find, corresponding to (6), for the effect of a layer of thickness dx,

$$2\pi n\,dx\,\{\tfrac{1}{6}kc^3 \sin k(at + x) - \tfrac{7}{36}k^4c^6 \cos k(at + x)\}. \quad\text{.........(18)}$$

* *Theory of Sound*, 2nd ed. § 334.

† [1902. μ here denotes the sine of the latitude.]

To this is to be added the expression (16) for the primary wave. The coefficient of $\cos k(at+x)$ is thus altered by the particles in the layer dx from unity to $(1-\frac{7}{18}k^4c^6\pi n\,dx)$, and the coefficient of $\sin k(at+x)$ from 0 to $\frac{1}{3}kc^3\pi n\,dx$. Thus, if E be the energy of the primary wave,

$$dE/E = -\tfrac{7}{9}k^4c^6\pi n\,dx\,;$$

so that if, as in (4), $E = E_0 e^{-hx}$,

$$h = \tfrac{7}{9}\pi n k^4 c^6. \qquad \text{.............................(19)}$$

The same result may be obtained indirectly from the *first* term of (15). For the whole energy emitted from one sphere may be reckoned as

$$\frac{k^4c^6}{9r^2}\int_{-1}^{+1} 2\pi r^2 (1+\tfrac{3}{2}\mu)^2\,d\mu = \frac{7\pi k^4c^6}{9}, \qquad \text{..............(20)}$$

unity representing the energy of the primary wave per unit area of wave-front. From (20) we deduce the same value of h as in (19).

The first term of (18) gives the refractivity of the medium. If δ be the retardation due to the spheres of the stratum dx,

$$\sin k\delta = \tfrac{1}{3}kc^3\pi n\,dx,$$

or

$$\delta = \tfrac{1}{3}\pi n c^3 dx. \qquad \text{.............................(21)}$$

Thus, if μ be the refractive index as modified by the spheres, that of the original medium being unity,

$$\mu - 1 = \tfrac{1}{3}\pi n c^3 = \tfrac{1}{4}p, \qquad \text{..........................(22)}$$

where p denotes the (small) ratio of the volume occupied by the spheres to the whole volume. This result agrees with equations formerly obtained for the refractivity of a medium containing spherical obstacles disposed in cubic order*.

Let us now inquire what degree of transparency of air is admitted by its molecular constitution, *i.e.*, in the absence of all foreign matter. We may take $\lambda = 6\times 10^{-5}$ centim., $\mu - 1 = \cdot 0003$; whence from (14) we obtain as the distance x, equal to $1/h$, which light must travel in order to undergo attenuation in the ratio $e:1$,

$$x = 4\cdot 4\times 10^{-13}\times n. \qquad \text{..........................(23)}$$

The completion of the calculation requires the value of n. Unfortunately this number—according to Avogadro's law the same for all gases—can hardly be regarded as known. Maxwell† estimates the number of molecules under standard conditions as 19×10^{18} per cub. centim. If we use this value of n, we find

$$x = 8\cdot 3\times 10^6 \text{ cm.} = 83 \text{ kilometres},$$

* *Phil. Mag.* Vol. XXXIV. p. 499 (1892). [Vol. IV. p. 35.] Suppose $m=\infty$, $\sigma=\infty$.

† "Molecules," *Nature*, VIII. p. 440 (1873).

as the distance through which light must pass in air at atmospheric pressure before its intensity is reduced in the ratio of 2·7 : 1.

Although Mount Everest appears fairly bright at 100 miles distance as seen from the neighbourhood of Darjeeling, we cannot suppose that the atmosphere is as transparent as is implied in the above numbers; and of course this is not to be expected, since there is certainly suspended matter to be reckoned with. Perhaps the best data for a comparison are those afforded by the varying brightness of stars at various altitudes. Bouguer and others estimate about ·8 for the transmission of light through the entire atmosphere from a star in the zenith. This corresponds to 8·3 kilometres of air at standard pressure. At this rate the transmission through 83 kilometres would be $(\cdot 8)^{10}$, or ·11, instead of $1/e$ or ·37. It appears then that the actual transmission through 83 kilometres is only about 3 times less than that calculated (with the above value of n) from molecular diffraction without any allowance for foreign matter at all. And we may conclude that the light scattered from the molecules would suffice to give us a blue sky, not so very greatly darker than that actually enjoyed.

If n be regarded as altogether unknown, we may reverse our argument, and we then arrive at the conclusion that n cannot be greatly less than was estimated by Maxwell. A lower limit for n, say 7×10^{18} per cubic centimetre, is somewhat sharply indicated. For a still smaller value, or rather the increased individual efficacy which according to the observed refraction would be its accompaniment, must lead to a less degree of transparency than is actually found. When we take into account the known presence of foreign matter, we shall probably see no ground for any reduction of Maxwell's number.

The results which we have obtained are based upon (14), and are as true as the theories from which that equation was derived. In the electromagnetic theory we have treated the molecules as spherical continuous bodies differing from the rest of the medium merely in the value of their dielectric constant. If we abandon the restriction as to sphericity, the results will be modified in a manner that cannot be precisely defined until the shape is specified. On the whole, however, it does not appear probable that this consideration would greatly affect the calculation as to transparency, since the particles must be supposed to be oriented in all directions indifferently. But the theoretical conclusion that the light diffracted in a direction perpendicular to the primary rays should be *completely* polarized may well be seriously disturbed. If the view, suggested in the present paper, that a large part of the light from the sky is diffracted from the molecules themselves, be correct, the observed incomplete polarization at 90° from the Sun may be partly due to the molecules behaving rather as elongated bodies with indifferent orientation than as spheres of homogeneous material.

Again, the suppositions upon which we have proceeded give no account of *dispersion*. That the refraction of gases increases as the wave-length diminishes is an observed fact; and it is probable that the relation between refraction and transparency expressed in (14) holds good for each wave-length. If so, the falling off of transparency at the blue end of the spectrum will be even more marked than according to the inverse fourth power of the wave-length.

An interesting question arises as to whether (14) can be applied to highly compressed gases and to liquids or solids. Since approximately $(\mu - 1)$ is proportional to n, so also is h according to (14). We have no reason to suppose that the purest water is any more transparent than (14) would indicate; but it is more than doubtful whether the calculations are applicable to such a case, where the fundamental supposition, that the phases are entirely at random, is violated. When the volume occupied by the molecules is no longer very small compared with the whole volume, the fact that two molecules cannot occupy the same space detracts from the random character of the distribution. And when, as in liquids and solids, there is some approach to a regular spacing, the scattered light must be much less than upon a theory of random distribution.

Hitherto we have considered the case of obstacles small compared to the wave-length. In conclusion it may not be inappropriate to make a few remarks upon the opposite extreme case and to consider briefly the obstruction presented, for example, by a shower of rain, where the diameters of the drops are large multiples of the wave-length of light.

The full solution of the problem presented by spherical drops of water would include the theory of the rainbow, and if practicable at all would be a very complicated matter. But so far as the direct light is concerned, it would seem to make little difference whether we have to do with a spherical refracting drop, or with an opaque disk of the same diameter. Let us suppose then that a large number of small disks are distributed at random over a plane parallel to a wave-front, and let us consider their effect upon the direct light at a great distance behind. The plane of the disks may be divided into a system of Fresnel's zones, each of which will by hypothesis include a large number of disks. If α be the area of each disk, and ν the number distributed per unit of area of the plane, the efficiency of each zone is diminished in the ratio $1 : 1 - \nu\alpha$, and, so far as the direct wave is concerned, this is the only effect. The *amplitude* of the direct wave is accordingly reduced in the ratio $1 : 1 - \nu\alpha$, or, if we denote the relative opaque area by m, in the ratio $1 : 1 - m$*. A second operation of the same kind will reduce the

* The *intensity* of the direct wave is $1 - 2m$, and that of the scattered light m, making altogether $1 - m$.

amplitude to $(1-m)^2$, and so on. After x passages the amplitude is $(1-m)^x$, which if m be very small may be equated to e^{-mx}. Here mx denotes the whole opaque area passed, reckoned per unit area of wave-front; and it would seem that the result is applicable to any sufficiently sparse random distribution of obstacles.

It may be of interest to give a numerical example. If the unit of length be the centimetre and x the distance travelled, m will denote the projected area of the drops situated in one cubic centimetre. Suppose now that a is the radius of a drop, and n the number of drops per cubic centimetre, then $m = n\pi a^2$. The distance required to reduce the *amplitude* in the ratio $e : 1$ is given by

$$x = 1/n\pi a^2.$$

Suppose that $a = \frac{1}{20}$ centim., then the above-named reduction will occur in a distance of one kilometre ($x = 10^5$) when n is about 10^{-3}, *i.e.* when there is about one drop of one millimetre diameter per litre.

It should be noticed that according to this theory a distant point of light seen through a shower of rain ultimately becomes invisible, not by failure of definition, but by loss of intensity either absolutely or relatively to the scattered light.