Stabilising the triple inverted pendulum by variable gain linear quadratic regulator

Article *in* International Journal of Systems Control and Communications · August 2012

DOI: 10.1504/IJSCC.2012.048616

Modelling, Controller Design and Implementation for Spherical Double Inverted Pendulum System

Yongli Zhang
School of Automation and
Electrical Engineering
TianJin Keylaboratory of Information Sensing &
Intelligent Control
Tianjin University of Technology and Education
Tianjin 300222, China
Email: zylzhang@126.com

Guoliang Zhao
Faculty of Science
Heilongjiang Institute of Technology
Harbin 150027, China
Email: ocnzhao@gmail.com

Jiayin Wang
School of Mathematical Sciences
School of Mathematical Sciences
Beijing 100875, China
Email: wjy@bnu.edu.cn

***Abstract*—This paper design a nonlinear control law for stabilizing a spherical double inverted pendulum system (SDIPS). SDIPS is a typical 6DOF underactuated nonlinear system. The model of the SDIPS is builded and a new variable feedback gain nonlinear controller is designed to achieve high precision control based on Lyapunov function. The design of the controller is exhaustively discussed in this paper. The major feature of this control scheme is online solving the nonlinear algebraic Riccati equation and obtaining the dynamic feedback gain. Finally, the stabilization of the SDIPS demonstrates the effectiveness of the proposed control scheme. The results of simulation and physical experiments illustrate the proposed nonlinear regulator has good adaptivity and strong robustness.**

I. INTRODUCTION

The research on inverted pendulum systems relate to many fields, such as dynamic modelling, mechanic and electric design, and kinds of control problems, etc. Many contributions exist for the control of different types of inverted pendulum systems. Some famous experiments about inverted pendulum have been done in the past decades, such as [1]–[5] et al. In [5], [6] Åström discussed the problems of swing-up a pendulum in detail. In recent years, some new results relate to inverted pendulum systems have emerged, such as [7]–[16] et.al. There are many control methods applied to pendulum systems, such as gain scheduling control [17], variable universe adaptive fuzzy control [15], SDRE technique [18], dynamic inversion control [7], nested saturation control technique [16], and so on.

As is well known, the inverted pendulum offers a very good example for control engineers to verify a control theory and attracts the attention of control researchers in recent decades as a benchmark for testing and evaluating a wide range of classical and contemporary nonlinear control method. Recently, some new methods are presented some solutions for stabilization of spherical inverted pendulums. The spherical inverted pendulum represents a more challenging control problem, which continually moves toward an uncontrolled state.

The SDIPS is a more high sensitive nonlinear underactuated mechanical system, which has six degrees of freedom and only two control inputs are exerted on the cart. Therefore, the SDIPS is a marginally unstable system. This means that standard linear techniques cannot model the nonlinear dynamics of the system. The modelling of a simplified spherical single inverted pendulum (i.e., a bob with mass m supported by a massless beam) on a cart was considered in [19], [20]. The modelling for a slim cylindrical beam, a rigid body, was given explicitly in [11], [12]. However, the 6DOF SDIPS consists a cart and two rods (i.e. two pendulums). The two rods are cascade linked and attached to the cart via universal joint (see Fig. 2). In this paper, the motion of rigid bodies in three dimensions, which constrained either to rotate about a fixed axis or to move parallel to a fixed plane, is analyzed by using mechanics of rigid body and coordinate transformation principle, and then the model of the spherical double inverted pendulum is derived by using Lagrange method.

In this paper, we develop a nonlinear regulator, variable gain LQR controller (VGLQR), for a class of nonlinear systems. The control method is derived by using the Lyapunov function approach, and the linear quadratic regulation (LQR) theory is employed to address nonlinear regulation problem. This control scheme can be viewed as a new version of gain scheduling controller. The significant feature of the VGLQR method is that feedback gain is changing with state variables, and the adaptivity of the control scheme is guaranteed. In practice, all measurements are only taken at discrete sampling instants, and the control action must be piecewise constant, and actually the VGLQR control method is equivalent to using LQR method to obtain optimal feedback gain in every sampling interval. Therefore, the VGLQR method can guarantee asymptotic stability of the nonlinear control system, and to some extent has the desirable suboptimal property.

The key of the VGLQR method is computing the Jacobi matrixes of the nonlinear system and solving the nonlinear algebraic Riccati equation (ARE) online. Hence, the Schur method [21] for solving ARE is outlined in the paper, and a fast algorithm based on Schur method is developed to solve ARE online during the present research. Finally, the simulation and physical experiments are conducted, and the spherical double inverted pendulum is stabilized by using the VGLQR method. The results of simulations and physical experiments are displayed, which demonstrate the potential of the control scheme.

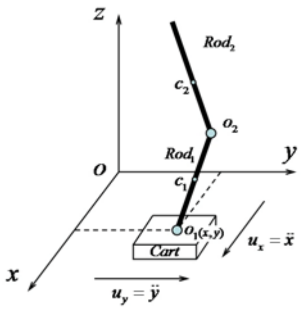

Fig. 1. The simplified structure of the spherical double inverted pendulum system.

TABLE I. MECHANICAL PARAMETERS OF THE THE SPHERICAL DOUBLE INVERTED PENDULUM SYSTEM.

Link	Rod_1	Rod_2
m_i	$0.278(Kg)$	$0.171(Kg)$
J_i	$0.0022153(Kg\cdot m^2)$	$0.0098123(Kg\cdot m^2)$
l_i	$0.128(m)$	$0.228(m)$
L_i	$0.205(m)$	$0.563(m)$

The rest of the paper is organized as follows. In Section II, the modelling for SDIPS is derived. In Section III, a nonlinear regulator , the VGLQR controller, for a class of nonlinear systems is proposed. In Section IV, a fast algorithm for the solution of algebraic Riccati equation based on Schur method is developed. In Section V, a spherical double inverted pendulum is stabilized by utilizing VGLQR method, and the experimental results are displayed. In Section VI, some concluding remarks are made.

II. MODELLING FOR SPHERICAL DOUBLE INVERTED PENDULUM

As shown in Fig.1, the spherical double pendulum system (SDIPS) consists of two rods with different lengths L_i. The first rod is attached to a cart moving on a plane with the position coordinate $(x(t), y(t))$. The two rods are cascade linked by using universal joint, and each rod (e.g. rod$_i$) can rotate in three-dimensional Euclidean space around its joint (e.g. O_i). The rotation of the rod$_i$ is given by the rotation angles ψ_i, θ_i and $\varphi_i = 0$. The other mechanical parameters are denoted as: c_i the centroid of rod$_i$, m_i the mass of rod$_i$, J_i the moment of inertia of rod$_i$ around c_i (It is designed that the moment of inertia $J_{xx} = J_{yy} = J_i$ and the moment of inertia about the $z-axis$ J_{zz} is neglected), l_i the distance from O_i to c_i, $(i = 1, 2)$. Here all the friction are neglected. The equilibrium point of the double inverted pendulum is that the rod$_1$ and rod$_2$ are all in upright position and the cart's position is designated as zero point (i.e., $x = 0, y = 0, \psi_i = 0, \theta_i = 0$). The values of the mechanical parameters are listed in Table 1. The acceleration $\ddot{x}(t)$ and $\ddot{y}(t)$ of the cart serves as input $u_x(t) = \ddot{x}(t)$ and $u_y(t) = \ddot{y}(t)$ to the system respectively. In addition, we define that the clockwise rotation angular is positive in a two dimensional right hand coordinate system. In the experimental set-up (see Section 5), $\ddot{x}(t)$ and $\ddot{y}(t)$ are controlled by a variable gain LQR controller.

The mathematical model for the SDIPS is constructed by using Lagrange method under the following assumptions: (i) the pendulums (rod$_1$, rod$_2$) are rigid bodies, (ii) there is no frictional force in the system. As shown in Fig.1,the coordinate of the centroid of rod$_i$ in $O-xyz$ is derived as

$$\begin{aligned} x_{c_i} &= x + \sum_{k=1}^{i-1} L_k \sin\theta_k + l_i \sin\theta_i, \\ y_{c_i} &= y + \sum_{k=1}^{i-1} L_k \sin\psi_k \cos\theta_k + l_i \sin\psi_i \cos\theta_i, \\ z_{c_i} &= \sum_{k=1}^{i-1} L_k \cos\psi_k \cos\theta_k + l_i \cos\psi_i \cos\theta_i, \end{aligned} \tag{1}$$

where $i = 1,\ 2.$

Together with the cart dynamics and , the overall model of the spherical double inverted pendulum can be formally written as a system of second-order ODEs

$$M(\boldsymbol{z}, \dot{\boldsymbol{z}})\ddot{\boldsymbol{z}} + C(\boldsymbol{z}, \dot{\boldsymbol{z}})\dot{\boldsymbol{z}} + G(\boldsymbol{z}) = \tau, \tag{2}$$

where

$$M(\boldsymbol{z}, \dot{\boldsymbol{z}}) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ f_1 & p_1 & b_1 + J_1 & d_{12} & 0 & e_{12} \\ f_2 & p_2 & d_{12} & b_2 + J_2 & \bar{e}_{12} & 0 \\ 0 & h_1 & 0 & \bar{e}_{12} & c_{11} + J_1 & c_{12} \\ 0 & h_2 & e_{12} & 0 & c_{12} & c_{22} + J_2 \end{bmatrix},$$

$$C(\boldsymbol{z}, \dot{\boldsymbol{z}}) = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & u_{12}\dot{\psi}_2 & v_{11}\dot{\psi}_1 & \bar{v}_{12}\dot{\psi}_2 \\ 0 & 0 & -u_{12}\dot{\psi}_1 & 0 & v_{12}\dot{\psi}_1 & v_{22}\dot{\psi}_2 \\ 0 & 0 & -v_{11}\dot{\psi}_1 & -v_{12}\dot{\psi}_2 & 0 & w_{12}\dot{\psi}_2 \\ 0 & 0 & -\bar{v}_{12}\dot{\psi}_1 & -v_{22}\dot{\psi}_2 & -w_{12}\dot{\psi}_1 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -r_{12}\dot{\theta}_2 & 0 & u_{12}\dot{\theta}_2 \\ 0 & 0 & -\bar{r}_{12}\dot{\theta}_1 & 0 & -u_{12}\dot{\theta}_1 & 0 \\ 0 & 0 & 0 & w_{12}\dot{\theta}_2 & -v_{11}\dot{\theta}_1 & -v_{12}\dot{\theta}_2 \\ 0 & 0 & -w_{12}\dot{\theta}_1 & 0 & -\bar{v}_{12}\dot{\theta}_1 & -v_{22}\dot{\theta}_2 \end{bmatrix},$$

$$G(\boldsymbol{z}) = \begin{bmatrix} 0 \\ 0 \\ -a_1 g \cos\psi_1 \sin\theta_1 \\ -a_2 g \cos\psi_2 \sin\theta_2 \\ -a_1 g \sin\psi_1 \cos\theta_1 \\ -a_2 g \sin\psi_2 \cos\theta_2 \end{bmatrix}, \qquad \tau = \begin{bmatrix} u_x \\ u_y \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix},$$

And where we define that

$$a_i = m_i l_i + \sum_{j=i+1}^{n} m_j L_i, \quad b_i = m_i l_i^2 + \sum_{j=i+1}^{n} m_j L_i^2,\ (n = 2),$$

$$f_i = a_i \cos\theta_i, \quad h_i = a_i \cos\theta_i \cos\psi_i, \quad p_i = -a_i \sin\theta_i \sin\psi_i,$$

$$c_{ij} = \begin{cases} L_i a_j \cos\theta_i \cos\theta_j \cos(\psi_i - \psi_j), & i \neq j, \\ b_i \cos\theta_i \cos\theta_j \cos(\psi_i - \psi_j), & i = j, \end{cases}$$

$$d_{ij} = \begin{cases} L_i a_j (\cos\theta_i \cos\theta_j + \sin\theta_i \sin\theta_j \cos(\psi_i - \psi_j)), & i \neq j, \\ b_i (\cos\theta_i \cos\theta_j + \sin\theta_i \sin\theta_j \cos(\psi_i - \psi_j)), & i = j, \end{cases}$$

$$e_{ij} = \begin{cases} -L_i a_j \sin\theta_i \cos\theta_j \sin(\psi_i - \psi_j), & i \neq j, \\ -b_i \sin\theta_i \cos\theta_j \sin(\psi_i - \psi_j), & i = j, \end{cases}$$

$$\bar{e}_{ij} = \begin{cases} L_i a_j \sin\theta_j \cos\theta_i \sin(\psi_i - \psi_j), & i \neq j, \\ b_i \sin\theta_j \cos\theta_i \sin(\psi_i - \psi_j), & i = j, \end{cases}$$

$$r_{ij} = \begin{cases} L_i a_j (\cos\theta_i \sin\theta_j - \sin\theta_i \cos\theta_j \cos(\psi_i - \psi_j)), & i \neq j, \\ b_i (\cos\theta_i \sin\theta_j - \sin\theta_i \cos\theta_j \cos(\psi_i - \psi_j)), & i = j, \end{cases}$$

$$\bar{r}_{ij} = \begin{cases} L_i a_j (\cos\theta_j \sin\theta_i - \sin\theta_j \cos\theta_i \cos(\psi_i - \psi_j)), & i \neq j, \\ b_i (\cos\theta_j \sin\theta_i - \sin\theta_j \cos\theta_i \cos(\psi_i - \psi_j)), & i = j, \end{cases}$$

$$u_{ij} = \begin{cases} L_i a_j \sin\theta_i \sin\theta_j \sin(\psi_i - \psi_j), & i \neq j, \\ b_i \sin\theta_i \sin\theta_j \sin(\psi_i - \psi_j), & i = j, \end{cases}$$
$$v_{ij} = \begin{cases} L_i a_j \cos\theta_i \sin\theta_j \cos(\psi_i - \psi_j), & i \neq j, \\ b_i \cos\theta_i \sin\theta_j \cos(\psi_i - \psi_j), & i = j, \end{cases}$$
$$\bar{v}_{ij} = \begin{cases} L_i a_j \cos\theta_j \sin\theta_i \cos(\psi_i - \psi_j), & i \neq j, \\ b_i \cos\theta_j \sin\theta_i \cos(\psi_i - \psi_j), & i = j, \end{cases}$$
$$w_{ij} = \begin{cases} L_i a_j \cos\theta_i \cos\theta_j \sin(\psi_i - \psi_j), & i \neq j, \\ b_i \cos\theta_i \cos\theta_j \sin(\psi_i - \psi_j), & i = j, \end{cases}$$

where $i = 1, 2;\ j = 1, 2.$

Let $\boldsymbol{x} = [x, y, \theta_1, \theta_2, \psi_1, \psi_2, \dot{x}, \dot{y}, \dot{\theta}_1, \dot{\theta}_2, \dot{\psi}_1, \dot{\psi}_2]^T$. According to the equation (2), we can obtain the state-space model

$$\begin{aligned} \dot{\boldsymbol{x}} &= \boldsymbol{f}(\boldsymbol{x}, u), \\ \boldsymbol{y} &= \boldsymbol{x}. \end{aligned} \tag{3}$$

where $\boldsymbol{f}(\boldsymbol{x}, u) = [f_1, f_2, \cdots, f_{12}]^T$.

III. The nonlinear regulator design for a class of nonlinear systems

Consider the following nonlinear control system

$$\begin{aligned} \dot{\boldsymbol{x}} &= \boldsymbol{f}(\boldsymbol{x}, \boldsymbol{u}), \\ \boldsymbol{y} &= C\boldsymbol{x}, \qquad C = I, \end{aligned} \tag{4}$$

where $\boldsymbol{x} \in \mathbb{R}^n$ is the state, $\boldsymbol{u} \in \mathbb{R}^m$ the unconstrained control input, $\boldsymbol{y}$ the measured output. $f(\boldsymbol{x}, \boldsymbol{u})$ is a continuously differentiable vector function of $\boldsymbol{x}$ and $\boldsymbol{u}$, and we simplify $f(\boldsymbol{x}, \boldsymbol{u})$ as $\boldsymbol{f}$ in the following sections. It is assumed that the system (4) has an equilibrium point at origin, and the neighborhood of the equilibrium point is denoted as $\mathbb{B}(\boldsymbol{0}, \varepsilon)$. Our control objective is to seek a feedback control $\boldsymbol{u} = \boldsymbol{\phi}(\boldsymbol{x})$ to stabilize all the state variables of (4) to zero. For system (4), we have the following assumptions:

Assumption 1. The whole state $\boldsymbol{x}$ can be measured and is available for control.

Assumption 2. $\boldsymbol{\phi}(\boldsymbol{x})$ is a continuously differentiable function of $\boldsymbol{x}$.

Assumption 3. $A(\boldsymbol{x}, \boldsymbol{u})$ and $B(\boldsymbol{x}, \boldsymbol{u})$ are detectable.

Assumption 4. The pair$(A(\boldsymbol{x}, \boldsymbol{u}), C)$ is observable for all $\boldsymbol{x} \in \mathbb{B}(\boldsymbol{0}, \varepsilon)$.

Assumption 5. The pair$(A(\boldsymbol{x}, \boldsymbol{u}), B(\boldsymbol{x}, \boldsymbol{u}))$ is controllable for all $\boldsymbol{x} \in \mathbb{B}(\boldsymbol{0}, \varepsilon)$.

The Jacobi matrixes are denoted as

$$A(\boldsymbol{x}, \boldsymbol{u}) \triangleq \frac{\partial \boldsymbol{f}}{\partial \boldsymbol{x}}, \quad B(\boldsymbol{x}, \boldsymbol{u}) \triangleq \frac{\partial \boldsymbol{f}}{\partial \boldsymbol{u}}, \quad \Gamma(\boldsymbol{x}) \triangleq \frac{\partial \boldsymbol{\phi}}{\partial \boldsymbol{x}}. \tag{5}$$

Theorem 1 If $\boldsymbol{x} \in \mathbb{B}(\boldsymbol{0}, \varepsilon)$, the pair $(A(\boldsymbol{x}, \boldsymbol{u}), B(\boldsymbol{x}, \boldsymbol{u}))$ is controllable and the pair $(A(\boldsymbol{x}, \boldsymbol{u}), C)$, where $C = I_{n \times n}$, is observable, then the origin of the nonlinear system (4) is stable.

Proof Assume that $P(\boldsymbol{x})$ is a symmetric positive definite matrix, the elements of $P(\boldsymbol{x})$ are function of $\boldsymbol{x}$, and $P(\boldsymbol{x})$ satisfies $\dot{p} = \frac{dP(\boldsymbol{x})}{dt} = \boldsymbol{0}$. We define the Lyapunov function V as $V(x) = \boldsymbol{f}^T P(\boldsymbol{x}) \boldsymbol{f}$. Since

$$\begin{aligned} \dot{\boldsymbol{f}} = \frac{d\boldsymbol{f}}{dt} &= \frac{\partial \boldsymbol{f}}{\partial \boldsymbol{x}} \frac{d\boldsymbol{x}}{dt} + \frac{\partial \boldsymbol{f}}{\partial \boldsymbol{u}} \frac{\partial \boldsymbol{u}}{\partial \boldsymbol{x}} \frac{d\boldsymbol{x}}{dt} \\ &= [A(\boldsymbol{x}, \boldsymbol{u}) + B(\boldsymbol{x}, \boldsymbol{u})\Gamma(\boldsymbol{x})]\dot{\boldsymbol{x}} \\ &= [A(\boldsymbol{x}, \boldsymbol{u}) + B(\boldsymbol{x}, \boldsymbol{u})\Gamma(\boldsymbol{x})]\boldsymbol{f}, \end{aligned}$$

we have

$$\begin{aligned} \dot{V} &= \frac{dV(\boldsymbol{x})}{dt} = \boldsymbol{f}^T P(\boldsymbol{x}) \dot{\boldsymbol{f}} + \dot{\boldsymbol{f}}^T P(\boldsymbol{x}) \boldsymbol{f} + \boldsymbol{f}^T \dot{P}(\boldsymbol{x}) \dot{\boldsymbol{f}} \\ &= \boldsymbol{f}^T P(\boldsymbol{x})[A(\boldsymbol{x}, \boldsymbol{u}) + B(\boldsymbol{x}, \boldsymbol{u})\Gamma(\boldsymbol{x})]\boldsymbol{f} \\ &\quad + \boldsymbol{f}^T [A(\boldsymbol{x}, \boldsymbol{u})^T + \Gamma(\boldsymbol{x})^T B(\boldsymbol{x}, \boldsymbol{u})^T] P(\boldsymbol{x}) \boldsymbol{f}. \end{aligned} \tag{6}$$

Let

$$\begin{aligned} J = \; & P(\boldsymbol{x})A(\boldsymbol{x}, \boldsymbol{u}) + P(\boldsymbol{x})B(\boldsymbol{x}, \boldsymbol{u})\Gamma(\boldsymbol{x}) \\ & + A(\boldsymbol{x}, \boldsymbol{u})^T P(\boldsymbol{x}) + \Gamma(\boldsymbol{x})^T B(\boldsymbol{x}, \boldsymbol{u})^T P(\boldsymbol{x}). \end{aligned}$$

We obtain $\dot{V} = \boldsymbol{f}^T J \boldsymbol{f}$. If J is a negative semidefinite matrix (ie. $J \leq 0$), we have

$$\dot{V} = \boldsymbol{f}^T J \boldsymbol{f} \leq 0. \tag{7}$$

In this case, we can conclude that the origin of the nonlinear system (4) will be stable. Therefore, the core of the subject is if there exists a positive definite solution $P(\boldsymbol{x}) > 0$, such that

$$\begin{aligned} J = \; & P(\boldsymbol{x})A(\boldsymbol{x}, \boldsymbol{u}) + P(\boldsymbol{x})B(\boldsymbol{x}, \boldsymbol{u})\Gamma(\boldsymbol{x}) \\ & + A(\boldsymbol{x}, \boldsymbol{u})^T P(\boldsymbol{x}) + \Gamma(\boldsymbol{x})^T B(\boldsymbol{x}, \boldsymbol{u})^T P(\boldsymbol{x}) \leq 0. \end{aligned} \tag{8}$$

Since $\boldsymbol{u}$ is unconstrained control input of the system, the $\Gamma(\boldsymbol{x})$ can be set as

$$\Gamma(\boldsymbol{x}) = -K(\boldsymbol{x}), \tag{9}$$

where $K(\boldsymbol{x})$ is given by

$$K(\boldsymbol{x}) = R^{-1} B(\boldsymbol{x}, \boldsymbol{u})^T P(\boldsymbol{x}), \tag{10}$$

and R is an $m \times m$ symmetric positive definite matrix. Substituting (9) through (10) into (8) yields

$$\begin{aligned} & P(\boldsymbol{x})A(\boldsymbol{x}, \boldsymbol{u}) + A(\boldsymbol{x}, \boldsymbol{u})^T P(\boldsymbol{x}) \\ & -2P(\boldsymbol{x})B(\boldsymbol{x}, \boldsymbol{u})R^{-1}B(\boldsymbol{x}, \boldsymbol{u})^T P(\boldsymbol{x}) \leq 0. \end{aligned} \tag{11}$$

Consider the following equation

$$\begin{aligned} & P(\boldsymbol{x})A(\boldsymbol{x}, \boldsymbol{u}) + A(\boldsymbol{x}, \boldsymbol{u})^T P(\boldsymbol{x}) \\ & -2P(\boldsymbol{x})B(\boldsymbol{x}, \boldsymbol{u})R^{-1}B(\boldsymbol{x}, \boldsymbol{u})^T P(\boldsymbol{x}) = -C^T Q C, \end{aligned} \tag{12}$$

where Q is a symmetric positive semidefinite matrix. We can rewrite (12) as

$$\begin{aligned} & P(\boldsymbol{x})A(\boldsymbol{x}, \boldsymbol{u}) + A(\boldsymbol{x}, \boldsymbol{u})^T P(\boldsymbol{x}) \\ & -P(\boldsymbol{x})B(\boldsymbol{x}, \boldsymbol{u})(\tfrac{1}{2}R)^{-1}B(\boldsymbol{x}, \boldsymbol{u})^T P(\boldsymbol{x}) + C^T Q C = 0. \end{aligned} \tag{13}$$

Obviously, the equation (13) is an algebraic Riccati equation. Since the pair $(A(\boldsymbol{x}, \boldsymbol{u}), B(\boldsymbol{x}, \boldsymbol{u}))$ is controllable and the pair $(A(\boldsymbol{x}, \boldsymbol{u}), C)$ is observable, there exists a unique positive definite solution $P(\boldsymbol{x})$ to the equation (13) according to optimal regulation theory, and the inequality (8) will be satisfied. Hence, the inequality (7) will be satisfied.

This completes the proof.

Theorem 2 Assumed that $\boldsymbol{x} \in \mathbb{B}(\boldsymbol{0}, \varepsilon)$, the pair $(A(\boldsymbol{x}, \boldsymbol{u}), B(\boldsymbol{x}, \boldsymbol{u}))$ is controllable and the pair $(A(\boldsymbol{x}, \boldsymbol{u}), C)$,where $C = I_{n \times n}$, is observable, then the origin of the nonlinear system (4) can be stabilized if there exists a feedback gain

$$K(\boldsymbol{x}) = R^{-1} B(\boldsymbol{x}, \boldsymbol{u})^T P(\boldsymbol{x}), \tag{14}$$

where $P(\boldsymbol{x})$ is the unique solution for the Riccati equation

$$\begin{aligned} & P(\boldsymbol{x})A(\boldsymbol{x}, \boldsymbol{u}) + A(\boldsymbol{x}, \boldsymbol{u})^T P(\boldsymbol{x}) \\ & -P(\boldsymbol{x})B(\boldsymbol{x}, \boldsymbol{u})R^{-1}B(\boldsymbol{x}, \boldsymbol{u})^T P(\boldsymbol{x}) + C^T Q C = 0. \end{aligned} \tag{15}$$

Proof We define the Lyapunov function V as (III), and let $\Gamma(\boldsymbol{x}) = -K(\boldsymbol{x})$. Rewrite (15) as

$$\begin{array}{l} P(\boldsymbol{x})A(\boldsymbol{x},\boldsymbol{u}) + A(\boldsymbol{x},\boldsymbol{u})^T P(\boldsymbol{x}) \\ -2P(\boldsymbol{x})B(\boldsymbol{x},\boldsymbol{u})R^{-1}B(\boldsymbol{x},\boldsymbol{u})^T P(\boldsymbol{x}) = -M, \end{array} \tag{16}$$

where $M = C^T QC + P(\boldsymbol{x})B(\boldsymbol{x},\boldsymbol{u})R^{-1}B(\boldsymbol{x},\boldsymbol{u})^T P(\boldsymbol{x})$.

Note that $M \geq 0$. Therefor, if there exists a solution matrix $P(\boldsymbol{x})$ for equation (15), the $\dot{V} \leq 0$ will be satisfied. Since the pair $(A(\boldsymbol{x},\boldsymbol{u}), B(\boldsymbol{x},\boldsymbol{u}))$ is controllable and the pair $(A(\boldsymbol{x},\boldsymbol{u}), C)$ is observable, there exists a unique positive definite solution $P(\boldsymbol{x})$ to the equation (15) according to optimal regulation theory.

This completes the proof.

Theorem 3 With regard to nonlinear (4), assumed that the equilibrium point is $(\boldsymbol{x}(0) = \mathbf{0}, \boldsymbol{u}(0) = \mathbf{0})$. If there exists a feedback gain (14), a stabilizing state feedback control law will be

$$\boldsymbol{u} = -K(\boldsymbol{x})\boldsymbol{x}. \tag{17}$$

Proof According Theorem 1 and Theorem 2, we can obtain a stabilizing state feedback control law

$$\boldsymbol{u}(t) = \int_{\boldsymbol{x}(t_0)}^{\boldsymbol{x}(t)} -K(\boldsymbol{\xi})d\boldsymbol{\xi}, \quad \boldsymbol{x}(t_0) = \mathbf{0}, \boldsymbol{u}(t_0) = \mathbf{0}, \tag{18}$$

where $K(\boldsymbol{x})$ is taken as (14), which can be rewritten as

$$K(\boldsymbol{x}(t)) = R^{-1}B(\boldsymbol{x}(t), \phi(\boldsymbol{x}(t)))^T P(\boldsymbol{x}(t)). \tag{19}$$

Spouse that the discrete interval $[t_0, t]$ is divided into

$$0 = t_0 < t_1 < \cdots < t_n = t.$$

And we have the sequence of $\boldsymbol{x}(t_i)$ and $K(x(t_i))$ as follow

$$\begin{array}{llll} \boldsymbol{x}(t_0) = 0, & \boldsymbol{x}(t_1), & \cdots, & \boldsymbol{x}(t_n) = \boldsymbol{x}(t), \\ K(\boldsymbol{x}(t_0)) = \mathbf{0}, & K(\boldsymbol{x}(t_1)), & \cdots, & K(\boldsymbol{x}(t_n)) = K(\boldsymbol{x}(t)). \end{array}$$

Note that the sequence of $\boldsymbol{x}(t_i)$ and $K(x(t_i))$ don't have monotonicity. Since $K(\boldsymbol{x})$ is continuous for $\boldsymbol{x}$, we have

$$\begin{array}{rcl} \boldsymbol{u}(t) & = & \int_{\boldsymbol{x}(t_0)}^{\boldsymbol{x}(t)} -K(\boldsymbol{\xi})d\boldsymbol{\xi} \\ & = & -\left(\int_{\boldsymbol{x}(t_0)}^{\boldsymbol{x}(t_1)} K(\boldsymbol{\xi})d\boldsymbol{\xi} + \int_{\boldsymbol{x}(t_1)}^{\boldsymbol{x}(t_2)} K(\boldsymbol{\xi})d\boldsymbol{\xi} + \cdots \right. \\ & & \left. + \int_{\boldsymbol{x}(t_{n-1})}^{\boldsymbol{x}(t_n)} K(\boldsymbol{\xi})d\boldsymbol{\xi} \right). \end{array}$$

Applying the mean value theorem, yields

$$\begin{array}{rl} \boldsymbol{u}(t) = & -\left(K(\boldsymbol{\eta}_1)(\boldsymbol{x}_1 - \boldsymbol{x}_0) + K(\boldsymbol{\eta}_2)(\boldsymbol{x}_2 - \boldsymbol{x}_1) \right. \\ & \left. + \cdots + K(\boldsymbol{\eta}_n)(\boldsymbol{x}_n - \boldsymbol{x}_{n-1})\right), \end{array} \tag{20}$$

where $\boldsymbol{x}(t_i)$ is denoted as $\boldsymbol{x}_i$ and $\boldsymbol{\eta}_i$ is between $\boldsymbol{x}_i$ and $\boldsymbol{x}_{i-1}$. we take $\tau_i \triangleq t_i - t_{i-1}$. When $\max(\tau_i)$ is sufficient small, the $\Delta\boldsymbol{x}_i \triangleq \boldsymbol{x}_i - \boldsymbol{x}_{i-1}$ is sufficient small, and $\Delta K(\boldsymbol{x}_i) \triangleq K(\boldsymbol{x}_i) - K(\boldsymbol{x}_{i-1})$ is also sufficient small. Therefore, when n tends to the infinity(i.e., $n \to \infty$), and the $\max(\tau_i)$ is sufficient small (i.e., $\max(\tau_i) \to 0$), we have

$$\begin{array}{lll} \boldsymbol{\eta}_i \to \boldsymbol{x}_i, & \text{and} & \boldsymbol{\eta}_i \to \boldsymbol{x}_{i-1}, \\ K(\boldsymbol{\eta}_i) \to K(\boldsymbol{x}_i), & \text{and} & K(\boldsymbol{\eta}_i) \to K(\boldsymbol{x}_{i-1}), \end{array}$$

then yields

$$K(\boldsymbol{\eta}_i)(\boldsymbol{x}_i - \boldsymbol{x}_{i-1}) \approx K(\boldsymbol{x}_i)\boldsymbol{x}_i - K(\boldsymbol{x}_{i-1})\boldsymbol{x}_{i-1}. \tag{21}$$

Substituting (21) into (20) yields

$$\begin{array}{rcl} \boldsymbol{u}(t) & \approx & -((K(\boldsymbol{x}_1)\boldsymbol{x}_1 - K(\boldsymbol{x}_0)\boldsymbol{x}_0) \\ & & +(K(\boldsymbol{x}_2)\boldsymbol{x}_2 - K(\boldsymbol{x}_1)\boldsymbol{x}_1) \\ & & + \cdots + (K(\boldsymbol{x}_n)\boldsymbol{x}_n - K(\boldsymbol{x}_{n-1})\boldsymbol{x}_{n-1})) \\ & \approx & -(K(\boldsymbol{x}_n)\boldsymbol{x}_n - K(\boldsymbol{x}_0)\boldsymbol{x}_0) \\ & \approx & -K(\boldsymbol{x}(t_n))\boldsymbol{x}(t_n). \end{array}$$

Therefore, when $\max(\tau_i) \to 0$, we have

$$\boldsymbol{u}(t) = \int_{\boldsymbol{x}(t_0)}^{\boldsymbol{x}(t)} -K(\boldsymbol{\xi})d\boldsymbol{\xi} = -K(\boldsymbol{x}(t))\boldsymbol{x}(t). \tag{22}$$

This completes the proof.

In practice, the control action is piecewise constant, and all measurements are only taken at discrete sampling intervals. Therefore, the solution of $P(\boldsymbol{x}(t))$ for equation (15) can be obtained in every sampling interval. We can obtain the feedback matrix $K(\boldsymbol{x}(t))$ at sampling point $\boldsymbol{x}(t)$.

From the foregoing analysis, the present control law is derived by using the Lyapunov function approach, and the linear quadratic regulation (LQR) theory is employed to address nonlinear regulation problem. The significant feature of the controller is that feedback gain is changing with state variables. Hence we call the nonlinear regulator "variable gain LQR (VGLQR)" controller. Because $K(\boldsymbol{x})$ is changing with the plant state, the present controller has high adaptivity and robustness.

In addition, Theorem 1, Theorem 2 and Theorem 3 are obtained under the Assumption 3, Assumption 4 and Assumption 5. Therefore, the controllability of $(A(\boldsymbol{x},\boldsymbol{u}), B(\boldsymbol{x},\boldsymbol{u}))$ and the observability of $(A(\boldsymbol{x},\boldsymbol{u}), C)$ need to be verified in region $\mathbb{B}(\mathbf{0}, \varepsilon)$. If the nonlinear system (4) have an equilibrium, the existence of the region $\mathbb{B}(\mathbf{0}, \varepsilon)$ can be guaranteed by the following conclusions.

At the origin (i.e., equilibrium point) system (4) can be linearized as

$$\begin{array}{l} \dot{\boldsymbol{x}} = A^{\mathrm{o}}\boldsymbol{x} + B^{\mathrm{o}}\boldsymbol{u}, \\ \boldsymbol{y} = C^{\mathrm{o}}\boldsymbol{x}, \end{array} \tag{23}$$

where $A^{\mathrm{o}} = A(\boldsymbol{x},\boldsymbol{u})\big|_{\substack{\boldsymbol{x}=0 \\ \boldsymbol{u}=0}}, B^{\mathrm{o}} = B(\boldsymbol{x},\boldsymbol{u})\big|_{\substack{\boldsymbol{x}=0 \\ \boldsymbol{u}=0}}, C^{\mathrm{o}} = I$.

Theorem 4 Assume that the pair $(A^{\mathrm{o}}, B^{\mathrm{o}})$ is controllable. There exists some $\varepsilon > 0$ such that for any $(\boldsymbol{x},\boldsymbol{u}) \in \mathbb{B}(\mathbf{0}, \varepsilon)$, the pair $(A(\boldsymbol{x},\boldsymbol{u}), B(\boldsymbol{x},\boldsymbol{u}))$ is also controllable.

Proof Take the controllability matrix

$$\begin{array}{rcl} W_{\text{ctrl}}(\boldsymbol{x},\boldsymbol{u}) & = & [B(\boldsymbol{x},\boldsymbol{u}) \;\; A(\boldsymbol{x},\boldsymbol{u})B(\boldsymbol{x},\boldsymbol{u}) \\ & & \cdots \;\; A(\boldsymbol{x},\boldsymbol{u})^{n-1}B(\boldsymbol{x},\boldsymbol{u})]. \end{array} \tag{24}$$

The elements of matrix $W_{\text{ctrl}}(\boldsymbol{x},\boldsymbol{u})$ are continuous functions of $\boldsymbol{x}$ and $\boldsymbol{u}$. In addition, since $(A^{\mathrm{o}}, B^{\mathrm{o}})$ is controllable, the controllability matrix W_{ctrl} is of full row rank, i.e., $\operatorname{rank}\left(W_{\text{ctrl}}(\boldsymbol{x},\boldsymbol{u})\big|_{\substack{\boldsymbol{x}=0 \\ \boldsymbol{u}=0}}\right) = n$. Hence, an $n \times n$ submatrix $W^*_{\text{ctrl}}(\boldsymbol{x},\boldsymbol{u})$, which satisfies $\det\left(W^*_{\text{ctrl}}(\boldsymbol{x},\boldsymbol{u})\big|_{\substack{\boldsymbol{x}=0 \\ \boldsymbol{u}=0}}\right) \neq 0$, can be found in $W_{\text{ctrl}}(\boldsymbol{x},\boldsymbol{u})$.

Let $W^{\mathrm{o}}_{\text{ctrl}} \triangleq W^*_{\text{ctrl}}(\boldsymbol{x},\boldsymbol{u})\big|_{\substack{\boldsymbol{x}=0 \\ \boldsymbol{u}=0}}, \quad \omega(\boldsymbol{x},\boldsymbol{u}) \triangleq \det(W^*_{\text{ctrl}}(\boldsymbol{x},\boldsymbol{u}))$. Since $\det(W^{\mathrm{o}}_{\text{ctrl}}) \neq 0$, we have $\omega(\boldsymbol{x},\boldsymbol{u})\big|_{\substack{\boldsymbol{x}=0 \\ \boldsymbol{u}=0}} \neq \mathbf{0}$. Because

$\omega(\boldsymbol{x},\boldsymbol{u})$ is continuous function of $\boldsymbol{x}$ and $\boldsymbol{u}$, there exists $\varepsilon > 0$ such that for any $(\boldsymbol{x},\boldsymbol{u}) \in \mathbb{B}(\mathbf{0},\varepsilon)$, $\omega(\boldsymbol{x},\boldsymbol{u}) \neq \mathbf{0}$. Therefore, for any $(\boldsymbol{x},\boldsymbol{u}) \in \mathbb{B}(\mathbf{0},\varepsilon)$, rank $(W_{\text{ctrl}}(\boldsymbol{x},\boldsymbol{u})) = n$, i.e., the pair $(A(\boldsymbol{x},\boldsymbol{u}), B(\boldsymbol{x},\boldsymbol{u}))$ is controllable.

This completes the proof.

Theorem 5 Assume that the pair $(A^{\text{o}}, C^{\text{o}})$ is observable. There exists some $\varepsilon > 0$ such that for any $(\boldsymbol{x},\boldsymbol{u}) \in \mathbb{B}(\mathbf{0},\varepsilon)$, the pair $(A(\boldsymbol{x},\boldsymbol{u}), C)$, is also observable.

Proof Consider the observability matrix

$$W_{\text{obsv}}(\boldsymbol{x},\boldsymbol{u}) = [C \quad CA(\boldsymbol{x},\boldsymbol{u}) \cdots CA(\boldsymbol{x},\boldsymbol{u})^{n-1}]^T.$$

The rest proof is similar to that of Theorem 4.

To sum up, the VGLQR control method is summarized as follows:

i) Caculate the Jacobi matrixes $A(\boldsymbol{x},\boldsymbol{u})$ and $B(\boldsymbol{x},\boldsymbol{u})$ according to the the model of the nonlinear system.

ii) Solve the Riccati equation (15) to obtain $P(\boldsymbol{x}) > 0$.

iii) Construct the nonlinear feedback controller via

$$\boldsymbol{u} = -K(\boldsymbol{x})\boldsymbol{x}, \quad K(\boldsymbol{x}) = R^{-1}B(\boldsymbol{x},\boldsymbol{u})^T P(\boldsymbol{x}). \tag{25}$$

Remark 1 The above steps are finished online, and the time-variant feedback gain $\boldsymbol{K}(\boldsymbol{x})$ is changing with respect to the plant state.

The control law (25) will come down to online solving the Riccati equation (15). In this paper, a fast algorithm based on Schur method for the solution of algebraic Riccati equation, which is suitable for computing on line, is developed by using Borland C++ and successfully implemented in the spherical double inverted pendulum control system.

IV. The implementation for stabilization of SDIPS

A. Analysis and controller design

According to the model of SDIPS (3), the Jacobi matrix $A(\boldsymbol{x},u)$ and $B(\boldsymbol{x},u)$ can be easily derived via Matlab function diff$(\cdot)$, and we can find that $A(\boldsymbol{x},u)$, $B(\boldsymbol{x},u)$ are only related to the plant state variables $\boldsymbol{x}$ from the derived results if the infinitesimal of higher order is omitted. Hence, the Jacobi matrixes $A(\boldsymbol{x},u)$, $B(\boldsymbol{x},u)$ are simplified as $A(\boldsymbol{x})$, $B(\boldsymbol{x})$ in the following sections.

At the equilibrium point, the nonlinear system (3) can be linearized as

$$\begin{aligned} \dot{\boldsymbol{x}} &= A^{\text{o}}\boldsymbol{x} + B^{\text{o}}u, \\ \boldsymbol{y} &= C^{\text{o}}\boldsymbol{x}, \qquad\qquad C^{\text{o}} = I, \end{aligned} \tag{26}$$

where

$$A^{\text{o}} = \begin{bmatrix} 0_{6\times 6} & I_{6\times 6} \\ a_{21} & 0_{6\times 6} \end{bmatrix},$$

$$B^{\text{o}} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -5.12 & 0.11 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0v & -5.12 & 0.11 \end{bmatrix}^T,$$

$$a_{21} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 65.676 & -15.492 & 0 & 0 \\ 0 & 0 & -28.068 & 27.051 & 0 & 0 \\ 0 & 0 & 0 & 0 & 65.676 & -15.492 \\ 0 & 0 & 0 & 0 & -28.068 & 27.051 \end{bmatrix}.$$

It is easy to verify that the pair $(A^{\text{o}}, B^{\text{o}})$ is controllable, and the pair $(A^{\text{o}}, C^{\text{o}})$ is observable. Hence, according to the dynamical model of the SDIPS (3), the controller law is established as

$$\boldsymbol{u} = -K(\boldsymbol{x})\boldsymbol{x}, \quad K(\boldsymbol{x}) = R^{-1}B(\boldsymbol{x})^T P(\boldsymbol{x}), \tag{27}$$

where $P(\boldsymbol{x})$ is the unique solution for the Riccati equation

$$\begin{aligned} &P(\boldsymbol{x})A(\boldsymbol{x}) + A(\boldsymbol{x})^T P(\boldsymbol{x}) \\ &-P(\boldsymbol{x})B(\boldsymbol{x})R^{-1}B(\boldsymbol{x})^T P(\boldsymbol{x}) + Q = 0, \end{aligned} \tag{28}$$

where $Q = \text{diag}([10, 10, 500, 500, 500, 500, 0, 0, 0, 0, 0, 0])$, $R = \text{diag}([0.1, 0.1])$.

B. The numerical experiments

In order to illustrate the effectiveness of the control scheme proposed in this paper, numerical experiments have been comparatively conducted by using the VGLQR control scheme and the classical LQR controller. The LQR controller is based on the linearization model (26). The initial values of the displacement of the cart, and the angles of the rods are taken as

$$\begin{aligned} x(0) &= 0, & y(0) &= 0, \\ \theta_1 &= -0.20137(\text{rad}), & \psi_1 &= -0.20137(\text{rad}), \\ \theta_2 &= 0.20137(\text{rad}), & \psi_2 &= 0.20137(\text{rad}). \end{aligned}$$

This condition is difficult to control. The results of the simulation are shown in Fig. 2. The simulation results show that the performance of VGLQR controller is distinctly better than that of the LQR controller.

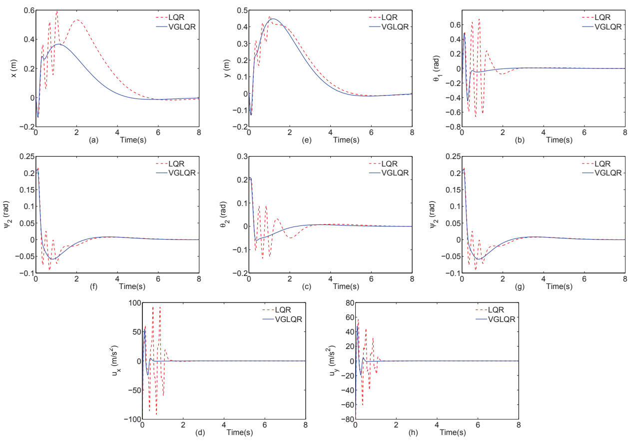

Fig. 2. The simulation results of x,θ_i,y,ψ_i,u_x,u_y,$(i = 1, 2)$.

C. The physical experiments

The implementation for stabilization of the SDIPS is conducted by the present nonlinear regulator. The implementation scheme is based an IPC-based control system.The experimental apparatus, as shown in Fig. 3, consist of three parts: (i) the mechanical structure of the inverted pendulum, (ii) the AC servo motor and its driver and (iii) IPC (Industrial Personal Computer). The signals of encoders and analogue outputs are handled by a 4-Axis GM-400 motion control card. The sampling period is 0.006s.

Fig. 3. The SDIP is being stabilized in the up position

The results of real-time experiments are shown in Fig. 7. Because the friction and other features of our nonlinear simulation model can approximate but are unable to duplicate exactly the physical system's behaviour, the real-time curves of x, y, ψ_i, θ_i fluctuate in a small range and can not converge to zero. In practice, since the uncertain factors always exert on the system during the control process, stabilizing the two pendulums around the unstable equilibrium state is a dynamic control process, and the cart and pendulums fluctuate near the equilibrium point.

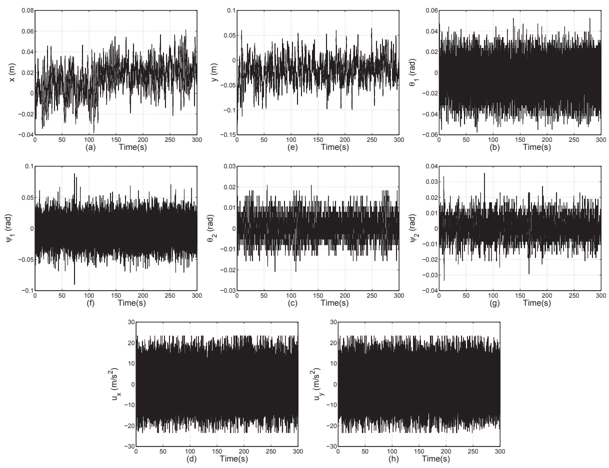

Fig. 4. The real-time curves of $x, \theta_i, y, \psi_i, u_x, u_y, (i = 1, 2)$.

V. Conclusions

In this paper, the general form of the model for the spherical double inverted pendulum system (SDIPS) is presented, and a nonlinear regulator, variable gain LQR controller (VGLQR), is designed for a class of nonlinear systems. By using the proposed control scheme the stabilization of the SDIPS is implemented. The experiment results show that the effectiveness and the potential of the VGLQR control method.

Acknowledgment

The authors would like to thank the anonymous reviewers for their helpful suggestions to improve the quality of this article; and the authors are very pleased to acknowledge the support from the Scientific Research Foundation of Tianjin University of Technology and Education (KYQD03010), the Scientific Research Foundation of Tianjin municiprl education commission(20130824).

References

[1] S. Mori, H. Nishihara, K. Furuta. Control of unstable mechanical system control of pendulum. *International Journal of Control*, vol. 23, no. 5, pp. 673–692, 1976.

[2] M. K. Bouafoura and N. B. Braiek. A State Feedback Control Design for Generalized Fractional Systems Through Orthogonal Functions: Application to a Fractional Inverted Pendulum. *Asian journal of control*. vol. 15, no. 4, pp. 773C782, 2013.

[3] Chih-Chen Yih . Sliding Mode Control for Swing-Up and Stabilization of the Cart-Pole Underactuated System. *Asian journal of control*. vol. 15, no. 3, pp. 1201C1214, 2013.

[4] K. Furuta, K. Hiroyuki, K. Kazuhiro. Digital control of a double inverted pendulum on an inclined rail. *International Journal of Control*, vol. 32, no. 5, pp. 907–924, 1980.

[5] K.J. Åström, and K. Furuta. Swinging up a Pendulum by Energy Control. *Automatica*, vol. 36, no. 2, pp. 287–295, 2000.

[6] K.J. Åström, J. Aracil, F. Gordillo. A family of smooth controllers for swinging up a pendulum. *Automatica*, vol. 44, no. 7, pp. 1841–1848, 2008.

[7] K. Graichen, M. Treuer, M. Zeitz, Swing-up of the double pendulum on a cart by feedforward and feedback control with experimental validation. *Automatica*, vol. 43, no. 1, pp. 63–71, 2007.

[8] S.J. Huang, C.L. Huang, Control of an inverted pendulum using grey prediction model. *IEEE transactions on industry applications*, vol. 36, no. 2, pp. 452–458, 2000.

[9] Khaled, G.E. and Chen-Yuan, Kuo. Nonlinear optimal control of a triple link inverted pendulum with single control input. *Interna-tional Journal of Control*, vol. 69, no. 2, pp. 239–256, 1998.

[10] K. Graichen, V. Hagenmeyer & M. Zeitz, A new approach to inversion-based feedforward control design for nonlinear systems. *Automatica*, vol. 41, no. 12, pp. 2033–2041, 2005.

[11] G. Liu, D. Nešiá and I. Mareels, Modelling and stabilisation of a spherical inverted pendulum. *In Proceeding of IFAC, Prague, Czech Republic*, 2005.

[12] G. Liu, D. Nešić and I. Mareels, Non-local stabilization of a spherical inverted pendulum, *International Journal of Control*, vol. 81, no. 7, pp. 1035–1053, 2008.

[13] L. Consolinia,and M. Tosquesb. On the exact tracking of the spherical inverted pendulum via an homotopy method. *Systems & Control Letters*, vol. 58, no. 1, pp. 1–6, 2009. 15

[14] T. K. Liu, C. H. Chen, Z. S. Li, J. H. Chou. Method of inequalities-based multiobjective genetic algorithm for optimizing a cart-double-pendulum system. *International Journal of Automation and Computing*, vol. 06, no. 1, pp. 29–37, 2009.

[15] H.X. Li, Z.H. Miao, J.Y. Wang. Variable universe adaptive fuzzy control on the quadruple inverted pendulum. *Science in China*, vol. 45, no. 2, pp. 213-224, 2002.

[16] C. A. Ibañez, O. G. Frias. Controlling the inverted pendulum by means of a nested saturation function. *Nonlinear Dynamics*, vol. 53, no. 4, pp. 273–280, 2008.

[17] W.J. Rugh, & J.S. Shamma. Research on gain scheduling. *Automatica*, vol. 36, no. 10, pp. 1401–1425, 2000.

[18] J.R. Cloutier, C.N. D'Souza & C.P. Mracek. Nonlinear regulation and nonlinear H_∞ control via the state-dependent Riccati equation technique. *In Proceedings of the International Conference on Nonlinear Problems in Aviation and Aerospace, Daytona Beach, FL*, pp. 117–141, May, 1996.

[19] A. Bloch, N. Leonard and J. Marsden. Controlled Lagragians and the stabilisation of mechanical systems I: the first matching theorem. *IEEE Trans. Autom. Contr.*, vol. 45, pp. 2253–2269, 2000.

[20] A. Bloch, D. Chang, N. Leonard and J. Marsden. Controlled Lagragians and the stabilisation of mechanical systems II: potential shaping. *IEEE transaction on Automatic Control*, vol. 46, 1556–1571, 2001.

[21] A.J. Laub. A Schur method for solving algebraic Riccati equations. *IEEE transaction on Automatic Control*, vol. 24, no. 6, 913–921, 1979.