§3.1　线性方程组的消元解法

数学系　梁卓滨

2016 - 2017 学年 I 暑修班

记号

考虑 n 个未知量 m 个方程的线性方程组：

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m \end{cases}$$

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可以，等价地，改写成矩阵形式

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增广矩阵 $$(A \vdots b) = \left(\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{array}\right)$$

消元法求解线性方程组——示例

例 解方程组：
$$\begin{cases} x_1 + x_2 - x_3 = 2 \\ x_1 + 2x_2 = -1 \\ 4x_1 + 7x_2 - x_3 = -1 \\ -3x_1 - 4x_2 + 2x_3 = -3 \end{cases}$$

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$$(A \vdots b) = \left(\begin{array}{ccc|c} 1 & 1 & -1 & 2 \\ 1 & 2 & 0 & -1 \\ 4 & 7 & -1 & -1 \\ -3 & -4 & 2 & -3 \end{array}\right)$$

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$$\Updownarrow \qquad\qquad\qquad\qquad \Updownarrow$$

$$(A \vdots b) = \left(\begin{array}{ccc|c} 1 & 1 & -1 & 2 \\ 1 & 2 & 0 & -1 \\ 4 & 7 & -1 & -1 \\ -3 & -4 & 2 & -3 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|c} 1 & 1 & -1 & 2 \\ 0 & 1 & 1 & -3 \\ 0 & 3 & 3 & -9 \\ 0 & -1 & -1 & 3 \end{array}\right) \quad r_4 + 3r_1$$

消元法求解线性方程组——示例

例 解方程组：
$$\begin{cases} x_1+x_2-x_3=2 \\ x_1+2x_2=-1 \\ 4x_1+7x_2-x_3=-1 \\ -3x_1-4x_2+2x_3=-3 \end{cases}$$

消元法步骤示例：

$$\begin{cases} x_1+x_2-x_3=2 \\ x_1+2x_2=-1 \\ 4x_1+7x_2-x_3=-1 \\ -3x_1-4x_2+2x_3=-3 \end{cases} \longrightarrow \begin{cases} x_1+x_2-x_3=2 \\ x_2+x_3=-3 \\ 0=0 \\ -x_2-x_3=3 \end{cases} \quad (3)-3(2)$$

$$\Updownarrow \qquad\qquad \Updownarrow$$

$$(A \vdots b)=\left(\begin{array}{ccc|c} 1 & 1 & -1 & 2 \\ 1 & 2 & 0 & -1 \\ 4 & 7 & -1 & -1 \\ -3 & -4 & 2 & -3 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|c} 1 & 1 & -1 & 2 \\ 0 & 1 & 1 & -3 \\ 0 & 3 & 3 & -9 \\ 0 & -1 & -1 & 3 \end{array}\right) \quad r_4+3r_1$$

消元法求解线性方程组——示例

例 解方程组：
$$\begin{cases} x_1 + x_2 - x_3 = 2 \\ x_1 + 2x_2 = -1 \\ 4x_1 + 7x_2 - x_3 = -1 \\ -3x_1 - 4x_2 + 2x_3 = -3 \end{cases}$$

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$$\Updownarrow \qquad\qquad \Updownarrow$$

$$(A \vdots b) = \left(\begin{array}{ccc|c} 1 & 1 & -1 & 2 \\ 1 & 2 & 0 & -1 \\ 4 & 7 & -1 & -1 \\ -3 & -4 & 2 & -3 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|c} 1 & 1 & -1 & 2 \\ 0 & 1 & 1 & -3 \\ 0 & 3 & 3 & -9 \\ 0 & -1 & -1 & 3 \end{array}\right) \quad r_3 - 2r_2$$

消元法求解线性方程组——示例

例 解方程组：
$$\begin{cases} x_1+x_2-x_3=2 \\ x_1+2x_2=-1 \\ 4x_1+7x_2-x_3=-1 \\ -3x_1-4x_2+2x_3=-3 \end{cases}$$

消元法步骤示例：

$$\begin{cases} x_1+x_2-x_3=2 \\ x_1+2x_2=-1 \\ 4x_1+7x_2-x_3=-1 \\ -3x_1-4x_2+2x_3=-3 \end{cases} \longrightarrow \begin{cases} x_1+x_2-x_3=2 \\ x_2+x_3=-3 \\ 0=0 \\ -x_2-x_3=3 \end{cases} \quad (3)-3(2)$$

$$\Updownarrow \qquad\qquad \Updownarrow$$

$$(A\,\vdots\,b)=\left(\begin{array}{ccc|c} 1 & 1 & -1 & 2 \\ 1 & 2 & 0 & -1 \\ 4 & 7 & -1 & -1 \\ -3 & -4 & 2 & -3 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|c} 1 & 1 & -1 & 2 \\ 0 & 1 & 1 & -3 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & -1 & 3 \end{array}\right) \quad r_3-2r_2$$

消元法求解线性方程组——示例

例 解方程组：
$$\begin{cases} x_1+x_2-x_3=2 \\ x_1+2x_2=-1 \\ 4x_1+7x_2-x_3=-1 \\ -3x_1-4x_2+2x_3=-3 \end{cases}$$

消元法步骤示例：

$$\begin{cases} x_1+x_2-x_3=2 \\ x_1+2x_2=-1 \\ 4x_1+7x_2-x_3=-1 \\ -3x_1-4x_2+2x_3=-3 \end{cases} \longrightarrow \begin{cases} x_1+x_2-x_3=2 \\ x_2+x_3=-3 \\ 0=0 \\ -x_2-x_3=3 \end{cases} \quad (4)+(2)$$

$$\Updownarrow \qquad\qquad\qquad \Updownarrow$$

$$(A\,\vdots\,b)=\left(\begin{array}{ccc|c} 1 & 1 & -1 & 2 \\ 1 & 2 & 0 & -1 \\ 4 & 7 & -1 & -1 \\ -3 & -4 & 2 & -3 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|c} 1 & 1 & -1 & 2 \\ 0 & 1 & 1 & -3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \quad r_3-2r_2$$

消元法求解线性方程组——示例

例 解方程组：
$$\begin{cases} x_1+\ \ x_2-\ \ x_3=2\\ x_1+2x_2\qquad\ =-1\\ 4x_1+7x_2-\ \ x_3=-1\\ -3x_1-4x_2+2x_3=-3\end{cases}$$

消元法步骤示例：

$$\begin{cases} x_1+\ \ x_2-\ \ x_3=2\\ x_1+2x_2\qquad\ =-1\\ 4x_1+7x_2-\ \ x_3=-1\\ -3x_1-4x_2+2x_3=-3\end{cases}\longrightarrow\begin{cases} x_1+x_2-x_3=2\\ \qquad x_2+x_3=-3\\ \qquad\qquad 0=0\\ \qquad\qquad 0=0\end{cases}\quad (4)+(2)$$

$$\Updownarrow\qquad\qquad\qquad\qquad\Updownarrow$$

$$(A\vdots b)=\left(\begin{array}{ccc|c} 1&1&-1&2\\ 1&2&0&-1\\ 4&7&-1&-1\\ -3&-4&2&-3\end{array}\right)\longrightarrow\left(\begin{array}{ccc|c} 1&1&-1&2\\ 0&1&1&-3\\ 0&0&0&0\\ 0&0&0&0\end{array}\right)\quad r_3-2r_2$$

消元法求解线性方程组——示例

例 解方程组：
$$\begin{cases} x_1+\ \ x_2-\ \ x_3=2 \\ x_1+2x_2\qquad\ =-1 \\ 4x_1+7x_2-\ \ x_3=-1 \\ -3x_1-4x_2+2x_3=-3 \end{cases}$$

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$$\Updownarrow \qquad\qquad \Updownarrow$$

$$(A\,\vdots\,b)=\left(\begin{array}{ccc|c} 1 & 1 & -1 & 2 \\ 1 & 2 & 0 & -1 \\ 4 & 7 & -1 & -1 \\ -3 & -4 & 2 & -3 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|c} 1 & 1 & -1 & 2 \\ 0 & 1 & 1 & -3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \qquad r_4+r_2$$

消元法求解线性方程组——示例

例 解方程组：
$$\begin{cases} x_1 + x_2 - x_3 = 2 \\ x_1 + 2x_2 = -1 \\ 4x_1 + 7x_2 - x_3 = -1 \\ -3x_1 - 4x_2 + 2x_3 = -3 \end{cases}$$

消元法步骤示例：

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$$\Updownarrow \qquad\qquad \Updownarrow$$

$$(A \vdots b) = \left(\begin{array}{ccc|c} 1 & 1 & -1 & 2 \\ 1 & 2 & 0 & -1 \\ 4 & 7 & -1 & -1 \\ -3 & -4 & 2 & -3 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|c} 1 & 1 & -1 & 2 \\ 0 & 1 & 1 & -3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \quad r_4 + r_2$$

消元法求解线性方程组——示例

例 解方程组：
$$\begin{cases} x_1 + x_2 - x_3 = 2 \\ x_1 + 2x_2 = -1 \\ 4x_1 + 7x_2 - x_3 = -1 \\ -3x_1 - 4x_2 + 2x_3 = -3 \end{cases}$$

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$$\Updownarrow \qquad\qquad\qquad\qquad \Updownarrow$$

$$(A \vdots b) = \left(\begin{array}{ccc|c} 1 & 1 & -1 & 2 \\ 1 & 2 & 0 & -1 \\ 4 & 7 & -1 & -1 \\ -3 & -4 & 2 & -3 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|c} 1 & 1 & -1 & 2 \\ 0 & 1 & 1 & -3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \quad r_4 + r_2$$

消元法求解线性方程组——示例

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$$\Updownarrow \qquad\qquad \Updownarrow$$

$$(A \vdots b) = \left(\begin{array}{ccc|c} 1 & 1 & -1 & 2 \\ 1 & 2 & 0 & -1 \\ 4 & 7 & -1 & -1 \\ -3 & -4 & 2 & -3 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|c} 1 & 1 & -1 & 2 \\ 0 & 1 & 1 & -3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \quad r_4 + r_2$$

消元法求解线性方程组——示例

例 解方程组：
$$\begin{cases} x_1 + x_2 - x_3 = 2 \\ x_1 + 2x_2 = -1 \\ 4x_1 + 7x_2 - x_3 = -1 \\ -3x_1 - 4x_2 + 2x_3 = -3 \end{cases}$$

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$$\Updownarrow \qquad\qquad \Updownarrow$$

$$(A \vdots b) = \left(\begin{array}{ccc|c} 1 & 1 & -1 & 2 \\ 1 & 2 & 0 & -1 \\ 4 & 7 & -1 & -1 \\ -3 & -4 & 2 & -3 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|c} 1 & 1 & -1 & 2 \\ 0 & 1 & 1 & -3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \quad r_1 - r_2$$

消元法求解线性方程组——示例

例 解方程组:
$$\begin{cases} x_1 + x_2 - x_3 = 2 \\ x_1 + 2x_2 = -1 \\ 4x_1 + 7x_2 - x_3 = -1 \\ -3x_1 - 4x_2 + 2x_3 = -3 \end{cases}$$

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$$\Updownarrow \qquad\qquad \Updownarrow$$

$$(A \vdots b) = \left(\begin{array}{ccc|c} 1 & 1 & -1 & 2 \\ 1 & 2 & 0 & -1 \\ 4 & 7 & -1 & -1 \\ -3 & -4 & 2 & -3 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|c} 1 & 0 & -2 & 5 \\ 0 & 1 & 1 & -3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \quad r_1 - r_2$$

消元法求解线性方程组——示例

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$$\Updownarrow \qquad\qquad\qquad \Updownarrow$$

$$(A \vdots b) = \left(\begin{array}{ccc|c} 1 & 1 & -1 & 2 \\ 1 & 2 & 0 & -1 \\ 4 & 7 & -1 & -1 \\ -3 & -4 & 2 & -3 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|c} 1 & 0 & -2 & 5 \\ 0 & 1 & 1 & -3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

消元法求解线性方程组——示例

例 解方程组：
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$$\Updownarrow \qquad\qquad \Updownarrow$$

$$(A \vdots b) = \left(\begin{array}{ccc|c} 1 & 1 & -1 & 2 \\ 1 & 2 & 0 & -1 \\ 4 & 7 & -1 & -1 \\ -3 & -4 & 2 & -3 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|c} 1 & 0 & -2 & 5 \\ 0 & 1 & 1 & -3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \Rightarrow \begin{cases} x_1 = 5 + 2x_3 \\ x_2 = -3 - x_3 \end{cases}$$

消元法求解线性方程组——示例

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$$\begin{cases} x_1+x_2-x_3=2 \\ x_1+2x_2=-1 \\ 4x_1+7x_2-x_3=-1 \\ -3x_1-4x_2+2x_3=-3 \end{cases}$$

消元法步骤示例：

$$\begin{cases} x_1+x_2-x_3=2 \\ x_1+2x_2=-1 \\ 4x_1+7x_2-x_3=-1 \\ -3x_1-4x_2+2x_3=-3 \end{cases} \longrightarrow \begin{cases} x_1-2x_3=5 \\ x_2+x_3=-3 \\ 0=0 \\ 0=0 \end{cases} \Rightarrow \begin{cases} x_1=5+2x_3 \\ x_2=-3-x_3 \end{cases}$$

$$\Updownarrow \qquad\qquad\qquad \Updownarrow$$

$$(A\,\vdots\,b)=\left(\begin{array}{ccc|c} 1 & 1 & -1 & 2 \\ 1 & 2 & 0 & -1 \\ 4 & 7 & -1 & -1 \\ -3 & -4 & 2 & -3 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|c} 1 & 0 & -2 & 5 \\ 0 & 1 & 1 & -3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \Rightarrow \begin{cases} x_1=5+2x_3 \\ x_2=-3-x_3 \end{cases}$$

初等行变换求解线性方程组

$$Ax = b \quad \Longrightarrow \quad (A \vdots b) \xrightarrow{\text{初等行变换}} \text{简化的阶梯型矩阵}$$

回忆：阶梯形矩阵

形如：

$$A=\begin{pmatrix} 0 & \cdots & 0 & b_1 & \cdots & * & * & \cdots & * & * & \cdots & \cdots & * & * & \cdots & * \\ 0 & \cdots & \cdots & \cdots & \cdots & 0 & b_2 & \cdots & * & * & \cdots & \cdots & * & * & \cdots & * \\ 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & b_3 & \cdots & \cdots & * & * & \cdots & * \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & b_r & \cdots & * \\ 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \end{pmatrix}$$

的矩阵，其中 $b_1, b_2, \ldots, b_r \neq 0$，称为阶梯型矩阵。

简化的阶梯型矩阵

形如：

$$
A=\begin{pmatrix}
0 & \cdots & 0 & 1 & \cdots & * & 0 & \cdots & * & 0 & \cdots & \cdots & * & 0 & \cdots & * \\
0 & \cdots & \cdots & \cdots & \cdots & 0 & 1 & \cdots & * & 0 & \cdots & \cdots & * & 0 & \cdots & * \\
0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & 1 & \cdots & \cdots & * & 0 & \cdots & * \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & 1 & \cdots & * \\
0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0
\end{pmatrix}
$$

的矩阵，其中 $b_1, b_2, \ldots, b_r \neq 0$，称为简化的阶梯型矩阵。

例 1 解方程组：$\begin{cases} x_1 + x_2 - x_3 = 2 \\ x_1 + 2x_2 = -1 \\ 2x_1 + 5x_2 + x_3 = -5 \\ -2x_1 - 3x_2 + x_3 = -1 \end{cases}$

例 1 解方程组：$\begin{cases} x_1 + x_2 - x_3 = 2 \\ x_1 + 2x_2 = -1 \\ 2x_1 + 5x_2 + x_3 = -5 \\ -2x_1 - 3x_2 + x_3 = -1 \end{cases}$

解 $$(A \vdots b) = \left(\begin{array}{rrr|r} 1 & 1 & -1 & 2 \\ 1 & 2 & 0 & -1 \\ 2 & 5 & 1 & -5 \\ -2 & -3 & 1 & -1 \end{array}\right)$$

例 1 解方程组：
$$\begin{cases} x_1 + x_2 - x_3 = 2 \\ x_1 + 2x_2 = -1 \\ 2x_1 + 5x_2 + x_3 = -5 \\ -2x_1 - 3x_2 + x_3 = -1 \end{cases}$$

解
$$(A \vdots b) = \left(\begin{array}{rrr|r} 1 & 1 & -1 & 2 \\ 1 & 2 & 0 & -1 \\ 2 & 5 & 1 & -5 \\ -2 & -3 & 1 & -1 \end{array}\right) \xrightarrow[\substack{r_3 - 2r_1 \\ r_4 + 2r_1}]{r_2 - r_1}$$

例 1 解方程组：
$$\begin{cases} x_1+x_2-x_3=2 \\ x_1+2x_2=-1 \\ 2x_1+5x_2+x_3=-5 \\ -2x_1-3x_2+x_3=-1 \end{cases}$$

解
$$(A\vdots b)=\left(\begin{array}{ccc|c} 1 & 1 & -1 & 2 \\ 1 & 2 & 0 & -1 \\ 2 & 5 & 1 & -5 \\ -2 & -3 & 1 & -1 \end{array}\right) \xrightarrow[\substack{r_3-2r_1 \\ r_4+2r_1}]{r_2-r_1} \left(\begin{array}{ccc|c} 1 & 1 & -1 & 2 \\ & & & \\ & & & \\ & & & \end{array}\right)$$

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$$(A \vdots b) = \left(\begin{array}{ccc|c} 1 & 1 & -1 & 2 \\ 1 & 2 & 0 & -1 \\ 2 & 5 & 1 & -5 \\ -2 & -3 & 1 & -1 \end{array}\right) \xrightarrow[\substack{r_3 - 2r_1 \\ r_4 + 2r_1}]{r_2 - r_1} \left(\begin{array}{ccc|c} 1 & 1 & -1 & 2 \\ 0 & 1 & 1 & -3 \\ & & & \\ & & & \end{array}\right)$$

例 1 解方程组：
$$\begin{cases} x_1 + x_2 - x_3 = 2 \\ x_1 + 2x_2 = -1 \\ 2x_1 + 5x_2 + x_3 = -5 \\ -2x_1 - 3x_2 + x_3 = -1 \end{cases}$$

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$$(A \vdots b) = \left(\begin{array}{ccc|c} 1 & 1 & -1 & 2 \\ 1 & 2 & 0 & -1 \\ 2 & 5 & 1 & -5 \\ -2 & -3 & 1 & -1 \end{array}\right) \xrightarrow[\substack{r_3-2r_1 \\ r_4+2r_1}]{r_2-r_1} \left(\begin{array}{ccc|c} 1 & 1 & -1 & 2 \\ 0 & 1 & 1 & -3 \\ 0 & 3 & 3 & -9 \end{array}\right)$$

例 1 解方程组:
$$\begin{cases} x_1 + x_2 - x_3 = 2 \\ x_1 + 2x_2 = -1 \\ 2x_1 + 5x_2 + x_3 = -5 \\ -2x_1 - 3x_2 + x_3 = -1 \end{cases}$$

解
$$(A \vdots b) = \left(\begin{array}{ccc|c} 1 & 1 & -1 & 2 \\ 1 & 2 & 0 & -1 \\ 2 & 5 & 1 & -5 \\ -2 & -3 & 1 & -1 \end{array}\right) \xrightarrow[\substack{r_3-2r_1 \\ r_4+2r_1}]{r_2-r_1} \left(\begin{array}{ccc|c} 1 & 1 & -1 & 2 \\ 0 & 1 & 1 & -3 \\ 0 & 3 & 3 & -9 \\ 0 & -1 & -1 & 3 \end{array}\right)$$

例 1 解方程组：$\begin{cases} x_1 + x_2 - x_3 = 2 \\ x_1 + 2x_2 = -1 \\ 2x_1 + 5x_2 + x_3 = -5 \\ -2x_1 - 3x_2 + x_3 = -1 \end{cases}$

解

$$(A \vdots b) = \left(\begin{array}{rrr|r} 1 & 1 & -1 & 2 \\ 1 & 2 & 0 & -1 \\ 2 & 5 & 1 & -5 \\ -2 & -3 & 1 & -1 \end{array}\right) \xrightarrow[\substack{r_3-2r_1 \\ r_4+2r_1}]{r_2-r_1} \left(\begin{array}{rrr|r} 1 & 1 & -1 & 2 \\ 0 & 1 & 1 & -3 \\ 0 & 3 & 3 & -9 \\ 0 & -1 & -1 & 3 \end{array}\right)$$

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$$\begin{cases} x_1 + \quad - 2x_3 = 5 \\ x_2 + x_3 = -3 \end{cases}$$

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$$\begin{cases} x_1 - 2x_3 = 5 \\ x_2 + x_3 = -3 \end{cases} \Leftrightarrow \begin{cases} x_1 = 5 + 2x_3 \\ x_2 = -3 - x_3 \end{cases}$$

所以通解是：

$$\begin{cases} \\ \\ x_3 = c_1 \end{cases} \qquad (c_1\text{为任意常数})$$

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$$\begin{cases} x_1 + 2x_2 + 4x_3 = 28 \\ -2x_1 - 3x_2 - 9x_3 = -53 \\ 3x_1 + 6x_2 + 13x_3 = 88 \\ 5x_1 + 9x_2 + 22x_3 = 141 \end{cases}$$

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$$\xrightarrow{r_4+r_2} \left(\begin{array}{ccc|c} 1 & 2 & 4 & 28 \\ 0 & 1 & -1 & 3 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 1 & 4 \end{array}\right) \xrightarrow{r_4-r_3} \left(\begin{array}{ccc|c} 1 & 2 & 4 & 28 \\ 0 & 1 & -1 & 3 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$$\xrightarrow[r_1-4r_3]{r_2+r_3}$$

例 2 解方程组：
$$\begin{cases} x_1+2x_2+4x_3=28 \\ -2x_1-3x_2-9x_3=-53 \\ 3x_1+6x_2+13x_3=88 \\ 5x_1+9x_2+22x_3=141 \end{cases}$$

解

$$(A\vdots b)=\left(\begin{array}{ccc|c} 1 & 2 & 4 & 28 \\ -2 & -3 & -9 & -53 \\ 3 & 6 & 13 & 88 \\ 5 & 9 & 22 & 141 \end{array}\right) \xrightarrow[\substack{r_3-3r_1 \\ r_4-5r_1}]{r_2+2r_1} \left(\begin{array}{ccc|c} 1 & 2 & 4 & 28 \\ 0 & 1 & -1 & 3 \\ 0 & 0 & 1 & 4 \\ 0 & -1 & 2 & 1 \end{array}\right)$$

$$\xrightarrow{r_4+r_2} \left(\begin{array}{ccc|c} 1 & 2 & 4 & 28 \\ 0 & 1 & -1 & 3 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 1 & 4 \end{array}\right) \xrightarrow{r_4-r_3} \left(\begin{array}{ccc|c} 1 & 2 & 4 & 28 \\ 0 & 1 & -1 & 3 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

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例 2 解方程组：
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所以原方程组等价于
$$\begin{cases} x_1 = -2 \\ x_2 = 7 \\ x_3 = 4 \end{cases}$$

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例 3 解方程组：$\begin{cases} 4x_1 + 2x_2 - 7x_3 = -3 \\ 2x_1 + x_2 - 4x_3 = -1 \\ 5x_1 + 3x_2 - 11x_3 = 2 \\ x_1 + x_2 - 4x_3 = 2 \end{cases}$

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$$\left(\begin{array}{ccc|c} 1 & 1 & -4 & 2 \\ 0 & -1 & 4 & -5 \\ 0 & -2 & 9 & -8 \\ 0 & -2 & 9 & -11 \end{array}\right) \xrightarrow[r_4 - 2r_2]{r_3 - 2r_2} \left(\begin{array}{ccc|c} 1 & 1 & -4 & 2 \\ 0 & -1 & 4 & -5 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 1 & -1 \end{array}\right) \xrightarrow{r_4 - r_3} \left(\begin{array}{ccc|c} 1 & 1 & -4 & 2 \\ 0 & -1 & 4 & -5 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & -3 \end{array}\right)$$

所以原方程组等价于

例 3 解方程组:
$$\begin{cases} 4x_1 + 2x_2 - 7x_3 = -3 \\ 2x_1 + x_2 - 4x_3 = -1 \\ 5x_1 + 3x_2 - 11x_3 = 2 \\ x_1 + x_2 - 4x_3 = 2 \end{cases}$$

解

$$(A \vdots b) = \left(\begin{array}{ccc|c} 4 & 2 & -7 & -3 \\ 2 & 1 & -4 & -1 \\ 5 & 3 & -11 & 2 \\ 1 & 1 & -4 & 2 \end{array}\right) \xrightarrow{r_1 \leftrightarrow r_4} \left(\begin{array}{ccc|c} 1 & 1 & -4 & 2 \\ 2 & 1 & -4 & -1 \\ 5 & 3 & -11 & 2 \\ 4 & 2 & -7 & -3 \end{array}\right) \xrightarrow[\substack{r_3 - 5r_1 \\ r_4 - 4r_1}]{r_2 - 2r_1}$$

$$\left(\begin{array}{ccc|c} 1 & 1 & -4 & 2 \\ 0 & -1 & 4 & -5 \\ 0 & -2 & 9 & -8 \\ 0 & -2 & 9 & -11 \end{array}\right) \xrightarrow[r_4 - 2r_2]{r_3 - 2r_2} \left(\begin{array}{ccc|c} 1 & 1 & -4 & 2 \\ 0 & -1 & 4 & -5 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 1 & -1 \end{array}\right) \xrightarrow{r_4 - r_3} \left(\begin{array}{ccc|c} 1 & 1 & -4 & 2 \\ 0 & -1 & 4 & -5 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & -3 \end{array}\right)$$

所以原方程组等价于
$$\begin{cases} x_1 + x_2 - 4x_3 = 2 \\ -x_2 + 4x_3 = -5 \\ x_3 = 2 \\ 0 = -3 \end{cases}$$

例 3 解方程组：
$$\begin{cases} 4x_1+2x_2-7x_3=-3 \\ 2x_1+x_2-4x_3=-1 \\ 5x_1+3x_2-11x_3=2 \\ x_1+x_2-4x_3=2 \end{cases}$$

解

$$(A \vdots b)=\left(\begin{array}{ccc|c} 4 & 2 & -7 & -3 \\ 2 & 1 & -4 & -1 \\ 5 & 3 & -11 & 2 \\ 1 & 1 & -4 & 2 \end{array}\right) \xrightarrow{r_1 \leftrightarrow r_4} \left(\begin{array}{ccc|c} 1 & 1 & -4 & 2 \\ 2 & 1 & -4 & -1 \\ 5 & 3 & -11 & 2 \\ 4 & 2 & -7 & -3 \end{array}\right) \xrightarrow[\substack{r_3-5r_1 \\ r_4-4r_1}]{r_2-2r_1}$$

$$\left(\begin{array}{ccc|c} 1 & 1 & -4 & 2 \\ 0 & -1 & 4 & -5 \\ 0 & -2 & 9 & -8 \\ 0 & -2 & 9 & -11 \end{array}\right) \xrightarrow[r_4-2r_2]{r_3-2r_2} \left(\begin{array}{ccc|c} 1 & 1 & -4 & 2 \\ 0 & -1 & 4 & -5 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 1 & -1 \end{array}\right) \xrightarrow{r_4-r_3} \left(\begin{array}{ccc|c} 1 & 1 & -4 & 2 \\ 0 & -1 & 4 & -5 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & -3 \end{array}\right)$$

所以原方程组等价于
$$\begin{cases} x_1+x_2-4x_3=2 \\ -x_2+4x_3=-5 \\ x_3=2 \\ 0=-3 \end{cases} \Rightarrow \quad \text{无解！}$$

例 3 解方程组：$\begin{cases} 4x_1+2x_2-7x_3=-3 \\ 2x_1+x_2-4x_3=-1 \\ 5x_1+3x_2-11x_3=2 \\ x_1+x_2-4x_3=2 \end{cases}$

解

$$(A\vdots b)=\left(\begin{array}{ccc|c} 4 & 2 & -7 & -3 \\ 2 & 1 & -4 & -1 \\ 5 & 3 & -11 & 2 \\ 1 & 1 & -4 & 2 \end{array}\right) \xrightarrow{r_1\leftrightarrow r_4} \left(\begin{array}{ccc|c} 1 & 1 & -4 & 2 \\ 2 & 1 & -4 & -1 \\ 5 & 3 & -11 & 2 \\ 4 & 2 & -7 & -3 \end{array}\right) \xrightarrow[\substack{r_3-5r_1 \\ r_4-4r_1}]{r_2-2r_1}$$

$$\left(\begin{array}{ccc|c} 1 & 1 & -4 & 2 \\ 0 & -1 & 4 & -5 \\ 0 & -2 & 9 & -8 \\ 0 & -2 & 9 & -11 \end{array}\right) \xrightarrow[r_4-2r_2]{r_3-2r_2} \left(\begin{array}{ccc|c} 1 & 1 & -4 & 2 \\ 0 & -1 & 4 & -5 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 1 & -1 \end{array}\right) \xrightarrow{r_4-r_3} \left(\begin{array}{ccc|c} 1 & 1 & -4 & 2 \\ 0 & -1 & 4 & -5 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & -3 \end{array}\right)$$

所以原方程组等价于 $\begin{cases} x_1+x_2-4x_3=2 \\ -x_2+4x_3=-5 \\ x_3=2 \\ 0=-3 \end{cases} \Rightarrow$ 无解！

总结

$$\begin{cases} x_1 + x_2 - x_3 = 2 \\ x_1 + 2x_2 = -1 \\ 2x_1 + 5x_2 + x_3 = -5 \\ -2x_1 - 3x_2 + x_3 = -1 \end{cases} \qquad \begin{cases} x_1 + 2x_2 + 4x_3 = 28 \\ -2x_1 - 3x_2 - 9x_3 = -53 \\ 3x_1 + 6x_2 + 13x_3 = 88 \\ 5x_1 + 9x_2 + 22x_3 = 141 \end{cases} \qquad \begin{cases} 4x_1 + 2x_2 - 7x_3 = -3 \\ 2x_1 + x_2 - 4x_3 = -1 \\ 5x_1 + 3x_2 - 11x_3 = 2 \\ x_1 + x_2 - 4x_3 = 2 \end{cases}$$

总结

$$\begin{cases} x_1+ & x_2- & x_3 & =2 \\ x_1+ & 2x_2 & & =-1 \\ 2x_1+ & 5x_2+ & x_3 & =-5 \\ -2x_1- & 3x_2+ & x_3 & =-1 \end{cases}$$

$$\Downarrow$$

$$(A \vdots b)$$

$$\begin{cases} x_1+ & 2x_2+ & 4x_3= & 28 \\ -2x_1- & 3x_2- & 9x_3= & -53 \\ 3x_1+ & 6x_2+ & 13x_3= & 88 \\ 5x_1+ & 9x_2+ & 22x_3= & 141 \end{cases}$$

$$\Downarrow$$

$$(A \vdots b)$$

$$\begin{cases} 4x_1+ & 2x_2- & 7x_3= & -3 \\ 2x_1+ & x_2- & 4x_3= & -1 \\ 5x_1+ & 3x_2- & 11x_3= & 2 \\ x_1+ & x_2- & 4x_3= & 2 \end{cases}$$

$$\Downarrow$$

$$(A \vdots b)$$

总结

$$\begin{cases} x_1 + x_2 - x_3 = 2 \\ x_1 + 2x_2 = -1 \\ 2x_1 + 5x_2 + x_3 = -5 \\ -2x_1 - 3x_2 + x_3 = -1 \end{cases} \qquad \begin{cases} x_1 + 2x_2 + 4x_3 = 28 \\ -2x_1 - 3x_2 - 9x_3 = -53 \\ 3x_1 + 6x_2 + 13x_3 = 88 \\ 5x_1 + 9x_2 + 22x_3 = 141 \end{cases} \qquad \begin{cases} 4x_1 + 2x_2 - 7x_3 = -3 \\ 2x_1 + x_2 - 4x_3 = -1 \\ 5x_1 + 3x_2 - 11x_3 = 2 \\ x_1 + x_2 - 4x_3 = 2 \end{cases}$$

$$\Downarrow \qquad\qquad \Downarrow \qquad\qquad \Downarrow$$

$$(A \vdots b) \qquad\qquad (A \vdots b) \qquad\qquad (A \vdots b)$$

初等$\Downarrow$行变换　　初等$\Downarrow$行变换　　初等$\Downarrow$行变换

总结

$$\begin{cases} x_1+ & x_2- & x_3 & =2 \\ x_1+ & 2x_2 & & =-1 \\ 2x_1+ & 5x_2+ & x_3 & =-5 \\ -2x_1- & 3x_2+ & x_3 & =-1 \end{cases}$$

$$\Downarrow$$

$$(A \vdots b)$$

初等 $\Downarrow$ 行变换

$$\left(\begin{array}{ccc|c} 1 & 0 & -2 & 5 \\ 0 & 1 & 1 & -3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$$\begin{cases} x_1+ & 2x_2+ & 4x_3 = & 28 \\ -2x_1- & 3x_2- & 9x_3 = & -53 \\ 3x_1+ & 6x_2+ & 13x_3 = & 88 \\ 5x_1+ & 9x_2+ & 22x_3 = & 141 \end{cases}$$

$$\Downarrow$$

$$(A \vdots b)$$

初等 $\Downarrow$ 行变换

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & 7 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$$\begin{cases} 4x_1+ & 2x_2- & 7x_3 = & -3 \\ 2x_1+ & x_2- & 4x_3 = & -1 \\ 5x_1+ & 3x_2- & 11x_3 = & 2 \\ x_1+ & x_2- & 4x_3 = & 2 \end{cases}$$

$$\Downarrow$$

$$(A \vdots b)$$

初等 $\Downarrow$ 行变换

$$\left(\begin{array}{ccc|c} 1 & 1 & -4 & 2 \\ 0 & -1 & 4 & -5 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & -3 \end{array}\right)$$

总结

$$\begin{cases} x_1+x_2-x_3=2 \\ x_1+2x_2=-1 \\ 2x_1+5x_2+x_3=-5 \\ -2x_1-3x_2+x_3=-1 \end{cases}
\qquad
\begin{cases} x_1+2x_2+4x_3=28 \\ -2x_1-3x_2-9x_3=-53 \\ 3x_1+6x_2+13x_3=88 \\ 5x_1+9x_2+22x_3=141 \end{cases}
\qquad
\begin{cases} 4x_1+2x_2-7x_3=-3 \\ 2x_1+x_2-4x_3=-1 \\ 5x_1+3x_2-11x_3=2 \\ x_1+x_2-4x_3=2 \end{cases}$$

$$\Downarrow \qquad\qquad \Downarrow \qquad\qquad \Downarrow$$

$$(A \vdots b) \qquad\qquad (A \vdots b) \qquad\qquad (A \vdots b)$$

初等⇓行变换　　初等⇓行变换　　初等⇓行变换

$$\left(\begin{array}{ccc|c} 1 & 0 & -2 & 5 \\ 0 & 1 & 1 & -3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)
\qquad
\left(\begin{array}{ccc|c} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & 7 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{array}\right)
\qquad
\left(\begin{array}{ccc|c} 1 & 1 & -4 & 2 \\ 0 & -1 & 4 & -5 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & -3 \end{array}\right)$$

无穷多解　　唯一解　　无解

总结

$$\begin{cases} x_1+x_2-x_3=2 \\ x_1+2x_2=-1 \\ 2x_1+5x_2+x_3=-5 \\ -2x_1-3x_2+x_3=-1 \end{cases} \quad \begin{cases} x_1+2x_2+4x_3=28 \\ -2x_1-3x_2-9x_3=-53 \\ 3x_1+6x_2+13x_3=88 \\ 5x_1+9x_2+22x_3=141 \end{cases} \quad \begin{cases} 4x_1+2x_2-7x_3=-3 \\ 2x_1+x_2-4x_3=-1 \\ 5x_1+3x_2-11x_3=2 \\ x_1+x_2-4x_3=2 \end{cases}$$

$$\Downarrow \qquad\qquad \Downarrow \qquad\qquad \Downarrow$$

$$(A\vdots b) \qquad\qquad (A\vdots b) \qquad\qquad (A\vdots b)$$

初等 $\Downarrow$ 行变换　　初等 $\Downarrow$ 行变换　　初等 $\Downarrow$ 行变换

$$\left(\begin{array}{ccc|c} 1 & 0 & -2 & 5 \\ 0 & 1 & 1 & -3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \quad \left(\begin{array}{ccc|c} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & 7 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{array}\right) \quad \left(\begin{array}{ccc|c} 1 & 1 & -4 & 2 \\ 0 & -1 & 4 & -5 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & -3 \end{array}\right)$$

无穷多解　　唯一解　　无解

总结

$$\begin{cases} x_1 + x_2 - x_3 = 2 \\ x_1 + 2x_2 = -1 \\ 2x_1 + 5x_2 + x_3 = -5 \\ -2x_1 - 3x_2 + x_3 = -1 \end{cases} \qquad \begin{cases} x_1 + 2x_2 + 4x_3 = 28 \\ -2x_1 - 3x_2 - 9x_3 = -53 \\ 3x_1 + 6x_2 + 13x_3 = 88 \\ 5x_1 + 9x_2 + 22x_3 = 141 \end{cases} \qquad \begin{cases} 4x_1 + 2x_2 - 7x_3 = -3 \\ 2x_1 + x_2 - 4x_3 = -1 \\ 5x_1 + 3x_2 - 11x_3 = 2 \\ x_1 + x_2 - 4x_3 = 2 \end{cases}$$

$$\Downarrow \qquad\qquad \Downarrow \qquad\qquad \Downarrow$$

$$(A \vdots b) \qquad\qquad (A \vdots b) \qquad\qquad (A \vdots b)$$

初等 $\Downarrow$ 行变换　　初等 $\Downarrow$ 行变换　　初等 $\Downarrow$ 行变换

$$\left(\begin{array}{ccc|c} 1 & 0 & -2 & 5 \\ 0 & 1 & 1 & -3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \qquad \left(\begin{array}{ccc|c} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & 7 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{array}\right) \qquad \left(\begin{array}{ccc|c} 1 & 1 & -4 & 2 \\ 0 & -1 & 4 & -5 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & -3 \end{array}\right)$$

无穷多解　　唯一解　　无解

$r(A) \quad r(A \vdots b) \qquad r(A) \quad r(A \vdots b) \qquad r(A) \quad r(A \vdots b)$

总结

$$\begin{cases} x_1 + x_2 - x_3 = 2 \\ x_1 + 2x_2 = -1 \\ 2x_1 + 5x_2 + x_3 = -5 \\ -2x_1 - 3x_2 + x_3 = -1 \end{cases} \qquad \begin{cases} x_1 + 2x_2 + 4x_3 = 28 \\ -2x_1 - 3x_2 - 9x_3 = -53 \\ 3x_1 + 6x_2 + 13x_3 = 88 \\ 5x_1 + 9x_2 + 22x_3 = 141 \end{cases} \qquad \begin{cases} 4x_1 + 2x_2 - 7x_3 = -3 \\ 2x_1 + x_2 - 4x_3 = -1 \\ 5x_1 + 3x_2 - 11x_3 = 2 \\ x_1 + x_2 - 4x_3 = 2 \end{cases}$$

$$\Downarrow \qquad\qquad \Downarrow \qquad\qquad \Downarrow$$

$$(A \vdots b) \qquad\qquad (A \vdots b) \qquad\qquad (A \vdots b)$$

初等 $\Downarrow$ 行变换 　　初等 $\Downarrow$ 行变换 　　初等 $\Downarrow$ 行变换

$$\left(\begin{array}{ccc|c} 1 & 0 & -2 & 5 \\ 0 & 1 & 1 & -3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \qquad \left(\begin{array}{ccc|c} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & 7 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{array}\right) \qquad \left(\begin{array}{ccc|c} 1 & 1 & -4 & 2 \\ 0 & -1 & 4 & -5 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & -3 \end{array}\right)$$

无穷多解 　　唯一解 　　无解

$$r(A) \quad r(A \vdots b) \qquad\qquad r(A) \quad r(A \vdots b) \qquad\qquad r(A) \neq r(A \vdots b)$$

总结

$$\begin{cases} x_1 + x_2 - x_3 = 2 \\ x_1 + 2x_2 = -1 \\ 2x_1 + 5x_2 + x_3 = -5 \\ -2x_1 - 3x_2 + x_3 = -1 \end{cases}$$

$$\Downarrow$$

$$(A \vdots b)$$

初等 $\Downarrow$ 行变换

$$\begin{pmatrix} 1 & 0 & -2 & 5 \\ 0 & 1 & 1 & -3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

无穷多解

$$r(A) = r(A \vdots b)$$

$$\begin{cases} x_1 + 2x_2 + 4x_3 = 28 \\ -2x_1 - 3x_2 - 9x_3 = -53 \\ 3x_1 + 6x_2 + 13x_3 = 88 \\ 5x_1 + 9x_2 + 22x_3 = 141 \end{cases}$$

$$\Downarrow$$

$$(A \vdots b)$$

初等 $\Downarrow$ 行变换

$$\begin{pmatrix} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & 7 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

唯一解

$$r(A) = r(A \vdots b)$$

$$\begin{cases} 4x_1 + 2x_2 - 7x_3 = -3 \\ 2x_1 + x_2 - 4x_3 = -1 \\ 5x_1 + 3x_2 - 11x_3 = 2 \\ x_1 + x_2 - 4x_3 = 2 \end{cases}$$

$$\Downarrow$$

$$(A \vdots b)$$

初等 $\Downarrow$ 行变换

$$\begin{pmatrix} 1 & 1 & -4 & 2 \\ 0 & -1 & 4 & -5 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & -3 \end{pmatrix}$$

无解

$$r(A) \neq r(A \vdots b)$$

总结

$$\begin{cases} x_1 + x_2 - x_3 = 2 \\ x_1 + 2x_2 = -1 \\ 2x_1 + 5x_2 + x_3 = -5 \\ -2x_1 - 3x_2 + x_3 = -1 \end{cases} \quad \begin{cases} x_1 + 2x_2 + 4x_3 = 28 \\ -2x_1 - 3x_2 - 9x_3 = -53 \\ 3x_1 + 6x_2 + 13x_3 = 88 \\ 5x_1 + 9x_2 + 22x_3 = 141 \end{cases} \quad \begin{cases} 4x_1 + 2x_2 - 7x_3 = -3 \\ 2x_1 + x_2 - 4x_3 = -1 \\ 5x_1 + 3x_2 - 11x_3 = 2 \\ x_1 + x_2 - 4x_3 = 2 \end{cases}$$

$\Downarrow$ $\Downarrow$ $\Downarrow$

$(A \vdots b)$ $(A \vdots b)$ $(A \vdots b)$

初等 $\Downarrow$ 行变换 初等 $\Downarrow$ 行变换 初等 $\Downarrow$ 行变换

$$\left(\begin{array}{ccc|c} 1 & 0 & -2 & 5 \\ 0 & 1 & 1 & -3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \quad \left(\begin{array}{ccc|c} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & 7 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{array}\right) \quad \left(\begin{array}{ccc|c} 1 & 1 & -4 & 2 \\ 0 & -1 & 4 & -5 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & -3 \end{array}\right)$$

无穷多解 唯一解 无解

$r(A) = r(A \vdots b)$ $r(A) = r(A \vdots b)$ $r(A) < r(A \vdots b)$

总结

$$\begin{cases} x_1 + x_2 - x_3 = 2 \\ x_1 + 2x_2 = -1 \\ 2x_1 + 5x_2 + x_3 = -5 \\ -2x_1 - 3x_2 + x_3 = -1 \end{cases} \Downarrow (A \vdots b) \text{ 初等} \Downarrow \text{行变换} \begin{pmatrix} 1 & 0 & -2 & 5 \\ 0 & 1 & 1 & -3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

无穷多解

$$r(A) = r(A \vdots b) < n$$

$$\begin{cases} x_1 + 2x_2 + 4x_3 = 28 \\ -2x_1 - 3x_2 - 9x_3 = -53 \\ 3x_1 + 6x_2 + 13x_3 = 88 \\ 5x_1 + 9x_2 + 22x_3 = 141 \end{cases} \Downarrow (A \vdots b) \text{ 初等} \Downarrow \text{行变换} \begin{pmatrix} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & 7 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

唯一解

$$r(A) = r(A \vdots b)$$

$$\begin{cases} 4x_1 + 2x_2 - 7x_3 = -3 \\ 2x_1 + x_2 - 4x_3 = -1 \\ 5x_1 + 3x_2 - 11x_3 = 2 \\ x_1 + x_2 - 4x_3 = 2 \end{cases} \Downarrow (A \vdots b) \text{ 初等} \Downarrow \text{行变换} \begin{pmatrix} 1 & 1 & -4 & 2 \\ 0 & -1 & 4 & -5 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & -3 \end{pmatrix}$$

无解

$$r(A) < r(A \vdots b)$$

总结

$$\begin{cases} x_1 + x_2 - x_3 = 2 \\ x_1 + 2x_2 = -1 \\ 2x_1 + 5x_2 + x_3 = -5 \\ -2x_1 - 3x_2 + x_3 = -1 \end{cases}$$

$$\Downarrow$$

$$(A \vdots b)$$

初等 $\Downarrow$ 行变换

$$\left(\begin{array}{ccc|c} 1 & 0 & -2 & 5 \\ 0 & 1 & 1 & -3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

无穷多解

$$r(A) = r(A \vdots b) < n$$

$$\begin{cases} x_1 + 2x_2 + 4x_3 = 28 \\ -2x_1 - 3x_2 - 9x_3 = -53 \\ 3x_1 + 6x_2 + 13x_3 = 88 \\ 5x_1 + 9x_2 + 22x_3 = 141 \end{cases}$$

$$\Downarrow$$

$$(A \vdots b)$$

初等 $\Downarrow$ 行变换

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & 7 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

唯一解

$$r(A) = r(A \vdots b) = n$$

$$\begin{cases} 4x_1 + 2x_2 - 7x_3 = -3 \\ 2x_1 + x_2 - 4x_3 = -1 \\ 5x_1 + 3x_2 - 11x_3 = 2 \\ x_1 + x_2 - 4x_3 = 2 \end{cases}$$

$$\Downarrow$$

$$(A \vdots b)$$

初等 $\Downarrow$ 行变换

$$\left(\begin{array}{ccc|c} 1 & 1 & -4 & 2 \\ 0 & -1 & 4 & -5 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & -3 \end{array}\right)$$

无解

$$r(A) < r(A \vdots b)$$

总结

定理 方程组
$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m \end{cases} \Leftrightarrow Ax = b$$
的解有如下情形：

总结

定理 方程组
$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m \end{cases} \Leftrightarrow Ax = b$$
的解有如下情形：

1. 有解

2. 无解

总结

定理 方程组
$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m \end{cases} \Leftrightarrow Ax = b$$
的解有如下情形：

1. 有解
   - 有无穷多解
   - 只有唯一解
2. 无解

总结

定理 方程组
$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m \end{cases} \Leftrightarrow Ax = b$$
的解有如下情形：

1. 有解 $\Leftrightarrow r(A \vdots b) = r(A)$
   - 有无穷多解
   - 只有唯一解
2. 无解

总结

定理 方程组
$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m \end{cases} \Leftrightarrow Ax = b$$
的解有如下情形：

1. 有解$\Leftrightarrow r(A \vdots b) = r(A)$
   - 有无穷多解
   - 只有唯一解
2. 无解 $\Leftrightarrow r(A) \neq r(A \vdots b)$

总结

定理 方程组
$$\begin{cases} a_{11}x_1+a_{12}x_2+\cdots+a_{1n}x_n=b_1\\ a_{21}x_1+a_{22}x_2+\cdots+a_{2n}x_n=b_2\\ \vdots\\ a_{m1}x_1+a_{m2}x_2+\cdots+a_{mn}x_n=b_m \end{cases} \Leftrightarrow Ax=b$$
的解有如下情形：

1. 有解$\Leftrightarrow r(A\,\vdots\,b)=r(A)$
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总结

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的解有如下情形:

1. 有解$\Leftrightarrow r(A\,\vdots\,b) = r(A)$
   - 有无穷多解 $\Leftrightarrow r(A) = r(A\,\vdots\,b) < n$
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总结

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注

- $r(A \vdots b) = r(A)$ 的值，相当于方程组中“独立”方程个数；此时

总结

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注

- $r(A \vdots b) = r(A)$ 的值，相当于方程组中“独立”方程个数；此时
- $n - r(A)$ 为自由变量的个数

练习 求解 $\begin{cases} x_1+2x_2+x_3+x_4+x_5=1 \\ 2x_1+4x_2+3x_3+x_4+x_5=3 \\ -x_1-2x_2+x_3+3x_4-3x_5=7 \\ 2x_3+5x_4-2x_5=9 \end{cases}$

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解

$$(A \vdots b)=\left(\begin{array}{ccccc|c} 1 & 2 & 1 & 1 & 1 & 1 \\ 2 & 4 & 3 & 1 & 1 & 3 \\ -1 & -2 & 1 & 3 & -3 & 7 \\ 0 & 0 & 2 & 5 & -2 & 9 \end{array}\right)$$

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解

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$$\xrightarrow[r_4-2r_2]{r_3-2r_2}$$

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$$\xrightarrow[r_4-2r_2]{r_3-2r_2} \left(\begin{array}{ccccc|c} 1 & 2 & 1 & 1 & 1 & 1\\ 0 & 0 & 1 & -1 & -1 & 1\\ 0 & 0 & 0 & 6 & 0 & 6\\ 0 & 0 & 0 & 7 & 0 & 7 \end{array}\right)$$

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$$\xrightarrow[r_4-2r_2]{r_3-2r_2} \left(\begin{array}{ccccc|c} 1 & 2 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & -1 & -1 & 1 \\ 0 & 0 & 0 & 6 & 0 & 6 \\ 0 & 0 & 0 & 7 & 0 & 7 \end{array}\right) \xrightarrow[\frac{1}{7}\times r_4]{\frac{1}{6}\times r_3}$$

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解

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$$\xrightarrow[r_4-2r_2]{r_3-2r_2} \left(\begin{array}{ccccc|c} 1 & 2 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & -1 & -1 & 1 \\ 0 & 0 & 0 & 6 & 0 & 6 \\ 0 & 0 & 0 & 7 & 0 & 7 \end{array}\right) \xrightarrow[\frac{1}{7}\times r_4]{\frac{1}{6}\times r_3} \left(\begin{array}{ccccc|c} 1 & 2 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & -1 & -1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 \end{array}\right)$$

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$$\xrightarrow[r_4-2r_2]{r_3-2r_2} \left(\begin{array}{ccccc|c} 1 & 2 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & -1 & -1 & 1 \\ 0 & 0 & 0 & 6 & 0 & 6 \\ 0 & 0 & 0 & 7 & 0 & 7 \end{array}\right) \xrightarrow[\frac{1}{7}\times r_4]{\frac{1}{6}\times r_3} \left(\begin{array}{ccccc|c} 1 & 2 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & -1 & -1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 \end{array}\right)$$

$$\xrightarrow{r_4-r_3} \left(\begin{array}{ccccc|c} 1 & 2 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & -1 & -1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

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$$\xrightarrow{r_4-r_3} \left(\begin{array}{ccccc|c} 1 & 2 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & -1 & -1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right) \xrightarrow[r_1-r_3]{r_2+r_3}$$

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$$\xrightarrow[r_4-2r_2]{r_3-2r_2} \left(\begin{array}{ccccc|c} 1 & 2 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & -1 & -1 & 1 \\ 0 & 0 & 0 & 6 & 0 & 6 \\ 0 & 0 & 0 & 7 & 0 & 7 \end{array}\right) \xrightarrow[\frac{1}{7}\times r_4]{\frac{1}{6}\times r_3} \left(\begin{array}{ccccc|c} 1 & 2 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & -1 & -1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 \end{array}\right)$$

$$\xrightarrow{r_4-r_3} \left(\begin{array}{ccccc|c} 1 & 2 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & -1 & -1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right) \xrightarrow[r_1-r_3]{r_2+r_3} \left(\begin{array}{ccccc|c} 1 & 2 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & -1 & 2 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

练习 求解 $\begin{cases} x_1+2x_2+x_3+x_4+x_5=1 \\ 2x_1+4x_2+3x_3+x_4+x_5=3 \\ -x_1-2x_2+x_3+3x_4-3x_5=7 \\ 2x_3+5x_4-2x_5=9 \end{cases}$

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$$\xrightarrow[r_4-2r_2]{r_3-2r_2} \left(\begin{array}{ccccc|c} 1 & 2 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & -1 & -1 & 1 \\ 0 & 0 & 0 & 6 & 0 & 6 \\ 0 & 0 & 0 & 7 & 0 & 7 \end{array}\right) \xrightarrow[\frac{1}{7}\times r_4]{\frac{1}{6}\times r_3} \left(\begin{array}{ccccc|c} 1 & 2 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & -1 & -1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 \end{array}\right)$$

$$\xrightarrow{r_4-r_3} \left(\begin{array}{ccccc|c} 1 & 2 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & -1 & -1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right) \xrightarrow[r_1-r_3]{r_2+r_3} \left(\begin{array}{ccccc|c} 1 & 2 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & -1 & 2 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

$$\xrightarrow{r_1-r_2}$$

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$$\xrightarrow[r_4-2r_2]{r_3-2r_2} \left(\begin{array}{ccccc|c} 1 & 2 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & -1 & -1 & 1 \\ 0 & 0 & 0 & 6 & 0 & 6 \\ 0 & 0 & 0 & 7 & 0 & 7 \end{array}\right) \xrightarrow[\frac{1}{7}\times r_4]{\frac{1}{6}\times r_3} \left(\begin{array}{ccccc|c} 1 & 2 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & -1 & -1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 \end{array}\right)$$

$$\xrightarrow{r_4-r_3} \left(\begin{array}{ccccc|c} 1 & 2 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & -1 & -1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right) \xrightarrow[r_1-r_3]{r_2+r_3} \left(\begin{array}{ccccc|c} 1 & 2 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & -1 & 2 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

$$\xrightarrow{r_1-r_2} \left(\begin{array}{ccccc|c} 1 & 2 & 0 & 0 & 2 & -2 \\ 0 & 0 & 1 & 0 & -1 & 2 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

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解

$$(A \vdots b) = \left(\begin{array}{ccccc|c} 1 & 2 & 1 & 1 & 1 & 1 \\ 2 & 4 & 3 & 1 & 1 & 3 \\ -1 & -2 & 1 & 3 & -3 & 7 \\ 0 & 0 & 2 & 5 & -2 & 9 \end{array}\right) \xrightarrow[r_3+r_1]{r_2-2r_1} \left(\begin{array}{ccccc|c} 1 & 2 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & -1 & -1 & 1 \\ 0 & 0 & 2 & 4 & -2 & 8 \\ 0 & 0 & 2 & 5 & -2 & 9 \end{array}\right)$$

$$\xrightarrow[r_4-2r_2]{r_3-2r_2} \left(\begin{array}{ccccc|c} 1 & 2 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & -1 & -1 & 1 \\ 0 & 0 & 0 & 6 & 0 & 6 \\ 0 & 0 & 0 & 7 & 0 & 7 \end{array}\right) \xrightarrow[\frac{1}{7}\times r_4]{\frac{1}{6}\times r_3} \left(\begin{array}{ccccc|c} 1 & 2 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & -1 & -1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 \end{array}\right)$$

$$\xrightarrow{r_4-r_3} \left(\begin{array}{ccccc|c} 1 & 2 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & -1 & -1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right) \xrightarrow[r_1-r_3]{r_2+r_3} \left(\begin{array}{ccccc|c} 1 & 2 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & -1 & 2 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

$$\xrightarrow{r_1-r_2} \left(\begin{array}{ccccc|c} 1 & 2 & 0 & 0 & 2 & -2 \\ 0 & 0 & 1 & 0 & -1 & 2 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

练习 求解 $\begin{cases} x_1+2x_2+x_3+x_4+x_5=1 \\ 2x_1+4x_2+3x_3+x_4+x_5=3 \\ -x_1-2x_2+x_3+3x_4-3x_5=7 \\ 2x_3+5x_4-2x_5=9 \end{cases}$

解

$$(A \vdots b) \longrightarrow \left(\begin{array}{ccccc|c} 1 & 2 & 0 & 0 & 2 & -2 \\ 0 & 0 & 1 & 0 & -1 & 2 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

练习 求解 $\begin{cases} x_1+2x_2+x_3+x_4+x_5=1 \\ 2x_1+4x_2+3x_3+x_4+x_5=3 \\ -x_1-2x_2+x_3+3x_4-3x_5=7 \\ 2x_3+5x_4-2x_5=9 \end{cases}$

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- 可见 $r(A)=r(A \vdots b)=3<5$，有无穷多的解，

练习 求解 $\begin{cases} x_1 + 2x_2 + x_3 + x_4 + x_5 = 1 \\ 2x_1 + 4x_2 + 3x_3 + x_4 + x_5 = 3 \\ -x_1 - 2x_2 + x_3 + 3x_4 - 3x_5 = 7 \\ 2x_3 + 5x_4 - 2x_5 = 9 \end{cases}$

解

$$(A \vdots b) \longrightarrow \left(\begin{array}{ccccc|c} 1 & 2 & 0 & 0 & 2 & -2 \\ 0 & 0 & 1 & 0 & -1 & 2 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

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解

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- 由既约阶梯形矩阵可知，原方程组等价于

$$\begin{cases} x_1+2x_2+2x_5=-2 \\ x_3-x_5=2 \\ x_4=1 \end{cases}$$

练习 求解 $\begin{cases} x_1+2x_2+x_3+x_4+x_5=1 \\ 2x_1+4x_2+3x_3+x_4+x_5=3 \\ -x_1-2x_2+x_3+3x_4-3x_5=7 \\ 2x_3+5x_4-2x_5=9 \end{cases}$

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$$\begin{cases} x_1+2x_2+2x_5=-2 \\ x_3-x_5=2 \\ x_4=1 \end{cases} \Rightarrow \begin{cases} x_1=-2-2x_2-2x_5 \\ x_3=2+x_5 \\ x_4=1 \end{cases}$$

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所以通解是

$$\begin{cases} x_1= \\ x_2= \\ x_3= \\ x_4= \\ x_5= \end{cases}$$

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解

$$(A \vdots b) \longrightarrow \left(\begin{array}{ccccc|c} 1 & 2 & 0 & 0 & 2 & -2 \\ 0 & 0 & 1 & 0 & -1 & 2 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

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所以通解是

$$\begin{cases} x_1= \\ x_2=c_1 \\ x_3= \\ x_4= \\ x_5=c_2 \end{cases} \qquad (c_1,\ c_2\text{为任意常数})$$

练习 求解 $\begin{cases} x_1+2x_2+x_3+x_4+x_5=1 \\ 2x_1+4x_2+3x_3+x_4+x_5=3 \\ -x_1-2x_2+x_3+3x_4-3x_5=7 \\ 2x_3+5x_4-2x_5=9 \end{cases}$

解

$$(A\vdots b) \longrightarrow \left(\begin{array}{ccccc|c} 1 & 2 & 0 & 0 & 2 & -2 \\ 0 & 0 & 1 & 0 & -1 & 2 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

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所以通解是

$$\begin{cases} x_1=-2-2c_1-2c_2 \\ x_2=c_1 \\ x_3= \\ x_4= \\ x_5=c_2 \end{cases} \qquad (c_1,\ c_2\text{为任意常数})$$

练习 求解 $\begin{cases} x_1 + 2x_2 + x_3 + x_4 + x_5 = 1 \\ 2x_1 + 4x_2 + 3x_3 + x_4 + x_5 = 3 \\ -x_1 - 2x_2 + x_3 + 3x_4 - 3x_5 = 7 \\ 2x_3 + 5x_4 - 2x_5 = 9 \end{cases}$

解

$$(A \vdots b) \longrightarrow \left(\begin{array}{ccccc|c} 1 & 2 & 0 & 0 & 2 & -2 \\ 0 & 0 & 1 & 0 & -1 & 2 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

- 可见 $r(A) = r(A \vdots b) = 3 < 5$，有无穷多的解，含 $5 - 3 = 2$ 个自由变量
- 由既约阶梯形矩阵可知，原方程组等价于

$$\begin{cases} x_1 + 2x_2 + 2x_5 = -2 \\ x_3 - x_5 = 2 \\ x_4 = 1 \end{cases} \Rightarrow \begin{cases} x_1 = -2 - 2x_2 - 2x_5 \\ x_3 = 2 + x_5 \\ x_4 = 1 \end{cases}$$

所以通解是

$$\begin{cases} x_1 = -2 - 2c_1 - 2c_2 \\ x_2 = c_1 \\ x_3 = 2 + c_2 \\ x_4 = \\ x_5 = c_2 \end{cases} \qquad (c_1, c_2\text{为任意常数})$$

练习 求解 $\begin{cases} x_1+ & 2x_2+ & x_3+ & x_4+ & x_5 = & 1 \\ 2x_1+ & 4x_2+ & 3x_3+ & x_4+ & x_5 = & 3 \\ -x_1- & 2x_2+ & x_3+ & 3x_4- & 3x_5 = & 7 \\ & & 2x_3+ & 5x_4- & 2x_5 = & 9 \end{cases}$

解

$$(A \vdots b) \longrightarrow \left(\begin{array}{ccccc|c} 1 & 2 & 0 & 0 & 2 & -2 \\ 0 & 0 & 1 & 0 & -1 & 2 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

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$$\begin{cases} x_1 + 2x_2 + 2x_5 = -2 \\ x_3 - x_5 = 2 \\ x_4 = 1 \end{cases} \Rightarrow \begin{cases} x_1 = -2 - 2x_2 - 2x_5 \\ x_3 = 2 + x_5 \\ x_4 = 1 \end{cases}$$

所以通解是

$$\begin{cases} x_1 = -2 - 2c_1 - 2c_2 \\ x_2 = c_1 \\ x_3 = 2 + c_2 \\ x_4 = 1 \\ x_5 = c_2 \end{cases} \qquad (c_1, c_2\text{为任意常数})$$

例 讨论 a, b 取何值时，方程组

$$\begin{cases} x_1+x_2+x_3+x_4=0 \\ x_2+2x_3+2x_4=1 \\ -x_2+(a-3)x_3-2x_4=b \\ 3x_1+2x_2+x_3+ax_4=-1 \end{cases}$$

有无穷解、唯一解，及无解？

例 讨论 a, b 取何值时，方程组

$$\begin{cases} x_1 + x_2 + x_3 + x_4 = 0 \\ x_2 + 2x_3 + 2x_4 = 1 \\ -x_2 + (a-3)x_3 - 2x_4 = b \\ 3x_1 + 2x_2 + x_3 + ax_4 = -1 \end{cases}$$

有无穷解、唯一解，及无解？

解

$$(A \vdots b) = \left(\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 2 & 2 & 1 \\ 0 & -1 & a-3 & -2 & b \\ 3 & 2 & 1 & a & -1 \end{array}\right)$$

例 讨论 a, b 取何值时，方程组

$$\begin{cases} x_1 + x_2 + x_3 + x_4 = 0 \\ x_2 + 2x_3 + 2x_4 = 1 \\ -x_2 + (a-3)x_3 - 2x_4 = b \\ 3x_1 + 2x_2 + x_3 + ax_4 = -1 \end{cases}$$

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解

$$(A \vdots b) = \left(\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 2 & 2 & 1 \\ 0 & -1 & a-3 & -2 & b \\ 3 & 2 & 1 & a & -1 \end{array}\right) \xrightarrow{r_4 - 3r_1}$$

例 讨论 a, b 取何值时，方程组

$$\begin{cases} x_1 + x_2 + x_3 + x_4 = 0 \\ x_2 + 2x_3 + 2x_4 = 1 \\ -x_2 + (a-3)x_3 - 2x_4 = b \\ 3x_1 + 2x_2 + x_3 + ax_4 = -1 \end{cases}$$

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解

$$(A \vdots b) = \left(\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 2 & 2 & 1 \\ 0 & -1 & a-3 & -2 & b \\ 3 & 2 & 1 & a & -1 \end{array}\right) \xrightarrow{r_4 - 3r_1} \left(\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 2 & 2 & 1 \\ 0 & -1 & a-3 & -2 & b \\ 0 & -1 & -2 & a-3 & -1 \end{array}\right)$$

例 讨论 a, b 取何值时，方程组

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解

$$(A \vdots b) = \left(\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 2 & 2 & 1 \\ 0 & -1 & a-3 & -2 & b \\ 3 & 2 & 1 & a & -1 \end{array}\right) \xrightarrow{r_4 - 3r_1} \left(\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 2 & 2 & 1 \\ 0 & -1 & a-3 & -2 & b \\ 0 & -1 & -2 & a-3 & -1 \end{array}\right)$$

$$\xrightarrow[r_4 + r_2]{r_3 + r_2}$$

例 讨论 a, b 取何值时，方程组

$$\begin{cases} x_1+ & x_2+ & x_3+ & x_4 = 0 \\ & x_2+ & 2x_3+ & 2x_4 = 1 \\ & -x_2+ & (a-3)x_3- & 2x_4 = b \\ 3x_1+ & 2x_2+ & x_3+ & ax_4 = -1 \end{cases}$$

有无穷解、唯一解，及无解？

解

$$(A \vdots b) = \left(\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 2 & 2 & 1 \\ 0 & -1 & a-3 & -2 & b \\ 3 & 2 & 1 & a & -1 \end{array}\right) \xrightarrow{r_4-3r_1} \left(\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 2 & 2 & 1 \\ 0 & -1 & a-3 & -2 & b \\ 0 & -1 & -2 & a-3 & -1 \end{array}\right)$$

$$\xrightarrow[r_4+r_2]{r_3+r_2} \left(\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 2 & 2 & 1 \\ 0 & 0 & a-1 & 0 & b+1 \\ 0 & 0 & 0 & a-1 & 0 \end{array}\right)$$

例 讨论 a, b 取何值时，方程组

$$\begin{cases} x_1 + x_2 + x_3 + x_4 = 0 \\ x_2 + 2x_3 + 2x_4 = 1 \\ -x_2 + (a-3)x_3 - 2x_4 = b \\ 3x_1 + 2x_2 + x_3 + ax_4 = -1 \end{cases}$$

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解

$$(A \vdots b) \longrightarrow \left(\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 2 & 2 & 1 \\ 0 & 0 & a-1 & 0 & b+1 \\ 0 & 0 & 0 & a-1 & 0 \end{array}\right)$$

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解

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- 当 $a \neq 1$ 时
- 当 $a = 1$ 时

例 讨论 a, b 取何值时，方程组

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解

$$(A \vdots b) \longrightarrow \left(\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 2 & 2 & 1 \\ 0 & 0 & a-1 & 0 & b+1 \\ 0 & 0 & 0 & a-1 & 0 \end{array}\right)$$

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解

$$(A \vdots b) \longrightarrow \left(\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 2 & 2 & 1 \\ 0 & 0 & a-1 & 0 & b+1 \\ 0 & 0 & 0 & a-1 & 0 \end{array}\right)$$

- 当 $a \neq 1$ 时（b 为任意数），$r(A) = r(A \vdots b) = 4$，
- 当 $a = 1$ 时

例 讨论 a, b 取何值时，方程组

$$\begin{cases} x_1 + x_2 + x_3 + x_4 = 0 \\ x_2 + 2x_3 + 2x_4 = 1 \\ -x_2 + (a-3)x_3 - 2x_4 = b \\ 3x_1 + 2x_2 + x_3 + ax_4 = -1 \end{cases}$$

有无穷解、唯一解，及无解？

解

$$(A \vdots b) \longrightarrow \left(\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 2 & 2 & 1 \\ 0 & 0 & a-1 & 0 & b+1 \\ 0 & 0 & 0 & a-1 & 0 \end{array}\right)$$

- 当 $a \neq 1$ 时（b 为任意数），$r(A) = r(A \vdots b) = 4$，有唯一解；
- 当 $a = 1$ 时

例 讨论 a, b 取何值时，方程组

$$\begin{cases} x_1 + x_2 + x_3 + x_4 = 0 \\ x_2 + 2x_3 + 2x_4 = 1 \\ -x_2 + (a-3)x_3 - 2x_4 = b \\ 3x_1 + 2x_2 + x_3 + ax_4 = -1 \end{cases}$$

有无穷解、唯一解，及无解？

解

$$(A \vdots b) \longrightarrow \left(\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 2 & 2 & 1 \\ 0 & 0 & a-1 & 0 & b+1 \\ 0 & 0 & 0 & a-1 & 0 \end{array}\right)$$

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$$(A \vdots b) \longrightarrow \left(\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 2 & 2 & 1 \\ 0 & 0 & 0 & 0 & b+1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

例 讨论 a, b 取何值时，方程组

$$\begin{cases} x_1 + \quad x_2 + \qquad\qquad x_3 + \quad x_4 = \quad 0 \\ \qquad\quad x_2 + \qquad\quad 2x_3 + \ 2x_4 = \quad 1 \\ \quad\ -x_2 + (a-3)x_3 - \ 2x_4 = \quad b \\ 3x_1 + \ 2x_2 + \qquad\qquad x_3 + \ ax_4 = -1 \end{cases}$$

有无穷解、唯一解，及无解？

解

$$(A \vdots b) \longrightarrow \left(\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 2 & 2 & 1 \\ 0 & 0 & a-1 & 0 & b+1 \\ 0 & 0 & 0 & a-1 & 0 \end{array}\right)$$

- 当 $a \neq 1$ 时（b 为任意数），$r(A) = r(A \vdots b) = 4$，有唯一解；
- 当 $a = 1$ 时

$$(A \vdots b) \longrightarrow \left(\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 2 & 2 & 1 \\ 0 & 0 & 0 & 0 & b+1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

  - $a = 1, b = -1$ 时
  - $a = 1, b \neq -1$ 时

例 讨论 a, b 取何值时，方程组

$$\begin{cases} x_1 + x_2 + x_3 + x_4 = 0 \\ x_2 + 2x_3 + 2x_4 = 1 \\ -x_2 + (a-3)x_3 - 2x_4 = b \\ 3x_1 + 2x_2 + x_3 + ax_4 = -1 \end{cases}$$

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解

$$(A \vdots b) \longrightarrow \left(\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 2 & 2 & 1 \\ 0 & 0 & a-1 & 0 & b+1 \\ 0 & 0 & 0 & a-1 & 0 \end{array}\right)$$

- 当 $a \neq 1$ 时（b 为任意数），$r(A) = r(A \vdots b) = 4$，有唯一解；
- 当 $a = 1$ 时

$$(A \vdots b) \longrightarrow \left(\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 2 & 2 & 1 \\ 0 & 0 & 0 & 0 & b+1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

  - $a = 1, b = -1$ 时
  - $a = 1, b \neq -1$ 时

例 讨论 a, b 取何值时，方程组

$$\begin{cases} x_1 + x_2 + x_3 + x_4 = 0 \\ x_2 + 2x_3 + 2x_4 = 1 \\ -x_2 + (a-3)x_3 - 2x_4 = b \\ 3x_1 + 2x_2 + x_3 + ax_4 = -1 \end{cases}$$

有无穷解、唯一解，及无解？

解

$$(A \vdots b) \longrightarrow \left(\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 2 & 2 & 1 \\ 0 & 0 & a-1 & 0 & b+1 \\ 0 & 0 & 0 & a-1 & 0 \end{array}\right)$$

- 当 $a \neq 1$ 时（b 为任意数），$r(A) = r(A \vdots b) = 4$，有唯一解；
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- 当 $a \neq 7$ 时（b 为任意数），$r(A\vdots b) = r(A) = 3$，有唯一解；
- 当 $a = 7$ 时
$$(A\vdots b) \longrightarrow \left(\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 0 & -1 & -1 & -3 \\ 0 & 0 & 0 & b+3 \end{array}\right)$$
  - $a = 7, b = -3$ 时，$r(A\vdots b) = r(A) = 2 < 3$，有无穷多解
  - $a = 7, b \neq -3$ 时

例 讨论 a, b 取何值时，方程组 $\begin{cases} x_1+2x_2+3x_3=1 \\ 2x_1+3x_2+5x_3=-1 \\ 3x_1+4x_2+ax_3=b \end{cases}$ 有无穷解、唯一解，及无解？

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线性方程组解的分类

- 一般线性方程组 $A_{m\times n}x = b$（m 个方程，n 个未知量）

$Ax = b$	有无穷解	有唯一解	无解
	$r(A) = r(A \vdots b) < n$	$r(A) = r(A \vdots b) = n$	$r(A) < r(A \vdots b)$

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$$(A \vdots b) \longrightarrow \left(\begin{array}{cccc|c} 1 & 0 & 3/2 & 1 & 0 \\ 0 & 1 & -7/2 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

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$$\begin{cases} x_1 + \frac{3}{2}x_3 + x_4 = 0 \\ x_2 - \frac{7}{2}x_3 + 2x_4 = 0 \end{cases} \Leftrightarrow \begin{cases} x_1 = -\frac{3}{2}x_3 - x_4 \\ x_2 = \frac{7}{2}x_3 - 2x_4 \end{cases}$$

所以 $\begin{cases} x_1 = -\frac{3}{2}c_1 - c_2 \\ x_2 = \frac{7}{2}c_1 - 2c_2 \\ x_3 = c_1 \\ x_4 = c_2 \end{cases}$

例 解齐次线性方程组 $\begin{cases} x_1 - x_2 + 5x_3 - x_4 = 0 \\ x_1 + x_2 - 2x_3 + 3x_4 = 0 \\ 3x_1 - x_2 + 8x_3 + x_4 = 0 \\ x_1 + 3x_2 - 9x_3 + 7x_4 = 0 \end{cases}$

解

$$(A \vdots b) \longrightarrow \left(\begin{array}{cccc|c} 1 & 0 & 3/2 & 1 & 0 \\ 0 & 1 & -7/2 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

所以原方程组等价于

$$\begin{cases} x_1 + \frac{3}{2}x_3 + x_4 = 0 \\ x_2 - \frac{7}{2}x_3 + 2x_4 = 0 \end{cases} \Leftrightarrow \begin{cases} x_1 = -\frac{3}{2}x_3 - x_4 \\ x_2 = \frac{7}{2}x_3 - 2x_4 \end{cases}$$

所以 $\begin{cases} x_1 = -\frac{3}{2}c_1 - c_2 \\ x_2 = \frac{7}{2}c_1 - 2c_2 \\ x_3 = c_1 \\ x_4 = c_2 \end{cases}$ （注自由变量个数 $= 2 = 4 - r(A)$）