§3.1 线性方程组的消元解法

数学系 梁卓滨

2016 - 2017 学年 I 暑修班



记号

考虑 n 个未知量 m 个方程的线性方程组:

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n = b_2 \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n = b_m \end{cases}$$

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可以, 等价地, 改写成矩阵形式

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$$(A \vdots b) = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{pmatrix}$$



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整个方程组的信息包含在:

增广矩阵
$$(A : b) = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{pmatrix}$$



消元法求解线性方程组——示例

例 解方程组:
$$\begin{cases} x_1 + x_2 - x_3 = 2 \\ x_1 + 2x_2 = -1 \\ 4x_1 + 7x_2 - x_3 = -1 \\ -3x_1 - 4x_2 + 2x_3 = -3 \end{cases}$$

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$$(A \vdots b) = \begin{pmatrix} 1 & 1 & -1 & 2 \\ 1 & 2 & 0 & -1 \\ 4 & 7 & -1 & -1 \\ -3 & -4 & 2 & -3 \end{pmatrix}$$

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(2)-(1)

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消元法步骤示例:
 $x_{1}+x_{2}-x_{3}=2$ $x_{1}+2x_{2}=-1$

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- - $(A:b) = \begin{pmatrix} 1 & 1 & -1 & 2 \\ 1 & 2 & 0 & -1 \\ 4 & 7 & -1 & -1 \\ -3 & -4 & 2 & -3 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 1 & -1 & 2 \\ 0 & 1 & 1 & -3 \\ 4 & 7 & -1 & -1 \\ -3 & -4 & 2 & -3 \end{pmatrix}$ $r_2 r_1$



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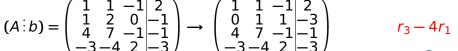
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线性方程组的消元解法

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(4) + 3(1)

列:
$$\begin{array}{ccc}
X_3 = 2 & & \\
 & = -1 & \\
\end{array}$$

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$$r_3 - 4r_1$$

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$$r_4 + 3r_1$$
(3.1 线性方程组的消元解法

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 (4) + 3(1)

线性方程组的消元解法

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 (3) - 3(2)

 $(A:b) = \begin{pmatrix} 1 & 1 & -1 & 2 \\ 1 & 2 & 0 & -1 \\ 4 & 7 & -1 & -1 \\ -3 & -4 & 2 & -3 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 1 & -1 & 2 \\ 0 & 1 & 1 & -3 \\ 0 & 3 & 3 & -9 \\ 0 & -1 & -1 & 3 \end{pmatrix}$

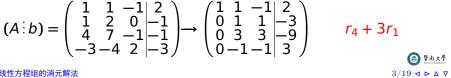
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例 解方程组:
$$\begin{cases} x_1 + x_2 - x_3 = 2 \\ x_1 + 2x_2 = -1 \\ 4x_1 + 7x_2 - x_3 = -1 \\ -3x_1 - 4x_2 + 2x_3 = -3 \end{cases}$$

$$\begin{cases} x_1 + x_2 - x_3 = 2 \\ x_1 + 2x_2 = -1 \\ 4x_1 + 7x_2 - x_3 = -1 \\ -3x_1 - 4x_2 + 2x_3 = -3 \end{cases} \rightarrow \begin{cases} x_1 + x_2 - x_3 = 2 \\ x_2 + x_3 = -3 \\ 0 = 0 \\ -x_2 - x_3 = 3 \end{cases}$$
 (3) - 3(2)

线性方程组的消元解法



例 解方程组:
$$\begin{cases} x_1 + x_2 - x_3 = 2 \\ x_1 + 2x_2 = -1 \\ 4x_1 + 7x_2 - x_3 = -1 \\ -3x_1 - 4x_2 + 2x_3 = -3 \end{cases}$$

$$\begin{cases} x_{1} + x_{2} - x_{3} = 2 \\ x_{1} + 2x_{2} = -1 \\ 4x_{1} + 7x_{2} - x_{3} = -1 \\ -3x_{1} - 4x_{2} + 2x_{3} = -3 \end{cases} \rightarrow \begin{cases} x_{1} + x_{2} - x_{3} = 2 \\ x_{2} + x_{3} = -3 \\ 0 = 0 \\ -x_{2} - x_{3} = 3 \end{cases}$$
(3) - 3(2)

 $(A : b) = \begin{pmatrix} 1 & 1 & -1 & 2 \\ 1 & 2 & 0 & -1 \\ 4 & 7 & -1 & -1 \\ 2 & -4 & 2 & -3 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 1 & -1 & 2 \\ 0 & 1 & 1 & -3 \\ 0 & 3 & 3 & -9 \\ 0 & -1 & -1 & 3 \end{pmatrix} \qquad r_3 - 2r_2$

4

3.1 线性方程组的消元解法

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例 解方程组:
$$\begin{cases} x_1 + x_2 - x_3 = 2 \\ x_1 + 2x_2 = -1 \\ 4x_1 + 7x_2 - x_3 = -1 \\ -3x_1 - 4x_2 + 2x_3 = -3 \end{cases}$$

$$\begin{cases} x_1 + x_2 - x_3 = 2 \\ x_1 + 2x_2 = -1 \\ 4x_1 + 7x_2 - x_3 = -1 \\ -3x_1 - 4x_2 + 2x_3 = -3 \end{cases} \rightarrow \begin{cases} x_1 + x_2 - x_3 = 2 \\ x_2 + x_3 = -3 \\ 0 = 0 \\ -x_2 - x_3 = 3 \end{cases}$$
 (3) - 3(2)

 $(A:b) = \begin{pmatrix} 1 & 1 & -1 & 2 \\ 1 & 2 & 0 & -1 \\ 4 & 7 & -1 & -1 \\ 2 & 4 & 2 & -1 \\ 3 & 4 & 2 & -1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 1 & -1 & 2 \\ 0 & 1 & 1 & -3 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & -1 & 3 \end{pmatrix} \qquad r_3 - 2r_2$

§3.1 线性方程组的消元解法



例 解方程组:
$$\begin{cases} x_1 + x_2 - x_3 = 2 \\ x_1 + 2x_2 = -1 \\ 4x_1 + 7x_2 - x_3 = -1 \\ -3x_1 - 4x_2 + 2x_3 = -3 \end{cases}$$

$$\begin{cases} x_{1} + x_{2} - x_{3} = 2 \\ x_{1} + 2x_{2} = -1 \\ 4x_{1} + 7x_{2} - x_{3} = -1 \\ -3x_{1} - 4x_{2} + 2x_{3} = -3 \end{cases} \rightarrow \begin{cases} x_{1} + x_{2} - x_{3} = 2 \\ x_{2} + x_{3} = -3 \\ 0 = 0 \\ -x_{2} - x_{3} = 3 \end{cases}$$
(4) + (2)

$$(A:b) = \begin{pmatrix} 1 & 1 & -1 & 2 \\ 1 & 2 & 0 & -1 \\ 4 & 7 & -1 & -1 \\ -3 & -4 & 2 & -3 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 1 & -1 & 2 \\ 0 & 1 & 1 & -3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$r_3 - 2r_2$$



例 解方程组:
$$\begin{cases} x_1 + x_2 - x_3 = 2 \\ x_1 + 2x_2 = -1 \\ 4x_1 + 7x_2 - x_3 = -1 \\ -3x_1 - 4x_2 + 2x_3 = -3 \end{cases}$$

$$\begin{cases} x_{1} + x_{2} - x_{3} = 2 \\ x_{1} + 2x_{2} = -1 \\ 4x_{1} + 7x_{2} - x_{3} = -1 \\ -3x_{1} - 4x_{2} + 2x_{3} = -3 \end{cases} \rightarrow \begin{cases} x_{1} + x_{2} - x_{3} = 2 \\ x_{2} + x_{3} = -3 \\ 0 = 0 \\ 0 = 0 \end{cases}$$

$$\downarrow (4) + (2)$$

$$(A:b) = \begin{pmatrix} 1 & 1 & -1 & 2 \\ 1 & 2 & 0 & -1 \\ 4 & 7 & -1 & -1 \\ -3 & -4 & 2 & -3 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 1 & -1 & 2 \\ 0 & 1 & 1 & -3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad r_3 - 2r_2$$



例 解方程组:
$$\begin{cases} x_1 + x_2 - x_3 = 2 \\ x_1 + 2x_2 = -1 \\ 4x_1 + 7x_2 - x_3 = -1 \\ -3x_1 - 4x_2 + 2x_3 = -3 \end{cases}$$

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$$\downarrow \qquad \qquad \downarrow \qquad \qquad \downarrow$$

$$(A:b) = \begin{pmatrix} 1 & 1 & -1 & 2 \\ 1 & 2 & 0 & -1 \\ 4 & 7 & -1 & -1 \\ -3 & -4 & 2 & -3 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 1 & -1 & 2 \\ 0 & 1 & 1 & -3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad r_4 + r_2$$



例 解方程组:
$$\begin{cases} x_1 + x_2 - x_3 = 2 \\ x_1 + 2x_2 = -1 \\ 4x_1 + 7x_2 - x_3 = -1 \\ -3x_1 - 4x_2 + 2x_3 = -3 \end{cases}$$

$$\begin{cases} x_{1} + x_{2} - x_{3} = 2 \\ x_{1} + 2x_{2} = -1 \\ 4x_{1} + 7x_{2} - x_{3} = -1 \\ -3x_{1} - 4x_{2} + 2x_{3} = -3 \end{cases} \rightarrow \begin{cases} x_{1} + x_{2} - x_{3} = 2 \\ x_{2} + x_{3} = -3 \\ 0 = 0 \\ 0 = 0 \end{cases}$$

$$\downarrow \qquad \qquad \downarrow \qquad \qquad \downarrow$$

$$(A:b) = \begin{pmatrix} 1 & 1 & -1 & 2 \\ 1 & 2 & 0 & -1 \\ 4 & 7 & -1 & -1 \\ -3 & -4 & 2 & -3 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 1 & -1 & 2 \\ 0 & 1 & 1 & -3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad r_4 + r_2$$



例 解方程组:
$$\begin{cases} x_1 + x_2 - x_3 = 2 \\ x_1 + 2x_2 = -1 \\ 4x_1 + 7x_2 - x_3 = -1 \\ -3x_1 - 4x_2 + 2x_3 = -3 \end{cases}$$

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(1)-(2)

 $(A : b) = \begin{pmatrix} 1 & 1 & -1 & 2 \\ 1 & 2 & 0 & -1 \\ 4 & 7 & -1 & -1 \\ 2 & 4 & 2 & -3 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 1 & -1 & 2 \\ 0 & 1 & 1 & -3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad r_4 + r_2$

例 解方程组:
$$\begin{cases} x_1 + x_2 - x_3 = 2 \\ x_1 + 2x_2 = -1 \\ 4x_1 + 7x_2 - x_3 = -1 \\ -3x_1 - 4x_2 + 2x_3 = -3 \end{cases}$$

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(1) - (2)

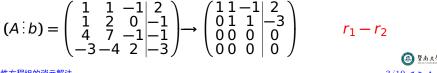
 $(A:b) = \begin{pmatrix} 1 & 1 & -1 & 2 \\ 1 & 2 & 0 & -1 \\ 4 & 7 & -1 & -1 \\ -3 & -4 & 2 & -3 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 1 & -1 & 2 \\ 0 & 1 & 1 & -3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad r_4 + r_2$



例 解方程组:
$$\begin{cases} x_1 + x_2 - x_3 = 2 \\ x_1 + 2x_2 = -1 \\ 4x_1 + 7x_2 - x_3 = -1 \\ -3x_1 - 4x_2 + 2x_3 = -3 \end{cases}$$

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(1)-(2)

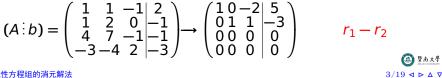
线性方程组的消元解法



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(1) - (2)

线性方程组的消元解法



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消元法步骤示例:

$$\begin{cases} x_{1} + x_{2} - x_{3} = 2 \\ x_{1} + 2x_{2} = -1 \\ 4x_{1} + 7x_{2} - x_{3} = -1 \\ -3x_{1} - 4x_{2} + 2x_{3} = -3 \end{cases} \rightarrow \begin{cases} x_{1} - 2x_{3} = 5 \\ x_{2} + x_{3} = -3 \\ 0 = 0 \end{cases} \Rightarrow \begin{cases} x_{1} = 5 + 2x_{3} \\ x_{2} = -3 - x_{3} \end{cases}$$

$$0 = 0$$

$$\downarrow \qquad \qquad \downarrow \qquad \downarrow$$

例 解方程组:
$$\begin{cases} x_1 + x_2 - x_3 = 2 \\ x_1 + 2x_2 = -1 \\ 4x_1 + 7x_2 - x_3 = -1 \\ -3x_1 - 4x_2 + 2x_3 = -3 \end{cases}$$

消元法步骤示例:

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 $(A:b) = \begin{pmatrix} 1 & 1 & -1 & 2 \\ 1 & 2 & 0 & -1 \\ 4 & 7 & -1 & -1 \\ -3 & -4 & 2 & -3 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & -2 & 5 \\ 0 & 1 & 1 & -3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \Rightarrow \begin{cases} x_1 = 5 + 2x_3 \\ x_2 = -3 - x_3 \\ x_3 = -3 - x_3 \end{cases}$

例 解方程组:
$$\begin{cases} x_1 + x_2 - x_3 = 2 \\ x_1 + 2x_2 = -1 \\ 4x_1 + 7x_2 - x_3 = -1 \\ -3x_1 - 4x_2 + 2x_3 = -3 \end{cases}$$

消元法步骤示例:

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线性方程组的消元解法

 $(A:b) = \begin{pmatrix} 1 & 1 & -1 & 2 \\ 1 & 2 & 0 & -1 \\ 4 & 7 & -1 & -1 \\ -3 & -4 & 2 & -3 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & -2 & 5 \\ 0 & 1 & 1 & -3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \Rightarrow \begin{cases} x_1 = 5 + 2x_3 \\ x_2 = -3 - x_3 \\ x_3 = -3 - x_3 \end{cases}$

初等行变换求解线性方程组

$$Ax = b \implies (A : b) \xrightarrow{instance{A}}$$
 简化的阶梯型矩阵



回忆: 阶梯形矩阵

形如:

的矩阵, 其中 $b_1, b_2, \ldots, b_r \neq 0$, 称为阶梯型矩阵。

简化的阶梯型矩阵

形如:

的矩阵, 其中 $b_1, b_2, \ldots, b_r \neq 0$, 称为简化的阶梯型矩阵。

例 1 解方程组:
$$\begin{cases} x_1 + x_2 - x_3 = 2 \\ x_1 + 2x_2 = -1 \\ 2x_1 + 5x_2 + x_3 = -5 \\ -2x_1 - 3x_2 + x_3 = -1 \end{cases}$$

$$(A : b) = \begin{pmatrix} 1 & 1 & -1 & 2 \\ 1 & 2 & 0 & -1 \\ 2 & 5 & 1 & -5 \\ -2 & -3 & 1 & -1 \end{pmatrix}$$

例 1 解方程组:
$$\begin{cases} x_1 + x_2 - x_3 = 2 \\ x_1 + 2x_2 = -1 \\ 2x_1 + 5x_2 + x_3 = -5 \\ -2x_1 - 3x_2 + x_3 = -1 \end{cases}$$

$$\mathbf{M} \quad (A:b) = \begin{pmatrix} 1 & 1 & -1 & | & 2 \\ 1 & 2 & 5 & 1 & | & -5 \\ -2 & -3 & 1 & | & -1 \end{pmatrix} \xrightarrow[r_4 + 2r_1]{r_2 - r_1} \xrightarrow[r_4 + 2r_1]{r_4 - 2r_1}$$

例 1 解方程组:
$$\begin{cases} x_1 + x_2 - x_3 = 2 \\ x_1 + 2x_2 = -1 \\ 2x_1 + 5x_2 + x_3 = -5 \\ -2x_1 - 3x_2 + x_3 = -1 \end{cases}$$

$$(A:b) = \begin{pmatrix} 1 & 1 & -1 & 2 \\ 1 & 2 & 0 & -1 \\ 2 & 5 & 1 & -5 \\ -2 & -3 & 1 & -1 \end{pmatrix} \xrightarrow[r_3 - 2r_1]{r_3 - 2r_1}{r_4 + 2r_1} \begin{pmatrix} 1 & 1 & -1 & 2 \\ 1 & 1 & -1 & 2 \\ 1 & 1 & 1 & -1 & 2 \end{pmatrix}$$

$$\begin{cases} 2x_1 + 3x_2 + x_3 = -1 \\ -2x_1 - 3x_2 + x_3 = -1 \end{cases}$$

$$\begin{cases} 1 & 1 - 1 \\ 2 & 0 - 1 \end{cases} \xrightarrow{r_2 - r_1}.$$

$$(A:b) = \begin{pmatrix} 1 & 1 & -1 & 2 \\ 1 & 2 & 0 & -1 \\ 2 & 5 & 1 & -5 \\ -2 & -3 & 1 & -1 \end{pmatrix} \xrightarrow[r_4+2r_1]{r_2-r_1} \begin{pmatrix} 1 & 1 & -1 & -2 \\ 0 & 1 & 1 & -3 \\ -2 & 3 & 1 & -3 \end{pmatrix}$$



$$\begin{cases} 2x_1 + 5x_2 + x_3 = -5 \\ -2x_1 - 3x_2 + x_3 = -5 \end{cases}$$

$$\begin{cases}
2x_1 + 5x_2 + x_3 = -5 \\
-2x_1 - 3x_2 + x_3 = -5
\end{cases}$$

$$\mathbf{H} \quad (A:b) = \begin{pmatrix} 1 & 1 & -1 & | & 2 \\ 1 & 2 & 0 & | & -1 \\ 2 & 5 & 1 & | & -5 \\ -2 & -3 & 1 & | & -1 \end{pmatrix} \xrightarrow[r_4+2r_1]{r_2-r_1} \begin{pmatrix} 1 & 1 & -1 & | & -2 \\ 0 & 1 & 1 & | & -3 \\ 0 & -1 & -1 & | & -3 \\ 0 & -1 & -1 & | & 3 \end{pmatrix}$$

例 1 解方程组:
$$\begin{cases} x_1 + x_2 - x_3 = 2 \\ x_1 + 2x_2 = -1 \\ 2x_1 + 5x_2 + x_3 = -5 \\ -2x_1 - 3x_2 + x_3 = -1 \end{cases}$$

$$\mathbf{H} \quad (A:b) = \begin{pmatrix} 1 & 1 & -1 & | & 2 \\ 1 & 2 & 0 & | & -1 \\ 2 & 5 & 1 & | & -5 \\ -2 & -3 & 1 & | & -1 \end{pmatrix} \xrightarrow[r_4+2r_1]{r_2-r_1} \begin{pmatrix} 1 & 1 & -1 & | & -2 \\ 0 & 1 & 1 & | & -3 \\ 0 & -1 & -1 & | & -3 \\ 0 & -1 & -1 & | & 3 \end{pmatrix}$$

$$r_3-3r_2$$



$$\mathbf{H} \quad (A:b) = \begin{pmatrix} 1 & 1 & -1 & 2 \\ 1 & 2 & 0 & -1 \\ 2 & 5 & 1 & -5 \\ -2 & -3 & 1 & -1 \end{pmatrix} \xrightarrow[r_4+2r_1]{r_2-r_1} \begin{pmatrix} 1 & 1 & -1 & 2 \\ 0 & 1 & 1 & -1 & -3 \\ 0 & -1 & -1 & 3 \end{pmatrix}$$

$$\underline{r_3-3r_2} \quad \begin{pmatrix} 1 & 1 & -1 & 2 \\ 0 & 1 & 1 & -3 \\ 0 & 1 & -1 & -3 \end{pmatrix}$$

$$\xrightarrow[r_4+r_2]{r_3-3r_2} \left(\begin{array}{ccc|c} 1 & 1 & -1 & 2 \\ 0 & 1 & 1 & -3 \end{array} \right)$$

$$(A : b) = \begin{pmatrix} -2x_1 - 3x_2 + x_3 = -1 \\ \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & \frac{2}{5} & 1 \\ -2 & -3 & 1 \end{vmatrix} \xrightarrow{r_3 - 2r_1} \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ 0 & \frac{1}{3} & \frac{1}{3} & -\frac{3}{2} \\ 0 & -1 & -1 \end{vmatrix} \xrightarrow{r_3}$$

$$\begin{array}{c|ccccc}
 & -2 & -3 & 1 & |-1 & 7 & 7 \\
\hline
 & \frac{r_3 - 3r_2}{r_4 + r_2} & \begin{pmatrix} 1 & 1 & -1 & | & -3 & | \\ 0 & 0 & & 0 & | & -3 & | \\ 0 & 0 & & 0 & | & 0 & | \\
\end{array}$$

$$(A:b) = \begin{pmatrix} -2x_1 - 3x_2 + x_3 = -1 \\ 1 & 2 & 0 \\ 2 & 5 & 1 \\ -2 & -3 & 1 \end{pmatrix} \xrightarrow[r_1+2r_1]{r_2-r_1} \begin{pmatrix} 1 & 1 & -1 \\ 0 & 1 & 1 \\ 0 & 3 & 3 \\ 0 & -1 & -1 \end{pmatrix} \xrightarrow[s_1+2r_2]{r_2-r_1}$$

$$\begin{pmatrix} -2 & -3 & 1 & | -1 & | & r_2 \\ -2 & -3 & 1 & | -1 & | & | & r_3 \\ \hline r_3 - 3r_2 & \begin{pmatrix} 1 & 1 & -1 & | & 2 \\ 0 & 0 & 0 & | & 0 \\ 0 & 0 & 0 & | & 0 \end{pmatrix}$$

$$(A:b) = \begin{pmatrix} -2x_1 - 3x_2 + x_3 = -1 \\ 1 & 1 & -1 \\ 2 & 5 & 1 \\ -2 & -3 & 1 \end{pmatrix} \xrightarrow[r_4+2r_1]{r_2-r_1} \begin{pmatrix} 1 & 1 & -1 \\ 0 & 1 & 1 \\ 0 & 3 & 3 \\ 0 & -1 & -1 \end{pmatrix} \xrightarrow[s_4+2r_1]{r_2-r_1}$$

$$\begin{pmatrix}
-2 & -3 & 1 & -3 \\
-2 & -3 & 1 & -1
\end{pmatrix}
\xrightarrow[r_4+2r_1]{r_3-2r_1}$$

$$\xrightarrow[r_4+r_2]{r_3-3r_2}
\begin{pmatrix}
1 & 1 & -1 & -3 \\
0 & 0 & 0 & 0
\end{pmatrix}
\xrightarrow[r_4+r_2]{r_1-r_2}$$

$$\begin{cases}
2x_1 + 5x_2 + x_3 = -5 \\
-2x_1 - 3x_2 + x_3 = -5
\end{cases}$$

$$\mathbf{H} \quad (A : b) = \begin{pmatrix} \begin{vmatrix} 1 & 1 & -1 \\ 1 & 2 & 0 \\ -2 & 5 & 1 \\ -2 & -3 & 1 \end{vmatrix} \begin{vmatrix} -1 \\ -5 \\ -1 \end{pmatrix} \xrightarrow[r_4+2r_1]{r_2-r_1} \begin{pmatrix} \begin{vmatrix} 1 & 1 & -1 \\ 0 & 1 & 1 \\ 0 & 3 & 3 \\ -9 & 3 \end{pmatrix}$$

$$\xrightarrow[r_4+r_2]{r_3-3r_2} \left(\begin{array}{ccc|c} 1 & 1 & -1 & 2 \\ 0 & 1 & 1 & -3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right) \xrightarrow[r_1-r_2]{r_1-r_2} \left(\begin{array}{ccc|c} 1 & 0 & -2 & 5 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right)$$

$$\begin{array}{c}
2x_1 + 5x_2 + x_3 = -5 \\
-2x_1 - 3x_2 + x_3 = -1
\end{array}$$

$$\mathbf{H} \quad (A:b) = \begin{pmatrix} 1 & 1 & -1 & | & 2 \\ 1 & 2 & 0 & | & -1 \\ 2 & 5 & 1 & | & -5 \\ -2 & -3 & 1 & | & -1 \end{pmatrix} \xrightarrow[r_4+2r_1]{r_2-r_1} \begin{pmatrix} 1 & 1 & -1 & | & -2 \\ 0 & 1 & 1 & | & -3 \\ 0 & -1 & -1 & | & -3 \end{pmatrix}$$

$$\xrightarrow[r_4+r_2]{r_3-3r_2} \left(\begin{array}{ccc|c} 1 & 1 & -1 & 2 \\ 0 & 1 & 1 & -3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right) \xrightarrow[r_1-r_2]{r_1-r_2} \left(\begin{array}{ccc|c} 1 & 0 & -2 & 5 \\ 0 & 1 & 1 & -3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right)$$

所以原方程组等价于

$$\begin{cases} 2x_1 + 5x_2 + x_3 = -5 \\ -2x_1 - 3x_2 + x_3 = -1 \end{cases}$$

$$\mathbf{H} \quad (A:b) = \begin{pmatrix} 1 & 1 & -1 & | & 2 \\ 1 & 2 & 0 & | & -1 \\ 2 & 5 & 1 & | & -5 \\ -2 & -3 & 1 & | & -1 \end{pmatrix} \xrightarrow[r_4+2r_1]{r_2-r_1} \begin{pmatrix} 1 & 1 & -1 & | & -2 \\ 0 & 1 & 1 & | & -3 \\ 0 & -1 & -1 & | & -3 \end{pmatrix}$$

$$\frac{r_3 - 3r_2}{r_4 + r_2} \left(\begin{array}{ccc|c} 1 & 1 & -1 & 2 \\ 0 & 1 & 1 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right) \xrightarrow{r_1 - r_2} \left(\begin{array}{ccc|c} 1 & 0 & -2 & 5 \\ 0 & 1 & 1 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right)$$

所以原方程组等价于

$$\begin{cases} x_1 + & -2x_3 = 5 \\ x_2 + & x_3 = -3 \end{cases}$$

$$\begin{cases} 2x_1 + 5x_2 + x_3 = -5 \\ -2x_1 - 3x_2 + x_3 = -1 \end{cases}$$

$$\mathbf{H} \quad (A:b) = \begin{pmatrix} 1 & 1 & -1 & | & 2 \\ 1 & 2 & 0 & | & -1 \\ 2 & 5 & 1 & | & -5 \\ -2 & -3 & 1 & | & -1 \end{pmatrix} \xrightarrow[r_4+2r_1]{r_2-r_1} \begin{pmatrix} 1 & 1 & -1 & | & 2 \\ 0 & 1 & 1 & | & -3 \\ 0 & -1 & -1 & | & -3 \\ 0 & 0 & 0 & | & -3 \end{pmatrix} \\
\xrightarrow[r_4+r_2]{r_4+r_2} \begin{pmatrix} 1 & 1 & -1 & | & 2 \\ 0 & 0 & 0 & | & -3 \\ 0 & 0 & 0 & | & -3 \\ 0 & 0 & 0 & | & 0 \end{pmatrix} \xrightarrow[r_1-r_2]{r_1-r_2} \begin{pmatrix} 1 & 0 & -2 & | & 5 \\ 0 & 0 & 0 & | & -3 \\ 0 & 0 & 0 & | & 0 \end{pmatrix}$$

所以原方程组等价于

$$\begin{cases} x_1 + & -2x_3 = 5 \\ x_2 + & x_3 = -3 \end{cases} \iff \begin{cases} x_1 + & = 5 + 2x_3 \\ x_2 = -3 - x_3 \end{cases}$$



例 1 解方程组: $\begin{cases} x_1 + x_2 - x_3 = 2 \\ x_1 + 2x_2 = -1 \\ 2x_1 + 5x_2 + x_3 = -5 \\ -2x_1 - 3x_2 + x_3 = -1 \end{cases}$ $\mathbf{(A:b)} = \begin{pmatrix}
1 & 1 & -1 & 2 \\
1 & 2 & 0 & -1 \\
2 & 5 & 1 & -5 \\
-2 & -3 & 1 & -1
\end{pmatrix} \xrightarrow[r_3+2r_1]{r_2-r_1} \begin{pmatrix}
1 & 1 & -1 & -3 \\
0 & 1 & 3 & 3 \\
0 & -1 & -1 & -3
\end{pmatrix}$ $\xrightarrow{r_3 - 3r_2} \left(\begin{array}{ccc|c} 1 & 1 & -1 & -2 \\ 0 & 1 & 1 & -3 \\ 0 & 0 & 0 & 0 \end{array} \right) \xrightarrow{r_1 - r_2} \left(\begin{array}{ccc|c} 1 & 0 & -2 & -5 \\ 0 & 1 & 1 & -3 \\ 0 & 0 & 0 & 0 \end{array} \right)$ $\begin{cases} x_{1} + -2x_{3} = 5 \\ x_{2} + x_{3} = -3 \end{cases} \Leftrightarrow \begin{cases} x_{1} + = 5 + 2x_{3} \\ x_{2} = -3 - x_{3} \end{cases}$ $(c_1$ 为任意常数)

例 1 解方程组: $\begin{cases} x_1 + x_2 - x_3 = 2 \\ x_1 + 2x_2 = -1 \\ 2x_1 + 5x_2 + x_3 = -5 \\ -2x_1 - 3x_2 + x_3 = -1 \end{cases}$

 $\mathbf{H} \quad (A:b) = \begin{pmatrix} 1 & 1 & -1 & 2 \\ 1 & 2 & 0 & -1 \\ 2 & 5 & 1 & -5 \\ -2 & -3 & 1 & -1 \end{pmatrix} \xrightarrow[r_{A}+2r_{1}]{r_{2}-r_{1}} \begin{pmatrix} 1 & 1 & -1 & -2 \\ 0 & 1 & 3 & 3 \\ 0 & -1 & -1 & -3 \end{pmatrix}$ $\xrightarrow{r_3 - 3r_2} \left(\begin{array}{ccc|c} 1 & 1 & -1 & -2 \\ 0 & 1 & 1 & -3 \\ 0 & 0 & 0 & 0 \end{array} \right) \xrightarrow{r_1 - r_2} \left(\begin{array}{ccc|c} 1 & 0 & -2 & -5 \\ 0 & 1 & 1 & -3 \\ 0 & 0 & 0 & 0 \end{array} \right)$ $\begin{cases} x_1 + & -2x_3 = 5 \\ & x_2 + & x_3 = -3 \end{cases} \iff \begin{cases} x_1 + & = 5 + 2x_3 \\ & x_2 = -3 - x_3 \end{cases}$ $\begin{cases} x_1 = 5 + 2c_1 \\ (c_1 为任意常数) \end{cases}$ $x_3 = c_1$

 $\mathbf{H} \quad (A:b) = \begin{pmatrix} 1 & 1 & -1 & 2 \\ 1 & 2 & 0 & -1 \\ 2 & 5 & 1 & -5 \\ -2 & -3 & 1 & -1 \end{pmatrix} \xrightarrow[r_1+2r_1]{r_2-r_1} \begin{pmatrix} 1 & 1 & -1 & -3 \\ 0 & 1 & 3 & 3 & -9 \\ 0 & -1 & -1 & -3 \end{pmatrix}$

例 1 解方程组: $\begin{cases} x_1 + x_2 - x_3 = 2 \\ x_1 + 2x_2 = -1 \\ 2x_1 + 5x_2 + x_3 = -5 \\ -2x_1 - 3x_2 + x_3 = -1 \end{cases}$

 $\xrightarrow{r_3 - 3r_2} \left(\begin{array}{ccc|c} 1 & 1 & -1 & -2 \\ 0 & 1 & 1 & -3 \\ 0 & 0 & 0 & 0 \end{array} \right) \xrightarrow{r_1 - r_2} \left(\begin{array}{ccc|c} 1 & 0 & -2 & -5 \\ 0 & 1 & 1 & -3 \\ 0 & 0 & 0 & 0 \end{array} \right)$ $\begin{cases} x_1 + & -2x_3 = 5 \\ & x_2 + & x_3 = -3 \end{cases} \iff \begin{cases} x_1 + & = 5 + 2x_3 \\ & x_2 = -3 - x_3 \end{cases}$

 $\begin{cases} x_1 = 5 + 2c_1 \\ x_2 = -3 - c_1 \end{cases}$ (c₁为任意常数)

 $\mathbf{H} \quad (A:b) = \begin{pmatrix} 1 & 1 & -1 & 2 \\ 1 & 2 & 0 & -1 \\ 2 & 5 & 1 & -5 \\ -2 & -3 & 1 & -1 \end{pmatrix} \xrightarrow[r_1+2r_1]{r_2-r_1} \begin{pmatrix} 1 & 1 & -1 & -3 \\ 0 & 1 & 3 & 3 & -9 \\ 0 & -1 & -1 & -3 \end{pmatrix}$

例 1 解方程组: $\begin{cases} x_1 + x_2 - x_3 = 2 \\ x_1 + 2x_2 = -1 \\ 2x_1 + 5x_2 + x_3 = -5 \\ -2x_1 - 3x_2 + x_3 = -1 \end{cases}$

 $\begin{cases} x_1 + & -2x_3 = 5 \\ & x_2 + & x_3 = -3 \end{cases} \iff \begin{cases} x_1 + & = 5 + 2x_3 \\ & x_2 = -3 - x_3 \end{cases}$

 $\begin{cases} x_1 = 5 + 2c_1 \\ x_2 = -3 - c_1 \end{cases}$ (c₁为任意常数)

例 2 解方程组:
$$\begin{cases} x_1 + 2x_2 + 4x_3 = 28 \\ -2x_1 - 3x_2 - 9x_3 = -53 \\ 3x_1 + 6x_2 + 13x_3 = 88 \\ 5x_1 + 9x_2 + 22x_3 = 141 \end{cases}$$

例 2 解方程组:
$$\begin{cases} x_1 + 2x_2 + 4x_3 = 28 \\ -2x_1 - 3x_2 - 9x_3 = -53 \\ 3x_1 + 6x_2 + 13x_3 = 88 \\ 5x_1 + 9x_2 + 22x_3 = 141 \end{cases}$$

$$\mathbf{H} (A:b) = \begin{pmatrix}
-\frac{1}{2} & -\frac{2}{3} & -\frac{4}{9} & -\frac{28}{53} \\
\frac{3}{5} & \frac{6}{9} & \frac{13}{22} & \frac{88}{141}
\end{pmatrix}$$

例 2 解方程组:
$$\begin{cases} x_1 + 2x_2 + 4x_3 = 28 \\ -2x_1 - 3x_2 - 9x_3 = -53 \\ 3x_1 + 6x_2 + 13x_3 = 88 \\ 5x_1 + 9x_2 + 22x_3 = 141 \end{cases}$$

$$\mathbf{A}:b) = \begin{pmatrix}
-\frac{1}{2} & -\frac{2}{3} & -\frac{4}{9} & -\frac{28}{53} \\
-\frac{1}{3} & -\frac{1}{6} & -\frac{1}{3} & \frac{1}{88} \\
-\frac{1}{3} & -\frac{1}{6} & -\frac{1}{3} & \frac{1}{6} \\
-\frac{1}{3} & -\frac{1}{3} & -\frac{1}{3} & -\frac{1}{3} \\
-\frac{1}{3} &$$

例 2 解方程组: $\begin{cases} x_1 + 2x_2 + 4x_3 = 28 \\ -2x_1 - 3x_2 - 9x_3 = -53 \\ 3x_1 + 6x_2 + 13x_3 = 88 \\ 5x_1 + 9x_2 + 22x_3 = 141 \end{cases}$

$$\begin{cases} 3x_1 + 6x_2 + 13x_3 = 8 \\ 5x_1 + 9x_2 + 22x_3 = 14 \end{cases}$$

$$\begin{cases} 1 & 2 & 4 & 28 \\ -2 & -3 & -9 & -53 \end{cases} \xrightarrow{r_2 + 2r_1} \begin{pmatrix} 1 & 2 & 4 & 28 \\ -2 & -3 & -9 & -53 \end{pmatrix}$$

例 2 解方程组:
$$\begin{cases} x_1 + 2x_2 + 4x_3 = 28 \\ -2x_1 - 3x_2 - 9x_3 = -53 \\ 3x_1 + 6x_2 + 13x_3 = 88 \\ 5x_1 + 9x_2 + 22x_3 = 141 \end{cases}$$

例 2 解方程组: $\begin{cases} x_1 + 2x_2 + 4x_3 = 28 \\ -2x_1 - 3x_2 - 9x_3 = -53 \\ 3x_1 + 6x_2 + 13x_3 = 88 \\ 5x_1 + 9x_2 + 22x_3 = 141 \end{cases}$

$$\begin{cases} 5x_1 + 6x_2 + 15x_3 = 6 \\ 5x_1 + 9x_2 + 22x_3 = 14 \end{cases}$$

$$\begin{cases} 1 & 2 & 4 & 28 \\ -2 & -3 & -9 & -53 \end{cases} x_2 + 2r_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

$$\mathbf{H} (A:b) = \begin{pmatrix} 1 & 2 & 4 & 28 \\ -2 & -3 & -9 & -53 \\ 3 & 6 & 13 & 88 \\ 5 & 9 & 22 & 141 \end{pmatrix} \xrightarrow[r_4-5r_1]{r_2+2r_1} \begin{pmatrix} 1 & 2 & 4 & 28 \\ 0 & 1 & -1 & 3 \\ 0 & 0 & 1 & 4 \end{pmatrix}$$

例 2 解方程组:
$$\begin{cases} x_1 + 2x_2 + 4x_3 = 28 \\ -2x_1 - 3x_2 - 9x_3 = -53 \\ 3x_1 + 6x_2 + 13x_3 = 88 \\ 5x_1 + 9x_2 + 22x_3 = 141 \end{cases}$$

$$\mathbf{H} (A:b) = \begin{pmatrix} 1 & 2 & 4 & 28 \\ -2 & -3 & -9 & -53 \\ 3 & 6 & 13 & 88 \\ 5 & 9 & 22 & 141 \end{pmatrix} \xrightarrow[r_4-5r_1]{r_2+2r_1} \begin{pmatrix} 1 & 2 & 4 & 28 \\ 0 & 1 & -1 & 3 \\ 0 & 0 & 1 & 4 \\ 0 & -1 & 2 & 1 \end{pmatrix}$$

例 2 解方程组:
$$\begin{cases} x_1 + 2x_2 + 4x_3 = 28 \\ -2x_1 - 3x_2 - 9x_3 = -53 \\ 3x_1 + 6x_2 + 13x_3 = 88 \\ 5x_1 + 9x_2 + 22x_3 = 141 \end{cases}$$

$$(A:b) = \begin{pmatrix} 5x_1 + 9x_2 + 22x_3 = 141 \\ -\frac{1}{2} & -\frac{2}{3} & -\frac{9}{9} \\ \frac{3}{5} & 6 & \frac{13}{13} \\ \frac{3}{5} & 9 & \frac{7}{22} & \frac{1}{13} \end{pmatrix} \xrightarrow{r_2 + 2r_1} \begin{pmatrix} 1 & 2 & 4 & 28 \\ 0 & 1 & -1 & 3 \\ 0 & 0 & 1 & 4 \\ 0 & -1 & 2 & 1 \end{pmatrix}$$

$$r_4+r_2$$

例 2 解方程组:
$$\begin{cases} x_1 + 2x_2 + 4x_3 = 28 \\ -2x_1 - 3x_2 - 9x_3 = -53 \\ 3x_1 + 6x_2 + 13x_3 = 88 \\ 5x_1 + 9x_2 + 22x_3 = 141 \end{cases}$$

$$(A:b) = \begin{pmatrix} 5x_1 + 9x_2 + 22x_3 = 141 \\ -\frac{1}{2} & -\frac{2}{3} & -\frac{4}{9} \\ \frac{7}{3} & 6 & \frac{13}{13} \\ \frac{7}{3} & 9 & \frac{7}{22} & \frac{1}{13} \\ \frac{7}{3} & -\frac{3}{2}r_1 \\ \frac{7}{3} & -\frac{3}$$

$$\xrightarrow{r_4+r_2} \left(\begin{array}{ccc|c} 1 & 2 & 4 & 28 \\ 0 & 1 & -1 & 3 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 1 & 4 \end{array}\right)$$



例 2 解方程组:
$$\begin{cases} x_1 + 2x_2 + 4x_3 = 28 \\ -2x_1 - 3x_2 - 9x_3 = -53 \\ 3x_1 + 6x_2 + 13x_3 = 88 \\ 5x_1 + 9x_2 + 22x_3 = 141 \end{cases}$$

 $\mathbf{H} (A:b) = \begin{pmatrix}
-\frac{1}{2} & -\frac{2}{3} & -\frac{4}{9} & -\frac{28}{53} \\
\frac{3}{5} & \frac{6}{9} & \frac{13}{22} & \frac{88}{141}
\end{pmatrix} \xrightarrow{r_2 + 2r_1} \begin{pmatrix}
1 & 2 & -\frac{4}{2} & 28 \\
0 & 1 & -\frac{1}{2} & \frac{28}{3} \\
0 & -1 & 2 & 1
\end{pmatrix}$

$$\xrightarrow{r_4+r_2} \left(\begin{array}{ccc|c} 1 & 2 & 4 & 28 \\ 0 & 1 & -1 & 3 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 1 & 4 \end{array}\right) \xrightarrow{r_4-r_3}$$

例 2 解方程组:
$$\begin{cases} x_1 + 2x_2 + 4x_3 = 28 \\ -2x_1 - 3x_2 - 9x_3 = -53 \\ 3x_1 + 6x_2 + 13x_3 = 88 \\ 5x_1 + 9x_2 + 22x_3 = 141 \end{cases}$$

$$\mathbf{H} (A:b) = \begin{pmatrix} -\frac{1}{2} & -\frac{2}{3} & -\frac{4}{9} & | & -\frac{28}{53} \\ \frac{3}{5} & \frac{6}{9} & \frac{13}{22} & | & \frac{1}{141} \end{pmatrix} \xrightarrow{r_2 + 2r_1} \begin{pmatrix} \frac{1}{0} & \frac{2}{1} & -\frac{4}{1} & | & 28\\ 0 & 0 & \frac{1}{1} & | & \frac{4}{1} \\ 0 & -1 & \frac{1}{2} & | & \frac{1}{1} \end{pmatrix}$$

$$\xrightarrow{r_4+r_2} \left(\begin{array}{ccc|c} 1 & 2 & 4 & 28 \\ 0 & 1 & -1 & 3 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 1 & 4 \end{array} \right) \xrightarrow{r_4-r_3} \left(\begin{array}{ccc|c} 1 & 2 & 4 & 28 \\ 0 & 1 & -1 & 3 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{array} \right)$$

例 2 解方程组:
$$\begin{cases} x_1 + 2x_2 + 4x_3 = 28 \\ -2x_1 - 3x_2 - 9x_3 = -53 \\ 3x_1 + 6x_2 + 13x_3 = 88 \\ 5x_1 + 9x_2 + 22x_3 = 141 \end{cases}$$

$$\mathbf{H} \quad (A : b) = \begin{pmatrix} -\frac{1}{2} & -\frac{2}{3} & -\frac{4}{9} & | & -\frac{28}{5} \\ \frac{3}{5} & \frac{6}{9} & \frac{13}{22} & | & \frac{1}{3} & \frac{7}{2} + \frac{2}{7_1} \\ \frac{3}{5} & \frac{1}{9} & \frac{2}{22} & | & \frac{1}{41} & \frac{2}{7_4 - 5r_1} & \frac{1}{9} & \frac{2}{9} & \frac{4}{3} \\ \frac{3}{6} & \frac{1}{1} & \frac{3}{4} & \frac{4}{1} & \frac{2}{9} & \frac{4}{3} & \frac{2}{9} \\ \frac{3}{6} & \frac{1}{1} & \frac{3}{4} & \frac{4}{1} & \frac{2}{9} & \frac{4}{3} & \frac{2}{9} \\ \frac{3}{6} & \frac{1}{1} & \frac{3}{4} & \frac{4}{1} & \frac{2}{9} & \frac{4}{1} & \frac{4}{1} & \frac{2}{9} & \frac{4}{1} & \frac{$$

$$\xrightarrow{r_4+r_2} \begin{pmatrix} 1 & 2 & 4 & 28 \\ 0 & 1 & -1 & 3 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 1 & 4 \end{pmatrix} \xrightarrow{r_4-r_3} \begin{pmatrix} 1 & 2 & 4 & 28 \\ 0 & 1 & -1 & 3 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$r_2 + r_3$$

 $r_1 - 4r_3$

$$\begin{array}{c}
(5x_1 + 9x_2 + 22x_3 = 141) \\
\text{(A:b)} = \begin{pmatrix} -\frac{1}{2} & -\frac{2}{3} & -\frac{4}{9} & | & -\frac{28}{53} \\ -\frac{2}{3} & -\frac{6}{6} & \frac{13}{13} & | & \frac{r_2 + 2r_1}{r_3 - 3r_1} \\ 5 & 9 & 22 & | & 141 \end{pmatrix} \xrightarrow{r_3 - 3r_1} \begin{pmatrix} 1 & 2 & -\frac{4}{3} & | & 28 \\ 0 & 1 & -\frac{1}{3} & | & \frac{3}{4} \\ 0 & -1 & 2 & | & 1 \end{pmatrix}$$

$$\xrightarrow{r_4+r_2} \begin{pmatrix} 1 & 2 & 4 & 28 \\ 0 & 1 & -1 & 3 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 1 & 4 \end{pmatrix} \xrightarrow{r_4-r_3} \begin{pmatrix} 1 & 2 & 4 & 28 \\ 0 & 1 & -1 & 3 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\frac{r_2 + r_3}{r_1 - 4r_3} \left(\begin{array}{c|c} 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{array} \right)$$

$$\mathbf{H} (A:b) = \begin{pmatrix}
-\frac{1}{2} & -\frac{2}{3} & -\frac{4}{9} & | & -\frac{28}{5} & | & -\frac{27}{9} & | & -\frac{28}{3} & | & -\frac{27}{1} & | & -\frac{27}{$$

$$\xrightarrow{r_4+r_2} \left(\begin{array}{ccc|c} 1 & 2 & 4 & 28 \\ 0 & 1 & -1 & 3 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 1 & 4 \end{array} \right) \xrightarrow{r_4-r_3} \left(\begin{array}{ccc|c} 1 & 2 & 4 & 28 \\ 0 & 1 & -1 & 3 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{array} \right)$$

$$\xrightarrow[r_1-4r_3]{r_2+r_3} \left(\begin{array}{ccc|c} 0 & 1 & 0 & 7 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{array} \right)$$

$$(A:b) = \begin{pmatrix} 5x_1 + 9x_2 + 22x_3 = 141 \\ -\frac{1}{2} & -\frac{3}{3} & -\frac{9}{6} & \frac{28}{131} \\ \frac{3}{5} & \frac{6}{9} & \frac{13}{22} & \frac{141}{141} \end{pmatrix} \xrightarrow{r_2 + 2r_1} \begin{pmatrix} 1 & 2 & 4 & 28 \\ 0 & 1 & -1 & \frac{1}{4} & \frac{3}{4} \\ 0 & -1 & 2 & 1 \end{pmatrix}$$

$$\xrightarrow{r_4+r_2} \begin{pmatrix} 1 & 2 & 4 & 28 \\ 0 & 1 & -1 & 3 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 1 & 4 \end{pmatrix} \xrightarrow{r_4-r_3} \begin{pmatrix} 1 & 2 & 4 & 28 \\ 0 & 1 & -1 & 3 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\xrightarrow[r_1-4r_3]{r_2+r_3} \left(\begin{array}{ccc|c} 1 & 2 & 0 & 12 \\ 0 & 1 & 0 & 7 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{array} \right)$$

$$\mathbf{H} (A:b) = \begin{pmatrix}
-\frac{1}{2} & -\frac{2}{3} & -\frac{4}{9} & | & -\frac{28}{53} & | & \frac{r_2 + 2r_1}{6} & | & \frac{1}{2} & | & \frac{4}{28} & | & \frac{28}{3} & | & \frac{r_2 + 2r_1}{6} & | & \frac{1}{2} & | & \frac{4}{28} & | & \frac{1}{2} & | & \frac{1}{2}$$

$$\xrightarrow[r_1-4r_3]{r_2+r_3} \left(\begin{array}{ccc|c} 1 & 2 & 0 & 12 \\ 0 & 1 & 0 & 7 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 1 & 0 \end{array} \right) \xrightarrow{r_1-2r_2}$$

$$\mathbf{H} (A:b) = \begin{pmatrix} 1 & 2 & 4 & 28 \\ -2 & -3 & -9 & -53 \\ 3 & 6 & 13 & 88 \\ 5 & 9 & 22 & 141 \end{pmatrix} \xrightarrow{r_2 + 2r_1} \begin{pmatrix} 1 & 2 & 4 & 28 \\ 0 & 1 & -1 & 3 \\ 0 & 0 & 1 & 4 \\ 0 & -1 & 2 & 1 \end{pmatrix}$$

$$\xrightarrow{r_4+r_2} \begin{pmatrix} 1 & 2 & 4 & 28 \\ 0 & 1 & -1 & 3 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 1 & 4 \end{pmatrix} \xrightarrow{r_4-r_3} \begin{pmatrix} 1 & 2 & 4 & 28 \\ 0 & 1 & -1 & 3 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\xrightarrow[r_1-4r_3]{\begin{array}{c}1 & 2 & 0 & 12\\0 & 1 & 0 & 7\\0 & 0 & 1 & 4\\0 & 0 & 0 & 0\end{array}} \xrightarrow[r_1-2r_2]{\begin{array}{c}1 & 0 & 0 & -2\\0 & 1 & 0 & 7\\0 & 0 & 1 & 4\\0 & 0 & 0 & 0\end{array}}$$

例 2 解方程组:
$$\begin{cases} x_1 + 2x_2 + 4x_3 = 28 \\ -2x_1 - 3x_2 - 9x_3 = -53 \\ 3x_1 + 6x_2 + 13x_3 = 88 \\ 5x_1 + 9x_2 + 22x_3 = 141 \end{cases}$$

 $\mathbf{H} (A:b) = \begin{pmatrix}
-\frac{1}{2} & -\frac{2}{3} & -\frac{4}{9} & | & -\frac{28}{53} \\
\frac{3}{5} & \frac{6}{9} & \frac{13}{22} & | & \frac{1}{141} & \frac{7}{12} & \frac{7}{$ $\xrightarrow{r_4+r_2} \left(\begin{array}{ccc|c} 1 & 2 & 4 & 28 \\ 0 & 1 & -1 & 3 \\ 0 & 0 & 1 & 4 \end{array} \right) \xrightarrow{r_4-r_3} \left(\begin{array}{ccc|c} 1 & 2 & 4 & 28 \\ 0 & 1 & -1 & 3 \\ 0 & 0 & 1 & 4 \end{array} \right)$

所以原方程组等价于 $\begin{cases} x_1 & = -2 \\ x_2 & = 7 \\ x_3 & = 4 \end{cases}$

例 2 解方程组:
$$\begin{cases} x_1 + 2x_2 + 4x_3 = 28 \\ -2x_1 - 3x_2 - 9x_3 = -53 \\ 3x_1 + 6x_2 + 13x_3 = 88 \\ 5x_1 + 9x_2 + 22x_3 = 141 \end{cases}$$

$$\begin{pmatrix} A:b \end{pmatrix} = \begin{pmatrix} -\frac{1}{2} & -\frac{4}{3} & -\frac{28}{5} \\ -\frac{2}{3} & -\frac{4}{5} & -\frac{28}{3} \\ 5 & 9 & 22 & 41 \end{pmatrix} \xrightarrow{r_2 + 2r_1} \begin{pmatrix} 1 & 2 & -\frac{4}{3} & 28 \\ 0 & 0 & 1 & 1 & 4 \\ 0 & -1 & 2 & 1 \end{pmatrix}$$

$$\frac{r_{2}+r_{3}}{r_{1}-4r_{3}} \begin{pmatrix} 1 & 2 & 0 & 12 \\ 0 & 1 & 0 & 7 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{pmatrix} \xrightarrow{r_{1}-2r_{2}} \begin{pmatrix} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & 7 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

所以原方程组等价于 $\begin{cases} x_1 & = -2 \\ x_2 & = 7 \\ x_3 & = 4 \end{cases}$

例 3 解方程组:
$$\begin{cases} 4x_1 + 2x_2 - 7x_3 = -3\\ 2x_1 + x_2 - 4x_3 = -1\\ 5x_1 + 3x_2 - 11x_3 = 2\\ x_1 + x_2 - 4x_3 = 2 \end{cases}$$

$$(A : b) = \begin{pmatrix} 4 & 2 & -7 & -3 \\ 2 & 1 & -4 & -1 \\ 5 & 3 & -11 & 2 \\ 1 & 1 & -4 & 2 \end{pmatrix}$$

$$(A : b) = \begin{pmatrix} 4 & 2 & -7 & | & -3 \\ 2 & 1 & -4 & | & -1 \\ 5 & 3 & -11 & | & 2 \\ 1 & 1 & -4 & | & 2 \end{pmatrix} \xrightarrow{r_1 \leftrightarrow r_4}$$

$$(A : b) = \begin{pmatrix} 4 & 2 & -7 & -3 \\ 2 & 1 & -4 & -1 \\ 5 & 3 & -11 & 2 \\ 1 & 1 & -4 & 2 \end{pmatrix} \xrightarrow{r_1 \leftrightarrow r_4} \begin{pmatrix} 1 & 1 & -4 & 2 \\ 2 & 1 & -4 & -1 \\ 5 & 3 & -11 & 2 \\ 4 & 2 & -7 & -3 \end{pmatrix}$$



$$(A : b) = \begin{pmatrix} 4 & 2 & -7 & | & -3 \\ 2 & 1 & -4 & | & -1 \\ 5 & 3 & -11 & | & 2 \\ 1 & 1 & -4 & | & 2 \end{pmatrix} \xrightarrow{r_1 \leftrightarrow r_4} \begin{pmatrix} 1 & 1 & -4 & | & 2 \\ 2 & 1 & -4 & | & -1 \\ 5 & 3 & -11 & | & 2 \\ 4 & 2 & -7 & | & -3 \end{pmatrix} \frac{r_2 - 2r_1}{r_3 - 5r_1}$$



$$(A:b) = \begin{pmatrix} 4 & 2 & -7 & -3 \\ 2 & 1 & -4 & -1 \\ 5 & 3 & -11 & 2 \\ 1 & 1 & -4 & 2 \end{pmatrix} \xrightarrow{r_1 \leftrightarrow r_4} \begin{pmatrix} 1 & 1 & -4 & -2 \\ 2 & 1 & -4 & -1 \\ 5 & 3 & -11 & 2 \\ 4 & 2 & -7 & -3 \end{pmatrix} \xrightarrow{r_2 - 2r_1} \begin{bmatrix} r_2 - 2r_1 & r_3 - 5r_1 & r_4 - 4r_1 & r_4 &$$



$$(A:b) = \begin{pmatrix} 4 & 2 & -7 & -3 \\ 2 & 1 & -4 & -1 \\ 5 & 3 & -11 & 2 \\ 1 & 1 & -4 & 2 \end{pmatrix} \xrightarrow{r_1 \leftrightarrow r_4} \begin{pmatrix} 1 & 1 & -4 & 2 \\ 2 & 1 & -4 & -1 \\ 5 & 3 & -11 & 2 \\ 4 & 2 & -7 & -3 \end{pmatrix} \xrightarrow{r_2 - 2r_1} \xrightarrow{r_3 - 5r_1} \begin{pmatrix} 1 & -1 & -4 & 2 \\ 5 & 3 & -11 & 2 \\ 4 & 2 & -7 & -3 \end{pmatrix} \xrightarrow{r_3 - 5r_1} \begin{pmatrix} 1 & -1 & -4 & 2 \\ -1 & 2 & 2 & -7 & -3 \end{pmatrix}$$



$$(A:b) = \begin{pmatrix} 4 & 2 & -7 & -3 \\ 2 & 1 & -4 & -1 \\ 5 & 3 & -11 & 2 \\ 1 & 1 & -4 & 2 \end{pmatrix} \xrightarrow{r_1 \leftrightarrow r_4} \begin{pmatrix} 1 & 1 & -4 & 2 \\ 2 & 1 & -4 & -1 \\ 5 & 3 & -11 & 2 \\ 4 & 2 & -7 & -3 \end{pmatrix} \xrightarrow{r_2 - 2r_1} \begin{bmatrix} r_2 - 2r_1 & r_3 - 5r_1 & r_4 - 4r_1 & r_4 & r_4$$



$$(A:b) = \begin{pmatrix} 4 & 2 & -7 & -3 \\ 2 & 1 & -4 & -1 \\ 5 & 3 & -11 & 2 \\ 1 & 1 & -4 & 2 \end{pmatrix} \xrightarrow{r_1 \leftrightarrow r_4} \begin{pmatrix} 1 & 1 & -4 & 2 \\ 2 & 1 & -4 & -1 \\ 5 & 3 & -11 & 2 \\ 4 & 2 & -7 & -3 \end{pmatrix} \xrightarrow{r_2 - 2r_1} \begin{bmatrix} r_2 - 2r_1 & r_3 - 5r_1 & r_4 - 4r_1 \\ r_4 - 4r_1 & r_4 - 4r_1 \end{bmatrix}$$

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$$(A:b) = \begin{pmatrix} 4 & 2 & -7 & -3 \\ 2 & 1 & -4 & -1 \\ 5 & 3 & -11 & 2 \\ 1 & 1 & -4 & 2 \end{pmatrix} \xrightarrow{r_1 \leftrightarrow r_4} \begin{pmatrix} 1 & 1 & -4 & 2 \\ 2 & 1 & -4 & -1 \\ 5 & 3 & -11 & 2 \\ 4 & 2 & -7 & -3 \end{pmatrix} \xrightarrow{r_2 - 2r_1} \begin{bmatrix} r_2 - 2r_1 & r_3 - 5r_1 & r_4 - 4r_1 \\ r_4 - 4r_1 & r_4 - 4r_1 \end{bmatrix}$$



$$(A:b) = \begin{pmatrix} 4 & 2 & -7 & -3 \\ 2 & 1 & -4 & -1 \\ 5 & 3 & -11 & 2 \\ 1 & 1 & -4 & 2 \end{pmatrix} \xrightarrow{r_1 \leftrightarrow r_4} \begin{pmatrix} 1 & 1 & -4 & 2 \\ 2 & 1 & -4 & -1 \\ 5 & 3 & -11 & 2 \\ 4 & 2 & -7 & -3 \end{pmatrix} \xrightarrow{r_2 - 2r_1} \begin{pmatrix} 1 & 1 & -4 & 2 \\ 5 & 3 & -11 & 2 \\ 4 & 2 & -7 & -3 \end{pmatrix} \xrightarrow{r_3 - 5r_1} \begin{pmatrix} 1 & -1 & -4 & 2 \\ 0 & -1 & 4 & -5 \\ 0 & -2 & 9 & -8 \end{pmatrix} \xrightarrow{r_3 - 2r_2} \begin{pmatrix} 1 & 1 & -4 & 2 \\ 0 & -1 & 4 & -5 \\ -8 & 1 & 1 & -4 & -5 \end{pmatrix}$$



$$(A:b) = \begin{pmatrix} 4 & 2 & -7 & | & -3 \\ 2 & 1 & -4 & | & -1 \\ 5 & 3 & -11 & | & 2 \\ 1 & 1 & -4 & | & 2 \end{pmatrix} \xrightarrow{r_1 \leftrightarrow r_4} \begin{pmatrix} 1 & 1 & -4 & | & 2 \\ 2 & 1 & -4 & | & -1 \\ 5 & 3 & -11 & | & 2 \\ 4 & 2 & -7 & | & -3 \end{pmatrix} \frac{r_2 - 2r_1}{r_3 - 5r_1}$$

$$\begin{pmatrix} 1 & 1-4 \\ 0-1 & 4 \\ 0-2 & 9 \\ 0-2 & 9 \\ -11 \end{pmatrix} \xrightarrow[r_4-2r_2]{r_3-2r_2} \begin{pmatrix} 1 & 1-4 \\ 0-1 & 4 \\ 0 & 0 & 1 \\ -\frac{5}{2} \end{pmatrix}$$



$$(A:b) = \begin{pmatrix} 4 & 2 & -7 & | & -3 \\ 2 & 1 & -4 & | & -1 \\ 5 & 3 & -11 & | & 2 \\ 1 & 1 & -4 & | & 2 \end{pmatrix} \xrightarrow{r_1 \leftrightarrow r_4} \begin{pmatrix} 1 & 1 & -4 & | & 2 \\ 2 & 1 & -4 & | & -1 \\ 5 & 3 & -11 & | & 2 \\ 4 & 2 & -7 & | & -3 \end{pmatrix} \xrightarrow{r_2 - 2r_1} \begin{pmatrix} 1 & 1 & -4 & | & 2 \\ 5 & 3 & -11 & | & 2 \\ 4 & 2 & -7 & | & -3 \end{pmatrix} \xrightarrow{r_3 - 5r_1} \begin{pmatrix} 1 & -1 & -4 & | & 2 \\ 0 & -1 & 4 & | & -5 \\ 0 & 0 & 1 & 2 \end{pmatrix}$$



解

$$(A:b) = \begin{pmatrix} 4 & 2 & -7 & -3 \\ 2 & 1 & -4 & -1 \\ 5 & 3 & -11 & 2 \\ 1 & 1 & -4 & 2 \end{pmatrix} \xrightarrow{r_1 \leftrightarrow r_4} \begin{pmatrix} 1 & 1 & -4 & 2 \\ 2 & 1 & -4 & -1 \\ 5 & 3 & -11 & 2 \\ 4 & 2 & -7 & -3 \end{pmatrix} \xrightarrow{r_2 - 2r_1} \begin{pmatrix} 1 & 1 & -4 & 2 \\ 5 & 3 & -11 & 2 \\ 4 & 2 & -7 & -3 \end{pmatrix} \xrightarrow{r_3 - 5r_1} \begin{pmatrix} 1 & -1 & 4 & 2 \\ 0 & -1 & 4 & -5 \\ 0 & -2 & 9 & -8 \end{pmatrix} \xrightarrow{r_3 - 2r_2} \begin{pmatrix} 1 & 1 & -4 & 2 \\ 0 & 1 & 4 & -5 \\ 0 & 0 & 1 & 2 \end{pmatrix} \xrightarrow{r_4 - r_3} \xrightarrow{r_4 - r_3}$$



$$(A : b) = \begin{pmatrix} 4 & 2 & -7 & | & -3 \\ 2 & 1 & -4 & | & -1 \\ 5 & 3 & -11 & | & 2 \\ 1 & 1 & -4 & | & 2 \end{pmatrix} \xrightarrow{r_1 \leftrightarrow r_4} \begin{pmatrix} 1 & 1 & -4 & | & 2 \\ 2 & 1 & -4 & | & -1 \\ 5 & 3 & -11 & | & 2 \\ 4 & 2 & -7 & | & -3 \end{pmatrix} \frac{r_2 - 2r_1}{r_4 - 4r_1}$$

$$\begin{pmatrix}
3 & 3 & -1 & 1 & 2 \\
1 & 1 & -4 & 2
\end{pmatrix}
\begin{pmatrix}
3 & 3 & -1 & 1 & 2 \\
4 & 2 & -7 & -3
\end{pmatrix}
\begin{pmatrix}
r_{3} - 5r_{1} \\
r_{4} - 4r_{1}
\end{pmatrix}$$

$$\begin{pmatrix}
1 & 1 - 4 & 2 \\
0 - 1 & 4 & -5 \\
0 & -2 & 9 & -8 \\
0 - 2 & 9 & -11
\end{pmatrix}
\begin{pmatrix}
1 & 1 - 4 & 2 \\
0 & 0 & 1 & 2 \\
0 & 0 & 1 & -1
\end{pmatrix}
\begin{pmatrix}
1 & 1 - 4 & 2 \\
0 & 0 & 1 & 2 \\
0 & 0 & 1 & -3
\end{pmatrix}$$



解

$$(A:b) = \begin{pmatrix} 4 & 2 & -7 & | & -3 \\ 2 & 1 & -4 & | & -1 \\ 5 & 3 & -11 & | & 2 \\ 1 & 1 & -4 & | & 2 \end{pmatrix} \xrightarrow{r_1 \leftrightarrow r_4} \begin{pmatrix} 1 & 1 & -4 & | & 2 \\ 2 & 1 & -4 & | & -1 \\ 5 & 3 & -11 & | & 2 \\ 4 & 2 & -7 & | & -3 \end{pmatrix} \frac{r_2 - 2r_1}{r_4 - 4r_1}$$

$$\begin{pmatrix} 1 & 1-4 & 2 \\ 0-1 & 4 & -5 \\ 0-2 & 9 & -8 \\ 0-2 & 9 & -11 \end{pmatrix} \xrightarrow{r_3-2r_2} \begin{pmatrix} 1 & 1-4 & 2 \\ 0-1 & 4 & -5 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 1 & -1 \end{pmatrix} \xrightarrow{r_4-r_3} \begin{pmatrix} 1 & 1-4 & 2 \\ 0 & 1 & 4 & -5 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & -3 \end{pmatrix}$$

所以原方程组等价于

$$(A:b) = \begin{pmatrix} 4 & 2 & -7 & | & -3 \\ 2 & 1 & -4 & | & -1 \\ 5 & 3 & -11 & | & 2 \\ 1 & 1 & -4 & | & 2 \end{pmatrix} \xrightarrow{r_1 \leftrightarrow r_4} \begin{pmatrix} 1 & 1 & -4 & | & 2 \\ 2 & 1 & -4 & | & -1 \\ 5 & 3 & -11 & | & 2 \\ 4 & 2 & -7 & | & -3 \end{pmatrix} \xrightarrow{r_2 - 2r_1} \xrightarrow{r_3 - 5r_1} \xrightarrow{r_4 - 4r_1}$$

$$\begin{pmatrix} 1 & 1 & -4 \\ 0 & -1 & 4 \\ 0 & -2 & 9 \\ 0 & -2 & 9 \\ -11 \end{pmatrix} \xrightarrow{r_3 - 2r_2} \begin{pmatrix} 1 & 1 & -4 \\ 0 & -1 & 4 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \\ -1 \end{pmatrix} \xrightarrow{r_4 - r_3} \begin{pmatrix} 1 & 1 & -4 \\ 0 & -1 & 4 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ -3 \end{pmatrix}$$
所以原方程组等价于
$$\begin{pmatrix} x_1 + x_2 - 4x_3 = 2 \end{pmatrix}$$

所以原方程组等价于
$$\begin{cases} x_1 + & x_2 - 4x_3 = 2 \\ & -x_2 + 4x_3 = -5 \\ & x_3 = 2 \\ & 0 = -3 \end{cases}$$



$$(A:b) = \begin{pmatrix} 4 & 2 & -7 & | & -3 \\ 2 & 1 & -4 & | & -1 \\ 5 & 3 & -11 & | & 2 \\ 1 & 1 & -4 & | & 2 \end{pmatrix} \xrightarrow{r_1 \leftrightarrow r_4} \begin{pmatrix} 1 & 1 & -4 & | & 2 \\ 2 & 1 & -4 & | & -1 \\ 5 & 3 & -11 & | & 2 \\ 4 & 2 & -7 & | & -3 \end{pmatrix} \xrightarrow{r_2 - 2r_1} \xrightarrow{r_3 - 5r_1} \xrightarrow{r_4 - 4r_1}$$

 $\begin{pmatrix} 1 & 1 - 4 & 2 \\ 0 - 1 & 4 & -5 \\ 0 - 2 & 9 & -8 \end{pmatrix} \xrightarrow{r_3 - 2r_2} \begin{pmatrix} 1 & 1 - 4 & 2 \\ 0 - 1 & 4 & -5 \\ 0 & 0 & 1 & 2 \end{pmatrix} \xrightarrow{r_4 - r_3} \begin{pmatrix} 1 & 1 - 4 & 2 \\ 0 & 1 & 4 & -5 \\ 0 & 0 & 1 & 2 \end{pmatrix}$

所以原方程组等价于
$$\begin{cases} x_1 + x_2 - 4x_3 = 2 \\ -x_2 + 4x_3 = -5 \\ x_3 = 2 \\ 0 = -3 \end{cases} \Rightarrow$$
 无解!



$$(A:b) = \begin{pmatrix} 4 & 2 & -7 & -3 \\ 2 & 1 & -4 & -1 \\ 5 & 3 & -11 & 2 \\ 1 & 1 & -4 & 2 \end{pmatrix} \xrightarrow{r_1 \leftrightarrow r_4} \begin{pmatrix} 1 & 1 & -4 & 2 \\ 2 & 1 & -4 & -1 \\ 5 & 3 & -11 & 2 \\ 4 & 2 & -7 & -3 \end{pmatrix} \xrightarrow{r_2 - 2r_1} \xrightarrow{r_3 - 5r_1} \xrightarrow{r_4 - 4r_1}$$

$$\begin{pmatrix}
1 & 1 & -4 & 2
\end{pmatrix}
\begin{pmatrix}
4 & 2 & -7 & -3
\end{pmatrix}
\begin{pmatrix}
r_{3}-5r_{1} \\
r_{4}-4r_{1}
\end{pmatrix}$$

$$\begin{pmatrix}
1 & 1 & -4 & 2 \\
0 & -1 & 4 & -5 \\
0 & -2 & 9 & -8
\end{pmatrix}
\begin{pmatrix}
r_{3}-2r_{2} \\
-8 \\
-8 \\
-8
\end{pmatrix}
\begin{pmatrix}
1 & 1 & -4 & 2 \\
0 & 0 & 1 & 2
\end{pmatrix}
\begin{pmatrix}
1 & 1 & -4 & 2 \\
0 & 0 & 1 & 2
\end{pmatrix}
\begin{pmatrix}
1 & 1 & -4 & 2 \\
0 & 0 & 1 & 2
\end{pmatrix}
\begin{pmatrix}
1 & 1 & -4 & 2 \\
0 & 0 & 1 & 2
\end{pmatrix}$$

所以原方程组等价于
$$\begin{cases} x_1 + x_2 - 4x_3 = 2 \end{cases}$$

所以原方程组等价于
$$\begin{cases} x_1 + x_2 - 4x_3 = 2 \\ -x_2 + 4x_3 = -5 \\ x_3 = 2 \\ 0 = -3 \end{cases} \Rightarrow$$
 无解!



$$\begin{cases} x_{1} + & x_{2} - & x_{3} = 2 \\ x_{1} + & 2x_{2} & = -1 \\ 2x_{1} + & 5x_{2} + & x_{3} = -5 \\ -2x_{1} - & 3x_{2} + & x_{3} = -1 \end{cases} \qquad \begin{cases} x_{1} + & 2x_{2} + & 4x_{3} = & 28 \\ -2x_{1} - & 3x_{2} - & 9x_{3} = & -53 \\ 3x_{1} + & 6x_{2} + & 13x_{3} = & 88 \\ 5x_{1} + & 9x_{2} + & 22x_{3} = & 141 \end{cases} \qquad \begin{cases} 4x_{1} + & 2x_{2} - & 7x_{3} = & -3 \\ 2x_{1} + & x_{2} - & 4x_{3} = & -1 \\ 5x_{1} + & 3x_{2} - & 11x_{3} = & 2 \\ x_{1} + & x_{2} - & 4x_{3} = & 2 \end{cases}$$

$$\begin{cases} x_{1}+ & x_{2}- & x_{3} = 2 \\ x_{1}+ & 2x_{2} & = -1 \\ 2x_{1}+ & 5x_{2}+ & x_{3} = -5 \\ -2x_{1}- & 3x_{2}+ & x_{3} = -1 \end{cases} \qquad \begin{cases} x_{1}+ & 2x_{2}+ & 4x_{3} = 28 \\ -2x_{1}- & 3x_{2}- & 9x_{3} = -53 \\ 3x_{1}+ & 6x_{2}+ & 13x_{3} = 88 \\ 5x_{1}+ & 9x_{2}+ & 22x_{3} = 141 \end{cases} \qquad \begin{cases} 4x_{1}+ & 2x_{2}- & 7x_{3}= -3 \\ 2x_{1}+ & x_{2}- & 4x_{3}= -1 \\ 5x_{1}+ & 3x_{2}- & 11x_{3}= 2 \\ x_{1}+ & x_{2}- & 4x_{3}= 2 \end{cases}$$

$$\downarrow \qquad \qquad \downarrow \qquad$$











$$\Leftrightarrow Ax = b$$



总结
$$\texttt{定理 方程组} \begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m \end{cases}$$

$$\Leftrightarrow Ax = b$$

1. 有解

2. 无解



总结
$$\texttt{定理 方程组} \begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m \end{cases}$$

$$\Leftrightarrow Ax = b$$

- 1. 有解
 - 有无穷多解
 - 只有唯一解
- 2. 无解

总结
定理 方程组
$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n = b_2 \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n = b_m \end{cases}$$

$$\Leftrightarrow Ax = b$$

- 1. 有解⇔r(A:b) = r(A)
 - 有无穷多解
 - 只有唯一解
- 2. 无解

 $\Leftrightarrow Ax = b$

- 1. 有解⇔r(A:b) = r(A)
 - 有无穷多解
 - 只有唯一解
- 2. 无解 \Leftrightarrow $r(A) \neq r(A : b)$

总结
定理 方程组
$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n = b_2 \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n = b_m \end{cases}$$

$$\Leftrightarrow Ax = b$$

- 1. 有解⇔r(A:b) = r(A)
 - 有无穷多解
 - 只有唯一解
- 2. 无解 \Leftrightarrow $r(A) \neq r(A : b) \Leftrightarrow r(A) < r(A : b)$

$$\Leftrightarrow Ax = b$$

- 1. 有解⇔r(A:b) = r(A)
 - 有无穷多解 $\Leftrightarrow r(A) = r(A : b) < n$
 - 只有唯一解
- 2. 无解 \Leftrightarrow $r(A) \neq r(A : b) \Leftrightarrow r(A) < r(A : b)$

$$\Leftrightarrow Ax = b$$

- 1. 有解⇔r(A:b) = r(A)
 - 有无穷多解 $\Leftrightarrow r(A) = r(A : b) < n$
 - 只有唯一解 \Leftrightarrow r(A) = r(A : b) = n
- 2. 无解 \Leftrightarrow $r(A) \neq r(A : b) \Leftrightarrow r(A) < r(A : b)$

总结
定理 方程组
$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n = b_2 \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n = b_m \end{cases}$$

$$\Leftrightarrow Ax = b$$

- 1. 有解⇔r(A:b) = r(A)
 - 有无穷多解 $\Leftrightarrow r(A) = r(A : b) < n$
 - 只有唯一解 \Leftrightarrow r(A) = r(A : b) = n
- 2. 无解 \Leftrightarrow $r(A) \neq r(A : b) \Leftrightarrow r(A) < r(A : b)$

总结 定理 方程组 $\begin{cases} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n = b_2 \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n = b_m \end{cases}$

$$\Leftrightarrow Ax = b$$

的解有如下情形:

- 1. 有解⇔r(A:b) = r(A)
 - 有无穷多解 $\Leftrightarrow r(A) = r(A : b) < n$
 - 只有唯一解 $\Leftrightarrow r(A) = r(A : b) = n$
- 2. 无解 \Leftrightarrow $r(A) \neq r(A : b) \Leftrightarrow r(A) < r(A : b)$

注

• r(A : b) = r(A) 的值,相当于方程组中"独立"方程个数;此时



总结
定理 方程组
$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n = b_2 \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n = b_m \end{cases}$$

$$\Leftrightarrow Ax = b$$

- 1. 有解⇔r(A:b) = r(A)
 - 有无穷多解 $\Leftrightarrow r(A) = r(A : b) < n$
 - 只有唯一解 \Leftrightarrow r(A) = r(A : b) = n
- 2. 无解 \Leftrightarrow $r(A) \neq r(A : b) \Leftrightarrow r(A) < r(A : b)$

注

- r(A:b) = r(A) 的值,相当于方程组中"独立"方程个数;此时
- n − r(A) 为自由变量的个数



练习 求解 $\begin{cases} x_1 + & 2x_2 + & x_3 + & x_4 + & x_5 = 1 \\ 2x_1 + & 4x_2 + & 3x_3 + & x_4 + & x_5 = 3 \\ -x_1 - & 2x_2 + & x_3 + & 3x_4 - & 3x_5 = 7 \\ & & 2x_3 + & 5x_4 - & 2x_5 = 9 \end{cases}$

练习 求解
$$\begin{cases} x_1 + & 2x_2 + & x_3 + & x_4 + & x_5 = 1 \\ 2x_1 + & 4x_2 + & 3x_3 + & x_4 + & x_5 = 3 \\ -x_1 - & 2x_2 + & x_3 + & 3x_4 - & 3x_5 = 7 \\ & & 2x_3 + & 5x_4 - & 2x_5 = 9 \end{cases}$$

$$(A:b) = \begin{pmatrix} 1 & 2 & 1 & 1 & 1 & 1 \\ 2 & 4 & 3 & 1 & 1 & 3 \\ -1 & -2 & 1 & 3 & -3 & 7 \\ 0 & 0 & 2 & 5 & -2 & 9 \end{pmatrix}$$

练习 求解
$$\begin{cases} x_1 + & 2x_2 + & x_3 + & x_4 + & x_5 = 1 \\ 2x_1 + & 4x_2 + & 3x_3 + & x_4 + & x_5 = 3 \\ -x_1 - & 2x_2 + & x_3 + & 3x_4 - & 3x_5 = 7 \\ & & 2x_3 + & 5x_4 - & 2x_5 = 9 \end{cases}$$

$$(A:b) = \begin{pmatrix} 1 & 2 & 1 & 1 & 1 & 1 \\ 2 & 4 & 3 & 1 & 1 & 3 \\ -1 & -2 & 1 & 3 & -3 & 7 \\ 0 & 0 & 2 & 5 & -2 & 9 \end{pmatrix} \xrightarrow{r_2 - 2r_1} r_3 + r_1$$

练习 求解
$$\begin{cases} x_1 + & 2x_2 + & x_3 + & x_4 + & x_5 = 1 \\ 2x_1 + & 4x_2 + & 3x_3 + & x_4 + & x_5 = 3 \\ -x_1 - & 2x_2 + & x_3 + & 3x_4 - & 3x_5 = 7 \\ & & 2x_3 + & 5x_4 - & 2x_5 = 9 \end{cases}$$

$$(A : b) = \begin{pmatrix} 1 & 2 & 1 & 1 & 1 & 1 \\ 2 & 4 & 3 & 1 & 1 & 3 \\ -1 & -2 & 1 & 3 & -3 & 7 \\ 0 & 0 & 2 & 5 & -2 & 9 \end{pmatrix} \xrightarrow{r_2 - 2r_1} \begin{pmatrix} 1 & 2 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & -1 & -1 & 1 \\ 0 & 0 & 2 & 4 & -2 & 8 \\ 0 & 0 & 2 & 5 & -2 & 9 \end{pmatrix}$$



练习 求解
$$\begin{cases} x_1 + 2x_2 + x_3 + x_4 + x_5 = 1 \\ 2x_1 + 4x_2 + 3x_3 + x_4 + x_5 = 3 \\ -x_1 - 2x_2 + x_3 + 3x_4 - 3x_5 = 7 \\ 2x_3 + 5x_4 - 2x_5 = 9 \end{cases}$$

$$(A \vdots b) = \begin{pmatrix} 1 & 2 & 1 & 1 & 1 & 1 \\ 2 & 4 & 3 & 1 & 1 & 3 \\ -1 & -2 & 1 & 3 & -3 & 7 \\ 0 & 0 & 2 & 5 & -2 & 9 \end{pmatrix} \xrightarrow{r_2 - 2r_1} \begin{pmatrix} 1 & 2 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & -1 & -1 & 1 \\ 0 & 0 & 2 & 4 & -2 & 8 \\ 0 & 0 & 2 & 5 & -2 & 9 \end{pmatrix}$$

$$\frac{r_3 - 2r_2}{r_4 - 2r_3}$$



练习 求解
$$\begin{cases} x_1 + & 2x_2 + & x_3 + & x_4 + & x_5 = 1 \\ 2x_1 + & 4x_2 + & 3x_3 + & x_4 + & x_5 = 3 \\ -x_1 - & 2x_2 + & x_3 + & 3x_4 - & 3x_5 = 7 \\ & & 2x_3 + & 5x_4 - & 2x_5 = 9 \end{cases}$$

$$(A : b) = \begin{pmatrix} 1 & 2 & 1 & 1 & 1 & 1 \\ 2 & 4 & 3 & 1 & 1 & 3 \\ -1 & -2 & 1 & 3 & -3 & 7 \\ 0 & 0 & 2 & 5 & -2 & 9 \end{pmatrix} \xrightarrow{r_2 - 2r_1} \begin{pmatrix} 1 & 2 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & -1 & -1 & 1 \\ 0 & 0 & 2 & 4 & -2 & 8 \\ 0 & 0 & 2 & 5 & -2 & 9 \end{pmatrix}$$

$$\frac{r_3 - 2r_2}{r_4 - 2r_2} \left(\begin{array}{ccccc}
1 & 2 & 1 & 1 & 1 & 1 \\
0 & 0 & 1 & -1 & -1 & 1 \\
0 & 0 & 0 & 6 & 0 & 6 \\
0 & 0 & 0 & 7 & 0 & 7
\end{array} \right)$$



练习 求解
$$\begin{cases} x_1 + & 2x_2 + & x_3 + & x_4 + & x_5 = 1 \\ 2x_1 + & 4x_2 + & 3x_3 + & x_4 + & x_5 = 3 \\ -x_1 - & 2x_2 + & x_3 + & 3x_4 - & 3x_5 = 7 \\ & & 2x_3 + & 5x_4 - & 2x_5 = 9 \end{cases}$$

$$(A:b) = \begin{pmatrix} 1 & 2 & 1 & 1 & 1 & 1 \\ 2 & 4 & 3 & 1 & 1 & 3 \\ -1 & -2 & 1 & 3 & -3 & 7 \\ 0 & 0 & 2 & 5 & -2 & 9 \end{pmatrix} \xrightarrow{r_2 - 2r_1} \begin{pmatrix} 1 & 2 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & -1 & -1 & 1 \\ 0 & 0 & 2 & 4 & -2 & 8 \\ 0 & 0 & 2 & 5 & -2 & 9 \end{pmatrix}$$

$$\xrightarrow[r_4-2r_2]{ \begin{array}{c} 1 & 2 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & -1 & -1 & 1 \\ 0 & 0 & 0 & 6 & 0 & 6 \\ 0 & 0 & 0 & 7 & 0 & 7 \end{array}} \xrightarrow[\bar{\tau} \times r_4]{ \begin{array}{c} \frac{1}{6} \times r_3 \\ \frac{1}{7} \times r_4 \end{array}}$$



练习 求解
$$\begin{cases} x_1 + & 2x_2 + & x_3 + & x_4 + & x_5 = 1 \\ 2x_1 + & 4x_2 + & 3x_3 + & x_4 + & x_5 = 3 \\ -x_1 - & 2x_2 + & x_3 + & 3x_4 - & 3x_5 = 7 \\ & & & 2x_3 + & 5x_4 - & 2x_5 = 9 \end{cases}$$

$$(A:b) = \begin{pmatrix} 1 & 2 & 1 & 1 & 1 & 1 \\ 2 & 4 & 3 & 1 & 1 & 3 \\ -1 & -2 & 1 & 3 & -3 & 7 \\ 0 & 0 & 2 & 5 & -2 & 9 \end{pmatrix} \xrightarrow{r_2 - 2r_1} \begin{pmatrix} 1 & 2 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & -1 & -1 & 1 \\ 0 & 0 & 2 & 4 & -2 & 8 \\ 0 & 0 & 2 & 5 & -2 & 9 \end{pmatrix}$$

$$\frac{r_{3-2r_{2}}}{r_{4-2r_{2}}} \left(\begin{array}{ccccccc}
1 & 2 & 1 & 1 & 1 & 1 \\
0 & 0 & 1 & -1 & -1 & 1 \\
0 & 0 & 0 & 6 & 0 & 6 \\
0 & 0 & 0 & 7 & 0 & 7
\end{array} \right) \xrightarrow{\frac{1}{6} \times r_{3}} \left(\begin{array}{cccccccc}
1 & 2 & 1 & 1 & 1 & 1 \\
0 & 0 & 1 & -1 & -1 & 1 \\
0 & 0 & 0 & 1 & 0 & 1 \\
0 & 0 & 0 & 1 & 0 & 1
\end{array} \right)$$



练习 求解
$$\begin{cases} x_1 + & 2x_2 + & x_3 + & x_4 + & x_5 = 1 \\ 2x_1 + & 4x_2 + & 3x_3 + & x_4 + & x_5 = 3 \\ -x_1 - & 2x_2 + & x_3 + & 3x_4 - & 3x_5 = 7 \\ & & & 2x_3 + & 5x_4 - & 2x_5 = 9 \end{cases}$$

$$(A:b) = \begin{pmatrix} 1 & 2 & 1 & 1 & 1 & 1 \\ 2 & 4 & 3 & 1 & 1 & 3 \\ -1 & -2 & 1 & 3 & -3 & 7 \\ 0 & 0 & 2 & 5 & -2 & 9 \end{pmatrix} \xrightarrow{r_2 - 2r_1} \begin{pmatrix} 1 & 2 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & -1 & -1 & 1 \\ 0 & 0 & 2 & 4 & -2 & 8 \\ 0 & 0 & 2 & 5 & -2 & 9 \end{pmatrix}$$

$$\frac{r_{3}-2r_{2}}{r_{4}-2r_{2}} \left(\begin{array}{cccc|c} 1 & 2 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & -1 & -1 & 1 \\ 0 & 0 & 0 & 6 & 0 & 6 \\ 0 & 0 & 0 & 7 & 0 & 7 \end{array} \right) \frac{\frac{1}{6} \times r_{3}}{\frac{1}{7} \times r_{4}} \left(\begin{array}{cccc|c} 1 & 2 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & -1 & -1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 \end{array} \right)$$

$$r_4 - r_3$$



练习 求解
$$\begin{cases} x_1 + & 2x_2 + & x_3 + & x_4 + & x_5 = 1 \\ 2x_1 + & 4x_2 + & 3x_3 + & x_4 + & x_5 = 3 \\ -x_1 - & 2x_2 + & x_3 + & 3x_4 - & 3x_5 = 7 \\ & & 2x_3 + & 5x_4 - & 2x_5 = 9 \end{cases}$$

$$(A:b) = \begin{pmatrix} 1 & 2 & 1 & 1 & 1 & 1 \\ 2 & 4 & 3 & 1 & 1 & 3 \\ -1 & -2 & 1 & 3 & -3 & 7 \\ 0 & 0 & 2 & 5 & -2 & 9 \end{pmatrix} \xrightarrow{r_2 - 2r_1} \begin{pmatrix} 1 & 2 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & -1 & -1 & 1 \\ 0 & 0 & 2 & 4 & -2 & 8 \\ 0 & 0 & 2 & 5 & -2 & 9 \end{pmatrix}$$

$$\frac{r_{3}-2r_{2}}{r_{4}-2r_{2}} \left(\begin{array}{ccccc} 1 & 2 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & -1 & -1 & 1 \\ 0 & 0 & 0 & 6 & 0 & 6 \\ 0 & 0 & 0 & 7 & 0 & 7 \end{array} \right) \frac{1}{6} \times r_{3} \xrightarrow{\frac{1}{6} \times r_{3}} \left(\begin{array}{cccccc} 1 & 2 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & -1 & -1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 \end{array} \right)$$

$$\xrightarrow{r_{4}-r_{3}} \left(\begin{array}{cccccc} 1 & 2 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & -1 & -1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array} \right)$$

练习 求解
$$\begin{cases} x_1 + & 2x_2 + & x_3 + & x_4 + & x_5 = 1 \\ 2x_1 + & 4x_2 + & 3x_3 + & x_4 + & x_5 = 3 \\ -x_1 - & 2x_2 + & x_3 + & 3x_4 - & 3x_5 = 7 \\ & & 2x_3 + & 5x_4 - & 2x_5 = 9 \end{cases}$$

$$(A:b) = \begin{pmatrix} 1 & 2 & 1 & 1 & 1 & 1 \\ 2 & 4 & 3 & 1 & 1 & 3 \\ -1 & -2 & 1 & 3 & -3 & 7 \\ 0 & 0 & 2 & 5 & -2 & 9 \end{pmatrix} \xrightarrow{r_2 - 2r_1} \begin{pmatrix} 1 & 2 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & -1 & -1 & 1 \\ 0 & 0 & 2 & 4 & -2 & 8 \\ 0 & 0 & 2 & 5 & -2 & 9 \end{pmatrix}$$

练习 求解
$$\begin{cases} x_1 + & 2x_2 + & x_3 + & x_4 + & x_5 = 1 \\ 2x_1 + & 4x_2 + & 3x_3 + & x_4 + & x_5 = 3 \\ -x_1 - & 2x_2 + & x_3 + & 3x_4 - & 3x_5 = 7 \\ & & 2x_3 + & 5x_4 - & 2x_5 = 9 \end{cases}$$

$$(A:b) = \begin{pmatrix} 1 & 2 & 1 & 1 & 1 & 1 \\ 2 & 4 & 3 & 1 & 1 & 3 \\ -1 & -2 & 1 & 3 & -3 & 7 \\ 0 & 0 & 2 & 5 & -2 & 9 \end{pmatrix} \xrightarrow{r_2 - 2r_1} \begin{pmatrix} 1 & 2 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & -1 & -1 & 1 \\ 0 & 0 & 2 & 4 & -2 & 8 \\ 0 & 0 & 2 & 5 & -2 & 9 \end{pmatrix}$$

$$\frac{r_{3}-2r_{2}}{r_{4}-2r_{2}} \left(\begin{array}{cccccc} 1 & 2 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & -1 & -1 & 1 \\ 0 & 0 & 0 & 6 & 0 & 6 \\ 0 & 0 & 0 & 7 & 0 & 7 \end{array} \right) \frac{1}{6} \times r_{3} \xrightarrow{\frac{1}{7} \times r_{4}} \left(\begin{array}{ccccccccc} 1 & 2 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & -1 & -1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 \end{array} \right) \xrightarrow{r_{2}+r_{3}} \left(\begin{array}{cccccccccc} 1 & 2 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array} \right)$$

练习 求解
$$\begin{cases} x_1 + & 2x_2 + & x_3 + & x_4 + & x_5 = 1 \\ 2x_1 + & 4x_2 + & 3x_3 + & x_4 + & x_5 = 3 \\ -x_1 - & 2x_2 + & x_3 + & 3x_4 - & 3x_5 = 7 \\ & & 2x_3 + & 5x_4 - & 2x_5 = 9 \end{cases}$$

$$(A:b) = \begin{pmatrix} 1 & 2 & 1 & 1 & 1 & 1 \\ 2 & 4 & 3 & 1 & 1 & 3 \\ -1 & -2 & 1 & 3 & -3 & 7 \\ 0 & 0 & 2 & 5 & -2 & 9 \end{pmatrix} \xrightarrow{r_2 - 2r_1} \begin{pmatrix} 1 & 2 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & -1 & -1 & 1 \\ 0 & 0 & 2 & 4 & -2 & 8 \\ 0 & 0 & 2 & 5 & -2 & 9 \end{pmatrix}$$

$$r_1-r_2$$



练习 求解
$$\begin{cases} x_1 + 2x_2 + x_3 + x_4 + x_5 = 1\\ 2x_1 + 4x_2 + 3x_3 + x_4 + x_5 = 3\\ -x_1 - 2x_2 + x_3 + 3x_4 - 3x_5 = 7\\ 2x_3 + 5x_4 - 2x_5 = 9 \end{cases}$$
解

$$(A:b) = \begin{pmatrix} 1 & 2 & 1 & 1 & 1 & 1 \\ 2 & 4 & 3 & 1 & 1 & 3 \\ -1 & -2 & 1 & 3 & -3 & 7 \\ 0 & 0 & 2 & 5 & -2 & 9 \end{pmatrix} \xrightarrow{r_2 - 2r_1} \begin{pmatrix} 1 & 2 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & -1 & -1 & 1 \\ 0 & 0 & 2 & 4 & -2 & 8 \\ 0 & 0 & 2 & 5 & -2 & 9 \end{pmatrix}$$

$$\xrightarrow{r_3 - 2r_2} \begin{pmatrix} 1 & 2 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & -1 & -1 & 1 \\ 0 & 0 & 0 & 6 & 0 \\ 0 & 0 & 0 & 7 & 0 & 7 \end{pmatrix} \xrightarrow{\frac{1}{7} \times r_4} \begin{pmatrix} 1 & 2 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & -1 & -1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 \end{pmatrix}$$

$$\xrightarrow{r_4 - r_3} \begin{pmatrix} 1 & 2 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & -1 & -1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \xrightarrow{r_2 + r_3} \begin{pmatrix} 1 & 2 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\xrightarrow{r_1-r_2} \left(\begin{array}{cccc|ccc} 1 & 2 & 0 & 0 & 2 & -2 \\ 0 & 0 & 1 & 0 & -1 & 2 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right)$$



练习 求解
$$\begin{cases} x_1 + 2x_2 + x_3 + x_4 + x_5 = 1\\ 2x_1 + 4x_2 + 3x_3 + x_4 + x_5 = 3\\ -x_1 - 2x_2 + x_3 + 3x_4 - 3x_5 = 7\\ 2x_3 + 5x_4 - 2x_5 = 9 \end{cases}$$
解

$$(A:b) = \begin{pmatrix} 1 & 2 & 1 & 1 & 1 & 1 \\ 2 & 4 & 3 & 1 & 1 & 3 \\ -1 & -2 & 1 & 3 & -3 & 7 \\ 0 & 0 & 2 & 5 & -2 & 9 \end{pmatrix} \xrightarrow{r_2 - 2r_1} \begin{pmatrix} 1 & 2 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & -1 & -1 & 1 \\ 0 & 0 & 2 & 4 & -2 & 8 \\ 0 & 0 & 2 & 5 & -2 & 9 \end{pmatrix}$$

$$\xrightarrow{r_3 - 2r_2} \begin{pmatrix} 1 & 2 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & -1 & -1 & 1 \\ 0 & 0 & 0 & 6 & 0 \\ 0 & 0 & 0 & 7 & 0 & 7 \end{pmatrix} \xrightarrow{\frac{1}{7} \times r_4} \begin{pmatrix} 1 & 2 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & -1 & -1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 \end{pmatrix}$$

$$\xrightarrow{r_4 - r_3} \begin{pmatrix} 1 & 2 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & -1 & -1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \xrightarrow{r_2 + r_3} \begin{pmatrix} 1 & 2 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$



练习 求解
$$\begin{cases} x_1 + 2x_2 + x_3 + x_4 + x_5 = 1 \\ 2x_1 + 4x_2 + 3x_3 + x_4 + x_5 = 3 \\ -x_1 - 2x_2 + x_3 + 3x_4 - 3x_5 = 7 \\ 2x_3 + 5x_4 - 2x_5 = 9 \end{cases}$$

$$(A:b) \longrightarrow \begin{pmatrix} 1 & 2 & 0 & 0 & 2 & -2 \\ 0 & 0 & 1 & 0 & -1 & 2 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

练习 求解
$$\begin{cases} x_1 + 2x_2 + x_3 + x_4 + x_5 = 1 \\ 2x_1 + 4x_2 + 3x_3 + x_4 + x_5 = 3 \\ -x_1 - 2x_2 + x_3 + 3x_4 - 3x_5 = 7 \\ 2x_3 + 5x_4 - 2x_5 = 9 \end{cases}$$

$$(A \mid b) \longrightarrow \begin{pmatrix} 1 & 2 & 0 & 0 & 2 & -2 \\ 0 & 0 & 1 & 0 & -1 & 2 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

• 可见 $r(A) = r(A \cdot b) = 3 < 5$,有无穷多的解,

练习 求解
$$\begin{cases} x_1 + & 2x_2 + & x_3 + & x_4 + & x_5 = 1 \\ 2x_1 + & 4x_2 + & 3x_3 + & x_4 + & x_5 = 3 \\ -x_1 - & 2x_2 + & x_3 + & 3x_4 - & 3x_5 = 7 \\ & & 2x_3 + & 5x_4 - & 2x_5 = 9 \end{cases}$$

$$(A:b) \longrightarrow \begin{pmatrix} 1 & 2 & 0 & 0 & 2 & -2 \\ 0 & 0 & 1 & 0 & -1 & 2 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

• 可见 r(A) = r(A : b) = 3 < 5,有无穷多的解,含 5 - 3 = 2 个自由变量

练习 求解
$$\begin{cases} x_1 + & 2x_2 + & x_3 + & x_4 + & x_5 = 1 \\ 2x_1 + & 4x_2 + & 3x_3 + & x_4 + & x_5 = 3 \\ -x_1 - & 2x_2 + & x_3 + & 3x_4 - & 3x_5 = 7 \\ & & 2x_3 + & 5x_4 - & 2x_5 = 9 \end{cases}$$

$$(A:b) \longrightarrow \begin{pmatrix} 1 & 2 & 0 & 0 & 2 & -2 \\ 0 & 0 & 1 & 0 & -1 & 2 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

- 可见 r(A) = r(A : b) = 3 < 5,有无穷多的解,含 5 3 = 2 个自由变量
- 由既约阶梯形矩阵可知,原方程组等价于

$$\begin{cases} x_1 + 2x_2 & + 2x_5 = -2 \\ x_3 & - x_5 = 2 \\ x_4 & = 1 \end{cases}$$

练习 求解
$$\begin{cases} x_1 + & 2x_2 + & x_3 + & x_4 + & x_5 = 1 \\ 2x_1 + & 4x_2 + & 3x_3 + & x_4 + & x_5 = 3 \\ -x_1 - & 2x_2 + & x_3 + & 3x_4 - & 3x_5 = 7 \\ & & 2x_3 + & 5x_4 - & 2x_5 = 9 \end{cases}$$

$$(A:b) \longrightarrow \begin{pmatrix} 1 & 2 & 0 & 0 & 2 & -2 \\ 0 & 0 & 1 & 0 & -1 & 2 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

- 可见 r(A) = r(A : b) = 3 < 5,有无穷多的解,含 5 3 = 2 个自由变量
- 由既约阶梯形矩阵可知,原方程组等价于

$$\begin{cases} x_1 + 2x_2 & + 2x_5 = -2 \\ x_3 & - x_5 = 2 \\ x_4 & = 1 \end{cases} \Rightarrow \begin{cases} x_1 = -2 - 2x_2 - 2x_5 \\ x_3 = 2 + x_5 \\ x_4 = 1 \end{cases}$$

练习 求解 $\begin{cases} x_1 + 2x_2 + x_3 + x_4 + x_5 = 1 \\ 2x_1 + 4x_2 + 3x_3 + x_4 + x_5 = 3 \\ -x_1 - 2x_2 + x_3 + 3x_4 - 3x_5 = 7 \\ 2x_3 + 5x_4 - 2x_5 = 9 \end{cases}$

解

$$(A:b) \longrightarrow \left(\begin{array}{c|cccc} 0 & 0 & 1 & 0 & -1 & 2 \\ \hline 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

- 可见 r(A) = r(A : b) = 3 < 5,有无穷多的解,含 5 3 = 2 个自由变量
- 由既约阶梯形矩阵可知,原方程组等价于

$$\begin{cases} x_1 + 2x_2 & + 2x_5 = -2 \\ x_3 & - x_5 = 2 \\ x_4 & = 1 \end{cases} \Rightarrow \begin{cases} x_1 = -2 - 2x_2 - 2x_5 \\ x_3 = 2 + x_5 \\ x_4 = 1 \end{cases}$$

所以通解是

$$\begin{cases} x_1 = \\ x_2 = \\ x_3 = \\ x_4 = \\ x_5 = \end{cases}$$



解

$$(A:b) \longrightarrow \left(\begin{array}{ccc|cccc} 0 & 0 & 1 & 0 & -1 & 2 \\ 0 & 0 & 1 & 0 & -1 & 2 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

- 可见 r(A) = r(A : b) = 3 < 5,有无穷多的解,含 5 3 = 2 个自由变量
- 由既约阶梯形矩阵可知,原方程组等价于

$$\begin{cases} x_1 + 2x_2 & + 2x_5 = -2 \\ x_3 & - x_5 = 2 \\ x_4 & = 1 \end{cases} \Rightarrow \begin{cases} x_1 = -2 - 2x_2 - 2x_5 \\ x_3 = 2 + x_5 \\ x_4 = 1 \end{cases}$$

$$\begin{cases} x_1 = \\ x_2 = c_1 \\ x_3 = \\ x_4 = \\ x_5 = c_2 \end{cases}$$
 $(c_1, c_2$ 为任意常数)



解

$$(A:b) \longrightarrow \left(\begin{array}{ccc|cccc} 0 & 0 & 1 & 0 & -1 & 2 \\ 0 & 0 & 1 & 0 & -1 & 2 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

- 可见 r(A) = r(A : b) = 3 < 5,有无穷多的解,含 5 3 = 2 个自由变量
- 由既约阶梯形矩阵可知,原方程组等价于

$$\begin{cases} x_1 + 2x_2 & + 2x_5 = -2 \\ x_3 & - x_5 = 2 \\ x_4 & = 1 \end{cases} \Rightarrow \begin{cases} x_1 = -2 - 2x_2 - 2x_5 \\ x_3 = 2 + x_5 \\ x_4 = 1 \end{cases}$$

記是
$$\begin{cases} x_1 = -2 - 2c_1 - 2c_2 \\ x_2 = c_1 \\ x_3 = \\ x_4 = \\ x_5 = c_2 \end{cases} (c_1, c_2 为任意常数)$$



解

$$(A:b) \longrightarrow \left(\begin{array}{c|cccc} 0 & 0 & 1 & 0 & -1 & 2 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

- 可见 r(A) = r(A : b) = 3 < 5,有无穷多的解,含 5 3 = 2 个自由变量
- 由既约阶梯形矩阵可知,原方程组等价于

$$\begin{cases} x_1 + 2x_2 & + 2x_5 = -2 \\ x_3 & - x_5 = 2 \\ x_4 & = 1 \end{cases} \Rightarrow \begin{cases} x_1 = -2 - 2x_2 - 2x_5 \\ x_3 = 2 + x_5 \\ x_4 = 1 \end{cases}$$

召是
$$\begin{cases} x_1 = -2 - 2c_1 - 2c_2 \\ x_2 = c_1 \\ x_3 = 2 + c_2 \\ x_4 = \\ x_5 = c_2 \end{cases} \quad (c_1, c_2 为任意常数)$$



解

$$(A:b) \longrightarrow \left(\begin{array}{c|cccc} 0 & 0 & 1 & 0 & -1 & 2 \\ 0 & 0 & 1 & 0 & -1 & 2 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

- 可见 r(A) = r(A : b) = 3 < 5,有无穷多的解,含 5 3 = 2 个自由变量
- 由既约阶梯形矩阵可知,原方程组等价于

$$\begin{cases} x_1 + 2x_2 & + 2x_5 = -2 \\ x_3 & - x_5 = 2 \\ x_4 & = 1 \end{cases} \Rightarrow \begin{cases} x_1 = -2 - 2x_2 - 2x_5 \\ x_3 = 2 + x_5 \\ x_4 = 1 \end{cases}$$

程是
$$\begin{cases} x_1 = -2 - 2c_1 - 2c_2 \\ x_2 = c_1 \\ x_3 = 2 + c_2 \\ x_4 = 1 \\ x_5 = c_2 \end{cases} \qquad (c_1, c_2 为任意常数)$$



$$\begin{cases} x_1 + x_2 + x_3 + x_4 = 0 \\ x_2 + 2x_3 + 2x_4 = 1 \\ -x_2 + (a-3)x_3 - 2x_4 = b \\ 3x_1 + 2x_2 + x_3 + ax_4 = -1 \end{cases}$$
 有无穷解、唯一解,及无

$$\begin{cases} x_{1}+ & x_{2}+ & x_{3}+ & x_{4}=0\\ & x_{2}+ & 2x_{3}+2x_{4}=1\\ & -x_{2}+(a-3)x_{3}-2x_{4}=b\\ 3x_{1}+2x_{2}+ & x_{3}+ax_{4}=-1 \end{cases}$$
 有无穷解、唯一解,及无

解?

$$(A : b) = \begin{pmatrix} 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 2 & 2 & 2 & 1 \\ 0 & -1 & a & -3 & -2 & b \\ 3 & 2 & 1 & a & -1 \end{pmatrix}$$

$$\begin{cases} x_1 + x_2 + x_3 + x_4 = 0 \\ x_2 + 2x_3 + 2x_4 = 1 \\ -x_2 + (\alpha - 3)x_3 - 2x_4 = b \\ 3x_1 + 2x_2 + x_3 + ax_4 = -1 \end{cases}$$
 有无穷解、唯一解,及无

解?

$$(A:b) = \begin{pmatrix} 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & a - 3 & -2 & b \\ 3 & 2 & 1 & a - 1 \end{pmatrix} \xrightarrow{r_4 - 3r_1}$$

$$\begin{cases} x_1 + x_2 + x_3 + x_4 = 0 \\ x_2 + 2x_3 + 2x_4 = 1 \\ -x_2 + (\alpha - 3)x_3 - 2x_4 = b \\ 3x_1 + 2x_2 + x_3 + ax_4 = -1 \end{cases}$$
 有无穷解、唯一解,及无

解?

$$(A : b) = \begin{pmatrix} 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & a - \frac{2}{3} & -\frac{2}{3} & b \\ 0 & -1 & a - \frac{3}{3} & -2 & b \\ 1 & a & -1 \end{pmatrix} \xrightarrow{r_4 - 3r_1} \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 2 & 2 & 2 & 1 \\ 0 & -1 & a - \frac{3}{3} & -2 & b \\ 0 & -1 & -2 & a - 3 & -1 \end{pmatrix}$$

$$\begin{cases} x_1 + x_2 + x_3 + x_4 = 0 \\ x_2 + 2x_3 + 2x_4 = 1 \\ -x_2 + (\alpha - 3)x_3 - 2x_4 = b \\ 3x_1 + 2x_2 + x_3 + \alpha x_4 = -1 \end{cases}$$
 有无穷解、唯一解,及无

解?

$$(A \vdots b) = \begin{pmatrix} 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & a - \frac{2}{3} & -\frac{2}{3} & | & b \\ 0 & -1 & a & | & -1 \end{pmatrix} \xrightarrow{r_4 - 3r_1} \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 2 & 2 & | & 1 \\ 0 & -1 & a - 3 & -2 & | & b \\ 0 & -1 & -2 & a - 3 & | & -1 \end{pmatrix}$$

$$r_3+r_2$$



$$\begin{cases} x_1 + x_2 + x_3 + x_4 = 0 \\ x_2 + 2x_3 + 2x_4 = 1 \\ -x_2 + (a-3)x_3 - 2x_4 = b \\ 3x_1 + 2x_2 + x_3 + ax_4 = -1 \end{cases}$$
 有无穷解、唯一解,及无

解?

$$(A:b) = \begin{pmatrix} 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 2 & 2 & 1 & 0 \\ 0 & -1 & a & -3 & -2 & b \\ 3 & 2 & a & a & -1 \end{pmatrix} \xrightarrow{r_4 - 3r_1} \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 2 & 2 & 2 & 1 \\ 0 & -1 & a & -3 & -2 & b \\ 0 & -1 & -2 & a & 3 & -1 \end{pmatrix}$$

$$\frac{r_3 + r_2}{r_4 + r_2} \left(\begin{array}{ccc|c} 1 & 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & a & 1 & 2 & 2 \\ 0 & 0 & a & 1 & 0 & a \\ 0 & 0 & a & 1 & 0 \\ 0 & 0 & a & 1 & 0 \end{array} \right) b + \frac{1}{1} b$$

无解?

无解?

$$(A:b) \longrightarrow \begin{pmatrix} 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 2 & 2 & 2 \\ 0 & 0 & a - 1 & 0 & b + 1 \\ 0 & 0 & a - 1 & b + 1 \end{pmatrix}$$

- 当 a ≠ 1 时
- 当 a = 1 时

$$\begin{cases} x_1 + & x_2 + & x_3 + & x_4 = & 0 \\ & x_2 + & 2x_3 + & 2x_4 = & 1 \\ & -x_2 + & (\alpha - 3)x_3 - & 2x_4 = & b \\ 3x_1 + & 2x_2 + & x_3 + & ax_4 = & -1 \end{cases}$$
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无解?

- 当 $a \neq 1$ 时 (b 为任意数), r(A) = r(A : b) = 4,
- 当 a = 1 时

无解?

 $(A : b) \longrightarrow \begin{pmatrix} \begin{vmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 & 1 \end{pmatrix} b + \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & -1 & 1 \end{pmatrix}$

- 当 $\alpha \neq 1$ 时(b 为任意数), $r(A) = r(A \cdot b) = 4$, 有唯一解;
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无解?

$$(A:b) \longrightarrow \begin{pmatrix} 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & a - 1 & 2 & 2 & 2 \\ 0 & 0 & a - 1 & 0 & a - 1 \end{pmatrix} b + 1 \\ 0 & 0 & 0 & a - 1 & 0 & a - 1 \end{pmatrix}$$

- 当 $a \neq 1$ 时(b 为任意数), $r(A) = r(A \cdot b) = 4$, 有唯一解;
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$$\begin{cases} x_1 + x_2 + x_3 + x_4 = 0 \\ x_2 + 2x_3 + 2x_4 = 1 \\ -x_2 + (a-3)x_3 - 2x_4 = b \end{cases}$$
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$$\mathbf{H} \qquad (A : b) \longrightarrow \begin{pmatrix} 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 2 & 2 & 2 \\ 0 & 0 & a - 1 & 0 & b + 1 \\ 0 & 0 & a - 1 & b + 1 \end{pmatrix}$$

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-a=1, b=-1 时, r(A)=r(A:b)=2<4,

- 当 $\alpha \neq 1$ 时 (b 为任意数), r(A) = r(A : b) = 4, 有唯一解;
- 当 a = 1 时

$$(A : b) \longrightarrow \begin{pmatrix} \begin{vmatrix} 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 2 & 2 & 1 \\ 0 & 0 & 0 & 0 & b+1 \\ 0 & 0 & 0 & 0 & b+1 \end{pmatrix}$$

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- $\alpha = 1$, b = -1 时, r(A) = r(A : b) = 2 < 4, 有无穷多解
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无解? 解

- $\alpha = 1$, b = -1 时, r(A) = r(A : b) = 2 < 4, 有无穷多解

-
$$a = 1$$
, $b \neq -1$ 时, $r(A) = 2 < 3 = r(A : b)$,



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 有无穷解、唯一解,及
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 $(A \vdots b) \longrightarrow \begin{pmatrix} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & a - 1 \\ 0 & 0 & a - 1 \\ 0 & 0 & a \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & a \end{bmatrix}$ 解

无解?

• 当
$$a \neq 1$$
 时 (b 为任意数), $r(A) = r(A : b) = 4$, 有唯一解;
• 当 $a = 1$ 时
$$(A : b) \longrightarrow \begin{pmatrix} 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 2 & 2 & 1 \\ 0 & 0 & 0 & 0 & b + 1 \\ 0 & 0 & 0 & 0 & b + 1 \end{pmatrix}$$

- $\alpha = 1$, b = -1 时, r(A) = r(A : b) = 2 < 4, 有无穷多解

-
$$a = 1, b \neq -1$$
 时, $r(A) = 2 < 3 = r(A : b)$, 无解

$$(A : b) = \begin{pmatrix} 1 & 2 & 3 & | & 1 \\ 2 & 3 & 5 & | & -1 \\ 3 & 4 & a & | & b \end{pmatrix}$$



$$(A \vdots b) = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 5 \\ 3 & 4 & a \end{pmatrix} - \begin{pmatrix} 1 \\ 1 \\ b \end{pmatrix} \xrightarrow{r_2 - 2r_1} r_3 - 3r_1$$

$$(A:b) = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 5 \\ 3 & 4 & a \end{pmatrix} - \frac{1}{b} \frac{r_2 - 2r_1}{r_3 - 3r_1} \begin{pmatrix} 1 & 2 & 3 \\ & & & \end{pmatrix}$$



例 讨论
$$a$$
, b 取何值时,方程组
$$\begin{cases} x_1 + 2x_2 + 3x_3 = 1 \\ 2x_1 + 3x_2 + 5x_3 = -1 \end{cases}$$
 有无 $3x_1 + 4x_2 + ax_3 = b$

$$(A:b) = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 5 \\ 3 & 4 & a \end{pmatrix} - \frac{1}{b} \xrightarrow{r_2 - 2r_1} \begin{pmatrix} 1 & 2 & 3 \\ 0 & -1 & -1 \\ 0 & -1 & -3 \end{pmatrix}$$



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例 讨论
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$$r_3 - 2r_2$$

例 讨论
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 有无 $3x_1 + 4x_2 + ax_3 = b$

$$(A:b) = \begin{pmatrix} 1 & 2 & 3 & | & -1 & | & \frac{r_2 - 2r_1}{r_3 - 3r_1} & \begin{pmatrix} 1 & -2 & | & -3 & | & -3 & | \\ 0 & -2 & | & | & -3 & | & -3 & | \\ \frac{r_3 - 2r_2}{r_3 - 2r_2} & \begin{pmatrix} 1 & 2 & | & -3 & | & -3 & | & -3 & | \\ 0 & -1 & | & -1 & | & -3 & | & -3 & | \\ 0 & | & a - 7 & | & b + 3 & | & -3 & | & -3 & | \\ \end{pmatrix}$$



例 讨论
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$$(A:b) = \begin{pmatrix} \frac{1}{2} & \frac{2}{3} & \frac{3}{5} & -\frac{1}{1} \\ \frac{2}{3} & \frac{3}{4} & \frac{5}{a} & -\frac{1}{b} \end{pmatrix} \xrightarrow{r_2 - 2r_1} \begin{pmatrix} \frac{1}{0} & -\frac{2}{1} & -\frac{3}{1} \\ 0 & -\frac{2}{2} & a - \frac{3}{9} & b - \frac{3}{3} \end{pmatrix}$$

$$\xrightarrow{r_3 - 2r_2} \begin{pmatrix} \frac{1}{0} & -\frac{2}{1} & -\frac{3}{1} \\ 0 & 0 & a - \frac{7}{1} & b + \frac{3}{3} \end{pmatrix}$$

- 当 a ≠ 7 时
- 当 a = 7 时

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$$\xrightarrow{r_3 - 2r_2} \begin{pmatrix} 1 \\ 0 \\ -1 \\ 0 \end{pmatrix} - \begin{pmatrix} 2 \\ 1 \\ 0 \\ a - 7 \end{pmatrix} \begin{pmatrix} 3 \\ b + 3 \end{pmatrix}$$

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$$(A:b) = \begin{pmatrix} 1 & 2 & 3 & | & -1 & | & \frac{r_2 - 2r_1}{r_3 - 3r_1} & \begin{pmatrix} 1 & -2 & | & -3 & | & -3 & | \\ 0 & -2 & | & | & -3 & | & | & -3 & | \end{pmatrix}$$

$$\xrightarrow{r_3 - 2r_2} \begin{pmatrix} 1 & -2 & | & -3 & | & -3 & | & -3 & | & -3 & | & -3 & | & -3 & | & -3 & | & -3 & | & -3 & | & -3 & | & -3 & | & -3 & | & -3 & | & -3 & | & -3 & | & -3 & | & -3 & | & -3 & | & -3 & | & -3 & | & -3 & | & -3 & | & -3 & | & -3 & | & -3 & | & -3 & | & -3 & | & -3 & | & -3 & | & -3 & | & -3 & | & -3 & | & -3 & | & -3 & | & -3 & | & -3 & | & -3 & | & -3 & | & -3 & | & -3 & | & -3 & | & -3 & | & -3 & | & -3 & | & -3 & | & -3 & | & -3 & | & -3 & | & -3 & | & -3 & | & -3 & | & -3 & | & -3 & | & -3 & | & -3 & | & -3 & | & -3 & | & -3 & | & -3 & | & -3 & | & -3 & | & -3 & | & -3 & | & -3 & | & -3 & | & -3 & | & -3 & | & -3 & | & -3 & | & -3 & | & -3 & | & -3 & | & -3 & | & -3 & | & -3 & | & -3 & | & -3 & | & -3 & | & -3 & | & -3 & | & -3 & | & -3 & | & -3 & | & -3 & | & -3 & | & -3 & | & -3 & | & -3 & | & -3 & | & -3 & | & -3 & | & -3 & | & -3 & | & -3 & | & -3 & | & -3 & | & -3 & | & -3 & | & -3 & | & -3 & | & -3 & | & -3 & | & -3 & | & -3 & | & -3 & | & -3 & | & -3 & | & -3 & | & -3 & | & -3 & | & -3 & | & -3 & | & -3 & | & -3 & | & -3 & | & -3 & | & -3 & | & -3 & | & -3 & | & -3 & | & -3 & | & -3 & | & -3 & | & -3 & | & -3 & | & -3 & | & -3 & | & -3 & | & -3 & | & -3 & | & -3 & | & -3 & | & -3 & | & -3 & | & -3 & | & -3 & | & -3 & | & -3 & | & -3 & | & -3 & | & -3 & | & -3 & | & -3 & | & -3 & | & -3 & | & -3 & | & -3 & | & -3 & | & -3 & | & -3 & | & -3 & | & -3 & | & -3 & | & -3 & | & -3 & | & -3 & | & -3 & | & -3 & | & -3 & | & -3 & | & -3 & | & -3 & | & -3 & | & -3 & | & -3 & | & -3 & | & -3 & | & -3 & | & -3 & | & -3 & | & -3 & | & -3 & | & -3 & | & -3 & | & -3 & | & -3 & | & -3 & | & -3 & | & -3 & | & -3 & | & -3 & | & -3 & | & -3 & | & -3 & | & -3 & | & -3 & | & -3 & | & -3 & | & -3 & | & -3 & | & -3 & | & -3 & | & -3 & | & -3 & | & -3 & | & -3 & | & -3 & | & -3 & | & -3 & | & -3 & | & -3 & | & -3 & | & -3 & | & -3 & | & -3 & | & -3 & | & -3 & | & -3 & | & -3 & |$$

- 当 $\alpha \neq 7$ 时 (b 为任意数), r(A : b) = r(A) = 3,
- 当 a = 7 时

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 有无 $3x_1 + 4x_2 + \alpha x_3 = b$

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$$(A:b) = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 5 \\ 3 & 4 & a \end{pmatrix} - \begin{pmatrix} 1 \\ b \end{pmatrix} \xrightarrow{r_2 - 2r_1} \begin{pmatrix} 1 \\ 0 \\ -1 \\ -2 \end{pmatrix} \xrightarrow{r_3 - 3r_1} \begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix} \begin{pmatrix} -1 \\ a - 9 \end{pmatrix} \begin{pmatrix} 1 \\ b - 3 \end{pmatrix}$$

$$\xrightarrow{r_3 - 2r_2} \begin{pmatrix} 1 \\ 0 \\ -1 \\ a - 7 \end{pmatrix} \begin{pmatrix} 2 \\ -1 \\ b + 3 \end{pmatrix}$$

- 当 α ≠ 7 时(b 为任意数), r(A : b) = r(A) = 3, 有唯一解;
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- 当 $\alpha \neq 7$ 时 (b 为任意数), r(A : b) = r(A) = 3, 有唯一解;
- $\exists \alpha = 7 \text{ pt}$ $(A:b) \longrightarrow \begin{pmatrix} 1 & 2 & 3 & 1 \\ 0 & -1 & -1 & -3 \\ 0 & 0 & 0 & b+3 \end{pmatrix}$



$$(A:b) = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 5 \\ 3 & 4 & a \end{pmatrix} - \begin{pmatrix} 1 \\ -1 \\ b \end{pmatrix} \xrightarrow{r_2 - 2r_1} \begin{pmatrix} 1 \\ 0 & -1 \\ 0 & -2 & a - 9 \end{pmatrix} \begin{pmatrix} 1 \\ b - 3 \end{pmatrix} \\
 \xrightarrow{r_3 - 2r_2} \begin{pmatrix} 1 \\ 0 & -1 \\ 0 & a - 7 \end{pmatrix} \begin{pmatrix} 1 \\ b + 3 \end{pmatrix}$$

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- $\exists a = 7 \text{ ft}$ $(A : b) \longrightarrow \begin{pmatrix} 1 & 2 & 3 & 1 \\ 0 & -1 & -1 & -3 \\ 0 & 0 & 0 & b + 3 \end{pmatrix}$
 - a = 7, b = -3 时
 - a = 7, b ≠ −3 时



$$(A:b) = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 5 \\ 3 & 4 & a \end{pmatrix} - \begin{pmatrix} 1 \\ b \end{pmatrix} \xrightarrow{r_2 - 2r_1} \begin{pmatrix} 1 \\ 0 & -1 \\ 0 & -2 & a - 9 \end{pmatrix} \begin{pmatrix} -3 \\ b - 3 \end{pmatrix}$$

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- 当 α ≠ 7 时(b 为任意数), r(A · b) = r(A) = 3, 有唯一解;
- $\exists a = 7 \text{ pt}$ $(A : b) \longrightarrow \begin{pmatrix} \begin{vmatrix} 1 & 2 & 3 & 1 \\ 0 & -1 & -1 & -3 \\ 0 & 0 & b + 3 \end{pmatrix}$
 - -a=7, b=-3 时
 - $a = 7, b \neq -3$ 时

$$(A:b) = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 5 \\ 3 & 4 & a \end{pmatrix} - \begin{pmatrix} 1 \\ -1 \\ b \end{pmatrix} \xrightarrow{r_2 - 2r_1} \begin{pmatrix} 1 \\ 0 \\ -1 \\ -2 \\ a - 9 \end{pmatrix} \begin{pmatrix} 3 \\ b - 3 \end{pmatrix}$$

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 - a = 7, b = -3 时, r(A : b) = r(A) = 2 < 3,
 - $a = 7, b \neq -3$ 时

$$(A:b) = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 5 \\ 3 & 4 & a \end{pmatrix} - \begin{pmatrix} 1 \\ -1 \\ b \end{pmatrix} \xrightarrow{r_2 - 2r_1} \begin{pmatrix} 1 \\ 0 & -1 \\ 0 & -2 & a - 9 \end{pmatrix} \begin{pmatrix} 1 \\ -3 \\ b - 3 \end{pmatrix}$$

$$\xrightarrow{r_3 - 2r_2} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} - \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} - \begin{pmatrix} 3 \\ -1 \\ 0 \end{pmatrix} - \begin{pmatrix} 1 \\ -3 \\ b + 3 \end{pmatrix}$$

- 当 α ≠ 7 时(b 为任意数), r(A : b) = r(A) = 3, 有唯一解;
- - $-\alpha = 7, b = -3$ 时, r(A : b) = r(A) = 2 < 3, 有无穷多解
 - a = 7, $b \neq -3$ 时



$$\mathbf{(A:b)} = \begin{pmatrix} 1 & 2 & 3 & | & 1 \\ 2 & 3 & 5 & | & -1 \\ 3 & 4 & a & | & b \end{pmatrix} \xrightarrow{r_2 - 2r_1} \begin{pmatrix} 1 & 2 & 3 & | & -1 \\ 0 & -1 & -1 & | & -3 \\ 0 & -2 & a - 9 & | & b - 3 \end{pmatrix}$$

$$\xrightarrow{r_3 - 2r_2} \begin{pmatrix} 1 & 2 & 3 & | & -1 \\ 0 & 0 & a - 7 & | & b + 3 \end{pmatrix}$$

- 当 $\alpha \neq 7$ 时(b 为任意数), r(A : b) = r(A) = 3, 有唯一解;
- - $\alpha = 7$, b = -3 时,r(A : b) = r(A) = 2 < 3,有无穷多解
 - $a = 7, b \neq -3$ 时

$$(A:b) = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 5 \\ 3 & 4 & a \end{pmatrix} - \begin{pmatrix} 1 \\ -1 \\ b \end{pmatrix} \xrightarrow{r_2 - 2r_1} \begin{pmatrix} 1 \\ 0 & -1 \\ 0 & -2 & a - 9 \end{pmatrix} \begin{pmatrix} -3 \\ b - 3 \end{pmatrix}$$

$$\xrightarrow{r_3 - 2r_2} \begin{pmatrix} 1 \\ 0 & -1 \\ 0 & a - 7 \end{pmatrix} \begin{pmatrix} -3 \\ b + 3 \end{pmatrix}$$

- 当 a ≠ 7 时(b 为任意数), r(A · b) = r(A) = 3, 有唯一解;
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 - a = 7, $b \neq -3$ 时



$$(A:b) = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 5 \\ 3 & 4 & a \end{pmatrix} - \begin{pmatrix} 1 \\ b \end{pmatrix} \xrightarrow{r_2 - 2r_1} \begin{pmatrix} 1 \\ 0 & -1 \\ 0 & -2 & a - 9 \end{pmatrix} \begin{pmatrix} -3 \\ b - 3 \end{pmatrix}$$

$$\xrightarrow{r_3 - 2r_2} \begin{pmatrix} 1 \\ 0 & -1 \\ 0 & a - 7 \end{pmatrix} \begin{pmatrix} 3 \\ b + 3 \end{pmatrix}$$

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 - a = 7, $b \neq -3$ 时, $r(A : b) = 3 \neq 2 = r(A)$,



$$(A:b) = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 5 \\ 3 & 4 & a \end{pmatrix} - \begin{pmatrix} 1 \\ b \end{pmatrix} \xrightarrow{r_2 - 2r_1} \begin{pmatrix} 1 \\ 0 & -1 \\ 0 & -2 & a - 9 \end{pmatrix} \begin{pmatrix} -3 \\ b - 3 \end{pmatrix}$$

$$\xrightarrow{r_3 - 2r_2} \begin{pmatrix} 1 \\ 0 & -1 \\ 0 & a - 7 \end{pmatrix} \begin{pmatrix} 3 \\ b + 3 \end{pmatrix}$$

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- - a = 7, b = -3 时, r(A : b) = r(A) = 2 < 3, 有无穷多解
 - a = 7, $b \neq -3$ 时, $r(A : b) = 3 \neq 2 = r(A)$, 无解



• 一般线性方程组 $A_{m \times n} x = b$ (m 个方程, n 个未知量)

Ax = b	有无穷解	有唯一解	无解
	r(A) = r(A : b) < n	r(A) = r(A : b) = n	r(A) < r(A : b)

• 一般线性方程组 $A_{m \times n} x = b$ (m 个方程, n 个未知量)

Ax = b	有无穷解	有唯一解	无解
	r(A) = r(A : b) < n	$r(A) = r(A \vdots b) = n$	r(A) < r(A : b)

• 齐次线性方程组 $A_{m \times n} x = 0$,一定有 $r(A) = r(A \stackrel{!}{\cdot} 0)$,所以一定有解(至少有零解)

• 一般线性方程组 $A_{m \times n} x = b$ (m 个方程, n 个未知量)

Ax = b	有无穷解	有唯一解	无解
	r(A) = r(A : b) < n	r(A) = r(A : b) = n	r(A) < r(A : b)

• 齐次线性方程组 $A_{m\times n}x=0$,一定有 r(A)=r(A:0),所以一定 有解(至少有零解)

Ax = 0	有无穷解	有唯一解(零解)

• 一般线性方程组 $A_{m \times n} x = b$ (m 个方程, n 个未知量)

Ax = b	有无穷解	有唯一解	无解
	r(A) = r(A : b) < n	r(A) = r(A : b) = n	r(A) < r(A : b)

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• 一般线性方程组 $A_{m \times n} x = b$ (m 个方程, n 个未知量)

Ax = b	有无穷解	有唯一解	无解
	r(A) = r(A : b) < n	r(A) = r(A : b) = n	r(A) < r(A : b)

• 齐次线性方程组 $A_{m\times n}x=0$,一定有 $r(A)=r(A \vdots 0)$,所以一定有解(至少有零解)

Ax = 0	有无穷解	有唯一解(零解)
	r(A) < n	r(A) = n



例 解齐次线性方程组
$$\begin{cases} x_1- & x_2+ 5x_3- x_4=0 \\ x_1+ & x_2- 2x_3+ 3x_4=0 \\ 3x_1- & x_2+ 8x_3+ x_4=0 \\ x_1+ 3x_2- 9x_3+ 7x_4=0 \end{cases}$$

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$$\begin{cases} x_1- & x_2+ 5x_3- x_4=0\\ x_1+ & x_2- 2x_3+ 3x_4=0\\ 3x_1- & x_2+ 8x_3+ x_4=0\\ x_1+ 3x_2- 9x_3+ 7x_4=0 \end{cases}$$

$$(A:b) = \begin{pmatrix} 1 & -1 & 5 & -1 & 0 \\ 1 & 1 & -2 & 3 & 0 \\ 3 & -1 & 8 & 1 & 0 \\ 1 & 3 & -9 & 7 & 0 \end{pmatrix}$$

例 解齐次线性方程组
$$\begin{cases} x_1- & x_2+ 5x_3- & x_4=0\\ x_1+ & x_2- 2x_3+ 3x_4=0\\ 3x_1- & x_2+ 8x_3+ x_4=0\\ x_1+ 3x_2- 9x_3+ 7x_4=0 \end{cases}$$

$$(A:b) = \begin{pmatrix} 1 & -1 & 3 & -1 & 0 \\ 1 & 1 & -2 & 3 & 0 \\ 3 & -1 & 8 & 1 & 0 \\ 1 & 3 & -9 & 7 & 0 \end{pmatrix} \xrightarrow[r_4 - r_1]{r_2 - r_1} \xrightarrow[r_4 - r_1]{r_3 - 3r_1}$$

例 解齐次线性方程组
$$\begin{cases} x_1-&x_2+&5x_3-&x_4&=0\\ x_1+&x_2-&2x_3+&3x_4&=0\\ 3x_1-&x_2+&8x_3+&x_4&=0\\ x_1+&3x_2-&9x_3+&7x_4&=0 \end{cases}$$

$$(A : b) = \begin{pmatrix} 1 & -1 & 5 & -1 & 0 \\ 1 & 1 & -2 & 3 & 0 \\ 3 & -1 & 8 & 1 & 0 \\ 1 & 3 & -9 & 7 & 0 \end{pmatrix} \xrightarrow[r_3 - 3r_1]{r_3 - 3r_1} \begin{pmatrix} 1 & -1 & 5 & -1 & 0 \\ r_3 - 3r_1 & r_4 - r_1 & r_4 - r_1 & r_4 - r_1 & r_4 - r_1 \end{pmatrix}$$



例 解齐次线性方程组
$$\begin{cases} x_1-&x_2+&5x_3-&x_4&=0\\ x_1+&x_2-&2x_3+&3x_4&=0\\ 3x_1-&x_2+&8x_3+&x_4&=0\\ x_1+&3x_2-&9x_3+&7x_4&=0 \end{cases}$$

$$(A \vdots b) = \begin{pmatrix} 1 & -1 & 5 & -1 & 0 \\ 1 & 1 & -2 & 3 & 0 \\ 3 & -1 & 8 & 1 & 0 \\ 1 & 3 & -9 & 7 & 0 \end{pmatrix} \xrightarrow[r_3 - 3r_1]{r_3 - 3r_1} \begin{pmatrix} 1 & -1 & 5 & -1 & 0 \\ 0 & 2 & -7 & 4 & 0 \end{pmatrix}$$

例 解齐次线性方程组
$$\begin{cases} x_1- & x_2+ 5x_3- x_4=0\\ x_1+ & x_2- 2x_3+ 3x_4=0\\ 3x_1- & x_2+ 8x_3+ x_4=0\\ x_1+ 3x_2- 9x_3+ 7x_4=0 \end{cases}$$

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$$(A:b) = \begin{pmatrix} 1 & -1 & 5 & -1 & 0 \\ 1 & 1 & -2 & 3 & 0 \\ 3 & -1 & 8 & 1 & 0 \\ 1 & 3 & -9 & 7 & 0 \end{pmatrix} \xrightarrow[r_3 - 3r_1]{r_3 - 3r_1} \begin{pmatrix} 1 & -1 & 5 & -1 & 0 \\ 0 & 2 & -7 & 4 & 0 \\ 0 & 2 & -7 & 4 & 0 \\ 0 & 4 & -14 & 8 & 0 \end{pmatrix}$$

例 解齐次线性方程组
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$$(A:b) = \begin{pmatrix} 1 & -1 & 5 & -1 & 0 \\ 1 & 1 & -2 & 3 & 0 \\ 3 & -1 & 8 & 1 & 0 \\ 1 & 3 & -9 & 7 & 0 \end{pmatrix} \xrightarrow[r_4-r_1]{r_2-r_1} \begin{pmatrix} 1 & -1 & 5 & -1 & 0 \\ 0 & 2 & -7 & 4 & 0 \\ 0 & 2 & -7 & 4 & 0 \\ 0 & 4 & -14 & 8 & 0 \end{pmatrix}$$

$$\frac{r_3 - r_2}{r_4 - 2r_2}$$

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$$(A:b) = \begin{pmatrix} 1 & -1 & 5 & -1 & 0 \\ 1 & 1 & -2 & 3 & 0 \\ 3 & -1 & 8 & 1 & 0 \\ 1 & 3 & -9 & 7 & 0 \end{pmatrix} \xrightarrow[r_{3}-3r_{1}]{r_{3}-3r_{1}} \begin{pmatrix} 1 & -1 & 5 & -1 & 0 \\ 0 & 2 & -7 & 4 & 0 \\ 0 & 2 & -7 & 4 & 0 \\ 0 & 4 & -14 & 8 & 0 \end{pmatrix}$$

$$\xrightarrow[r_{3}-r_{2}]{r_{4}-2r_{2}} \begin{pmatrix} 1 & -1 & 5 & -1 & 0 \\ 0 & 2 & -7 & 4 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\frac{r_3 - r_2}{r_4 - 2r_2} \left(\begin{array}{cccc}
1 & -1 & 5 & -1 & 0 \\
0 & 2 & -7 & 4 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0
\end{array} \right)$$

例 解齐次线性方程组
$$\begin{cases} x_1- & x_2+ 5x_3- x_4=0\\ x_1+ & x_2- 2x_3+ 3x_4=0\\ 3x_1- & x_2+ 8x_3+ x_4=0\\ x_1+ 3x_2- 9x_3+ 7x_4=0 \end{cases}$$

$$\frac{1}{2} \times r_2$$

例 解齐次线性方程组
$$\begin{cases} x_1- & x_2+ 5x_3- x_4=0 \\ x_1+ & x_2- 2x_3+ 3x_4=0 \\ 3x_1- & x_2+ 8x_3+ x_4=0 \\ x_1+ 3x_2- 9x_3+ 7x_4=0 \end{cases}$$



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$$\begin{cases} x_1- & x_2+ 5x_3- x_4=0\\ x_1+ & x_2- 2x_3+ 3x_4=0\\ 3x_1- & x_2+ 8x_3+ x_4=0\\ x_1+ 3x_2- 9x_3+ 7x_4=0 \end{cases}$$



例 解齐次线性方程组
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$$\xrightarrow{\frac{1}{2} \times r_2} \begin{pmatrix}
1 & -1 & 5 & -1 & | & 0 \\
0 & 1 & -7/2 & 2 & | & 0 \\
0 & 0 & 0 & 0 & | & 0 \\
0 & 0 & 0 & 0 & | & 0
\end{pmatrix} \xrightarrow{r_1 + r_2} \begin{pmatrix}
1 & 0 & 3/2 & 1 & | & 0 \\
0 & 1 & -7/2 & 2 & | & 0 \\
0 & 0 & 0 & 0 & | & 0 \\
0 & 0 & 0 & 0 & | & 0
\end{pmatrix}$$



例 解齐次线性方程组 $\begin{cases} x_1-&x_2+&5x_3-&x_4&=0\\ x_1+&x_2-&2x_3+&3x_4&=0\\ 3x_1-&x_2+&8x_3+&x_4&=0\\ x_1+&3x_2-&9x_3+&7x_4&=0 \end{cases}$

解

例 解齐次线性方程组 $\begin{cases} x_1- & x_2+ 5x_3- x_4=0 \\ x_1+ & x_2- 2x_3+ 3x_4=0 \\ 3x_1- & x_2+ 8x_3+ x_4=0 \\ x_1+ 3x_2- 9x_3+ 7x_4=0 \end{cases}$

解

$$(A:b) \longrightarrow \begin{pmatrix} 1 & 0 & 3/2 & 1 & | & 0 \\ 0 & 1 & -7/2 & 2 & | & 0 \\ 0 & 0 & & 0 & 0 & | & 0 \\ 0 & 0 & & 0 & 0 & | & 0 \end{pmatrix}$$

例 解齐次线性方程组
$$\begin{cases} x_1-&x_2+&5x_3-&x_4&=0\\ x_1+&x_2-&2x_3+&3x_4&=0\\ 3x_1-&x_2+&8x_3+&x_4&=0\\ x_1+&3x_2-&9x_3+&7x_4&=0 \end{cases}$$

解

$$\begin{cases} x_1 + & \frac{3}{2}x_3 + x_4 = 0 \\ x_2 - \frac{7}{2}x_3 + 2x_4 = 0 \end{cases}$$

例 解齐次线性方程组
$$\begin{cases} x_1- & x_2+ 5x_3- x_4=0\\ x_1+ & x_2- 2x_3+ 3x_4=0\\ 3x_1- & x_2+ 8x_3+ x_4=0\\ x_1+ 3x_2- 9x_3+ 7x_4=0 \end{cases}$$

解

$$\begin{cases} x_1 + & \frac{3}{2}x_3 + x_4 = 0 \\ x_2 - \frac{7}{2}x_3 + 2x_4 = 0 \end{cases} \iff \begin{cases} x_1 + & = -\frac{3}{2}x_3 - x_4 \\ x_2 = \frac{7}{2}x_3 - 2x_4 \end{cases}$$

例 解齐次线性方程组 $\begin{cases} x_1-&x_2+&5x_3-&x_4&=0\\ x_1+&x_2-&2x_3+&3x_4&=0\\ 3x_1-&x_2+&8x_3+&x_4&=0\\ x_1+&3x_2-&9x_3+&7x_4&=0 \end{cases}$

解

所以原方程组等价于
$$\begin{cases} x_1 + \frac{3}{2}x_3 + x_4 = 0 \\ x_2 - \frac{7}{2}x_3 + 2x_4 = 0 \end{cases} \iff \begin{cases} x_1 + \frac{3}{2}x_3 - x_4 \\ x_2 = \frac{7}{2}x_3 - 2x_4 \end{cases}$$

所以 $\begin{cases} x_3 = c_1 \\ x_4 = c_2 \end{cases}$



例 解齐次线性方程组
$$\begin{cases} x_1- & x_2+ 5x_3- x_4=0\\ x_1+ & x_2- 2x_3+ 3x_4=0\\ 3x_1- & x_2+ 8x_3+ x_4=0\\ x_1+ 3x_2- 9x_3+ 7x_4=0 \end{cases}$$

解

$$\begin{cases} x_{1} + \frac{3}{2}x_{3} + x_{4} = 0 \\ x_{2} - \frac{7}{2}x_{3} + 2x_{4} = 0 \end{cases} \iff \begin{cases} x_{1} + \frac{3}{2}x_{3} - x_{4} \\ x_{2} = \frac{7}{2}x_{3} - 2x_{4} \end{cases}$$
所以
$$\begin{cases} x_{1} = -\frac{3}{2}c_{1} - c_{2} \\ x_{3} = c_{1} \\ x_{4} = c_{2} \end{cases}$$

例 解齐次线性方程组
$$\begin{cases} x_1- & x_2+ 5x_3- x_4=0\\ x_1+ & x_2- 2x_3+ 3x_4=0\\ 3x_1- & x_2+ 8x_3+ x_4=0\\ x_1+ 3x_2- 9x_3+ 7x_4=0 \end{cases}$$

解

$$\begin{cases} x_{1} + \frac{3}{2}x_{3} + x_{4} = 0 \\ x_{2} - \frac{7}{2}x_{3} + 2x_{4} = 0 \end{cases} \iff \begin{cases} x_{1} + \frac{3}{2}x_{3} - x_{4} \\ x_{2} = \frac{7}{2}x_{3} - 2x_{4} \end{cases}$$
所以
$$\begin{cases} x_{1} = -\frac{3}{2}c_{1} - c_{2} \\ x_{2} = \frac{7}{2}c_{1} - 2c_{2} \\ x_{3} = c_{1} \\ x_{4} = c_{2} \end{cases}$$

例 解齐次线性方程组 $\begin{cases} x_1- & x_2+ 5x_3- x_4=0\\ x_1+ & x_2- 2x_3+ 3x_4=0\\ 3x_1- & x_2+ 8x_3+ x_4=0\\ x_1+ 3x_2- 9x_3+ 7x_4=0 \end{cases}$ 解

所以原方程组等价于

所以
$$\begin{cases} x_1 + \frac{7}{2}x_3 + x_4 - \frac{7}{2}x_3 + 2x_4 = x_2 - \frac{7}{2}x_3 + 2x_4 = x_2 - \frac{7}{2}x_1 - x_2 \\ x_2 = \frac{7}{2}c_1 - 2c_2 \\ x_3 = c_1 \\ x_4 = c_2 \end{cases}$$

 $\begin{cases} x_1 + \frac{3}{2}x_3 + x_4 = 0 \\ x_2 - \frac{7}{2}x_3 + 2x_4 = 0 \end{cases} \iff \begin{cases} x_1 + \frac{3}{2}x_3 - x_4 \\ x_2 = \frac{7}{2}x_3 - 2x_4 \end{cases}$



(注自由变量个数 = 2 = 4 - r(A))