2

Analytic hierarchy process

2.1 Introduction

This chapter explains the theory behind and practical uses of the analytic hierarchy process (AHP) method as well as its extensions. *MakeItRational*, a software package that helps to structure problems and calculate priorities using AHP, is described. Section 2.3 is designed for readers interested in the methodological background of AHP. Section 2.4 covers the extensions of AHP in group decision, sorting, scenarios with incomparability and large size problems.

The companion website provides illustrative examples with *Microsoft Excel*, and case studies and examples with *MakeItRational*.

2.2 Essential concepts of AHP

AHP was developed by Saaty (1977, 1980). It is a particularly useful method when the decision maker is unable to construct a utility function, otherwise MAUT is recommended (Chapter 4). To use AHP the user needs to complete four steps to obtain the ranking of the alternatives. As with any other MCDA method, the problem first has to be structured (Section 2.2.1). Following this, scores – or priorities, as they are known in AHP – are calculated based on the pairwise comparisons provided by the user (Section 2.2.2). The decision maker does not need to provide a numerical judgement; instead a relative verbal appreciation, more familiar to our daily live, is sufficient. There are two additional steps that can be carried out: a consistency check (Section 2.2.3) and a sensitivity analysis (Section 2.2.4). Both steps are optional but recommended as confirmation of the robustness of the results. The consistency check is common in all methods based on pairwise comparisons like AHP. The supporting software of *MakeItRational* facilitates the sensitivity analysis.

Multi-Criteria Decision Analysis: Methods and Software, First Edition. Alessio Ishizaka and Philippe Nemery.

2.2.1 Problem structuring

AHP is based on the motto *divide and conquer*. Problems that require MCDA techniques are complex and, as a result, it is advantageous to break them down and solve one 'sub-problem' at a time. This breakdown is done in two phases of the decision process during:

- the problem structuring and
- the elicitation of priorities through pairwise comparisons.

The problem is structured according to a hierarchy (e.g. Figure 2.2) where the top element is the goal of the decision. The second level of the hierarchy represents the criteria, and the lowest level represents the alternatives. In more complex hierarchies, more levels can be added. These additional levels represent the sub-criteria. In any case, there are a minimum of three levels in the hierarchy.

Throughout this chapter, a shop location problem (Case Study 2.1) will be considered to illustrate the different steps of the AHP process.

Case Study 2.1

A businessman wants to open a new sports shop in one of three different locations:

(a) **A shopping centre.** The shopping centre has a high concentration of a variety of shops and restaurants. It is a busy area, with a mix of customers and people walking around. Shops regularly use large displays and promotions to attract potential customers. As demand for these retail units is low, the rental costs are reasonable.

(b) **The city centre.** The city centre is a busy area, and a meeting point for both young people and tourists. Attractions such as dance shows, clowns and market stalls are often organized, which attract a variety of visitors. The city centre has several small shops located at ground level in historical buildings, which suggests high rental costs. These shops have a high number of customers and are often in competition.

(c) **A new industrial area.** The new industrial estate is in the suburbs of the city, where several businesses have recently been set up. Some buildings have been earmarked for small shops, but on the whole it has been difficult to attract tenants, which means that rental costs are currently low. Customers of the existing shops mainly work in the area and only a few customers come from the surrounding towns or cities to shop here.

Given the description of the problem, four criteria will be considered in making the final decision (Table 2.1).

Table 2.1 Criteria for shop location decision.

Criterion	Explanation
Visibility	Probability that a random passer-by notices the shop
Competition	Level of competition in the area
Frequency	Average number of customers in similar shops in the area
Rental cost	Average rental cost by square metre

Figure 2.1 represents the hierarchy of Case Study 2.1. It has three levels, the minimum required to solve a problem with AHP. Other sub-criteria could be considered, for example, the competition criterion could be broken down into two sub-criteria: direct and indirect competition. Direct competition would be the number of other sports shops. Indirect competition would represent other types of shop, which could distract potential customers. To keep the example simple, additional levels will not be considered at this stage.

Each lower level is prioritized according to its immediate upper level. The appropriate question to ask with regard to prioritization depends on the context and sometimes on the decision maker. For example, in order to prioritize the criteria of level 2 with regard to the goal 'location of a sports shop', an appropriate question would be: 'Which criterion is most important for choosing the location of the sports shop and to what extent?' On the other hand, the alternatives in level 3 must be prioritized with regard to each criterion in level 2. In this case, an appropriate question would

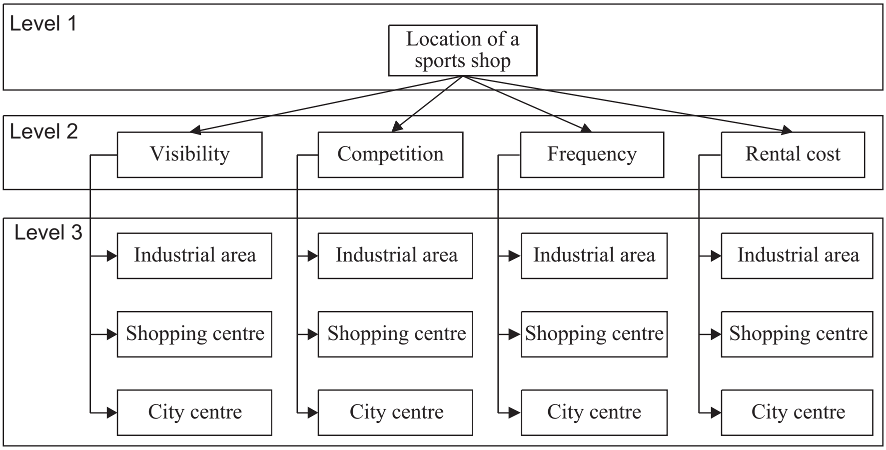

Figure 2.1 Hierarchy of decision levels for Case Study 2.1.

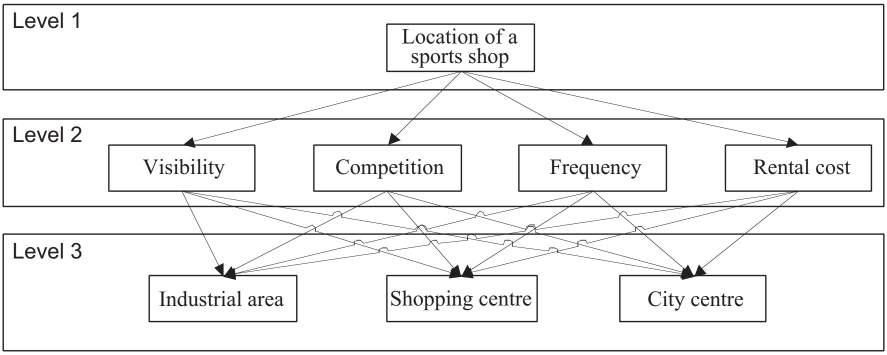

Figure 2.2 Traditional representation of the hierarchy.

be: ‘Which alternative is preferable to fulfil the given criterion and to what extent?’ In Case Study 2.1, five different prioritizations are required:

- four local prioritizations of alternatives with regard to each criterion and
- one criteria prioritization.

The aggregation of the local and criteria prioritizations leads to global prioritizations.

As Figure 2.1 contains redundant information at the lowest level, the alternatives in the hierarchy are often not repeated or are connected as in Figure 2.2.

2.2.2 Priority calculation

A priority is a score that ranks the importance of the alternative or criterion in the decision. Following the problem-structuring phase (see Section 2.2.1), three types of priorities have to be calculated:

- **Criteria priorities**. Importance of each criterion (with respect to the top goal).
- **Local alternative priorities**. Importance of an alternative with respect to one specific criterion.
- **Global alternative priorities**. Priority criteria and local alternative priorities are intermediate results used to calculate the global alternative priorities. The global alternative priorities rank alternatives with respect to all criteria and consequently the overall goal.

The criteria and local alternatives priorities are calculated using the same technique. Instead of directly allocating performances to alternatives (or criteria) as in the other techniques from the American school (see MAUT, Chapter 4), AHP uses

Table 2.2 The 1–9 fundamental scale.

Degree of importance	Definition
1	**Equal importance**
2	Weak
3	**Moderate importance**
4	Moderate plus
5	**Strong importance**
6	Strong plus
7	**Very strong or demon-strated importance**
8	Very, very strong
9	**Extreme importance**

pairwise comparisons. Psychologists often use this technique (Yokoyama 1921; Thurstone 1927), for example, to evaluate the food preference of a cat by presenting two dishes at a time. The cat indicates its preference by eating one dish. The psychologists argue that it is easier and more accurate to express a preference between only two alternatives than simultaneously among all the alternatives. The use of pairwise comparisons (called paired comparisons by psychologists) is generally evaluated on the fundamental 1–9 scale. The conversion from verbal to numerical scale is given in Table 2.2. Psychologists suggest that a smaller scale, say 1–5, would not give the same level of detail in a data set, and that the decision maker would be lost in a larger scale: for example, on a scale of 1–100, it is difficult for the decision maker to distinguish between a score of 62 and 63. In practice, there is no fixed rule and other scales have been proposed (Section 2.4.2).

The comparisons are collected in a matrix (Example 2.1).

Example 2.1 The comparison matrix in Figure 2.3 gathers the pairwise comparisons between the criteria. All comparisons are positive. The comparisons on the main diagonal are 1 because a criterion is compared with itself. The matrix is reciprocal because the upper triangle is the reverse of the lower triangle, for example visibility is 1/4 as important as competition and competition is 4 times as important as visibility.

The advantage of precision requires more effort, especially when there are a large number of criteria or alternatives. The number of necessary comparisons for each comparison matrix is

$$\frac{n^2 - n}{2} \tag{2.1}$$

	Visibility	Competition	Frequency	Rental costs
Visibility	**1**	1/4	1/5	2
Competition	4	**1**	1/2	1
Frequency	5	2	**1**	4
Rental costs	1/2	1	1/4	**1**

Figure 2.3 Comparison matrix.

where n is the number of alternatives/criteria. This formula can be explained as follows:

- n^2 is the total number of comparisons that can be written in a matrix.
- n of these represent the comparison of the alternative with itself (on the main diagonal). The evaluation is 1 and therefore not required (shown in bold in Figure 2.3).
- As the matrix is reciprocal, only half of the comparisons are required. The other half are automatically calculated from the first half.

For example, in Figure 2.3 we have $n = 4$, therefore the number of comparisons to provide is $(4^2 - 4)/2 = 6$.

Even though the squared number is reduced by n and divided by 2, the required number of comparisons can be very high. For example, 10 alternatives lead to 45 questions for each criterion. The effort required to complete the matrix is time-consuming and can be discouraging. In Section 2.5.3, ways to deal with this quadratic increase in the number of comparisons will be discussed.

From these comparison matrices, the software will calculate the local and criteria priorities; see Section 2.4.4, where the calculation of these priorities is explained. Finally, it aggregates these two priorities to establish the global priority. Priorities only make sense if they are derived from consistent or near-consistent matrices, and as a result a consistency check must be performed, to which we now turn.

2.2.3 Consistency check

When the matrix is complete, a consistency check may be performed to detect possible contradictions in the entries. When several successive pairwise comparisons are presented, they may contradict each other. The reasons for these contradictions could be, for example, vaguely defined problems, a lack of sufficient information (known as bounded rationality), uncertain information or lack of concentration. Suppose that the decision maker, as an example, gives the following pairwise comparisons:

- The shopping centre is **two times** more visible than the city centre.
- The city centre is **three times** more visible than the industrial area.
- The industrial area is **four times** more visible than the shopping centre.

The third assertion is inconsistent as determined from the two first assertions; the industrial area is six times more visible than the shopping centre (2×3). Human nature is often inconsistent, for example, in football it is possible for the team at the top of the table to lose against the team at the bottom of the table. To allow this inconsistent reality, AHP allows up to a 10% inconsistency compared to the average inconsistency of 500 randomly filled matrices. A calculation is done by the supporting software and indicates if a matrix needs to be reconsidered due to its high inconsistency (a detailed description is available in the black box Section 2.4.3).

2.2.4 Sensitivity analysis

The last step of the decision process is the sensitivity analysis, where the input data is slightly modified to observe the impact on the results. As complex decision models are often inherently ill defined, the sensitivity analysis allows different scenarios to be generated. These different scenarios may result in other rankings, and further discussion may be needed to reach a consensus. If the ranking does not change, the results are said to be *robust* – otherwise they are sensitive. The sensitivity analysis in *MakeItRational* is performed by varying the weight of the criteria and observing the impact on the global alternative priority.

Exercise 2.1

The following multiple-choice questions test your knowledge of the basics of AHP. Only one answer is correct. Answers can be found on the companion website.

1. What does AHP stand for?
    a) Analytic Hierarchy Program
    b) Analytic Hierarchy Process
    c) Analytic Hierarchical Programming
    d) Analytical Hierarchy Partitioning
2. What is the typical Saaty scale?
    a) A 1–5 scale
    b) A 1–9 scale
    c) A 1–10 scale
    d) A 1–100 scale
3. What is the main purpose of AHP?
    a) AHP prioritizes alternatives based on criteria and constraints
    b) AHP assigns goals to alternatives

   c) AHP prioritizes alternatives based on criteria
   d) AHP assigns criteria to alternatives

4. Pairwise comparisons in AHP are based on which scale?
   a) Ratio scale
   b) Interval scale
   c) Ordinal scale
   d) Nominal scale

5. How many pairwise comparisons are required to rank five criteria?
   a) 25
   b) 20
   c) 15
   d) 10

2.3 AHP software: MakeItRational

The available AHP software has greatly contributed to the success of the AHP method. The software incorporates intuitive graphical user interfaces, automatic calculation of priorities and inconsistencies, and provides several ways of processing a sensitivity analysis (Ishizaka and Labib 2009). At the time of writing there are several software packages: Expert Choice, Decision Lens, HIPRE 3+, RightChoiceDSS, Criterium, EasyMind, Questfox, ChoiceResults and 123AHP, as well as the option of adapting a template in *Microsoft Excel* (e.g. see Exercise 2.3).

This section describes the AHP web software *MakeItRational*, available from http://makeitrational.com. This software was chosen because of its simplicity and the free trial version available (Kaspar and Ossadnik 2013). The disadvantage of the free version is that models cannot be saved, but as *MakeItRational* is an online software package, it is automatically updated. Data is stored on the web server, although a server edition can be purchased, which allows the data to be saved on computer. When using *MakeItRational*, it is not necessary to know *how* priorities are calculated, only *what* should be ranked. This section describes the graphical user interface. The four steps introduced in Section 2.2 will be followed by navigating between the top tabs (Figure 2.5) of the software.

2.3.1 Problem structuring

For problem structuring, three tabs are necessary.

- Project tab: Name the project (this is needed to save it) and enter a description (optional).

- Alternatives tab: Enter a minimum of two alternatives.
- Criteria tab: Enter a minimum of two criteria.

2.3.2 Preferences and priority calculation

The Evaluation tab in Figure 2.4 displays the pairwise comparisons needed to calculate the priorities. The user first has to select the Goal in the left panel, and the right panel will ask for pairwise evaluations of the criteria. For example, in Figure 2.4, *Competition* has been evaluated as twice as important as *Frequency*. When this step is complete, the user will need to select the first criterion from the left panel, where again the right panel will ask for pairwise comparisons. In Figure 2.5, the *City centre* has been evaluated as 5 times as important as *the Industrial area* with regard to *Competition*. This process is repeated for each criterion.

MakeItRational allows a direct rating of the alternatives/criteria if they are already known. For example, in Figure 2.6, the exact frequency of people per hour for each alternative is known; therefore the precise amount can be entered. Note that the criterion needs to be maximized. For criteria to minimize, the score needs to be inverted, for example x becomes $1/x$. If all evaluation preferences are rated directly, then the weighted sum is used. In this case, the decision maker needs to know the utility function either implicitly or explicitly. *MakeItRational* is able to support both methods because they share several common features.

Priorities will be automatically calculated by *MakeItRational* after a consistency check.

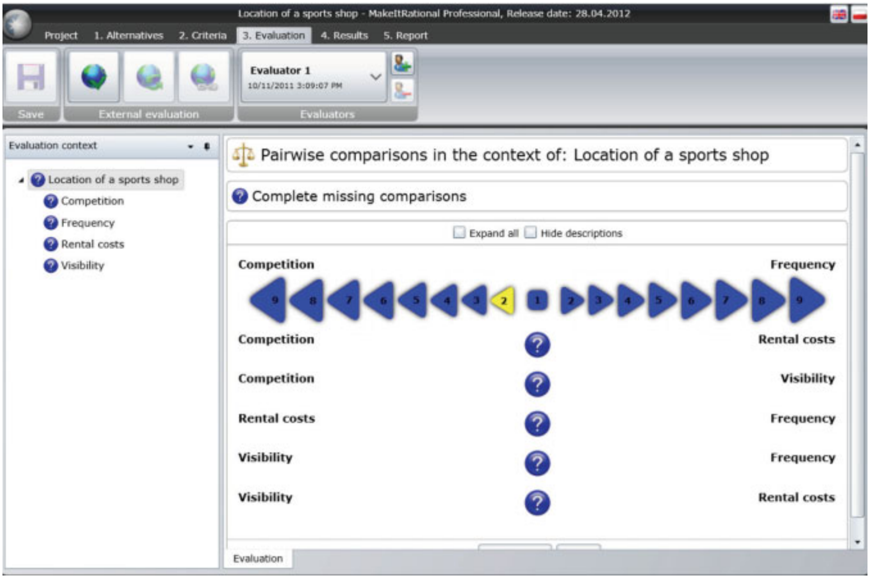

Figure 2.4 Pairwise comparisons of criteria in MakeItRational. *Reproduced by permission of BS Consulting Dawid Opydo.*

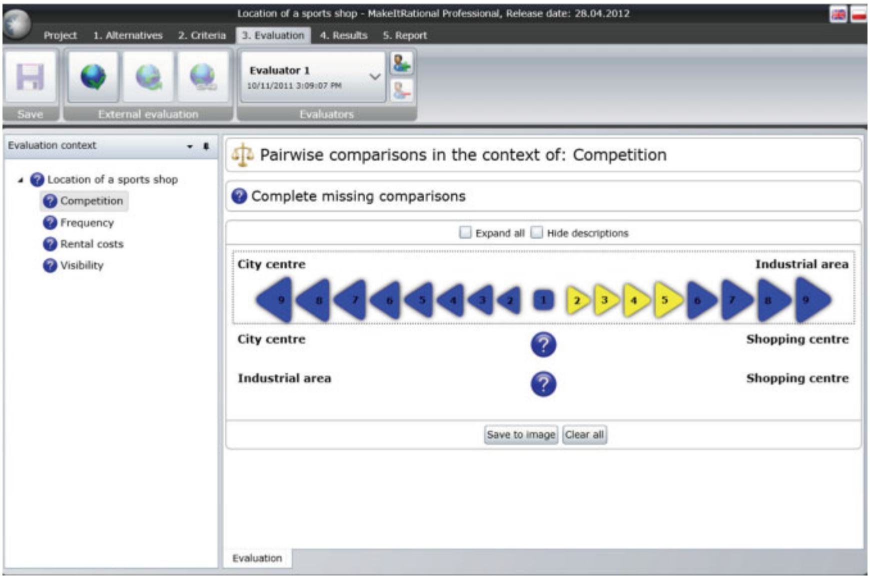

Figure 2.5 Pairwise comparisons of alternatives in MakeItRational. *Reproduced by permission of BS Consulting Dawid Opydo.*

2.3.3 Consistency check

MakeItRational has various consistency checks represented by the icons on the left pane of the tab (Figure 2.6). Table 2.3 explains the status.

- The *Complete* status means that all pairwise comparisons have been consistently entered.
- In the *Enough* status, not all pairwise comparisons are entered but those provided can be used to estimate the missing ones (Section 2.4.4.1). This status can be used when a large number of alternatives are evaluated in order to decrease the number of required pairwise comparisons. Therefore, comfortable comparisons should be entered first.

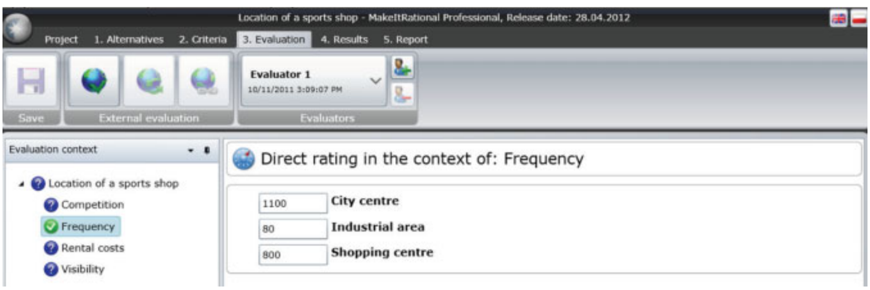

Figure 2.6 Direct rating of alternatives in MakeItRational. *Reproduced by permission of BS Consulting Dawid Opydo.*

Table 2.3 Preference status. Reproduced by permission of BS Consulting Dawid Opydo.

Icon	Status	Description
	Complete	All judgements in the context of this criterion have been provided. The entered pairwise comparisons are consistent (CR $<$ 10%).
	Enough	There are some empty judgements in the context of this criterion but weights/scores can be calculated.
	Inconsistency	The entered pairwise comparisons are inconsistent (CR $>$ 10%).
	Contradictory	Contradictory pairwise comparisons in the context of this criterion.
	Missing	There is not enough data to calculate weights/scores.
	Error	The decision problem contains only one criterion or one alternative.

- The *Inconsistency* status recommends the revision of pairwise comparisons in order to decrease inconsistency. The consistency ratio (CR) should be lower than 10% to be considered acceptable. *MakeItRational* will recommend which comparison to modify. For example, in Figure 2.7, the most inconsistent comparison is between visibility and rental costs. *MakeItRational* recommends modifying the comparison to 4.

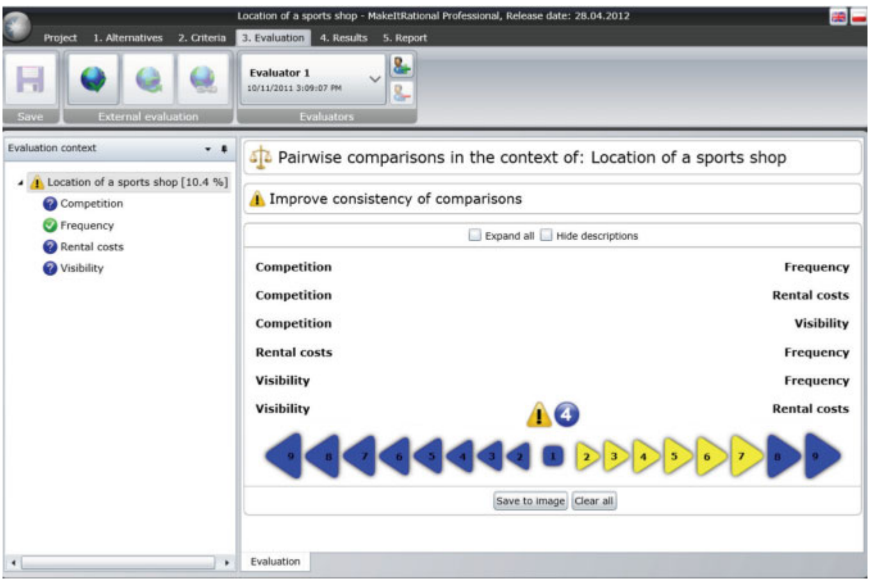

Figure 2.7 Inconsistency and recommended comparisons. Reproduced by permission of BS Consulting Dawid Opydo.

- The *Contradictory* status indicates logically impossible cardinal preferences. For example, I prefer the *shopping centre* to the *city centre*, I prefer the *city centre* to the *industrial area*, and I prefer the *industrial area* to the *shopping centre*, which induces an impossible preference cycle:

$$\textit{shopping centre} > \textit{city centre} > \textit{industrial area} > \textit{shopping centre}.$$

- The *Missing* status indicates that not enough data has been provided to calculate priorities.
- The *Error* status indicates an error in the problem structuring: a criterion contains only one sub-criterion or the problem contains only one alternative.

Priorities will be calculated for the first four matrix statuses, but it is strongly recommended to revise pairwise comparisons for the *Inconsistency* and *Contradictory* status.

2.3.4 Results

Figure 2.8 shows the global priorities of the alternatives with regard to the goal 'Location selection for a sports shop'. The results are displayed with scores and stacked bar diagrams for better visualization. It can be seen that the city centre is the preferred alternative, especially because of its high frequency. In the chart data

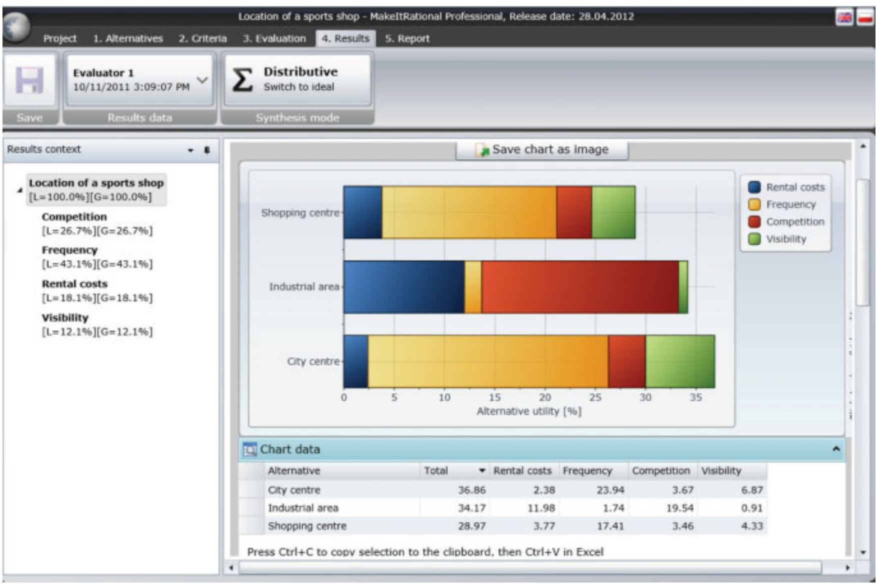

Figure 2.8 Global priorities in MakeItRational. *Reproduced by permission of BS Consulting Dawid Opydo.*

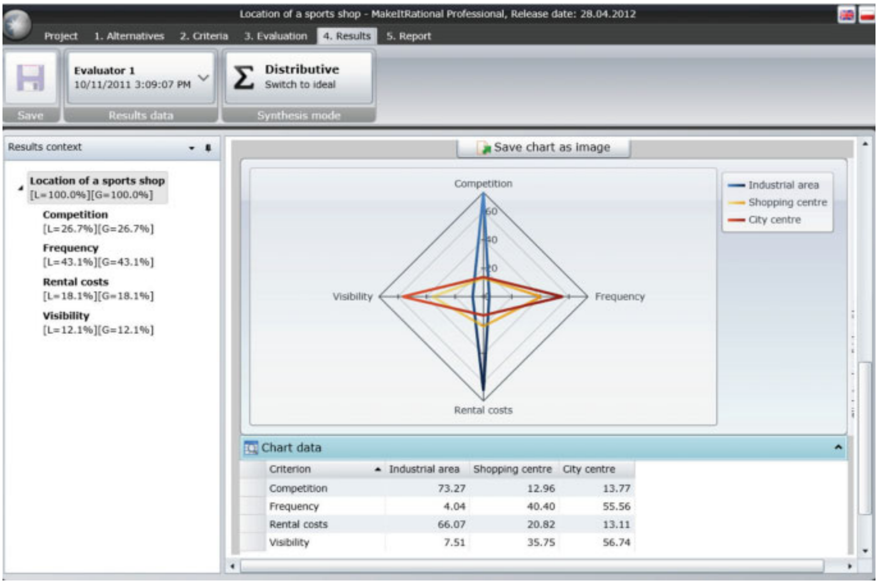

Figure 2.9 Local priority in MakeItRational. *Reproduced by permission of BS Consulting Dawid Opydo.*

of Figure 2.8, it can be seen that frequency contributes 23.94 towards the total score of 36.86.

MakeItRational allows the local (Figure 2.9) and criteria priorities (Figure 2.10) to be seen.

Figure 2.9 displays the unweighted local priorities in a spider diagram and the scores in the table immediately below. This representation allows a visualization of the strengths and weaknesses of each alternative. In this case, the industrial area is very strong on the competition criterion but very weak on the visibility and frequency criteria. The shopping centre scores very high on the frequency and visibility criteria. Figure 2.10 displays the criteria priorities in a pie chart and the scores in the table beneath it.

2.3.5 Sensitivity analysis

On the same Results tab, a sensitivity analysis in *MakeItRational* allows the impact of the changes of one criterion weight over the global priority to be seen. For example, in Figure 2.11, if the current rental costs weight of 18.1% is increased to over 22.35%, then the preferred alternative is no longer the city centre but the industrial area.

Finally, the results can be collected in a report and downloaded in different formats. An example of a report can be downloaded from the companion website (Report_MakeItRational.pdf).

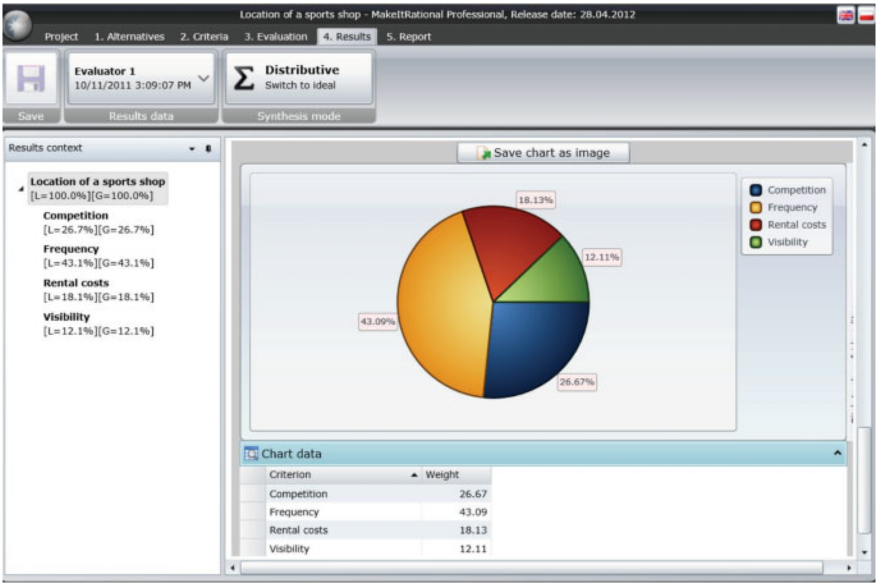

Figure 2.10 Criterion priorities in MakeItRational. *Reproduced by permission of BS Consulting Dawid Opydo.*

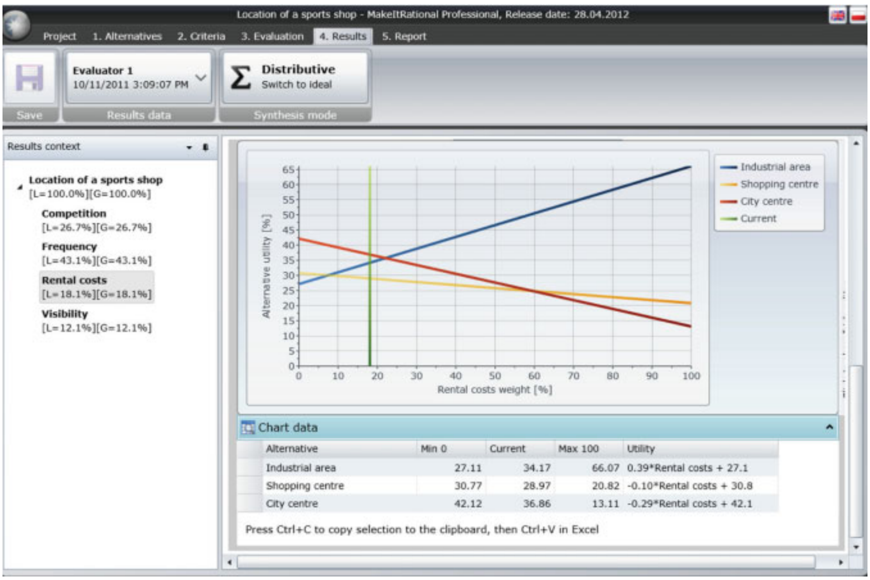

Figure 2.11 Sensitivity analysis in MakeItRational. *Reproduced by permission of BS Consulting Dawid Opydo.*

Exercise 2.2

In this exercise, the sports shop problem in Case Study 2.1 will be solved with the *MakeItRational* software.

Learning Outcomes

- Structure a problem in *MakeItRational*
- Enter pairwise comparisons
- Understand the results
- Conduct a sensitivity analysis

Tasks

a) Open the webpage http://makeitrational.com/demo. The free version has the full functionalities but the problem cannot be saved.

b) Read the description of Case Study 2.1, on page 14.

c) Give your decision project a name (*Project* tab).

d) Enter the alternatives (*Alternatives* tab).

e) Enter the criteria (*Criteria* tab).

f) Enter the pairwise comparisons (*Evaluation* tab). Are they consistent?

g) Read your global ranking and conduct a sensitivity analysis (*Results* tab).

2.4 In the black box of AHP

2.4.1 Problem structuring

In most cases, the problem is not as well defined as in Case Study 2.1. The decision maker may have a vague idea of wanting to open a shop but without knowing the precise alternatives and criteria. A structure must be formed through brainstorming sessions, analysing similar problem studies and organizing focus groups etc. Saaty and Forman (1992) have written a book describing hierarchical structures in various AHP applications, which may be of use in the structuring process.

This hierarchization of decision elements is important because a different structure may lead to a different final ranking. Several authors (Pöyhönen et al. 1997; Stillwell et al. 1987; Weber et al. 1988) have observed that criteria with a large number of sub-criteria tend to receive more weight than when they are less detailed.

Table 2.4 Food and drink quantities in two menus.

	Food [kg]	Drinks [l]
Menu A	0.80	1
Menu B	2	0.5

2.4.2 Judgement scales

The use of verbal comparisons (Table 2.2) is intuitively appealing, user-friendly and more common in our everyday lives than numbers. It may also allow for some fuzziness in difficult comparisons – a verbal comparison is not as precise as a number. However, this ambiguity in the English language has also been criticized (Donegan et al. 1992).

AHP, due to its pairwise comparisons, needs ratio scales, which, contrary to methods using interval scales (Kainulainen et al. 2009), require no units of comparison. The judgement is a relative value or a quotient a / b of two quantities a and b having the same units (intensity, utility, etc.). Barzilai (2005) claims that preferences cannot be represented with ratio scales, because in his opinion an absolute zero does not exist, for example, temperature or electrical tension. Similarly, Dodd and Donegan (1995) have criticized the absence of zero in the preference scale in Table 2.2. On the contrary, Saaty (1994a) states that ratio scales are the only possibility for aggregating measurements in a commensurate (i.e. same units) way (Example 2.2).

Example 2.2 Consider two lunch menus evaluated on two criteria of the quantity of food and quantity of drinks (Table 2.4). The food quantity is considered twice as important as the drinks quantity. The two menus can be compared on a ratio scale:

$$\frac{\text{Menu B}}{\text{Menu A}} = 2 \cdot \frac{2}{0.8} + \frac{0.5}{1} = 5.5.$$

Therefore, menu B is five and half times as good as menu A. On an interval scale,

$$\text{Menu B} - \text{Menu A} = 2 \cdot (2 - 0.8) + (0.5 - 1) = 1.9.$$

This result also indicates that menu B is better.

However, if the scale is changed from litres to decilitres, the results change for the interval scale,

$$\text{Menu B} - \text{Menu A} = 2 \cdot (2 - 0.8) + (5 - 10) = -2.6,$$

but not for the ratio scale,

$$\frac{\text{Menu B}}{\text{Menu A}} = 2 \cdot \frac{2}{0.8} + \frac{5}{10} = 5.5.$$

In order to correct this change, the weights should be adjusted as well:

$$\text{Menu B} - \text{Menu A} = 2 \cdot (2 - 0.8) + 0.1\ (5 - 10) = 1.9.$$

If mathematically the weight adjustment is feasible, can the decision maker be expected to adjust the weight preferences when a change on the scale is adopted?

To derive priorities, verbal comparisons must be converted to numerical ones. In Saaty's AHP the verbal statements are converted into integers 1–9. Theoretically, there is no reason to be restricted to these numbers and this verbal gradation. If the verbal gradation has been little investigated, various other numerical scales have been proposed (Table 2.5, Figure 2.12 and Figure 2.13). Harker and Vargas (1987) evaluated a quadratic and a square root scale in only one simple example and argued in favour of Saaty's 1–9 scale. However, one example is not enough to conclude the superiority of the 1–9 linear scale. Lootsma (1989) argued that the geometric scale is preferable to the 1–9 linear scale. Salo and Hämäläinen (1997) point out that the integers 1–9 yield local weights that are unevenly dispersed so that there is lack of sensitivity

Table 2.5 Different scales for comparing two alternatives.

Scale types	Equal importance	Weak	Moderate importance	Moderate plus	Strong importance	Strong plus	Very strong importance	Very very strong	Extreme importance
Linear (Saaty 1977)	1	2	3	4	5	6	7	8	9
Power (Harker and Vargas 1987)	1	4	9	16	25	36	49	64	81
Geometric (Lootsma 1989)	1	2	4	8	16	32	64	128	256
Logarithmic (Ishizaka et al. 2006)	1	1.58	2	2.32	2.58	2.81	3	3.17	3.32
Square root (Harker and Vargas 1987)	1	1.41	1.73	2	2.23	2.45	2.65	2.83	3
Asymptotical (Dodd and Donegan 1995)	0	0.12	0.24	0.36	0.46	0.55	0.63	0.70	0.76
Inverse linear (Ma and Zheng 1991)	1	1.13	1.29	1.5	1.8	2.25	3	4.5	9
Balanced (Salo and Hämäläinen 1997)	1	1.22	1.5	1.86	2.33	3	4	5.67	9

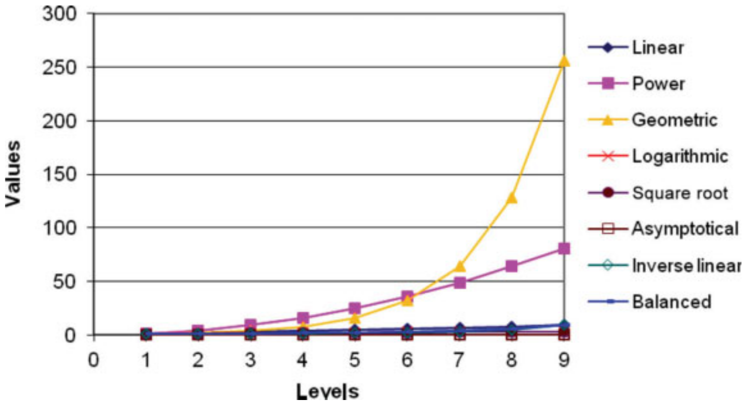

Figure 2.12 Graph of judgement scales.

when comparing elements which are preferentially close to each other. Based on this observation, they propose a balanced scale where the local weights are evenly dispersed over the weight range [0.1, 0.9]. Earlier, Ma and Zheng (1991) calculated a scale where the inverse elements x of the scale $1/x$ are linear instead of the x in the Saaty scale. Donegan et al. (1992) proposed an asymptotic scale avoiding the boundary problem (e.g. if the decision maker enters the pairwise comparison $a_{ij} = 3$ and $a_{jk} = 4$, they are forced into an intransitive relation because the upper limit of the scale is 9 and they cannot enter $a_{ik} = 12$). Ji and Jiang (2003) propose a mixture of verbal scale and geometric scale. The possibility of integrating negative values into the scale has also been explored (Millet and Schoner 2005; Saaty and Ozdemir 2003a).

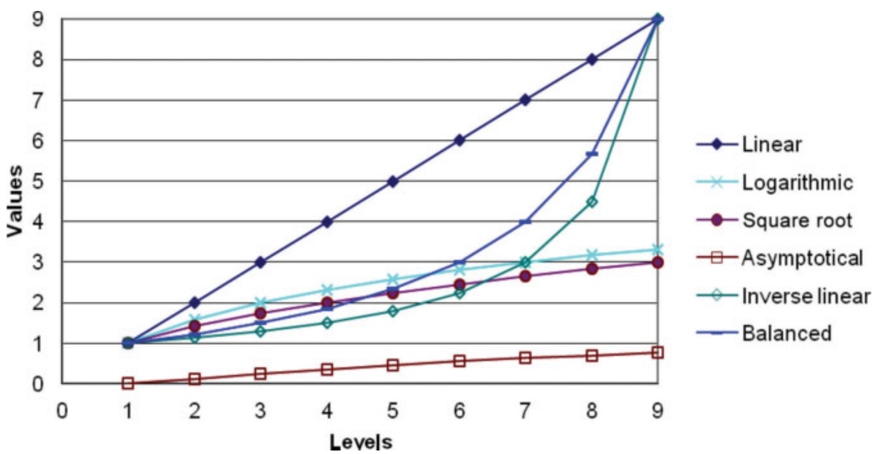

Figure 2.13 Graph of the judgement scales with the geometric and power scales omitted.

Figure 2.12 and Figure 2.13 represent graphically the different scales of Table 2.5. Large differences are noted, which imply different final results (Ishizaka et al. 2010).

Among the proposed scales, the linear scale with the integers 1–9 and their reciprocals have been used most often in applications. It is also the only one implemented in *Expert Choice* and *MakeItRational*. Saaty (1980, 1991) advocates it as the best scale to represent weight ratios. However, the cited examples deal with objective measurable alternatives like the areas of figures, whereas AHP treats mainly decision processes on subjective issues. It is technically much more difficult to verify the effectiveness of scales through subjective issues. Salo and Hämäläinen (1997) demonstrate the superiority of the balanced scale when comparing two elements. The choice of the 'best' scale is a hotly debated issue. Some scientists argue that the choice depends on the person and the decision problem (Harker and Vargas 1987; Pöyhönen et al. 1997). Therefore, other scales would be welcomed in the AHP software.

2.4.3 Consistency

In Section 2.2.3, minimal consistency was necessary to calculate meaningful priorities. A matrix filled by the pairwise comparison a_{ij} is called consistent if the transitivity (2.2) and the reciprocity (2.3) rules are respected.

Transitivity Rule:

$$a_{ij} = a_{ik} \cdot a_{kj} \tag{2.2}$$

where a_{ij} is the comparison of alternative i with j.

Suppose a person likes an apple twice as much as an orange ($a_{12} = 2$) and an orange three times as much as a banana ($a_{23} = 3$). If the person likes an apple six times as much as a banana ($a_{13} = 6$), the transitivity rule is respected.

Reciprocity Rule:

$$a_{ij} = \frac{1}{a_{ji}} \tag{2.3}$$

where i, j and k are any alternatives of the matrix.

If a person likes an apple twice as much as an orange ($a_{12} = 2$), then they like an orange half as much as an apple ($a_{21} = 1/2$).

If we suppose that preferences p_i are known, a perfectly consistent matrix

$$\mathbf{A} = \begin{bmatrix} p_1/p_1 & \cdots & p_1/p_j & \cdots & p_1/p_n \\ \cdots & 1 & \cdots & \cdots & \cdots \\ p_i/p_1 & \cdots & 1 & \cdots & p_i/p_n \\ \cdots & \cdots & \cdots & 1 & \cdots \\ p_n/p_1 & \cdots & p_n/p_j & \cdots & p_n/p_n \end{bmatrix} \tag{2.4}$$

can be constructed because all the comparisons a_{ij} obey the equality

$$a_{ij} = \frac{p_i}{p_j}, \tag{2.5}$$

where p_i is the priority of the alternative i.

Priorities are not known in advance in AHP. As priorities only make sense if derived from consistent or near-consistent matrices, a consistency check must be applied. The threshold for defining an intolerably inconsistent matrix is not clear. Several methods have been proposed to measure consistency. Peláez and Lamata (2003) describe a method based on the determinant of the matrix. Crawford and Williams (1985) prefer to add the difference between the ratio of the calculated priorities and the given comparisons. The transitivity rule (2.2) was used by Salo and Hämäläinen (1997) and later by Ji and Jiang (2003). Stein and Mizzi (2007) use the normalized column of the comparison matrix. However, the most commonly used method (including in *MakeItRational*) was developed by Saaty (1977), who proposed a consistency index (CI), which is related to the eigenvalue method (Section 2.4.4):

$$\mathrm{CI} = \frac{\lambda_{\max} - n}{n - 1}, \tag{2.6}$$

where $\lambda_{\max}$ is the maximal eigenvalue. The consistency ratio (CR) is given by

$$\mathrm{CR} = \mathrm{CI}/\mathrm{RI}, \tag{2.7}$$

where RI is the random index (the average CI of 500 randomly filled matrices). If CR is less than 10% (the inconsistency is less than 10% of 500 randomly filled matrices), then the matrix is of an acceptable consistency.

Saaty (1977) calculated the random indices shown in Table 2.6. Other researchers have run simulations with different numbers of matrices (Alonso and Lamata 2006; Lane and Verdini 1989; Tummala and Wan 1994) or incomplete matrices (Forman 1990). Their random indices are different than but close to Saaty's.

Alonso and Lamata (2006) have computed a regression of the random indices and propose the formulation

$$\lambda_{\max} < n + 0.1(1.7699n - 4.3513). \tag{2.8}$$

Table 2.6 Random indices from Saaty (1977).

n	3	4	5	6	7	8	9	10
RI	0.58	0.9	1.12	1.24	1.32	1.41	1.45	1.49

For all proposed consistency checking methods, some questions remain: where is the cut-off to declare the matrix inconsistent? Should this rule depend on the size of the matrix? How should the consistency definition be adapted when using another judgement scale?

2.4.4 Priorities derivation

Priorities derivation is the corner-stone of the mathematics behind the AHP method, otherwise rankings could not be produced. Various methods have been proposed to calculate priorities from a pairwise comparison matrix (Lin 2007; Cho and Wedley 2004). In this section, three methods will be introduced: an approximate method, the eigenvalue method and the geometric mean. The approximate method requires only additions and averages. The eigenvalue method calculates not only the priorities but also the degree of inconsistency. The geometric mean has been proposed to solve the problem of the right–left rank reversal of the eigenvalue method. Each method calculates identical priorities when matrices are consistent.

2.4.4.1 Approximate method

This method is based on two simple steps:

1. Summation of the elements of row i:

$$r_i = \sum_i a_{ij}. \tag{2.9}$$

2. Normalization of the sums:

$$pi = \frac{r_i}{\sum_i r_i}. \tag{2.10}$$

Example 2.3 Suppose the decision maker has provided the comparisons in Table 2.7. The two steps are thus as follows:

1. The rows are summed as in the final column of Table 2.7.
2. Normalization the sums gives the criteria priorities as in Table 2.8

Table 2.7 Sum of the elements of the rows.

	Industrial area	Shopping centre	City centre	Total
Industrial area	1	6	2	9.00
Shopping centre	1/6	1	1/2	1.67
City centre	1/2	2	1	3.50

Table 2.8 Criteria priorities.

	Industrial area	Shopping centre	City centre	Total
Industrial area	1	6	2	0.64
Shopping centre	1/6	1	1/2	0.12
City centre	1/2	2	1	0.25

The approximate method does not calculate the consistency of the matrices. Therefore, all the AHP software packages that have been tested prefer to use a more intensive calculation method, which allows the calculation of the inconsistency rate.

2.4.4.2 Eigenvalue method

In the eigenvalue method, the vector of the priorities **p** is calculated by solving the equation

$$\mathbf{A}\mathbf{p} = n\mathbf{p} \tag{2.11}$$

where n is the dimension of $\mathbf{A}$ and $\mathbf{p} = (p_1, \ldots, p_j, \ldots, p_n)$.

First, the validity of the eigenvalue method on a consistent matrix $\mathbf{A}$ is demonstrated. Let us suppose that the priorities are known. It is easy to deduce a consistent comparison matrix from the priorities as follows. Let $a_{ij} = p_i/p_j$. Multiplying $\mathbf{A}$ by the priority vector $\mathbf{p}$ gives the right-hand side of equation (2.11). To simplify the calculation, only row i of $\mathbf{A}$ is first considered:

$$\frac{p_i}{p_1}p_1 + \frac{p_i}{p_2}p_2 + \ldots + \frac{p_i}{p_j}p_j + \ldots + \frac{p_i}{p_n}p_n = p_i + p_i + \ldots + p_i + \ldots + p_i = np_i$$

or

$$\sum_j \frac{p_i}{p_j} \cdot p_j = np_i. \tag{2.12}$$

Thus the product of row i by the priority vector $\mathbf{p}$ gives n times the priority p_i. By multiplying all the elements of the comparison matrix $\mathbf{A}$ by the priority vector $\mathbf{p}$, the following equality is obtained:

$$\begin{bmatrix} p_1/p_1 & p_1/p_2 & \cdots & p_1/p_n \\ p_2/p_1 & p_2/p_2 & \cdots & p_2/p_n \\ \cdots & \cdots & \cdots & \cdots \\ p_n/p_1 & p_n/p_2 & \cdots & p_n/p_n \end{bmatrix} \begin{bmatrix} p_1 \\ p_2 \\ \cdots \\ p_n \end{bmatrix} = n \begin{bmatrix} p_1 \\ p_2 \\ \cdots \\ p_n \end{bmatrix},$$

Table 2.9 Consistent comparison matrix.

	Industrial area	Shopping centre	City centre
Industrial area	1	6	3
Shopping centre	1/6	1	1/2
City centre	1/3	2	1

which is equation (2.11). Therefore, for a consistent matrix, the priority vector is obtained by solving equation (2.11).

Example 2.4 The eigenvalue problem is illustrated by the comparison matrix in Table 2.9. Because the comparison matrix is consistent, the priorities can be calculated by solving (2.11):

$$\mathbf{Ap} = \begin{bmatrix} 1 & 6 & 3 \\ 1/6 & 1 & 1/2 \\ 1/3 & 2 & 1 \end{bmatrix} \mathbf{p} = 3.\mathbf{p}$$

The priority vector **p** is the solution of following the linear system:

$$\begin{aligned} 1 \quad \cdot p_1 + 6 \cdot p_2 + 3 \quad \cdot p_3 &= 3 \cdot p_1 \\ 1/6 \cdot p_1 + 1 \cdot p_2 + 1/2 \cdot p_3 &= 3 \cdot p_1 \\ 1/6 \cdot p_1 + 1 \cdot p_2 + 1/2 \cdot p_3 &= 3 \cdot p_1. \end{aligned}$$

Solving this system for the unknowns p_1, p_2 and p_3 results in

$$\mathbf{p} = \begin{bmatrix} p_1 \\ p_2 \\ p_3 \end{bmatrix} = \begin{bmatrix} 0.667 \\ 0.111 \\ 0.222 \end{bmatrix}.$$

For an inconsistent matrix, this relation is no longer valid, as the comparison between element i and j is not necessarily given by formula (2.5). Therefore, the dimension n is replaced by the unknown λ. The calculation of λ and **p**, from an equation such as $\mathbf{Ap} = \lambda\mathbf{p}$ is called, in linear algebra, an eigenvalue problem. Any value λ satisfying this equation is called an eigenvalue and **p** is its associated eigenvector.

According to the Perron theorem, a positive matrix has a unique positive eigenvalue. The non-trivial eigenvalue is called the maximum eigenvalue $\lambda_{\max}$. If $\lambda_{\max} = n$, then the matrix is perfectly consistent. Otherwise, the difference between $\lambda_{\max} - n$ is a measure of the inconsistency (Section 2.4.3).

Table 2.10 First iteration of the power method.

	Industrial area	Shopping centre	City centre	Sum of the row	Eigenvector
Industrial area	3	16	7	26	0.62
Shopping centre	0.583	3	1.333	4.916	0.12
City centre	1.333	7	3	11.333	0.27
				42.249	1

Is the eigenvalue still valid for inconsistent matrices? Saaty (1977, 1980) justifies the eigenvalue approach for slightly inconsistent matrices with perturbation theory, which says that slight variations in a consistent matrix imply only slight variations of the eigenvector and eigenvalue.

In order to calculate the eigenvector associated to the maximum eigenvalue, most software packages, including *MakeItRational*, use the power method, which is an iterative process:

1. The pairwise matrix is squared: $\mathbf{A}_{n+1} = \mathbf{A}_n \mathbf{A}_n$.
2. The row sums are then calculated and normalized. This is the first approximation of the eigenvector.
3. Using the matrix $\mathbf{A}_{n+1}$, steps 1 and 2 are repeated.
4. Step 3 is repeated until the difference between these sums in two consecutive priorities calculations is smaller than a given stop criterion.

Example 2.5 Considering the inconsistent matrix **B** of Table 2.7, the first iteration of the power method is given in Table 2.10.

The power method is not fully transparent, much less than the approximate method. Several articles have highlighted this irrationality. According to Johnson et al. (1979), the 'aggregation process embedded in the eigenvector ... is not fully understood mathematically'. Chu et al. (1979) say that 'the weighted least squares method ... is conceptually easier to understand than the eigenvalue method'.

If the matrix of Table 2.11 is considered, it is known that a comparison can be estimated indirectly by the transitivity rule (2.2). Using (2.2), the comparison a_{13} is calculated as follows:

$$
\begin{aligned}
a_{13} &= a_{11} \cdot a_{13} = 1 \cdot 2 = 2, \\
a_{13} &= a_{12} \cdot a_{23} = 1/7 \cdot 6 = 0.857, \\
a_{13} &= a_{13} \cdot a_{33} = 2 \cdot 1 = 2, \\
a_{13} &= a_{14} \cdot a_{43} = 1/2 \cdot 1 = 1/2.
\end{aligned}
$$

Table 2.11 Criteria comparison matrix with illustrative values.

	Visibility	Competition	Frequency	Rental cost
Visibility	1	1/7	2	1/2
Competition	7	1	6	2
Frequency	1/5	1/6	1	1
Rental cost	2	1/2	1	1

It can be seen that the estimations of comparison a_{13} are very different. The matrix in Table 2.11 is thus inconsistent.

By squaring the matrix, the power method combines all the estimates of a_{13} due to the scalar product of the first column and the third row of the matrix. In general, each element a'_{ij} of $\mathbf{A}^2$ is given by the sum $a'_{ij} = \sum_k a_{ik} \cdot a_{kj}$. Each additional squaring refines the estimation of the comparison. Therefore, the eigenvalue method is based on a procedure of averaging the direct and indirect estimations of the comparisons.

The eigenvalue method has a major drawback: the right–left inconsistency, which leads to a rank reversal phenomenon after an inversion of the scale, was discovered two years after the publication of the original AHP method (Johnson et al. 1979). If all comparisons are replaced by their reciprocal values (e.g., 5 becomes 1/5), then the derived ranking should logically also be reversed. This is not always the case for the eigen value method; however, if a matrix is perfectly consistent or of rank $n = 3$, then a rank reversal is impossible. For inconsistent matrices of rank $n \geq 4$, rankings can be different after a scale inversion. The following example illustrates this phenomenon. Consider the matrix in Table 2.12, asking 'Which alternative is most economical?'. The calculated priorities give the following ranking of the alternatives: D > B > C > A > E.

If the question is inverted: 'Which alternative is most expensive?', then the comparisons are simply inverted (Table 2.13). The calculated priorities give the following ranking of the alternatives: B > D > C > A > E. In this case, alternative B is preferred, but before it was alternative D. This rank reversal is due to the formulation of the problem (which is different from and independent of rank reversal due to the introduction or deletion of an alternative, discussed in Section 2.4.5).

Table 2.12 Comparison matrix.

	A	B	C	D	E	Priority	Rank
A	1	1/7	1/2	1/8	2	0.061	4
B	7	1	3	1	8	0.374	2
C	2	1/3	1	1/4	5	0.134	3
D	8	1	4	1	5	0.387	**1**
E	1/2	1/8	1/5	1/5	1	0.043	5

Table 2.13 Comparisons of Table 2.11 inverted.

	A	B	C	D	E	Priority	Rank
A	1	7	2	8	1/2	0.299	4
B	1/7	1	1/3	1	1/8	0.047	**1**
C	1/2	3	1	4	1/5	0.140	3
D	1/8	1	1/4	1	1/5	0.051	2
E	2	8	5	5	1	0.462	5

2.4.4.3 Geometric mean

In order to avoid the left–right rank reversal, Crawford and Williams (1985) adopted another approach by minimizing the multiplicative error (2.13):

$$a_{ij} = \frac{p_i}{p_j} e_{ij}, \tag{2.13}$$

where a_{ij} is the comparison between object i and j, p_i is the priority of object I, and e_{ij} is the error.

The multiplicative error is commonly accepted to be log-normally distributed (similarly, the additive error would be assumed to be normally distributed). The geometric mean,

$$p_i = \sqrt[n]{\prod_{j=1}^{n} a_{ij}}, \tag{2.14}$$

is the one, which minimizes the sum of these errors,

$$\min \sum_{i=1}^{n} \sum_{j=1}^{n} \left(1\text{n}(a_{ij}) - 1\text{n} \left(\frac{p_i}{p_j} \right) \right)^2. \tag{2.15}$$

The geometric mean (also sometimes known as logarithmic least squares method) can be easily calculated by hand and has been supported by a large segment of the AHP community (Aguarón and Moreno-Jiménez 2000, 2003; Barzilai 1997; Barzilai and Lootsma 1997; Budescu 1984; Escobar and Moreno-Jiménez 2000; Fichtner 1986; Leskinen and Kangas 2005; Lootsma 1993, 1996). Its main advantage is the absence of rank reversals due to right–left inconsistency: the row and column geometric means provide the same ranking in an inverse order (this is not necessarily the case with the eigenvalue method).

Example 2.6 The priorities of Table 2.12 are

$$p_1 = \sqrt[5]{1 \cdot \tfrac{1}{7} \cdot \tfrac{1}{2} \cdot \tfrac{1}{8} \cdot 2} = 0.447, \quad p_2 = 2.787,$$
$$p_3 = .0.964,\ p_4 = 2.759,\ p_5 = 0.302.$$

Note that these priorities are in in the reverse order compared to the priorities of Table 2.13: $p_1 = 2.237$; $p_2 = 0.359$; $p_3 = 1.037$, $p_4 = 0.362$, $p_5 = 3.314$. Therefore the geometric mean does not have any rank reversal due to an inversion of the scale.

2.4.5 Aggregation

The last necessary step is the synthesis of the local priorities across all criteria in order to determine the global priority. The historical AHP approach adopts an additive aggregation with normalization of the sum of the local priorities to unity. This type of normalization is called distributive mode. This additive aggregation is expressed as

$$P_i = \sum_j w_j \cdot p_{ij} \tag{2.16}$$

where P_i is the global priority of alternative i, p_{ij} is the local priority with regard to criterion j, and w_j is the weight of the criterion j.

If priorities are known, the distributive mode is the only approach that will retrieve these priorities. However, if a copy (Belton and Gear 1983), or near-copy (the pairwise comparison is almost the same as the original) (Dyer 1990b), of an alternative is introduced or removed (Troutt 1988), a rank reversal of the alternatives may occur. This phenomenon has been criticized by some (Dyer 1990a, 1990b; Holder 1990, 1991; Stam and Duarte Silva 2003) and accepted by others (Harker and Vargas 1987, 1990; Pérez 1995; Saaty 1986, 1990, 1991, 1994b, 2006). This rank reversal phenomenon is not unique to AHP but common to all additive models having a normalization step (Triantaphyllou 2001; Wang and Luo 2009). In fact, when the number of alternatives is changed, the denominator for the normalization is also changed, which implies a change of scale and possible rank reversal.

To avoid this rank reversal problem, priorities should be normalized by dividing them by the same denominator in any configuration of the problem for which the ideal mode was proposed. This normalization is done by dividing the score of each alternative by the score of the best alternative under each criterion.

When should the distributive or ideal mode be used? Millet and Saaty (2000) give some guidance on which normalization to use. If in a *closed system* (i.e. no alternative will be added or removed), then the distributive mode should be used. If in an *open system* (i.e. alternatives can be added or removed) and the preference is allowed for alternatives to be dependent on other alternatives (in other words, the rank reversal phenomenon is accepted), then the distributive mode is indicated. If we are in an open system and do not want other alternatives to affect the outcome, then the ideal mode is recommended.

Multiplicative aggregation has been proposed to prevent the rank reversal phenomenon observed in the distributive mode (Lootsma 1993; Barzilai and Lootsma 1997). In the notation of (2.16), this form of aggregation is expressed as

$$p_i = \prod_j p_{ij}^{w_j} \tag{2.17}$$

Multiplicative aggregation has non-linear properties allowing a superior compromise to be selected; this is not the case with additive aggregation (Stam and Duarte Silva 2003; Ishizaka et al. 2006, 2010). Vargas (1997) demonstrated that additive aggregation is the only way to retrieve exact weights of known objects. *MakeItRational* offers only the additive distributive and ideal mode of aggregation.

Exercise 2.3

First you will learn to calculate priorities for one criterion step by step. Then you will be given the opportunity to complete the spreadsheet for the other criteria.

Learning Outcomes

- Understand the calculation of priorities with the approximate method in *Microsoft Excel*
- Understand the calculation of priorities with the eigenvalue method in *Excel*
- Understand the calculation of priorities with the geometric mean method in *Excel*

Tasks

Open the file Sport Shop.xls. It contains three spreadsheets with the three different priority calculation methods.

Complete the following tasks:

a) Describe the meaning of each calculation cell and its formula. Read the comments in the red squares in case of difficulties.

b) The spreadsheets are incomplete because they calculate only one local alternative. Complete them in order to calculate the other local alternatives.

2.5 Extensions of AHP

In this section, four extensions of AHP are presented. The analytic hierarchy process ordering method was introduced to separately analyse criteria that have to be minimized and maximized. The group analytic hierarchy process is used for group decisions. The clusters and pivots technique is applied to large problems to reduce the number of pairwise comparisons. AHPSort is implemented to solve sorting problems.

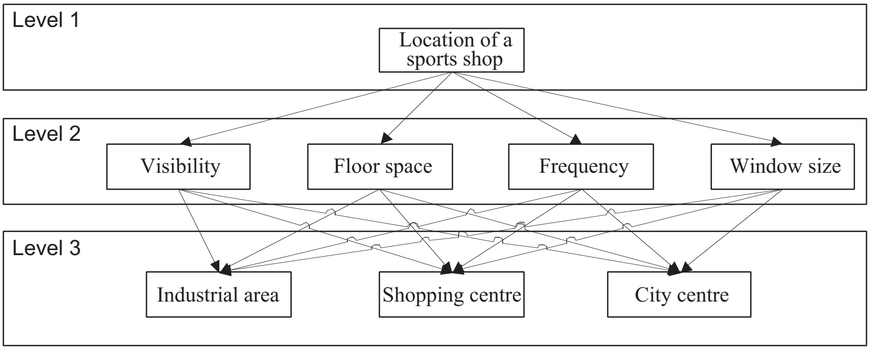

Figure 2.14 Benefit hierarchy.

2.5.1 Analytic hierarchy process ordering

The analytic hierarchy process ordering method, which considers incomparability, was first proposed by Ishizaka and Labib (2011). In line with the philosophy of AHP, some researchers have proposed deconstructing the model into sub-problems (Azis 1990; Clayton et al. 1993; Wedley et al. 2001). They propose separating the criteria in opposite directions in different hierarchies: benefits versus cost. The reason for this additional decomposition is that criteria in the same direction are much easier to compare than in opposite directions, such as a criterion to be minimized and another maximized.

For example, suppose that three more criteria are added to Case Study 2.1: vandalism, floor space and window size. We now have seven criteria in total. Some have to be maximized (visibility, floor space, frequency and window size) and others minimized (competition, rental costs, vandalism). Two hierarchies can be constructed: a benefit hierarchy (Figure 2.14) and a cost hierarchy (Figure 2.15). These two hierarchies are then solved separately. As a result the output of the method is a partial ranking (and not priorities).

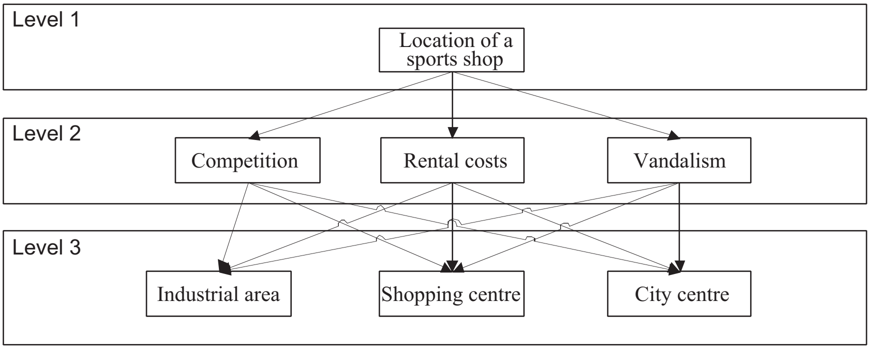

Figure 2.15 Cost hierarchy.

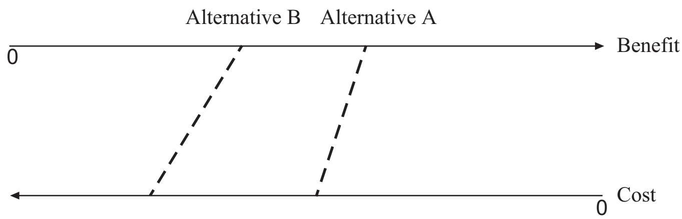

Figure 2.16 Graphical representation of the preference relation.

If cost and benefit rankings are not aggregated, we have a partial ordinal ranking (Ishizaka and Labib 2011). A partial ordinal ranking can be derived from the cost and benefit analysis, where:

1. Alternative A is *better than* alternative B if alternative A is ranked better than alternative B in the cost and benefit analysis (Figure 2.16).
2. Alternative A and alternative B are *indifferent* if alternative A has the same score as alternative B in the cost and benefit analysis (Figure 2.17).
3. Alternative A is *incomparable to* alternative B if alternative A is better in one analysis and worse in the other analysis (Figure 2.18).

Incomparability does not exist in standard AHP. This status is important as it reveals that a decision maker cannot decide which of the two alternatives is the best, while not being indifferent: an alternative is better in some aspects but worse in others. To decide which alternative is better, further discussion between the decision makers and moderation by the analyst are needed. This debate may require additional information.

However, if a debate cannot be held (e.g. if the decision maker is unavailable), the cost and benefit analysis can be merged into a complete ranking. First, the importance scores of the benefits and costs are weighted and then the weighted score of the cost analysis divides the weighted score of the benefit analysis. This produces the complete cardinal ranking.

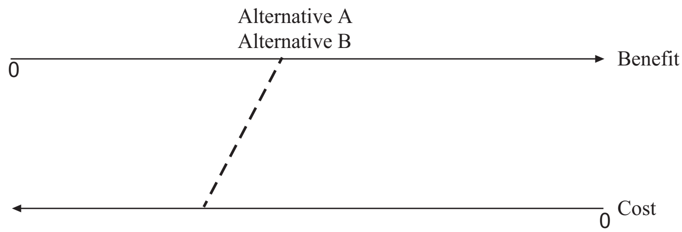

Figure 2.17 Graphical representation of the indifference relation.

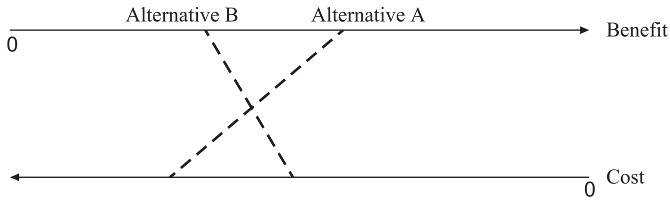

Figure 2.18 Graphical representation of incomparability.

Researchers have proposed four hierarchies: benefit, cost, opportunity and risk (Saaty and Ozdemir 2003a; Saaty 1994b). Each hierarchy is solved separately and the local priorities are aggregated according to

$$p_i = \frac{a \cdot p_o + b \cdot p_b}{c \cdot p_r + d \cdot p_c} \tag{2.18}$$

where a, b, c, d are weights, p_o is the priority of the opportunity hierarchy, p_b is the priority of the benefit hierarchy, p_r is the priority of the risk hierarchy, and p_c is the priority of the cost hierarchy.

Exercise 2.4

In this exercise, you will be able to solve the sports shop problem with the Analytic Hierarchy Process Ordering.

Prerequisites

Exercise 2.2

Learning Outcomes

- Structure the benefit and cost hierarchy in *MakeItRational*
- Understand the partial ordinal ranking
- Understand the complete cardinal ranking

Tasks

a) In *MakeItRational*, construct the two hierarchies as in Figure 2.14 and Figure 2.15.

b) Evaluate the pairwise comparisons and calculate the priorities for each hierarchy.

c) Draw a graph similar to that in Figure 2.16, Figure 2.17 or Figure 2.18. Confirm whether alternatives are preferred, indifferent or incomparable.

d) Aggregate the priorities of the two alternatives to obtain a complete cardinal ranking.

Table 2.14 Four ways to combine preferences (Ishizaka and Labib 2011).

		Mathematical aggregation	
		Yes	No
Aggregation on	Judgements (Figure 2.19)	Geometric mean on comparisons	Consensus vote on comparisons
	Priorities (Figure 2.20)	Weighted arithmetic mean on priorities	Consensus vote on priorities

2.5.2 Group analytic hierarchy process

2.5.2.1 Group aggregation

As a decision often affects several people, standard AHP has been adapted so that it can be applied to group decisions. By consulting various experts, the bias often present when judgements are accepted from a single expert is eliminated. There are four ways to combine the preferences into a consensus rating (Table 2.14).

The consensus vote is used when there is a synergistic group and not a collection of individuals. In this case, the hierarchy of the problem must be the same for all decision makers. On the judgement level, this method requires the group to reach an agreement on the value of each entry in a matrix of pairwise comparisons. A consistent agreement is usually difficult to obtain, the difficulty increasing with the number of comparison matrices and related discussions. To bypass this difficulty, the consensus vote can be postponed until after the calculation of the priorities for each participant. O'Leary (1993) recommends this version because an early aggregation could result 'in a meaningless average performance measure'. An aggregation after the calculation of priorities allows decision makers with different opinions to be detected and further discussion over any disagreement.

If a consensus cannot be reached (e.g. with a large number of people or people in different locations), a mathematical aggregation can be adopted. Two synthesizing methods exist and provide the same results in the case of perfect consistency in the pairwise matrices (Saaty and Vargas 2005). In the first method, the geometric mean of the individual evaluations is used as an element in the pairwise matrices and then priorities are calculated from that (Figure 2.19). The geometric mean method must be adopted instead of the arithmetical mean to preserve the reciprocal property (Aczél and Saaty 1983). For example, if person A enters comparison 9 and person B enters 1/9, then by intuition the mathematical consensus should be $\sqrt{9 \cdot \frac{1}{9}} = 1$, which is a geometric mean and not $(9 + 1/9)/2 = 4.56$, which is an arithmetic mean. However, Ramanathan and Ganesh (1994) give an example where the Pareto optimality (i.e. if all group members prefer A to B, then the group decision should prefer A) is not satisfied with the geometric mean method. Van den Honert and Lootsma (1997) argue

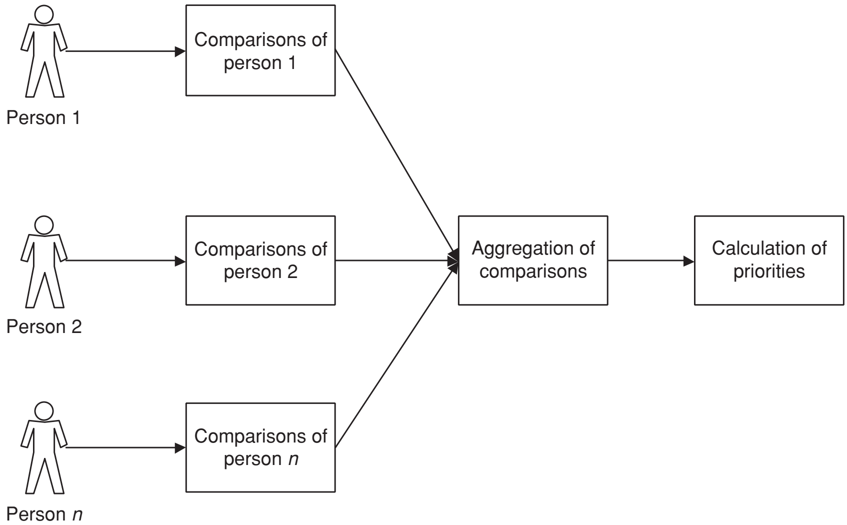

Figure 2.19 Aggregation at the comparison level.

that this violation is expected because the pairwise assessments are a compromise of all the group members' assessments and, therefore, the compromise does not represent any one opinion of the group members. Madu and Kuei (1995) and Saaty and Vargas (2007) introduce a measure of the dispersion of the judgements to avoid this problem. If the group is not homogeneous, further discussions are required to reach a consensus.

In the second method (Figure 2.20), decision makers constitute the first level below the goal of the AHP hierarchy (Figure 2.21). Priorities are computed and then aggregated using the weighted arithmetic mean method. Applications can be found in Labib and Shah (2001) and Labib et al. (1996). Arbel and Orgler (1990) introduced a level above that of the stakeholders representing the several economic scenarios. This extra level determines the priorities (weights) of the stakeholders.

In a compromise method an individual's derived priorities can be aggregated at each node. According to Forman and Peniwati (1998), this method is 'less meaningful and not commonly used'. Aggregation methods with linear programming (Mikhailov 2004) and the Bayesian approach (Altuzarra et al. 2007) have been proposed to make a decision even when comparisons are missing, for example, when a stakeholder does not have the expertise to judge a particular comparison.

A group decision may be skewed due to collusion or distortion in the judgements to secure the preferred outcome. This problem does not arise when there is a single decision maker because the first choice will always remain the first. In a group decision, a participant does not have this certitude as the results are aggregated with those of the other stakeholders. One decision maker may overweight their preferred alternative and bias the group decision. As individual identities are lost with an aggregation, early aggregation is not recommended.

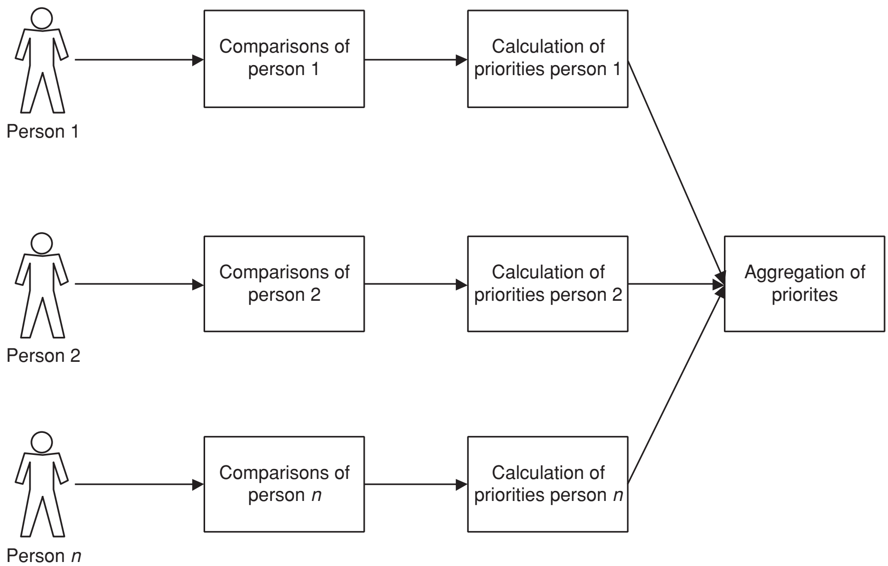

Figure 2.20 Aggregation at the priorities level.

2.5.2.2 Weight of stakeholders

If the decision makers do not have equal weight, their priorities must be determined as discussed in Ishizaka and Labib (2011). The weights should reflect the expertise of a decision maker (Weiss and Rao 1987) or the importance of the impact of the

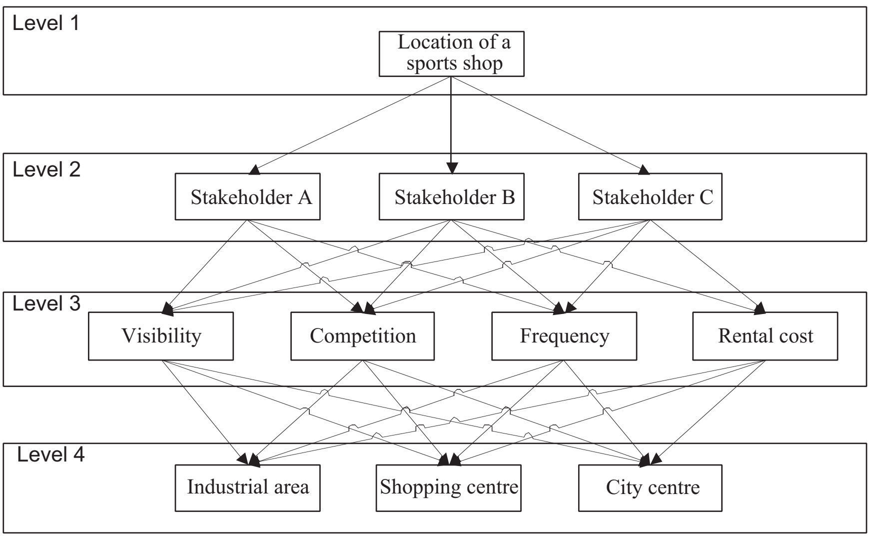

Figure 2.21 Hierarchy including the stakeholders.

decision on the decision maker. The weights can be allocated by a supra decision maker or by a participatory approach. Finding a supra decision maker or benevolent dictator, accepted by everybody, may be difficult.

Ramanathan and Ganesh (1994) proposed a method based on pairwise comparisons to calculate the weights. All n members complete a comparison matrix with the relative importance of each member. A vector of priorities is calculated for each member. The n vectors of priorities are gathered in an $n \times n$ matrix and the final weight of each member is given by the eigenvector of this matrix. To incorporate the uncertainty of the expertise of the participants, AHP has been combined with variable precision rough sets (Xie et al. 2008) and fuzzy logic (Jaganathan et al. 2007).

Ishizaka and Labib (2011) also use pairwise comparisons to judge other members of the group, with each evaluated member allowed the possibility of a veto on the received evaluation. This technique can be viewed as fairer and is applied in situations such as sporting competitions, for example ice-skating, where judges cannot evaluate competitors of the same nationality. The consistency of the weights given by the appraisers is checked with the consistency ratio formula (2.6).

Cho and Cho (2008) have a surprising way of determining weights with levels of inconsistency: decision makers with less inconsistency receive more weight. We do not support this method because inconsistency is useful feedback for the user (Section 2.2.3): it highlights the decision maker's consistency and gives a hint as to revisions of comparisons that could be manual errors in setting the comparisons, sometimes forced due to the upper limitation of the comparison scale (e.g. if the user first enters $a_{12} = 4$ and $a_{23} = 5$, they should enter $a_{13} = 20$ to be consistent, but they can only enter $a_{13} = 9$ due to the maximal value of the measurement scale). The consistency index is therefore not a measure of the quality or expertise of the decision maker.

Exercise 2.5

In this group exercise, you will be able to solve a group decision with the group analytic hierarchy process.

Prerequisites

Exercise 2.2

Learning Outcomes

- Structure a group hierarchy in *MakeItRational*
- Understand the aggregation of individual priorities
- Understand the final group ranking

Tasks

a) Form a group of three or four people.

b) In *MakeItRational* construct the hierarchy shown in Figure 2.21.

c) Each stakeholder evaluates the pairwise comparisons.

d) Evaluate pairwise the weight of each stakeholder.

e) Discuss the final ranking. Is everybody satisfied?

2.5.3 Clusters and pivots for a large number of alternatives

The main drawback of AHP is the high number of pairwise evaluations required for completing large matrices; see expression (2.1). To bypass this problem, a cluster and pivots method has been proposed (Saaty 2001; Ishizaka 2012). It is based on four steps:

a) *For each criterion, all alternatives are ordinally ranked*. If all criteria produce the same order of alternatives, they would be a replica of themselves and the problem would be a mono-criterion one.

b) *For each criterion, alternatives are divided into clusters.* The classical cluster analysis cannot be used in this case because AHP incorporates qualitative criteria in the model. It is a delicate and subjective operation, for which no algorithm exists. The decision maker must evaluate which alternatives are close enough and therefore easy to compare. A heuristic way to construct the clusters is to compare the best ordered alternative successively with the next ones, from the second best to the worst, until:

   – either the cluster contains seven elements. Psychologists have observed that it is difficult to evaluate more than seven elements (Saaty and Ozdemir 2003b). It is recommended that clusters do not contain more than seven elements.

   – or the comparison entered is 9 (if a 9-point scale is used). As no higher strength of preference is available on the comparison scale (Table 2.2), it is appropriate to close the cluster.

   The last compared alternative becomes the pivot (which becomes now the best amongst the remaining ones) at the boundary of both clusters. The same process is repeated with the pivot until seven elements are in the cluster, a comparison value of 9 is entered, or all entries are provided. In Figure 2.22, alternative D is the pivot.

c) *All alternatives of the same cluster are compared and then priorities are calculated.*

d) *Priorities of the clusters are joined with a common element: 'the pivot'*. The pivot is used for the conversion rate between two clusters.

In AHP, all alternatives are compared to each other in a unique comparison matrix, which can be perceived as a one-cluster problem. In a scoring model, direct judgements are used. Each element can be considered a separate cluster. The AHP

	A	B	C	D	E	F
A	1			**9**		
B		1				
C			1			
D				1		
E					1	
F						1

Figure 2.22 Building clusters.

and scoring model represent the two extremes, where 1 or n clusters are used. This model is the middle way between the two methods.

Example 2.7 Suppose that in Case Study 2.1, the businessman has to choose where to situate the sports shop from a choice of 12 cities. In this example, only one criterion is considered (e.g. quality of life), as the process is identical with several criteria.

a) *Alternatives are preordered*. The 12 cities are preordered according to their quality of life: A, B, C, D, E, F, G, H, I, J, K, L.

b) *Alternatives are divided into clusters*. The best city for quality of life, A, is compared successively with the next cities until the comparison entered is 9 (Table 2.15). As city A is 9 times better than F for its quality, alternative F is declared the pivot: the last element in the first cluster and the first element in the second cluster.

c) *Comparisons are entered in clustered matrices and priorities are calculated*. Notice that the process requires 30 comparisons (highlighted in grey in

Table 2.15 Comparison matrix for the criterion quality – the unnecessary comparisons are shaded grey.

	A	B	C	D	E	F	G	H	I	J	K	L
A	1	2	3	5	8	9						
B		1	2	3	6	7						
C			1	3	5	6						
D				1	4	5						
E					1	2						
F						1	1	3	4	5	5	7
G							1	3	3	4	5	7
H								1	2	3	4	6
I									1	2	2	4
J										1	1	3
K											1	3
L												1

Table 2.16 First cluster of cities.

Cities	Priority
F	0.311
G	0.289
H	0.157
I	0.097
J	0.061
K	0.057
L	0.028

Table 2.17 Second cluster of cities.

Cities	Priorities	Priorities linked with the first cluster
A	0.404	4.333
B	0.249	2.670
C	0.178	1.909
D	0.101	1.083
E	0.040	0.429
F	0.029	0.311

Table 2.15) less than the classical AHP approach. Priorities are calculated for both clusters (Tables 2.16 and 2.17).

d) *Priorities of both clusters are joined with a common element: 'the pivot'*. Results of the second cluster (Table 2.17) are linked to the first (Table 2.16) by dividing them by the ratio of the scores of pivot F in the two clusters: 0.311/0.029. Final results are given in Table 2.18.

2.5.4 AHPSort

Whilst AHP solves ranking problems, AHPSort has been developed for sorting problems (Ishizaka et al. 2012). This method is a variant of AHP used when alternatives have to be sorted into categories predefined with central limiting profiles. Suppose that the businessman in Case Study 2.1 aims to open several sports shops in different cities. The cities will be sorted into three classes (Figure 2.23):

a) cities where sports shops will be highly profitable;

b) cities where sports shop may be profitable;

c) cities where sports shops will not be profitable.

Table 2.18 Priorities of the cities.

Cities	Priorities
A	4.333
B	2.670
C	1.909
D	1.083
E	0.429
F	**0.311**
G	0.289
H	0.157
I	0.097
J	0.061
K	0.057
L	0.028

AHPSort is based on eight steps:

A) Problem definition

1) Define the goal, criteria and alternatives of the problem.
2) Define the categories C_i, $i = 1, \ldots, n$.
3) Define the profiles of each class. This can be done with limiting profiles lp_i, which indicate the minimum performance needed on each criterion to belong to class C_i, or with central profiles cp_i, given by a typical example of an element belonging to class C_i. To define each class, $n - 1$ limiting profiles or n central profiles are needed.

B) Evaluations

4) Evaluate pairwise the importance of the criteria and derive weights (Section 2.4.4).

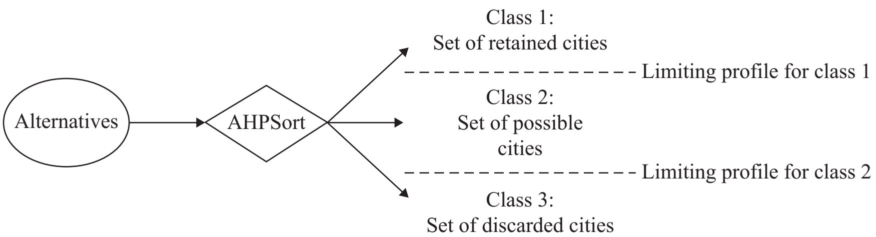

Figure 2.23 AHPSort for the sorting process.

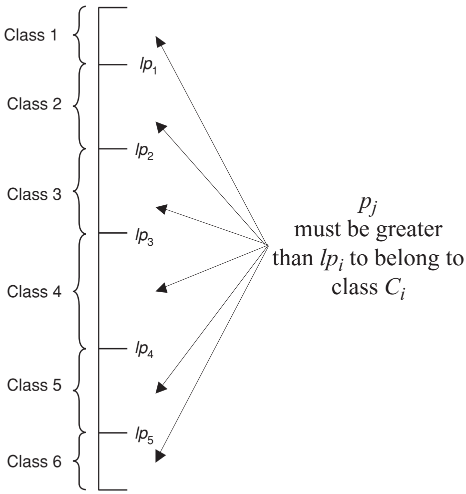

Figure 2.24 Sorting with limiting profiles.

5) Compare in a pairwise comparison matrix a single candidate to be sorted with limiting or central profiles for each criterion to derive local priorities for each criterion (Section 2.4.4).

6) Aggregate the local weighted priorities, which provide a global priority p_j of alternative a_j.

C) Assignment to classes

7) The global priority p_j is used to assign the alternative a_j to class C_i.

   a) *Limiting profiles*. If limiting profiles have been defined, then alternative a_j is assigned to class C_i which has the lp_i just below the global priority p_j (Figure 2.24).

$$
\begin{aligned}
p_j \geq lp_1 \quad &\Rightarrow \quad a_j \in C_1 \\
lp_2 \leq p_j < lp_1 \quad &\Rightarrow \quad a_j \in C_2 \\
\ldots \\
p_j < lp_{n-1} \quad &\Rightarrow \quad a_j \in C_n
\end{aligned}
$$

   b) *Central profiles*. Alternative a_j is assigned to class C_i which has the nearest central profile cp_i to p_j (Figure 2.25). In the case of equal distance between two central profiles, the optimistic assignment vision

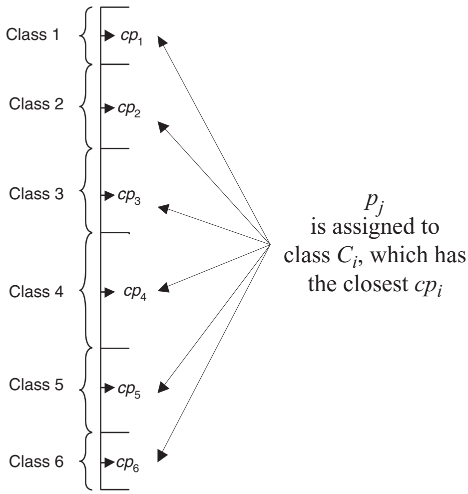

Figure 2.25 Sorting with central profiles.

allocates a_j to the upper class, whilst the pessimistic assignment vision allocates a_j to the lower class.

$$
\begin{aligned}
&p_i \geq cp_1 && \Rightarrow \mathrm{a}_j \in C_1\\
&cp_2 \leq p_j < cp_1 \text{ AND } (cp_1 - p_j) < (cp_2 - p_j) && \Rightarrow \mathrm{a}_j \in C_1\\
&cp_2 \leq p_j < cp_1 \text{ AND } (cp_1 - p_j) = (cp_2 - p_j) && \Rightarrow \mathrm{a}_j \in C_1 \text{ in the}\\
& && \text{optimistic vision}\\
&cp_2 \leq p_j < cp_1 \text{ AND } (cp_1 - p_j) = (cp_2 - p_j) && \Rightarrow \mathrm{a}_j \in C_2 \text{ in the}\\
& && \text{pessimistic vision}\\
&cp_2 \leq p_j < cp_1 \text{ AND } (cp_1 - p_j) > (cp_2 - p_j) && \Rightarrow \mathrm{a}_j \in C_2\\
&\ldots\\
&p_j < cp_n \Rightarrow \mathrm{a}_j \in C_n
\end{aligned}
\tag{2.19}
$$

8) Repeat steps 4–7 for each alternative to be classified.

Exercise 2.6

In this exercise, you will sort cities into three categories:

a) cities where sports shops must be open;

b) cities where sports shops may be opened later;

c) cities where sports shops would not be profitable.

Prerequisites

Exercise 2.2

Learning Outcomes

- ➢ Structure a sorting problem in *MakeItRational*
- ➢ Define classes
- ➢ Understand the sorting results

Tasks

a) Choose 10 potential cities as alternatives for opening a sports shop.

b) Define two limiting profiles for each class.

c) Model the problem in *MakeItRational*.

d) Enter the pariwise comparison for each city.

e) Sort the city into a category according to its score.

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