Ex 5.3

We know from 5.2:

$$F_2(\hat{x},\hat{y}) = \begin{pmatrix} 1-\hat{y}/2 \\ \frac{1}{2}+\frac{\hat{x}}{2} \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$$

$$F_2 : \hat{T} \longrightarrow T_2$$

$$\Rightarrow \begin{cases} y = \frac{1}{2}+\frac{\hat{x}}{2} & \Rightarrow \quad \hat{x} = 2y-1 \\ x = 1-\hat{y}/2 & \Rightarrow \quad \hat{y} = 2(1-x) \end{cases}$$

Now we calculate the $\varphi_m = \hat{\varphi}_{\hat{m}} \cdot F_2^{-1}$ (and $\nabla\varphi_m$)

global index

$$\varphi_2^T = \hat{\varphi}_1 \circ F_2^{-1} = \varphi_1(\hat{x}) \cdot \varphi_0(\hat{y}) = \varphi_1(2y-1)\cdot\varphi_0(2(1-x)) =$$

$$= (2y-1)(\underbrace{1-2(1-x)}_{1-2+2x=2x-1}) = \cancel{(1-y)(1-x)}\ \cancel{2(y-1)(1-x)}\ (2y-1)(2x-1)$$

$$\varphi_6^T = \hat{\varphi}_2 \circ F_2^{-1} = \varphi_1(\hat{x})\,\varphi_1(\hat{y}) = \varphi_1(2y-1)\,\varphi_1(2(1-x)) =$$

$$\cancel{=(1-2y+1)(2(1-x)) = 4(1-y)(1-x)} = 2(2y-1)(1-x)$$

$$\varphi_8^T = \hat{\varphi}_3 \circ F_2^{-1} = \varphi_0(\hat{x})\,\varphi_1(\hat{y}) = \varphi_0(2y-1)\,\varphi_1(2(1-x)) =$$

$$= (\underbrace{1-2y+1}_{2-2y})\,(2(1-x)) = 4(1-y)(1-x)$$

$$\varphi_5^T = \hat{\varphi}_0 \circ F_2^{-1} = \varphi_0(2y-1)\,\varphi_0(2(1-x)) = (1-2y+1)(1-2+2x) =$$

$$= 2(1-y)(2x-1)$$

$$\nabla\varphi_2^T = \begin{pmatrix} 2(2y-1) \\ 2(2x-1) \end{pmatrix}, \quad \nabla\varphi_6^T = \begin{pmatrix} -2(2y-1) \\ 4(1-x) \end{pmatrix},$$

$$\nabla\varphi_8^T = \begin{pmatrix} -4(1-y) \\ -4(1-x) \end{pmatrix}, \quad \nabla\varphi_5^T = \begin{pmatrix} 4(1-y) \\ -2(2x-1) \end{pmatrix}$$

~~Next~~ Next we calculate $(A_T)_{mn} = \int_{T_2} 6 \nabla\varphi_n \nabla\varphi_m$

$\to$ Identify local φ_i with the global φ_j^T from before

$$\varphi_0 = \varphi_5^T, \quad \varphi_1 = \varphi_2^T, \quad \varphi_2 = \varphi_6^T, \quad \varphi_3 = \varphi_8^T$$

$$(A_T)_{00} = 6\int_{T_2} \nabla\varphi_0 \nabla\varphi_0 = 6 \int_{[0.5,1]^2} \text{"}\begin{pmatrix} 4(1-y) \\ -2(2x-1) \end{pmatrix}^2\text{"} = 6\int_{[0.5,1]} \left[16(1-y)^2 + 4(2x-1)^2\right] dx\,dy =$$

$$= 6 \cdot 4 \cdot \tfrac{1}{2} \cdot \int_{s=0.5}^{1} \underbrace{4 \cdot (1 - 2s + s^2)}_{4-8s+4s^2} + 4s^2 - 4s + 1 \, ds =$$

$$= 12 \int_{0.5}^{1} 5 - 12s + 8s^2 \, ds = 12 \left[5s - 6s^2 + \frac{8s^3}{3}\right]_{1/2}^{1} =$$

$$= 12\left[\underbrace{5-6}_{-1} + \frac{8}{3} \underbrace{- \frac{5}{2} + \frac{6}{4}}_{= -\frac{10}{4} + \frac{6}{4} = -\frac{4}{4} = -1} - \frac{1}{3}\right] = 12\left[-2 + \frac{7}{3}\right] = \cancel{12}^{\,4} \cdot \frac{7-6}{\cancel{3}} = \underline{\underline{4}}$$

... $\to$ Jupyter with sympy

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d) T_2:

$$\begin{pmatrix} \varphi_0 \\ \varphi_1 \\ \varphi_2 \\ \varphi_3 \end{pmatrix} \doteq \begin{pmatrix} \varphi_5^T \\ \varphi_2^T \\ \varphi_6^T \\ \varphi_8^T \end{pmatrix} = C^t_{T_2} \begin{pmatrix} \varphi_0^T \\ \varphi_1^T \\ \vdots \\ \varphi_8^T \end{pmatrix}$$

$$C^t_{T_2} = \begin{array}{c} \\ \\ \\ \\ \\ \end{array} \begin{array}{c} \begin{array}{ccccccccc} \varphi_0^T & \varphi_1^T & \varphi_2^T & \varphi_3^T & \varphi_4^T & \varphi_5^T & \varphi_6^T & \varphi_7^T & \varphi_8^T \end{array} \\ \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \end{array} \begin{array}{l} \\ \to \varphi_0 \\ \to \varphi_1 \\ \to \varphi_2 \\ \to \varphi_3 \end{array}$$