NUMPDE Exercise 5

1 Example 5.1

2 Example 5.2

3 Example 5.3

4 Example 5.4

Let $\hat{T}$ be the reference simplex in $\mathbb{R}^d$. Every d-dimensional simplex in $\mathbb{R}^d$ is affine equivalent to $\hat{T}$, i.e there exists $F : \hat{T} \to T, F(x) = Ax + b$ with $A \in \mathbb{R}^{d\times d}$,$b \in \mathbb{R}^d$ and $\det(A) \neq 0$. F is invertible and there holds (for c,$C > 0$):

$$||DF|| = ||A|| \leq \frac{h_T}{\rho_{\hat{T}}} \tag{1}$$

$$||DF^{-1}|| = ||A^{-1}|| \leq \frac{h_{\hat{T}}}{\rho_T} \tag{2}$$

$$c\rho_T^d \leq |\det(DF)| = |\det(A)| \leq Ch_T^d \tag{3}$$

To proof 3) we start with integrate over T and use the coordinate transformations determinant.

$$|T| = \int_T dx = \int_{\hat{T}} |\det(A)| d\hat{x} = |\det(A)||\hat{T}|$$

$$|\det(A)| = \frac{|T|}{|\hat{T}|}$$

For proof 1) and 2) we use the hint in the description.

$$||A|| = \sup_{\zeta\in\mathbb{R}^d} \frac{||A\zeta||}{||\zeta||} = \sup_{||\zeta||=c} \frac{||A\zeta||}{||\zeta||}$$

where $c = \rho_{\hat{T}}$

$$\Rightarrow \sup_{||\zeta||=c} \frac{||A\zeta||}{||\zeta||} = \frac{1}{\rho_{\hat{T}}} \sup_{||\zeta||=\rho_{\hat{T}}} ||A\zeta||$$

Now we could define a proper $\zeta = \hat{x}_1 - \hat{x}_2 \in \hat{T}$ and
use the affine property of A where $A\zeta = F(\hat{x}_1) - F(\hat{x}_2) = x_1 - x_2$

$$||A\zeta|| \leq |x_1 - x_2| \leq h_T$$

$$||A|| = \sup_{||\zeta||=\rho_{\hat{T}}} \frac{1}{\rho_{\hat{T}}} ||A\zeta|| \leq \frac{h_T}{\rho_{\hat{T}}}$$

For 2) we start first with that

$$||A^{-1}|| = \frac{1}{\rho_T} \sup_{||\varepsilon||=\rho_T} ||A^{-1}\varepsilon||$$

Now we could define a proper $\varepsilon = x_1 - x_2 \in T$ and
use the affine property of A^{-1} where $A^{-1}\varepsilon = F^{-1}(x_1) - F^{-1}(x_2) = \hat{x}_1 - \hat{x}_2$

$$||A^{-1}\varepsilon|| \leq |\hat{x}_1 - \hat{x}_2| \leq h_{\hat{T}}$$

$$||A^{-1}|| = \sup_{||\varepsilon||=\rho_T} \frac{1}{\rho_T} ||A^{-1}\varepsilon|| \leq \frac{h_{\hat{T}}}{\rho_T}$$