Part II, Chapter 8

Local interpolation in affine meshes

We have seen in the previous chapter how to build finite elements and local interpolation operators in each cell K of a mesh $\mathcal{T}_h$. In this chapter, we analyze the local interpolation error for smooth $\mathbb{R}^q$-valued functions, $q \geq 1$. We restrict the material to affine meshes and to transformations ψ_K such that ♡♡

$$\psi_K(v) = \mathbb{A}_K(v \circ \boldsymbol{T}_K), \tag{8.1}$$

where $\mathbb{A}_K$ is a matrix in $\mathbb{R}^{q\times q}$. Non-affine meshes are treated in Chapter 9. We introduce the notion of shape-regular families of affine meshes, we study the transformation of Sobolev norms using (8.1), and we present important approximation results collectively known as the Bramble–Hilbert lemmas. We finally prove the main result of this chapter, which is an upper bound on the local interpolation error over each mesh cell for smooth functions.

8.1 Shape-regularity for affine meshes

Let $\mathcal{T}_h$ be an affine mesh. Let $K \in \mathcal{T}_h$. Since the geometric map $\boldsymbol{T}_K$ is affine, there is a matrix $\mathbb{J}_K \in \mathbb{R}^{d\times d}$ such that

$$\boldsymbol{T}_K(\widehat{\boldsymbol{x}}) - \boldsymbol{T}_K(\widehat{\boldsymbol{y}}) = \mathbb{J}_K(\widehat{\boldsymbol{x}} - \widehat{\boldsymbol{y}}), \qquad \forall \widehat{\boldsymbol{x}}, \widehat{\boldsymbol{y}} \in \widehat{K}. \tag{8.2}$$

The matrix $\mathbb{J}_K$ is invertible since the map $\boldsymbol{T}_K$ is bijective. Moreover, the (Fréchet) derivative of the geometric map is such that $D\boldsymbol{T}_K(\widehat{\boldsymbol{x}})(\widehat{\boldsymbol{h}}) = \mathbb{J}_K\widehat{\boldsymbol{h}}$ for all $\widehat{\boldsymbol{h}} \in \mathbb{R}^d$. We denote the Euclidean norm in $\mathbb{R}^d$ by $\|\cdot\|_{\ell^2(\mathbb{R}^d)}$, or $\|\cdot\|_{\ell^2}$ when the context is unambiguous. We abuse the notation by using the same symbol for the induced matrix norm.

Lemma 8.1 (Bound on $\mathbb{J}_K$). *Let $\mathcal{T}_h$ be an affine mesh and let $K \in \mathcal{T}_h$. Let ρ_K be the diameter of the largest ball that can be inscribed in K and let h_K*

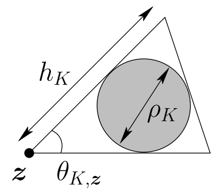

Fig. 8.1. Triangular cell K with vertex $\boldsymbol{z}$, angle $\theta_{K,\boldsymbol{z}}$, and largest inscribed ball.

be the diameter of K (see Figure 8.1). Let $\widehat{\rho}_{\widehat{K}}$ and $\widehat{h}_{\widehat{K}}$ be defined similarly. Then, the following holds:

$$|\det(\mathbb{J}_K)| = \frac{|K|}{|\widehat{K}|}, \qquad \|\mathbb{J}_K\|_{\ell^2} \le \frac{h_K}{\rho_{\widehat{K}}}, \qquad \|\mathbb{J}_K^{-1}\|_{\ell^2} \le \frac{h_{\widehat{K}}}{\rho_K}. \tag{8.3}$$

Proof. The first equality results from the fact that

$$|K| = \int_K \mathrm{d}x = \int_{\widehat{K}} |\det(\mathbb{J}_K)|\,\mathrm{d}\widehat{x} = |\det(\mathbb{J}_K)||\widehat{K}|.$$

Regarding the bound on $\|\mathbb{J}_K\|_{\ell^2}$, we observe that

$$\|\mathbb{J}_K\|_{\ell^2} = \sup_{\widehat{\boldsymbol{h}}\neq 0} \frac{\|\mathbb{J}_K\widehat{\boldsymbol{h}}\|_{\ell^2}}{\|\widehat{\boldsymbol{h}}\|_{\ell^2}} = \frac{1}{\rho_{\widehat{K}}} \sup_{\|\widehat{\boldsymbol{h}}\|_{\ell^2}=\rho_{\widehat{K}}} \|\mathbb{J}_K\widehat{\boldsymbol{h}}\|_{\ell^2}.$$

Write $\widehat{\boldsymbol{h}} = \widehat{\boldsymbol{x}}_1 - \widehat{\boldsymbol{x}}_2$ with $\widehat{\boldsymbol{x}}_1, \widehat{\boldsymbol{x}}_2 \in \widehat{K}$ and use $\mathbb{J}_K\widehat{\boldsymbol{h}} = \boldsymbol{T}_K(\widehat{\boldsymbol{x}}_1) - \boldsymbol{T}_K(\widehat{\boldsymbol{x}}_2) = \boldsymbol{x}_1 - \boldsymbol{x}_2$ to infer that $\|\mathbb{J}_K\widehat{\boldsymbol{h}}\|_{\ell^2} \le h_K$. This proves the bound on $\|\mathbb{J}_K\|_{\ell^2}$. The bound on $\|\mathbb{J}_K^{-1}\|_{\ell^2}$ is obtained by exchanging the roles of K and $\widehat{K}$. □

Since the analysis of the interpolation error (implicitly) invokes sequences of successively refined meshes, we henceforth denote by $(\mathcal{T}_h)_{h>0}$ a sequence of meshes approximating a domain $D \subset \mathbb{R}^d$, where the index h takes values in a countable set having zero as the only accumulation point. A mesh sequence is said to be *affine* if all the meshes in the sequence are affine.

Definition 8.2 (Shape-regularity). *A sequence of affine meshes $(\mathcal{T}_h)_{h>0}$ is said to be* shape-regular *if there exists $\sigma_\sharp$ such that*

$$\sigma_K := \frac{h_K}{\rho_K} \le \sigma_\sharp, \qquad \forall K \in \mathcal{T}_h,\ \forall h > 0. \tag{8.4}$$

Owing to Lemma 8.1, a shape-regular sequence of affine meshes satisfies

$$\|\mathbb{J}_K\|_{\ell^2}\|\mathbb{J}_K^{-1}\|_{\ell^2} \le \sigma_\sharp\sigma_{\widehat{K}}, \qquad \forall K \in \mathcal{T}_h,\ \forall h > 0. \tag{8.5}$$

Example 8.3 (Dimension 1). Every sequence of one-dimensional meshes is shape-regular, since $h_K = \rho_K$ when $d = 1$. □

Example 8.4 (Triangulations). A shape-regular sequence of affine triangulations can be obtained from an initial triangulation by connecting all the edge midpoints and repeating this procedure as many times as needed. □

Remark 8.5 (Angles). Let $(\mathcal{T}_h)_{h>0}$ be a shape-regular sequence of affine simplicial meshes and let $K \in \mathcal{T}_h$. Assume that $d = 2$, let K be a triangle in $\mathcal{T}_h$ and let $\boldsymbol{z}$ be a vertex of K; then, the angle $\theta_{K,\boldsymbol{z}} \in (0, 2\pi)$ formed by the (two) edges of K sharing $\boldsymbol{z}$ is uniformly bounded away from zero. Indeed, the angular sector centered at $\boldsymbol{z}$ of angle $\theta_{K,\boldsymbol{z}}$ and radius h_K covers the ball of diameter ρ_K that is inscribed in K (see Figure 8.1); as a result, $\frac{1}{2}h_K^2\theta_{K,\boldsymbol{z}} \geq \frac{1}{4}\pi\rho_K^2$, which in turn implies $\theta_{K,\boldsymbol{z}} \geq \frac{1}{2}\pi\sigma_\sharp^{-2}$. Assume now that $d = 3$, let K be a tetrahedron and let $\boldsymbol{z}$ be a vertex of K; then, the solid angle $\omega_{K,\boldsymbol{z}} \in (0, 4\pi)$ formed by the (three) faces of K sharing $\boldsymbol{z}$ is uniformly bounded away from zero. Reasoning as above, with volumes instead of surfaces, leads to $\frac{1}{3}h_K^3\omega_{K,\boldsymbol{z}} \geq \frac{1}{6}\pi\rho_K^3$, thereby implying that $\omega_{K,\boldsymbol{z}} \geq \frac{1}{2}\pi\sigma_\sharp^{-3}$. □

8.2 Transformation of Sobolev seminorms

The question we investigate now is the following: given a function $v \in W^{m,p}(K;\mathbb{R}^q)$, how does the seminorm of $\psi_K(v)$ in $W^{m,p}(\widehat{K};\mathbb{R}^q)$ compare to that of v in $W^{m,p}(K;\mathbb{R}^q)$ when ψ_K is defined by (8.1)?

Lemma 8.6 (Norm scaling by ψ_K). *Let $\mathcal{T}_h$ be a mesh. Let $m \in \mathbb{N}$ and $p \in [1,\infty]$ (with $z^{\pm\frac{1}{p}} = 1$, $\forall z > 0$ if $p = \infty$). There exists c depending only on m and d such that the following bounds hold for all $K \in \mathcal{T}_h$ and all $v \in W^{m,p}(K;\mathbb{R}^q)$:*

$$|\psi_K(v)|_{W^{m,p}(\widehat{K};\mathbb{R}^q)} \leq c\,\|\mathbb{A}_K\|_{\ell^2}\|\mathbb{J}_K\|_{\ell^2}^m\,|\det(\mathbb{J}_K)|^{-\frac{1}{p}}\,|v|_{W^{m,p}(K;\mathbb{R}^q)}, \tag{8.6a}$$

$$|v|_{W^{m,p}(K;\mathbb{R}^q)} \leq c\,\|\mathbb{A}_K^{-1}\|_{\ell^2}\|\mathbb{J}_K^{-1}\|_{\ell^2}^m\,|\det(\mathbb{J}_K)|^{\frac{1}{p}}\,|\psi_K(v)|_{W^{m,p}(\widehat{K};\mathbb{R}^q)}. \tag{8.6b}$$

Proof. Let α be a multi-index with length $|\alpha| = m$, i.e., $\alpha = (\alpha_1, \ldots, \alpha_d) \in \{1{:}l\}^d$ with $\alpha_1 + \ldots + \alpha_d = m$. Let $\widehat{\boldsymbol{x}} \in \widehat{K}$. Owing to (B.8), we infer that

$$\partial^\alpha(\psi_K(v))(\widehat{\boldsymbol{x}}) = \mathbb{A}_K D^m(v \circ \boldsymbol{T}_K)(\widehat{\boldsymbol{x}})(\underbrace{\boldsymbol{e}_1, \ldots, \boldsymbol{e}_1}_{\times\alpha_1}, \ldots, \underbrace{\boldsymbol{e}_d, \ldots, \boldsymbol{e}_d}_{\times\alpha_d}),$$

with $D^m(v\circ\boldsymbol{T}_K)(\widehat{\boldsymbol{x}})$ the m-th Fréchet derivative of $v\circ\boldsymbol{T}_K$ at $\widehat{\boldsymbol{x}}$ and $(\boldsymbol{e}_1, \ldots, \boldsymbol{e}_d)$ the canonical Cartesian basis of $\mathbb{R}^d$. We now apply the chain rule (see Lemma B.13) to $v \circ \boldsymbol{T}_K$. Since $\boldsymbol{T}_K$ is affine, the Fréchet derivative of $\boldsymbol{T}_K$ is independent of $\widehat{\boldsymbol{x}}$ and its higher-order Fréchet derivatives vanish. Hence,

$$D^m(v \circ \boldsymbol{T}_K)(\widehat{\boldsymbol{x}})(\boldsymbol{h}_1, \ldots, \boldsymbol{h}_m) = \sum_{\sigma \in \mathcal{S}_m} \frac{1}{m!} (D^m v)(\boldsymbol{T}_K(\widehat{\boldsymbol{x}}))(D\boldsymbol{T}_K(\boldsymbol{h}_{\sigma(1)}), \ldots, D\boldsymbol{T}_K(\boldsymbol{h}_{\sigma(m)})),$$

for all $\boldsymbol{h}_1, \ldots, \boldsymbol{h}_m \in \mathbb{R}^d$, where $\mathcal{S}_m$ is the set of the permutations of $\{1{:}m\}$. Since $D\boldsymbol{T}_K(\boldsymbol{h}) = \mathbb{J}_K \boldsymbol{h}$ for all $\boldsymbol{h} \in \mathbb{R}^d$ owing to (8.2), we infer that

$$|\partial^\alpha (v \circ \boldsymbol{T}_K)(\widehat{\boldsymbol{x}})| \le \|\mathbb{J}_K\|_{\ell^2}^m \, \|(D^m v)(\boldsymbol{T}_K(\widehat{\boldsymbol{x}}))\|_{\mathcal{M}_m(\mathbb{R}^d,\ldots,\mathbb{R}^d;\mathbb{R}^q)},$$

where we have set $\|A\|_{\mathcal{M}_m(\mathbb{R}^d,\ldots,\mathbb{R}^d;\mathbb{R}^q)} := \sup_{(\boldsymbol{y}_1,\ldots,\boldsymbol{y}_m)\in\mathbb{R}^d\times\ldots\times\mathbb{R}^d} \frac{\|A(\boldsymbol{y}_1,\ldots,\boldsymbol{y}_m)\|_{\ell^2}}{\|\boldsymbol{y}_1\|_{\ell^2}\cdots\|\boldsymbol{y}_m\|_{\ell^2}}$ for any multilinear map $A \in \mathcal{M}_m(\mathbb{R}^d, \ldots, \mathbb{R}^d; \mathbb{R}^q)$. Owing to the multilinearity of $D^m v$ and using again (B.8), we infer that (see Exercise 8.1)

$$\|(D^m v)(\boldsymbol{T}_K(\widehat{\boldsymbol{x}}))\|_{\mathcal{M}_m(\mathbb{R}^d,\ldots,\mathbb{R}^d;\mathbb{R}^q)} \le c \sum_{|\beta|=m} \|(\partial^\beta v)(\boldsymbol{T}_K(\widehat{\boldsymbol{x}}))\|_{\ell^2},$$

where c only depends on m and d. As a result,

$$\|\partial^\alpha(\psi_K(v))(\widehat{\boldsymbol{x}})\|_{\ell^2} \le c \, \|\mathbb{A}_K\|_{\ell^2} \|\mathbb{J}_K\|_{\ell^2}^m \sum_{|\beta|=m} \|(\partial^\beta v)(\boldsymbol{T}_K(\widehat{\boldsymbol{x}}))\|_{\ell^2},$$

and (8.6a) follows by taking the $L^p(\widehat{K})$-norm of the functions on both sides of the inequality. The proof of (8.6b) is similar. $\square$

Remark 8.7 (Seminorms). The upper bounds in (8.6a) and (8.6b) involve only seminorms because the geometric maps are affine. $\square$

8.3 Bramble–Hilbert Lemmas

This section contains important results for the analysis of the approximation properties of finite elements. Lemma 8.8 is the key result. The estimate (8.7) is proved in Bramble and Hilbert [78, Thm. 1] and in Ciarlet and Raviart [149, Lem. 7], see also Deny and Lions [190]. The more general estimate (8.8), rarely mentioned in the literature, is useful to estimate the local interpolation error for a larger class of functions and polynomial degrees; see Theorem 8.12 below. We state the results for scalar-valued functions; they readily extend to the vector-valued case by working componentwise.

Lemma 8.8 ($\mathbb{P}_k$-Bramble–Hilbert/Deny–Lions). *Let S be a connected, open, bounded Lipschitz set in $\mathbb{R}^d$. Let p be a real number such that $p \in [1, \infty]$. Let $k \in \mathbb{N}$. Then, there exists $c > 0$ (depending on k, p, S) such that*

$$\inf_{q \in \mathbb{P}_{k,d}} \|v + q\|_{W^{k+1,p}(S)} \le c \, |v|_{W^{k+1,p}(S)}, \quad \forall v \in W^{k+1,p}(S). \tag{8.7}$$

More generally, for all $l \in \mathbb{N}$ with $l \geq k+1$, there exists $c > 0$ such that

$$\inf_{q \in \mathbb{P}_{k,d}} \|v+q\|_{W^{l,p}(S)} \leq c \sum_{m=k+1}^{l} |v|_{W^{m,p}(S)}, \quad \forall v \in W^{l,p}(S). \tag{8.8}$$

Proof. It suffices to prove (8.8), since (8.7) is just (8.8) with $l = k+1$.
(1) Let $N_{k,d} = \dim \mathbb{P}_{k,d}$, and introduce the $N_{k,d} = \binom{k+d}{d}$ continuous linear forms

$$f_\alpha : W^{l,p}(S) \ni v \longmapsto f_\alpha(v) = \int_S \partial^\alpha v \, \mathrm{d}x \in \mathbb{R}, \qquad \forall \alpha \in \mathbb{N}^d, \ |\alpha| \leq k,$$

where $\alpha = (\alpha_1, \ldots, \alpha_d)$ is a multi-index. The key property of these linear forms is that the restriction to $\mathbb{P}_{k,d}$ of the map $\Phi_k : W^{l,p}(S) \ni q \mapsto (f_\alpha(q))_{|\alpha| \leq k} \in \mathbb{R}^{N_{k,d}}$ is an isomorphism (it is injective and $\dim \mathbb{P}_{k,d} = N_{k,d}$).
(2) Let us prove that there is $c > 0$, depending on S, k and l, such that

$$c \, \|v\|_{W^{l,p}(S)} \leq \sum_{m=k+1}^{l} |v|_{W^{m,p}(S)} + \sum_{|\alpha| \leq k} |f_\alpha(v)|, \quad \forall v \in W^{k+1,p}(S). \tag{8.9}$$

Reasoning by contradiction, let $(v_n)_{n \in \mathbb{N}}$ be a sequence such that

$$\|v_n\|_{W^{l,p}(S)} = 1, \lim_{n \to \infty} |v_n|_{W^{m,p}(S)} = 0, \lim_{n \to \infty} \|\Phi_k(v_n)\|_{\ell^1(\mathbb{R}^{N_{k,d}})} = 0, \tag{8.10}$$

for all $m \in \{k+1{:}l\}$. Owing to the Rellich–Kondrachov Theorem B.104, we infer that, up to a subsequence (not renumbered for simplicity), the sequence $(v_n)_{n \in \mathbb{N}}$ converges strongly to a function v in $W^{l-1,p}(S)$. Moreover, $(v_n)_{n \in \mathbb{N}}$ is a Cauchy-sequence in $W^{l,p}(S)$ since

$$\|v_n - v_m\|_{W^{l,p}(S)} \leq \|v_n - v_m\|_{W^{l-1,p}(S)} + |v_n - v_m|_{W^{l,p}(S)}.$$

Hence, $(v_n)_{n \in \mathbb{N}}$ converges to v strongly in $W^{l,p}(S)$ (that the limit is indeed v comes from the uniqueness of the limit in $W^{l-1,p}(S)$). Owing to (8.10) we infer that $\|v\|_{W^{l,p}(S)} = 1$, $|v|_{W^{m,p}(S)} = 0$ for all $m \in \{k+1{:}l\}$, and $\Phi_k(v) = 0$. Repeated applications of Lemma B.50 (stating that in an open connected set S, $\nabla v = 0$ implies that v is constant in S) allows us to infer from $|v|_{W^{k+1,p}(S)} = 0$ that $v \in \mathbb{P}_{k,d}$. Since the restriction of Φ_k to $\mathbb{P}_{k,d}$ is an isomorphism, this yields $v = 0$, which is a contradiction since $\|v\|_{W^{l,p}(S)} = 1$.
(3) Let $v \in W^{l,p}(S)$ and define $\pi(v) \in \mathbb{P}_{k,d}$ such that $\Phi_k(\pi(v)) = -\Phi_k(v)$. This is possible since the restriction of Φ_k to $\mathbb{P}_{k,d}$ is an isomorphism. Then,

$$\begin{aligned} \inf_{q \in \mathbb{P}_{k,d}} \|v+q\|_{W^{l,p}(S)} &\leq \|v+\pi(v)\|_{W^{l,p}(S)} \leq c \sum_{m=k+1}^{l} |v+\pi(v)|_{W^{m,p}(S)} \\ &\quad + c \, \|\Phi_k(v+\pi(v))\|_{\ell^1(\mathbb{R}^{N_{k,d}})} = c \, (|v|_{W^{k+1,p}(S)} + \ldots + |v|_{W^{l,p}(S)}) \end{aligned}$$

since $\partial^\alpha \pi(v) = 0$ for all α such that $k+1 \leq |\alpha|$. This completes the proof. □

Remark 8.9 (Peetre-Tartar Lemma). Step (2) in the above proof is similar to the Peetre–Tartar Lemma A.53. For instance, take $l = k+1$ and define $X = W^{k+1,p}(S)$, $Y = [L^p(D)]^{N_{k+1,d}-N_{k,d}} \times \mathbb{R}^{N_{k,d}}$, and $Z = W^{k,p}(S)$. Define the operator

$$A : X \ni v \longmapsto ((\partial^\alpha v)_{|\alpha|=k+1}, \Phi_k(v)) \in Y.$$

Then, A is bounded and injective, and the embedding $X \subset Z$ is compact. Property (8.9) then results from the Peetre–Tartar Lemma. □

Lemma 8.10 ($\mathbb{P}_k$-Bramble–Hilbert for linear functionals). *Under the hypotheses of Lemma* 8.8 *(with* $l = k+1$*), there is* $c > 0$ *such that the following holds for all* $f \in (W^{k+1,p}(S))' := \mathcal{L}(W^{k+1,p}(S); \mathbb{R})$ *vanishing on* $\mathbb{P}_{k,d}$*,*

$$|f(v)| \leq c \, \|f\|_{(W^{k+1,p}(S))'} |v|_{W^{k+1,p}(S)}, \qquad \forall v \in W^{k+1,p}(S). \tag{8.11}$$

Proof. Left as an exercise. The estimate (8.11) is proved in Bramble and Hilbert [78, Thm. 2] and in Ciarlet and Raviart [149, Lem. 6]. □

Remark 8.11 (Terminology). It seems that there is some variability in the literature regarding the exact statement of the Bramble–Hilbert Lemma. For instance, Lemma 8.8 is attributed to Bramble and Hilbert in Brenner and Scott [96, p. 102] and Ciarlet and Raviart [147, p. 219], whereas this lemma is attributed to Deny and Lions in Ciarlet [146, p. 111], and it is not given any name in Braess [75, p. 77]. Lemma 8.10 is called Bramble–Hilbert Lemma in Ciarlet [146, p. 192] and Braess [75, p. 78]. Incidentally, there are two additional Lemmas by Bramble and Hilbert that are the counterparts of Lemma 8.8 and Lemma 8.10 for $\mathbb{Q}_{k,d}$ polynomials, see Lemma 9.7 and Lemma 9.8 below. □

8.4 Local finite element interpolation

This section contains our main result on local finite element interpolation. Recall the construction of §7.3 to generate a finite element and a local interpolation operator in each mesh cell $K \in \mathcal{T}_h$. Our goal is now to bound the interpolation error $v - \mathcal{I}_K v$ for a smooth function v. The key point is that we want this bound to depend on K only through its size h_K, assuming shape-regularity of the mesh sequence. We can see that the Bramble–Hilbert/Deny–Lions Lemma 8.8 cannot be used directly on K since it will lead to a constant depending on K. The crucial idea is then to use the fact that $\mathcal{I}_K = \psi_K^{-1} \circ \mathcal{I}_{\widehat{K}} \circ \psi_K$ owing to Proposition 7.9 and to apply Lemma 8.8 on the fixed reference cell $\widehat{K}$. To allow for a bit more generality, we do not assume that $\mathcal{I}_{\widehat{K}}$ is a finite element interpolation operator, and we assume that $\mathcal{I}_K$ is defined by the relation $\mathcal{I}_K = \psi_K^{-1} \circ \mathcal{I}_{\widehat{K}} \circ \psi_K$ instead of resulting from the construction of §7.3.

Theorem 8.12 (Local interpolation). *Let $\mathcal{I}_{\widehat{K}} \in \mathcal{L}(V(\widehat{K}); \widehat{P})$ with $\widehat{P}$ finite dimensional. Let $p \in [1, \infty]$, let $k, l \in \mathbb{N}$ and assume that the following holds:*

(i) $[\mathbb{P}_{k,d}]^q \subset \widehat{P} \subset W^{k+1,p}(\widehat{K}; \mathbb{R}^q)$.
(ii) $[\mathbb{P}_{k,d}]^q$ *is pointwise invariant under* $\mathcal{I}_{\widehat{K}}$.
(iii) $W^{l,p}(\widehat{K}; \mathbb{R}^q) \hookrightarrow V(\widehat{K})$ *(i.e.,* $\|\widehat{v}\|_{V(\widehat{K})} \leq \widehat{c}\, \|\widehat{v}\|_{W^{l,p}(\widehat{K};\mathbb{R}^q)}$ *for all* $\widehat{v} \in V(\widehat{K})$*).*

Let $\mathcal{T}_h$ be a shape-regular sequence of affine meshes, let $K \in \mathcal{T}_h$, and define the operator $\mathcal{I}_K := \psi_K^{-1} \circ \mathcal{I}_{\widehat{K}} \circ \psi_K$, with ψ_K defined by (8.1), and assume that

$$\|\mathbb{A}_K\|_{\ell^2} \|\mathbb{A}_K^{-1}\|_{\ell^2} \leq \gamma\, \|\mathbb{J}_K\|_{\ell^2} \|\mathbb{J}_K^{-1}\|_{\ell^2}, \tag{8.12}$$

where $\gamma > 0$ is uniform with respect to K and h. Then, there exists $c > 0$ such that the following local error estimate holds:

$$|v - \mathcal{I}_K v|_{W^{m,p}(K;\mathbb{R}^q)} \leq c\, h_K^{r-m} (|v|_{W^{r,p}(K;\mathbb{R}^q)} + \ldots + h_K^{s-r} |v|_{W^{s,p}(K;\mathbb{R}^q)}), \tag{8.13}$$

for all $r \in \{0{:}k+1\}$, all $m \in \{0{:}r\}$, $s := \max(l, r)$, all $v \in W^{s,p}(K; \mathbb{R}^q)$, and all $K \in \mathcal{T}_h$.

Proof. Let $r \in \{0{:}k+1\}$, $m \in \{0{:}r\}$, and set $s = \max(l, r)$. Let c denote a generic constant whose value can change at each occurrence as long as it is independent of K and $v \in W^{s,p}(K; \mathbb{R}^q)$.

(1) For all $\widehat{w} \in W^{s,p}(\widehat{K}; \mathbb{R}^q)$, we set $\mathcal{G}(\widehat{w}) := \widehat{w} - \mathcal{I}_{\widehat{K}}(\widehat{w})$. The operator $\mathcal{G}$ is linear and well-defined (i.e., $\mathcal{I}_{\widehat{K}}(\widehat{w})$ is well-defined since $s \geq l$ and $W^{l,p}(\widehat{K}; \mathbb{R}^q) \hookrightarrow V(\widehat{K})$ imply $W^{s,p}(\widehat{K}; \mathbb{R}^q) \hookrightarrow V(\widehat{K})$). Moreover, $\mathcal{G}$ maps $W^{s,p}(\widehat{K}; \mathbb{R}^q)$ boundedly to $W^{m,p}(\widehat{K}; \mathbb{R}^q)$, i.e., $\mathcal{G} \in \mathcal{L}(W^{s,p}(\widehat{K}; \mathbb{R}^q); W^{m,p}(\widehat{K}; \mathbb{R}^q))$. Assume first that $r \geq 1$. Then, the operator $\mathcal{G}$ vanishes on $\mathbb{P}_{r-1,d}$, since $r - 1 \leq k$ and $\mathbb{P}_{r-1,d} \subset \mathbb{P}_{k,d}$, which implies that $[\mathbb{P}_{r-1,d}]^q$ is pointwise invariant under $\mathcal{I}_{\widehat{K}}$. As a consequence, we infer that

$$\begin{aligned}
|\widehat{w} - \mathcal{I}_{\widehat{K}} \widehat{w}|_{W^{m,p}(\widehat{K};\mathbb{R}^q)} &= |\mathcal{G}(\widehat{w})|_{W^{m,p}(\widehat{K};\mathbb{R}^q)} = \inf_{\widehat{p} \in [\mathbb{P}_{r-1,d}]^q} |\mathcal{G}(\widehat{w} + \widehat{p})|_{W^{m,p}(\widehat{K};\mathbb{R}^q)} \\
&\leq \|\mathcal{G}\|_{\mathcal{L}(W^{s,p}(\widehat{K};\mathbb{R}^q); W^{m,p}(\widehat{K};\mathbb{R}^q))} \inf_{\widehat{p} \in [\mathbb{P}_{r-1,d}]^q} \|\widehat{w} + \widehat{p}\|_{W^{s,p}(\widehat{K};\mathbb{R}^q)} \\
&\leq c \inf_{\widehat{p} \in [\mathbb{P}_{r-1,d}]^q} \|\widehat{w} + \widehat{p}\|_{W^{s,p}(\widehat{K};\mathbb{R}^q)} \leq c (|\widehat{w}|_{W^{r,p}(\widehat{K};\mathbb{R}^q)} + \ldots + |\widehat{w}|_{W^{s,p}(\widehat{K};\mathbb{R}^q)}),
\end{aligned}$$

for all $\widehat{w} \in W^{s,p}(\widehat{K}; \mathbb{R}^q)$, where we used the estimate (8.8) from the Bramble–Hilbert/Deny–Lions Lemma 8.8 componentwise with with $r-1$ in lieu of k and s in lieu of l (this is possible since $s \geq r$). Furthermore, the above inequality is trivial if $r = 0$.

(2) Now let $v \in W^{s,p}(K; \mathbb{R}^q)$. We infer that

$$\begin{aligned}
|v - \mathcal{I}_K v|_{W^{m,p}(K;\mathbb{R}^q)} &\leq c\, \|\mathbb{A}_K^{-1}\|_{\ell^2} \|\mathbb{J}_K^{-1}\|_{\ell^2}^m\, |\det(\mathbb{J}_K)|^{\frac{1}{p}}\, |\psi_K(v - \mathcal{I}_K v)|_{W^{m,p}(\widehat{K};\mathbb{R}^q)} \\
&\leq c\, \|\mathbb{A}_K^{-1}\|_{\ell^2} \|\mathbb{J}_K^{-1}\|_{\ell^2}^m\, |\det(\mathbb{J}_K)|^{\frac{1}{p}}\, |\psi_K(v) - \mathcal{I}_{\widehat{K}}(\psi_K(v))|_{W^{m,p}(\widehat{K};\mathbb{R}^q)} \\
&\leq c\, \|\mathbb{A}_K^{-1}\|_{\ell^2} \|\mathbb{J}_K^{-1}\|_{\ell^2}^m\, |\det(\mathbb{J}_K)|^{\frac{1}{p}}\, (|\psi_K(v)|_{W^{r,p}(\widehat{K};\mathbb{R}^q)} + \ldots + |\psi_K(v)|_{W^{s,p}(\widehat{K};\mathbb{R}^q)}) \\
&\leq c\, \|\mathbb{J}_K^{-1}\|_{\ell^2}^m\, \|\mathbb{J}_K\|_{\ell^2}^r (|v|_{W^{r,p}(K;\mathbb{R}^q)} + \ldots + \|\mathbb{J}_K\|_{\ell^2}^{s-r} |v|_{W^{s,p}(K;\mathbb{R}^q)}),
\end{aligned}$$

where we have used the bound (8.6b) in the first line, the linearity of ψ_K and the assumption $\mathcal{I}_K := \psi_K^{-1} \circ \mathcal{I}_{\widehat{K}} \circ \psi_K$ on the second line, the result of Step (1) in the third line, and the bound (8.6a) together with (8.12) in the fourth line. The estimate (8.13) follows by using (8.3) and the fact that $\sigma_K = \frac{h_K}{\rho_K}$ is uniformly bounded owing to shape-regularity. □

In the context of finite elements, assumption (i) in Theorem 8.12 is easy to satisfy since $\widehat{P}$ is in general composed of polynomial functions, and assumption (ii) follows from (i) since $\widehat{P}$ is pointwise invariant under $\mathcal{I}_{\widehat{K}}$.

Definition 8.13 (Degree of a finite element). *The largest integer k such that $[\mathbb{P}_{k,d}]^q \subset \widehat{P} \subset W^{k+1,p}(\widehat{K};\mathbb{R}^q)$ is called the* degree *of the finite element.*

Remark 8.14 (Simpler version). Theorem 8.12 is usually formulated in the literature under the assumption that the degree k of the finite element is large enough so that $k+1 \geq l$, where l is the integer from assumption (iii). In this case, the statement of the theorem can be simplified by taking $r \geq l$ so that $s = r$. The estimate (8.7) from Lemma 8.8 is then sufficient to complete the proof, and only one term is present in the error bound. The statement is

$$|v - \mathcal{I}_K v|_{W^{m,p}(K;\mathbb{R}^q)} \leq c\, h_K^{r-m} |v|_{W^{r,p}(K;\mathbb{R}^q)}, \tag{8.14}$$

for all $r \in \{l{:}k+1\}$, all $m \in \{0{:}r\}$, all $v \in W^{r,p}(K;\mathbb{R}^q)$, and all $K \in \mathcal{T}_h$. □

Example 8.15 (Lagrange elements). The choice (8.1) with $\mathbb{A}_K = 1$ for ψ_K is legitimate for scalar-valued Lagrange elements, see §7.4.1. Taking $V(\widehat{K}) = C^0(\widehat{K})$, assumption (iii) is satisfied if we take l to be the smallest integer such that $l > \frac{d}{p}$; indeed, this implies that $W^{l,p}(\widehat{K}) \hookrightarrow V(\widehat{K})$ owing to Theorem B.104. Assuming that $k+1 > \frac{d}{p}$ (so that $k+1 \geq l$), the simplified statement (8.14) becomes

$$|v - \mathcal{I}_K^{\mathrm{L}} v|_{W^{m,p}(K)} \leq c h_K^{r-m} |v|_{W^{r,p}(K)}, \tag{8.15}$$

for all $r \in \{l{:}k+1\}$, all $m \in \{0{:}r\}$, all $v \in W^{r,p}(K)$, and all $K \in \mathcal{T}_h$. Instead, the more general estimate (8.13) has to be used whenever $k+1 \leq \frac{d}{p}$. For instance, assume $k=1$ and $p \in [1, \frac{d}{2}]$, $d \geq 2$, so that $k+1 = 2 \leq \frac{d}{p}$. For $d=2$ and $p=1$ or for $d=3$ and $p \in (1, \frac{3}{2}]$, we can take $l=3$ in assumption (iii) (since $l > \frac{d}{p}$). For $m=0$ and $r=k+1=2$, we get $s = \max(l,r) = 3$, which yields $\|v - \mathcal{I}_K^{\mathrm{L}} v\|_{L^p(K)} \leq c h_K^2 (|v|_{W^{2,p}(K)} + h_K |v|_{W^{3,p}(K)})$. □

Example 8.16 (Modal elements). In this case, it is legitimate to take $l = 0$ in assumption (iii) since $V(\widehat{K}) = L^1(\widehat{K};\mathbb{R}^q)$, see §7.4.2. Hence, the simplified bound (8.14) can always be used. □

Example 8.17 (Piola transformations). The Piola transformations defined in (7.12) and(7.13) satisfy (8.1) with $\mathbb{A}_K := \det(\mathbb{J}_K)\mathbb{J}_K^{-1}$ and $\mathbb{A}_K := \mathbb{J}_K^{\mathsf{T}}$, respectively. Note that in both cases, the assumption (8.12) holds with $c = 1$. The interpolation error analysis is detailed in Chapter 12. □

Exercises

Exercise 8.1 (High-order derivative). Let two integers $m, d \geq 2$. Consider the map $\Phi : \{1{:}d\}^m \ni \mathtt{j} \longmapsto (\Phi_1(\mathtt{j}), \ldots, \Phi_d(\mathtt{j})) \in \mathbb{N}^d$ where $\Phi_i(\mathtt{j}) = \operatorname{card}\{k \in \{1{:}m\} \mid \mathtt{j}_k = i\}$ for all $i \in \{1{:}d\}$. Observe that $|\Phi(\mathtt{j})| = m$ by construction. Define $C_{m,d} = \max_{\alpha\in\mathbb{N}^d,|\alpha|=m} \operatorname{card}\{\mathtt{j} \in \{1{:}d\}^m \mid \Phi(\mathtt{j}) = \alpha\}$. Let v be a smooth (scalar-valued) function.

(i) Show that $\|D^m v\|_{\mathcal{M}_m(\mathbb{R}^d,\ldots,\mathbb{R}^d;\mathbb{R})} \leq C_{m,d}^{\frac{1}{2}} \left(\sum_{\alpha\in\mathbb{N}^d,|\alpha|=m} |\partial^\alpha v|^2\right)^{\frac{1}{2}}$.
(ii) Show that for $d = 2$, $C_{m,d} = \max_{0\leq l\leq m} \binom{m}{l} = 2^m$.
(iii) Evaluate $C_{m,d}$ for $d = 3$ and $m \in \{2, 3\}$.
(iv) Show conversely that $\sum_{\alpha\in\mathbb{N}^d,|\alpha|=m} |\partial^\alpha v| \leq \check{C}_{m,d}\|D^m v\|_{\mathcal{M}_m(\mathbb{R}^d,\ldots,\mathbb{R}^d;\mathbb{R})}$ with $\check{C}_{m,d} = \binom{d+m-1}{d-1}$.

Exercise 8.2 (Flat triangle). Let K be a triangle with vertices $(0, 0)$, $(1, 0)$ and $(-1, \epsilon)$ with $0 < \epsilon \ll 1$. Consider the function $v(x_1, x_2) = x_1^2$. Evaluate the $\mathbb{P}_1$ Lagrange interpolant $\mathcal{I}_K^{\mathrm{L}} v$, see (7.11), and show that $|v - \mathcal{I}_K^{\mathrm{L}} v|_{H^1(K)} \geq \epsilon^{-1}|v|_{H^2(K)}$. (*Hint*: use a direct calculation of $\mathcal{I}_K^{\mathrm{L}} v$.)

Exercise 8.3 (Barycentric coordinate). Let K be a simplex with barycentric coordinates $\{\lambda_i\}_{0\leq i\leq d}$. Prove that $|\lambda_i|_{W^{1,\infty}(K)} \leq \rho_K^{-1}$ for all $i \in \{0{:}d\}$.

Exercise 8.4 (L^p-norm of shape functions). Let $\theta_{K,i}$, $i \in \mathcal{N}$, be a local shape function. Let $p \in [1, \infty]$. Assume that the mesh sequence is shape-regular. Prove that $\|\theta_{K,i}\|_{L^p(K)}$ is equivalent to $h_K^{d/p}$ uniformly with respect to K.

Exercise 8.5 (Mapping Φ_k). Let $k \in \mathbb{N}$ and set $N_{k,d} = \dim(\mathbb{P}_{k,d}) = \binom{k+d}{d}$. Let S be a connected, open, bounded Lipschitz set in $\mathbb{R}^d$. Consider the mapping $\Phi_k : \mathbb{P}_{k,d} \ni q \mapsto (f_\alpha(q))_{|\alpha|\leq k} \in \mathbb{R}^{N_{k,d}}$, where $f_\alpha(q) = \int_S \partial^\alpha v \, \mathrm{d}x$. Show that Φ_k is an isomorphism.

Exercise 8.6 (Bramble–Hilbert). Prove Lemma 8.10.

Exercise 8.7 (Taylor polynomial). Let K be a convex cell. Consider a Lagrange finite element of degree $k \geq 1$ with nodes $\{\boldsymbol{a}_i\}_{i\in\mathcal{N}}$ and associated shape functions $\{\theta_i\}_{i\in\mathcal{N}}$. Consider a sufficiently smooth function v. For all $\boldsymbol{x}, \boldsymbol{y} \in K$, consider the Taylor polynomial of order k and the exact remainder such that

$$\mathbb{T}_k(\boldsymbol{x}, \boldsymbol{y}) = v(\boldsymbol{x}) + Dv(\boldsymbol{x})(\boldsymbol{y} - \boldsymbol{x}) + \ldots + \frac{1}{k!} D^k v(\boldsymbol{x})(\underbrace{\boldsymbol{y} - \boldsymbol{x}, \ldots, \boldsymbol{y} - \boldsymbol{x}}_{k \text{ times}}),$$

$$R_k(v)(\boldsymbol{x}, \boldsymbol{y}) = \frac{1}{(k+1)!} D^{k+1} v(\eta\boldsymbol{x} + (1-\eta)\boldsymbol{y})(\underbrace{\boldsymbol{y} - \boldsymbol{x}, \ldots, \boldsymbol{y} - \boldsymbol{x}}_{(k+1) \text{ times}}),$$

so that $v(\boldsymbol{y}) = \mathbb{T}_k(\boldsymbol{x}, \boldsymbol{y}) + R_k(v)(\boldsymbol{x}, \boldsymbol{y})$ for a certain $\eta \in [0, 1]$.

(i) Prove that $v(\boldsymbol{x}) = \mathcal{I}_K^{\mathrm{L}} v(\boldsymbol{x}) - \sum_{i\in\mathcal{N}} R_k(v)(\boldsymbol{x}, \boldsymbol{a}_i)\theta_i(\boldsymbol{x})$, where $\mathcal{I}_K^{\mathrm{L}}$ is the Lagrange interpolant, see (7.11). (*Hint*: interpolate with respect to $\boldsymbol{y}$.)

(ii) Prove that $D^m v(\boldsymbol{x}) = D^m(\mathcal{I}_K^{\mathrm{L}} v)(\boldsymbol{x}) - \sum_{i=1}^{n_{\mathrm{sh}}} R_k(v)(\boldsymbol{x}, \boldsymbol{a}_i) D^m\theta_i(\boldsymbol{x})$ for all $m \le k$.

(iii) Infer that $|v - \mathcal{I}_K^{\mathrm{L}} v|_{W^{m,\infty}(K)} \le c\sigma_K^m h_K^{k+1-m} |v|_{W^{k+1,\infty}(K)}$ with constant $c = \frac{1}{(k+1)!} c_* h_{\widehat{K}}^m \sum_{i\in\mathcal{N}} |\widehat{\theta}_i|_{W^{k+1,\infty}(\widehat{K})}$ where c_* stems from (8.6b) for $s = m$ and $p = \infty$.

Exercise 8.8 (L^p-stability of Lagrange interpolant). Let $\alpha \in (0,1)$. Consider the $\mathbb{P}_1$ finite element space spanned by $\theta_1(x) = 1 - x$ and $\theta_2(x) = x$. Consider the sequence of continuous functions $\{u_n\}_{n>0}$ defined as follows over the interval $[0,1]$: $u_n(x) = n^\alpha - 1$ if $0 \le x \le \frac{1}{n}$ and $u_n(x) = x^{-\alpha} - 1$ otherwise.

(i) Prove that the sequence is uniformly bounded in $L^p(0,1)$ for all p such that $p\alpha < 1$.

(ii) Compute $\mathcal{I}_K^{\mathrm{L}} u_n$. Is the operator $\mathcal{I}_K^{\mathrm{L}}$ stable in the L^p-norm?

(iii) Is the operator $\mathcal{I}_K^{\mathrm{L}}$ stable in any L^r-norm with $r \in [1,\infty)$