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**Numerical methods for partial differential equations**

Exercise 9 – 19. May 2020

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**Example 9.1**

Use the notations and the assumptions of Theorem 101 (Brezzi). Now we also assume that the bilinear form $a(\cdot,\cdot)$ is symmetric. Consider the constrained minimization problem

$$\min_{\substack{v\in V\\ b(v,q)=g(q)\ \forall q\in Q}} \frac{1}{2}a(v,v) - f(v)$$

Show that there exists a unique solution and that it is given by the variational problem: Find $u \in V$ and $p \in Q$ such that

$$\begin{aligned} a(u,v) + b(v,p) &= f(v) \quad \forall v \in V,\\ b(u,q) &= g(q) \quad \forall q \in Q. \end{aligned}$$

Hint: Check for the definition of Lagrange functions/multipliers to solve minimization problems with constraints. Check the proof of the first example in the first exercise.

**Example 9.2** (Prager Synge)

Let $u$ be the solution of $-\Delta u = f$ with Dirichlet boundary conditions $u = u_D$ on $\Gamma_D$ and $\frac{\partial u}{\partial n} = g$ on $\Gamma_N$ (with $\partial\Omega = \Gamma_N \cup \Gamma_D$). Further, let $v_h \in H^1(\Omega)$ and $\tau_h \in H(\operatorname{div},\Omega)$ be two arbitrary finite element functions with $v_h = u_D$ on $\Gamma_D$ and $\operatorname{div}\tau_h = -f$ and $\tau_h \cdot n = -g$ on $\Gamma_N$. Show the following orthogonality (with respect to the $L^2(\Omega)$ inner product)

$$\nabla v_h - \nabla u \perp^{L^2} \nabla u - \tau_h$$

and

$$\|\nabla v_h - \nabla u\|_{L^2}^2 + \|\nabla u - \tau_h\|_{L^2}^2 = \|\nabla v_h - \tau_h\|_{L^2}^2.$$

Why (and where) could this equation be useful?

**Example 9.3**

Let $\Omega = (0,1)^2$, $\Gamma_D = (0,1) \times \{0\}$, $\Gamma_N = \{0\} \times (0,1)$, and $\Gamma_R = \partial\Omega \setminus (\Gamma_N \cup \Gamma_D)$. Solve the problem

$$\begin{aligned} -\Delta u &= f && \text{in } \Omega,\\ u &= u_D && \text{on } \Gamma_D,\\ \frac{\partial u}{\partial n} &= g && \text{on } \Gamma_N,\\ \alpha u + \frac{\partial u}{\partial n} &= g && \text{on } \Gamma_R, \end{aligned}$$

with $f = 1$, $u_D = x$, $\alpha = 5$, $g = y$ using a primal method and using a mixed method for the flux. Compare the difference $u_{\text{prim}} - u_{\text{mix}}$ and $\nabla u_{\text{prim}} - \sigma_{\text{mix}}$ on a series of meshes with decreasing mesh size and different polynomial orders. Be careful with the different use of essential and natural boundary conditions.

**Example 9.4** (Revised Reissner Mindlin)

Let $\Omega = (0,1)^2$ and $f = 1$. Again we want to solve the Reissner Mindlin plate equation: Find a $(w, \beta) \in V \subset [H^1(\Omega)]^3$ such that

$$\int_\Omega \nabla\beta \cdot \nabla\delta \,\mathrm{d}x + \frac{1}{t^2}\int_\Omega (\nabla w - \beta)\cdot(\nabla v - \delta)\,\mathrm{d}x = \int_\Omega f v \,\mathrm{d}x \quad \forall (v,\delta) \in V.$$

In example 4.3 we realized that for $t \to 0$ there appears a so called locking phenomena, i.e. although the discrete system is solvable the error of the solution is big (due to the $t$-dependence in the constant of Cea's-Lemma). A solution to this problem is given by reformulating the method as a mixed method. To this end we define a new variable $p = 1/t^2(\nabla w - \beta) \in Q = L^2(\Omega)$. Testing this with "sufficient" enough test functions gives

$$\int_\Omega p \cdot q = \int_\Omega 1/t^2(\nabla w - \beta)\cdot q \quad \forall q \in Q.$$

The mixed method then reads as: Find a $((w,\beta),p) \in V \times Q$ such that

$$\int_\Omega \nabla\beta \cdot \nabla\delta \,\mathrm{d}x + \int_\Omega (\nabla v - \delta)\cdot p \,\mathrm{d}x = \int_\Omega f v \,\mathrm{d}x \quad \forall (v,\delta) \in V. \tag{1a}$$

$$\int_\Omega (\nabla w - \beta)\cdot q \,\mathrm{d}x - t^2 \int_\Omega p \cdot q \,\mathrm{d}x = 0 \qquad \forall q \in Q. \tag{1b}$$

For the discretization we choose the spaces $(W_h, \Theta_h) \subset V$, where $W_h$ is a scalar space for the discretization of $w_h$ and $\Theta_h$ is a vector valued space for $\beta_h$. Further we choose a $Q_h \subset Q_h$. A proper analysis of the mixed system needs an "extended" Brezzi-Theorem that includes the bilinear form at the lower right corner on the left hand side ($-t^2 \int_\Omega pq$). We take the following choice: $W_h = P^2(\mathcal{T}) \cap H^1$ (standard Lagrangian space of polynomial order 2), $\Theta_h = [P^1(\mathcal{T}) \cap H^1 \oplus \mathcal{B}(\mathcal{T})]^2$ where $\mathcal{B}(\mathcal{T})$ is the space of the local cubic polynomials on each element that vanish on $\partial T$ (also called bubbles), and $Q_h = P^0(\mathcal{T})$ (constants on each element). In NGSolve this reads as

```
Wh = H1(mesh, order = 2, dirichlet = ...)
Thetah = H1(mesh, order = 1, orderinner = 3, dirichlet = ...)
Qh = L2(mesh, order = 0)
```

Note, that `Thetah` and `Qh` are only the scalar components of $\Theta_h$ and $Q_h$. Unfortunately, the system (1) with the above choice is still not solvable in the discrete setting. To this end we add the stabilizing bilinear form

$$\alpha \int_\Omega (\nabla w_h - \beta_h)\cdot(\nabla v_h - \delta_h)\,\mathrm{d}x,$$

to the system, where $\alpha$ is a parameter.

1. Implement the mixed method with $t = 0.1$ and $\alpha = 1, \frac{1}{t^2+h^2}, 0$ (the last one should not be stable) and the boundary conditions $w_h = 0$ on $\partial\Omega$. (Choose `maxh` $= 0.1$)

2. Present a plot where you evaluate the displacement $w_h$ at the point $(0.5, 0.5)$ for values $t = 10^{-i}$ with $i = 1, \dots, 6$. What do you observe?

Remark: To solve the mixed system use the umpack-solver:

```
inv = a.mat.Inverse(fes.FreeDofs(), inverse = "umfpack")
```