**Ex 6.2**

Let $Q_h(v) := \sum_{T \in \mathcal{T}} Q_T(v)$ and $v \in H^3(\Omega)$

Show that

$$\left| \int_\Omega v\,dx - Q_h(v) \right| \lesssim h^3 |v|_{H^3(\Omega)}$$

Hint: Use Theorem 69

$$\hookrightarrow \quad |v - I_T v|^2_{H^1(\Omega)} < C \sum_{T \in \mathcal{T}} h_T^2 |v|_{H^2(T^2)}$$

$I_T : H^3 \to P^2$

For quasi-uniform triangulations:

$$|v - I_T v|_{H^1(\Omega)} \lesssim h |u|_{H^2}$$

Ex 6.1: $= \int_T I_T v\,dx$

$$\underbrace{\left| \int_\Omega v\,dx - Q_h(v) \right|}_{\text{const}}^2 \cancel{\leq \left\| \int_\Omega v\,dx - Q_h(v) \right\|^2_{L^2(\mathcal{T})} = \sum_{T \in \mathcal{T}} \left\| \int_T v\,dx - Q_T(v) \right\|^2_{L^2(T)}}$$

$$\cancel{= \sum_T \left\| \int (id - I_T) v_T\,dx \right\|^2_{L^2(T)} \lesssim \sum \det B_T \cdot \|B_T^{-1}\|}$$

$$= \left| \sum_T \left[ \int (id - I_T) v \right] \right|^2 \underset{\text{triangle}}{\leq} \sum_T \left| \int (id - I_T) v\,dx \right|^2 \leq$$

bound by $L^2$-norm

$$\leq \sum_T \left\| \int_T (id - I_T) v\,dx \right\|^2_{L^2(T)} \cdot \frac{\det B}{\det B} \underset{L66}{=} \sum_T \det B_T \left\| \int_{\hat{T}} (id - I_T)(v \circ F_T)\,d\hat{x} \right\|^2_{L^2(\hat{T})}$$

**Lemma 66:**

$$\| u \circ F_T \|_{L^2(\hat{T})} = (\underset{\text{reg}}{\det B_T})^{-\frac{1}{2}} \|u\|_{L^2(T)} \qquad (4.2)$$

$F_T : \hat{T} \to T, \; x \to a + Bx$

Lagrangian FE are equivalent

$\Leftrightarrow \quad I_T = I_{\hat{T}}$

$$\ldots = \sum_T \det B_T \, \Big\| \int_{\hat{T}} \underbrace{(id - I_{\hat{T}})}_{L} \underbrace{(v \circ F_T)}_{u} \, d\hat{x} \Big\|^2_{L^2(\hat{T})} \leq$$

**Theorem 54:** $\|Lu\|_U \leq |v|_{H^k}$

$L: H^k \to U$

$Lq = 0$ for $q \in P^{k-1}$ $\rightarrow$ here $k-1=2$ (Ex 6.1)

$\Rightarrow k = 3$

$$\leq \sum_T \det B_T \cdot \big| v \circ F_T \big|^2_{H^3(\hat{T})} \lesssim \sum_T \Big( \underbrace{\|B_T\|^3}_{\approx h_T \approx h} \cdot |v|_{H^3(T)} \Big)^2 =$$

L66: (4.4) uniform

$$= h^6 \sum_{T \in \mathcal{T}} |v|^2_{H^3(T)} = h^6 \, |v|^2_{H^3(\Omega)} \checkmark$$

**Lemma 66:** $|u \circ F|_{H^m(\hat{T})} \lesssim (\det B)^{-\frac{1}{2}} \|B\|^m |u|_{H^m(T)}$ (4.4)