Ex5.1 (Raviart-Thomas element)

$\hat{T}$ ... ref triangle $\to \hat{T} = \operatorname{conv}((0,0)^T, (1,0)^T, (0,1)^T)$

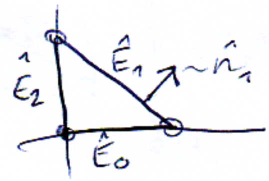
with edges $E_0 = \operatorname{conv}((0,0)^T, (1,0)^T)$, E_1 ..., E_2 ...

With $V_{\hat{T}} := \left\{ \begin{pmatrix} a(\hat{x},\hat{y}) \\ b(\hat{x},\hat{y}) \end{pmatrix} + \begin{pmatrix} \hat{x} \\ \hat{y} \end{pmatrix} c(\hat{x},\hat{y}) \;\middle|\; \underbrace{a,b,c \in P^0(T)}_{} \right\}$

↳ so... constants?

and $\Psi_T = \{ \psi_0^{\hat{T}}, \psi_1^{\hat{T}}, \psi_2^{\hat{T}} \}$, $\psi_i^{\hat{T}}(v) := \int_{\hat{E}_i} v \cdot \hat{n} \, ds$, $i = 0,1,2$

where $\hat{n}$ is the outer normal to $\hat{T}$



1) The triplet $(\hat{T}, V_{\hat{T}}, \Psi_{\hat{T}})$ is a FE in sense of Ciarlet.
Determine the dual basis $\{\hat{\varphi}_i\}_{i=0,1,2}$ and draw a sketch

$\psi_i^{\hat{T}}(\varphi_j^{\hat{T}}) = \delta_{ij}$

↳ $\psi_0(\varphi_0) = \int_{E_0} \varphi_0 \hat{n} \, ds = \int_0^1 \varphi_0(\cancel{x}, \cancel{y}) \cdot \begin{pmatrix} 0 \\ -1 \end{pmatrix} ds \overset{\uparrow}{=} \int_0^1 \begin{pmatrix} a_0 + c_0 x \\ b_0 \end{pmatrix} \begin{pmatrix} 0 \\ -1 \end{pmatrix} dx = \ldots$

$\varphi_i(x,y) = \begin{pmatrix} a_i \\ b_i \end{pmatrix} + c_i \begin{pmatrix} x \\ y \end{pmatrix}$

$\ldots = -b_0 \int_0^1 dx = -b_0 \overset{!}{=} 1 \implies b_0 = -1$

$\psi_1(\varphi_0) = \int_{E_1} \varphi_0 \hat{n} \, ds = \int_{(1,0)}^{(0,1)} \varphi_0 \hat{n} \, ds = \frac{1}{\sqrt{2}} \int_1^0 \begin{pmatrix} a_0 + c_0 x \\ -1 + c_0(1-x) \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} dx =$

↳ $y(x) = 1 - x$

mit ~~x~~ x von 1 bis 0

$= \frac{1}{\sqrt{2}} \int_1^0 a_0 + \cancel{c_0 x} - 1 + c_0 - \cancel{c_0 x} \, dx =$

$= \frac{1}{\sqrt{2}} (\underset{\uparrow b_0}{1} - a_0 - c_0) \overset{!}{=} 0 \implies \overset{b_0}{1} = a_0 + c_0$

↳ $a_0 = 1 - c_0$

$$\Psi_2(\varphi_0) = \int_{E_2} \varphi_0 \cdot n \, ds = \int_{y=1}^{0} \varphi_0(0,y) \begin{pmatrix}-1\\0\end{pmatrix} dy = \int_{1}^{0} \begin{pmatrix} a_0 \\ \cdots \end{pmatrix} \begin{pmatrix}-1\\0\end{pmatrix} dy =$$

$$= -a_0 \int_{1}^{0} dy = a_0 \overset{!}{=} 0 \quad \Rightarrow \quad c_0 = 1$$

$$\hookrightarrow \varphi_0(x,y) = \begin{pmatrix}0\\-1\end{pmatrix} + \begin{pmatrix}x\\y\end{pmatrix}$$

$$\varphi_1 = \begin{pmatrix}a_1\\b_1\end{pmatrix} + c_1 \begin{pmatrix}x\\y\end{pmatrix}$$

$$\Psi_0(\varphi_1) = \int_{x=0}^{1} \begin{pmatrix}\cdots\\b_1\end{pmatrix}\begin{pmatrix}0\\-1\end{pmatrix} = \ldots = -b_1 \overset{!}{=} 0$$

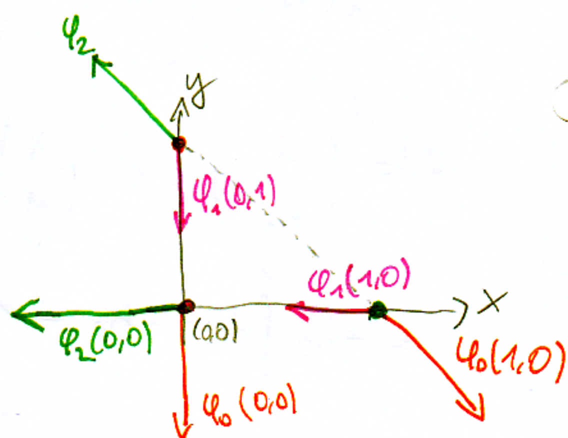
$$\Rightarrow b_1 = 0$$

$$\Psi_1(\varphi_1) = \ldots = \frac{1}{\sqrt{2}}(-a_1 - c_1) \overset{!}{=} 1 \quad \Rightarrow \quad c_1 = -\sqrt{2} - a_1$$

$$\Psi_2(\varphi_1) = \int_{y=1}^{0} \varphi_1(0,y) \begin{pmatrix}-1\\0\end{pmatrix} dy = \int \ldots = -a_1 \overset{!}{=} 0 \quad \Rightarrow \quad a_1 = 0$$

$$\Rightarrow c_1 = -\sqrt{2}$$

$$\hookrightarrow \varphi_1(x,y) = -\sqrt{2} \begin{pmatrix}x\\y\end{pmatrix}$$



$$\varphi_2(x,y) = \begin{pmatrix}a_2\\b_2\end{pmatrix} + c_2 \begin{pmatrix}x\\y\end{pmatrix}$$

$$\Psi_0(\varphi_2) = \ldots = -b_2 \overset{!}{=} 0$$

$$\Psi_2(\varphi_2) = \ldots = -a_2 \overset{!}{=} 1 \quad \Rightarrow \quad a_2 = -1$$

$$\Psi_1(\varphi_2) = \ldots = \frac{1}{\sqrt{2}}(b_2 - a_2 - c_2) \overset{!}{=} 0$$

$$\Leftrightarrow \quad b_2 - a_2 = c_2 = 0 - (-1) = 1$$

$$\hookrightarrow \varphi_2(x,y) = \begin{pmatrix}-1\\0\end{pmatrix} + \begin{pmatrix}x\\y\end{pmatrix}$$

What actually happened: $\qquad \hat{v} = \begin{pmatrix} a \\ b \end{pmatrix} + c \begin{pmatrix} \hat{x} \\ \hat{y} \end{pmatrix}$

$$\hat{\varphi}_0(\hat{v}) = \int_{\hat{E}_0} \hat{v} \cdot \hat{n}_0 \, ds = \ldots = -b$$

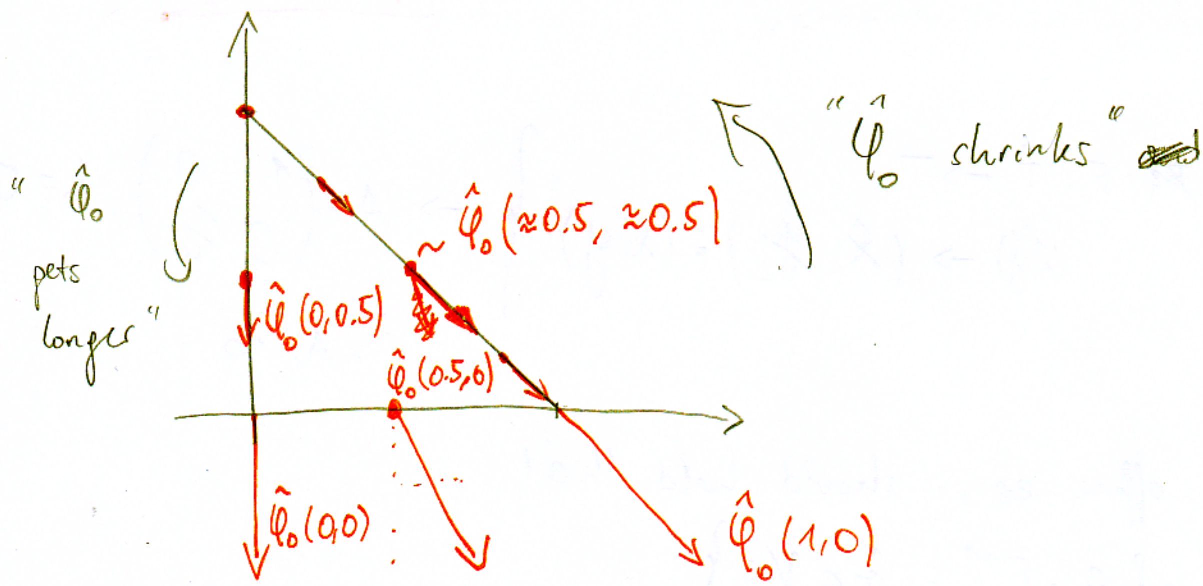
$$\hat{\varphi}_1(\hat{v}) = \ldots = -\frac{1}{\sqrt{2}}(a+b+c)$$

$$\hat{\varphi}_2(\hat{v}) = \ldots = a$$

Now assume $\hat{\varphi}_i = \begin{pmatrix} a_i \\ b_i \end{pmatrix} + c_i \begin{pmatrix} x \\ y \end{pmatrix}$ and insert above + use duality $\hat{\varphi}_i(\hat{\varphi}_j) = \delta_{ij}$

$\Rightarrow$ rules for construction of the $\hat{\varphi}_i$

$-\hat{\varphi}_0$

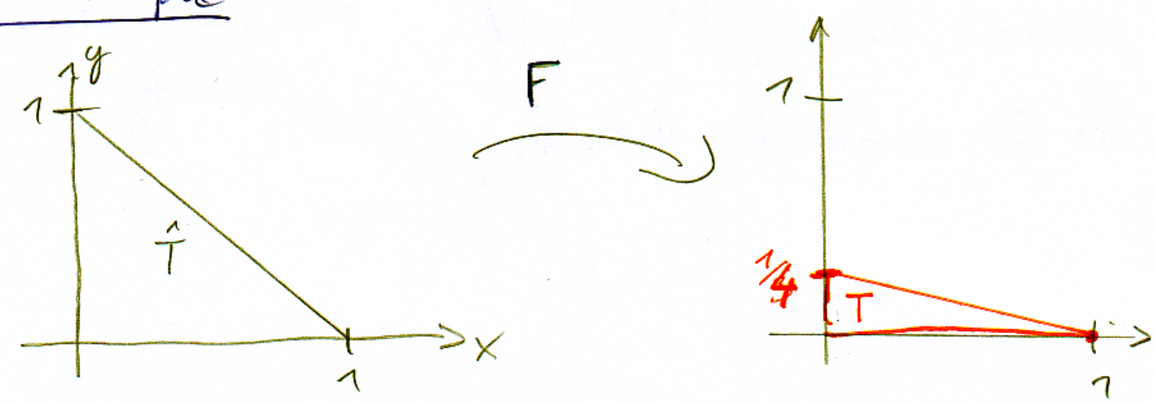


" $\hat{\varphi}_0$ gets longer + changes direction"

2) Let $F: \hat{T} \to T$, F... linear map $\Bigg\} \Leftrightarrow$ $T, \hat{T}$ are affine eq. triangles

Show that $(\hat{T}, V_{\hat{T}}, \hat{\psi}_{\hat{T}})$ and (T, V_T, ψ_T) are not affine eq. FE.

<u>Counterexample</u>



$$\Rightarrow \left.\begin{array}{l} F: \hat{T} \to T, \\ (\hat{x}, \hat{y}) \to (\hat{x}, \hat{y}/2) = (x, y) \end{array}\right\} \to A = \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{4} \end{pmatrix}, \; b = 0$$

$F = Ax + b$

For 2 affine eq, it should hold that

∘) $V_T = \{ \hat{v} \circ F^{-1} : \hat{v} \in V_{\hat{T}} \}$

$$F^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & 4 \end{pmatrix} x = \begin{pmatrix} \hat{x} \\ \hat{y} \end{pmatrix}$$

$$v(x,y) = (\hat{v} \circ F^{-1})(x,y) = \hat{v}(x, 4y) = \begin{pmatrix} a \\ b \end{pmatrix} + c \begin{pmatrix} x \\ 4y \end{pmatrix} \notin V_{\hat{T}}$$

"The problem"

<u>Side note:</u> Use $F(\hat{T}) = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix} \begin{pmatrix} \hat{x} \\ \hat{y} \end{pmatrix}$ $\to$ then the triples are aff eq.

3.) Let $F: \hat{T} \to T$ be affine linear with $F(\hat{x}) = A\hat{x} + b$,

~~sign~~ $\operatorname{sign}(\det(DF)) = \operatorname{sign}(\det A) > 0$

Define Piola trafo:

$$\varphi_i \circ F = \frac{(DF)\hat{\varphi}_i}{\det(DF)}, \quad i = 0,1,2$$

Show that $\{\varphi_i\}_{i=0,1,2}$ is dual basis to Ψ_T:

$$\Psi_i(\varphi_j) = \delta_{ij}$$

$$\underbrace{\Psi_i(\varphi_j)}_{T} = \int_{E_i} \overbrace{\varphi_j \cdot n_i}^{f} \, ds \overset{\text{Hint 1}}{=} \int_{\hat{E}_i} J_F\big((\varphi_j \cdot n_i) \circ F\big) d\hat{x}$$

corresponding

$$= \int_{\hat{E}_i} \det(DF) \, \| DF^{-T} \hat{n} \| \, (\varphi_j \circ F \cdot n_i \circ F) \, d\hat{x} \underset{\text{Hint 2}}{=}$$

Piola

$$= \int_{\hat{E}_i} \cancel{\det(DF)} \, \cancel{\| DF^{-T} \hat{n} \|} \left(\frac{(DF)\hat{\varphi}_j}{\cancel{\det(DF)}} \cdot \frac{(DF)^{-T}\hat{n}_i}{\cancel{\| DF^{-T} \hat{n} \|}} \right) =$$

$$= \int_{\hat{E}_i} (DF)\hat{\varphi}_j \cdot (DF)^{-T} \hat{n}_i = \underbrace{\int_{\hat{E}_i} \hat{\varphi}_j \hat{n}_i}_{\hat{\Psi}_i(\hat{\varphi}_j)} = \delta_{ij} \checkmark$$

There is probably a better argument...

$$r \, \cancel{x} = \underline{\underline{A}} \, \underline{x} \, \underline{\underline{A}}^{-1} \underline{y} \quad / \underline{\underline{A}}^{-1} \cdot$$

$$\hookrightarrow \underline{\underline{A}}^{-1} r = \underline{x} \, \underline{\underline{A}}^{-1} \underline{y} = \underline{\underline{A}}^{-1} \underline{y} \cdot \underline{x} \quad / \underline{\underline{A}} \cdot$$

$$\hookrightarrow r = \underline{y} \, \underline{x}$$

Hint 1: Let E be edge of T and $f \in L^1(E)$. Then

$$\int_E f \, dx = \int_{\hat{E}} J_F (f \circ F) \, d\hat{x},$$

$$J_F = \det(DF) \cdot \| (DF)^{-T} \hat{n} \|$$

Hint 2: n ... unit outer normal vector of T and $\hat{n}$ ... of $\hat{T}$. Then

$$n \circ F = \frac{(DF)^{-T} \hat{n}}{\| (DF)^{-T} \hat{n} \|}$$