Ex5.2 | $\hat{\varphi}_0(\hat{x},\hat{y}) = \varphi_0(\hat{x})\,\varphi_0(\hat{y})$,

$\hat{\varphi}_1(\hat{x},\hat{y}) = \varphi_1(\hat{x})\,\varphi_0(\hat{y})$

$\hat{\varphi}_2(\hat{x},\hat{y}) = \varphi_1(\hat{x})\,\varphi_1(\hat{y})$

$\hat{\varphi}_3(\hat{x},\hat{y}) = \varphi_0(\hat{x})\,\varphi_1(\hat{x})$

where $\varphi_0(s) = 1 - s$, $\varphi_1(s) = s$

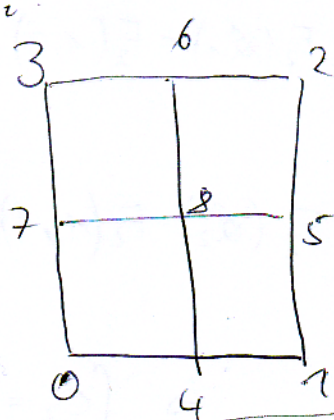
a.) $u(x,y) = x + y = \sum_i c_i \varphi_i$

Since point eval functionals $\rightarrow$ $\varphi_i(a_i) = 1$

$\varphi_i(a_j) = 0,\ j \neq i$

$\rightarrow u(a_i) = x_i + y_i = c_i \cdot \varphi_i(x_i, y_i)$

a_i:



$a_0 = (0,0)$

$\rightarrow u(0,0) = 0 = c_0 \cdot \overbrace{\varphi_0(a_0)}^{=1} \Rightarrow c_0 = 0$

a_1: $u(1,0) = 1 = c_1 \cdot 1$

a_2: $u(1,1) = 2 = c_2$

a_3: $u(0,1) = 1 = c_3$

a_4: $u(0.5, 0) = 0.5 = c_4$

a_5: $u(1, 0.5) = 1.5 = \frac{3}{2} = c_5$

a_6: $u(0.5, 1) = 1.5 = \frac{3}{2} = c_6$

a_7: $u(0, 0.5) = 0.5 = c_7$

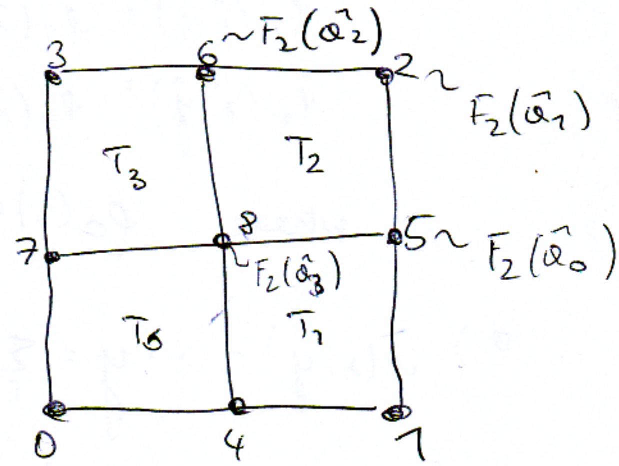
a_8: $u(0.5, 0.5) = 1 = c_8$

$\Rightarrow c = \left(0, 1, 2, 1, \frac{1}{2}, \frac{3}{2}, \frac{3}{2}, \frac{1}{2}, 1\right)^T$

b) Determine the linear map $F_2 : \hat{T} \to T_2$,

s.t. $\quad F_2(\hat{a}_0) = a_5$; $F_2(\hat{a}_1) = a_2$, $F_2(\hat{a}_2) = a_6$,

$F_2(\hat{a}_3) = a_8$



$\underline{F_2 \in Q^1}$

$$\Rightarrow F_2(\hat{x},\hat{y}) = \begin{pmatrix} c_0 + c_1\hat{x} + c_2\hat{y} + c_3\hat{x}\hat{y} \\ d_0 + d_1\hat{x} + d_2\hat{y} + d_3\hat{x}\hat{y} \end{pmatrix}$$

$$F_2(\hat{a}_0) = F_2(0,0) = \begin{pmatrix} c_0 \\ d_0 \end{pmatrix} \overset{!}{=} \begin{pmatrix} 1 \\ 0.5 \end{pmatrix} = a_5 \Rightarrow \begin{cases} \underline{c_0 = 1} \\ \underline{d_0 = 0.5} \end{cases}$$

$$F_2(\hat{a}_1) = F_2(1,0) = \begin{pmatrix} 1 + c_1 \\ 0.5 + d_1 \end{pmatrix} \overset{!}{=} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = a_2 \Rightarrow \begin{cases} \underline{c_1 = 0} \\ \underline{d_1 = 0.5} \end{cases}$$

$$F_2(\hat{a}_2) = F_2(1,1) = \begin{pmatrix} 1 + 0 + c_2 + c_3 \\ 0.5 + 0.5 + d_2 + d_3 \end{pmatrix} \overset{!}{=} \begin{pmatrix} 0.5 \\ 1 \end{pmatrix} = a_6$$

$$\hookrightarrow \begin{cases} c_2 = -(c_3 + \tfrac{1}{2}) \quad \text{bzw} \quad c_3 = -\tfrac{1}{2} - c_2 \\ d_2 = -d_3 \end{cases}$$

$$F_2(\hat{a}_3) = F_2(0,1) = \begin{pmatrix} 1 + 0 + c_2 \\ \tfrac{1}{2} + 0 + d_2 \end{pmatrix} \overset{!}{=} \begin{pmatrix} \tfrac{1}{2} \\ \tfrac{1}{2} \end{pmatrix} = a_8$$

$$\hookrightarrow \begin{cases} \underline{c_2 = -\tfrac{1}{2}} \Rightarrow \underline{c_3} = -c_2 - \tfrac{1}{2} = \tfrac{1}{2} - \tfrac{1}{2} = \underline{0} \\ \underline{d_2 = 0 = d_3} \end{cases}$$

$$\Rightarrow F_2(\hat{x},\hat{y}) = \begin{pmatrix} 1 - \tfrac{1}{2}\hat{y} \\ \tfrac{1}{2} + \tfrac{1}{2}\hat{x} \end{pmatrix} \checkmark$$

Test: $F_2(1,0) = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \checkmark$, $F_2(1,1) = \begin{pmatrix} 0.5 \\ 1 \end{pmatrix} \checkmark$, $F_2(0,0) = \begin{pmatrix} 1 \\ \tfrac{1}{2} \end{pmatrix} \checkmark$, $F_2(0,1) = \begin{pmatrix} \tfrac{1}{2} \\ \tfrac{1}{2} \end{pmatrix}$

Compute DF_2 and $\det(DF_2)$

$$F_2(x,y) = \begin{pmatrix} 1 - \frac{1}{2}y \\ \frac{1}{2} + \frac{1}{2}x \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 2 - y \\ 1 + x \end{pmatrix} = \cancel{1}\begin{pmatrix} F_{2x} \\ F_{2y} \end{pmatrix}$$

$$DF_2 = \begin{pmatrix} \partial_x F_{2x} & \partial_y F_{2x} \\ \partial_x F_{2y} & \partial_y F_{2y} \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

$$\det(DF_2) = \left(\frac{1}{2}\right)^2 \cdot \left(0 - (-1)\right) = \frac{1}{4}$$