NUMPDE Exercise 6

1 Example 6.1

Proof that:

$$Q_T(v) = \int_T v\,dx \quad \forall v \in \mathbb{P}^2(T)$$

First we define a $\hat{v}(\hat{x},\hat{y})$ Element and integral over it.

$$\hat{v}(\hat{x},\hat{y}) = c_0 + c_1\hat{x} + c_2\hat{y} + c_3\hat{x}^2 + c_4\hat{y}^2 + c_5\hat{x}\hat{y}$$

$$\int_{\hat{x}=0}^{1}\int_{\hat{y}=0}^{1-\hat{x}} \hat{v}d\hat{x} = \frac{c_0}{2} + \frac{c_1+c_2}{6} + \frac{c_3+c_4}{12} + \frac{c_5}{24}$$

Now we can use the midpoint rule to calculate the area $\in V$.

$$Q_{\hat{T}}(v) := \frac{|\hat{T}|}{3}\left[v(\hat{x}_1)v(\hat{x}_2)v(\hat{x}_3)\right]$$

Also the are of $|\hat{T}| = \frac{1}{2}|T|$ and

$\hat{v}(\hat{x_1}) = \hat{v}(\frac{1}{2},0) = c_0 + \frac{c_1}{2} + \frac{c_3}{4}$
$\hat{v}(\hat{x_2}) = \hat{v}(\frac{1}{2},\frac{1}{2}) = c_0 + \frac{c_1}{2} + \frac{c_3}{4}$
$\hat{v}(\hat{x_3}) = \hat{v}(0,\frac{1}{2}) = c_0 + \frac{c_1}{2} + \frac{c_2}{2} + \frac{c_3}{4} + \frac{c_4}{4} + \frac{c_5}{4}$

$Q_{\hat{T}}(\hat{v}) = \frac{c_0}{2} + \frac{c_1+c_2}{6} + \frac{c_3+c_4}{12} + \frac{c_5}{24}$
We showed finally that $Q_{\hat{T}}(\hat{v}) = \int_{\hat{T}}\hat{v}$ and we know that lagrange FE are equivalent. We define $F : \hat{T} \to T$ as an affine linear mapping.

$$\int_T v(x)\,dx = \int_T \hat{v} \circ F^{-1}(x)\,dx = \int_{\hat{T}} \Big(\hat{v} \circ F^{-1}\Big) \circ F(\hat{x})|\det DF|\,d\hat{x} = |\det DF|\int_{\hat{T}}\hat{v}(\hat{x})\,d\hat{x}$$

At last we only have to show that $\int_{\hat{T}}\hat{v}(\hat{x})\,d\hat{x} = Q_{\hat{T}(\hat{v})}$

$$|\det DF|Q_{\hat{T}(\hat{v})} = |\det DF|\frac{|\hat{T}|}{3}\Big(\hat{v}(\hat{x_1}) + \hat{v}(\hat{x_2}) + \hat{v}(\hat{x_3})\Big)$$
$$= \frac{|\hat{T}|}{3}\Bigg((v\circ F)(F^{-1}(x_1)) + (v\circ F)(F^{-1}(x_2)) + (v\circ F)(F^{-1}(x_3))\Bigg)$$
$$= \frac{|\hat{T}|}{3}\Big(v(x_1) + v(x_2) + v(x_3)\Big)$$

2 Example 6.2

This proof stats with Theorem 69 and we start with where $Q_h(v) = \sum_{T\in\mathbb{T}} Q_T(v) = \sum_{T\in\mathbb{T}} \int_T v\,dx$:

$$
\begin{aligned}
&|\int_\Omega v\,dx - Q_h(v)|^2 = |\int_\Omega v\,dx - \sum_{T\in T_h}\int_T I_T v\,dx|^2 = |\sum_{T\in T_h}\int_T (v - I_T v) dx|^2 \\
&= \sum_{T\in\mathbb{T}} |\int (id - I_T) v_T\,dx|^2 \leq \sum_{T\in T_h} ||\int_T (id - I_T) v\,dx||^2_{L^2(T)}
\end{aligned}
$$

Now using (4.2)

$$\sum_{T\in T_h} \det(B)||\int (id - I_T) v\,dx||^2_{L_2(\hat{T})}$$

and after the transformation to the reference triangle where $I_T = I_{\hat{T}}$

$$\sum_{T\in T_h} \det(B)||\int \det(B)(id - I_{\hat{T}})(v \circ F_T)\,d\hat{x}||^2_{L_2(\hat{T})}$$

After that we apply the Bramble Hilbert lemma. The Quatrature rule is exact for polynominals for decreas 2 as we seen in 6.1 $\Rightarrow \int (id - I_{\hat{T}}) q\,dx = Lq = 0$ for $q \in \mathbb{P}^2(\hat{T})$
Theorem 54: $||Lu||_a \leq ||u_{H^k}$

$$\leq \sum_{T\in\mathbb{T}} \det(B)|v \circ F_T|^2_{H^3(\hat{T})}$$

$$\preceq \sum_{T\in\mathbb{T}} \det(B)\det(B)^{-1}||B||^6|v|^2_{H^3(T)}$$

In the end we use the quasi-uniform $||B||^6 \simeq h^6$ propperty:

$$||\int_\Omega v\,dx - Q_h(v)||_{L_2(\Omega)} \preceq h^3|v|_{H^3(\Omega)}$$

3 Example 6.3

We first proof that $v_h \in V^k_{h,0}$ there holds: $||e_h'||^2_{L^2(\Omega)} = (e_h', e_h' - v_h')_{L^2(\Omega)}$

$$\int u'v'\,d\Omega = A(u,v) \qquad \int f\,v\,d\Omega = f(v)$$

$$
\begin{aligned}
&A(e_h, e_h - v_h) = A(u - u_h, u - u_h) - A(u - u_h, v_h) \\
&= A(u - u_h, u - u_h) - A(u, v_h) + A(u_h, v_h) \\
&= A(u - u_h, u - u_h) - f(v_h) + f(v_h) = A(e_h, e_h) \\
&= ||e_h'||^2_{L^2(\Omega)}
\end{aligned}
$$

By applay the integration by parts:

$$\int e_h'(e_h' - v_h')\,dT = \frac{d}{dx}\int e_h'(e_h - v_h)\,dT - \int e_h''(e_h - v_h)\,dT = e_h'' = f + u_h''$$

where $\frac{d}{dx}\int e_h'(e_h - v_h)\,dT = 0$ And so:

$$\begin{aligned}
||e_h'||_{L^2(\Omega)} &= \sum_{T\in T_h} ||e_h'||^2_{L^2(T)} = \sum_{T\in T_h} (e_h', e_h' - v_h')_{L^2(T)} \\
&= \sum_{T\in T_h} (-e_h'', e_h - v_h)_{L^2(T)} \\
&\leq \sum_{T\in T_h} (-e_h'', e_h - I_h e_h)_{L^2(T)} \\
&= \sum_{T\in T_h} (f + u_h'', e_h - I_h e_h)_{L^2(T)}
\end{aligned}$$

For the second proof we start:

$$\begin{aligned}
&||e_h'||^2_{L^2(\Omega)} = |u - u_h|_{H_1(\Omega)} \\
&e_h' - I_h e_h' = u' - u_h' - I_h u' + I_h u_h' = u' - I_h u'
\end{aligned}$$

$$\begin{aligned}
&||e_h'||^2_{L^2(\Omega)} = \sum_{T\in T_h} ||e_h'||^2_{L^2(T)} = \sum_{T\in T_h} (u' - u_h', u' - u_h')_{L^2(T)} \\
&\leq \sum_{T\in T_h} (u' - I_h u', u' - I_h u')_{L^2(T)} = \sum_{T\in T_h} (e_h' - I_h e_h', e_h' - I_h e_h')_{L^2(T)} \\
&c \sum_{T\in T_h} h_T^2 |e_h'|^2_{H_1} = c \sum_{T\in T_h} h_T^2 || - (f + u_h'')^2_{L^2} = c \sum_{T\in T_h} h_T^2 ||f + u_h''||^2_{L^2} \\
&|u - u_h|_{H^1(\Omega)} \leq \sum_{T\in T_h} h_T^2 ||f + u_h''||^2_{L^2}
\end{aligned}$$

$$y = a_0 + a_1 x$$

$$J = \frac{1}{n}\sum_{i=1}^{n} (\hat{y}_i - y_i)^2$$

$$J_L = \sum_{i=1}^{n} (\hat{y}_i - y_i)^2 + \alpha \sum_{j=1}^{m} |\omega_j|$$