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Numerical methods for partial differential equations

Exercise 10 – 26. May 2020

Example 10.1

1. Implement the weak formulation of the Dirichlet boundary conditions on the unit square with non homogenous Dirichlet boundary conditions $u_D = y(1-y)(1-x)$ and a zero right hand side. In NGSolve use the spaces

```
V = H1(mesh, order = k)
Q = SurfaceL2(mesh, order = k-1)
```

2. Follow the same steps as in the derivation of the mixed methods for the weak formulation of the Dirichlet boundary conditions in the lecture to derive a mixed method for the Poisson problem with Robin boundary conditions

$$\begin{aligned} -\Delta u &= f \quad \text{in } \Omega \\ u - u_D &= \varepsilon \frac{\partial u}{\partial n} \quad \text{on } \partial\Omega \end{aligned}$$

with a given ε and u_D. Implement this method with $f = 100e^{-100((x-0.5)^2+(y-0.5)^2)}$, $\varepsilon = 1$ and $u_D = 1$.

Example 10.2

We continue with the findings of the previous example. Let $\Omega_1 = (0,1) \times (0,1)$ and $\Omega_2 = (1,2) \times (0,1)$. Further let $\Gamma_{int} := \{1\} \times (0,1)$ and $\Gamma_{out} := (\partial\Omega_1 \cup \partial\Omega_2) \setminus \Gamma_{int}$. Solve the problem

$$\begin{aligned} -\Delta u_1 &= 10 \quad \text{in } \Omega_1 \\ -\Delta u_2 &= 0 \quad \text{in } \Omega_2 \\ u_1 = u_2 &= 0 \quad \text{on } \Gamma_{out} \\ u_1 - u_2 &= \varepsilon \frac{\partial u_1}{\partial n_1} \quad \text{on } \Gamma_{int} \\ \frac{\partial u_1}{\partial n_1} &= -\frac{\partial u_2}{\partial n_2} \quad \text{on } \Gamma_{int}, \end{aligned}$$

with $\varepsilon = 0.1$. We approximate both solutions in H^1-conforming finite element spaces defined on Ω_1 and Ω_2 and use a mixed method to incorporate the boundary conditions. Use the file `jump.py` as starting point.

Example 10.3

Consider the instationary Navier-Stokes equations with homogeneous Dirichlet boundary conditions ($u_D = 0$ on $\partial\Omega$) and $f = 0$. Show that the kinematic energy

$$\frac{1}{2}\int_{\Omega} |u|^2 \,\mathrm{d}x$$

is monotone decreasing (in time). Give a physical interpretation.
Hint: Try to reformulate the convective term and use $\operatorname{div}(u) = 0$. Note that the i-th component of $(u \cdot \nabla)u$ is given by

$$[(u \cdot \nabla)u]_i = \sum_{j=1}^{d} u_j \frac{\partial u_i}{\partial x_j}.$$

Does your proof also hold for a discrete method? (How is the divergence constraint incorporated...)

Example 10.4 (Right inverse of divergence)

Let Ω be star shaped with respect to $\omega \subset \Omega$, and let $a \in \omega$. Let $p \in L^2_0(\Omega)$ and extend it trivially by zero to $L^2(\mathbb{R}^d)$. We define

$$u_a(x) := -(x-a)\int_1^{\infty} t^{d-1} p(a + t(x-a)) \,\mathrm{d}t \quad x \neq a,$$

and $u_a(a) = 0$. Show: if $\int_\Omega p = 0$, then we have

$$\operatorname{div}(u_a) = p \quad \text{in } \Omega, \quad \text{and} \quad u = 0 \quad \text{on } \partial\Omega.$$

Where do we need the assumption $\int_\Omega p = 0$? You can assume that p is smooth enough such that all integrals and derivations exist.

Remark: Using the above result one can then define an averaging over all star-points

$$u := \frac{1}{|\omega|}\int_{\omega} u_a \,\mathrm{d}a.$$

Then we still have $\operatorname{div}(u) = p$ and it is possible to prove that $\|u\|_{H^1} \preceq \|p\|_{L^2}$. Thus above constructions proves the continuous Stokes-LBB if Ω is star shaped.