PDEs | 2.3.2020

§

→ course homepage: asc-wiki page

→ PW: numpde20

→ lecture:

↳ Mo., ~~...~~ 14:00 - 16:00

↳ Tu:, 14:00 - 15:00

→ (oral) exam: before 1. Aug (philip gone for 5 months afterwards)

→ exercise: date → tbd

↳ 1x per week ... maybe after lecture on monday

Chapter 1: Introduction

Assume a given domain $\Omega \in \mathbb{R}^d$. A general form of a linear PDE is given by: find a function $u: \Omega \to \mathbb{R}$, $u \in ?$ s.t.

$$\sum_{i,j=1}^{d} -\frac{\partial}{\partial x_i}\left(a_{ij}(x)\frac{\partial u}{\partial x_j}\right) + \sum_{i=1}^{d} b_i(x)\frac{\partial u(x)}{\partial x_i} + c(x)\,u(x) = f(x)$$

where a_{ij}, b_i, c and f are given functions. Depending on the values ($=0$ or $\neq 0$) of the coeffs, the second order diff operator,

$$L := \sum -\frac{\partial}{\partial x_i} a_{ij} \frac{\partial}{\partial x_j} + \sum^{d} b_i \frac{\partial}{\partial x_i} + c,$$ describes the behaviour of the PDE.

In general, second order PDEs can be divided into elliptic, parabolic and ~~by~~ hyperbolic systems.

Some ex: • poisson eq (elliptic)

$$(1) \quad -\Delta u(x) = f(x) \qquad \left(\Delta := \sum_{i}^{d} \frac{\partial^2}{\partial x_i^2}\right)$$

• Hyperbolic equations

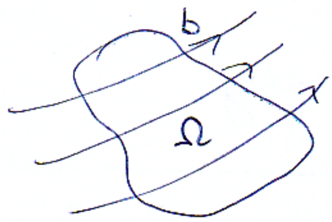
$$-\sum_{i,j}^{d} \frac{\partial^2 u}{\partial x_i \partial x_j} + \frac{\partial^2 u}{\partial x_d^2} = f \qquad (x_d = t \Rightarrow \partial_t^2 u - \Delta u = f \ldots \text{ wave eq})$$

- Heat eq (parabol)

$$\partial_t u - \Delta u = f$$

- Transport eq

$$\sum_i^d b_i \frac{\partial u}{\partial x_i} \; (+c\cdot u) = \underset{\substack{\uparrow \\ \text{often called „wind"}}}{b} \cdot \nabla u \; (+c \cdot u) = f$$

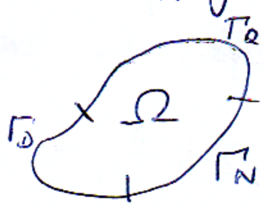


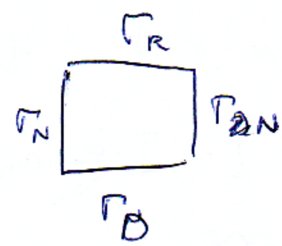
Note: all equations need BCs in order to close the system

1.1.) Weak formulations of the Poisson equation

Suppose an open domain (open set) $\Omega \in \mathbb{R}^d$ s.t. the boundary $(\partial\Omega)$ is split into three non overlapping parts

$$\partial\Omega := \Gamma_D \cup \Gamma_N \cup \Gamma_R$$



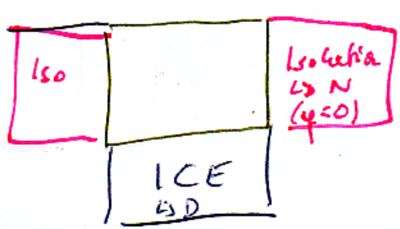


We consider the poisson eq $-\Delta u = f$ in Ω with the boundary condition

$u = u_D$ on Γ_D (Dirichlet: function value)

$\frac{\partial u}{\partial n} = \nabla u \cdot n = g$ on Γ_N (Neumann $\hat{=}$ normal flux)

$\left(\frac{\partial u}{\partial n} + \alpha u\right) = g$ on Γ_R (Robin: lin comb of both $\hat{=}$)



There exits different solution ~~d~~ (classes) to the Poisson eq.
A classical solution would require regularity $u \in C^2(\Omega) \cap C(\Omega)$.
We want to construct a weak formulation of (1) with its BC. For this we choose a smooth function v ~~for~~ (for a $\lambda \in C_0^\infty(\Omega) = \{ v \in C^\infty : \operatorname{supp}(v) \text{ is compact set} \}$) and
$\operatorname{supp}(v) := \{ x \in \Omega : v(x) \text{ non zero} \}$

multiply (1) with v and integrate over Ω:

$$\int_\Omega -\Delta u \, v = \int_\Omega f \cdot v \, (dx)$$

Applying integration by parts gives (1D: $\int f' g = fg\big|_{\partial\Omega} - \int f g'$)

$$\int_\Omega \nabla u \cdot \nabla v - \int_{\partial\Omega} \frac{\partial u}{\partial n} \cdot v \overset{\text{apply "natural bc"}}{=} \int_\Omega \nabla u \cdot \nabla v + \int_{\Gamma_R} \alpha u v - \int_{\Gamma_D} \frac{\partial u}{\partial n} v =$$

$$= \int f v + \int_{\Gamma_N} g \, v + \int_{\Gamma_R} g v$$

Dirichlet bc are incorporated in a "strong" sense ($\hat{=}$ essential bc).
At the same time we also assume that $v = 0$ on Γ_D. The weak form of the Poisson eq is:

Find $u \in ?$, with $u = u_D$ on Γ_D s.t.

$$(2) \qquad \int_\Omega \nabla u \cdot \nabla v + \int_{\Gamma_R} \alpha u \cdot v = \int_\Omega f \cdot v + \int_{\Gamma_N \cup \Gamma_R} g v, \quad \forall v \in ? \text{ with } v = 0 \text{ on } \Gamma_D$$

The "proper" space, such that all the above terms are well defined, is given by $V := \{ v \in L^2(\Omega) : \nabla v \in L^2(\Omega, \mathbb{R}^d), v|_{\partial\Omega} \in L^2(\partial\Omega) \}$

$\Rightarrow$ V is a complete space ~~with~~ and with
$(u,v)_V := (u,v)_{L^2(\Omega)} + (\nabla u, \nabla v)_{L^2(\Omega)} + (u,v)_{L^2(\partial\Omega)}$ a Hilbert space.

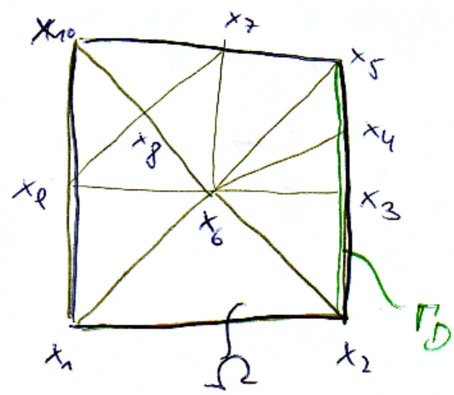
Further we need $f \in L^2(\Omega)$, $g \in L^2(\Gamma_N \cup \Gamma_R)$

The Dirichlet data has to be chosen s.t. there exists an $u \in V$ with $u = u_D$ on Γ_D

1.2. Finite Element Method

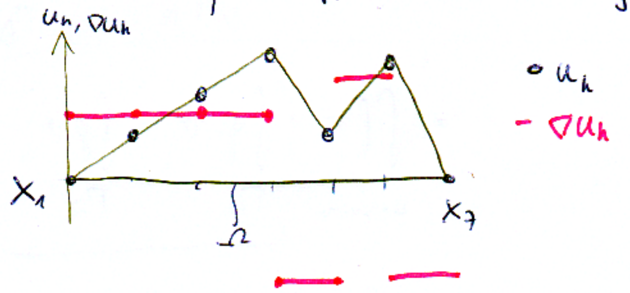
Find a num. method for approximation ~~the~~ of the solution of the weak form (2). Decompose the domain Ω into triangles or tetrahedrons for 2,3 dim respectively.

We call $\mathcal{T} := \{T\}$ a triangulation with the corresponding nodes (vertices) $\mathcal{N} = \{x_i\}$



Define a finite element space

$$V_h := \{ v \in C(\Omega) : \overbrace{v|_T \text{ is affine linear}}^{\text{"a linear polynomial"}} \ \forall T \in \mathcal{T} \}$$



The derivatives of u_h are p.w. constant $\Rightarrow L^2(\Omega)$.

Every function in V_h is determined by its values in $\mathcal{N}$.

Next we decompose $\underbrace{\mathcal{N}_D}_{\text{Dirichlet}} = \text{"}\mathcal{N} \cap \Gamma_D\text{"}$ and $\underbrace{\mathcal{N}_f}_{\text{free}} = \mathcal{N} \setminus \mathcal{N}_D$

$\rightarrow$ $\mathcal{N}_D = \{x_2, x_3, x_4, x_5\}$

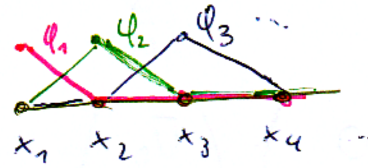
The finite element approximation now reads as:

Find $u_h \in V_h$, $u(x_i) = u_D(x_i)$ $\forall x_i \in \mathcal{N}_D$ s.t.

$$(3) \quad \int_\Omega \nabla u_h \nabla v_h + \int \alpha u_h v_h = \int f v_h + \int_{\Gamma_N + \Gamma_R} g v_h \qquad \forall v_h \in V_h, \ v_h(x_i) = 0 \ \ \forall x_i \in \mathcal{N}_D$$

Now we define a basis for V_h: The most convenient is the nodal basis $\{\varphi_e\}$ given by

$$\varphi_i(x_j) = \delta_{ij}$$



$\Rightarrow$ We can write u_h in terms of φ_i

$$(4)\quad u_h(x) = \sum_{i=1}^{|\mathcal{N}|} \underbrace{u_i}_{\text{coeffs}} \varphi_i(x) \qquad (u_h(x_j) = u_j),\ N = |\mathcal{N}| \text{ – number of vertices/nodes}$$

Solving (3) shows that we just have to find the coefficients $\bar{u} = (u_1, u_2, \dots, u_N)$, $N = |\mathcal{N}|$.

Note that the coeffs on $\mathcal{N}_D$ are already defined (because ~~property~~ $u(x_i) = u_D$, $\forall x_i \in \mathcal{N}_D$)

$$\rightarrow \sum_i \left\{ \int_\Omega \nabla\varphi_i \nabla\varphi_j + \alpha \int_{\Gamma_R} \varphi_i \varphi_j \right\} u_i = \int f \varphi_j + \int g \varphi_j \qquad \forall \varphi_j,\ x_j \in \mathcal{N}_f$$

[$u_h(x) = \sum_i u_i \cdot \varphi_i(x)$

$v_h(x) = \varphi_j(x)$ ... "arbitrary" test function]

We define $(A_{ij}) \in \mathbb{R}^{N\times N}$, $\vec{f} = \vec{f}_j \in \mathbb{R}^N$, $N = |\mathcal{N}|$ ... number of nodes

$$A_{ij} = \int \nabla\varphi_i \nabla\varphi_j + \alpha \int \varphi_i \varphi_j \ ; \qquad \vec{f}_j = \int_\Omega f \varphi_j + \int_{\Gamma_N \cup \Gamma_R} g \varphi_j$$

$$\Rightarrow A = \begin{pmatrix} A_{DD} & A_{Df} \\ A_{fD} & A_{ff} \end{pmatrix}, \quad \vec{f} = \begin{pmatrix} f_D \\ f_f \end{pmatrix}$$

2. Lecture 3.3.2020

$\bar{u} = (\bar{u}_D, \bar{u}_f)$

$u_i, x_i \in \mathcal{N}_D$

$$\begin{array}{ccc} A & \cdot \; \vec{u} & = \vec{f} \end{array}$$

$$\begin{pmatrix} I & 0 \\ A_{fD} & A_{ff} \end{pmatrix} \begin{pmatrix} \bar{u}_D \\ \bar{u}_f \end{pmatrix} = \begin{pmatrix} \bar{u}_D \\ \vec{f}_f \end{pmatrix} \quad f_D = u_D$$

$$\Rightarrow \quad A_{ff}\,\vec{\bar{u}}_f = \vec{f}_f - A_{fD}\,\vec{\bar{u}}_D$$

2. Abstract theory

2.1. Basic properties

Define lemmas/theorems ≙ framework to derive solvability conditions (LAX-Milgram) for var. formulations

Def.:
A functional or linear form $l(\cdot)$ on V (normal vector space) is a linear mapping $l(\cdot): V \to \mathbb{R}$.

- The norm for lin. forms is the dual norm

$$(\|l\|_{V'} =) \; \|l\|_{V^*} := \sup_{0 \neq v \in V} \frac{l(v)}{\|v\|_V}$$

- l is bounded $\Leftrightarrow$ $\|l\|_{V^*} < \infty$
- V^* (dual space) ≙ vector space of all bounded l.f.
- For $f \in V^*$ we also write $\langle f, u \rangle_V = f(u)$

Ex: $l(\cdot): L^2(\Omega) \to \mathbb{R}$, Ω bounded

$$l(v) = \int_\Omega v \, dx \overset{C.S.}{\leq} \underbrace{\left(\int_\Omega v^2\right)^{\frac{1}{2}}}_{\|v\|_{L^2}} \cdot \underbrace{\left(\int_\Omega 1\right)^{\frac{1}{2}}}_{|\Omega|}$$

Def.: A bilinear form $A(\cdot,\cdot)$ on V is a mapping $A: V \times V \to \mathbb{R}$, which is linear in each component. It is called symmetric if

- $A(u,v) = A(v,u) \quad \forall u,v \in V$ (ex: $A(u,v) = \int u \cdot v$
- A is called inner prod if
  - $A(u,u) \geq 0 \quad \forall u \in V$
  - $A(u,u) = 0 \Leftrightarrow u = 0$

Lemma Cauchy-Schwarz inequality

If $A(\cdot,\cdot)$ is a sym. blf s.t. $A(u,u) \geq 0 \quad \forall u \in V$:

$$A(u,v) \leq A(u,u)^{\frac{1}{2}} A(v,v)^{\frac{1}{2}} \quad \forall u,v \in V$$

Theorem 2.1: Let S be a closed subspace of a Hilbert space V. Let $u \in V$, then there exists a unique closest point $u_0 \in S$, thus

$$\|u - u_0\|_V \leq \|u - v\| \quad \forall v \in S$$

Riesz representation theorem:

Let $u \in V$, V... Hilbert space, we define

$$l_u(\cdot) \in V^* \quad \text{by} \quad l_u(v) := (u, v)_V \qquad \forall v \in V$$

Theorem 2.5

$$\forall l \in V^* \quad \exists u_l \in V \quad \text{s.t.}$$

$$l(v) = (u_l, v)_V \qquad \forall v \in V$$

and $\|l\|_{V^*} = \|u_l\|_V$

2.4. Symmetric variational formulations

We take the function space $C^1(\Omega)$, $\Omega \in \mathbb{R}^d$ and bounded, and we define the blf $A(u,v) = \int_\Omega \nabla u \nabla v + \int_\Gamma u v$

and the lf $f(v) = \int_\Omega f v$

As [$A(u,v)$ is non negative] and [$A(u,u) = 0 \Leftrightarrow u = 0$], we get that $A(\cdot,\cdot)$ is an inner product with the norm

$$\|v\|_A := A(v,v)^{\frac{1}{2}}$$

Since the vector space $(C^1(\Omega), \|\cdot\|_A)$ is not a complete space, we define the space $V := \overline{C^1(\Omega)}^{\|\cdot\|_A}$ ($\hat{=}$ the Sobolev space H^1)

If we can show that there exists a constant (which we will do) s.t.

$$f(v) = \int f \cdot v \leq C \|v\|_A \quad \forall v \in V$$

$\Rightarrow f(\cdot)$ is a continous (bounded) lf on V.

Rāiesz-theorem

↳ ~~Riesz-theorem~~ => $\exists$ an unique $u_f \in V$ s.t.

$$A(u_f, v) = f(v)$$

=> week formulation has ~~an~~ on unique solution

FEM-Approx: Choose $V_h \subset V$ (V_h finite dim => closed)

The FEM-approx was given by

$$A(u_h, v_h) = f(v_h) \qquad \forall u_h \in V_h$$

$$A(u-u_h, v_h) = A(u, v_h) - A(u_h, v_h) = f(v_h) - f(v_h) = 0 \quad \forall v_h \in V_h$$

=> $\|u - u_h\|_A \leq \|u - v_h\|_A \qquad \forall v_h \in V_h$

=> $u - u_h$ is orth. on V_h