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Numerical methods for partial differential equations
Exercise 11 – 9. June, 2020

Example 11.1

Simulate the Navier-Stokes equations on a channel with a step-like cross-section extension: $\Omega = (0,2] \times (0,1) \cup (2,10) \times (-1,1)$. On the left side we have a parabolic inflow profile, on the right side natural boundary conditions (outlet), thus $(-\nu \nabla u + pI) \cdot n = 0$. Everywhere else we have wall boundary conditions (homogeneous Dirichlet bc). Choose a viscosity $\nu = 10^{-3}$ and a zero right hand side and use the Taylor-Hood element and the IMEX scheme as in the tutorial example `navierstokes.py`. Increase the inflow velocity and present an interpretation of your results. For this you might google "backward facing step, navier stokes". Further note, that you might need to adapt the time step size τ.

Example 11.2 (discrete LBB)

Let (P^2-bubble and P^1 discontinuous element)

$$\begin{aligned} V_h &:= \{v_h \in H_0^1(\Omega, \mathbb{R}^d) : (v_h)|_T \in P^3(T), (v_h)|_E \in P^2(E), \forall T \in \mathcal{T}_h, \forall E \subset \partial T\}, \\ Q_h &:= \{q_h \in L_0^2(\Omega) : (q_h)|_T \in P^1(T)\}. \end{aligned}$$

Prove the discrete Stokes-LBB for $V_h \times Q_h$.

Hint: An arbitrary polynomial q on an element T can be written as $q = q_1 + q_0$ with $\int_T q_0 = 0$. Use a combination of a Fortin operator introduced in the lecture and a new one where you control the mean value of q using the additional bubble functions in V_h.

Example 11.3

Let j be a divergence free current density in the sense of $(j, \nabla\psi)_{L^2} = 0$ for all $\psi \in H^1$. Let $A = A^0$ be the solution of the mixed formulation (WF2) of the lecture notes. Let A^ε be the solution of the regularized problem: Find $A^\varepsilon \in H(\mathrm{curl})$ such that:

$$\int_\Omega \mu^{-1} \operatorname{curl} A^\varepsilon \cdot \operatorname{curl} v + \varepsilon A^\varepsilon \cdot v = \int_\Omega j \cdot v,$$

where $\varepsilon > 0$. Show that $\|A - A^\varepsilon\|_{H(\mathrm{curl})} = \mathcal{O}(\varepsilon)$.

Hint: Write down both solutions in a mixed form and subtract them. Further assume that the mixed system is solveable and that the stability estimate in the Brezzi theorem holds true.

Example 11.4

Extend the example `maxwell.py`. The coil should consist of 1000 windings with a current of 1 Ampere. We do not want to resolve the wires within the mesh but want to replace it using a current density (as in the lecture). Calculate this density for the coil with the geometry as in the python file.
Now, the coil should be loaded with an alternating current with a frequency of 50 Hz. Which (peak) voltage is needed in order to get a current whose peak value is 1 Ampere? The voltage is given by Faraday's induction law: Let ∂S describe a conductor loop (a loop of a wire in the coil). The induced voltage is

$$\frac{d}{dt}\int_S B \cdot n.$$

Define a linear form b, to evaluate (with an inner product) the voltage from the solution (in NGSolve)). You can check your result with the integrate function. You might need to increase the order of integration (`Integrate(..., order = ..  )`). Also try to change the order of the curving of the mesh (`mesh.Curve(..)`. What do you observe?

Hint: Use the Stokes theorem. The line integrals over the 1000 conductor loops can be approximated by a volume integral. Further use the ansatz for the vector potential $A(x,t) = \hat{A}(x)\cos(...\, t)$, where $\hat{A}$ is the peak value.