Ex 7.2

linear functionals $b: V \to \mathbb{R}$, where $V = H_0^1(\Omega)$

Design a method, s.t. the error in the goal b is small, i.e. $b(u) - b(u_h) \to$ min/small

Let $f: V \to \mathbb{R}$ be (lin + cont) RHS and

$$A(u,v) := \int_\Omega \nabla u \cdot \nabla v \, dx \quad \forall u, v \in V$$

Using a FES $V_h \subset V$:

PRIMAL	cont: (a) Find	$u \in V$	s.t. $A(u,v) = f(v)$	$\forall v \in V$
	disc (b)	$u_h \in V_h$	s.t. $A(u_h, v_h) = f(v_h)$	$\forall v_h \in V_h$
DUAL	cont (c)	$w \in V$	s.t. $A(v,w) = b(v)$	$\forall v \in V$
	disc: d.)	$w_h \in V_h$	s.t. $A(v_h, w_h) = b(v_h)$	$\forall v_h \in V_h$

$\boxed{V_h \subset V}$ $\boxed{V_h \subset V}$

Show that

$$|b(u) - b(u_h)| \leq \eta^1(u_h)\,\eta^2(w_h), \quad \eta^1, \eta^2 \text{ reliable error estimates for (a), (c)}$$

When in doubt $\to$ add zero:

$$|b(u) - b(u_h)| = |b(\underbrace{u}_{\in V}) - b(\underbrace{u_h}_{\in V_h \subset V}) + f(\underbrace{w_h}_{\in V_h \subset V}) - f(\underbrace{w_h}_{\in V_h})| = \quad \text{Galerkin}$$

$$= |\underbrace{A(u,w) - A(u_h,w)}_{(c),\,(d)} + A(u,w_h) - A(u_h,w_h)| \overset{\text{linearity}}{=}$$

$$\overset{\text{linearity}}{=} |A(u-u_h, w) + A(u-u_h, w_h)| \overset{\text{lin}}{=} |A(u-u_h, w-w_h)| \leq$$

$$\overset{A \text{ cont}}{\leq} \alpha_2 \underbrace{\|u-u_h\|_{H^1}}_{\leq \eta^1} \cdot \underbrace{\|w-w_h\|_{H^1}}_{\leq \eta^2} \leq \alpha_2 \eta^1 \eta^2 \lesssim \eta^1 \eta^2$$