Ex 7.1

$\Omega \subset \mathbb{R}^d$ ... convex domain with Lipschitz boundary

$V = H^1(\Omega)$

$f \in L^2(\Omega)$

$\mathcal{T}_h$ ... admissible triangulation of Ω

V_h ... Lagrange finite element space of order k

Consider:

$$A(u,v) \longleftrightarrow \|\cdot\|_{H^1}$$

Find $u \in V$ s.t. $\overbrace{\langle \nabla u, \nabla v \rangle_{L^2(\Omega)} + \langle u, v \rangle_{L^2}}^{A(u,v)} = f(v) \quad \forall v \in V$

→ with $u_h \in V_h$ / $\tilde{u} \in H^1$ we define the residual $R(\tilde{u}) \in H^{-1}$

$$R(\tilde{u}) := \Delta \tilde{u} - \tilde{u} + f$$

$$R(\tilde{u})(v) = \langle R(\tilde{u}), v \rangle_{H_0^1} = \int -\nabla \tilde{u} \cdot \nabla v - \tilde{u} v + f v \, dx \quad \forall v \in H_0^1(\Omega)$$

1.) $R(u_h)(v_h) \overset{!}{=} 0$ holds for all $v_h \in V_h$

$$\to R(u_h)(v_h) = \int_\Omega \underbrace{-\nabla u_h \nabla v_h - u_h v_h}_{-A(u_h, v_h) = -f(v_h)} + f v_h \, dx = 0 \quad \checkmark$$

use triangulation

2.) $$R(u_h)(v) = \sum_{T \in \mathcal{T}_h} \int_T -\nabla u_h \nabla v - u_h v + f v \, dx \overset{-0 = -R(u_h, v_h)}{=}$$

$$= \sum_T \int_T \underbrace{-\nabla u_h \cdot \nabla (v - v_h)} - u_h \cdot (v - v_h) + f \cdot (v - v_h) \, dx$$

$$-\int_T \nabla u \cdot \nabla v = \int_T \Delta u \, v - \int_{\partial T} (\nabla u) v \vec{n}$$

IBP

$$= \sum_T \left[\int_T (\Delta u_h - u_h + f) \cdot (v - v_h) - \int_{\partial T} \frac{\partial u_h}{\partial n} (v - v_h) \right] =$$

$$= \sum_T \Big[\int_T (\Delta u_h - u_h + f)(v - v_h)\,dx + \underbrace{\sum_{F \in \partial T \setminus \partial\Omega}}_{\text{"inner"}} \int_F -\frac{\partial u_h}{\partial n}(v - v_h)\,ds \, +$$

$$+ \underbrace{\sum_{F \in \partial T \cap \partial\Omega}}_{\text{"outer"}} \int_F -\frac{\partial u_h}{\partial n}(v - v_h)\,ds \Big]$$

$$\cancel{\Rightarrow R(u_h)(v) = \sum_{T \in \mathcal{T}_h} \Big[\int_T (\Delta u_h - u_h + f)(v - v_h)\,dx + \sum_{F \in \partial T \setminus \partial\Omega} \int_F \frac{\partial u_h}{\partial n}(v - v_h)\,ds}$$

3) Show that

$$\|u - u_h\|_{H^1(\Omega)} \leq c \, \|R(u_h)\|_{V^*} \qquad \text{with } c = \text{const} = 1$$

↳ residual error estimator is reliable with $c = 1$

$A \ldots$ SPD → induces norm:

↳ $\|u - u_h\|^2_{H^1} = A(u - u_h, u - u_h)$

↳ $$\|u - u_h\|^2 = \frac{A(u - u_h, u - u_h)}{\|u - u_h\|} \leq \sup_{v \in H^1} \frac{A(u - u_h, v)}{\|v\|}$$

Since $A(u - u_h, v) = A(u, v) - A(u_h, v) = f(v) - A(u_h, v) = R(u_h)(v)$

3) $\|R(u_n)\|_{V^*} = \sup\limits_{v \in H_0^1} \frac{|R(u_n)(v)|}{\|v\|_{H^1}} \overset{v = u - u_n \in H_0^1}{\geq} \frac{|R(u_n)(u-u_n)|}{\|u-u_n\|_{H^1}} =$

$= \frac{|A(u_n, u-u_n) + f(u-u_n)|}{\| \cdots \|} = \frac{|A(\overset{-u_n}{\cancel{u_n}}, u-u_n) + A(u, u-u_n)|}{\| \cdots \|} =$

$= \frac{|A(u, u-u_n) + A(-u_n, u-u_n)|}{\| \cdots \|} = \frac{|A(u-u_n, u-u_n)|}{\| \cdots \|} \geq$

$A(u,u) \geq \alpha_1 \|u\|^2 \ldots$ coercivity

$\geq \frac{\alpha_1 \|u-u_n\|^2}{\|u-u_n\|} = \alpha_1 \|u-u_n\|_{H^1}$

$\Rightarrow \|u-u_n\|_{H^1} \leq \frac{1}{\alpha_1} \|R(u_n)\|_{V^*}$

But why does $\alpha_1 = 1$ hold?

A is SPD $\Leftrightarrow A(u,v) = \langle u, v \rangle_{H^1} \Leftrightarrow$ A induces norm $\overset{\text{of } \lim ?}{H^1}$

$\hookrightarrow$ Coercivity: $A(u,u) = \langle u, u \rangle_{H^1} = \|u\|_{H^1}^2 \geq \alpha_1 \|u\|_{H^1}^2$

$\hookrightarrow$ only true if $\alpha_1 = 1$