1 Exercise 5.4

1.1 Description

Let $\hat{T}$ be the reference simplex in $\mathbb{R}^d$. Every d-dimensional simplex in $\mathbb{R}^d$ is affine equivalent to $\hat{T}$ i.e there exists $F : \hat{T} \to T, F(x) = Ax + b$ with $A \in \mathbb{R}^{d \times d}, b \in \mathbb{R}^d$ and $\det(A) \neq 0$.F is invertible and there holds (for $c, C > 0$):

$$\|DF\| = \|A\| \leq \frac{h_T}{\rho_T} \tag{1}$$

$$\left\|DF^{-1}\right\| = \left\|A^{-1}\right\| \leq \frac{h_{\hat{T}}}{\rho_T} \tag{2}$$

$$c\rho_T^d \leq |\det(DF)| = |\det(A)| \leq Ch_T^d \tag{3}$$

Hint: For the first equation try to integrate the constant 1-function on T. The operator norm is given by

$$\|A\| = \sup_{\xi \in \mathbb{R}^d} \frac{\|A\xi\|}{\|\xi\|} = \sup_{\|\xi\|=c} \frac{\|A \leqslant \|}{\|\xi\|} \tag{4}$$

with a fixed constant c. Try to set $c = \rho_t$ for the second estimate, and choose a proper ξ

1.2 Proofs

1.2.1 1)

For proof 1) and 2) we use the hint in the description.

$$\|A\| = \sup_{\zeta \in \mathbb{R}^d} \frac{\|A\zeta\|}{\|\zeta\|} = \sup_{\substack{c \\ \|\mathbb{K}\|=c}} \frac{\|A\zeta\|}{\|\zeta\|} \tag{5}$$

where $c = \rho_T$ and pull out the constant term to the front

$$\Rightarrow \sup_{\|\mathbb{K}\|=c} \frac{\|A\zeta\|}{\|\zeta\|} = \frac{1}{\rho_{\tilde{r}}} \sup_{\|\mathbb{S}\|=\rho_T} \|A\zeta\| \tag{6}$$

Now we define $\zeta = \hat{x}_1 - \hat{x}_2 \in \hat{T}$and use the affine property of A where

$$A\zeta = F(\hat{x}_1) - F(\hat{x}_2) = x_1 - x_2 \tag{7}$$

and use the fact that $hatx_1 - \hat{x}_2 \leq h_T$ with h_T being the mesh-size, to get

$$\|A\zeta\| \leq |x_1 - x_2| \leq h_T \tag{8}$$

Now we just plug this into eq. (6) to get our final result

$$\|A\| = \sup_{\|\leqslant\|=\rho_T} \frac{1}{\rho_{\tilde{T}}} \|A\zeta\| \leq \frac{h_T}{\rho_{\hat{T}}} \tag{9}$$

1.2.2 2)

For 2), we follow a similar path and start first with

$$\left\|A^{-1}\right\| = \frac{1}{\rho_T} \sup_{\|\varepsilon\|=\rho_T} \left\|A^{-1}\varepsilon\right\| \tag{10}$$

Now we define $\varepsilon = x_1 - x_2 \in T$ and use the inverse map $F^{-1}(x) = A^{-1}x + b' = \hat{x}$ to get

$$A^{-1}\varepsilon = F^{-1}\left(x_1\right) - F^{-1}\left(x_2\right) = \hat{x}_1 - \hat{x}_2 \tag{11}$$

and

$$\left\|A^{-1}\varepsilon\right\| \leq \left|\hat{x}_1 - \hat{x}_2\right| \leq h_T \tag{12}$$

and

$$\left\|A^{-1}\right\| = \sup_{\|\varepsilon\|=\rho_T} \frac{1}{\rho_T} \left\|A^{-1}\varepsilon\right\| \leq \frac{h_{\hat{T}}}{\rho_T} \tag{13}$$

1.2.3 3)

To proof 3) we start by integrating over T (= "area" or "volume") and use the rule for integral transformations,

$$area(T) = \int_T dx = \int_{\hat{T}} |\det(A)| d\hat{x} = |\det(A)| area(\hat{T}) \tag{14}$$

$$|\det(A)| = \frac{area(T)}{area(\hat{T})} \tag{15}$$

Now we want to bound the "area" of our simplices from above and below by circles via

$$V_d(r) = \frac{\pi^{\frac{d}{2}}}{\Gamma\left(\frac{d}{2}+1\right)} r^d \tag{16}$$

$$\frac{\pi^{d/2}}{T\left(\frac{d}{2}+1\right)} \left(\frac{\rho_T}{2}\right)^d = V_d\left(\frac{\rho_T}{2}\right) \leq area(T) \leqslant V_d\left(\frac{h_T}{2}\right) = \frac{\pi^{d_{12}}}{T\left(\frac{d}{2}+1\right)} \left(\frac{h}{2}T\right)^d \tag{17}$$

So, we inscribed one sphere with radius ρ_T and "excribed" one sphere with radius h_T (mesh-size) and get

$$\Rightarrow \frac{\pi^{d/2}}{T\left(\frac{d}{2}+1\right)} \left(\frac{\rho_T}{2}\right)^d \leqslant \frac{area(T)}{area(\hat{T})} \leqslant \frac{\pi^{d/2}}{T\left(\frac{d}{2}+1\right)} \left(\frac{h_r}{2}\right)^d \tag{18}$$

Rearrange a bit

$$\underbrace{\frac{1}{area(\hat{T})} \cdot \frac{\pi^{d/2}}{\Gamma(\frac{d}{2}+1) \cdot 2^d}}_{c} \cdot \hat{\rho}_T^d \leqslant |\det(A)| \leqslant \underbrace{\frac{1}{area(\hat{T})} \cdot \frac{\pi^{d/2}}{\Gamma(\frac{d}{2}+1) \cdot 2^d}}_{C} h_T^d \tag{19}$$

$$A(S_{h_T}) \geq A(T) \geq A(S_{\rho_T})$$