NUMPDE Exercise 6

1 Example 6.1

Proof that:

$$Q_T(v) = \int_T v\,dx \quad \forall v \in \mathbb{P}^2(T)$$

Forst we difin a $\hat{v}(\hat{x},\hat{y})$ Element and integrat over it.

$$\hat{v}(\hat{x},\hat{y}) = c_0 + c_1\hat{x} + c_2\hat{y} + c_3\hat{x}^2 + c_4\hat{y}^2 + c_5\hat{x}\hat{y}$$

$$\int_{\hat{x}=0}^{1}\int_{\hat{y}=0}^{1-\hat{x}} \hat{v}d\hat{x} = \frac{c_0}{2} + \frac{c_1+c_2}{6} + \frac{c_3+c_4}{12} + \frac{c_5}{24}$$

Now we kan use the midpoint rule to calculate the area $\in V$.

$$Q_{\hat{T}}(v) := \frac{|\hat{T}|}{3}\big[v(\hat{x}_1)v(\hat{x}_2)v(\hat{x}_3)\big]$$

Also the are of $|\hat{T}| = \frac{1}{2}|T|$ and $\hat{v}(\hat{x_1}) = c_0+\frac{c_1}{2}+\frac{c_3}{4}$, $\hat{v}(\hat{x_2}) = c_0+\frac{c_1}{2}+\frac{c_3}{4}$, $\hat{v}(\hat{x_3}) = c_0+\frac{c_1}{2}+\frac{c_2}{2}+\frac{c_3}{4}+\frac{c_4}{4}+\frac{c_5}{4}$
$Q_{\hat{T}}(\hat{v}) = \frac{c_0}{2} + \frac{c_1+c_2}{6} + \frac{c_3+c_4}{12} + \frac{c_5}{24}$
We sowed finally that $Q_{\hat{T}}(\hat{v}) = \int_{\hat{T}} \hat{v}$ and we know that lagrange FR are equivalent. We define $F : \hat{T} \to T$ as an affine linear mapping.

$$\int_T v(x)\,dx = \int_T \hat{v} \circ F^{-1}(x)\,dx = \int_{\hat{T}} \left(\hat{v} \circ F^{-1}\right) \circ F(\hat{x})|\det DF|\,d\hat{x} = |\det DF| \int_{\hat{T}} \hat{v}(\hat{x})\,d\hat{x}$$

At last we only have to show that $\int_{\hat{T}} \hat{v}(\hat{x})\,d\hat{x} = Q_{\hat{T}(\hat{v})}$

$$|\det DF| Q_{\hat{T}(\hat{v})} = |\det DF|\frac{|\hat{T}|}{3}\Big(\hat{v}(\hat{x_1}) + \hat{v}(\hat{x_2}) + \hat{v}(\hat{x_3})\Big)$$

$$= \frac{|\hat{T}|}{3}\left((v\circ F)(F^{-1}(x_1)) + (v\circ F)(F^{-1}(x_2)) + (v\circ F)(F^{-1}(x_3))\right) = \frac{|\hat{T}|}{3}\Big(v(x_1) + v(x_2) + v(x_3)\Big)$$

2 Example 6.2

3 Example 6.3

4 Example 6.4