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NUMPDE Exercise 6

1 Example 6.1

Proof that:

$$Q_T(v) = \int_T v\, dx \quad \forall v \in \mathbb{P}^2(T)$$

Forst we difin a $\hat{v}(\hat{x}, \hat{y})$ Element and integrat over it.

$$\hat{v}(\hat{x}, \hat{y}) = c_0 + c_1\hat{x} + c_2\hat{y} + c_3\hat{x}^2 + c_4\hat{y}^2 + c_5\hat{x}\hat{y}$$

$$\int_{\hat{x}=0}^{1}\int_{\hat{y}=0}^{1-\hat{x}} \hat{v} d\hat{x} = \frac{c_0}{2} + \frac{c_1+c_2}{6} + \frac{c_3+c_4}{12} + \frac{c_5}{24}$$

Now we kan use the midpoint rule to calculate the area $\in V$.

$$Q_{\hat{T}}(v) := \frac{|\hat{T}|}{3}\big[v(\hat{x}_1)v(\hat{x}_2)v(\hat{x}_3)\big]$$

Also the are of $|\hat{T}| = \frac{1}{2}|T|$ and $\hat{v}(\hat{x_1}) = c_0 + \frac{c_1}{2} + \frac{c_3}{4}$, $\hat{v}(\hat{x_2}) = c_0 + \frac{c_1}{2} + \frac{c_3}{4}$, $\hat{v}(\hat{x_3}) = c_0 + \frac{c_1}{2} + \frac{c_2}{2} + \frac{c_3}{4} + \frac{c_4}{4} + \frac{c_5}{4}$
$Q_{\hat{T}}(\hat{v}) = \frac{c_0}{2} + \frac{c_1+c_2}{6} + \frac{c_3+c_4}{12} + \frac{c_5}{24}$
We sowed finally that $Q_{\hat{T}}(\hat{v}) = \int_{\hat{T}} \hat{v}$ and we know that lagrange FR are equivalent. We define $F : \hat{T} \to T$ as an affine linear mapping.

$$\int_T v(x)\, dx = \int_T \hat{v} \circ F^{-1}(x)\, dx = \int_{\hat{T}} \Big(\hat{v} \circ F^{-1}\Big) \circ F(\hat{x})|\det DF|\, d\hat{x} = |\det DF| \int_{\hat{T}} \hat{v}(\hat{x})\, d\hat{x}$$

At last we only have to show that $\int_{\hat{T}} \hat{v}(\hat{x})\, d\hat{x} = Q_{\hat{T}(\hat{v})}$

$$|\det DF| Q_{\hat{T}(\hat{v})} = |\det DF| \frac{|\hat{T}|}{3}\Big(\hat{v}(\hat{x_1}) + \hat{v}(\hat{x_2}) + \hat{v}(\hat{x_3})\Big)$$

$$= \frac{|\hat{T}|}{3}\left((v \circ F)(F^{-1}(x_1)) + (v \circ F)(F^{-1}(x_2)) + (v \circ F)(F^{-1}(x_3))\right) = \frac{|\hat{T}|}{3}\Big(v(x_1) + v(x_2) + v(x_3)\Big)$$

2 Example 6.2

This proof stats with Theorem 69 and we start with where $Q_h(v) = \sum_{T\in\mathbb{T}} Q_T(v) = \sum_{T\in\mathbb{T}} \int_T v\, dx$:

$$|| \int_\Omega v\, dx - Q_h(v)||_{L_2(T)} = \sum_{T\in\mathbb{T}} || \int (id - I_T)v_T\, dx||^2_{L_2(T)}$$

Now using (4.2)

$$\sum_{T\in\mathbb{T}} \det(B)|| \int (id - I_T)v_T\, dx||^2_{L_2(\hat{T})}$$

and after the transformation to the reference triangle:

$$\sum_{T\in\mathbb{T}} \det(B)|| \int \det(B)(id - I_{\hat{T}})(v_T \circ F_T)\, d\hat{x}||^2_{L_2(\hat{T})}$$

After that we apply the Bramble Hilbert lemma. The Quatrature rule is exact for polynominals for decreas 2 as we seen in 6.1 $\Rightarrow \int (id - I_{\hat{T}})q\,dx = 0$ for $q \in \mathbb{P}^2(\hat{T})$

$$|| \int_\Omega v\,dx - Q_h(v)||_{L_2(T)} \leq \sum_{T \in \mathbb{T}} \det(B)|v_T \circ F_T|^2_{H^3(\hat{T})}$$

$$\preceq \sum_{T \in \mathbb{T}} \det(B) \det(B)^{-1} ||B||^6 |v_T|^2_{H^3(T)}$$

In the end we use the quasi-uniform $||B||^6 \simeq h^6$ propperty:

$$|| \int_\Omega v\,dx - Q_h(v)||_{L_2(\Omega)} \preceq h^3 ||v_T||_{H^3(\Omega)}$$

3 Example 6.3

4 Example 6.4