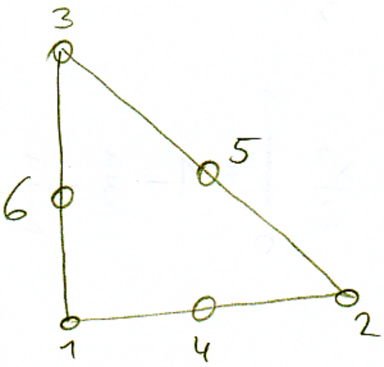
Ex 6.1

Let's first look at our ref triangle $(\hat{T}, \hat{V}, \hat{\varphi}_{\hat{T}})$



$\hat{V} = \{ax^2 + by^2 + cxy + dx + ey + f \mid a, \dots, f \in \mathbb{R}\}$

$\hookrightarrow \hat{V} = P^2(\hat{T})$ (Q^2 would include x^2y^2-term)

$\to$ Nodal basis: $\hat{\psi}_i^{\hat{T}}(\varphi_j) = \delta_{ij}$, $\hat{\psi}_i^{\hat{T}}(v) = v(\hat{x}_i)$

$\hat{\varphi}_1 = 2(x^2+y^2) + 4xy - 3(x+y) + 1$

$\hat{\varphi}_2 = (2x-1)x$

$\hat{\varphi}_3 = (2y-1)y$

$\hat{\varphi}_4 = -4x(x+y-1)$

$\hat{\varphi}_5 = 4xy$

$\hat{\varphi}_6 = -4y(x+y-1)$

~~take~~ the quadrature rule

$$Q_T(f) := \frac{|T|}{3}\left(f(x_1^e) + f(x_2^e) + f(x_3^e)\right)$$

Show $Q_T(v) = \int_T v\,dx$, $v \in P^2(T)$

On $\hat{T}$:

$$Q_{\hat{T}}(\underbrace{\hat{v} = \sum \hat{v}_i \hat{\varphi}_i}) = \frac{|\hat{T}|}{3}\left(\underbrace{\hat{v}(\hat{x}_4)}_{\hat{v}_4} + \underbrace{\hat{v}(\hat{x}_5)}_{\hat{v}_5} + \underbrace{\hat{v}(\hat{x}_6)}_{\hat{v}_6}\right)$$

$\overset{!}{=}$

$$\int_{\hat{T}} \hat{v}\,dx = \sum_i \hat{v}_i \int_{\hat{T}} \hat{\varphi}_i\,dx \quad \xrightarrow{\text{"should"}} \quad \int_{\hat{T}} \hat{\varphi}_i\,dx = \begin{cases} 0 & \text{for } i = 1,2,3 \\ \frac{1}{6} & \text{for } i = 4,5,6 \end{cases}$$

I will only show for one $\rightarrow \hat{\varphi}_2$ because it is easier

$$\int_{\hat{T}} \hat{\varphi}_2(x,y)\,dx = \underbrace{\int_{x=0}^{1} \int_{0}^{1-x}}_{\rightarrow \text{Triangle}} (2x-1)x \,dy\,dx = \int_{x=0}^{1} (2x^2-x)(1-x)\,dx =$$

$$= \int_0^1 3x^2 - x - 2x^3 \,dx = x^3 - \frac{x^4}{2} - \frac{x^2}{2} \Big|_0^1 = 1 - \frac{1}{2} - \frac{1}{2} = 0$$

The others follow analogously.

We now compute $\int_{\hat{T}} \hat{\varphi}_4 \,dx = \int_{\hat{T}} \hat{\varphi}_5 \,dx = \int_{\hat{T}} \hat{\varphi}_6 \,dx$

Again, only for $\hat{\varphi}_5$:

$$\int_{\hat{T}} \hat{\varphi}_5 \,dx = \int_{x=0}^{1} \int_{0}^{1-x} 4xy \,dy\,dx = \cancel{\int_0^1 4x - 4x^2 \,dx = 2x^2 - \frac{4x^3}{3} \Big|_0^1 = 2 - \frac{4}{3} = \frac{2}{3}}$$

$$= \int_0^1 2x \underbrace{(1-x)^2}_{1-2x+x^2} \,dx = \int_0^1 2x - 4x^2 + 2x^3 \,dx = x^2 - \frac{4x^3}{3} + \frac{x^4}{2} \Big|_0^1 =$$

$$= \frac{1}{2} - \frac{4}{3} + 1 = \frac{9}{6} - \frac{8}{6} = \frac{1}{6}$$

$$\Rightarrow \int_{\hat{T}} \hat{v}\,dx = \overbrace{\frac{1}{6}}^{\frac{1}{2}\cdot\frac{1}{3}} (\hat{v}_4 + \hat{v}_5 + \hat{v}_6) = \frac{|\hat{T}|}{3} (\hat{v}_4 + \hat{v}_5 + \hat{v}_6) = Q_{\hat{T}}(v)$$

Since $|\hat{T}| = \text{area}(\hat{T}) = \text{"}\frac{a \cdot b}{2}\text{"} = \frac{1}{2}$

We now use our result from Ex 5.4

$$|\det(A)| = \frac{|T|}{|\hat{T}|} \Rightarrow |T| = |\hat{T}| \det(A)$$

Additionally: We know that 2 Lagrange FE are equivalent.

$F: \hat{T} \to T$

transformation rule

$$\int_T v\,dx = \sum_{i=1}^{6} v_i \int_T \varphi_i\,dx = \sum_{i=1}^{6} v_i \int_{\hat{T}} \hat{\varphi}_i \underbrace{|\det(A)|}_{=\text{const}}\, d\hat{x} =$$

$$= \sum_{i=4}^{6} v_i\, |\det A|\, |\hat{T}| \frac{1}{3} = \frac{|T|}{3}(v_4 + v_5 + v_6)$$