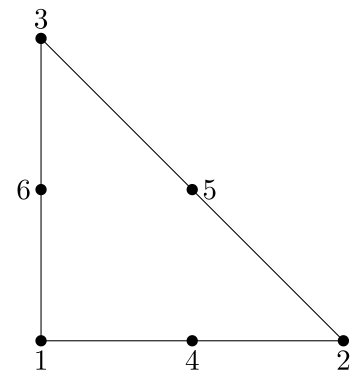
Exercise 6.1

$\hat{V} = \left\{ax^2 + by^2 + cxy + dx + ey + f | a, b, c, d, e, f \in \mathbb{R}\right\}$



Basisfunctions for reference triangle:

$$\begin{aligned}
\hat{\varphi}_1 &= 2(x^2 + y^2) + 4xy - 3(x + y) + 1 \\
\hat{\varphi}_2 &= (2x - 1)x \\
\hat{\varphi}_3 &= (2y - 1)y \\
\hat{\varphi}_4 &= -4x(x + y - 1) \\
\hat{\varphi}_5 &= 4xy \\
\hat{\varphi}_6 &= -4y(x + y - 1)
\end{aligned}$$

Let's assume we have a function $u = \sum_i^6 u_i \hat{\varphi}_i(x, y)$, then $\int u d\hat{A} = \sum_i^6 u_i \int \hat{\varphi}_i(x, y) d\hat{A}$. So we need to show $\int \hat{\varphi}_1 d\hat{A} = \int \hat{\varphi}_2 d\hat{A} = \int \hat{\varphi}_3 d\hat{A} = 0$. I just show it for $\hat{\varphi}_2$, the rest is analogous.

$$\begin{aligned}
\int\int \hat{\varphi}_2 d\hat{A} &= \int_0^1 \int_0^{1-x} (2x - 1)xdydx \\
&= \int_0^1 (2x - 1)x(1 - x)dx = \int_0^1 3x^2 - 2x^3 - xdx \\
&= x^2 - \frac{2}{4}x^4 - \frac{x^2}{2}\bigg|_0^1 = 0
\end{aligned}$$

If we now compute $\int \hat{\varphi}_4 d\hat{A} = \int \hat{\varphi}_5 d\hat{A} = \int \hat{\varphi}_6 d\hat{A} = \frac{1}{6}$. Again I show it only for $\int \hat{\varphi}_5 d\hat{A}$

$$\begin{aligned}
\int\int \hat{\varphi}_5 d\hat{A} &= \int_0^1 \int_0^{1-x} 4xy\, dydx \\
&= \int_0^1 2x(1 - x)^2\, dx = \int_0^1 2x - 4x^2 + 2x^3\, dx \\
&= x^2 - \frac{4}{3}x^3 + \frac{1}{2}x^4\bigg|_0^1 = \frac{1}{6}
\end{aligned}$$

Therefore we have shown that $\int \hat{\varphi}_1 d\hat{A} = \int \hat{\varphi}_2 d\hat{A} = \int \hat{\varphi}_3 d\hat{A} = 0$ and $\int \hat{\varphi}_4 d\hat{A} = \int \hat{\varphi}_5 d\hat{A} = \int \hat{\varphi}_6 d\hat{A} = \frac{1}{6}$. So the Integral yields:

$$\int u d\hat{A} = \sum_i^6 u_i \int \hat{\varphi}_i(x,y) d\hat{A} = \frac{1}{6}(u_4 + u_5 + u_6)$$

We know that $|\hat{T}| = \frac{1}{2}$, therefore we can rewrite it as:

$$\int u d\hat{A} = \sum_i^6 u_i \int \hat{\varphi}_i(x,y) d\hat{A} = \frac{|\hat{T}|}{3}(u_4 + u_5 + u_6)$$

By performing a transformation from the triangle T to $\hat{T}$, the only thing that changes is the additional determinant $\det(B)$, due to the integral transformation. And from the definition of the determinant $|\hat{T}| \det(B) = |T|$.

$$\int u dA = \sum_i^6 u_i \int \varphi_i(x,y) dA = \frac{|\hat{T}| \det(B)}{3}(u_4 + u_5 + u_6) = \frac{|T|}{3}(u_4 + u_5 + u_6)$$

Exercise 6.2

I followed pretty straightforward Theorem 69. Since $\left|\int_\Omega v dx - Q_h(v)\right|$ is a constant i bound it by the L_2 norm.

$$\begin{aligned}
\| \int_\Omega v dx - Q_h(v) \|^2_{L_2(T)} &= \sum_{T \in \mathbb{T}} \| \int (id - I_T) v_T dx \|^2_{L_2(T)} \\
\text{using (4.2)} &= \sum_{T \in \mathbb{T}} \det(B) \| \int (id - I_T) v_T dx \|^2_{L_2(\hat{T})} \\
\text{transforming} &= \sum_{T \in \mathbb{T}} \det(B) \| \int \det(B)(id - I_{\hat{T}})(v_T \circ F_T) d\hat{x} \|^2_{L_2(\hat{T})}
\end{aligned}$$

Now we apply the Bramble Hilbert lemma. We have shown in 6.1 that the Quadrature rule is exact for polynomials of degree 2. $Lq = \int (id - I_{\hat{T}}) q dx = 0$ for $q \in \mathcal{P}^2(\hat{T})$

$$\begin{aligned}
\| \int_\Omega v dx - Q_h(v) \|^2_{L_2(T)} &\leq \sum_{T \in \mathbb{T}} \det(B) |v_T \circ F_T|^2_{H^3(\hat{T})} \\
\text{using 4.4} &\preceq \sum_{T \in \mathbb{T}} \det(B) \det(B)^{-1} \|B\|^6 |v_T|^2_{H^3(T)} \\
\text{quasi-uniform} &\simeq h^6 \|v_T\|^2_{H^3(\Omega)}
\end{aligned}$$

Therefore:

$$\| \int_\Omega v dx - Q_h(v) \|_{L_2(\Omega)} \preceq h^3 \|v_T\|_{H^3(\Omega)}$$

Exercise 6.3

First, we prove that for $v_h \in V_{h,0}^k$ there holds : $\|e_h^{'}\|_{L^2(\Omega)}^2 = (e_h^{'}, e_h^{'} - v_h^{'})_{L^2(\Omega)}$.
For that we use the Galerkin orthogonality:

$$\int u'v'd\Omega = A(u,v) \qquad \int fvd\Omega = f(v)$$

$$\begin{aligned} A(e_h, e_h - v_h) &= A(u-u_h, u-u_h) - A(u-u_h, v_h) \\ &= A(u-u_h, u-u_h) - A(u, v_h) + A(u_h, v_h) \\ &= A(u-u_h, u-u_h) - f(v_h) + f(v_h) = A(e_h, e_h) = \|e_h^{'}\|_{L^2(\Omega)}^2 \end{aligned}$$

We exploit chain rule and the second derivative of e_h:

$$\begin{aligned} \int e_h^{'}(e_h^{'} - v_h{'})dT &= \underbrace{\frac{d}{dx}\int e_h^{'}(e_h - v_h)dT}_{=0} - \int e_h^{''}(e_h - v_h)dT \\ -e_h^{''} &= f + u_h^{''} \end{aligned}$$

Therefore:

$$\begin{aligned} \|e_h^{'}\|_{L^2(\Omega)}^2 &= \sum_{T\in T_h} \|e_h^{'}\|_{L^2(T)}^2 = \sum_{T\in T_h} (e_h^{'}, e_h^{'} - v_h{'})_{L^2(T)} \\ &= \sum_{T\in T_h} (-e_h^{''}, e_h - v_h)_{L^2(T)} \\ &\le \sum_{T\in T_h} (-e_h^{''}, e_h - I_h e_h)_{L^2(T)} \\ &= \sum_{T\in T_h} (f + u_h^{''}, e_h - I_h e_h)_{L^2(T)} \end{aligned}$$

For the second estimate we use:

$$\begin{aligned} &\|e_h^{'}\|_{L^2(\Omega)}^2 = |u - u_h|_{H_1(\Omega)} \\ &e_h^{'} - I_h e_h^{'} = u^{'} - u_h^{'} - I_h u^{'} + \underbrace{I_h u_h^{'}}_{=u_h^{'}} = u^{'} - I_h u^{'} \end{aligned}$$

$$\begin{aligned} \|e_h^{'}\|_{L^2(\Omega)}^2 &= \sum_{T\in T_h} \|e_h^{'}\|_{L^2(T)}^2 = \sum_{T\in T_h} (u^{'} - u_h^{'}, u^{'} - u_h^{'})_{L^2(T)} \\ &\le \sum_{T\in T_h} (u^{'} - I_h u^{'}, u^{'} - I_h u^{'})_{L^2(T)} = \sum_{T\in T_h} (e_h^{'} - I_h e_h^{'}, e_h^{'} - I_h e_h^{'})_{L^2(T)} \\ &\le c \sum_{T\in T_h} h_T^2 |e_h^{'}|_{H_1}^2 = c \sum_{T\in T_h} h_T^2 \| - (f + u_h^{''})\|_{L_2}^2 = c \sum_{T\in T_h} h_T^2 \|f + u_h^{''}\|_{L_2}^2 \\ |u - u_h|_{H_1(\Omega)} &\le c \sum_{T\in T_h} h_T^2 \|f + u_h^{''}\|_{L_2}^2 \end{aligned}$$