

# NUMPDE Exercise 5

## 1 Example 5.1

## 2 Example 5.2

## 3 Example 5.3

## 4 Example 5.4

Let  $\hat{T}$  be the reference simplex in  $\mathbb{R}^d$ . Every d-dimensional simplex in  $\mathbb{R}^d$  is affine equivalent to  $\hat{T}$ , i.e there exists  $F : \hat{T} \rightarrow T, F(x) = Ax + b$  with  $A \in \mathbb{R}^{d \times d}, b \in \mathbb{R}^d$  and  $\det(A) \neq 0$ . F is invertible and there holds (for  $c, C > 0$ ):

$$\|DF\| = \|A\| \leq \frac{h_T}{\rho_{\hat{T}}} \quad (1)$$

$$\|DF^{-1}\| = \|A^{-1}\| \leq \frac{h_{\hat{T}}}{\rho_T} \quad (2)$$

$$c\rho_T^d \leq |\det(DF)| = |\det(A)| \leq Ch_T^d \quad (3)$$

To proof 3) we start with integrate over  $T$  and use the coordinate transformations determinant.

$$|T| = \int_T dx = \int_{\hat{T}} |\det(A)| d\hat{x} = |\det(A)| |\hat{T}|$$

$$|\det(A)| = \frac{|T|}{|\hat{T}|}$$

For proof 1) and 2) we use the hint in the description.

$$\|A\| = \sup_{\zeta \in \mathbb{R}^d} \frac{\|A\zeta\|}{\|\zeta\|} = \sup_{\|\zeta\|=c} \frac{\|A\zeta\|}{\|\zeta\|}$$

where  $c = \rho_{\hat{T}}$

$$\Rightarrow \sup_{\|\zeta\|=c} \frac{\|A\zeta\|}{\|\zeta\|} = \frac{1}{\rho_{\hat{T}}} \sup_{\|\zeta\|=\rho_{\hat{T}}} \|A\zeta\|$$

Now we could define a proper  $\zeta = \hat{x}_1 - \hat{x}_2 \in \hat{T}$  and use the affine property of  $A$  where  $A\zeta = F(\hat{x}_1) - F(\hat{x}_2) = x_1 - x_2$

$$\|A\zeta\| \leq |x_1 - x_2| \leq h_T$$

$$\|A\| \leq \frac{1}{\rho_{\hat{T}}} \|A\zeta\| \leq \frac{h_T}{\rho_{\hat{T}}}$$

For 2) we start first with that

$$\|A^{-1}\| = \frac{1}{\rho_T} \sup_{\|\zeta\|=\rho_T} \|A^{-1}\zeta\|$$

Now we could define a proper  $\zeta = x_1 - x_2 \in T$  and use the affine property of  $A$  where  $A\zeta = F(x_1) - F(x_2) = \hat{x}_1 - \hat{x}_2$

$$\|A^{-1}\| \leq \frac{1}{\rho_T} \|A^{-1}\zeta\| \leq \frac{h_{\hat{T}}}{\rho_T}$$