Notes on Advanced Electrodynamics
Electromagnetic Theory

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Contents

Chapter 1

Basic Electrodynamics Theory

1.1 Maxwell Equation

1.1.1 Maxwell Equation in Matter

Maxwell's equations in matter is written:

$$\begin{cases} \oint_{\partial\Omega} \mathbf{D} \cdot \mathrm{d}\mathbf{S} = \iiint_{\Omega} \rho_{\mathrm{f}} \, \mathrm{d}V \\ \oint_{\partial\Sigma} \mathbf{H} \cdot \mathrm{d}\boldsymbol{\ell} = \iint_{\Sigma} \mathbf{J}_{\mathrm{f}} \cdot \mathrm{d}\mathbf{S} + \frac{d}{dt} \iint_{\Sigma} \mathbf{D} \cdot \mathrm{d}\mathbf{S} \\ \oint_{\partial\Omega} \mathbf{B} \cdot \mathrm{d}\mathbf{S} = 0 \\ \oint_{\partial\Sigma} \mathbf{E} \cdot \mathrm{d}\boldsymbol{\ell} = -\frac{d}{dt} \iint_{\Sigma} \mathbf{B} \cdot \mathrm{d}\mathbf{S} \end{cases} \tag{1.1}$$

which is in differential view:

$$\begin{cases} \nabla \cdot \mathbf{D} = \rho_{\mathrm{f}} \\ \nabla \times \mathbf{H} = \mathbf{J}_{\mathrm{f}} + \frac{\partial \mathbf{D}}{\partial t} \\ \nabla \cdot \mathbf{B} = 0 \\ \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \end{cases} \tag{1.2}$$

with definition of auxiliary fields, electrical displacement and magnetizing field:

$$\mathbf{D} = \varepsilon_0 \mathbf{E} + \mathbf{P} = \varepsilon_0 \mathbf{E} + \varepsilon_0 \chi \mathbf{E} = \varepsilon_0 \varepsilon_r \mathbf{E} = \varepsilon \mathbf{E} \tag{1.3}$$

$$\mathbf{H} = \frac{\mathbf{B}}{\mu_0} - \mathbf{M} \Rightarrow \mathbf{B} = \mu_0(\mathbf{H} + \mathbf{M}) = \mu_0(1 + \chi_m)\mathbf{H} = \mu\mathbf{H} \tag{1.4}$$

and for current we have Ohm's Law[1]:

$$\mathbf{J}_\mathrm{f} = \mathbf{J}' + \mathbf{J}_o = \mathbf{J}' + \sigma\mathbf{E} \tag{1.5}$$

where $\mathbf{J}'$ is external current and σ is conductivity of the medium, and charge conservation says:

$$\nabla \cdot \mathbf{J} + \frac{\partial \rho}{\partial t} = 0 \tag{1.6}$$

Relative Permittivity & Permeability Some times relative permittivity and permeability are not simply a scalar but a second order tenser:

$$\mathbf{D} = \overleftrightarrow{\varepsilon}\, \mathbf{E}, \quad \mathbf{B} = \overleftrightarrow{\mu}\, \mathbf{H} \tag{1.7}$$

and for some special mediums, they even have:

$$\mathbf{D} = \overleftrightarrow{\varepsilon}\, \mathbf{E} + \overleftrightarrow{\xi}\, \mathbf{H}, \quad \mathbf{B} = \overleftrightarrow{\mu}\, \mathbf{H} + \overleftrightarrow{\zeta}\, \mathbf{E} \tag{1.8}$$

Boundary Conditions boundary conditions of Maxwell equations writes:

$$\begin{cases} \mathbf{n}_{21} \cdot (\mathbf{D}_2 - \mathbf{D}_1) = \sigma_\mathrm{f} \\ \mathbf{n}_{21} \cdot (\mathbf{H}_2 - \mathbf{H}_1) = 0 \\ \mathbf{n}_{21} \times (\mathbf{D}_2 - \mathbf{D}_1) = 0 \\ \mathbf{n}_{21} \times (\mathbf{H}_2 - \mathbf{H}_1) = \mathbf{K}_\mathrm{f} \end{cases} \tag{1.9}$$

1.1.2 Maxwell Equation of Simple Harmonic Field

Wave Equation and Helmholtz Equation Wave function is:

$$\frac{\partial^2 u}{\partial t^2} = v^2 \nabla^2 u, \quad u = u(\mathbf{R}, t) \tag{1.10}$$

and separate the variables the equation becomes:

$$u = A(\mathbf{R})T(t) \Rightarrow A\ddot{T} = v^2 T \nabla^2 A \tag{1.11}$$

that is:

$$\frac{\nabla^2 A}{A} = \frac{\ddot{T}}{vT^2} = -k^2 \tag{1.12}$$

in which k is taken for simplicity, and we get Helmholtz equation:

$$\left(\nabla^2 + k^2\right) A(\mathbf{R}) = 0 \tag{1.13}$$

[1]Refer to Eq.(3.49) of notes on General Physics for comments on free current $\mathbf{J}_\mathrm{f}$.

Maxwell Equations in the form of Wave Equations From Eq.1.2 we can reshape Maxwell Equations into the form of wave equations:

$$\begin{cases} \nabla \times (\nabla \times \mathbf{H}) = \nabla \times \mathbf{J}_\mathrm{f} + \nabla \times \dfrac{\partial \mathbf{D}}{\partial t} \\ \nabla \times (\nabla \times \mathbf{E}) = -\nabla \times \dfrac{\partial \mathbf{B}}{\partial t} \end{cases} \tag{1.14}$$

notice that $\mathbf{J}_\mathrm{f}$ is set to 0 in this situation and using Eq.1.3&1.4, additionally:

$$\nabla \times (\nabla \times \mathbf{A}) = \nabla (\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A} \tag{1.15}$$

we have:

$$\begin{cases} \left(\nabla^2 - \mu\varepsilon \dfrac{\partial^2}{\partial t^2} \right) \mathbf{E} = 0 \\ \left(\nabla^2 - \mu\varepsilon \dfrac{\partial^2}{\partial t^2} \right) \mathbf{B} = 0 \end{cases} \tag{1.16}$$

Simple Harmonic Field Taking the simplest form of solves of the above equations:

$$\begin{cases} \mathbf{E}(\mathbf{R}, t) = \mathbf{E}(\mathbf{R}) \mathrm{e}^{-i\omega t} \\ \mathbf{B}(\mathbf{R}, t) = \mathbf{B}(\mathbf{R}) \mathrm{e}^{-i\omega t} \end{cases} \tag{1.17}$$

It's trivial to find out that:

$$\frac{\partial}{\partial t} = -i\omega \tag{1.18}$$

so Eq.1.16 can be rewrite into Helmholtz equations:

$$\begin{cases} \left(\nabla^2 + k^2 \right) \mathbf{E} = 0 \\ \left(\nabla^2 + k^2 \right) \mathbf{B} = 0 \end{cases} \tag{1.19}$$

where:

$$k^2 = \omega^2 \varepsilon\mu = \frac{\omega^2}{c^2}, \quad c = \frac{1}{\sqrt{\varepsilon\mu}} \tag{1.20}$$

and again take the simplest form of solutions of Eq.1.19, we have the wave function of **plane wave**:

$$\begin{cases} \mathbf{E}(\mathbf{R}) = \mathbf{E}_0 \mathrm{e}^{i\mathbf{k}\cdot\mathbf{R}} \\ \mathbf{B}(\mathbf{R}) = \mathbf{B}_0 \mathrm{e}^{i\mathbf{k}\cdot\mathbf{R}} \end{cases} \tag{1.21}$$

From Maxwell equations we have:

$$\begin{cases} \nabla \times \mathbf{E} = -\dfrac{\partial \mathbf{B}}{\partial t} = i\omega \mathbf{B} \\ \nabla \times \mathbf{H} = 0 + \dfrac{\partial \mathbf{D}}{\partial t} = -i\omega\varepsilon \mathbf{E} \end{cases}$$

$$\Rightarrow \begin{cases} \mathbf{B}(\mathbf{R}) =_{\text{(time harmonic)}} -\dfrac{i}{\omega}\nabla \times \mathbf{E}(\mathbf{R}) =_{\text{(plane wave)}} \dfrac{\mathbf{k}}{\omega} \times \mathbf{E}(\mathbf{R}) \\ \mathbf{E}(\mathbf{R}) = \dfrac{i}{\varepsilon\mu\omega}\nabla \times \mathbf{B}(\mathbf{R}) = \dfrac{ic}{k}\nabla \times \mathbf{B}(\mathbf{R}) = v\mathbf{B}(\mathbf{R}) \times \mathbf{e}_{\mathrm{k}} \end{cases} \tag{1.22}$$

If take **impedance**:

$$Z = \sqrt{\frac{\mu}{\varepsilon}} \tag{1.23}$$

can one rewrite:

$$Z\mathbf{H} = \mathbf{e_k} \times \mathbf{E} \tag{1.24}$$

Sommerfeld radiation condition Arnold Sommerfeld defined the condition of radiation for a scalar field satisfying the Helmholtz equation as

> "the sources must be sources, not sinks of energy. The energy which is radiated from the sources must scatter to infinity; no energy may be radiated from infinity into ... the field."

specifically:

$$\lim_{|\mathbf{r}| \to \infty} |x|^{\frac{n-1}{2}} \left(\frac{\partial}{\partial |\mathbf{r}|} - ik \right) u(x) = 0 \tag{1.25}$$

which implies at infinite far from the source, that in every direction in space, the wave has to tend to a plane wave propagating in that direction, and the difference between the actual wave and a plane wave propagatin in that direction has to decrease faster than $|x|^{\frac{n-1}{2}}$.

1.1.3 Energy

Energy density The energy density of any electromagnetic field is:

$$w = \frac{1}{2}(\mathbf{E} \cdot \mathbf{D} + \mathbf{B} \cdot \mathbf{H}) \tag{1.26}$$

of which proof is at 3.2.3 of Notes on General Physics.

Poynting Vector Energy conservation says:

$$\frac{\partial}{\partial t}\int_{\Omega} w\mathrm{d}V + \int_{\Sigma\Omega} \mathbf{s}\mathrm{d}\mathbf{S} + \int_{\Omega} p_l \mathrm{d}V = 0 \tag{1.27}$$

where p_l is the power density of loss and $\mathbf{s}$ is energy density flux vector, Poynting Vector. Using Eq.1.27 and Ohm's Law Eq.1.5:

$$\frac{\partial}{\partial t}\int_{\Omega}\frac{1}{2}\left(\mathbf{E}\cdot\mathbf{D}+\mathbf{B}\cdot\mathbf{H}\right)\mathrm{d}V + \int_{\Omega}\sigma E^2 \mathrm{d}V = -\int_{\Omega}\left(\nabla\cdot\mathbf{s}\right)\mathrm{d}V \tag{1.28}$$

thus:

$$\nabla\cdot\mathbf{s} = -(\varepsilon E\dot{E} + \mu H\dot{H}) - \sigma E^2 = \mathbf{H}\cdot\nabla\times\mathbf{E} - \mathbf{E}\cdot\nabla\times\mathbf{H} = \nabla\cdot(\mathbf{E}\times\mathbf{H}) \tag{1.29}$$

then Poynting Vector is:

$$\mathbf{s} = \mathbf{E}\times\mathbf{H} \tag{1.30}$$

Momentum Flux

$$\mathbf{G} = \rho\mathbf{v} = \frac{1}{c^2}wv = \frac{1}{c^2}\mathbf{s} \tag{1.31}$$

1.1.4 Energy of Harmonic Plane Wave (Complex Poynting Th.)

Using the solution in Eq.1.17&1.21 we have the simplest wave:

$$\begin{cases} \mathbf{E}(\mathbf{r},t) = \mathbf{E}_0 \mathrm{e}^{i(\varphi_E(\mathbf{R})-\omega t)} = \tilde{\mathbf{E}}(\mathbf{r})\mathrm{e}^{-i\omega t} \\ \mathbf{H}(\mathbf{r},t) = \mathbf{H}_0 \mathrm{e}^{i(\varphi_H(\mathbf{R})-\omega t)} = \tilde{\mathbf{H}}(\mathbf{r})\mathrm{e}^{-i\omega t} \end{cases} \tag{1.32}$$

for plane waves:

$$\varphi_i(\mathbf{r}) = \varphi_{i0} + \mathbf{k}\cdot\mathbf{R} \tag{1.33}$$

and define the actual field which can be measured as the real parts of those solution, can one describe the wave in complex forms, which means:

$$\begin{aligned} \mathbf{E}^{(\mathrm{Real})}(t) &= \Re\left[\tilde{\mathbf{E}}e^{-i\omega t}\right] = \frac{1}{2}\left(\tilde{\mathbf{E}}e^{-i\omega t} + \tilde{\mathbf{E}}^* e^{i\omega t}\right) \\ \mathbf{H}^{(\mathrm{Real})}(t) &= \Re\left[\tilde{\mathbf{H}}e^{-i\omega t}\right] = \frac{1}{2}\left(\tilde{\mathbf{H}}e^{-i\omega t} + \tilde{\mathbf{H}}^* e^{i\omega t}\right) \end{aligned} \tag{1.34}$$

and the energy is:

$$w = \frac{1}{2}\left(\varepsilon\Re\left[\tilde{\mathbf{E}}e^{-i\omega t}\right]^2 + \mu\Re\left[\tilde{\mathbf{H}}e^{-i\omega t}\right]\right) \tag{1.35}$$

and its average[2]:

$$\langle w\rangle = \frac{1}{4}\left(\varepsilon\tilde{\mathbf{E}}\tilde{\mathbf{E}}^* + \mu\tilde{\mathbf{H}}\tilde{\mathbf{H}}^*\right) = \frac{1}{4}\left(\varepsilon\mathbf{E}_0\mathbf{E}_0^* + \mu\mathbf{H}_0\mathbf{H}_0^*\right) \tag{1.36}$$

[2] $\Re\left[\tilde{\mathbf{E}}e^{-i\omega t}\right]^2 = \frac{1}{4}\langle\tilde{\mathbf{E}}^2\mathrm{e}^{-2i\omega} + \tilde{\mathbf{E}}^{*2}\mathrm{e}^{2i\omega} + 2\tilde{\mathbf{E}}\tilde{\mathbf{E}}^*\rangle = \frac{1}{2}\tilde{\mathbf{E}}\tilde{\mathbf{E}}^*$

Complex Poynting Vector The instantaneous Poynting vector can be defined:

$$\mathbf{s} = \frac{1}{2}\tilde{\mathbf{E}} \times \tilde{\mathbf{H}}^* \tag{1.37}$$

which stands for average energy flux vector, and can be proved by method similar to the one above.

1.2 Strum-Liouville Theory

Chapter 2

Plane Wave

2.1 Plane Wave in Dispersive Medium

2.1.1 Dispersive relation

For the influence of the external electromagnetic field on the medium is continuous, the electronic displacement is written:

$$\mathbf{D}(\mathbf{R}, t) = \int_{-\infty}^{\infty} \mathrm{d}t' \, \varepsilon(t - t')\mathbf{E}(\mathbf{R}, t') \tag{2.1}$$

with $\varepsilon(t - t') = 0$ when $t - t' < 0$ as a result of cause-and-effect relationship. Thus for a simple harmonic wave:

$$\begin{bmatrix} \mathbf{E}(\mathbf{R}, t) \\ \mathbf{H}(\mathbf{R}, t) \end{bmatrix} = \begin{bmatrix} \mathbf{E}(\mathbf{R}) \\ \mathbf{D}(\mathbf{R}) \end{bmatrix} \mathrm{e}^{-i\omega t} \tag{2.2}$$

substitute Eq.2.2 into Eq.2.1 we have:

$$\mathbf{D}(\mathbf{R}) = \mathbf{E}(\mathbf{R}) \int_{-\infty}^{\infty} \mathrm{d}t' \, \varepsilon(t - t')\mathrm{e}^{-i\omega(t'-t)} = \varepsilon(\omega)\mathbf{E}(\mathbf{R}) \tag{2.3}$$

with dispersion relation:

$$\varepsilon(\omega) = \int_{-\infty}^{\infty} \varepsilon(t)\mathrm{e}^{i\omega t} \, \mathrm{d}t \tag{2.4}$$

and one can also wirte:

$$\begin{bmatrix} \mathbf{B}(\mathbf{R}) \\ \mathbf{J}_f(\mathbf{R}) \end{bmatrix} = \begin{bmatrix} \mu(\omega) & \sigma(\omega) \end{bmatrix} \begin{bmatrix} \mathbf{H}(\mathbf{R}) \\ \mathbf{E}(\mathbf{R}) \end{bmatrix} \tag{2.5}$$

2.1.2 The propagation of a monochromatic plane wave in a dispersive medium

Recall the technic used in Eq.1.22 we get:

$$\mathbf{k} \times \mathbf{k} \times \mathbf{E} = -\omega^2 \mu_0 \varepsilon(\omega)\mathbf{E} \tag{2.6}$$

then:

$$k = \omega\sqrt{\mu_0\varepsilon(\omega)} = k_0\sqrt{\frac{\varepsilon(\omega)}{\varepsilon_0}} \tag{2.7}$$

Conductive Media In a conductive media, the Maxwell Eq. changes into:

$$\nabla \times \mathbf{B} = \mu_0\left(\sigma_c\mathbf{E} + \varepsilon_b\frac{\partial \mathbf{E}}{\partial t}\right) \tag{2.8}$$

thus:

$$\mathbf{k} \times \mathbf{B} = -\omega\mu_0\left(\varepsilon_b + i\frac{\sigma_c}{\omega}\right)\mathbf{E} \tag{2.9}$$

then can one define a complex permittivity:

$$\varepsilon_{\text{eff}} = \varepsilon_b + i\frac{\sigma_c}{\omega} \tag{2.10}$$

and complex $k - \omega$ dispersive relation.

One can also wirte:

$$\varepsilon(\omega) = \varepsilon' + i\varepsilon'' \tag{2.11}$$

accordingly:

$$\mathbf{k}(\omega) = \mathbf{k}' + i\mathbf{k}'' \tag{2.12}$$

and with $\mathbf{k}\cdot\mathbf{k} = \omega^2\mu\varepsilon$ can derive $\mathbf{k}'$ & $\mathbf{k}''$ easily.

Recall the definition of wave vactor $\mathbf{k}$:

$$\mathbf{E} = \mathbf{E}_0\mathrm{e}^{-\mathbf{k}''\cdot\mathbf{R}}\mathrm{e}^{i(\mathbf{k}'\cdot\mathbf{R}-\omega t)} \tag{2.13}$$

It is obvious that the complex wave vector represents a plane wave (real part) with exponential decay (imaginary part).

2.2 Polarization of Monochromatic Plane Wave

2.2.1 Polarization Ellipse

For monochromatic plane wave:

$$\begin{aligned}\mathbf{E}(\mathbf{R},t) &= (\mathbf{e}_1E_{10} + \mathbf{e}_2E_{20})\mathrm{e}^{i(\mathbf{kR}-\omega t)}\\ &= (\mathbf{e}_1|E_{10}|\mathrm{e}^{i\delta_1} + \mathbf{e}_2|E_{20}|\mathrm{e}^{i\delta_2})\mathrm{e}^{i(\mathbf{kR}-\omega t)}\end{aligned} \tag{2.14}$$

take the real part of the field and reorganize the expression using the substitution $\phi = \mathbf{kR} - \omega t$:

$$\begin{aligned}\Re(\mathbf{E}) =& |E_{10}|(\cos\delta_1\cos\phi - \sin\delta_1\sin\phi)\mathbf{e}_1\\ &+ |E_{20}|(\cos\delta_2\cos\phi - \sin\delta_2\sin\phi)\mathbf{e}_2\\ =& E_1\mathbf{e}_1 + E_2\mathbf{e}_2\end{aligned} \tag{2.15}$$

Reform the red and blue equality, we have:

$$\begin{bmatrix} \dfrac{E_1}{|E_{10}|} \\ \\ \dfrac{E_2}{|E_{20}|} \end{bmatrix} = \begin{bmatrix} \cos\delta_1 \cos\phi - \sin\delta_1 \sin\phi \\ \\ \cos\delta_2 \cos\phi - \sin\delta_2 \sin\phi \end{bmatrix} \tag{2.16}$$

and right multiplied by $\begin{bmatrix} \sin\delta_2 & -\sin\delta_1 \\ \cos\delta_2 & -\cos\delta_2 \end{bmatrix}$, and the magnitude squared of the 2 sides of the equation should be the same:

$$\left(\frac{E_1}{|E_{10}|}\right)^2 + \left(\frac{E_2}{|E_{20}|}\right)^2 - 2\left(\frac{E_1}{|E_{10}|}\right)\left(\frac{E_2}{|E_{20}|}\right)\cos\delta = \sin^2\delta \tag{2.17}$$

where the substitution $\delta_2 - \delta_1 = \delta$ is used. And Eq.2.17 is the polarization ellipse of the wave.

2.2.2 Linear Polarization

If:

$$\delta = m\pi,\ m \in \mathbb{Z} \tag{2.18}$$

then the polarization ellipse degenerates into a line:

$$\frac{E_2}{|E_{20}|} = (-1)^m \frac{E_1}{|E_{10}|} \tag{2.19}$$

2.2.3 Circular Polarization

If:

$$\delta = \left(m + \frac{1}{2}\right)\pi,\ m \in \mathbb{Z}, \quad |E_{10}| = |E_{20}| = E_0 \tag{2.20}$$

then the polarization ellipse turns into a circle:

$$E_1^2 + E_2^2 = E_0^2 \tag{2.21}$$

Appendix A

Ex.1 Basic Electrodynamics Theory

Ex.1.1 Deriving the Frequency Domain Form of Maxwell's Equations Using Fourier Transform.

Solution: Assuming:

$$\begin{aligned} \mathcal{F}[\mathbf{E}](\mathbf{r},\omega) &= \mathbf{e}, & \mathcal{F}[\mathbf{B}](\mathbf{r},\omega) &= \mathbf{b} \\ \mathcal{F}[\mathbf{D}](\mathbf{r},\omega) &= \mathbf{d} = \varepsilon\mathbf{e}, & \mathcal{F}[\mathbf{H}](\mathbf{r},\omega) &= \mathbf{h} = \frac{1}{\mu}\mathbf{b} \\ \mathcal{F}[\rho](\mathbf{r},\omega) &= \varrho, & \mathcal{F}[\mathbf{J}](\mathbf{r},\omega) &= \mathbf{j} \end{aligned} \tag{A.1}$$

Notice that:

$$\begin{aligned} \mathcal{F}[\dot{A}(t)](\mathbf{r},\omega) &= \int_{-\infty}^{\infty} \dot{A}(t)\mathrm{e}^{-i\omega t}\mathrm{d}t = i\omega\int_{-\infty}^{\infty} A(t)\mathrm{e}^{-i\omega t}\mathrm{d}t + 0 \\ &= i\omega\mathcal{F}[A(t)](\mathbf{r},\omega) \end{aligned} \tag{A.2}$$

and Laplacian obviously has no impact on the Fourier transformation, thus if Fourier transformation is applied to Maxwell equations, we have:

$$\begin{cases} \nabla\cdot\mathbf{d} = \varrho_{\mathrm{f}} \\ \nabla\times\mathbf{h} = \mathbf{j}_{\mathrm{f}} + i\omega\mathbf{d} \\ \nabla\cdot\mathbf{b} = 0 \\ \nabla\times\mathbf{e} = -i\omega\mathbf{b} \end{cases} \tag{A.3}$$

Ex.1.5 The dielectric constant of a linear isotropic medium is $\varepsilon(\omega) = \varepsilon'(\omega) - i\varepsilon''(\omega)$, prove:

$$\begin{aligned} \varepsilon'(\omega) &= \varepsilon_0 - \frac{2}{\pi}\int_0^\infty \frac{z\varepsilon''(z)}{z^2-\omega^2}\mathrm{d}z \\ \varepsilon''(\omega) &= \frac{2\omega}{\pi}\int_0^\infty \frac{\varepsilon'(z)}{z^2-\omega^2}\mathrm{d}z \end{aligned} \tag{A.4}$$

Proof: First prove Kramers-Kronig relations relations, by Cauchy's residue theorem for complex integration:

$$0 = \oint \frac{\varepsilon(z)}{z-\omega}\,dz = \int_{-\infty}^{\infty} \frac{\varepsilon(z)}{z-\omega}\,dz - i\pi\varepsilon(\omega). \tag{A.5}$$

take $\varepsilon(\omega) = \varepsilon'(\omega) - i\varepsilon''(\omega)$ and rearrange it:

$$\begin{aligned} \varepsilon'(\omega) &= \frac{-1}{\pi}\int_{-\infty}^{\infty} \frac{\varepsilon''(z)}{z-\omega}\,dz = \frac{-2}{\pi}\int_0^\infty \frac{(z+\omega)\varepsilon''(z)}{z^2-\omega^2}\,dz \\ &= \varepsilon_0 - \frac{2}{\pi}\int_0^\infty \frac{z\varepsilon''(z)}{z^2-\omega^2}\,dz \end{aligned} \tag{A.6}$$

where ω is omitted for ε'' is an even fuction, and ε_0 is add to make sure $\varepsilon'(0) = \varepsilon_0$.

$$\begin{aligned} \varepsilon''(\omega) &= \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\varepsilon'(z)}{z-\omega}dz = \frac{2}{\pi}\int_0^\infty \frac{(z+\omega)\varepsilon'(z)}{z^2-\omega^2}dz \\ &= \frac{2\omega}{\pi}\int_0^\infty \frac{\varepsilon'(z)}{z^2-\omega^2}dz \end{aligned} \tag{A.7}$$

where z is omitted for ε' is an odd fuction.

Ex.1.8 (Complex Poynting Th.)

Proof: Notice:

$$\nabla \cdot (\mathbf{E} \times \mathbf{H}^*) = \mathbf{H}^* \cdot \nabla \times \mathbf{E} - \mathbf{E} \cdot \nabla \times \mathbf{H}^* \tag{A.8}$$

and using the Maxwell Eq.:

$$\begin{cases} \nabla \times \mathbf{E} = -i\omega \mathbf{B} \\ \nabla \times \mathbf{H}^* = -i\omega \mathbf{D}^* + \mathbf{J}^* \end{cases} \tag{A.9}$$

using it simplify Eq.A.8 with $\mathbf{B} = \boldsymbol{\mu} \cdot \mathbf{H}$, $\mathbf{D}^* = \boldsymbol{\varepsilon}^* \cdot \mathbf{E}^*$:

$$\begin{aligned} \nabla \cdot (\mathbf{E} \times \mathbf{H}^*) &= -i\omega \mathbf{H}^* \cdot \mathbf{B} + i\omega \mathbf{E} \cdot \mathbf{D}^* - \mathbf{E} \cdot \mathbf{J}^* \\ &= -i\omega \mathbf{H}^* \cdot \boldsymbol{\mu} \cdot \mathbf{H} + i\omega \mathbf{E} \cdot \boldsymbol{\varepsilon}^* \cdot \mathbf{E}^* - \mathbf{E} \cdot \mathbf{J}^* \end{aligned} \tag{A.10}$$

thus:

$$\begin{aligned} &\int_S d\mathbf{S} \cdot (\mathbf{E} \times \mathbf{H}^*) \\ &= -i\omega \int_V dV \left(\mathbf{H}^* \cdot \boldsymbol{\mu} \cdot \mathbf{H} - \mathbf{E} \cdot \boldsymbol{\varepsilon}^* \cdot \mathbf{E}^*\right) - \int_V dV \mathbf{E} \cdot \mathbf{J}^* \end{aligned} \tag{A.11}$$

take:

$$\boldsymbol{\mu} = \mu' - i\mu, \quad \boldsymbol{\varepsilon}^* = \varepsilon' - i\varepsilon'', \quad \mathbf{J}^* = \sigma^* \mathbf{E} \tag{A.12}$$

then comes:

$$\begin{aligned} &\int_S d\mathbf{S} \cdot (\mathbf{E} \times \mathbf{H}^*) \\ =& -\omega \int_V dV \left(\mu' \mathbf{H}^* \cdot \mathbf{H} + \varepsilon' \mathbf{E} \cdot \mathbf{E}^*\right) - \int_V dV \sigma^* \mathbf{E} \cdot \mathbf{E}^* \\ &- i2\omega \int_V \mathrm{d}V (w_{\mathrm{wav}} - w_{\mathrm{eav}}) \end{aligned} \tag{A.13}$$

Ex.1.9 If take $\varepsilon = \varepsilon(\mathbf{r})$, prove:

$$\nabla^2 \mathbf{E} + k^2 \mathbf{E} = -\nabla \left(\mathbf{E} \cdot \frac{\nabla \varepsilon}{\varepsilon} \right) \tag{A.14}$$

Proof: Notice:

$$\nabla \times \nabla \times \mathbf{E} = \omega^2 \mu \varepsilon \mathbf{E} \tag{A.15}$$

and:

$$\nabla \times \nabla \times \mathbf{E} = \nabla(\nabla \cdot \mathbf{E}) - \nabla^2 \mathbf{E} \tag{A.16}$$

From Maxwell Eq.:

$$\nabla \cdot \mathbf{D} = 0 = \nabla \cdot (\varepsilon \mathbf{E}) \Rightarrow \nabla \cdot \mathbf{E} = \mathbf{E} \cdot \frac{\nabla \varepsilon}{\varepsilon} \tag{A.17}$$

thus:

$$\omega^2 \mu \varepsilon \mathbf{E} = \nabla \times \nabla \times \mathbf{E} = \nabla \left(\mathbf{E} \cdot \frac{\nabla \varepsilon}{\varepsilon} \right) - \nabla^2 \mathbf{E} \tag{A.18}$$

take $k^2 = \omega^2 \varepsilon \mu$:

$$\left(\nabla^2 + k^2\right) \mathbf{E} = \nabla \left(\mathbf{E} \cdot \frac{\nabla \varepsilon}{\varepsilon} \right) \tag{A.19}$$