~~2, 3, 6~~ 1, 4, 5

6.81 6.83

2) a) $\underset{z=-1}{\text{Res}} \dfrac{z^{1/4}}{(z+1)^1} = \dfrac{1+i}{\sqrt{2}}$

$\phi(z) = z^{1/4} = z^{-4}$

$\phi(z_0) = (-1)^{-4} = \sqrt{i}$

$\cancel{a^2+b^2 = i}$ $(a+bi)^2 = i$

$(a^2-b^2)+(2ab)i = i$

$a^2-b^2=0$ $2ab=1$

$a=b$ $2a^2=1$

$a=\dfrac{1}{\sqrt{2}} \to \sqrt{i} = \dfrac{1+i}{\sqrt{2}} = \phi(z_0)$

~~b) Res~~ $\cancel{\dfrac{\text{Log } z}{(z^2+1)^2} = \dfrac{\pi+2i}{8}}$

$\cancel{\dfrac{\text{Log } z}{(z^2+1)^2} = \dfrac{\text{Log } z}{[(z+i)(z-i)]^2} = \dfrac{\frac{\text{Log}(z)}{(z+i)^2}}{(z-i)^2} \quad \phi = \dfrac{\text{Log } z}{(z+i)^2} \quad m=2}$

$\cancel{\phi'(i) = \lim_{z\to i} \dfrac{\phi(z)-\phi(i)}{z-i} = \dfrac{\frac{\text{Log } z}{(z+i)^2} - \frac{\text{Log}(i)}{-4}}{z-i}}$

$\cancel{\dfrac{\frac{4\text{ Log } z}{4(z+i)^2} + \frac{\text{Log}(i)(z+i)^2}{4(z+i)^2}}{z-i} = \dfrac{4\text{Log}(z)+\text{Log}(i)(z+i)^2}{4(z+i)^2(z-i)}}$

$\cancel{\dfrac{4\text{Log}(z) + i\frac{\pi}{2} - \frac{\pi}{2}\pi i}{4(z+i)^2(z-i)}}$

b) $\operatorname*{Res}_{z=i} \dfrac{\operatorname{Log} z}{(z^2+1)^2} = \dfrac{\frac{\operatorname{Log} z}{(z+i)^2}}{(z-i)^2}$ $\quad \phi = \dfrac{\operatorname{Log} z}{(z+i)^2} \quad m=2$

$$\phi'(z_0) = \frac{\frac{1}{z}(z+i)^2 - 2(z+i)\operatorname{Log} z}{(z+i)^4}$$

$$\frac{\frac{1}{i}(2i)^2 - 2(2i)\, i\frac{\pi}{2}}{(2i)^4} = \frac{4i + 2\pi}{(2i)^4} = \frac{4i+2\pi}{16}$$

$$= \frac{\pi + 2i}{8}$$

c) $\operatorname*{Res}_{z=i} \dfrac{z^{1/2}}{(z^2+1)^2}$

$$\frac{z^{1/2}}{(z^2+1)^2} = \frac{\frac{z^{1/2}}{(z+i)^2}}{(z-i)^2} \quad m=2 \quad \phi = \frac{z^{1/2}}{(z+i)^2}$$

$$\phi'(z_0) \cancel{= \frac{2(z+i)^2 z^{-3} - 2z^{-2}(z+i)}{(z+i)^4}}$$

$$\cancel{\frac{-2(2i)^2 \frac{1}{i^3} - 2\frac{1}{i^2}(2i)}{(2i)^4}}$$

$$\cancel{\phi'(z_0) = \frac{(z+i)}{2z^{\frac{1}{2}}}}$$

$\longrightarrow$

$$\phi'(z_0) = \frac{(z+i)^2 \cdot \frac{1}{2} z^{-\frac{1}{2}} - 2z^{\frac{1}{2}}(z+i)}{(z+i)^4}$$

$$= \frac{\frac{(2i)^2}{2}\left(i^{\frac{1}{2}}\right)^{-1} - 2(2i)(i)^{\frac{1}{2}}}{16}$$

$$= \underline{-2\left(\frac{1-i}{\sqrt{2}}\right) - 4i\left(\frac{1+i}{\sqrt{2}}\right)}$$

$$= \frac{1}{\sqrt{2}} \quad \frac{-2(1-i) - 4i(1+i)}{16}$$

$$= \frac{(-1+i) + -2i - 2i^2}{8\sqrt{2}}$$

$$= \frac{1-i}{8\sqrt{2}}$$

5) $\int_C \frac{dz}{z^3(z+4)}$

$f(z) = \frac{1}{z^3(z+4)}$ Residues at -4 & 0

$\underset{z=0}{\text{Res}} \frac{1 \cdot \frac{1}{(z+4)}}{(z-0)^3}$ $m=3$ $\phi = \frac{1}{(z+4)}$

~~$\phi''(0)/2$~~

$\phi'(z) = \frac{-1}{(z+4)^2}$

$\phi''(z) = \frac{\cancel{(z+4)}}{2(z+4)^{\cancel{3}2}}$ at $z=0$ $\frac{1}{2\cdot 4^2} = \frac{1}{32}$

$\underset{z=0}{\text{Res}} = \frac{\frac{1}{32}}{2!} = \frac{1}{64}$

$\underset{z=-4}{\text{Res}} = \frac{\frac{1}{z^3}}{(z+4)}$ $\phi = \frac{1}{z^3}$ $m=1$

$\underset{z=-4}{\text{Res}} = \phi(z_0) = \frac{1}{(-4)^3} = \frac{-1}{64}$

The cirlle $|z|=2$ contains the pole $z=0$

$\frac{2\pi i}{64} = \frac{\pi i}{32} = \int_C \frac{1}{z^3(z+4)} dz$

$|z+2|=3$ contains both $z=0$ & $z=-4$

$\frac{2\pi i}{64} + \frac{-2\pi i}{64} = 0 = \int_C \frac{1}{z^3(z+4)} dz$

b) $\int_C \frac{\cosh(\pi z)}{z(z^2+1)}\,dz$ has poles at $z=\pm i$, $z=0$

$$\int_C \frac{\cosh(\pi z)}{z(z^2+1)}\,dz = \operatorname*{Res}_{z=i} \frac{\cosh(\pi z)}{z(z^2+1)} + \operatorname*{Res}_{z=0} \frac{\cosh(\pi z)}{z(z^2+1)}$$

$$\operatorname*{Res}_{z=i} \frac{\frac{\cosh(\pi z)}{z(z+i)}}{(z-i)} \qquad \phi(z) = \frac{\cosh(\pi z)}{z(z+i)} \qquad m=1$$

$$\operatorname*{Res}_{z=i} f(z) = \phi(z_0) = \frac{\cosh(\pi i)}{i(2i)} = \frac{-\cosh(\pi i)}{2}$$

$$\cosh(i\pi) = \frac{e^{i\pi}+e^{-i\pi}}{2} = \frac{-1+-1}{2} = -1$$

$$\frac{-(-1)}{2} = \frac{1}{2}$$

$$\operatorname*{Res}_{z=0} \frac{\cosh(\pi z)}{z(z^2+1)} = \operatorname*{Res}_{z=0} \frac{\frac{\cosh(\pi z)}{(z^2+1)}}{z} \qquad \phi = \frac{\cosh(z\pi)}{(z^2+1)}$$

$$\phi(0) = \frac{\cosh(0)}{1} = 1$$

$$\operatorname*{Res}_{z=-i} \frac{\cosh(\pi z)}{z(z^2+1)} = \operatorname*{Res}_{z=-i} \frac{\frac{\cosh(z\pi)}{z(z-i)}}{(z+i)} \qquad \phi = \frac{\cosh(\pi z)}{z(z-i)}$$

$m=1$

$$\phi(-i) = \frac{\cosh(-i\pi)}{-i(-2i)} = \frac{-1}{-2} = \frac{1}{2}$$

~~Res~~ ~~Res~~

$$f(z) = \frac{\cosh(\pi z)}{z(z^2+1)}$$

$$\int_C \frac{\cosh \pi z}{z(z^2+1)}\,dz = \left(\operatorname*{Res}_{z=i} f(z) + \operatorname*{Res}_{z=-i} f(z) + \operatorname*{Res}_{z=0} f(z)\right)2\pi i$$

$$\operatorname*{Res}_{z=i} f(z) = \frac{1}{2}$$

$$\operatorname*{Res}_{z=-i} f(z) = \frac{1}{2}$$

$$\operatorname*{Res}_{z=0} f(z) = 1$$

$$\left(\frac{1}{2}+\frac{1}{2}+1\right)2\pi i = 4\pi i$$

6.83

1) $\sin(\theta)$ is analytic around 0, let $\frac{P(z)}{q(z)} = \frac{1}{\sin(z)}$

$$\operatorname*{Res}_{z=z_0} \frac{P(z)}{q(z)} = \frac{P(z_0)}{q'(z_0)} = \frac{1}{\cos(0)} = \frac{1}{1}$$ = Unity, pole of order 1

4a) $$\operatorname*{Res}_{z=z_n} z\sec z = (-1)^{n+1} z_n$$

$\frac{z}{\cos(z)}$ This function has a pole at every $\left(\frac{\pi}{2}+\pi n\right)$

let $p(z) = z$ $q(z) = \cos(z)$

$$\operatorname*{Res}_{z=z_n} = \frac{z_n}{-\sin(z_n)} = \frac{\frac{\pi}{2}+n\pi}{-\sin(\frac{\pi}{2}+n\pi)} = \frac{\pi}{2}+n\pi(\sin(\frac{\pi}{2}+n\pi))$$

if n is even, then $\sin(z_n) = 1$

else, $\sin(z_n) = -1$.

$= z_n(-1)^{1+n}$

b) $\operatorname{Res}_{z=z_n} \tanh(z) = 1$ $\quad z_n = \left(\frac{\pi}{2} + n\pi\right) i$

$\frac{p(z)}{q(z)} = \frac{\sinh(z)}{\cosh(z)}$ $\quad \operatorname{Res}_{z=z_n} = \frac{\sinh(z)}{\sinh(z)} = 1$ for any

z_n poles occur at

$z_n = \left(\frac{\pi}{2} + \pi n\right) i$

5) a) $\int_C \tan z \, dz = \int_C \frac{\sin(z)}{\cos(z)} dz$ has poles at

$z = \pm \frac{\pi}{2}$

$\operatorname{Res}_{z=\frac{\pi}{2}} \frac{\sin z}{\cos z} = \frac{\sin\left(\frac{\pi}{2}\right)}{-\sin\left(\frac{\pi}{2}\right)} = -1$

$\operatorname{Res}_{z=-\frac{\pi}{2}} \frac{\sin(z)}{\cos(z)} = \frac{\sin\left(-\frac{\pi}{2}\right)}{-\sin\left(-\frac{\pi}{2}\right)} = -1$

$(-1 + -1) 2\pi i = -4\pi i$

b) $\int_C \frac{1}{\sinh(2z)} dz$ has poles at 0 & $\pm i\frac{\pi}{2}$

$\operatorname{Res}_{z=0} \frac{1}{\sinh(2z)} = \frac{1}{2\cosh(2 \cdot 0)} = \frac{1}{2}$

$\operatorname{Res}_{z=\pm i\frac{\pi}{2}} = \frac{1}{2\cosh(i\pi)} = -\frac{1}{2}$ $\quad 2\pi i\left(\frac{1}{2} - \frac{1}{2} - \frac{1}{2}\right) = -\pi i$