~~1, 2, 3, 4~~

5.72 $\boxed{}$

1, 3, 4, 5

5.73 $\boxed{}$

1) $\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n$

take derrivative of both sides

$$\frac{d}{dz}\left(\frac{1}{1-z}\right) = \frac{1}{(1-z)^2}$$

$$\frac{d}{dz}\left(\sum_{n=0}^{\infty} z^n\right) = \sum_{n=1}^{\infty} n z^{n-1}$$ let $n \Rightarrow n+1$

$$\sum_{n+1=1}^{\infty} (n+1) z^{n+1-1} = \boxed{\sum_{n=0}^{\infty} (n+1) z^n = \frac{1}{(1-z)^2}}$$

ditto

$$\frac{d}{dz}\left(\frac{1}{(1-z)^2}\right) = \frac{(1-z)^2 \cdot 0 - (-2 \cdot 1) \cdot \cancel{(1-z)}}{(1-z)^{\cancel{4}}} = \frac{2}{(1-z)^3}$$

$$\frac{d}{dz}\left(\sum_{n=0}^{\infty} (n+1) z^n\right) = \sum_{n=1}^{\infty} n(n+1) z^{n-1}$$ let $n \Rightarrow n+1$

$$\boxed{= \sum_{n=0}^{\infty} (n+1)(n+2) z^n = \frac{2}{(1-z)^3}}$$

2) $\frac{1}{(1-z)^2} = \sum_{n=0}^{\infty} (n+1) z^n$ let $z \Rightarrow \frac{1}{1-z}$

$$\frac{1}{\left(1-\frac{1}{1-z}\right)^2} = \sum_{n=0}^{\infty} \frac{(n+1)}{(1-z)^n} = \sum_{n=0}^{\infty} \frac{(n+1)}{(1-z)^n}$$

$$= \frac{1}{\left(1-\frac{1}{1-z}\right)^2} = \frac{1}{1-\frac{2}{1-z}+\frac{1}{(1-z)^2}} = \frac{1}{\frac{(1-z)^2}{(1-z)^2} - \frac{2(1-z)}{(1-z)^2} + \frac{1}{1-z^2}}$$

$$= \frac{1-2z+z^2-2+2z+1}{(1-z)^2} = \frac{z^2}{(1-z)^2} = \left(1-\frac{1}{1-z}\right)^2 \therefore$$

$$\frac{1}{\frac{z^2}{(1-z)^2}} = \frac{(1-z)^2}{z^2} = \sum_{n=0}^{\infty} \frac{(n+1)}{(1-z)^n}$$ divide both by $(1-z)^2$, yeilding

$$\frac{1}{z^2} = \sum_{n=0}^{\infty} \frac{(n+1)}{(1-z)^{n+2}}$$ let $n = n-2$

$$\boxed{\frac{1}{z^2} = \sum_{n=2}^{\infty} \frac{(n-2+1)}{(1-z)^{n-2+2}} = \sum_{n=2}^{\infty} \frac{(n-1)(-1)^n}{(z-1)^n}}$$

Note: $(-1)^n$ comes from $(1-z)^n = [(-1)\cdot(z-1)]^n = (-1)^n(z-1)^n$

since $(-1)^n = (-1)^{-n}$, subsitute.

3) $\frac{1}{z} = \frac{1}{2+z-2} = \frac{1}{2} \cdot \frac{1}{1+\frac{z-2}{2}}$

$\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n$ sub $-z$ for z

$\frac{1}{1+z} = \sum_{n=0}^{\infty} (-1)^n z^n$ sub $\frac{z-2}{2}$ for z

$\frac{1}{1+\frac{z-2}{2}} = \sum_{n=0}^{\infty} (-1)^n \left(\frac{z-2}{2}\right)^n$ multiply both by $\frac{1}{2}$

$\frac{1}{2} \cdot \frac{1}{1+\frac{z-2}{2}} = \frac{1}{2} \sum_{n=0}^{\infty} (-1)^n \left(\frac{z-2}{2}\right)^n$

Term by term

$n=0, \quad 1, \quad 2, \quad 3$

$f(z)$: $1, \; -\left(\frac{z-2}{2}\right), \; \left(\frac{z-2}{2}\right)^2, \; -\left(\frac{z-2}{2}\right)^3 \; \ldots$

$f'z$: $0, \; -1, \; 2\left(\frac{z-2}{2}\right), \; -3\left(\frac{z-2}{2}\right)^2 \; \ldots$

yeilding $\frac{1}{4} \sum_{n=\phi}^{\infty} (-1)^{n} (n) \left(\frac{z-2}{2}\right)^{n-1}$ let $n = n+1$

$= \frac{1}{4} \sum_{n=0}^{\infty} (-1)^n (n+1) \left(\frac{z-2}{2}\right)^n = \frac{d}{dz} \frac{1}{z} = \frac{1}{z^2}$

4) $\cos z = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n}}{2n!} = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \frac{z^6}{6!} + \ldots$

mult both sides by -1

$-\cos z = -1 + \frac{z^2}{2!} - \frac{z^4}{4!} + \ldots$ add 1 to both sides

$1 - \cos z = \frac{z^2}{2!} - \frac{z^4}{4!} + \frac{z^6}{6!} \ldots$ divide by z^2

$\frac{1-\cos z}{z^2} = \frac{1}{2} - \frac{z^2}{4!} + \frac{z^4}{6!} \ldots$

Since this clearly converges at $z=0$

$\lim_{z \to 0} \frac{1-\cos z}{z^2} = \lim_{z \to 0} f(z) = f(0) = \frac{1}{2}$

$f(z) = \frac{1-\cos z}{z^2}$ can be represented as the convergent series, $\therefore$ is entire because it is convergent.

1) $e^z = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} \dots$

$\frac{1}{z^2+1} = 1 - z^2 + z^4 - z^6 \dots$

$e^z \cdot \frac{1}{z^2+1} = 1 + z - \frac{1}{2}z^2 + \left(\frac{1}{6} - 1\right) z^3 + \dots$

$\frac{e^z}{z(z^2+1)} \cancel{=} 1 + z - \frac{1}{2}z^2 - \frac{5}{6}z^3 \dots$ divide by z

$\rightarrow \frac{1}{z} + 1 - \frac{1}{2}z - \frac{5}{6}z^2$

3) $\csc(z) = \frac{1}{\sin(z)}$

$\sin(z) = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n+1}}{(2n+1)!} = z - \frac{z^3}{3!} + \frac{z^5}{5!} \dots$

$\frac{1}{\sin z} = \frac{1}{z}\left(\frac{1}{1 - \frac{z^2}{3!} + \frac{z^4}{5!} \dots}\right)$

$= 1 + \frac{z^2}{3!} + \left[+\frac{1}{(3!)^2} - \frac{1}{5!}\right] z^4 \dots = 1 + \frac{z^2}{3!} + \frac{7}{360} z^4 + \dots$

divide by z again

$= \frac{1}{z} + \frac{1}{3!}z + \left[\frac{1}{(3!)^2} - \frac{1}{5!}\right] z^3 + \dots$

4) $e^z - 1 = \sum_{n=0}^{\infty} \frac{z^n}{n!} - 1 = \sum_{n=1}^{\infty} \frac{z^n}{n!} = z\sum_{n=0}^{\infty} \frac{z^n}{(n+1)!} = z\left(1 + \sum_{n=1}^{\infty} \frac{z^n}{(n+1)!}\right)$

Note pattern, yeilding

$$\frac{1}{e^z-1} = \frac{1}{z}\left(1+\sum_{n=1}^{\infty}\frac{z^n}{(n+1)!}\right)^{-1} = \frac{1}{z}\sum_{k=0}^{\infty}(-1)^k\left[\sum_{n=1}^{\infty}\frac{z^n}{(n+1)!}\right]^k$$

$$= \frac{1}{z} - \frac{1}{z}\left(\frac{z}{2!}+\frac{z^2}{3!}+\frac{z^3}{4!}+\dots\right) + \frac{1}{z}\left(\frac{z}{2!}+\frac{z^2}{3!}+\frac{z^3}{4!}+\dots\right)^2 + \dots$$

$$= \frac{1}{z} - \frac{1}{2!} + \left(-\frac{1}{3!}+\frac{1}{2!^2}\right)z + \left(-\frac{1}{4!}+\frac{2}{2!\cdot 3!}-\frac{1}{2!^3}\right)z^2$$
$$+ \left(-\frac{1}{5!}+\frac{2}{2!\cdot 4!}+\frac{1}{3!^2}-\frac{3}{2!^2\cdot 3!}+\frac{1}{2!^4}\right)z^3 + \dots$$

$$= \frac{1}{z} - \frac{1}{2} + \frac{z}{12} - \frac{z^3}{720} + \dots$$

5) $\dfrac{1}{\frac{\sinh z}{z}} = 1 - \frac{1}{6}z^2 + \frac{7}{360}z^4 + \dots$

$$\frac{1}{z^2\sinh z} = \frac{1}{z^3}\cdot\frac{1}{\frac{\sinh z}{z}} = \frac{1}{z^3}\left(1 - \frac{1}{6}z^2 + \frac{7}{360}z^4 + \dots\right)$$

$$= \left(\frac{1}{z^3} - \frac{1}{6z} + \frac{7z}{360} + \dots\right) = \frac{1}{z^2\sinh z}$$

implies $\dfrac{1}{z^2 \sinh z} = \sum_{n=-\infty}^{\infty} c_n z^n$

$$c_n = \frac{1}{2\pi i} \int_C \left(\frac{1}{z^2 \sinh z} \right) \frac{1}{z^{n+1}} \, dz \quad n = 0, \pm 1, \pm 2 \ldots$$

for $n = -1$, we have

$$\frac{1}{2\pi i} \int_C \frac{1}{z^2 \sinh z} \, dz = c_{-1} = \frac{-1}{6}$$

$$\int_C \frac{1}{z^2 \sinh z} = \frac{-2\pi i}{6} = \frac{-\pi i}{3}$$