5.61 ~~1~~, ~~2~~, ~~4~~

1) $\lim_{n \to \infty} \left(\frac{1}{n^2} + i \right) = i \qquad z_n = \frac{1}{n^2} + i$

$|z_n - i| = \left| \frac{1}{n^2} \right| = \frac{1}{n^2} < \varepsilon$ whenever $n > \frac{1}{\sqrt{\varepsilon}}$

2) $z_n = 1 + \frac{i(-1)^n}{n^2}$

$z_n = re^{i\theta}$ since $-\pi < \theta < \pi$

$r_n = |z_n| = \sqrt{1^2 + \frac{(-1)^{2n}}{n^4}} = \sqrt{1 + \frac{1}{n^4}}$

$\lim_{n \to \infty} \sqrt{1 + \frac{1}{n^4}} = 1$

$\theta_n = \tan^{-1}\left(\frac{\frac{(-1)^n}{n^2}}{\frac{1}{1}} \right) = \tan^{-1}\left(\frac{(-1)^n}{n^2} \right)$

$\lim_{n \to \infty} \tan^{-1}\left(\frac{(-1)^n}{n^2} \right) \qquad \lim_{n \to \infty} \theta_{2n} = 0$ and $\lim_{n \to \infty} \theta_{2n-1} = 0$

$= \lim \tan^{-1}(0)$

$\lim_{n \to \infty} \theta_n = 0$

4) $\sum_{n=1}^{\infty} z_n = \frac{1}{1-z}$ $\qquad z = re^{i\theta}$

$= \frac{1}{1-re^{i\theta}}$ $\qquad re^{i\theta} = r\cos\theta + ri\sin\theta$

$= \frac{1}{1-r\cos\theta - ri\sin\theta} \cdot \frac{(1-r\cos\theta + ri\sin\theta)}{(1-r\cos\theta + ri\sin\theta)}$

$= \frac{1-r\cos\theta + ri\sin\theta}{(1-r\cos\theta)^2 + r^2\sin^2\theta} = \frac{1-r\cos\theta + ri\sin\theta}{1-2r\cos\theta + r^2\cos^2\theta + r^2\sin^2}$

$= \frac{1-r\cos\theta + ri\sin\theta}{1-2r\cos\theta + r^2(\cos^2\theta + \sin\theta)} = \frac{1-r\cos\theta + ri\sin\theta}{1-2r\cos\theta + r^2}$

$\frac{1-r\cos\theta}{1-2r\cos\theta + r^2} + i\,\frac{r\sin\theta}{1-2r\cos\theta + r^2} = \sum_{n=1}^{\infty} z_n$

$\sum_{n=0}^{\infty} r^n e^{in\theta} = \sum_{n=0}^{\infty} r^n(\cos n\theta + i\sin n\theta)$

$= \sum_{n=0}^{\infty} r^n \cos n\theta + i\sum_{n=0}^{\infty} r^n \sin n\theta$

$\sum_{n=0} r^n \cos n\theta = R\left(\sum_{n=0}^{\infty} r^n e^{in\theta}\right)$

$= R\left(\frac{1}{1-re^{i\theta}}\right)$

$= \frac{1-r\cos\theta}{1-2r\cos\theta + r^2}$.

$\sum_{n=0}^{\infty} r^n \sin n\theta = Im\left(\sum_{n=0}^{\infty} r^n e^{i\theta}\right)$

$= Im\left(\frac{1}{1-re^{i\theta}}\right)$

$= \frac{r\sin\theta}{1-2r\cos\theta + r^2}$

taking out $n=0$, we get

$$\sum_{n=0}^{0} r^n \cos n\theta = 1 \quad \& \quad \sum_{n=0}^{0} r^n \sin n\theta = 0$$

$$\sum_{n=1}^{\infty} r^n \cos n\theta = \frac{1-r\cos\theta}{1-2r\cos\theta+r^2} - 1$$

$$= \frac{1-r\cos\theta}{1-2r\cos\theta+r^2} - \frac{1-2\cos\theta+r^2}{1-2r\cos\theta+r^2}$$

$$= \frac{r\cos\theta-r^2}{1-2r\cos\theta+r^2} = \sum_{n=1}^{\infty} r^n \cos n\theta$$

$$\sum_{n=1}^{\infty} r^n \sin n\theta = \frac{r\sin\theta}{1-2r\cos\theta r^2} + 0$$

5.65 ~~1~~, ~~3~~, ~~8~~, 11

1) From the series $\cosh z = \sum_{n=0}^{\infty} \frac{z^{2n}}{2n!}$

~~then~~ replace z with z^2

$$\cosh z^2 = \sum_{n=0}^{\infty} \frac{z^{4n}}{2n!}$$

$$z \cosh z^2 = \sum_{n=0}^{\infty} \left(\frac{z^{4n}}{2n!}\right) \cdot z = \sum_{n=0}^{\infty} \frac{z^{4n+1}}{2n!} \quad \checkmark$$

3) $f(z) = \frac{z}{z^4+4} = \frac{z}{4} \cdot \frac{1}{1+\frac{z^4}{4}}$

we know $\frac{1}{1-z} = 1 + z + z^2 + \ldots \quad |z| < 1$

replace z with $-\frac{z^4}{4}$, we get

$$\frac{1}{1-\frac{z^4}{4}} = 1 - \frac{z^4}{4} + \frac{z^8}{4^2} - \frac{z^{12}}{4^3} \ldots$$

Therefore

$$\frac{z}{4} \cdot \left(1 - \frac{z^4}{4} + \frac{z^8}{4^2} - \frac{z^{12}}{4^3} \ldots\right) = \frac{z}{4} - \frac{z^5}{4^2} + \frac{z^9}{4^3} - \frac{z^{13}}{4^4} \ldots$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n \cdot z^{4n+1}}{4^{n+1}} = \sum_{n=0}^{\infty} \frac{(-1)^n}{2^{2n+2}} z^{4n+1}$$

$|z| <$ (converges

9) $\sin z = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n+1}}{(2n+1)!}$

replace z with z^2

$$\sin z^2 = \sum_{n=0}^{\infty} (-1)^n \frac{z^{4n+2}}{(2n+1)!}$$

$$\frac{d}{dz} \sin z^2 = 2z \cos z^2$$

$$\frac{d^2}{dz^2} \sin z^2 = 2\cos z^2 - 4 z^2 \sin z^2$$

$$\frac{d^3}{dz^3} \sin z^2 = -4z \sin z^2 - 8 z \sin z^2 - 8 z^3 \cos z^2$$

$$= -12 z \sin z^2 - 8 z^3 \cos z^2$$

$$\frac{d^4}{dz^4} = -12 \sin z^2 - 24 z^2 \cos z^2 - 32 z^2 \cos z^2 + 16 z^4 \sin z^2$$

generalizing

$$\left.\frac{d^{4n}}{dz^{4n}} \sin z^2\right|_{z=0} = 0$$ ($z=0$, $\sin 0 = 0$) all parts of derivative become 0, $\sin 0 \to 0$

$$\left.\frac{d^{2n+1}}{dz^{2n+1}} \sin z^2\right|_{z=0} = 0$$ ($z=0$) all parts of derivative become 0

11) $\dfrac{1}{4z-z^2} = \dfrac{1}{4z} \cdot \dfrac{1}{1-\frac{z}{4}} \rightarrow \dfrac{1}{1-z} = 1 + z + z^2 + z^3 \ldots$

replace z with $\frac{z}{4}$

$$\frac{1}{4z}\left(1 + \frac{z}{4} + \frac{z^2}{16} + \frac{z^3}{4^3} \cdots\right)$$

$$\frac{1}{4z} + \left(\frac{z}{4^2 z} + \frac{z^2}{4^3 z} + \frac{z^3}{4^4 z} \cdots\right) = \frac{1}{4z} + \left(\frac{1}{4^2} + \frac{z}{4^3} + \frac{z^2}{4^4} \cdots\right.$$

$$\frac{1}{4z} + \sum_{n=0}^{\infty} \frac{z^n}{4^{2+n}} \qquad 0 < |z| < 4$$

5.68 $\cancel{1}, \cancel{2}, 6, 7$

1) using the maclaurin series $\sin(z) = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n+1}}{(2n+1)!}$

replace z with $\frac{1}{z^2}$

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!\, z^{4n+2}} = \sin \frac{1}{z^2}$$

$$z^2 \cdot \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!\, z^{4n} z^2} = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} \cdot \frac{1}{z^{4n}}$$

Solve when $n=0$

$$\frac{(-1)^0}{(2\cdot 0+1)!} \cdot \frac{1}{z^0} = 1$$ yeilding $$1 + \sum_{n=1}^{\infty} \left(\frac{(-1)^n}{(2n+1)!}\right) \cdot \frac{1}{z^{4n}}$$

2) $f(z) = \frac{1}{z} \cdot \frac{1}{1+\frac{1}{z}}$

using $\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n = 1 + z + z^2 + \ldots$

replace z with $-\frac{1}{z} \rightarrow$ $1 - \frac{1}{z} + \frac{1}{z^2} - \frac{1}{z^3} \ldots$

multiply by $\frac{1}{z}$ $= \frac{1}{z} - \frac{1}{z^2} + \frac{1}{z^3} - \frac{1}{z^4} \ldots$

into series, $\sum_{n=0}^{\infty} \frac{(-1)^{n+2}}{z^{n+1}}$ let $n = n-1$

$$\sum_{n-1=0}^{\infty} \frac{(-1)^{n-1+2}}{z^{n-1+1}} = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{z^n}$$

$-1+1=0$

6) $\frac{z}{(z-1)(z-3)} = \frac{1}{(z-3)} \cdot \frac{z}{(z-1)} = \frac{1}{z-3} \cdot \frac{z-1+1}{z-1}$

$\frac{1}{z-3} = -\frac{1}{2+1-z} = -\frac{1}{2-(z-1)}$

$= -\frac{1}{2} \cdot \frac{1}{1-\frac{z-1}{2}} = -\frac{1}{2}\sum_{n=0}^{\infty} \frac{(z-1)^n}{2^n} = -\sum_{n=0} \frac{(z-1)^n}{2^{n+1}}$

$\left(1+\frac{1}{z-1}\right)(-1)\sum_{n=0}^{\infty} \frac{(z-1)^n}{2^{n+1}} = -\sum_{n=0}^{\infty} \frac{(z-1)^n}{2^{n+1}} - \sum_{n=0}^{\infty} \frac{(z-1)^n}{2^{n+1}(z-1)}$

~~Pt~~ Solve for n=1 of second ~~equation~~ Sumation

$-\sum_{n=0}^{0} \frac{(z-1)^0}{2^1(z-1)} = \frac{-1}{2(z-1)}$

yeilding $\frac{-1}{2(z-1)} - \sum_{n=0}^{\infty} \frac{(z-1)^n}{2^{n+1}} - \sum_{n=1}^{\infty} \frac{(z-1)^{n-1}}{2^{n+1}}$

let $n-1=n$ in the second sumation, yeilding

$\frac{-1}{2(z-1)} - \sum_{n=0}^{\infty} \frac{(z-1)^n}{2^{n+1}} - \sum_{n=0}^{\infty} \frac{(z-1)^n}{2^{n+2}}$

$= \frac{-1}{2(z-1)} - 3\sum^{\infty} \frac{(z-1)^n}{2^{n+2}}$

7) a) $\frac{1}{1-z} = \sum_{n=1}^{\infty} z^n$ which implies

$$\frac{1}{1-z} = -\frac{1}{(z)(1-\frac{1}{z})} = -z\sum_{n=1}^{\infty}\left(\frac{1}{z}\right)^n$$

$$= \sum_{n=0}^{\infty}\left(\frac{1}{z}\right)^{n+1} = \sum_{n=0}^{\infty} z^{-n-1}$$

insert a, getting

$$\frac{a}{z-a} = \frac{a}{a(1-\frac{z}{a})} = \frac{1}{1-\frac{z}{a}} = \sum_{n=0}^{\infty}\left(\frac{z}{a}\right)^{-n-1}$$

$$= \sum_{n=0}^{\infty}\left(\frac{a}{z}\right)^{n-1} \quad \text{let } n=$$

$$= \sum_{n=1}^{\infty}\frac{a^n}{z^n} \quad (|a|<|z|<\infty$$

(a) proven.

b) $$\frac{a}{z-a} = \frac{a}{e^{i\theta}-a} = \frac{a}{e^{i\theta}-a}\cdot\frac{\overline{e^{i\theta}-a}}{\overline{e^{i\theta}-a}} = \frac{a(e^{-i\theta}-a)}{|e^{i\theta}-a|^2}$$

$$= \frac{a(e^{-i\theta}-a)}{(\cos\theta-a)^2+\sin^2\theta} = \frac{a(e^{-i\theta}-a)}{a^2-2a\cos\theta+\cos^2\theta+\sin^2\theta} = \frac{a(e^{-i\theta}-a)}{a^2-2a\cos\theta+}$$

$$\frac{a(e^{-i\theta}-a)}{a^2-2a\cos\theta+1} = \frac{a}{z-a} = \sum_{n=1}^{\infty}\frac{a^n}{e^{i\theta n}} = \sum_{n=1}^{\infty} a^n e^{-in\theta}$$

$$\mathrm{Re}\left(\frac{a(e^{-i\theta}-a)}{a^2-2a\cos\theta+1}\right) = \mathrm{Re}\left(\sum_{n=1}^{\infty} a^n e^{-in\theta}\right)$$

$$\frac{a\cos\theta-a^2}{a^2-2a\cos\theta+1} = \sum_{n=1}^{\infty} a^n\cos n\theta$$

Likewise, $\operatorname{Im}\left(\frac{a(e^{-i\theta}-a)}{a^2-2a\cos\theta+1}\right) = \operatorname{Im}\left(\sum_{n=1}^{\infty} a^n e^{-i\theta n}\right)$

$$\frac{-a\sin\theta}{a^2-2a\cos\theta+1} = \sum_{n=1}^{\infty} a^n \sin n\theta$$