$\cancel{1}, \cancel{2}, 4, 5$ $\qquad$ 1, 2, 3, 4

6.77 $\boxed{}$ $\qquad$ 6.80 $\boxed{}$

1)
a) $\frac{1}{z+z^2} = \frac{1}{z} \cdot \frac{1}{z+1} = \frac{1}{z} \sum_{n=0}^{\infty} (-1)^n z^n$

$= \sum_{n=0}^{\infty} (-1)^n z^{n-1} \Rightarrow \frac{1}{z}$ coef at $n=0 \rightarrow$ 1 = residue

b) $z \cos\left(\frac{1}{z}\right) = z \sum_{n=0}^{\infty} (-1)^n \frac{z^{-2n}}{2n!} = \sum_{n=0}^{\infty} (-1)^n \frac{z^{-2n+1}}{2n!}$

$\frac{1}{z}$ coef @ $\cancel{0} = n$, $= -\frac{1}{2}$ = residue

c) $\frac{z - \sin z}{z} = 1 - \frac{\sin z}{z} = -\frac{1}{z} \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n+1}}{(2n+1)!}$

$= \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n}}{2n+1}$ $\quad$ no negative powers of z, $\therefore$ residue at $z=0$ is 0

d) $\frac{\cot z}{z^4} = \frac{\cos z}{z^4 \sin z} = \frac{1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \frac{z^6}{6!} + \dots}{z^4\left(z - \frac{z^3}{3!} + \frac{z^5}{5!} + \dots\right)}$

So, $\frac{\cot z}{z^4} = \frac{1}{z^5}\left(1 - \frac{z^2}{2!} + \frac{z^4}{4!} - + \dots\right)\left(1 + \frac{z^2}{3!} + \frac{1}{3!^2} - \frac{1}{5!} z^4 + \dots\right)$

$= \frac{1}{z^5} - \left(\frac{1}{2!} - \frac{1}{3!}\right) \cdot \frac{1}{z^3} + \left[\frac{1}{3!^2} - \frac{1}{5!} + \frac{1}{4!} - \frac{1}{2!\,3!}\right] \cdot \frac{1}{z} \rightarrow$

$$\therefore \operatorname*{Res}_{z=0} \frac{\cot z}{z^4} = \frac{1}{3!^2} - \frac{1}{5!} + \frac{1}{4!} - \frac{1}{3!\,2!} = -\frac{1}{45}$$

c) $$\frac{\sinh z}{z^4(1-z^2)} \cancel{= \frac{1}{z^4}\left[\frac{z^{2n+1}}{(2n+1)!}\right]\sum_{n=0}^{\infty}\left[z^{2n}\right]}$$

$$\cancel{= \frac{1}{z^4}\sum_{n=0}^{\infty}\frac{z^{2n+1}}{(2n+1)!}}$$

$$= \frac{1}{z^3}\left(1 + \frac{z^2}{3!} + \frac{z^4}{5!} + \dots\right)\left(1 + z^2 + z^2 + \dots\right)$$

$$\frac{\sinh z}{z^4(1-z^2)} = \frac{1}{z^3}\left\{1 + \left(1 + \frac{1}{3!}\right)z^2 + \dots\right\}$$

$$= \frac{1}{z^3} + \frac{7}{6}\cdot\frac{1}{z} + \dots$$

$$\operatorname*{res}_{z=0}\left(\frac{\sinh z}{z^4(1-z^2)}\right) = \frac{7}{6}$$

2) a) $\dfrac{e^{-z}}{z^2}$ $e^z = \sum_{n=0}^{\infty} \dfrac{z^n}{n!}$

sub $-z$ for z

$$\frac{1}{z^2}\sum_{n!}^{\infty}\frac{-1^n z^n}{n!} = \frac{1}{z^2}\left(1 - \frac{z}{1} + \frac{z^2}{2} - \frac{z^3}{6} + \dots\right)$$

$$= \frac{1}{z^2} - \frac{1}{z} + \frac{1}{2} - + \dots$$

$\frac{1}{z}$ coef $= -1$ $i2\pi\cdot(-1) = -2\pi i$

b) $\dfrac{e^{-z}}{(z-1)^2} = \cancel{\dfrac{1}{z^2-2z+1}\sum_{n=0}^{\infty}\dfrac{(-1)^n z^n}{n!}} \; \cancel{\sum_{n=0}^{\infty}\dfrac{(-1)^n z^n}{(z^2-2z+1)\,n!}}$

$e^{-z}\cdot\dfrac{1}{(z-1)^2} = \cancel{e^{-z}\sum_{n=0}^{\infty}\dfrac{(-1)^n (z-1)^{n-2}}{n!}} = \cancel{e^{-z}}$

$$\frac{e^{-z}}{(z-1)^2} = \frac{1}{e(z-1)^2} - \frac{e}{z-1} + \frac{1}{2e} - \dots$$

coef $= -e$, $\therefore -2\pi i e$

c) $z^2 e^{z^{-1}} =$

$$z^2\sum_{n=0}^{\infty}\frac{z^{-n}}{n!} = \frac{z^2}{1} + \frac{z}{1} + \frac{1}{2} + \frac{1}{6z}$$

$\dfrac{\pi i}{3}$

2) a) $\frac{e^{-z}}{z^2}$ $\quad e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}$

sub $-z$ for z

$$\frac{1}{z^2}\sum_{n!}^{\infty}\frac{-1^n z^n}{n!} = \frac{1}{z^2}\left(1-\frac{z}{1}+\frac{z^2}{2}-\frac{z^3}{6}+\dots\right)$$

$$= \frac{1}{z^2} - \frac{1}{z} + \frac{1}{2} - + \dots$$

$\frac{1}{z}$ coef $= -1$ $\quad i2\pi\cdot(-1) = -2\pi i$

b) $\frac{e^{-z}}{(z-1)^2} = \cancel{\frac{1}{z^2-2z+1}\sum_{n=0}\frac{(-1)^n z^n}{n!}\sum_{n=0}\frac{(-1)^n z^n}{(z^2-2z+1)n!}}$

$e^{-z}\cdot\frac{1}{(z-1)^2} = \cancel{e^{-z}\sum_{n=0}\frac{(-1)^n(z-1)^{n-2}}{n!}} = \cancel{e^{-z}}$

$$\frac{e^{-z}}{(z-1)^2} = \frac{1}{e(z-1)^2} - \frac{e}{z-1} + \frac{1}{2e} - \dots$$

coef $= -e$, $\therefore$ $-2\pi i e$

c) $z^2 e^{z^{-1}} =$

$$z^2\sum_{n=0}^{\infty}\frac{z^{-n}}{n!} = \frac{z^2}{1} + \frac{z}{1} + \frac{1}{2} + \frac{1}{6z}$$

$\frac{\pi i}{3}$

(at $z = 0$)

d) $\dfrac{z+1}{z^2-2z} = -\dfrac{1}{2z} - \dfrac{3}{4} + -\dfrac{3z}{8} + \dots$

at $z = 2$

$\dfrac{3}{2z-2} - \dfrac{1}{4} + \dfrac{z-2}{8} + \dots$

$-\dfrac{1}{2}$ & $\dfrac{3}{2}$ $\quad -\dfrac{1}{2} + \dfrac{3}{2} = 1,\ 1 \cdot 2\pi i$

$= 2\pi i$

4) a) $\dfrac{z^5}{1-z^3} \Rightarrow$

~~$\frac{z^5}{1-z^3}$ becomes the Laurent series $\sum_{n=0} -z^n$~~

$\dfrac{z^5}{1-z^3} = \dfrac{-z^2}{1-\frac{1}{z^3}} = -z^2\sum_{n=0}^{\infty} z^{-3n} \to \sum_{n=0}^{\infty} -z^{-3n+2}$

$\to \sum_{n=0}^{\infty} \dfrac{-z^{3n-2}}{z^2} = \sum \dfrac{-z^{3n}}{z^{4}} \quad b_1 = -1$

$-2\pi i$

4b) $\frac{1}{1+z^2} = \sum_{n=0}^{\infty} z^{2n} \to \sum_{n=0}^{\infty} z^{-2n-2}$

This clearly has no $\frac{1}{z}$ coef. $\therefore 0$

c) $\frac{1}{z}$ $\quad \frac{1}{z^2} f\left(\frac{1}{z}\right) = \frac{1}{z} \therefore \int f(z)\,dz = 2\pi i$

5) Show $\int_C e^{z+\frac{1}{z}}\,dz = 2\pi i \sum_{n=0}^{\infty} \frac{1}{n!(n+1)!}$

$e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}$ $\quad \sum_{n=0}^{\infty} \frac{1}{n!} \int_C z^n e^{-z}\,dz$

$z^n e^{z^{-1}} = z^n \sum \frac{z^{-n}}{n!} = \sum_{n=0}^{\infty} \frac{1}{n!(n+1)!}$

6.74 ~~1~~, 2, 3, 4

1) a) principal part of ze^{-z} is $\sum_{n=\cancel{0}}^{\infty} \frac{1}{n!\,z^n}$
$z=0$

~~Since there is an~~ essential pole because it is never bounded near ~~0~~.

b) principal: $\frac{1}{1+z}$
$z=-1$
simple because it can be bounded

c) principal: 0 it is removable
$z=0$ $\lim\left(\frac{\sin z}{z}\right)=1$

d) $\frac{\cos z}{z}$ principal $\frac{1}{z}$, simple pole
$z=0$ $\lim\left(\frac{\cos z}{z}\right)=0$

e) $\frac{1}{(2-z)^3}$ ~~Simple~~ pole of order of 3
$z=0$

principal part is $\frac{1}{(2-z)^3}$

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2) a) $\frac{1-\cosh \cancel{z}}{z^3} = \frac{1}{z^3}\sum_{n=0}^{\infty} \frac{-z^{2n}}{2n!} = \sum_{n=\cancel{0}}^{\infty} \frac{-z^{2n-3}}{2n!}$ ①

① first term of $\cosh z = 1$, $\therefore$ $1-\cosh z = \sum_{n=1}^{\infty} \frac{-z^{2n}}{2n!}$

$= \frac{-1}{2z} - \frac{z}{24} - \frac{z^3}{720}$

$m=1$, $B=\frac{-1}{2}$

b) $\frac{1-e^{2z}}{z^4} = \frac{1+\sum_{n=0}^{\infty} \frac{-z^n}{n!}}{z^4} = \sum_{n=1}^{\infty} \frac{-z^{n-4}}{n!} \cdot 2^n$

$= -\frac{2}{z^3} - \frac{4}{2z^2} - \frac{8}{6z}$ 3rd order, $m = -\frac{4}{3}$

c) $\cancel{\frac{e^{2z}}{(z-1)^2} = \frac{\sum_{n=0}^{\infty} \frac{2^n z^n}{n!}}{(z-1)^2}}$ principal part is

$$\frac{e^2}{(z-1)^2} + \frac{2e^2}{(z-1)}$$

the order is then 2, and $B = 2e^2$

$\frac{e^{2z}}{(z-1)^2}$ $\sum_{n=0}^{\infty} \frac{2^n e^2}{(z-1)^{-n+2}}$

3)

a) $g(z) = \frac{f(z)}{z-z_0} = \frac{\sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!}(z-z_0)^n}{z-z_0}$

$= \sum_{n=0}^{\infty} \frac{f^n(z_0)}{n!}(z-z_0)^{n-1} = \frac{\phi(z)}{(z-z_0)}$, residue of $f(z_0)$

b) if $f(z_0)=0$, then $\lim_{z \to 0} \frac{f(z)}{z-z_0} = 0$

4) $\phi(z)$ has the expansion

$$\sum_{n=0}^{\infty} \frac{\phi^n(z_0)}{n!}(z-ai)^{n-3} = \frac{\phi(ai)}{(z-ai)^3} + \frac{\phi'(ai)}{(z-ai)^2} + \frac{\phi''(ai)}{2(z-ai)}$$

$$\phi(ai) = \frac{8a^3(-a^2)}{-8a^3 i} = \frac{+a^2}{i} = -a^2 i = \phi(ai)$$

$$\phi'(ai) = \frac{(z+ai)^3 \cdot 16a^3 z - 8a^3 z^2 \cdot 3(z+ai)^2}{(z+ai)^6}$$

$$= \frac{-8a^3 i \cdot 16a^3 \cdot ai - 8a^3 \cdot (-a^2) \cdot 3(-4a^2)}{(2ai)^6}$$

$$= \frac{128a^7 - 96a^7}{-64a^6} = \frac{32a^7}{-64a^6} = \frac{1a}{2} = \phi'(ai)$$

$$\phi''(ai) = \frac{-16a^3(-z^2 + 4aiz + a^2)}{(z+ia)^5}$$

$$\frac{-16a^3(a^2 + -4a^2 + a^2)}{(2ai)^5} = \frac{32a^5 i}{32a^5} = i = \phi''(ai)$$

$$= \frac{a^2 i}{(z-ai)^3} - \frac{a/2}{(z-ai)^2} - \frac{i}{(z-ai)2}$$