JOS 4 Brown

1) $\frac{5-3i}{(2+i)(1-i)} = \frac{5-3i}{2-i+1} = \frac{5-3i}{3-i} \cdot \frac{(3+i)}{(3+i)}$

$= \cancel{\frac{15-5i-9i-3}{9}} = \frac{15+5i-9i+3}{9+1}$

$= \frac{18}{10} + \frac{-4i}{10}$

$= \frac{9}{5} + \left(-\frac{2i}{5}\right)$

2) $f(z) = u(r,\theta) + i v(r,\theta)$

According to CRE, $f'(z) = v_x + v_x$

let $z = re^{i\theta}$

Then $f'(z) = e^{-i\theta}\left(v_r(r,\theta) + i v_r(r,\theta)\right)$

or $f'(z) = \frac{e^{-i\theta}}{r}(v_\theta + i v_\theta)$

$|e^{-i\theta}(v_r + i v_r)| = \sqrt{(e^{-i\theta} v_r)^2 + (e^{-i\theta} v_r)^2}$

$= \sqrt{e^{-i2\theta}[(v_r)^2 + (v_r)^2]}$

$= e^{-i\theta}\sqrt{(v_r)^2 + (v_r)^2}$

likewise $\left|\frac{e^{-i\theta}}{r}(v_\theta + i v_\theta)\right| = \sqrt{\frac{e^{-i2\theta}}{r^2}\left[(v_\theta)^2 + (v_\theta)^2\right]}$

$= \frac{e^{-i\theta}}{r}\sqrt{(v_\theta)^2 + (v_\theta)^2}$

✓

3) $(1-\sqrt{3}i)^9 (-1+i)^4$

$1-\sqrt{3}i \to r = 4, \quad \theta = \frac{-\pi}{3}$

$$\left(4e^{-i\frac{\pi}{3}}\right)^9 = 262144e^{-i3\pi}$$

$-1+i \to r = \sqrt{2}, \quad \theta = \frac{3\pi}{4}$

$$\left(\sqrt{2}e^{i\frac{3\pi}{4}}\right)^4 = 4e^{i3\pi}$$

$$4e^{3\pi i} \cdot 262144e^{3\pi i} = 1048576e^{6\pi i}$$

$$x = 1048576\cos(6\pi)$$
$$y = 1048576\sin(6\pi)$$

$$z = 1048576 + 0i$$

$$z = x = 1048576$$

4) $\sqrt[4]{-16} \Rightarrow z^4 = -16$

$-16 = (-1)\,2^4 = 2^4 e^{i\pi}$ (euler)

$2^4 e^{i\pi + 2k\pi i} \quad k \in \mathbb{Z}$

$\sqrt[4]{z^4} = \sqrt[4]{2^4 e^{i\pi(2k+1)}}$

$z = 2e^{i\frac{\pi}{4}(2k+1)} \quad \underbrace{k = 0, 1, 2, 3}_{\text{4 roots}}$

$2e^{i\frac{\pi}{4}} = 2\left(\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i\right) = \sqrt{2} + \sqrt{2}i$

$2e^{i\frac{3\pi}{4}} = 2\left(-\frac{\sqrt{2}}{2} + \frac{\sqrt{2}i}{2}\right) = -\sqrt{2} + \sqrt{2}i$

$2e^{i\frac{5\pi}{4}} = 2\left(\frac{-\sqrt{2}}{2} - \frac{\sqrt{2}}{2}i\right) = -\sqrt{2} - \sqrt{2}i$

$2e^{i\frac{7\pi}{4}} = 2\left(\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}i\right) = \sqrt{2} - \sqrt{2}i$

5) F(z) differentiable means that the limit as $z \to z_0$ exists, while Analytic means the derivitive must exist in some neighborhood around z_0

6) $f'(z)$, $f(z) = \dfrac{e^{z^3+5z}}{z^4+1}$

$$f'(z) = \frac{e^{z^3+5z}(3z^2+5)(z^4+1) - 4z^3 e^{z^3+5}}{(z^4+1)^2}$$

~~but~~ from $\dfrac{(z^4+1)^2 \frac{d}{dz}\left(e^{z^3+5z}\right) - \left(e^{z^3+5z}\right)\frac{d}{dz}(z^4+1)}{(z^4+1)^2}$

7) $f(z) = \bar{z}$

$f(z) = u(x, y) + iv(x+y)$

$f(z) = x - iy$

$u_x = 1, \quad v_y = -1$

$u_x \neq v_y$, CRE violated

& $\therefore$ dnc

$$\lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z}$$

becomes $\frac{\Delta \bar{z}}{\Delta z}$, when real $= 1$,

when imaginary $= -1$

$1 \neq -1 \therefore$ DNC

let $A = -2$ & $B = -1$

$u_x = Ay + 1 \quad v_y = -2 + 2By$

$A + Ay = -2 + 2By$

$u_x = -2y + -2 \quad \cancel{v}_y = -2 + -2y$

$v_y = -2x \quad \cancel{u}_x = 2x$

$v_y = -\cancel{v}_x$ ✓ $\quad u_x = v_y$ ✓ ∴ analytic everywhere

by deff. entire

$$f'(z) = -2y - 2 + 2x$$

9) $v = xy + e^{-x}\cos y$ is harmonic

$v_x = y + -e^{-x}\cos y \qquad v_{xx} = e^{-x}\cos y$

$v_y = x - e^{-x}\sin y \qquad v_{yy} = -e^{-x}\cos y$

$v_{xx} + v_{yy} = 0$ harmonic

$u_x = v_y$

$u_y = -v_x$

$u_x = -x + e^{-x}\sin y$

$u_y = y + e^{-x}\cos y$

$u = \frac{x^2}{2} - \frac{y^2}{2} - e^{-x}\cos y$

$\frac{x^2}{2} - \frac{y^2}{2} - e^{-x}\cos y + i(xy + e^{-x}\cos y)$ is analytic

10) let z be real

$$f(z) = \frac{x \cancel{z} - i}{x + i}$$

$$\left| \frac{x - i}{x + i} \right| = \frac{\sqrt{x^2 + 1}}{\sqrt{x^2 + 1}} = 1$$

when $z \geq 0$, z maps onto the real unit circle aswell