Complex Final Josh Brown

1a) let C be the unit circle $|z|=1^+$

$$\int_C \frac{1}{2i}(z-\bar{z})$$

let $z=e^{it}$ $\quad \frac{dz}{dt} = ie^{it}$

$$\int_C f(z)\,dz = \int_0^{2\pi} \frac{e^{it}-e^{-it}}{2i} \cdot ie^{it}\,dt$$

$$= \frac{1}{2}\int_0^{2\pi} \left(e^{2it}-1\right)\,dt$$

$$\frac{1}{2}\left[\frac{e^{2it}}{2i} - t\right]_0^{2\pi} = -\pi$$

1b) $f(z) = \frac{1}{27z^3+1}$ → becomes the expansion $\sum_{n=0}^{\infty} \frac{(-1)^n (27z)^{3n}}{}$ which obviously shows

that the point is a removable singular point.

Therefore, the $\int_C \frac{1}{27z^3+1}\,dz = 0$

1c) $\int_C \frac{1}{z\sin z}\,dz$ $\quad z \neq 0, n\pi, \quad n = \pm1, \pm2, \pm3$

$n\pi \notin |z| = 1, \quad 0 \in |z| = 1$

to find $\int_C f(z)\,dz$, let $\frac{1}{z\sin z} = \frac{1}{z^2}\, g(z) = \frac{z}{\sin z}$

Since $g(z)$ is even, The taylor series will have no $\frac{1}{z}$ coef. $\therefore$, it follows That $\underset{z=0}{\mathrm{Res}}\, f(z) = 0$

$\int_C \frac{1}{z\sin z}\,dz = 0$

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2) $\int_0^{2\pi} \frac{d\theta}{5-4\cos\theta} = \int_C \frac{1}{iz(5-2z-\frac{2}{z})}\,dz \quad C: |z| = 1$

$\int_C \frac{1}{i(-2z^2+5z-2)}$ $\quad z = 2, \frac{1}{2}$ by quadratic

$\int_0^{2\pi} \frac{d\theta}{5-4\cos\theta} = 2\pi i\, f(z) = \int_C \frac{f(z)}{z-\frac{1}{2}}\,dz$

$f(z) = \frac{1}{i(-2z+4)}$ $\quad$ let $z = \frac{1}{2}$. $\quad z = 2$ is out of $|z| = 1$

$f(\frac{1}{2}) = \frac{1}{3i} \rightarrow \frac{1}{3i} \cdot 2\pi i = \frac{2\pi}{3}$

3a) $f(z) = \dfrac{z-1}{3-z}$ about $z=1$, $|z-1|<2$

let $w = z-1$

$$\frac{w}{2-w} = f(w) = \frac{w}{2}\cdot\frac{1}{1-\frac{w}{2}} = \frac{w}{2}\sum_{n=0}^{\infty}\frac{w^n}{2^n}$$

$$= \sum_{n=0}^{\infty}\frac{w^{n+1}}{2^{n+1}}$$

now, substituting back z for w,

$$\sum_{n=0}^{\infty}\frac{(z-1)^{n+1}}{2^{n+1}}$$

b) same as part a, but negate & reciprical.

This yeilds

$$-\frac{w}{2}\sum_{n=1}^{\infty}\frac{1}{\frac{w^n}{2^n}}$$

$$= -\frac{w}{2}\sum_{n=1}^{\infty}\frac{2^n}{w^n} = -\sum_{n=1}^{\infty}\frac{2^{n-1}}{(z-1)^{n-1}}$$

4) $\text{P.V} \int_0^\infty \frac{\cos x}{(x^2+4)(x^2+1)} dx = \int_0^\infty \frac{e^{iz}}{(z^2+4)(z^2+1)}$

$z = i2, i \rightarrow \operatorname{Res}_{z=i} \frac{e^{iz}}{(z^2+4)(z^2+1} = \frac{p(z)}{q'(z)}$

$= \frac{e^{i\cancel{z}}}{2z(z^2+4)+2z(z^2+1)} = \text{at } z = i = \frac{e^{-1}}{2i(3)}$

and at $z = 2i \rightarrow \frac{e^{-2}}{4i(-3)}$

since bounds are $[0, \infty)$,

$\int_0^\infty \frac{\cos x}{(x^2+4)(x^2+1)} dx = \left[\sum_{n=1}^{2} \operatorname{Res}_{z=z_k} \frac{e^{iz}}{(z^2+4)(z^2+1)}\right] \pi i$

$= \left(\frac{e^{-2}}{-12i} + \frac{e^{-1}}{6i}\right) \pi i = \frac{-\pi e^{-2} + 2\pi e^{-1}}{12}.$

$= \frac{-\pi e^{-2} + 2\pi e^{-1}}{12}$

5) $P.V \int_{-\infty}^{\infty} \frac{2x-3}{x^2(x^2+9)}\, dx = 2\pi i \operatorname*{Res}_{z=3i,0} f(z)$

$z=0,\ 3i,\ -3i.$ $-3i$ lies below the ~~x~~ axis,

$$\operatorname*{Res}_{z=3i} f(z) = \frac{p(z)}{q'(z)} = \frac{2z-3}{2z(z^2+9)+2x^3}$$

$$= \frac{2(3i)-3}{-2i27} = \frac{-6i+3}{54i} = \left(\frac{-6}{54} + \frac{3}{54i}\right)2\pi i$$

$$= \frac{\pi}{9} - \frac{2i\pi}{9}$$

$$\operatorname*{Res}_{z=0} f(z) = \phi'(z) = \frac{2(z^2+9) - 2z(2z-3)}{(z^2+9)}$$

$z_0 = 0$

$$= \frac{2i}{9}$$

$$\int_{-\infty}^{\infty} \frac{2x-3}{x^2(x^2+9)} = \frac{\pi}{9}$$

6) lourarnt expansion for $\frac{1}{z^5} \sin \frac{1}{z^2}$

a) start with $\sin(z) = \sum_{n=0}^{\infty} \frac{z^{2n+1}}{(2n+1)!}$

sub z^{-2} for z yeilding

$$\sum_{n=0}^{\infty} \frac{z^{-4n-2}}{(2n+1)!}$$

now mult $\frac{1}{z^5}$ $\rightarrow$ $\sum_{n=0}^{\infty} \frac{z^{-4n-7}}{(2n+1)!}$

b) ~~[illegible] with~~ $(-\infty, 0) \cup (0, \infty)$

c) essential singular point

d) ~~[illegible]~~

$$\cancel{\frac{z^{10}}{z^5} \sin\left(\frac{1}{z}\right) = \sum_{n=0}^{\infty} \frac{z^{-4n+3}}{(2n+1)!}}$$

~~None, since z is essential,~~

$n \geq 10$ $\sum_{n=0}^{\infty} \frac{z^{-4n+3}}{(2n+1)!}$ yeilds b_1 coef.

f) $\int_C z^{18} f(z)\, dz = \text{Res}\left(\sum_{n=0}^{\infty} \frac{z^{-4n+11}}{(2n+1)!}\right)$

$a_0^{11} + b_1^{7} + c_2^{3} + \frac{1}{z \cdot (7!)}$

$\int_C z^{18} f(z)\, dz = \frac{2\pi i}{7!}$

7) $g(z) = \frac{1}{z-3} + \frac{1}{(z-(2+3i))^5} + \sin\left(\frac{1}{z}\right)$

$g(z) = \frac{1}{z-3} + \frac{1}{[z-(2+3i)]^5} + \sin\left(\frac{1}{(z-1)}\right)$

$\text{Res}_{z=3}\, g(z) = \text{Res}_{z=3} \frac{1}{z-3} = \frac{1}{(z-3)} \cdot 2\pi i$

$\text{Res}_{z=3}\, g(z) = 2\pi i$

8) $\frac{1}{2\pi i}\int_C \frac{z^4}{\sin z}dz = \underset{z=\{c_k\}}{\text{Res}} \frac{z^4}{\sin z}$

$c_k = 0, \pi, -\pi$

$\underset{z=0}{\text{Res}} \frac{z^4}{\sin z} = 0 \cancel{\frac{p(z)}{q'(z)}}$ removable

$\underset{z=\pi}{\text{Res}} \frac{z^4}{\sin z} = \frac{p(z)}{q'(z)} = \frac{z^4}{\cos(z)} = \frac{\pi^4}{-1} = -\pi^4$

$\underset{z=-\pi}{\text{Res}} \frac{z^4}{\sin z} = \frac{p(z)}{q'(z)} = \frac{-\pi^4}{1} = -\pi^4$

$= -2\pi^4 \quad \therefore \int_C \frac{z^4}{\sin z}dz = (-2\pi^4)\, 2\pi i$

$\& \ \frac{1}{2\pi i}\int_C \frac{z^4}{\sin z} = -2\pi^4$

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9) $F(s) = \frac{1}{(s^2+2s+5)^2} \qquad F(s) \to f(t) = \frac{1}{2\pi i}\int_{-\infty}^{\infty} e^{zt}F(z)\,dz$

$s = -1 \pm 2i$

$\underset{s=(-1+2i)}{\text{Res}} \frac{\frac{e^{st}}{(s-(1+2i))^2}}{(s-(1-2i))^2} = \frac{te^{ts}(s-(1+2i))^2 - 2e^{st}(s-(1+2i)}{[s-(1+2i)]^4}$

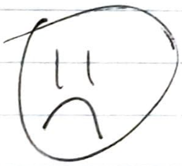
let $s = -1+2i$

$$\frac{te^{t(-1+2i)}(-2+4i)^2 - 2e^{(-1+2i)t}(-2+4i)}{(-2+4i)^4}$$

$$= e^{t(-1+2i)}\left[\frac{(-2+4i)t - 2}{(-2+4i)^3}\right]$$

$$\operatorname*{Res}_{z=-1-2i} = \frac{\frac{e^{st}}{(s-(1+2i))^2}}{(s-(-1+2i))} = e^{t(-1-2i)}\left[\frac{t(-2-4i)-2}{(-2-4i)^3}\right]$$

$$f(t) = e^{t(-1+2i)}\left[\frac{(-2+4i)t-2}{(-2+4i)^3}\right] + e^{t(-1-2i)}\left[\frac{t(-2-4i)-2}{(-2-4i)^3}\right]$$



10) a) $\int_C \frac{dz}{2z^{1/2}}$ let $z = 2e^{i\theta}$

$dz = 2ie^{i\theta}\,d\theta$ $[-\pi, \pi]$

$= \int_{-\pi}^{\pi} \frac{\cancel{2}ie^{i\theta}}{\cancel{2}\sqrt{2}\,e^{i\theta/2}}\,d\theta = \int_C \frac{ie^{i\theta/2}}{\sqrt{2}}\,d\theta$

$= \frac{2e^{i\theta/2}}{\sqrt{2}}\Big|_{-\pi}^{\pi} = \frac{-2i - 2i}{\sqrt{2}} = \frac{-4i}{\sqrt{2}}$

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b) $\int_C \frac{dz}{2z^{1/2}}$ let $z = 2e^{i\theta}$ branch cut on $[\pi, 2\pi]$

$dz = 2ie^{i\theta}\,d\theta$

$\int_{\pi}^{2\pi} \frac{2ie^{i\theta}}{2\sqrt{2}\,e^{i\theta/2}}\,d\theta = \int_{\pi}^{2\pi} \frac{ie^{i\theta/2}}{\sqrt{2}}\,d\theta$

$= \frac{2e^{i\theta/2}}{\sqrt{2}}\Big|_{\pi}^{2\pi} = \frac{2i + 2i}{\sqrt{2}} = \frac{4i}{\sqrt{2}}$