4.42, ~~1, 2, 3~~

4.43 5, 6

4.46 1, 3, 13

1) a) $\frac{d}{dt}[z_0 w(t)] = z_0 w'(t)$

Substitute $u(t) + i v(t)$ for $w(t)$

$$\frac{d}{dt}\left(z_0[u(t) + i v(t)]\right) = z_0[u'(t) + i v'(t)]$$

$$\frac{d}{dt}[z_0 u(t) + i v(t) z_0] = z_0 u'(t) + i v(t) z^0$$

$$z_0 u'(t) + i v'(t) z_0 = z_0 u'(t) + i v'(t) z_0$$

$u'(t)(x_0 + i y_0) + i v'(t)(x_0 + i y_0)$

$(u'(t) x_0 - y_0 v') + i(y_0 u'(t) + x_0 v')$ ✓

b) $\frac{d}{dt} w(-t) = \frac{d}{dt} u(-t) + \frac{d}{dt} i v(-t)$

$= -u'(-t) - i v'(-t) = -w'(-t)$

$-u'(-t) - i v'(-t) = -u'(-t) - i v'(-t)$ ✓

2) a) $\int_0^1 (1 + it)^2 \, dt \qquad (1 + it)^2 = 1 + 2it - t^2$

$$\int_0^1 1 + 2it - t^2 \, dt = t + it^2 - \frac{t^3}{3} \Big|_0^1$$

$$= 1 + i - \frac{1}{3} = \frac{2}{3} + i$$

b) $\int_1^2 \left(\frac{1}{t} - i\right)^2 dt \qquad \left(\frac{1}{t} - i\right)^2 = \frac{1}{t^2} - \frac{2i}{t} - 1$

$\int_1^2 \frac{1}{t^2} - \frac{2i}{t} - 1 \, dt = t^{-2} - \frac{2i}{t} - 1 \, dt$

$= -\frac{1}{t} - 2i\ln t - t \Big|_1^2$

$-\frac{1}{2} - 2i\ln(2) - 2 - (-1 - 1) = -\frac{1}{2} - i\ln 4$

c) $\int_0^{\pi/6} e^{i2t} dt = \frac{-ie^{i2t}}{2}\Big|_0^{\pi/6}$

$\frac{-ie^{i\frac{\pi}{3}}}{2} - \frac{i}{2}$

$= \frac{-i}{4} + \frac{\sqrt{3}}{4} - \frac{i}{2} = \frac{\sqrt{3}}{4} + \frac{i}{4}$

d) $\int_0^{\infty} e^{-zt} dt = \frac{-e^{-zt}}{z}\Big|_0^{\infty}$

$= 0 - \frac{1}{-z} = \frac{1}{z}$

3) $\int_0^{2\pi} e^{im\theta} - e^{in\theta}\, d\theta$ let $m=n$, then $m=n$

$$\int_0^{2\pi} e^{im\theta} - e^{im\theta}\, d\theta = \int_0^{2\pi} e^{\cancel{\ }0}\, d\theta = \int_0^{2\pi} 1\, d\theta$$

$$= \theta\Big|_0^{2\pi} = 2\pi \checkmark$$

$$\int_0^{2\pi} e^{im\theta} - e^{in\theta}\, d\theta$$

$$= \frac{-ie^{im\theta}}{m} + \frac{ie^{in\theta}}{n}\Big|_0^{2\pi}$$

$$= -\frac{i}{m} + \frac{i}{n} + \frac{i}{m} - \frac{i}{n} = 0 \checkmark$$

4) $\int_0^{\pi} e^{(1+i)x}\, dx = \cancel{\ } \quad \frac{\cancel{\ } e^{(1+i)x}}{(1+i)}$

$$\frac{e^{(1+i)x}}{(1+i)} \cdot \frac{(1-i)}{(1-i)} = \frac{e^{(1+i)x}(1-i)}{1+1}$$

$$\frac{(1-i)e^{(i+1)x}}{2}\Big|_0^{\pi} = \frac{(1-i)e^{\pi i+\pi}}{2} \qquad e^{\pi i} = -1$$

$$\frac{(-1+i)e^{\pi}}{2} - \left(\frac{(1-i)e^{0}}{2}\right)$$

$$= \pm\frac{1+e^{\pi}}{2}$$

4.43

~~5, 6~~

5) let $f(z) = u(x,y) + i v(x,y)$
$z(t) = x(t) + i y(t)$

$w(t) = u[x(t), y(t)] + i v[x(t), y(t)]$

$w'(t) = u_x x' + u_y y' + i v_x x' + v_y y'$

since analytic, riemann satisfied, ∴

$f'(z(t_0)) = u_x(x(t_0), y(t_0)) + i v_x(x(t_0), y(t_0))$

$\rightarrow w'(t) = u_x x' + u_y y' + i v_x x' + v_y y'$

$= u_x x' - v_x y' + i v_x x' + u_x y'$

$= (u_x + i v_x)(x'(t_0) + i y'(t_0))$

$= f'(z(t_0)) z'(t_0)$

6) a) $\frac{d}{dx} \operatorname{Re} z = 1$

$\frac{d}{dy} = 3x^2 \sin\left(\frac{\pi}{2}\right) + x^3 \cos\left(\frac{\pi}{x}\right) \cdot \frac{\pi}{x}$

Show continuous
show derivative exists everywhere

6a) $z = x + i y(x)$

b) crosses x axis at $y=0$, or when $x=0$ or

$\sin\frac{\pi}{x} = 0$, $x = \frac{1}{n}, n \in \mathbb{Z}$

Showing derivative exists

$Re'(z) = 1$

$Im'(z) = 3x^2\sin\left(\frac{\pi}{x}\right) + x^3\cos\left(\frac{\pi}{x}\right)\left(\frac{-\pi}{x^2}\right)$

~~the~~ derivative exists for $0 < x \leq 1$

for $x=0$, use squeeze theorem

$0 \leq |x^3 \sin\frac{\pi}{x}| \leq x^3$

$\therefore \lim_{x \to 0} x^3 \sin\frac{\pi}{x} = \lim_{x \to 0} x^3 = 0$

similarly,

$\lim_{x \to 0} (3x^2 \sin\frac{\pi}{x})i + 1$ & $\lim_{x \to 0} |x^2 \sin\frac{\pi}{x}| \leq x^2$

$= 1$ $\qquad = 0$

$\therefore \lim_{x \to 0} (z)' = 1 \rightarrow$ The derivative exists

4.46, ~~2~~, ~~3~~, 13

1) a) $\int_0^{\pi} \left(\frac{2e^{i\theta}+2}{2e^{i\theta}}\right) 2ie^{i\theta} = i\int_0^{\pi} 2e^{i\theta}+2$

$= 2e^{i\theta} + 2\theta i\Big|_0^{\pi}$

$= -4 + 2\pi i$

b) $\int_{\pi}^{2\pi}$ = from part a, $2e^{i\theta}+2\theta i\Big|_{\pi}^{2\pi} = 4+2\pi i$

c) (a) + (b) = $-4+2\pi i + 4+2\pi i = 4\pi i$

3) $f(z) = \pi e^{\pi \bar{z}} = \pi e^{\pi x - i y \pi}$

$y=0$ $\int_0^1 \pi e^{\pi x} + 1 \, dx$

$e^{\pi x}+1\Big|_0^1 \quad e^{\pi}+1-(e^0+1)$

$= e^{\pi}-1$

Since square, each side by deff. has same length. ∴ boundary of the square is 4x one side, or $4(e^{\pi}-1)$

13) $z = z_0 + Re^{i\theta}$

$\int_{C_0} (z - z_0)^{n-1}\, dz = \begin{cases} 0 & \text{when } n = \pm 1, 2, \\ 2\pi i & \text{when } n = 0 \end{cases}$

$\int_C (z - z_0)^{n-1}\, dz = \int_{-\pi}^{\pi} R^{n-1} e^{i\theta(n-1)} \cdot Rie^{i\theta}\, d\theta$

$= \int_{-\pi}^{\pi} R^n i e^{in\theta}\, d\theta = \left.\frac{R^n e^{in\theta}}{n}\right|_{-\pi}^{\pi} \quad n \neq 0 = 0$

$\left. i\theta \right|_{-\pi}^{\pi} \quad n = 0 = 2\pi i$