4.47, 4.49, 4.53

1) b) C $|z| = 2$, from $z = 2 \to z = 2i$

C circumfrence $= \pi$ $\quad (\frac{1}{4} \cdot 2\pi r)\ r=2$

$\left|\frac{1}{z^2-1}\right| \to |z^2-1| = |z^2| + |-1| = 3 \to \frac{1}{3}$

$\frac{1}{3} = M,\quad \pi = L.\ \therefore \left|\int_C \frac{dz}{z^2-1}\right| \leq \frac{\pi}{3}$

2) C is the line segment from $z = i$ to $z = 1$.

length of $C = \sqrt{2} = L$

Observe the midpoint is closest to the origin

& that $= \sqrt{2}/2 \quad \left|\left(\frac{\sqrt{2}}{2}\right)^4\right| = \frac{4}{16} \to \frac{1}{\frac{4}{16}} = \frac{16}{4} = 4$

$\therefore \left|\int_C \frac{dz}{z^4}\right| \leq 4\sqrt{2}$

3) C is the perimeter of the triangle with sides $3, 4, \&\ \sqrt{3^2+4^2} = 5$

$L = 12$

$|e^z - \bar{z}| \leq e^x + \sqrt{x^2+y^2}$ $\quad \left|\int_C (e^z - \bar{z})\, dz\right| \leq 60$

farthest point from the origin is -4. Since the $\operatorname{Re}(z) \leq 0$ for all points on the triangle, $|e^z| \leq 1$, $\therefore 1+4 = 5$, $ML = 60$

4.49

2)

a) $\int_0^{1+i} z^2\,dz = \frac{z^3}{3}\Big|_0^{1+i}$

$\frac{z^3}{3} = \frac{(1+i)^3}{3} = \frac{(1+i)^2(1+i)}{3} = \frac{2i(1+i)}{3}$

$= \frac{(2i-2)}{3} = \frac{2}{3}(-1+i)$

b) $\int_0^{\pi+2i} \cos\frac{z}{2}\,dz = 2\sin\left(\frac{z}{2}\right)\Big|_0^{\pi+2i}$

$2\sin\left(\frac{\pi+2i}{2}\right) = \left(\frac{e^{\frac{i(\pi+2i)}{2}} - e^{-i\left(\frac{\pi+2i}{2}\right)}}{2i}\right)2$

$= \frac{e^{\frac{i\pi}{2}}e^{-1} - e^{-\frac{i\pi}{2}}e}{i}$

$e^{i\frac{\pi}{2}} = \cos\frac{\pi}{2} + i\sin\left(\frac{\pi}{2}\right) = 0 + 1i = i$

$e^{-i\frac{\pi}{2}} = \cos -\frac{\pi}{2} + i\sin\left(-\frac{\pi}{2}\right) = -i$

$\frac{ie^{-1} + ie}{i} = e^{-1} + e = e + \frac{1}{e}$

c) $\int_1^3 (z-2)^3 \, dz = \left. \dfrac{(z-2)^4}{4} \right|_1^3$

$$\frac{(3-2)^4}{4} = \frac{1^4}{4} = \frac{1}{4}$$

$$\frac{(1-2)^4}{4} = \frac{1}{4}$$

$$\frac{1}{4} - \frac{1}{4} = 0$$

5) $\int_{-1}^{1} z^i \, dz = \dfrac{1+e^{-\pi}}{2}(1-i)$

$z^i = e^{i \operatorname{Log} z}$, $C = (z = -1 \text{ to } z = 1)$

$$\int_{-1}^{1} z^i \, dz = \frac{z^{i+1}}{(i+1)} = \left. \frac{1}{i+1} e^{(1+i)\operatorname{Log} z} \right|_{-1}^{1}$$

let $z=1$ $\quad \dfrac{1}{i+1} e^{(1+i)\operatorname{Log}(1)} = \dfrac{1}{i+1}$

let $z=-1$ $\quad \dfrac{1}{i+1} e^{(1+i)\operatorname{Log}(-1)} = \dfrac{e^{(1+i)\pi i}}{i+1} = \dfrac{e^{\pi i - \pi}}{i+1} = \dfrac{-e^{-\pi}}{1+i}$

$$\frac{1}{i+1} - \frac{-e^{-\pi}}{1+i} = \frac{1+e^{-\pi}}{1+i} \cdot \frac{(1-i)}{(1-i)} = \frac{1+e^{-\pi}}{2}(1-i)$$

4.53 1,4

1) a) $f(z) = \dfrac{z^2}{z+3}$ Singularity at $z = -3$.
z never $= 3$ at $|z| = 1$

~~Since~~ Since z is analytic & single valued through the circle $|z|=1$, $\int_c f(z)\,dz = 0$

$\cancel{\dfrac{(x+iy)^2}{(x+yi)+3} = \dfrac{x^2+2ixy-y^2}{x+iy+3}}$

b) ze^{-z} has no singularitys $\therefore$ analytic on $|z|=1$, because it is the product of two entire functions, z & e^{-z} $\therefore$

Cauchy-goursat $\rightarrow$ $\int_c f(z)\,dz = 0$

c) $f(z) = \dfrac{1}{z^2+2z+2}$

$= \dfrac{1}{(z+1+i)(z+1-i)}$

Singularity at $z = 1-i$ & $1+i$

$1 \pm i$ is $\sqrt{2}$ from 0,
since $|z|=1$, this is out of the range of z $\therefore$
entire & analytic

$\int_c f(z)\,dz = 0$

d) sech is never undef, analytic & entire.

$\therefore \int_C f(z)\,dz = 0$

e) tan z is undefined at nearest $\pm\frac{\pi}{2}$, which is

$> |z| = 1$, $\therefore$ entire and analytic

$\int_C f(z)\,dz = 0$

f) Log(z+2) undefined when $z \leq -2$

-2 is outside of range $|z| = 1$

analytic & entire $\therefore \int_C f(z)\,dz = 0$

4) $\int_0^\infty e^{-x^2} \cos 2bx\,dx = \frac{\sqrt{\pi}}{2} e^{-b^2}$

$$\int_{-a+bi}^{a+bi} e^{-z^2}\,dz = \int_{-a+bi}^{a+bi} e^{-x^2}\,dx + \int_{-a+bi}^{a+bi} e^{-x^2} e^{b^2} e^{2bix}\,dx$$

$$= 2\int_0^a e^{-x^2}\,dx - 2e^{b^2} \int_0^a e^{-x^2} \cos 2bx\,dx$$

$$\left(-\cos 2bx = e^{2bix}\right)$$

$\rightarrow$

$$\int_0^b ie^{-(a+iy)^2}\,dy + \int_b^0 ie^{-(-a+iy)^2}\,dy$$

$$= ie^{-a^2}\int_0^b e^{y^2-2iay}\,dx - ie^{-a^2}\int_0^b e^{y^2-i2ay}\,dy = \int_0^b e^{y}\sin 2ay\,dy$$

by Cauchy-Goursat, integral of $e^{-z^2}=0$

This $\to$

$$2\int_0^a e^{-x^2}\,dx - 2e^{b^2}\int_0^a e^{-x^2}\cos 2bx\,dx + 2e^{-a^2}\int_0^b e^{y^2}\sin 2ay$$

$$=0$$

$$\int_0^a e^{-x^2}\cos 2bx\,dx = \int_0^a e^{-x^2}\,dx + e^{-(a^2+b^2)}\int_0^b e^{y^2}\sin 2ay$$