

Energy Spectrum of the Dicke Hamiltonian

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In this paper we analyze the energy eigenvalues of the Dicke Hamiltonian which describes the interaction of N two-level atoms with an electromagnetic field. Using as a starting point the classification of the collective states proposed by Tavis and Cummings, we first derive approximate analytic expressions for the energy eigenvalues which are appropriate for special values of the collective parameters. We next evaluate a first-order correction which allows the application of our results to a wider class of collective states. An accurate description of the energy spectrum for arbitrary values of the collective parameters is provided using a method first introduced by Scharf for the case of resonant interaction, suitably generalized to account for possible frequency detuning between the radiation and the atomic resonant frequency. The limit of validity of the approximation is analyzed and the results compared with the numerical diagonalization of the Dicke Hamiltonian. With the exception of asymptotically small ranges of values of the parameters, the analytic expression for the density of energy eigenvalues derived in this paper compares very well with the results of the numerical diagonalization.

INTRODUCTION

The Dicke Hamiltonian¹ describing the interaction of N two-level atoms with an electromagnetic field has been used quite frequently as the basic working model to describe the collective interaction of radiation and matter. In addition to recent contributions dealing with the dynamics of coupled atom-field systems,²⁻⁷ several papers have focused on the eigenvalue structure of the Hamiltonian.⁸⁻¹³

Tavis and Cummings⁸⁻¹⁰ initiated an in-depth analysis of this problem suggesting a very convenient classification of the collective states that reduces the eigenvalue problem to the diagonalization of a symmetric tridiagonal matrix. While the exact diagonalization of this matrix for arbitrary values of the collective parameters is a formidable problem, Tavis and Cummings were able to provide a formal solution.

The result of their analysis indicates that the energy eigenvalues can be found, in principle, by solving *explicit* algebraic equations, the degree of which is a function of the parameters of the problem and is, therefore, quite large for macroscopic systems. Mallory¹¹ discussed the same problem allowing for an arbitrary detuning between the frequency of the electromagnetic mode and the atomic resonance. Using the coherent-state representation for the field variables and the traditional angular-momentum representation for the collective atomic states, he also reduced the eigenvalue problem to the diagonalization of a tridiagonal matrix for which he provided analytic as well as numerical solutions. Scharf¹² followed a different procedure. Starting from the difference

equation for the coefficients of the unknown collective eigenvectors, he introduced an exact second-order differential equation for the generating function of the expansion coefficients that he proceeded to solve using the Wentzel-Kramers-Brillouin (WKB) method. The limit of validity of the WKB approximation is not always clear because the turning points of the WKB integral fail to be real for some values of the physical parameters.

Adopting an ingenious trick (Kramers rule) to overcome this difficulty, he found, in general, a satisfactory agreement between the results of the exact numerical diagonalization and the eigenvalues calculated by means of the WKB integral. In addition he derived an analytic expression for the density of eigenvalues that shows very clearly that, for some special classes of collective states, the eigenvalues are not uniformly distributed.

In this paper we attempt an analysis of the eigenvalue spectrum of the Dicke Hamiltonian, both for resonant as well as nonresonant interactions, focusing our attention on the collective states which exhibit the largest deviations from a uniform density of eigenvalues.

Using as the starting point the classification of states introduced by Tavis and Cummings, we briefly review some of the known results and provide approximate analytic expressions for the eigenvalues in special regions of the spectrum of the collective states (Sec. I). We derive these results using a new method which lends itself very naturally to an improved approximation scheme. In Sec. II we derive approximate expressions for the energy eigenvalues that are appropriate for a larger class of collective states than those discussed in Sec. I.

In Sec. III we extend Scharf's method of approach based on the WKB integral to arbitrary values of the detuning parameter, derive an analytic expression for the density of eigenvalues, and study the limits of validity of the solution. Finally, the results of Sec. III are compared with the results of the numerical calculations (Sec. IV).

I. CLASSIFICATION OF THE COLLECTIVE STATES AND APPROXIMATE ENERGY EIGENVALUES

We consider an assembly of N identical two-level atoms with a resonant frequency ω_0 interacting with a single mode of the radiation field. As a consequence of the formal analogy between the operator algebra of two-level systems and spin- $\frac{1}{2}$ particles, the operators which are appropriate to describe the atomic system are the familiar angular-momentum operators satisfying the commutation relations

$$\begin{aligned} [R^+, R^-] &= 2R_z, \\ [R_z, R^\pm] &= \pm R^\pm \end{aligned} \quad (1.1)$$

and operating on a Hilbert space spanned by the eigenstates $|r, m\rangle$ of R^2 and R_z , where $(0, \frac{1}{2}) \leq r \leq \frac{1}{2}N$ and $|m| \leq r$. The lowest value of r is 0 or $\frac{1}{2}$, depending on whether N is even or odd.

The mathematical model of the interaction is characterized by a Hamiltonian which, with properly chosen units of energy, can be written in the form

$$H = a^\dagger a + \epsilon R_z + \kappa (a^\dagger R^- + a R^+). \quad (1.2)$$

The operators a^\dagger and a are the usual creation and annihilation operators of the electromagnetic-field mode, ϵ is a detuning parameter, equal to one for resonant interaction, and κ is the interaction strength.

The collective atomic operators are defined in terms of the individual spin operators for each atom in the familiar way

$$R_z = \frac{1}{2} \sum_{i=1}^N \sigma_i^z, \quad R^\pm = \sum_{i=1}^N \sigma_i^\pm. \quad (1.3)$$

The model Hamiltonian (1.2) contains only excitation-conserving interaction terms, in the sense that, for every unit of energy gained or lost by the field, one unit of energy is lost or gained by the collective atomic states. The so-called anti-resonant terms $a^\dagger R^+$ and $a R^-$, which describe excitation-nonconserving interactions, can be ignored when the atoms and the field are at, or near, resonance. This implies that, in practice, the detuning parameter ϵ cannot take arbitrary values, because for values of ϵ which are sufficiently dif-

ferent from unity the model Hamiltonian becomes invalid.

Following Tavis and Cummings, we observe that the Dicke Hamiltonian commutes with the square of the total angular-momentum operator R^2 and with the operator $C = a^\dagger a + R_z$. The square of the total angular-momentum operator measures the degree of quantum correlation among the individual spins. Its eigenvalues $r(r+1)$ depend on Dicke's "cooperation number" r .

The conserved operator C is related to the number of excitations in the collective system (i.e., the number of photons plus units of energy stored in the spin assembly). More precisely, if we let I be the appropriate identity operator, $C + rI$ can be interpreted as the quasiparticle number operator for a collective system characterized by a cooperation number r .

In view of the conserved nature of R^2 and C , it is convenient to decompose the complete Hilbert space of the N -spin field system into an infinite set of invariant subspaces spanned by the degenerate eigenstates of R^2 and C . The totality of the eigenstates of the Dicke Hamiltonian can be classified into groups, or multiplets, labeled by the pair of indices r and c , whose range of variation is given by

$$(0, \frac{1}{2}) \leq r \leq \frac{1}{2}, \quad N - r \leq c < \infty. \quad (1.4)$$

It is fairly obvious that if $c \leq r$, the (r, c) multiplets will contain $c + r + 1$ distinct eigenstates of the Hamiltonian, while for $c \geq r$, the number of eigenstates in the (r, c) multiplet will be $2r + 1$.

A convenient geometrical representation of the collective multiplets can be given in terms of the two-dimensional diagram shown in Fig. 1. Each point in the r - c plane, characterized by integer or half-integer coordinates (depending on the parity of N), is representative of a particular multiplet. The dimension of a multiplet located in the domain bounded by the diagonals $c = -r$ and $c = r$, and by the vertical line $r = \frac{1}{2}N$, is $c + r + 1$. The dimension of a multiplet lying on or above the line $c = r$ is $2r + 1$.

This classification is not completely descriptive of all possible microscopic configurations of the spin-field states. The Hamiltonian (1.2) is invariant under permutation among the identical spins. A simple argument^{1,11} shows that every collective eigenstate characterized by a cooperation number r can be constructed in $g(r)$ different ways by permutation of spins, where

$$g(r) = \frac{N! (2r + 1)}{(\frac{1}{2}N + r + 1)! (\frac{1}{2}N - r)!}. \quad (1.5)$$

This permutational-degeneracy factor is common to every state of a given (r, c) multiplet. Since

the energy eigenvalues are clearly unaffected by spin permutations, we shall ignore the permutational degeneracy throughout.

We now proceed to state the formal diagonalization problem. Following Tavis and Cummings, let $|r, c, l\rangle$ denote the arbitrary eigenstate of the Hamiltonian characterized by a cooperation number r and by $c+r$ excitations (quasiparticles).

Let l label the eigenstates of the (r, c) multiplet with

$$0 \leq l \leq r+c \text{ for } c \leq r,$$

$$0 \leq l \leq 2r \text{ for } c \geq r.$$

The eigenvalue equation to be solved is

$$H|r, c, l\rangle = E_l^{(r,c)}|r, c, l\rangle, \quad (1.6)$$

where H is given by Eq. (1.2)

The most general representation of the arbitrary eigenstate $|r, c, l\rangle$ consistent with the requirement that $|r, c, l\rangle$ be a simultaneous eigenstate of R^2 , C and the Hamiltonian is

$$|r, c, l\rangle = \sum_{n=\begin{cases} 0 \\ c-r \end{cases}}^{c+r} A_n^{(r,c,l)} |n\rangle |r, c-n\rangle, \quad (1.7)$$

where the lower limit of summation is 0 or $c-r$, depending on whether $c \leq r$ or $c \geq r$, respectively, and where the states $|n\rangle$ and $|r, c-n\rangle$ are the usual eigenstates of $a^\dagger a$ and R_z .

The coefficients $A_n^{(r,c,l)}$, if known, completely characterize the l th eigenvector in the (r, c) multiplet of the Hamiltonian. Using the representation (1.7), the eigenvalue equation can be cast in the form of a recursion relation for the coefficients $A_n^{(r,c,l)}$,

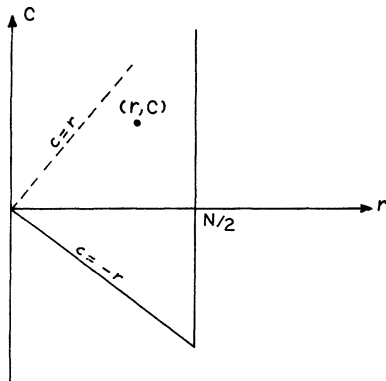


FIG. 1. Geometrical representation of the multiplets (r, c) . Each point in the r - c plane represents a set of $2r+1$ or $c+r+1$ states, respectively, for $c \geq r$ and $c \leq r$. For a system containing N two-level atoms, the boundaries of the domain of definition of the multiplets are $(0, \frac{1}{2}) \leq r \leq N/2$ and $-r \leq c < \infty$.

$$[n + \epsilon(c-n)] A_n^{(r,c,l)}$$

$$+ \kappa \sqrt{n} [(r-c+n)(r+c-n+1)]^{1/2} A_{n-1}^{(r,c,l)}$$

$$+ \kappa [(n+1)(r-c+n+1)(r+c-n)]^{1/2} A_{n+1}^{(r,c,l)}$$

$$= E_l^{(r,c)} A_n^{(r,c,l)} \quad (1.8)$$

where the index n ranges from 0 to $c+r$ if $c \leq r$ and from $c-r$ to $c+r$ if $c \geq r$.

If it is useful to visualize the three-term recursion relation (1.8) as the matrix equation

$$\mathcal{K} \underline{A}^l = E_l \underline{A}^l, \quad (1.9)$$

where the column vector \underline{A}^l is a $c+r+1$ (or $2r+1$) dimensional vector with components $A_n^{(r,c,l)}$, and \mathcal{K} is a tridiagonal symmetric matrix in a $c+r+1$ (or $2r+1$) dimensional space. We will find it convenient to return to the matrix representation (1.9) in connection with the first-order perturbation calculation to be presented in Sec. III.

The main problem to be discussed in this section concerns the diagonalization of the Hamiltonian in two special regions of the r - c plane, namely,

$$c \gg r, \quad (1.10a)$$

$$c = -r + p, \quad p \ll 2r \quad (1.10b)$$

In the first case we are considering collective states possessing a very large number of elementary excitations, whereas in the second we are interested in states with a small number of excitations relative to the number of accessible energy levels.

The parameter p , representing the vertical distance of the (r, c) multiplet from the lower boundary $c = -r$, is simply the number of excitations in the collective system. With the restrictions imposed by Eqs. (1.10a) and (1.10b), the eigenvalues and eigenvectors of the collective states can be calculated exactly.

$$A. \quad c \gg r \text{ [Eq. (1.10a)]}$$

The index n of the recursion relation (1.8) ranges over $2r+1$ values between $c-r$ and $c+r$. It will be convenient to introduce a new index i defined by

$$n = i + c, \quad |i| \leq r, \quad (1.11)$$

as well as a new label $j = l - r$ for the eigenvectors.

In terms of the new indices i and j , we now consider the equivalent set of homogeneous equations for the coefficients $A_i^{(r,c,j)}$,

$$\begin{aligned}
& [c + (1 - \epsilon)i] A_i^{(r, c, j)} \\
& + \kappa [(c + i)(r + i)(r - i + 1)]^{1/2} A_{i-1}^{(r, c, j)} \\
& + \kappa [(i + c + 1)(r + i + 1)(r - 1)]^{1/2} A_{i+1}^{(r, c, j)} \\
& = E_j A_i^{(r, c, j)}, \quad |i|, |j| \leq r. \quad (1.12)
\end{aligned}$$

Observing that for $c \gg r$, the factors $(c + i)^{1/2}$ and $(c + i + 1)^{1/2}$ are essentially constant over the entire range of variation of i , we proceed to replace Eq. (1.12) with its approximate form

$$\begin{aligned}
\frac{1 - \epsilon}{2\kappa} i A_i^{(r, c, j)} & \sim \sqrt{c} \left\{ \frac{1}{2} [(r - i)(r - i + 1)]^{1/2} A_{i-1}^{(r, c, j)} \right. \\
& \left. + \frac{1}{2} [(r + i + 1)(r - i)]^{1/2} \right\} \\
A_{i+1}^{(r, c, j)} & = \frac{E_j - c}{2\kappa} A_i^{(r, c, j)}. \quad (1.13)
\end{aligned}$$

It is now easy to recognize that, in a formal way, Eq. (1.13) can be obtained from the operator-eigenvalue problem

$$\left(\frac{1 - \epsilon}{2\kappa} S_z + \sqrt{c} S_x \right) |\psi\rangle_j = \frac{E_j - c}{2\kappa} |\psi\rangle_j, \quad |j| \leq r, \quad (1.14)$$

where S_z and S_x are angular-momentum operators in a $(2r + 1)$ -dimensional Hilbert space, and the unknown eigenvectors $|\psi\rangle_j$ are a linear superposition of the states which diagonalize S_z , i.e.,

$$|\psi\rangle_j = \sum_{i=-r}^r A_i^{(j)} |r, i\rangle. \quad (1.15)$$

Clearly, Eq. (1.13) is equivalent to Eq. (1.14) if one makes the identification $A_i^{(j)} \equiv A_i^{(r, c, j)}$. If we let

$$\begin{aligned}
\cos\theta & = \frac{1 - \epsilon}{2\kappa} \left[\left(\frac{1 - \epsilon}{2\kappa} \right)^2 + c \right]^{-1/2}, \\
\sin\theta & = \sqrt{c} \left[\left(\frac{1 - \epsilon}{2\kappa} \right)^2 + c \right]^{-1/2}, \quad (1.16)
\end{aligned}$$

the eigenvalue Eq. (1.14) takes the familiar form

$$\begin{aligned}
(\cos\theta S_z + \sin\theta S_x) |\psi\rangle_j \\
= \frac{E_j - c}{2\kappa} \left[\left(\frac{1 - \epsilon}{2\kappa} \right)^2 + c \right]^{-1/2} |\psi\rangle_j. \quad (1.17)
\end{aligned}$$

Performing the unitary transformation generated by the rotation operator $e^{i\theta S_y}$ and defining the transformed state

$$|\psi(\theta)\rangle_j \equiv e^{i\theta S_y} |\psi\rangle_j, \quad (1.18)$$

we finally have

$$S_z |\psi(\theta)\rangle_j = \frac{E_j - c}{2\kappa} \left[\left(\frac{1 - \epsilon}{2\kappa} \right)^2 + c \right]^{-1/2} |\psi(\theta)\rangle_j. \quad (1.19)$$

The eigenvalues of Eq. (1.19) are the $(2r + 1)$ inte-

ger or half-integer numbers j , ranging from $-r$ to $+r$, while the eigenstates $|\psi(\theta)\rangle_j$ are the customary $|r, j\rangle$ eigenstates of the angular-momentum theory. Thus we have

$$E_j^{(r, c)} = c + 2\kappa \left\{ [(1 - \epsilon)/2\kappa]^2 + c \right\}^{1/2} j, \quad |j| \leq r. \quad (1.20)$$

The expansion amplitudes of the original eigenstates are given by the matrix elements of the operator $e^{-i\theta S_y}$, as shown below

$$A_i^{(j)} = \langle r, i | e^{-i\theta S_y} | r, j \rangle \equiv A_i^{(r, c, j)}, \quad (1.21)$$

where

$$\theta = \arccos \left\{ \frac{1 - \epsilon}{2\kappa} \left[\left(\frac{1 - \epsilon}{2\kappa} \right)^2 + c \right]^{-1/2} \right\}. \quad (1.22)$$

Therefore, the diagonalization problem for $c \gg r$ is completely solved.

$$\text{B. } c = -r + p, \quad p \ll 2r \quad [\text{Eq. (1.10b)}]$$

In a completely similar way we now proceed to analyze the eigenvalues and eigenstates of the multiplets characterized by the values $c = -r + p$ with $p \ll 2r$. The restriction $p \ll 2r$ limits our analysis to the quantum systems represented by the points in the $r - c$ plane that lie well below the main diagonal $c = r$. These collective states are characterized by a very small number of elementary excitations compared to the number of accessible energy states. The formal diagonalization process has nothing new to offer but it will be repeated for the sake of completeness.

The starting point is the recursion relation (1.8). Since the space dimensionality in this case equals $c + r + 1 = p + 1$, we introduce the new indices

$$\begin{aligned}
i & = n - \frac{1}{2}p, \quad |i| \leq \frac{1}{2}p \\
j & = l - \frac{1}{2}p, \quad |j| \leq \frac{1}{2}p, \quad (1.23)
\end{aligned}$$

such that the transformed recursion relation can be easily cast in operator form after the suitable approximations are made. In terms of the new indices, the expansion amplitudes $A_i^{(r, c, j)}$ will be solutions of the exact recursion relation

$$\begin{aligned}
[(1 - \epsilon)/2\kappa] i A_i^{(r, c, j)} & + \frac{1}{2} [(i + \frac{1}{2}p)(2r - \frac{1}{2}p + i) \\
& \times (\frac{1}{2}p - i + 1)]^{1/2} A_{i-1}^{(r, c, j)} \\
& + \frac{1}{2} [(i + \frac{1}{2}p + 1)(2r - \frac{1}{2}p + i + 1)(\frac{1}{2}p - i)]^{1/2} \\
& \times A_{i+1}^{(r, c, j)} = (2\kappa)^{-1} (E_j + \epsilon r - \frac{1}{2}(1 + \epsilon)p) A_i^{(r, c, j)}. \quad (1.24)
\end{aligned}$$

If we restrict our attention to multiplets with $p \ll 2r$, it is apparent that the factors $(2r - \frac{1}{2}p + i)^{1/2}$ and $(2r - \frac{1}{2}p + i + 1)^{1/2}$ are approximately constant

and equal to $\sqrt{2r}$ for all values of i . Therefore, as in the previous case, we recognize that Eq. (1.24) is formally equivalent to the operator equation

$$\left(\frac{i-\epsilon}{2\kappa}S_x + (2r)^{1/2}S_y\right)|\psi\rangle_j = \frac{E_i + \epsilon r - \frac{1}{2}(1+\epsilon)p}{2\kappa}|\psi\rangle_j, \quad (1.25)$$

where S_x and S_y are angular-momentum operators in a $(p+1)$ -dimensional Hilbert space spanned by the eigenvectors $|\frac{1}{2}p, i\rangle$ of S_x .

By means of the rotation induced by the unitary operator $e^{i\theta S_y}$ with

$$\cos\theta = \frac{1-\epsilon}{2\kappa} \left[\left(\frac{1-\epsilon}{2\kappa} \right)^2 + 2r \right]^{-1/2}, \quad (1.26)$$

Eq. (1.25) can be transformed into the familiar form

$$S_x|\psi(\theta)\rangle_j = \frac{E_i + \epsilon r - \frac{1}{2}(1+\epsilon)p}{2\kappa} \left[\left(\frac{1-\epsilon}{2\kappa} \right)^2 + 2r \right]^{-1/2} \times |\psi(\theta)\rangle_j, \quad (1.27)$$

where

$$|\psi(\theta)\rangle_j = e^{i\theta S_y}|\psi\rangle_j, \quad |j| \leq \frac{1}{2}p. \quad (1.28)$$

Thus we conclude that the energy eigenvalues are given by

$$E_j^{(r,c)} = \frac{1}{2}(1+\epsilon)p - \epsilon r + 2\kappa \left[\left((1-\epsilon)/2\kappa \right)^2 + 2r \right]^{1/2} j, \quad |j| \leq \frac{1}{2}p/2, \quad (1.29)$$

while the eigenvectors of the (r, c) multiplet are of the form

$$|r, c, l\rangle = \sum_{n=0}^{c+r} A_n^{(r,c,l)} |n\rangle |r, c-n\rangle$$

with

$$A_n^{(r,c,l)} = \langle \frac{1}{2}p, n - \frac{1}{2}p | e^{i\theta S_y} | \frac{1}{2}p, l - \frac{1}{2}p \rangle, \quad 0 \leq n \leq p. \quad (1.30)$$

In summary, our results are given in Table I.

II. FIRST-ORDER CORRECTION TO THE ENERGY EIGENVALUES

To gain some insight into the range of validity of the asymptotic approximation derived in Sec. I, and to extend the class of multiplets for which an approximate diagonalization can be performed, we now discuss a perturbation scheme which provides a first-order correction to the energy eigenvalues when the asymptotic approximation is not strictly applicable. The present calculation is not based on the smallness of any physical parameter, but rather stems from the observation that, in the two asymptotic ranges of interest ($c \gg r$ and $c = -r + p$, $p \ll 2r$), the Dicke Hamiltonian can be accurately represented within the (r, c) multiplet of interest by effective Hamiltonian operators $\mathcal{H}_{\text{eff}}^{(1)}$ and $\mathcal{H}_{\text{eff}}^{(2)}$, which can be diagonalized exactly. The superscripts (1) and (2) refer to the two ranges of values of c .

TABLE I. Summary of the results of the approximate diagonalization in the asymptotic regions $c \gg r$ and $c = -r + p$, $p \ll 2r$.

$c \gg r$	
Eigenstates	$ r, c, l\rangle = \sum_{n=c-r}^{c+r} A_n^{(r,c,l)} n\rangle r, c-n\rangle, \quad l=0, \dots, 2r.$
Coefficients	$A_n^{(r,c,l)} = \langle r, n - c e^{-i\theta S_y} r, l - r \rangle$
Rotation angle	$\Theta = \arccos \left\{ \frac{1-\epsilon}{2\kappa} \left[\left(\frac{1-\epsilon}{2\kappa} \right)^2 + c \right]^{-1/2} \right\}$
Eigenenergies	$E_l^{(r,c)} = c + 2\kappa \left[(1-\epsilon)/2\kappa \right]^2 + c^{1/2} (l - r), \quad l=0, \dots, 2r.$
$c = -r + p$ $p \ll 2r$	
Eigenstates	$ r, c, l\rangle = \sum_{n=0}^{c+r} A_n^{(r,c,l)} n\rangle r, c-n\rangle, \quad l=0, \dots, r+c=p$
Coefficients	$A_n^{(r,c,l)} = \langle \frac{1}{2}p, n - \frac{1}{2}p e^{-i\theta S_y} \frac{1}{2}p, l - \frac{1}{2}p \rangle$
Rotation angle	$\Theta = \arccos \left\{ \frac{1-\epsilon}{2\kappa} \left[\left(\frac{1-\epsilon}{2\kappa} \right)^2 + 2r \right]^{-1/2} \right\}$
Eigenenergies	$E_l^{(r,c)} = \frac{1}{2}(1+\epsilon)p - \epsilon r + 2\kappa \left[(1-\epsilon)/2\kappa \right]^2 + 2r^{1/2} (l - \frac{1}{2}p), \quad l=0, \dots, p$

To be more specific, we can state that if \mathcal{H} is the Dicke Hamiltonian, it is possible to construct $\mathcal{H}_{\text{eff}}^{(1)}$ and $\mathcal{H}_{\text{eff}}^{(2)}$ such that the differences $\mathcal{H} - \mathcal{H}_{\text{eff}}^{(1)}$ and $\mathcal{H} - \mathcal{H}_{\text{eff}}^{(2)}$ become vanishingly small in the two asymptotic limits.

We now proceed to identify the "small corrections" upon which the perturbation calculation is based. Consider the formal decompositions.

$$\begin{aligned}\mathcal{H} &= \mathcal{H}_{\text{eff}}^{(1)} + (\mathcal{H} - \mathcal{H}_{\text{eff}}^{(1)}), \\ \mathcal{H} &= \mathcal{H}_{\text{eff}}^{(2)} + (\mathcal{H} - \mathcal{H}_{\text{eff}}^{(2)}).\end{aligned}\quad (2.1)$$

In view of our previous comments on the behavior of the differences $\mathcal{H} - \mathcal{H}_{\text{eff}}^{(i)}$ for $i = 1, 2$, we expect that the operators $\mathcal{H} - \mathcal{H}_{\text{eff}}^{(i)}$ can be treated as perturbations to the unperturbed "large" contributions $\mathcal{H}_{\text{eff}}^{(i)}$.

In the spirit of the first-order perturbation theory, let $\{|\psi\rangle_j\}$ be the complete collection of nondegenerate eigenstates of $\mathcal{H}_{\text{eff}}^{(i)}$. The required energy eigenvalues in first-order approximation will be given by

$$E_j^{(1,2)} = {}_j\langle\psi|\mathcal{H}_{\text{eff}}^{(1,2)}|\psi\rangle_j + {}_j\langle\psi|\mathcal{H} - \mathcal{H}_{\text{eff}}^{(1,2)}|\psi\rangle_j, \quad (2.2)$$

or, simply, by

$$E_j^{(1,2)} = {}_j\langle\psi|\mathcal{H}|\psi\rangle_j.$$

Hence, the approximate energy eigenvalues containing corrections of order $\mathcal{H} - \mathcal{H}_{\text{eff}}^{(i)}$ are given by the diagonal matrix elements of the Dicke Hamiltonian in the representation which diagonalizes $\mathcal{H}_{\text{eff}}^{(1,2)}$. At this point in our formal discussion we must remember that both $\mathcal{H}_{\text{eff}}^{(1)}$ and $\mathcal{H}_{\text{eff}}^{(2)}$, which, as we shall see, are related to the Hamiltonians of Eqs. (1.14) and (1.25), operate over a fictitious angular-momentum subspace (i.e., the space of the operators S_x , S_y , and S_z). The difficulty is only apparent, and will be removed by proceeding with the matrix representations of the operators of interest.

First we construct the explicit expressions for $\mathcal{H}_{\text{eff}}^{(1)}$ and $\mathcal{H}_{\text{eff}}^{(2)}$. For $c \geq r$ the matrix elements of the Dicke Hamiltonian in the subspace of the (r, c) multiplet are given by

$$\begin{aligned}\mathcal{H}_{ij} &= [c + (1 - \epsilon)i] \delta_{ij} \\ &+ \kappa[(c + i)(r + i)(r - i + 1)]^{1/2} \delta_{i-1,j} \\ &+ \kappa[(i + c + 1)(r + i + 1)(r - i)]^{1/2} \delta_{i+1,j}.\end{aligned}\quad (2.4)$$

In the limit $c \gg r$ we have

$$\begin{aligned}\lim_{c \gg r} \mathcal{H}_{ij} &= [c + (1 - \epsilon)i] \delta_{ij} \\ &\times \kappa \sqrt{c} \{[(r + i)(r - i + 1)]^{1/2} \delta_{i-1,j} \\ &+ [(r + i + 1)(r - i)]^{1/2} \delta_{i+1,j}\}.\end{aligned}\quad (2.5)$$

In analogy with the procedure followed in Sec. I, we can formally introduce an angular-momentum operator \tilde{S} such that

$$\mathcal{H}_{\text{eff}}^{(1)} \equiv \lim_{c \gg r} \mathcal{H} = c + (1 - \epsilon)S_z + 2\kappa \sqrt{c} S_x. \quad (2.6)$$

Similarly, for $c \leq r$ in the range $p \ll 2r$ we can formally write

$$\mathcal{H}_{\text{eff}}^{(2)} \equiv \lim_{2r \gg p} \mathcal{H} = -\epsilon r + \frac{1}{2}(1 + \epsilon)p + (1 - \epsilon)S_z + 2\kappa(2r)^{1/2} S_x. \quad (2.7)$$

The similarity transformation which diagonalizes $\mathcal{H}_{\text{eff}}^{(1)}$ and $\mathcal{H}_{\text{eff}}^{(2)}$ is effected by the rotation operator $e^{i\theta S_y}$, where the rotation angle θ is given in Table I.

The matrix elements of the rotation operator $e^{i\theta S_y}$ in the representation in which S_z is diagonal are given by the familiar elements of the Wigner rotation matrices $d(\theta)$. With an elementary calculation we can now conclude that the approximate eigenvalues are given by

$$E_j^{(1)} = \sum_{\rho, \sigma=-r}^r d_{\rho\sigma}^{(r)}(\theta) \mathcal{H}_{\rho\sigma} d_{\sigma j}^{(r)}(-\theta), \quad c > r, \quad (2.8)$$

$$E_j^{(2)} = \sum_{\rho, \sigma=-p/2}^{p/2} d_{\rho\sigma}^{(p/2)}(\theta) \mathcal{H}_{\rho\sigma} d_{\sigma j}^{(p/2)}(-\theta), \quad c < r, \quad (2.9)$$

where the matrix elements $\mathcal{H}_{\rho\sigma}$ are defined by Eq. (2.4).

III. BEHAVIOR OF THE DENSITY OF ENERGY EIGENVALUES

Considerable insight can be gained into the nature of the exact eigenvalue spectrum if we consider the density of energy levels of the collective states in the arbitrary multiplet (r, c) .

As shown in Sec. I, the energy eigenvalues of the Dicke Hamiltonian are uniformly distributed between their upper and lower bounds for multiplets located in the asymptotic regions $c \gg r$ and $c = -r + p$ with $p \ll 2r$. On the other hand, as has already been pointed out in the literature,^{9,12,13} the character of the energy spectrum changes rather drastically in the region of the $r - c$ plane, where $c \approx r$. In particular, it has been shown that for $c \approx r$, the energy levels of the resonant Hamiltonian ($\epsilon = 1$) are more densely distributed around the midpoint of the multiplet spectrum than they are in the neighborhood of the upper and lower bounds.

This feature of the spectrum has been emphasized by Scharf, who arrived at an analytic expression for the density of eigenvalues with a calculation based on the WKB approximation. In this section we summarize Scharf's argument

with a slight extension to include arbitrary values of the detuning parameter ϵ . We then proceed to investigate the accuracy of the method by direct comparison with the results of the numerical diagonalization. We also analyze the behavior of the density of eigenvalues in the interesting region $c \approx r$ for different values of the detuning parameter ϵ . It will be seen that the sharpest concentration of energy levels occurs around the midpoint of the multiplet spectrum for $c=r$. For $c \neq r$, on the other hand, the density of eigenvalues is a considerably smoother function of the energy. In the asymptotic regions $c \gg r$ and $c = -r + p$, with $p \ll 2r$, we recover the uniform distribution of energy levels described in Sec. II.

A surprising feature, in the case of nonresonant interaction ($\epsilon \neq 1$), is that the density of eigenvalues is not very different from that of the resonant case in the high-density region. There is, however, a small difference around the upper and lower bounds of the energy spectrum.

The present calculation can be summarized as follows: Starting from the difference equation (1.8) for the expansion coefficients $A_n^{(r, c, 1)}$, we define a generating function $f(z)$ in terms of the $\{A_n\}$, and construct the second-order differential equation for which $f(z)$ is a solution. Unlike more traditional eigenvalue equations of the Sturm-Liouville type, in this case we have no obvious boundary conditions for $f(z)$. The generating function, however, is required by definition to be a polynomial for all finite values of z . Hence, in principle, the problem is reduced to finding values of $E^{(r, c)}$ such that the differential equation admits a polynomial solution. In practice, this problem is equivalent to (and as difficult as) the original diagonalization problem. Following Scharf, we construct an implicit expression for the eigenvalues using the WKB approximation, and from the WKB integral we finally derive an analytic expression for the density of eigenvalues.

Our final result is in the form of a complete elliptic integral of the first kind, whose argument contains the roots of a cubic polynomial.

In detail, our calculation proceeds as follows: Consider the recursion relation (1.8) for the unknown coefficients of expansion $A_n^{(r, c, 1)}$ (henceforth the superscripts will be omitted)

$$[n + \epsilon(c - n)]A_n + \kappa[n(r - c + n)(r + c - n + 1)]^{1/2}A_{n-1} + \kappa[(n + 1)(r - c + n + 1)(r + c - n)]^{1/2}A_{n+1} = E_1 A_n, \quad (3.1)$$

and define a new set of expansion coefficients B_n such that

$$A_n = B_n [n!(r + c - n)!(n - c + r)!]^{1/2}. \quad (3.2)$$

The advantage of this transformation is that it eliminates the irrational factors in the recursion relation (3.1), which in terms of the new coefficients B_n takes the form

$$[n + \epsilon(c - n)]B_n + \kappa(r + c - n + 1)B_n + \kappa(n + 1)(r - c + n + 1)B_{n+1} = E_1 B_n. \quad (3.3)$$

It may be useful to emphasize that the above equation is valid for all values of r and c , provided we impose the boundary conditions

$$B_{c-r-1} = B_{c+r+1} = 0 \quad \text{for } c \geq r, \\ B_{-1} = B_{c+r+1} = 0 \quad \text{for } c \leq r, \quad (3.4)$$

which restrict the range of variation of the index n . The necessity for the conditions (3.4) can be understood if we compare the original recursion relation (3.1) with the transformed version (3.3). In the former equation, the absence of the amplitudes A_{c-r-1} and A_{c+r+1} (and similarly A_{-1} and A_{c+r+1}) is automatically ensured by the vanishing of the irrational factors at the boundaries of the range of variation of n . In the latter difference equation, the same result is accomplished by imposing Eqs. (3.4).

If we now define the generating function

$$f(z) = \sum_{n=\{0, c-r\}}^{c+r} B_n z^n, \quad (3.5)$$

where again the lower limit of summation is $c - r$ or 0, depending on whether $c \geq r$ or $c \leq r$, respectively, it is easy to verify that $f(z)$ will be a solution of the second-order differential equation

$$z \frac{d^2 f}{dz^2} + \left(\frac{1 - \epsilon}{2\kappa} z + (r - c + 1) - z^2 \right) \frac{df}{dz} + \left((r + c)z - \frac{E_1 - \epsilon c}{\kappa} \right) f = 0, \quad (3.6)$$

subject to the requirement that $f(z)$ be a polynomial. This requirement, in principle, defines the eigenvalues E_1 for any value of r and c .

As a trivial example, we see that for $c = -r$ the only solution which is consistent with Eq. (3.5) is $f(z) = \text{const}$. In this case the only eigenvalue is

$$E = \epsilon c = -\epsilon r.$$

In general, the solution of the eigenvalue equation (3.6) presents the same difficulties as the solution of the recursion relation. In this form, however, the problem is amenable to approximation schemes whose range of validity can be assessed more accurately.

It is convenient to transform equation (3.6) to the standard Sturm-Liouville form by introducing a new function $G(z)$ such that

$$f(z) = \exp(\frac{1}{4}z^2 - \delta z) z^\gamma G(z), \quad (3.7)$$

with

$$\delta = [(1 - \epsilon)/2\kappa], \quad \gamma = -\frac{1}{2}(r - c + 1). \quad (3.8)$$

The differential equation for $G(z)$ takes the form

$$G''(z) + [\mathcal{G} - V(z, q)]G(z) = 0, \quad (3.9)$$

where we have set

$$\mathcal{G} = \frac{1}{2}(3r + c + 2) - \delta^2, \quad (3.10)$$

$$V(z, q) = \frac{\gamma(\gamma + 1)}{z^2} + \frac{1}{4}z^2 + \frac{2q - 2\gamma\delta}{z} - \delta z,$$

$$2q = [(E_l - \epsilon c)/\kappa].$$

The parameter q is proportional to the eigenvalues E_l of the problem and, therefore, should be labelled by and index l . We choose to drop the index, since q will play the role of an essentially continuous variable. The formal similarity of Eq. (3.9) with a Schrödinger equation was recognized by Scharf in the case $\delta = 0$ (resonance condition). The same similarity exists in the non-resonant case but, as already pointed out, it has no direct physical significance. The most obvious difference between the usual Schrödinger eigenvalue problem and the one posed by Eq. (3.9) is that in this case the "potential energy" $V(z, q)$ varies as a function of q in such a way that, for specific values of E_l , the solution $G(z)$ has the required analytic behavior.

We now proceed to construct an approximate analytic expression for the density of eigenvalues using the WKB method. If we indicate with z_1 and z_2 ($z_1 > z_2$) the two classical turning points of the "potential" $V(z, q)$, the WKB integral takes the form

$$(l + \frac{1}{2})\pi = \int_{z_2}^{z_1} dz [\mathcal{G} - V(z, q)]^{1/2}, \quad (3.11)$$

where, as usual, the eigenvalue index l ranges from $c - r$ or 0 to $c + r$, depending on whether $c \geq r$ or $c \leq r$. If the eigenvalues are densely distributed as a function of the energy, we may differentiate both sides of Eq. (3.11) with respect to q .

Defining the density of eigenvalues as

$$\rho(q) = \frac{dl}{dq}, \quad (3.12)$$

we have

$$\rho(q) = \frac{1}{\pi} \int_{z_2}^{z_1} \frac{dz}{z [\mathcal{G} - V(z, q)]^{1/2}}, \quad (3.13)$$

or

$$\rho(q) = \frac{2}{\pi} \int_{z_2}^{z_1} dz [-z^4 + 4\delta z^3 - 8(q - \gamma\delta)z + 4\mathcal{G}z^2 - 4\gamma(\gamma + 1)]^{-1/2}. \quad (3.14)$$

A close examination of the zeros of the quartic equation in the integral of Eq. (3.14) reveals that, with the exception of a very small range of values of the parameters and for a fraction of the energy spectrum which becomes asymptotically negligible for large values of c and r , there always exist four real roots $z_1 > z_2 > z_3 > z_4$.

We postpone the discussion on the limits of validity of the procedure to Appendix A, and assume in the following that four real roots z_i , ($i = 1, 2, 3, 4$) exist such that by means of the standard transformation

$$z = \frac{z_2(z_1 - z_3) - z_3(z_1 - z_2) \sin^2 \Phi}{(z_1 - z_3) - (z_1 - z_2) \sin^2 \Phi}, \quad (3.15)$$

Eq. (3.14) can be cast in the form of the complete elliptic integral of the first kind

$$\rho(q) = \frac{4}{\pi [(z_1 - z_3)(z_2 - z_4)]^{1/2}} K \left(\frac{(z_1 - z_2)(z_3 - z_4)}{(z_1 - z_3)(z_2 - z_4)} \right). \quad (3.16)$$

For the purpose of numerical computations, we found it convenient to express the argument of the elliptic integral in terms of the roots of the cubic resolvent associated to the quartic equation. The details of the algebraic transformation are given in Appendix B. Here it will be sufficient to state that if y_1 , y_2 , and y_3 are the real roots of the cubic resolvent, one can express the four roots of the quartic equation as follows:

$$\begin{aligned} z_1 &= \frac{1}{4}(4\delta + \sqrt{y_1} + \sqrt{y_2} + \sqrt{y_3}), \\ z_2 &= \frac{1}{4}(4\delta + \sqrt{y_1} - \sqrt{y_2} - \sqrt{y_3}), \\ z_3 &= \frac{1}{4}(4\delta - \sqrt{y_1} + \sqrt{y_2} - \sqrt{y_3}), \\ z_4 &= \frac{1}{4}(4\delta - \sqrt{y_1} - \sqrt{y_2} + \sqrt{y_3}). \end{aligned} \quad (3.17)$$

Hence the density of eigenvalues can be conveniently cast into the form

$$\rho(q) = \frac{8}{\pi} \frac{1}{(y_1 - y_3)^{1/2}} K \left(\frac{y_2 - y_3}{y_1 - y_3} \right). \quad (3.18)$$

Although we have been able to construct explicit expressions for y_1 , y_2 , and y_3 that are useful in the discussion of special cases, in practice we have chosen to calculate the roots of the cubic equation by numerical methods. A direct comparison of the density of eigenvalues given by Eq. (3.18) and the results of the numerical diagonalization is given in Sec. IV. Before concluding the present discussion, we outline some of the general

features of the behavior of $\rho(q)$ for different values of the collective parameters. This discussion is confined to the resonant case in order to limit the algebraic labor involved. The more general case of $\epsilon \neq 1$ will be treated using the numerical results.

A. Asymptotic Behavior of the Density of Eigenvalues

As shown in Appendix B, the roots of the cubic resolvent associated to the quartic equation in (3.14) can be conveniently expressed in trigonometric form as follows:

$$\begin{aligned} y_1 &= 16 \left[\frac{2}{3} (\mathcal{G}'^2 + 3L'^2)^{1/2} \cos(\frac{1}{3}\varphi) + \frac{2}{3}\mathcal{G}' \right], \\ y_2 &= 16 \left[\frac{2}{3} (\mathcal{G}'^2 + 3L'^2)^{1/2} \cos(\frac{1}{3}\varphi + \frac{4}{3}\pi) + \frac{2}{3}\mathcal{G}' \right], \\ y_3 &= 16 \left[\frac{2}{3} (\mathcal{G}'^2 + 3L'^2)^{1/2} \cos(\frac{1}{3}\varphi + \frac{2}{3}\pi) + \frac{2}{3}\mathcal{G}' \right], \end{aligned} \quad (3.19)$$

where

$$\cos\varphi = - \frac{\frac{1}{27}\mathcal{G}'^3 - \frac{1}{3}\mathcal{G}'L'^2 - \frac{1}{8}(2q')^2}{[q^{-1}(\mathcal{G}'^2 + 3L'^2)]^{3/2}}. \quad (3.20)$$

In resonance ($\epsilon = 1$), we have $q' = q$ and the values of the parameters \mathcal{G}' and L'^2 can be expressed in terms of c and r as follows¹⁴:

$$\mathcal{G}' = \frac{1}{2}(3r + c + 2), \quad L'^2 = \frac{1}{4}[(c - r)^2 - 1]. \quad (3.21)$$

For future reference, it is also convenient to express the argument of the complete elliptic integral given by Eq. (3.18) in trigonometric form:

$$\frac{y_2 - y_3}{y_1 - y_3} = \frac{2 \sin(\frac{1}{3}\varphi)}{\sin(\frac{1}{3}\varphi) + \sqrt{3} \cos(\frac{1}{3}\varphi)}. \quad (3.22)$$

Equation (3.22) is valid without restrictions on the detuning parameter ϵ , but it will be used explicitly only in the present discussion, where $\epsilon = 1$ is assumed. For $c \gg r$, $\cos\varphi$ is approximately equal to unity up to terms of order $(r/c)^2$, as one can easily verify knowing that the maximum energy eigenvalue [Eq. (1.20)] is such that

$$(2q)_{\max}^2 = 4r^2c. \quad (3.23)$$

Hence the argument of the complete elliptic integral vanishes, and the limiting form of the density of eigenvalues is

$$\rho(q) = 1/\sqrt{c}, \quad c \gg r. \quad (3.24)$$

This result agrees with the inverse spacing of the consecutive eigenvalues found in Sec. I.

Similarly, for $c = -r + p$, with $p \ll 2r$, $\cos\varphi$ is also approximately equal to unity up to terms of order $(p/r)^2$. By Eq. (3.22) we see again that the argument of the elliptic integral vanishes and the asymptotic form of the density of eigenvalues becomes

$$\rho(q) = 1/(2r)^{1/2}, \quad c = -r + p, \quad p \ll 2r, \quad (3.25)$$

once again in agreement with the approximate results [Eq. (1.29)] of Sec. I.

B. Simple Analytic Form for the Density of Eigenvalues for $c = r \gg 1$

The multiplets characterized by the values $c = r$ are especially difficult to analyze by means of conventional diagonalization techniques, although Tavis and Cummings succeeded in deriving a simple approximation that describes a portion of the energy spectrum. The WKB integral given in Eq. (3.14), on the other hand, has been used successfully by Scharf to describe the density of eigenvalues even in this region of the $r - c$ plane.

It turns out, however, that while the major portion of the energy spectrum can be represented quite accurately, the WKB approximation fails in the immediate neighborhood of $q = 0$. The reason for this difficulty, which is only of minor consequence because it is limited to a very small number of eigenvalues, can be explained as follows. For $c = r \gg 1$, Eq. (3.20) takes the form

$$\cos\varphi = - \frac{[\frac{1}{3}(2r + 1)]^3 + \frac{1}{3}(2r + 1)\frac{1}{4} - \frac{1}{8}(2q)^2}{\{q^{-1}[(2r + 1)^2 - \frac{3}{4}]\}^{3/2}}. \quad (3.26)$$

A necessary condition for the validity of the approximation is $|\cos\varphi| \leq 1$, which is clearly not satisfied in the neighborhood of $q = 0$. Performing a series expansion of the right-hand side of Eq. (3.26), we find that $|\cos\varphi| \leq 1$ if the eigenvalues satisfy the restriction $q > (\frac{1}{2}r)^{1/2}$. This may appear as a rather stringent requirement, since one is interested in large values of r . In reality, considering that the upper bound of the eigenvalue spectrum is of the order of $(2r)^{3/2}$, we see that the ratio of the excluded range of the spectrum to the maximum energy eigenvalue is quite small for large values of r .

Furthermore, our numerical results indicate a bounded, although sharply peaked, behavior of the density of eigenvalues in the neighborhood of the origin. Thus, we feel confident that this restriction is of no practical consequence so long as large values of r are considered. As far as the limit of validity of the WKB approximation for large values of q is concerned, we have investigated the behavior of Eq. (3.26) in the neighborhood of the upper bound of the energy spectrum. Using variational arguments, Scharf pointed out that the maximum eigenvalues of the $c = r$ multiplets are in the range

$$\frac{2}{9}\sqrt{3}(2r + 1)^{3/2} \leq 2q_{\max} \leq \frac{4}{9}\sqrt{3}(2r + 1)^{3/2}. \quad (3.27)$$

Our numerical results consistently indicate that the upper bound is very close to $\frac{4}{3}\sqrt{3}(2r+1)^{3/2}$. In this range $|\cos\phi|$ is still less than unity. The results of the numerical diagonalization indeed show an excellent agreement with the results of the WKB approximation throughout the range $(\frac{1}{2}r)^{1/2} < q \leq q_{\max}$.

In what follows, we develop an approximate formula valid in the range

$$(\frac{1}{2}r)^{1/2} < q \ll q_{\max}. \quad (3.28)$$

Starting with Eq. (3.26) we perform a first-order expansion in the small parameter

$$\left(\frac{3}{2r+1}\right)^3 \left[\frac{(2q)^2}{8} - \frac{1}{4}\left(\frac{2r+1}{3}\right)\right] \ll 1, \quad (3.29)$$

and derive the approximate value of the angle ϕ ,

$$\phi = \pi - \delta\phi \equiv \pi - \frac{1}{2}\left(\frac{3}{2r+1}\right)\left((2q)^2\frac{3}{2r+1} - 3\right)^{1/2}, \quad (3.30)$$

which is valid in the range indicated by Eq. (3.28). Retaining only first-order contributions in $\delta\phi$, the argument of the elliptic integral given by Eq. (3.22) is approximately equal to

$$\frac{y_2 - y_3}{y_1 - y_3} = 1 - \frac{1}{2r+1} \left(\frac{(2q)^2}{2r+1} - 1\right)^{1/2}. \quad (3.31)$$

After a few minor manipulations, the approximate form of the density of eigenvalues in resonance ($\epsilon = 1$) and for $c = r$ is given by

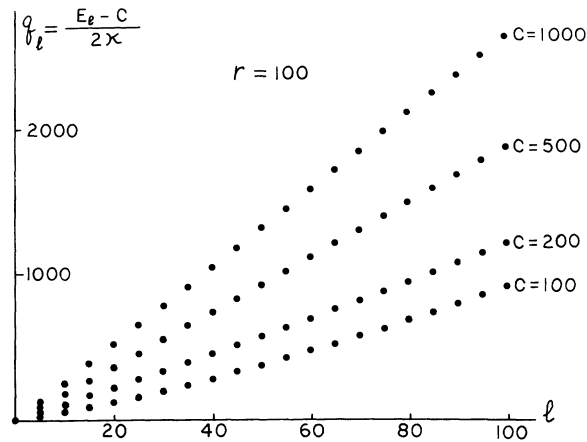


FIG. 2. Energy eigenvalues plotted as a function of the eigenstate quantum number l for $r=100$ and variable c . The detuning parameter δ equals zero. We observe that, while the energy eigenvalues corresponding to the eigenstates of the multiplet $r=100$, $c=1000$ are uniformly increasing, such is not the case for $r=c=100$. In particular, the energy eigenvalues are more closely spaced for low values of the index l .

$$\rho(q) = \frac{1}{\pi(2r+1)^{1/2}} \ln \frac{16(2r+1)}{[(2q)^2/(2r+1)] - 1}^{1/2}, \quad (3.32)$$

where the energy eigenvalues are restricted to the range stated in Eq. (3.28).

IV. NUMERICAL SOLUTIONS AND COMPARISON WITH THE ANALYTIC RESULTS

In this section we summarize the main conclusions of our numerical analysis and, whenever appropriate, we provide a comparison with available analytic results. In order to avoid excessive duplication, we limit our survey to a few typical cases, although the numerical investigation has been carried out for a wide range of values of the collective parameters.

We focus first on the energy eigenvalues of the Dicke Hamiltonian corresponding to a chosen set of multiplets characterized by a cooperation number $r=100$.

For resonant interaction ($\epsilon=1$), the energy spectrum is antisymmetric about $q=0$ (or $E^{(r,c)}=c$). Consequently, our diagrams display only one-half of the spectrum.

The curves shown in Fig. 2 represent the energy eigenvalues $q_l = (E_l - c)/2\kappa$ with l ranging from 100 to 200 for increasing values of c . These results, which reproduce the findings of Refs. 8, 9, and 11, show the transition of the energy spectrum from the region where exact results are not available to the region in which the asymptotic results given by Eq. (1.20) are well reproduced. For $c=r=100$ the eigenvalues are almost equally spaced in the neighborhood of the upper bound, but become more closely spaced around the midpoint

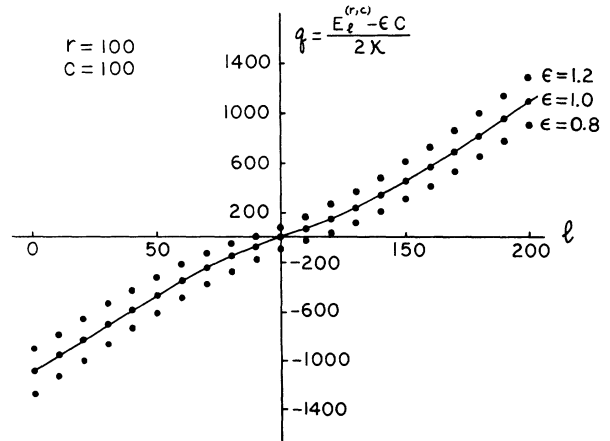


FIG. 3. Energy eigenvalues plotted as a function of the quantum number l for $r=c=100$ and variable detuning parameter ϵ .

of the spectrum where $q \approx 0$. In the asymptotic region ($c = 1000$), on the other hand, the energy spectrum is essentially uniform; i.e., the eigenvalues are equally spaced throughout the energy range.

For nonresonant interaction the distribution of energy eigenvalues in a given multiplet (r, c) is, in general, no longer symmetric about the midpoint of the spectrum, as we can see without much difficulty in Fig. 3, where the eigenvalues of the multiplet $r = c = 100$ are shown for different values of the detuning parameter. As we have observed in the resonant case, the eigenvalues exhibit the largest deviation from a uniform distribution for values of c equal to r even in the nonresonant case.

For increasing values of c , the eigenvalues are essentially distributed in accordance with Eq. (1.20), while for decreasing values of c they are accurately predicted by Eq. (1.29).

Although, as previously observed, the eigenvalues of a given (r, c) multiplet in the case $\epsilon \neq 1$ are not symmetrically distributed about $q = 0$, the numerical results displayed in Fig. 3 indicate that a certain symmetry exists among the positive and negative eigenvalues corresponding to the detuning parameters $\epsilon = 1.2$ and $\epsilon = 0.8$, respectively.

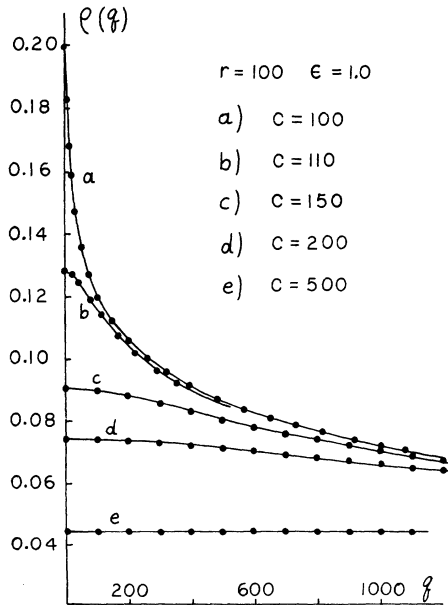


FIG. 4. Behavior of the density of eigenvalues $\rho(q)$ as a function of the energy for resonant interaction. The cooperation number r is fixed while c varies from $c = r$ to $c = 5r$. The dots represent the values of the density calculated from the numerical diagonalization of the Dicke Hamiltonian. The solid lines represent the analytic expression for $\rho(q)$ discussed in Sec. 3. Note that the density of eigenvalues is essentially uniform for $c = 5r$.

Indeed, we can readily prove, using the recursion relation (3.1), that if $\epsilon_1 = 1 + \delta\epsilon$ and $\epsilon_2 = 1 - \delta\epsilon$ are two given values of the detuning parameter, and if $q_l^{(r,c)}$ and $\{A_n^{(r,c,l)}\}$ are respectively the energy eigenvalue and the expansion coefficients of the l th eigenvector in the (r, c) multiplet corresponding to $\epsilon = \epsilon_1$, then $-q_l^{(r,c)}$ and $\{B_n^{(r,c,l)}\} = (-1)^n A_n^{(r,c,l)}$ are the energy eigenvalue and expansion amplitudes of the corresponding eigenvector for $\epsilon = \epsilon_2$.

In resonance ($\delta\epsilon = 0$), this symmetry property reduces to the already mentioned reflection symmetry of the eigenvalue spectrum about its midpoint.

For multiplets characterized by $c \approx r$, it is convenient to display the nonuniformity of the energy spectrum by plotting the density of eigenvalues as a function of the energy. In Fig. 4 we have assembled a series of discrete points (obtained from the numerical diagonalization) describing the behavior of the density of eigenvalues for a given value of r and increasing values of c in the case of resonant interaction. The solid curves passing through the points have been constructed using the analytic representation given by Eq. (3.18).

We have analyzed similar families of curves for values of r as large as 499 and found the same general behavior. For $c = r$ the density of eigenvalues is sharply peaked about $q = 0$. For increasing (or decreasing) values of c the peak value of the density function decreases rapidly until, in the asymptotic region, the entire curve reduces to a straight line with a constant value over the entire energy range. The agreement between the results of the numerical diagonalization and the calculation based on the explicit expression (3.18) is quite good.

For $c = r$, with the exception of the excluded region of the energy axis $|q| < (\frac{1}{2}r)^{1/2}$ discussed in Sec. III, the agreement between the numerical values of the density function and the values obtained from Eq. (3.18) is better than 1%. For $c > r$ we found essentially no difference between the two results beyond the standard roundoff error of the numerical calculation.

The constant asymptotic value of the density function for $c \gg r$ and for $c = -r + p$ with $p \ll 2r$ agrees with the predicted inverse spacing between the consecutive eigenvalues given by Eqs. (1.20) and (1.28) as well as with the values predicted by Eq. (3.18).

To test the accuracy of the logarithmic approximation to the elliptic integral given by Eq. (3.32), we have compared the prediction of Eq. (3.22) with the density of eigenvalues for $c = r = 499$ in resonance. The result is shown in Fig. 5. As ex-

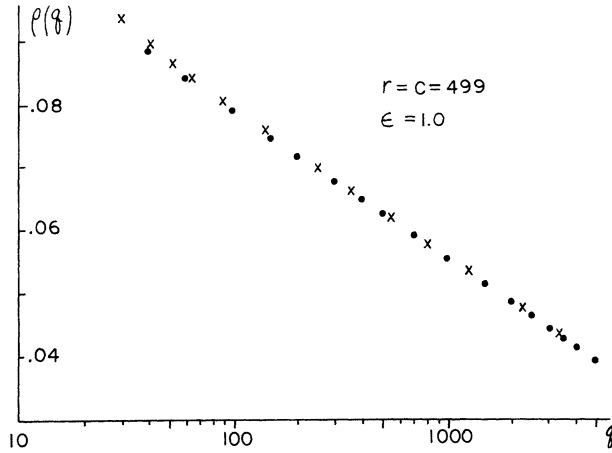


FIG. 5. Test of the accuracy of the logarithmic approximation for the density of eigenvalues. The density of eigenvalues is plotted as a function of the energy using the results of the numerical diagonalization (dots) and the logarithmic approximation of Eq. (3.32) (crosses). These results correspond to $c = r = 499$ and $\delta = 0$.

pected, in the energy range $(\frac{1}{2}r)^{1/2} < q < q_{\max}$ the agreement is quite satisfactory.

An example of the behavior of the density of eigenvalues for nonresonant interaction is shown in Fig. 6 for the special case of $c = r = 100$. We have not shown the results of our computer calculation for $c > r$ or $c < r$ because they are qualitatively similar to those discussed in the case of resonant interaction.

A somewhat surprising result of this investigation is that the peak of the density function is very insensitive to rather large variations of the detuning parameter, to the point that we have been forced to display the three cases shown in Fig. 6 using different systems of axes to avoid complete overlapping of the three curves. The numerical values of the densities corresponding to the different values of ϵ are only slightly different from one another around the midpoint of the spectrum. The relative displacement of the peaks, however, is such a small fraction of the entire energy spectrum that it is unobservable on any reasonable scale. More pronounced differences can be found in the neighborhood of the upper and lower bounds of the energy spectrum. These differences, it will be noticed, are consistent with the symmetry relation which we have discussed above in connection with the energy eigenvalues pertaining to detuning parameters of the form $\epsilon = 1 \pm \delta\epsilon$.

Note added in proof. One of the authors (M.O.) wishes to thank Professor G. Scharf for bringing to our attention that the Dicke Model in the non-resonant case has also been treated by Dr. R. Weiss in a paper appearing in the Helvetica Physica Acta.

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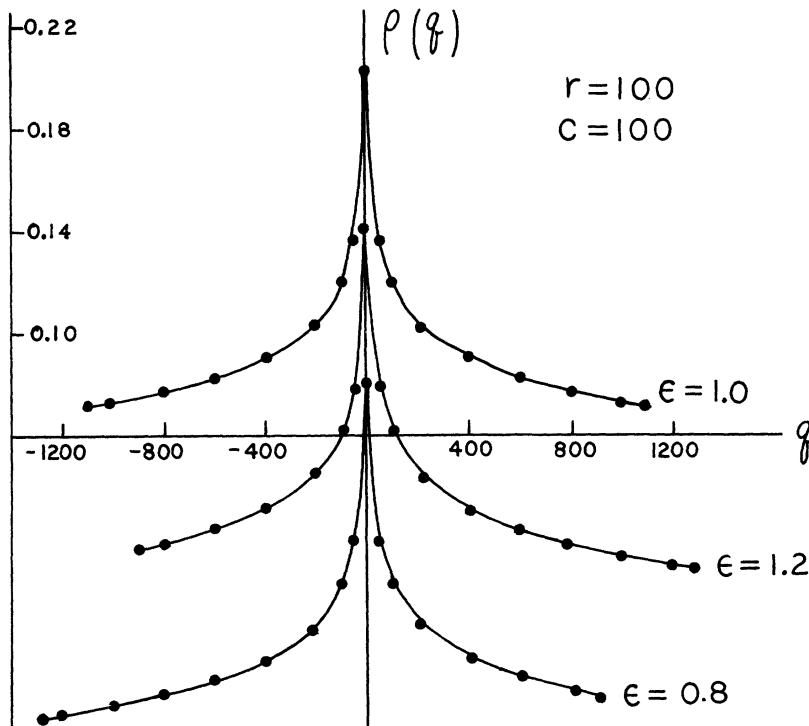


FIG. 6. Behavior of the density of eigenvalues for $c = r = 100$ and variable detuning parameters. The three curves shown above have been shifted along the vertical axis to allow a comparison of the different cases. The vertical scale refers to the curve with $\epsilon = 1.0$. There are very small differences among the three curves in the high-density region. More appreciable differences become apparent around the upper- and lower-energy ranges. As with Fig. 4, the dots represent the results of the numerical diagonalization while the solid lines are obtained using Eq. (3.18).

useful discussions and comments. Thanks are also due to Gordon L. Sturm for his generous help with some aspects of the numerical calculations.

APPENDIX A

The validity of the procedure outlined in Sec. III hinges on the behavior of the roots of the equation

$$V(z, q) = \mathcal{E}, \quad (\text{A1})$$

where

$$V(z, q) = \frac{(c-r)^2 - 1}{4} \frac{1}{z^2} + \frac{z^2}{4} + \frac{2q - 2\gamma\delta}{z} - \delta z, \quad (\text{A2})$$

and

$$\gamma = -\frac{1}{2}(r - c + 1), \quad \delta = (1 - \epsilon)/2\kappa,$$

$$\mathcal{E} = \frac{1}{2}(3r + c + 2) - \delta^2. \quad (\text{A3})$$

Except for the special cases in which $c = r$ and $|c - r| = 1$, the behavior of $V(z, q)$ is qualitatively shown in Fig. 7, where

$$V(z, q) \xrightarrow{z \rightarrow 0} \frac{1}{4}[(c-r)^2 - 1](z^2)^{-1},$$

$$V(z, q) \xrightarrow{z \rightarrow \infty} \frac{1}{4}z^2.$$

For all values of r and c except those mentioned above we can easily prove (e.g., by numerical methods) that there exist four real roots z_1, z_2, z_3 , and z_4 . The two classical turning points in question are z_1 and z_2 .

The two situations deserving special attention correspond to $c = r$ and $|c - r| = 1$.

1. $c = r$

From the form of the "potential function" (A2) for $c = r$, we recognize the existence of a maximum in a region of the z axis, where we can approximate $V(z, q)$ with

$$V(z, q) \approx -(4z^2)^{-1} + (2q + \delta)/z. \quad (\text{A4})$$

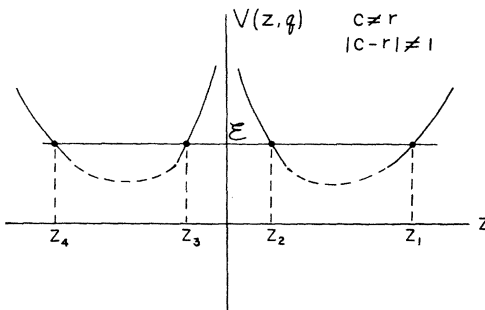


FIG. 7. Qualitative behavior of the "potential-energy function" $V(z, q)$ for $c \neq r$ and $c - r \neq 1$. The intercepts of $V(z, q)$ with the line $\mathcal{E} = \text{const.}$ are real. The classical turning points of the WKB integral are z_1 and z_2 .

The coordinate of the maximum is

$$z_{\text{max}} = [2(2q + \delta)]^{-1}, \quad (\text{A5})$$

and the value of $V(z, q)$ at $z = z_{\text{max}}$ is given with an excellent approximation by

$$V(z_{\text{max}}) = (2q + \delta)^2. \quad (\text{A6})$$

Two real turning points exist if we require that $\mathcal{E} \leq V(z_{\text{max}})$. This condition is satisfied provided we restrict the range of the eigenvalues as indicated below

$$q \geq |-\frac{1}{2}\delta + \frac{1}{2}[(2r + 1) - \delta^2]^{1/2}|. \quad (\text{A7})$$

We have thus identified a range of values of q for which the approximation of the WKB integral fails. The forbidden range of eigenvalues is

$$\Delta q = [(2r + 1) - \delta^2]^{1/2},$$

around $q = -\frac{1}{2}\delta$. As we pointed out in Sec. III, it may appear that for large values of r the restriction imposed by the WKB method is rather severe. In reality, for $c = r$ the largest eigenvalue is of the order of $r^{3/2}$. Therefore, the excluded range is only a fraction of order r^{-1} of the entire energy spectrum. The existence of two real turning points whenever the condition (A7) is fulfilled can be easily tested by numerical methods.

2. $|c - r| = 1$

Here we must distinguish two cases: $\gamma = 0$ and $\gamma = -1$. In the first case the "potential function" (A2) takes the form

$$V(z) = \frac{1}{4}z^2 + (2q/z) - \delta z. \quad (\text{A8})$$

The equation $V(z, q) = \mathcal{E}$ has a root $z = 0$. In addition, as shown qualitatively in Fig. 8, there are three other real roots, of which z_1 and z_2 are the

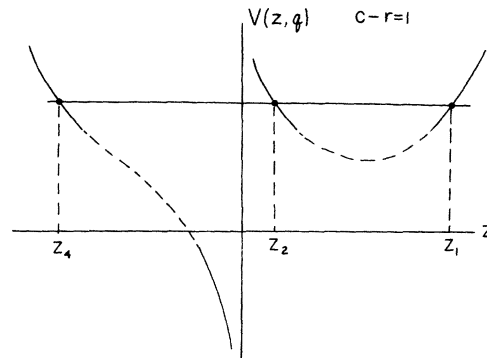


FIG. 8. Qualitative behavior of the "potential-energy function" $V(z, q)$ for $c - r = 1$. The intercepts z_1 and z_2 with the line $\mathcal{E} = \text{const.}$ are the classical turning points of the WKB integral.

turning points in question.

Concerning the reality of the intercepts of the line $\mathcal{E} = \frac{1}{2}(3\gamma + c + 2)$ with $V(z, q)$, this will always be the case if $q \neq 0$. Hence, with the possible exception of the eigenvalue $q = 0$, the two classical turning points of the WKB approximation exist and are real over the entire range of the energy eigenvalues.

In the case $\gamma = -1$, the potential takes the form

$$V(z, q) = \frac{1}{4}z^2 + [(2q + 2\delta)/z] - \delta z \quad (\text{A9})$$

Again we recognize that $z = 0$ is a root of the original equation $V(z, q) = \mathcal{E}$. In addition, as with the case $\gamma = 0$, the two real classical turning points exist for all values of q except $q = -\delta$.

APPENDIX B

In this Appendix we outline the method of solution of the quartic equation associated with the integral of Eq. (3.14). We want to find the roots of the equation

$$z^4 + Pz^3 + Qz^2 + Rz + S = 0, \quad (\text{B1})$$

where the coefficients are given in terms of the parameters of the problem as indicated below:

$$P = -4\delta, \quad Q = -4\mathcal{E}, \quad (\text{B2})$$

$$R = 8(q - \gamma\delta), \quad S = 4\gamma(\gamma + 1).$$

The present method of solution is based on the knowledge of the roots of the so-called cubic resolvent¹⁵

$$y^3 - (3P^2 - 8Q)y^2 + (3P^4 - 16P^2Q + 16Q^2 + 16PR - 64S)y - (P^3 - 4PQ + 8R)^2 = 0. \quad (\text{B3})$$

If the three roots of the cubic resolvent are y_1 , y_2 , and y_3 , it is proven in Ref. 15 that the four roots of the quartic equation take the form

$$\begin{aligned} z_1 &= \frac{1}{4}(-P + \sqrt{y_1} + \sqrt{y_2} + \sqrt{y_3}) \\ z_2 &= \frac{1}{4}(-P + \sqrt{y_1} - \sqrt{y_2} - \sqrt{y_3}) \\ z_3 &= \frac{1}{4}(-P - \sqrt{y_1} + \sqrt{y_2} - \sqrt{y_3}) \\ z_4 &= \frac{1}{4}(-P - \sqrt{y_1} - \sqrt{y_2} + \sqrt{y_3}). \end{aligned} \quad (\text{B4})$$

It follows that, in terms of the roots of the cubic resolvent, the density of eigenvalues can be expressed in the form

$$\rho(q) = \frac{8}{\pi} \frac{1}{(y_1 - y_3)^{1/2}} K\left(\frac{y_2 - y_3}{y_1 - y_3}\right), \quad (\text{B5})$$

where, as indicated in the main text, K stands for

the complete elliptic integral of the first kind.

Next we derive explicit expressions for the roots of the cubic resolvent. To simplify the comparison of the present results to the resonant case in which $\delta = 0$, we write the cubic resolvent in the form

$$y^3 - 32\mathcal{E}'y^2 + (16)^2(\mathcal{E}'^2 - L'^2)y - [32(2q')]^2 = 0, \quad (\text{B6})$$

where the primed parameters are defined as

$$\begin{aligned} \mathcal{E}' &= \mathcal{E} + \frac{3}{2}\delta^2, \\ L'^2 &= \gamma(\gamma + 1) + \delta(2q) - \delta^2(\mathcal{E} + 2\gamma) - \frac{3}{4}\delta^4, \\ 2q' &= 2q - 2\delta^3 - 2\delta(\mathcal{E} + \gamma). \end{aligned} \quad (\text{B7})$$

By means of the standard change of variable

$$x = \frac{1}{16}y - \frac{2}{3}\mathcal{E}', \quad (\text{B8})$$

the cubic equation is reduced to normal form

$$x^3 - (L'^2 + \frac{1}{3}\mathcal{E}'^2)x + [\frac{2}{27}(\mathcal{E}')^3 - \frac{2}{3}\mathcal{E}'L'^2 - \frac{1}{4}(2q')^2] = 0. \quad (\text{B9})$$

The roots of the cubic equation

$$x^3 + 3ax + 2b = 0, \quad (\text{normal form}) \quad (\text{B10})$$

where

$$\begin{aligned} a &= -q^{-1}(\mathcal{E}'^2 + 3L'^2), \\ b &= \frac{1}{27}\mathcal{E}'^3 - \frac{1}{3}(\mathcal{E}'L'^2) - \frac{1}{8}(2q')^2 \end{aligned} \quad (\text{B11})$$

can be easily cast in trigonometric form provided $a^3 + b^2 < 0$. If we let

$$\cos\phi = -(b^2/a^3)^{1/2},$$

the roots of Eq. (B10) become

$$\begin{aligned} x_1 &= 2(-a)^{1/2} \cos(\frac{1}{3}\phi), \\ x_2 &= 2(-a)^{1/2} \cos[\frac{1}{3}(\phi + 4\pi)], \\ x_3 &= 2(-a)^{1/2} \cos[\frac{1}{3}(\phi + 2\pi)]. \end{aligned} \quad (\text{B12})$$

In conclusion, the roots of the cubic resolvent are explicitly given by

$$\begin{aligned} y_1 &= 16[\frac{2}{3}(\mathcal{E}'^2 + 3L'^2)^{1/2} \cos(\frac{1}{3}\phi) + \frac{2}{3}\mathcal{E}'], \\ y_2 &= 16[\frac{2}{3}(\mathcal{E}'^2 + 3L'^2)^{1/2} \cos\frac{1}{3}(\phi + 4\pi) + \frac{2}{3}\mathcal{E}'], \\ y_3 &= 16[\frac{2}{3}(\mathcal{E}'^2 + 3L'^2)^{1/2} \cos\frac{1}{3}(\phi + 2\pi) + \frac{2}{3}\mathcal{E}'], \end{aligned}$$

where

$$\cos\phi = -\frac{\frac{1}{27}\mathcal{E}'^3 - \frac{1}{3}(\mathcal{E}'L'^2) - \frac{1}{8}(2q')^2}{[\frac{1}{3}(\mathcal{E}'^2 + 3L'^2)]^{3/2}}.$$

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