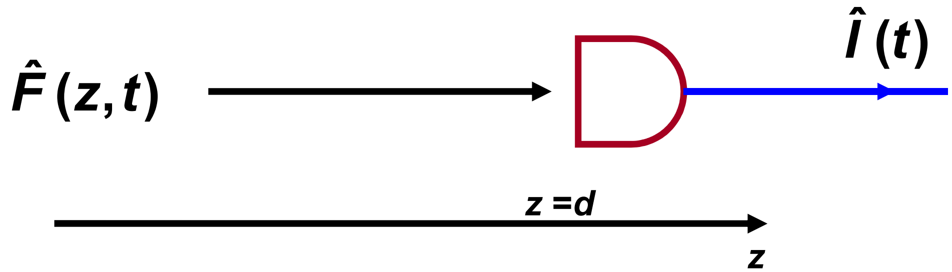
Chapter 11: Coherences, Correlation Functions, Measurement and Detection

In this lecture you will learn:

- **Direct Photodetection and Noise**
- **Coherence and Correlation Functions in Quantum Optics**
- **Photon Bunching and Anti-bunching**
- **Quantum Single Photon Emitters**
- **Hanbury Brown and Twiss Setup**
- **Photon Counting Distributions**
- **Homodyne and Heterodyne Detection**
- **Simultaneous Measurement of Two Non-Commuting Observables**
- **Optical Phase Detection and Standard Quantum Limit (SQL)**
- **Resonance Fluorescence and Mollow Spectrum**

Direct Photodetection



Assuming 100% detector efficiency, the detector current operator is defined as:

$$\hat{I}(t) = q\,\hat{F}(z_d,t) = q\,v_g\,\hat{a}^{+}(z_d,t)\,\hat{a}(z_d,t)$$

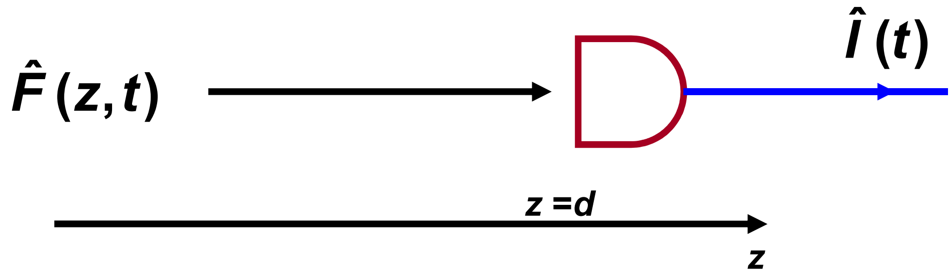
Example: A CW Coherent State:

$$\alpha(z) = \sqrt{\frac{P_o}{v_g \hbar \omega_o}}\, e^{i\phi}$$

$$\Rightarrow \left\langle \hat{F}(z,t) \right\rangle = \frac{P_o}{\hbar \omega_o}$$

$$\Rightarrow \left\langle \Delta \hat{F}(z,t_1)\,\Delta \hat{F}(z,t_2) \right\rangle = \frac{P_o}{h\omega_o}\,\delta(t_1 - t_2)$$

Direct Photodetection



Therefore:

$$\left\langle \hat{I}(t) \right\rangle = q \left\langle \hat{F}(z_d, t) \right\rangle = \frac{q}{\hbar\omega_o} P_o$$

$$\Rightarrow \left\langle \Delta\hat{I}(t_1)\Delta\hat{I}(t_2) \right\rangle = q^2 \left\langle \Delta\hat{F}(z,t_1)\ \Delta\hat{F}(z,t_2) \right\rangle = q^2 \frac{P_o}{h\omega_o} \delta(t_1 - t_2) = q \left\langle \hat{I}(t) \right\rangle \delta(t_1 - t_2)$$

Current Noise Spectral Density:

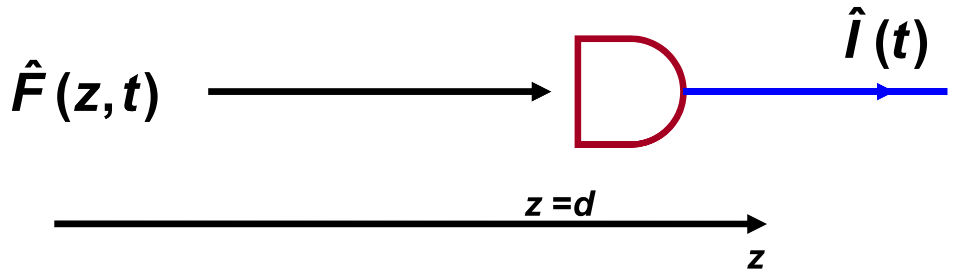
$$S_{\Delta I\,\Delta I}(\omega) = \int_{-\infty}^{\infty} d\tau\, e^{i\omega\tau} \left\langle \Delta\hat{I}(t+\tau)\Delta\hat{I}(t) \right\rangle$$

$$= q^2 \frac{P_o}{\hbar\omega_o} = q \left\langle \hat{I}(t) \right\rangle \longrightarrow \text{Shot Noise}$$

Symmetric correlation function is used when order matters:

$$\frac{\left\langle \hat{I}(t_1)\hat{I}(t_2) + \hat{I}(t_2)\hat{I}(t_1) \right\rangle}{2}$$

Direct Photodetection



Example: A CW Amplitude Squeezed State:

$$\left|\psi(t=0)\right\rangle = \left|\alpha(z),\varepsilon(z)\right\rangle = \hat{T}(\alpha)\,\hat{S}(\varepsilon)\left|0\right\rangle$$

$$\hat{S}(\varepsilon) = e^{\int_{-\infty}^{\infty} dz' \left(\frac{\varepsilon^*(z')}{2}\left(\hat{a}(z',0)\right)^2 - \frac{\varepsilon(z')}{2}\left(\hat{a}^+(z',0)\right)^2 \right)} \qquad \left\{ \varepsilon(z) = r\,e^{i2\phi} = \text{constant} \right.$$

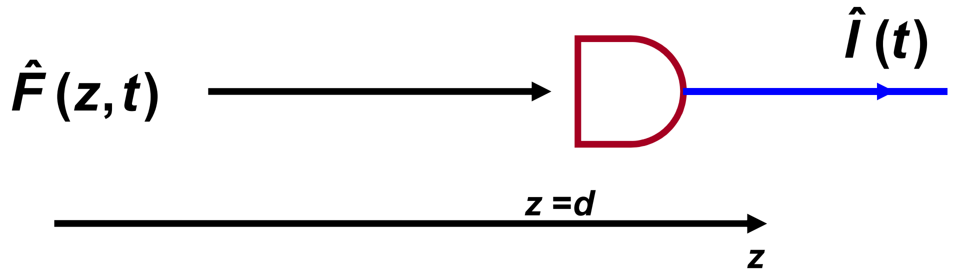
$$\hat{T}(\alpha) = e^{\int_{-\infty}^{\infty} dz' \left(\alpha(z')\,\hat{a}^+(z',0) - \alpha^*(z')\,\hat{a}(z',0) \right)} \qquad \left\{ \alpha(z) = |\alpha|\,e^{i\phi} = \text{constant} \right.$$

Average Current:

$$\left\langle \hat{I}(t) \right\rangle = q\left\langle \hat{F}(z_d,t)\right\rangle = qv_g|\alpha|^2 + qv_g \sinh^2(r)\frac{\Delta\beta}{2\pi} \approx qv_g|\alpha|^2$$

$$\left\{ \begin{array}{l} \text{If:} \\ |\alpha|^2 \gg \Delta\beta \end{array} \right.$$

Direct Photodetection



Current Noise:

$$\left\langle \Delta\hat{I}(t_1)\Delta\hat{I}(t_2)\right\rangle = q^2\left\langle \Delta\hat{F}(z,t_1)\ \Delta\hat{F}(z,t_2)\right\rangle = q^2 v_g |\alpha|^2 e^{-2r}\delta(t_1 - t_2)$$
$$+ 2q^2 v_g \sinh^2(r)\cosh^2(r)\frac{\Delta\beta}{2\pi}\delta(t_1 - t_2)$$

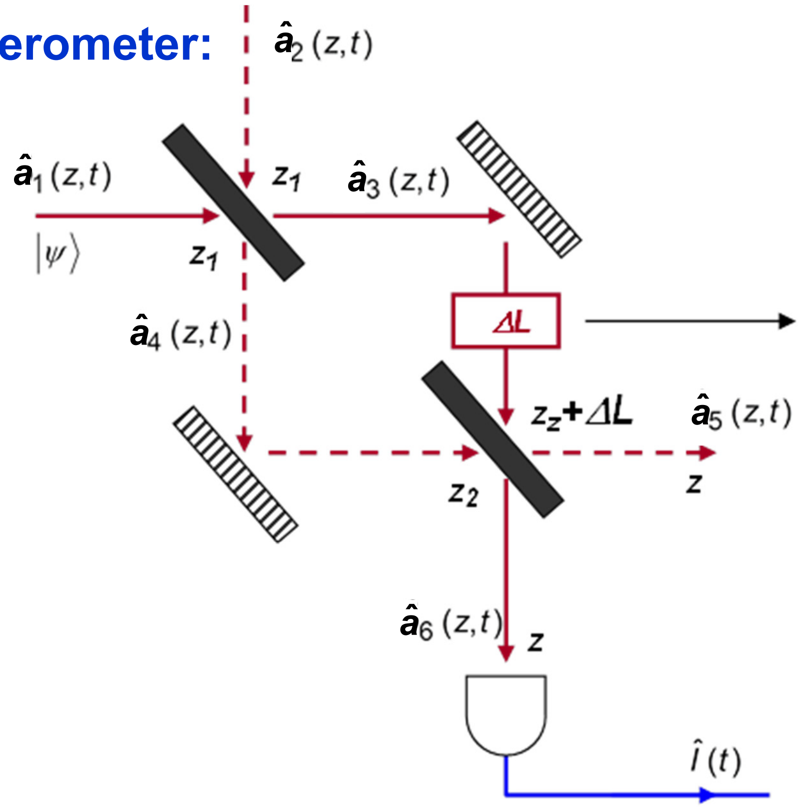
If: $|\alpha|^2 >> \Delta\beta$

$$\left\langle \Delta\hat{I}(t_1)\Delta\hat{I}(t_2)\right\rangle \approx \underbrace{q\left\langle \hat{I}(t)\right\rangle e^{-2r}}_{\text{Less than shot noise!}}\delta(t_1 - t_2) \longrightarrow \text{Sub-Poissonian photon statistics}$$

Coherence and Correlation Functions

Consider the following interferometer:

$$\langle \hat{F}_1(z_1,t)\rangle = F_o$$



$$\tau = \frac{\Delta L}{v_g}$$

Beam Splitter Relations:

1st:

$$\begin{bmatrix} \hat{a}_3(z_1,t)\, e^{i\beta_o z_1} \\ \hat{a}_4(z_1,t)\, e^{i\beta_o z_1} \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} \hat{a}_1(z_1,t)\, e^{i\beta_o z_1} \\ \hat{a}_2(z_1,t)\, e^{i\beta_o z_1} \end{bmatrix}$$

2nd:

$$\begin{bmatrix} \hat{a}_5(z_2,t)\, e^{i\beta_o z_2} \\ \hat{a}_6(z_2+\Delta L,t)\, e^{i\beta_o (z_2+\Delta L)} \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} \hat{a}_4(z_2,t)\, e^{i\beta_o z_2} \\ \hat{a}_3(z_2+\Delta L,t)\, e^{i\beta_o (z_2+\Delta L)} \end{bmatrix}$$

Coherence and Correlation Functions

The average current is a function of the path difference:

$$\frac{\left\langle \hat{I}_\tau\left(t+\frac{z_2-z_1}{v_g}+\tau\right)\right\rangle}{q}=\left\langle \hat{F}_6\left(z_2+\Delta L, t+\frac{z_2-z_1}{v_g}+\tau\right)\right\rangle$$

$$=\frac{1}{4}\left[\left\langle \hat{F}_1(z_1,t+\tau)\right\rangle+\left\langle \hat{F}_1(z_1,t)\right\rangle+v_g\left\langle \hat{a}_1^+(z_1,t+\tau)\hat{a}_1(z_1,t)\right\rangle+v_g\left\langle \hat{a}_1^+(z_1,t)\hat{a}_1(z_1,t+\tau)\right\rangle\right]$$

$$=\frac{1}{4}\left[2F_o+v_g\left\langle \hat{a}_1^+(z_1,t+\tau)\hat{a}_1(z_1,t)\right\rangle+v_g\left\langle \hat{a}_1^+(z_1,t)\hat{a}_1(z_1,t+\tau)\right\rangle\right]$$

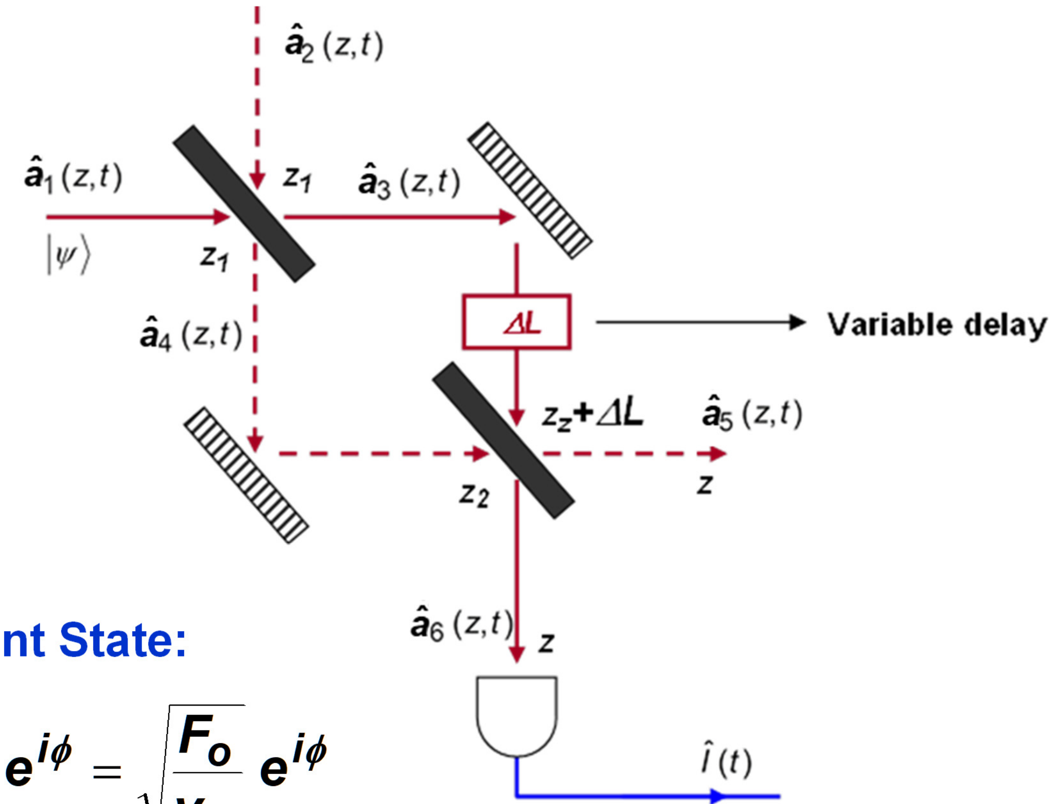
$$=\frac{F_o}{2}\left[1+\frac{v_g\left\langle \hat{a}_1^+(z_1,t+\tau)\hat{a}_1(z_1,t)\right\rangle}{2F_o}+\frac{v_g\left\langle \hat{a}_1^+(z_1,t)\hat{a}_1(z_1,t+\tau)\right\rangle}{2F_o}\right]$$

Where we have assumed that for free-space:

$$\omega_o=v_g\beta_o=c\beta_o$$

Coherence and Correlation Functions

$$\left\langle \hat{F}_1(z_1,t) \right\rangle = F_o$$



$$\tau = \frac{\Delta L}{v_g}$$

Consider a CW Coherent State:

$$\alpha(z) = \sqrt{\frac{P_o}{v_g \hbar \omega_o}}\, e^{i\phi} = \sqrt{\frac{F_o}{v_g}}\, e^{i\phi}$$

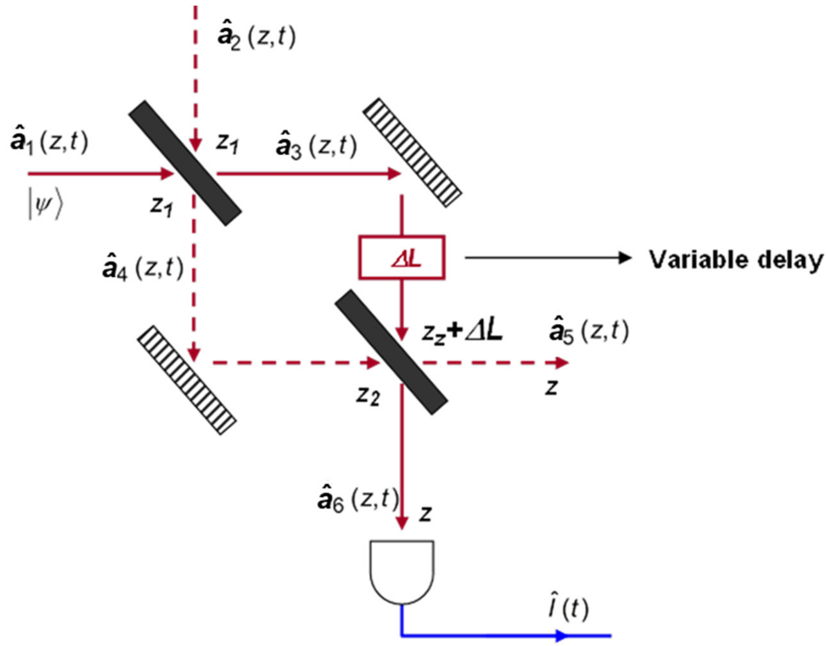
Then:

$$\left\langle \hat{a}_1^{+}(z_1, t+\tau)\, \hat{a}_1(z_1, t) \right\rangle = \frac{F_o}{v_g} e^{i\omega_o \tau}$$

$$\left\langle \hat{a}_1^{+}(z_1, t)\, \hat{a}_1(z_1, t+\tau) \right\rangle = \frac{F_o}{v_g} e^{-i\omega_o \tau}$$

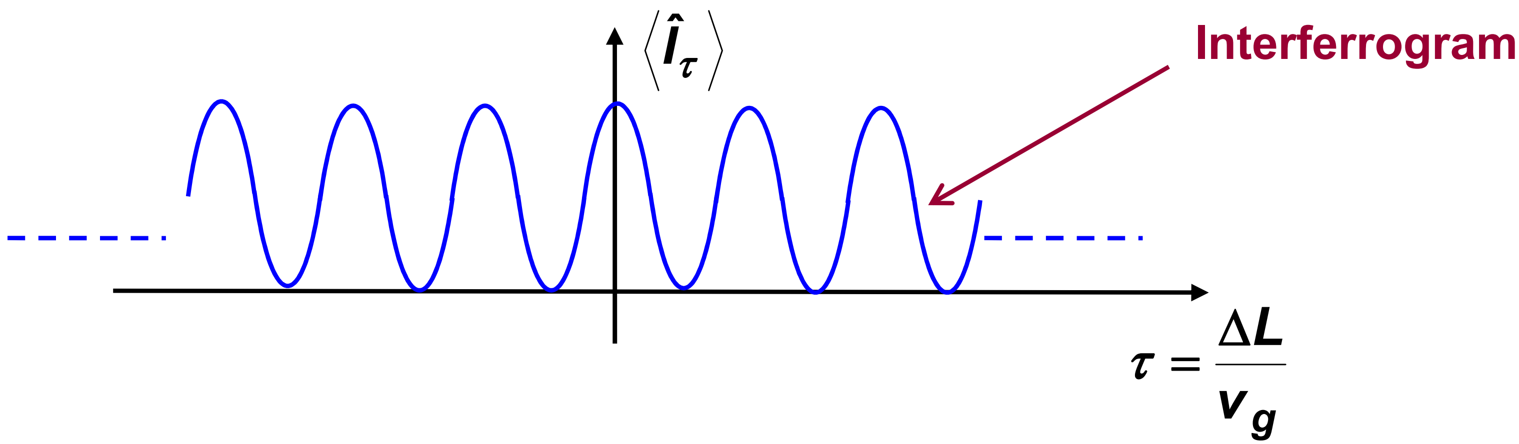
Coherence and Correlation Functions

$$\left\langle \hat{F}_1(z_1,t)\right\rangle = F_o$$

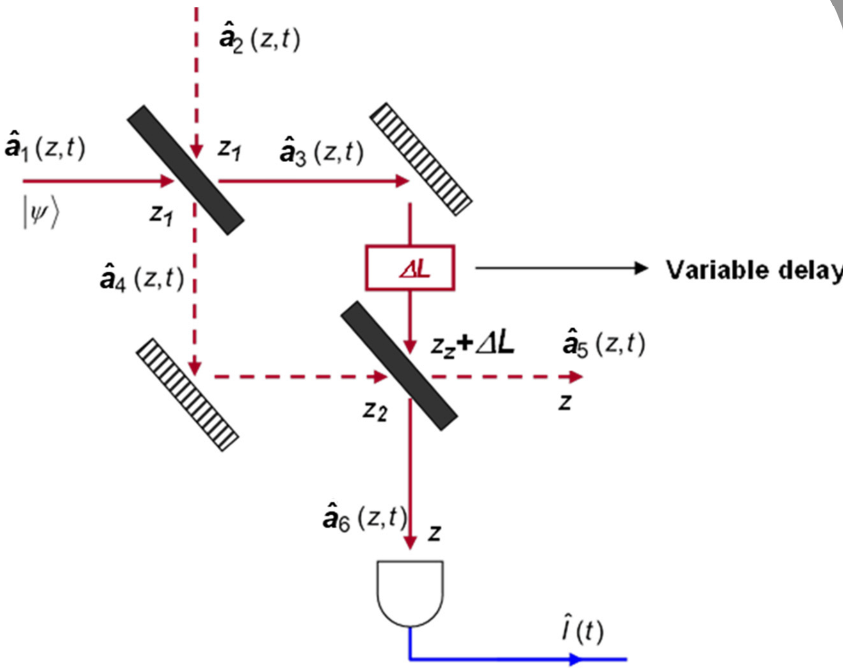


$$\tau = \frac{\Delta L}{v_g}$$

$$\frac{\left\langle \hat{I}_\tau\left(t+\frac{z_2-z_1}{v_g}+\tau\right)\right\rangle}{q} = \frac{F_o}{2}\left[1+\cos\left(\omega_o\tau\right)\right] = \frac{F_o}{2}\left[1+\cos\left(\omega_o\tau\right)\right]$$



Coherence and Correlation Functions



Now consider a random phase coherent state as the input:

$$\alpha(z) = \sqrt{\frac{F_o}{v_g}}\, e^{i\phi(z)}$$

Phase Correlation Function:

$$\left\langle e^{-i\phi(z)} e^{i\phi(z')} \right\rangle = e^{-\frac{|z-z'|}{L\phi}}$$

We get:

$$\left\langle \hat{a}_1^+(z_1, t+\tau)\, \hat{a}_1(z_1, t) \right\rangle = \left\langle \hat{a}_1^+\big(z_1 - v_g\tau, t\big)\, \hat{a}_1(z_1, t) \right\rangle e^{i\omega_o\tau} = \frac{F_o}{v_g} \left\langle e^{-i\varphi(z_1 - v_g\tau)} e^{i\varphi(z_1)} \right\rangle e^{i\omega_o\tau}$$

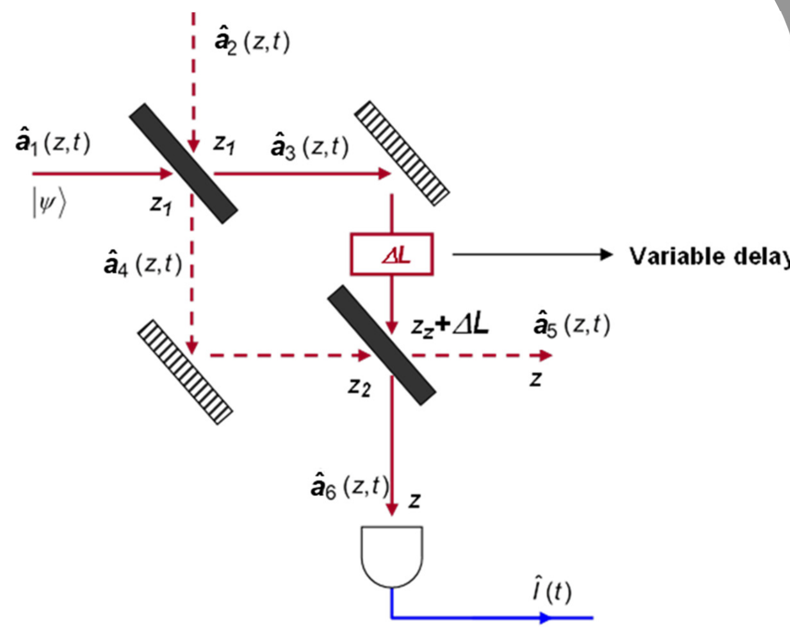
$$= \frac{F_o}{v_g} e^{-\frac{|\Delta L|}{L_\varphi}} e^{i\omega_o\tau}$$

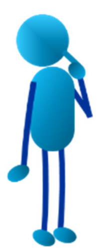
$$\left\langle \hat{b}_1^+(z_1, t)\, \hat{b}_1(z_1, t+\tau) \right\rangle = \frac{F_o}{v_g} e^{-\frac{|\Delta L|}{L_\varphi}} e^{i\omega_o\tau}$$

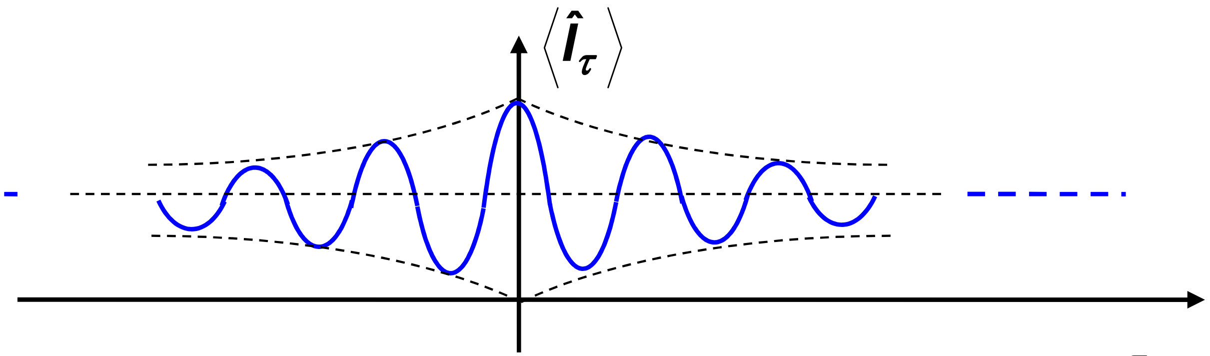
Coherence and Correlation Functions

The current is:

$$\frac{\left\langle \hat{I}_\tau\left(t+\frac{z_2-z_1}{v_g}+\tau\right)\right\rangle}{q}=\frac{F_o}{2}\left[1+e^{-\frac{v_g|\tau|}{L_\varphi}}\cos\left(\omega_o\tau\right)\right]$$







$$\tau=\frac{\Delta L}{v_g}$$

Measures temporal phase coherence:

$$\left\langle \hat{a}_1^{+}(z_1,t+\tau)\,\hat{a}_1(z_1,t)\right\rangle \qquad \left\langle \hat{a}_1^{+}(z_1,t)\,\hat{a}_1(z_1,t+\tau)\right\rangle$$

First Order Coherence Function

Start from:

$$\hat{\vec{E}}(\vec{r},t) = \hat{\vec{E}}_{+}(\vec{r},t) + \hat{\vec{E}}_{-}(\vec{r},t)$$

$$\hat{\vec{E}}_{+}(\vec{r},t) = i\sqrt{\frac{\hbar\omega(\beta_o)}{2\varepsilon_o\varepsilon}}\,\vec{\varphi}(x,y,\beta_o)\,\hat{a}(z,t)\,e^{i\beta_o z}$$

$$\hat{\vec{E}}_{-}(\vec{r},t) = \left(\hat{\vec{E}}_{+}(\vec{r},t)\right)^{+}$$

The first order coherence function is defined as ($\hat{n}$ is the polarization of the detector):

$$g_1(\vec{r}:t_1,t_2) = \frac{\left\langle \hat{n}.\hat{\vec{E}}_{-}(\vec{r},t_1)\,\hat{n}.\hat{\vec{E}}_{+}(\vec{r},t_2)\right\rangle}{\sqrt{\left\langle \hat{n}.\hat{\vec{E}}_{-}(\vec{r},t_1)\,\hat{n}.\hat{\vec{E}}_{+}(\vec{r},t_1)\right\rangle\left\langle \hat{n}.\hat{\vec{E}}_{-}(\vec{r},t_2)\,\hat{n}.\hat{\vec{E}}_{+}(\vec{r},t_2)\right\rangle}}$$

Or dropping the vector notation:

$$g_1(\vec{r}:t_1,t_2) = \frac{\left\langle \hat{E}_{-}(\vec{r},t_1)\,\hat{E}_{+}(\vec{r},t_2)\right\rangle}{\sqrt{\left\langle \hat{E}_{-}(\vec{r},t_1)\,\hat{E}_{+}(\vec{r},t_1)\right\rangle\left\langle \hat{E}_{-}(\vec{r},t_2)\,\hat{E}_{+}(\vec{r},t_2)\right\rangle}}$$

First Order Coherence Function

The first order coherence function can be written as:

$$g_1\left(z:t_1,t_2\right)=\frac{\left\langle \hat{a}^{+}\left(z,t_1\right)\hat{a}\left(z,t_2\right)\right\rangle}{\sqrt{\left\langle \hat{a}^{+}\left(z,t_1\right)\hat{a}\left(z,t_1\right)\right\rangle\left\langle \hat{a}^{+}\left(z,t_2\right)\hat{a}\left(z,t_2\right)\right\rangle}}$$

$$\left|\left\langle \hat{a}^{+}\left(z,t_1\right)\hat{a}\left(z,t_2\right)\right\rangle\right|^2 \le \left\langle \hat{a}^{+}\left(z,t_1\right)\hat{a}\left(z,t_1\right)\right\rangle \left\langle \hat{a}^{+}\left(z,t_2\right)\hat{a}\left(z,t_2\right)\right\rangle$$

$$\Rightarrow \left|g_1\left(z:t_1,t_2\right)\right| \le 1$$

Field has first order coherence if: $\left|g_1\left(z:t_1,t_2\right)\right| = 1$

Example: If:

$$\left\langle \hat{a}^{+}\left(z_1,t_1\right)\hat{a}\left(z_1,t_2\right)\right\rangle = \left\langle \hat{a}^{+}\left(z_1,t_1\right)\right\rangle \left\langle \hat{a}\left(z_1,t_2\right)\right\rangle$$

$$\Rightarrow \left|g_1\left(z:t_1,t_2\right)\right| = 1$$

First Order Coherence Function and Radiation Spectrum

Radiation spectrum is defined as:

$$S(\omega) = \int_{-\infty}^{\infty} d\tau \quad g_1(z:t,t+\tau)\; e^{i\omega\tau}$$

For a pure coherent state:

$$S(\omega) = \int_{-\infty}^{\infty} d\tau \quad g_1(z:t,t+\tau)\; e^{i\omega\tau}$$

$$= \int_{-\infty}^{\infty} d\tau \quad e^{-i\omega_o\tau}\; e^{i\omega\tau}$$

$$= 2\pi\,\delta(\omega - \omega_o)$$

For a coherent state with random phase:

$$S(\omega) = \int_{-\infty}^{\infty} d\tau \quad g_1(z:t,t+\tau)\, e^{i\omega\tau}$$

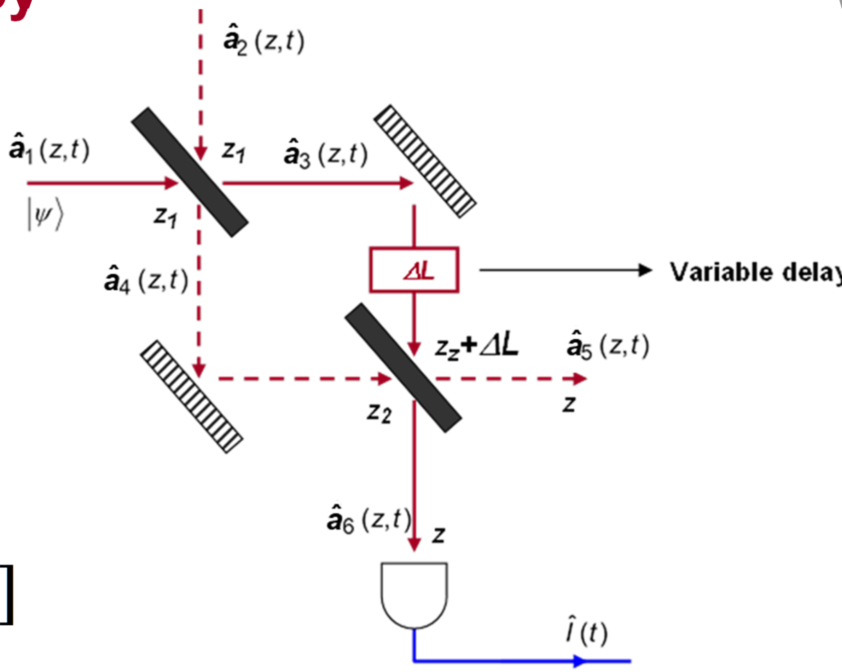
$$= \int_{-\infty}^{\infty} d\tau \quad e^{-\frac{v_g|\tau|}{L_\phi}}\; e^{-i\omega_o\tau}\; e^{i\omega\tau}$$

$$= 2\pi\,\frac{v_g/(L_\phi\pi)}{(\omega-\omega_o)^2 + (v_g/L_\phi)^2}$$

First Order Coherence Function and Fourier Transform Spectroscopy

Suppose:

$$\left\langle \hat{F}_1(z_1,t) \right\rangle = F_o$$



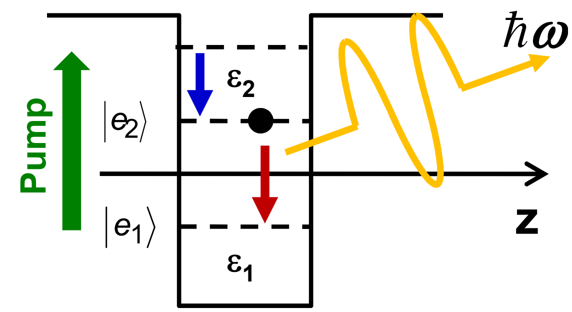
Then (assuming a non-dispersive medium):

$$\int_{-\infty}^{\infty} d\tau \ \left\langle \hat{I}_\tau \right\rangle e^{i\omega\tau} = \frac{F_o}{2}\left[2\pi\,\delta(\omega) + S(-\omega) + S(\omega)\right]$$

First Order Coherence Function of Spontaneous Emission

Consider a two-level system interaction with quantized radiation in free space

The pump is used to maintain an average non-zero population of the upper level (level 2)



$$\left\langle \hat{N}_2 \right\rangle_{\substack{steady\\state}} \neq 0$$

Suppose the Hamiltonian is:

$$\hat{H} = \varepsilon_1 |e_1\rangle\langle e_1| + \varepsilon_2 |e_2\rangle\langle e_2| - q\hat{\vec{E}}(\vec{r}).\hat{\vec{r}} + \hat{H}_{rad}$$

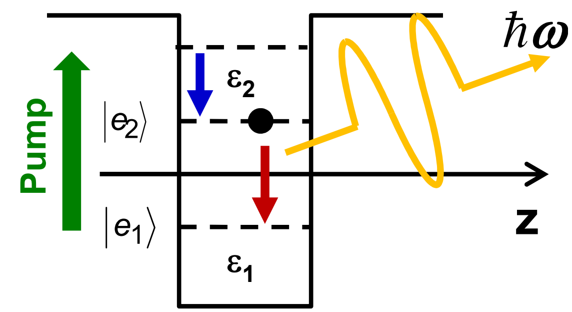
$$\hat{H}_{rad} = \sum_{j,\vec{k}} \hbar\omega_k \left[\hat{a}_j^+(\vec{k})\hat{a}_j(\vec{k}) + \frac{1}{2} \right]$$

$$\hat{\vec{E}}(\vec{r}) = \sum_{j,\vec{k}} i\sqrt{\frac{\hbar\omega_k}{2\varepsilon_o}} \left[\hat{a}_j(\vec{k}) - \hat{a}_j^+(-\vec{k}) \right] \frac{e^{i\vec{k}.\vec{r}}}{\sqrt{V}} \hat{\varepsilon}_j(\vec{k})$$

$$\hat{H}_{\text{int}} = \sum_{j,\vec{k}} \frac{\hbar}{\sqrt{V}} \left[\Pi_j(\vec{k})\hat{\sigma}_+ \hat{a}_j(\vec{k}) + \Pi_j^*(\vec{k})\hat{a}_j^+(\vec{k})\hat{\sigma}_- \right]$$

$$\Pi_j(\vec{k}) = -i\sqrt{\frac{\omega_k}{2\hbar\varepsilon_o}} \left[q\vec{d}.\hat{\varepsilon}_j(\vec{k}) \right] \qquad \left\{ q\vec{d} = q\hat{e}_z \langle e_2|\hat{z}|e_1\rangle \right.$$

First Order Coherence Function of Spontaneous Emission and Natural Linewidth



$$\omega_o = \frac{\Delta\varepsilon + \delta\varepsilon}{\hbar}$$

$$\Rightarrow \hat{a}_j^+\left(\vec{k},t\right) = \hat{a}_j^+\left(\vec{k}\right)e^{i\omega_k t} + i\frac{\Pi_j\left(\vec{k}\right)}{\sqrt{V}}\int_0^t dt'\ \hat{\sigma}_+(t')e^{i\omega_k(t-t')}$$

$$\Rightarrow \hat{a}_j\left(\vec{k},t\right) = \hat{a}_j\left(\vec{k}\right)e^{i\omega_k t} - i\frac{\Pi_j^*\left(\vec{k}\right)}{\sqrt{V}}\int_0^t dt'\ \hat{\sigma}_-(t')e^{-i\omega_k(t-t')}$$

Far-field:

$$\hat{\vec{E}}(\vec{r},t) = \sum_{j,\vec{k}} i\sqrt{\frac{\hbar\omega_k}{2\varepsilon_o}}\left[\hat{a}_j\left(\vec{k},t\right) - \hat{a}_j^+\left(-\vec{k},t\right)\right]\frac{e^{i\vec{k}.\vec{r}}}{\sqrt{V}}\hat{\varepsilon}_j\left(\vec{k}\right)$$

$$= \hat{\vec{E}}_+(\vec{r},t) + \hat{\vec{E}}_-(\vec{r},t)$$

Far-field radiation can be related to the dipole operator:

$$\hat{\vec{E}}_+(\vec{r},t) = \sum_{j,\vec{k}} i\sqrt{\frac{\hbar\omega_k}{2\varepsilon_o}}\left[\hat{a}_j\left(\vec{k}\right)e^{-i\omega_k t} - i\frac{\Pi_j^*\left(\vec{k}\right)}{\sqrt{V}}\int_0^t dt'\ \hat{\sigma}_-(t')e^{-i\omega_k(t-t')}\right]\frac{e^{i\vec{k}.\vec{r}}}{\sqrt{V}}\hat{\varepsilon}_j\left(\vec{k}\right)$$

$$\approx \hat{\vec{E}}_+^o(\vec{r},t) + \frac{\omega_o^2}{4\pi\varepsilon_o c^2 r}\left[\overline{\overline{1}} - \hat{r}\otimes\hat{r}\right].\left(q\vec{d}\right)\hat{\sigma}_-\left(t - \frac{r}{c}\right)$$

→ Far-field expression

Detour: Far-Field

$$\hat{\vec{E}}(\vec{r},t)=\sum_{j,\vec{k}} i\sqrt{\frac{\hbar\omega_k}{2\varepsilon_o}}\left[\hat{a}_j(\vec{k},t)-\hat{a}_j^{+}(-\vec{k},t)\right]\frac{e^{i\vec{k}.\vec{r}}}{\sqrt{V}}\hat{\varepsilon}_j(\vec{k})$$

$$=\hat{\vec{E}}_{+}(\vec{r},t)+\hat{\vec{E}}_{-}(\vec{r},t)$$

$$\hat{\vec{E}}_{+}(\vec{r},t)=\sum_{j,\vec{k}} i\sqrt{\frac{\hbar\omega_k}{2\varepsilon_o}}\left[\hat{a}_j(\vec{k})e^{i\omega_k t}-i\frac{\Pi_j^{*}(\vec{k})}{\sqrt{V}}\int_0^t dt'\ \hat{\sigma}_{-}(t')e^{-i\omega_k(t-t')}\right]\frac{e^{i\vec{k}.\vec{r}}}{\sqrt{V}}\hat{\varepsilon}_j(\vec{k})$$

Recall that: $\Pi_j(\vec{k})=-i\sqrt{\frac{\omega_k}{2\hbar\varepsilon_o}}q\vec{d}.\hat{\varepsilon}_j(\vec{k})$

Perform the k-space integration to get:

$$=\hat{\vec{E}}_{+}^{o}(\vec{r},t)+\int\frac{d^3\vec{k}}{(2\pi)^3}\frac{i\omega_k}{2\varepsilon_o}\int_0^t dt'\ \hat{\sigma}_{-}(t')e^{-i\omega_k(t-t')}e^{i\vec{k}.\vec{r}}\left[1-\hat{k}\otimes\hat{k}\right].q\vec{d}$$

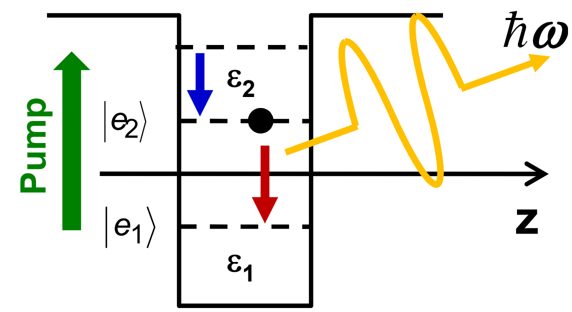
Detour: Far-Field

$$\hat{\vec{E}}_{+}(\vec{r},t) = \hat{\vec{E}}^{o}_{+}(\vec{r},t) + \int \frac{d^3\vec{k}}{(2\pi)^3} \frac{i\omega_k}{2\varepsilon_o} \int_0^t dt' \;\; \hat{\sigma}_{-}(t') e^{-i\omega_k(t-t')} e^{i\vec{k}.\vec{r}} \left[1 - \hat{k}\otimes\hat{k}\right].\left(q\vec{d}\right)$$

$$= \hat{\vec{E}}^{o}_{+}(\vec{r},t) + \frac{\omega_o^2}{4\pi\varepsilon_o c^2 r}\left[1 - \hat{r}\otimes\hat{r}\right].\left(q\vec{d}\right)\hat{\sigma}_{-}\left(t - \frac{r}{c}\right)$$

Far-field radiation can be related to the dipole operator as above

First Order Coherence Function of Spontaneous Emission and Natural Linewidth



First order field coherence function:

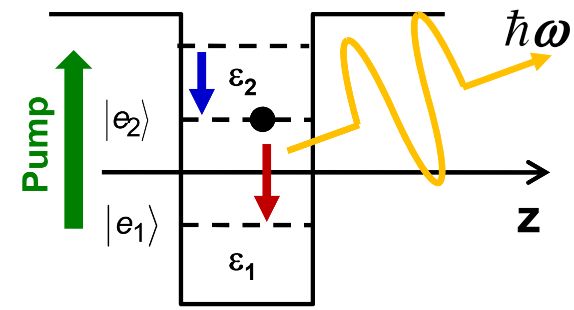
$$g_1(\vec{r}:t,t+\tau)=\frac{\left\langle \hat{E}_-(\vec{r},t)\hat{E}_+(\vec{r},t+\tau)\right\rangle}{\sqrt{\left\langle \hat{E}_-(\vec{r},t+\tau)\hat{E}_+(\vec{r},t+\tau)\right\rangle\left\langle \hat{E}_-(\vec{r},t)\hat{E}_+(\vec{r},t)\right\rangle}}$$

$$g_1(\vec{r}:t,t+\tau)=\frac{\left\langle \hat{\sigma}_+(t-r/c)\hat{\sigma}_-(t+\tau-r/c)\right\rangle}{\sqrt{\left\langle \hat{\sigma}_+(t+\tau-r/c)\hat{\sigma}_-(t+\tau-r/c)\right\rangle\left\langle \hat{\sigma}_+(t-r/c)\hat{\sigma}_-(t-r/c)\right\rangle}}$$

Assuming the correlation function is stationary:

$$g_1(\vec{r}:t,t+\tau)=\frac{\left\langle \hat{\sigma}_+(t)\hat{\sigma}_-(t+\tau)\right\rangle}{\sqrt{\left\langle \hat{\sigma}_+(t+\tau)\hat{\sigma}_-(t+\tau)\right\rangle\left\langle \hat{\sigma}_+(t)\hat{\sigma}_-(t)\right\rangle}}$$

First Order Coherence Function of Spontaneous Emission and Natural Linewidth



Start from:

$$\frac{d\hat{\sigma}_-(t)}{dt} = -\frac{i}{\hbar}(\Delta\varepsilon + \delta\varepsilon)\hat{\sigma}_-(t) - \frac{\hat{\sigma}_-(t)}{T_2} + \underbrace{\frac{i}{\sqrt{V}}\sum_{j,\vec{k}} \Pi_j(\vec{k})\,\hat{N}_d(t)\,\hat{a}_j(\vec{k})\,e^{-i\omega_k t}}_{\hat{G}_-(t)}$$

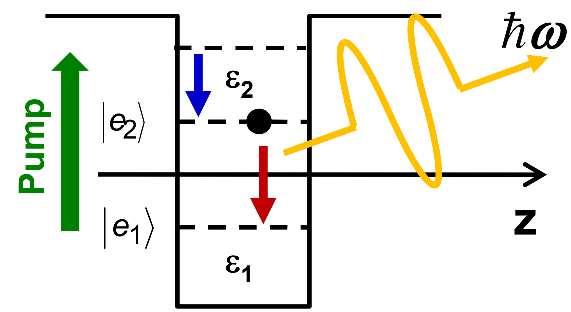
$$\frac{d\left\langle \hat{\sigma}_+(t)\hat{\sigma}_-(t+\tau)\right\rangle}{d\tau} = -\frac{i}{\hbar}(\Delta\varepsilon + \delta\varepsilon)\left\langle \hat{\sigma}_+(t)\hat{\sigma}_-(t+\tau)\right\rangle - \frac{\left\langle \hat{\sigma}_+(t)\hat{\sigma}_-(t+\tau)\right\rangle}{T_2} + \frac{i}{\sqrt{V}}\sum_{j,\vec{k}} \Pi_j(\vec{k})\cancelto{0}{\left\langle \hat{\sigma}_+(t)\,\hat{N}_d(t+\tau)\,\hat{a}_j(\vec{k})\,e^{-i\omega_k(t+\tau)}\right\rangle}$$

$$\frac{d\left\langle \hat{\sigma}_+(t)\hat{\sigma}_-(t+\tau)\right\rangle}{d\tau} = -\frac{i}{\hbar}(\Delta\varepsilon + \delta\varepsilon)\left\langle \hat{\sigma}_+(t)\hat{\sigma}_-(t+\tau)\right\rangle - \frac{\left\langle \hat{\sigma}_+(t)\hat{\sigma}_-(t+\tau)\right\rangle}{T_2}$$

$$\Rightarrow \left\langle \hat{\sigma}_+(t)\hat{\sigma}_-(t+\tau)\right\rangle = \left\langle \hat{\sigma}_+(t)\hat{\sigma}_-(t)\right\rangle e^{-\frac{i}{\hbar}(\Delta\varepsilon+\delta\varepsilon)\tau}\, e^{-\frac{\tau}{T_2}} \qquad \{\tau > 0\}$$

$$\Rightarrow \left\langle \hat{\sigma}_+(t)\hat{\sigma}_-(t+\tau)\right\rangle = \left\langle \hat{N}_2(t)\right\rangle_{\substack{\text{steady}\\ \text{state}}} e^{-\frac{i}{\hbar}(\Delta\varepsilon+\delta\varepsilon)\tau}\, e^{-\frac{|\tau|}{T_2}}$$

First Order Coherence Function of Spontaneous Emission and Natural Linewidth



$$\left\langle \hat{\sigma}_+(t)\hat{\sigma}_-(t+\tau)\right\rangle = \left\langle \hat{N}_2\right\rangle_{\substack{\text{steady}\\ \text{state}}} e^{-\frac{i}{\hbar}(\Delta\varepsilon+\delta\varepsilon)\tau} e^{-\frac{|\tau|}{T_2}}$$

Now use:

$$\hat{\vec{E}}_+(\vec{r},t) = \hat{\vec{E}}_+^o(\vec{r},t) + \frac{\omega_o^2}{4\pi\varepsilon_o c^2 r}\left[\overline{\overline{1}} - \hat{r}\otimes\hat{r}\right].\left(q\vec{d}\right)\hat{\sigma}_-\left(t-\frac{r}{c}\right)$$

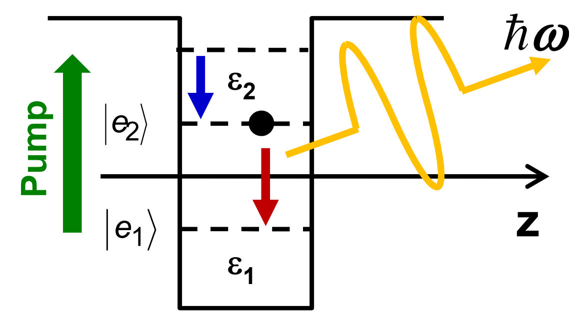
$$\hat{\vec{E}}_-(\vec{r},t) = \hat{\vec{E}}_-^o(\vec{r},t) + \frac{\omega_o^2}{4\pi\varepsilon_o c^2 r}\left[\overline{\overline{1}} - \hat{r}\otimes\hat{r}\right].\left(q\vec{d}\right)\hat{\sigma}_+\left(t-\frac{r}{c}\right)$$

To get:

$$\left\langle \hat{E}_-(\vec{r},t)\hat{E}_+(\vec{r},t+\tau)\right\rangle \propto \left\langle \hat{\sigma}_+(t)\hat{\sigma}_-(t+\tau)\right\rangle$$

$$= \left\langle \hat{N}_2\right\rangle_{\substack{\text{steady}\\ \text{state}}} e^{-\frac{i}{\hbar}(\Delta\varepsilon+\delta\varepsilon)\tau} e^{-\frac{|\tau|}{T_2}}$$

Spectral Density for Non-Stationary First Order Correlation Functions and Spontaneous Emission and Natural Linewidth



First order coherence function:

$$g_1(\vec{r}:t,t+\tau)=\frac{\left\langle \hat{E}_-(\vec{r},t)\hat{E}_+(\vec{r},t+\tau)\right\rangle}{\sqrt{\left\langle \hat{E}_-(\vec{r},t+\tau)\hat{E}_+(\vec{r},t+\tau)\right\rangle\left\langle \hat{E}_-(\vec{r},t)\hat{E}_+(\vec{r},t)\right\rangle}}$$

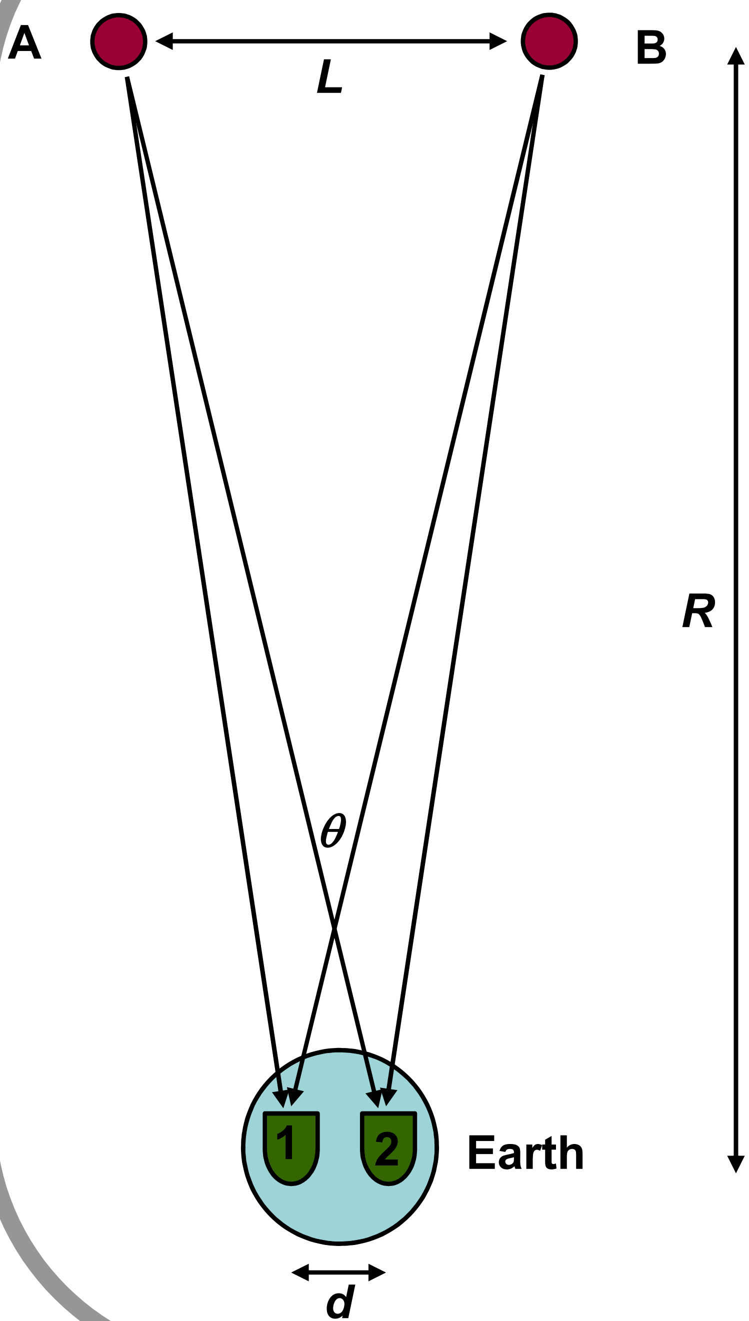
$$= e^{-\frac{i}{\hbar}(\Delta\varepsilon+\delta\varepsilon)\tau}\; e^{-\frac{|\tau|}{T_2}}$$

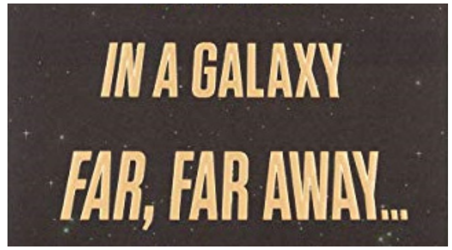
The above results in the following expression for the spectrum of the spontaneous emission:

$$S(\omega)=\int_{-\infty}^{\infty} d\tau \quad g_1(\vec{r}:t,t+\tau)\; e^{i\omega\tau} = \frac{\frac{2}{T_2}}{\left(\omega-\frac{(\Delta\varepsilon+\delta\varepsilon)}{\hbar}\right)^2+\left(\frac{1}{T_2}\right)^2}$$

Natural linewidth

Hanbury Brown and Twiss Setup: Intensity Interferometry





R. Hanbury Brown; R. Q. Twiss (1956). "A Test of a New Type of Stellar Interferometer on Sirius". Nature. 178 (4541): 1046–1048.

$$E_1 = Ae^{i\phi_A}e^{i\hat{n}_{A\to 1}\cdot(\vec{r}_1-\vec{r}_A)\frac{2\pi}{\lambda}} + Be^{i\phi_B}e^{i\hat{n}_{B\to 1}\cdot(\vec{r}_1-\vec{r}_B)\frac{2\pi}{\lambda}}$$

$$E_2 = Ae^{i\phi_A}e^{i\hat{n}_{A\to 2}\cdot(\vec{r}_2-\vec{r}_A)\frac{2\pi}{\lambda}} + Be^{i\phi_B}e^{i\hat{n}_{B\to 2}\cdot(\vec{r}_2-\vec{r}_B)\frac{2\pi}{\lambda}}$$

$$I_1 = E_1E_1 = A^2 + B^2 +$$

$$ABe^{i(\phi_A-\phi_B)}e^{i\left[\hat{n}_{A\to 1}\cdot(\vec{r}_1-\vec{r}_A)-\hat{n}_{B\to 1}\cdot(\vec{r}_1-\vec{r}_B)\right]\frac{2\pi}{\lambda}} + BAe^{-i(\phi_A-\phi_B)}e^{-i\left[\hat{n}_{A\to 1}\cdot(\vec{r}_1-\vec{r}_A)-\hat{n}_{B\to 1}\cdot(\vec{r}_1-\vec{r}_B)\right]\frac{2\pi}{\lambda}}$$

$$I_2 = E_2E_2 = A^2 + B^2 +$$

$$ABe^{i(\phi_A-\phi_B)}e^{i\left[\hat{n}_{A\to 2}\cdot(\vec{r}_2-\vec{r}_A)-\hat{n}_{B\to 2}\cdot(\vec{r}_2-\vec{r}_B)\right]\frac{2\pi}{\lambda}} + BAe^{-i(\phi_A-\phi_B)}e^{-i\left[\hat{n}_{A\to 2}\cdot(\vec{r}_2-\vec{r}_A)-\hat{n}_{B\to 2}\cdot(\vec{r}_2-\vec{r}_B)\right]\frac{2\pi}{\lambda}}$$

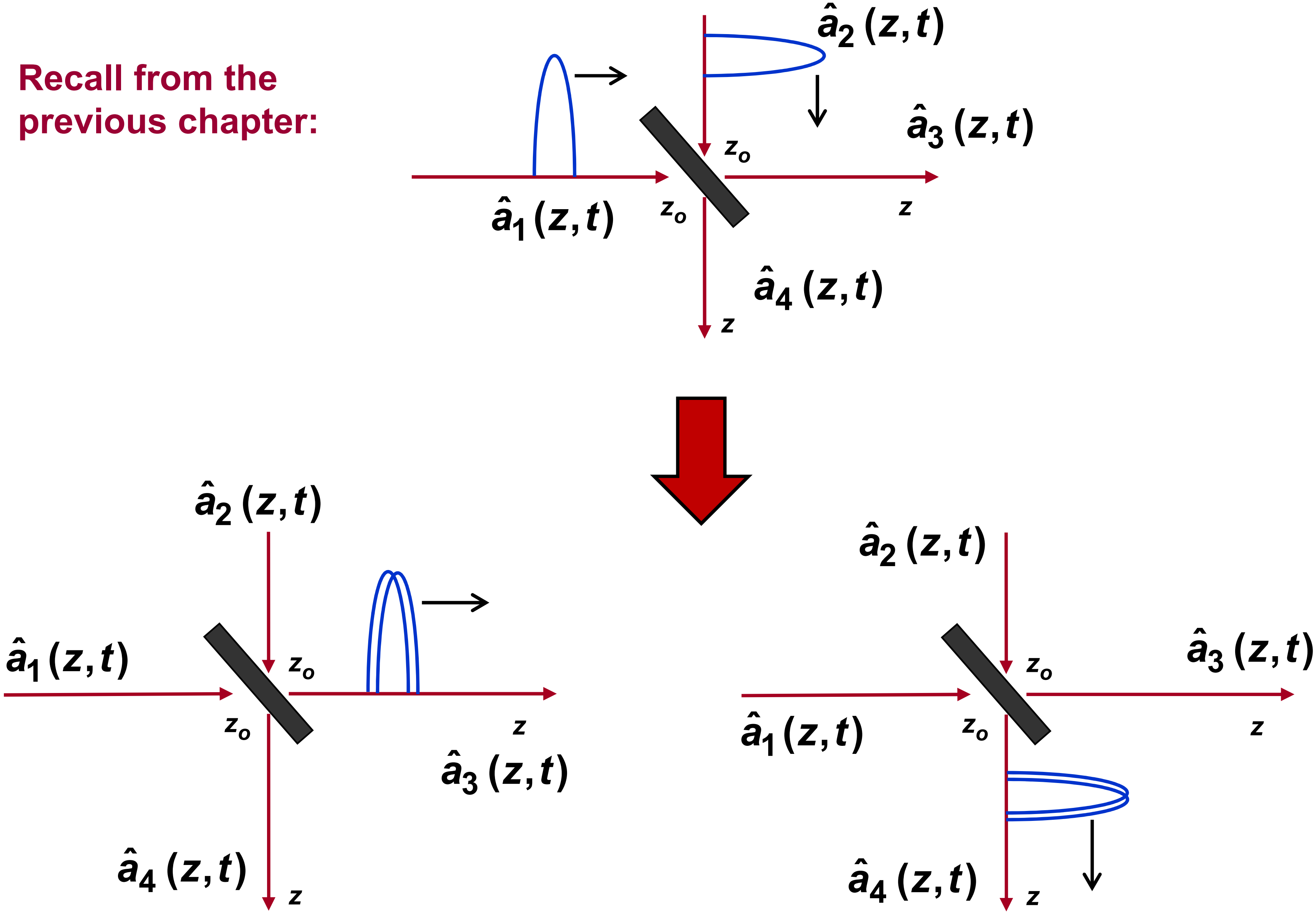
$$\langle I_1\rangle = A^2 + B^2 = \langle I_2\rangle$$

$$\langle I_1I_2\rangle - \langle I_1\rangle\langle I_2\rangle = 2A^2B^2\cos\left(\frac{2\pi}{\lambda}\frac{dL}{R}\right)$$

The interferogram gram allows precise measurement of *L* if *R* is know

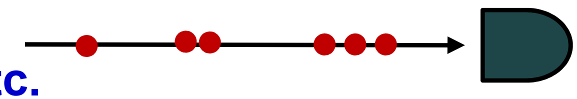
Photon Bunching: The Hong-Ou-Mandel Effect

Recall from the previous chapter:



Second Order Coherence Function

Suppose one wishes to see how photons are being emitted from a source and/or arriving at the detector, i.e. is there more or less likelihood of photons arriving together or arriving separately, etc.



One can see that by taking the quantum state $|\psi\rangle$ of light and attempting to destroy two photons at two different times at the same location:

$$\hat{a}(z,t_1)\hat{a}(z,t_2)|\psi\rangle$$

If the result is non-zero, then there a non-zero probability of detecting a photon at time t_1 AND at time t_2

How do we quantify this probability?
Let:

$$|\phi\rangle = \hat{a}(z,t_1)\hat{a}(z,t_2)|\psi\rangle$$

We find the norm of the state:

$$\langle\phi|\phi\rangle = \langle\psi|\hat{a}^+(z,t_2)\hat{a}^+(z,t_1)\hat{a}(z,t_1)\hat{a}(z,t_2)|\psi\rangle$$

The state $|\phi\rangle$ is not properly normalized and therefore $\langle\phi|\phi\rangle$ can be a very large number. One good way to proceed would be to normalize the above w.r.t. the average fluxes at the two times:

$$\frac{\langle\phi|\phi\rangle}{\langle\psi|\hat{a}^+(z,t_2)\hat{a}(z,t_2)|\psi\rangle\langle\psi|\hat{a}^+(z,t_1)\hat{a}(z,t_1)|\psi\rangle} = g_2(z:t_1,t_2)$$

Second Order Coherence Function

Second order coherence function is defined as:

$$g_1(\vec{r}:t_1,t_2) = \frac{\left\langle \hat{E}_-(\vec{r},t_2)\hat{E}_-(\vec{r},t_1)\hat{E}_+(\vec{r},t_1)\hat{E}_+(\vec{r},t_2)\right\rangle}{\left\langle \hat{E}_-(\vec{r},t_1)\hat{E}_+(\vec{r},t_1)\right\rangle\left\langle \hat{E}_-(\vec{r},t_2)\hat{E}_+(\vec{r},t_2)\right\rangle} = g_1(\vec{r}:t_2,t_1)$$

$$g_2(z:t_1,t_2) = \frac{\left\langle \hat{a}^+(z,t_2)\hat{a}^+(z,t_1)\ \hat{a}(z,t_1)\ \hat{a}(z,t_2)\right\rangle}{\left\langle \hat{a}_1^+(z,t_1)\hat{a}(z,t_1)\right\rangle\left\langle \hat{a}_1^+(z,t_2)\hat{a}(z,t_2)\right\rangle} = g(z:t_2,t_1)$$

Symmetry in time arguments:

$$\begin{aligned}
&g_2(z:t_1,t_2)\\
&=\left\langle \hat{a}^+(z,t_2)\hat{a}^+(z,t_1)\ \hat{a}(z,t_1)\ \hat{a}(z,t_2)\right\rangle\\
&=\left\langle \hat{a}^+(z-v_g t_2,0)\hat{a}^+(z-v_g t_1,0)\ \hat{a}(z-v_g t_1,0\ \ \hat{a}(z-v_g t_2,0)\right\rangle\\
&=\left\langle \hat{a}^+(z-v_g t_1,0)\hat{a}^+(z-v_g t_2,0)\ \hat{a}(z-v_g t_2,0)\ \ \hat{a}(z-v_g t_1,0)\right\rangle \quad \left\{\begin{array}{l}\text{Using equal-time}\\ \text{commutation relations}\end{array}\right.\\
&=\left\langle \hat{a}^+(z,t_1)\hat{a}^+(z,t_2)\ \hat{a}(z,t_2)\ \ \hat{a}(z,t_1)\right\rangle\\
&=g_2(z:t_2,t_1)
\end{aligned}$$

Second Order Coherence Function and Flux Correlation Function

$$g_2\left(z:t_1,t_2\right)=\frac{\left\langle \hat{a}^+(z,t_2)\,\hat{a}^+(z,t_1)\;\hat{a}(z,t_1)\;\;\hat{a}(z,t_2)\right\rangle}{\left\langle \hat{a}_1^+\left(z,t_1\right)\hat{a}\left(z,t_1\right)\right\rangle\left\langle \hat{a}_1^+\left(z,t_2\right)\hat{a}\left(z,t_2\right)\right\rangle}=g\left(z:t_2,t_1\right)$$

Relationship with the photon flux correlation function:

$$\left\langle \hat{F}(z,t_1)\;\hat{F}(z,t_2)\right\rangle=v_g^2\left\langle \hat{a}^+(z,t_1)\;\hat{a}(z,t_1)\;\hat{a}^+(z,t_2)\;\hat{a}(z,t_2)\right\rangle$$

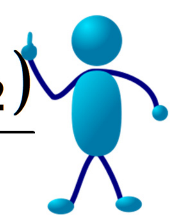
Recall that: $\left[\hat{a}\left(z,t\right),\hat{a}^+\left(z,t'\right)\right]=\dfrac{1}{v_g}\,\delta\left(t-t'\right)$

It follows that:

$$\begin{aligned}&\left\langle \hat{a}^+(z,t_2)\,\hat{a}^+(z,t_1)\;\hat{a}(z,t_1)\;\;\hat{a}(z,t_2)\right\rangle\\ &=\left\langle \hat{a}^+(z,t_1)\,\hat{a}^+(z,t_2)\;\hat{a}(z,t_1)\;\;\hat{a}(z,t_2)\right\rangle\\ &=\left\langle \hat{a}^+(z,t_1)\;\hat{a}(z,t_1)\;\hat{a}^+(z,t_2)\;\hat{a}(z,t_2)\right\rangle-\frac{\delta\left(t_1-t_2\right)}{v_g}\left\langle \hat{a}^+(z,t_1)\;\hat{a}(z,t_1)\right\rangle\end{aligned}$$

Therefore:

$$\Rightarrow g_2\left(z:t_1,t_2\right)=\frac{\left\langle \hat{F}(z,t_1)\;\;\hat{F}(z,t_2)\right\rangle-\left\langle \hat{F}\left(z,t_1\right)\right\rangle\delta\left(t_1-t_2\right)}{\left\langle \hat{F}\left(z,t_1\right)\right\rangle\left\langle \hat{F}\left(z,t_2\right)\right\rangle}$$



g_2 does not have a delta-function singularity at equal times

Second Order Coherence Function and Flux Noise Correlation

From the previous slide:

$$\left\langle \hat{F}(z,t_1)\ \hat{F}(z,t_2)\right\rangle = \left\langle \hat{F}(z,t_1)\right\rangle\left\langle \hat{F}(z,t_2)\right\rangle g_2(z:t_1,t_2) + \left\langle \hat{F}(z,t_1)\right\rangle \delta(t_1-t_2)$$

Therefore:

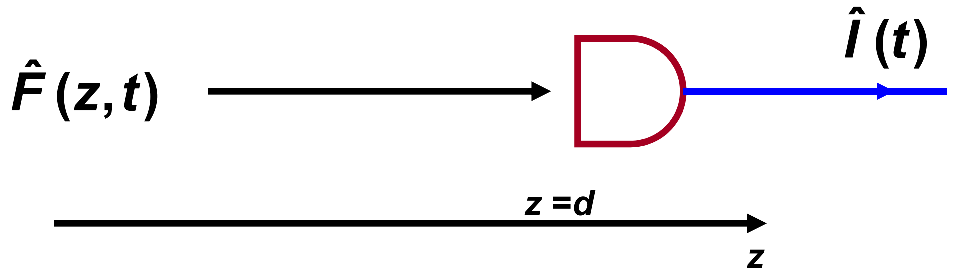
$$\left\langle \Delta\hat{F}(z,t_1)\ \Delta\hat{F}(z,t_2)\right\rangle = \left\langle \hat{F}(z,t_1)\right\rangle\left\langle \hat{F}(z,t_2)\right\rangle \left[g_2(z:t_1,t_2)-1\right] + \left\langle \hat{F}(z,t_1)\right\rangle \delta(t_1-t_2)$$

Problems with the flux noise correlation function:

- **For t_1 near t_2, the time delta function in the expression for the flux noise correlation function will mask any interesting feature in $g_2(t_1,t_2)$**

- **$g_2(t_1,t_2)$ is not affected by photon loss during the journey from the light source to the measuring apparatus whereas the flux noise correlation function changes with loss (picks up partition noise)**

Second Order Coherence Function: Measurement



Suppose:

$$\left\langle \hat{F}(z_d, t_1) \right\rangle = F_o$$

$$\left\langle \hat{I}(t) \right\rangle = qF_o$$

Then:

$$\left\langle \hat{I}(t_1)\, \hat{I}(t_2) \right\rangle = q^2 \left\langle \hat{F}(z_d, t_1)\, \hat{F}(z_d, t_2) \right\rangle$$

$$= q^2 v_g^2 \left\langle \hat{a}^+(z_d, t_1)\, \hat{a}(z_d, t_1)\, \hat{a}^+(z_d, t_2)\, \hat{a}(z_d, t_2) \right\rangle$$

$$= q^2 F_o \left[\delta(t_1 - t_2) + g_2(z_d : t_1, t_2) F_o \right]$$

And:

$$\left\langle \Delta\hat{I}(t_1)\, \Delta\hat{I}(t_2) \right\rangle = q^2 F_o \delta(t_1 - t_2) + q^2 F_o^2 \left[g_2(z_d : t_1, t_2) - 1 \right]$$

Second Order Coherence Function

$$g_2\left(z:t_1,t_2\right)=\frac{\left\langle \hat{a}^+(z,t_2)\,\hat{a}^+(z,t_1)\;\hat{a}(z,t_1)\;\hat{a}(z,t_2)\right\rangle}{\left\langle \hat{a}_1^+\left(z,t_1\right)\hat{a}\left(z,t_1\right)\right\rangle\left\langle \hat{a}_1^+\left(z,t_2\right)\hat{a}\left(z,t_2\right)\right\rangle}=g\left(z:t_2,t_1\right)$$

A state has complete second order coherence if:

$$g_2\left(z:t_1,t_2\right)=1$$

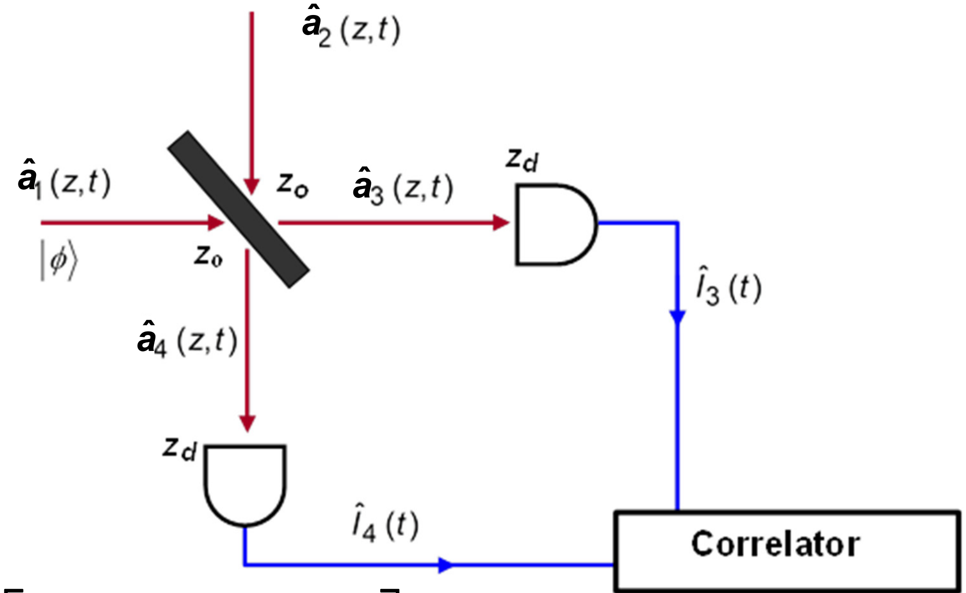
A state has complete second order coherence if the correlation function factorizes:

$$\left\langle \hat{a}^+(z,t_2)\,\hat{a}^+(z,t_1)\;\hat{a}(z,t_1)\;\hat{a}(z,t_2)\right\rangle=\left\langle \hat{a}^+(z,t_2)\right\rangle\left\langle \hat{a}^+(z,t_1)\right\rangle\left\langle \hat{a}(z,t_1)\right\rangle\;\left\langle \hat{a}(z,t_2)\right\rangle$$

$$\Rightarrow g_2\left(z:t_1,t_2\right)=1$$

Hanbury Brown and Twiss Setup for g_2

Consider the following setup:



$$|\psi(t=0)\rangle = |\phi\rangle_1 \otimes |0\rangle_2$$

$$\begin{bmatrix} \hat{a}_3(z_o,t)\, e^{i\beta_o z_o} \\ \hat{a}_4(z_o,t)\, e^{i\beta_o z_o} \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} \hat{a}_1(z_o,t)\, e^{i\beta_o z_o} \\ \hat{a}_2(z_o,t)\, e^{i\beta_o z_o} \end{bmatrix}$$

$$\Rightarrow \hat{a}_3(z_o,t) = \frac{1}{\sqrt{2}}\left[\hat{a}_1(z_o,t) + \hat{a}_2(z_o,t)\right]$$

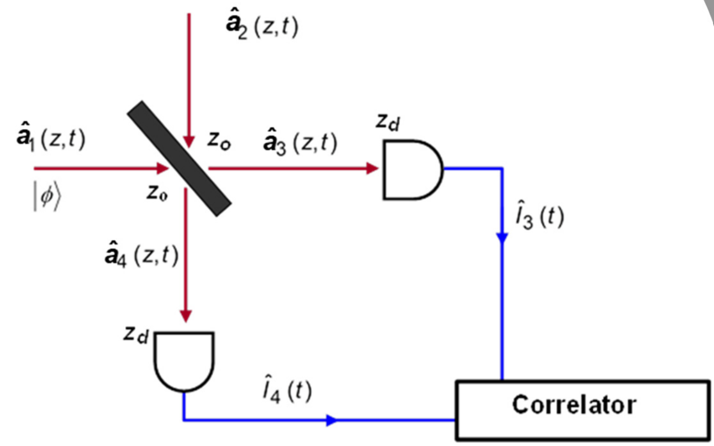
$$\Rightarrow \hat{a}_4(z_o,t) = \frac{1}{\sqrt{2}}\left[-\hat{a}_1(z_o,t) + \hat{a}_2(z_o,t)\right]$$

The detector currents are:

$$\hat{i}_3(t) = q v_g\, \hat{a}_3^{+}(z_d,t)\, \hat{a}_3(z_d,t)$$

$$\hat{i}_4(t) = q v_g\, \hat{a}_4^{+}(z_d,t)\, \hat{a}_4(z_d,t)$$

Hanbury Brown and Twiss Setup for g_2



$$\frac{\left\langle \psi(t=0) \right| \hat{i}_3\left(t_1+\dfrac{z_d-z_o}{v_g}\right) \hat{i}_4\left(t_2+\dfrac{z_d-z_o}{v_g}\right) \left| \psi(t=0) \right\rangle}{\left(qv_g/2\right)^2}$$

$$= {}_1\langle \varphi | \hat{a}_1^+(z_o,t_1)\, \hat{a}_1(z_o,t_1)\, \hat{a}_1^+(z_o,t_2)\, \hat{a}_1(z_o,t_2) | \varphi \rangle_1$$

$$- {}_1\langle \varphi | \otimes {}_2\langle 0 | \hat{a}_1^+(z_o,t_1)\, \hat{a}_2(z_o,t_1)\, \hat{a}_2^+(z_o,t_2)\, \hat{a}_1(z_o,t_1) | \varphi \rangle_1 \otimes | 0 \rangle_2$$

$$= {}_1\langle \varphi | \hat{a}_1^+(z_o,t_1)\, \hat{a}_1(z_o,t_1)\, \hat{a}_1^+(z_o,t_2)\, \hat{a}_1(z_o,t_2) | \varphi \rangle_1$$

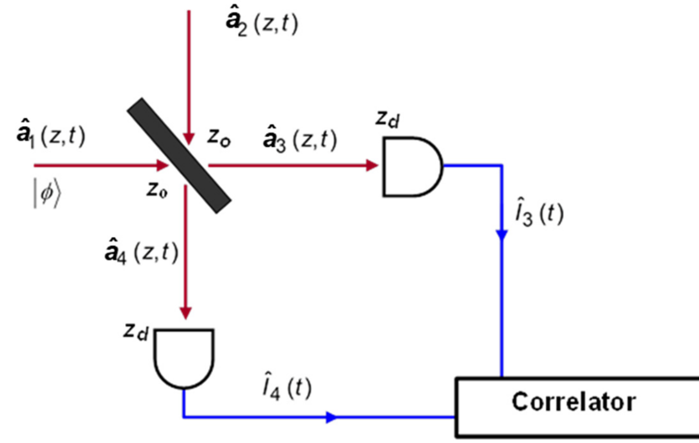
$$- {}_1\langle \varphi | \hat{a}_1^+(z_o,t_1)\, \hat{a}_1(z_o,t_2) | \varphi \rangle_1 \frac{\delta(t_1-t_2)}{v_g}$$

$$= {}_1\langle \varphi | \hat{a}_1^+(z_o,t_2)\, \hat{a}_1^+(z_o,t_1)\, \hat{a}_1(z_o,t_1)\, \hat{a}_1(z_o,t_2) | \varphi \rangle_1$$

$$= {}_1\langle \varphi | \hat{a}_1^+(z_o,t_1)\, \hat{a}_1(z_o,t_1) | \varphi \rangle_1 \langle \varphi | \hat{a}_1^+(z_o,t_2)\, \hat{a}_1(z_o,t_2) | \varphi \rangle_1\, g_2(z_o : t_1,t_2)$$

Hanbury Brown and Twiss Setup for g_2

$$\frac{\left\langle \hat{I}_3\left(t_1+\frac{z_d-z_o}{v_g}\right)\hat{I}_4\left(t_2+\frac{z_d-z_o}{v_g}\right)\right\rangle}{\left\langle \hat{I}_3\left(t_1+\frac{z_d-z_o}{v_g}\right)\right\rangle\left\langle \hat{I}_4\left(t_2+\frac{z_d-z_o}{v_g}\right)\right\rangle}=g_2(z_o:t_1,t_2)$$



Therefore, the Hanbury Brown and Twiss setup can measure the second order coherence function directly

Photon Bunching and Anti-Bunching

Suppose:

$$\left\langle \hat{F}(z_d, t_1) \right\rangle = F_o$$

Then:

$$\left\langle \hat{F}(z_d, t+\tau)\ \hat{F}(z_d, t) \right\rangle = F_o\ \delta(\tau) + F_o^2\ g_2(z_d : \tau)$$

Question: how does one quantify the tendency of the photons to arrive together, separately, or independently of each other?

$$\left\langle \hat{F}(z_d, t+\tau)\ \hat{F}(z_d, t) \right\rangle - \left\langle \hat{F}(z_d, t+\tau)\ \hat{F}(z_d, t) \right\rangle \Big|_{\tau=0^+} = F_o^2\ \left[g_2(z_d : \tau) - g_2(z_d : 0) \right]$$

Two Cases:

$g_2(z_d : \tau) < g_2(z_d : 0) \longrightarrow$ Photon bunching

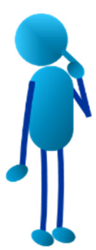
$g_2(z_d : \tau) > g_2(z_d : 0) \longrightarrow$ Photon anti-bunching

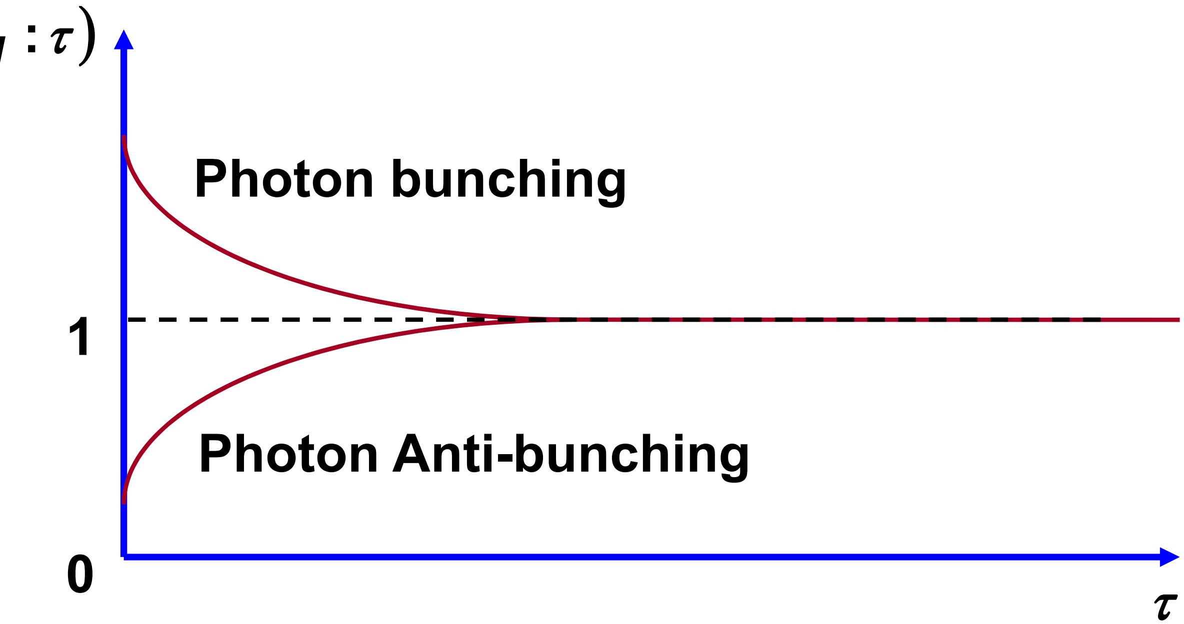
Photon Bunching and Anti-Bunching

Two Cases:

$$g_2(z_d:\tau) < g_2(z_d:0) \longrightarrow \text{Photon bunching}$$

$$g_2(z_d:\tau) > g_2(z_d:0) \longrightarrow \text{Photon anti-bunching}$$



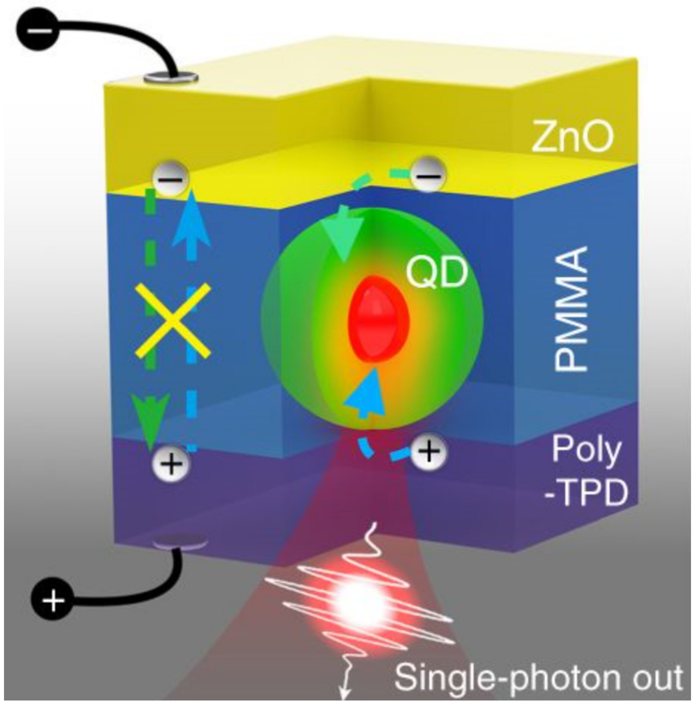


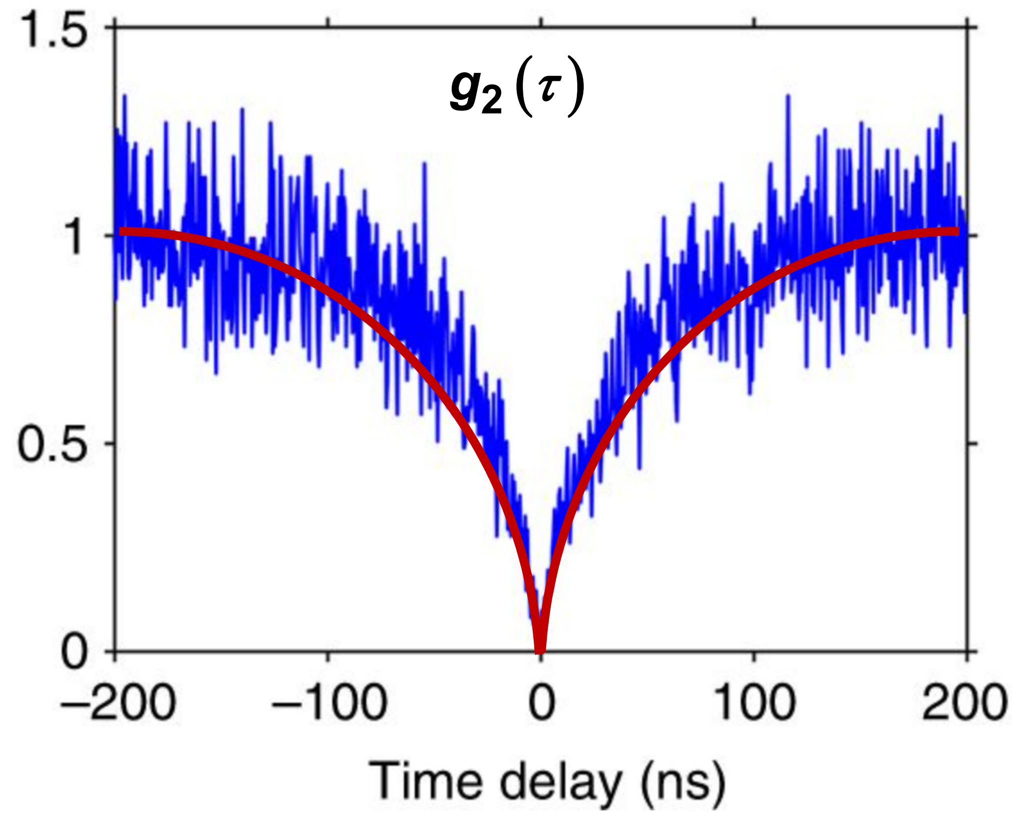
For most physical light sources:

$$g_2(z:\tau\rightarrow\infty)=1$$

Single-Photon Sources and Photon Anti-Bunching

Measured g_2 from an electrically driven single colloidal CdSe/CdS core/shell quantum dot





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Single-Photon Sources

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REVIEW ARTICLE

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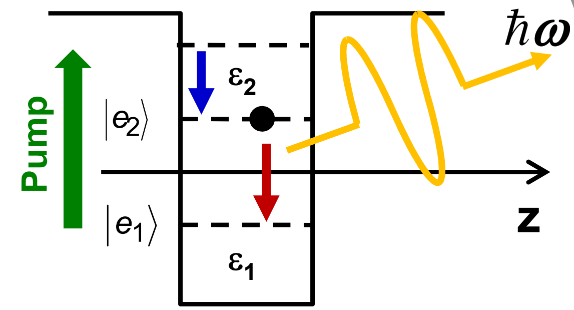
Solid-state single-photon emitters

Igor Aharonovich[1,2]*, Dirk Englund[3] and Milos Toth[1,2]

Single-photon emitters play an important role in many leading quantum technologies. There is still no 'ideal' on-demand single-photon emitter, but a plethora of promising material systems have been developed, and several have transitioned from proof-of-concept to engineering efforts with steadily improving performance. Here, we review recent progress in the race towards true single-photon emitters required for a range of quantum information processing applications. We focus on solid-state systems including quantum dots, defects in solids, two-dimensional hosts and carbon nanotubes, as these are well positioned to benefit from recent breakthroughs in nanofabrication and materials growth techniques. We consider the main challenges and key advantages of each platform, with a focus on scalable on-chip integration and fabrication of identical sources on photonic circuits.

Single-Photon Sources and Photon Anti-Bunching

$$\hat{\vec{E}}_{\pm}\left(\vec{r},t\right)=\hat{\vec{E}}^{o}_{-}\left(\vec{r},t\right)+\frac{\omega_o^2}{4\pi\varepsilon_o c^2 r}\left[1-\hat{r}\otimes\hat{r}\right].\left(q\vec{d}\right)\hat{\sigma}_{\mp}\left(t-\frac{r}{c}\right)$$



$$\boxed{\omega_o=\frac{\Delta\varepsilon+\delta\varepsilon}{\hbar}}$$

Need to calculate:

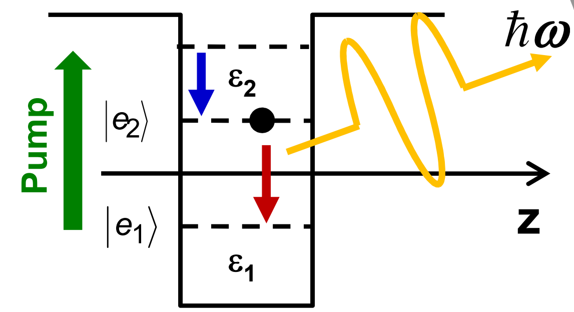
$$g_2\left(\vec{r}:t+\tau,t\right)=\frac{\left\langle \hat{E}_{-}\left(\vec{r},t\right)\hat{E}_{-}\left(\vec{r},t+\tau\right)\hat{E}_{+}\left(\vec{r},t+\tau\right)\hat{E}_{+}\left(\vec{r},t\right)\right\rangle}{\left\langle \hat{E}_{-}\left(\vec{r},t+\tau\right)\hat{E}_{+}\left(\vec{r},t+\tau\right)\right\rangle\left\langle \hat{E}_{-}\left(\vec{r},t+\tau\right)\hat{E}_{+}\left(\vec{r},t+\tau\right)\right\rangle}$$

$$=\frac{\left\langle \hat{\sigma}_{+}\left(t\right)\hat{\sigma}_{+}\left(t+\tau\right)\hat{\sigma}_{-}\left(t+\tau\right)\hat{\sigma}_{-}\left(t\right)\right\rangle}{\left\langle \hat{\sigma}_{+}\left(t+\tau\right)\hat{\sigma}_{-}\left(t+\tau\right)\right\rangle\left\langle \hat{\sigma}_{+}\left(t+\tau\right)\hat{\sigma}_{-}\left(t+\tau\right)\right\rangle}$$

$$=\frac{\left\langle \hat{\sigma}_{+}\left(t\right)\hat{N}_2\left(t+\tau\right)\hat{\sigma}_{-}\left(t\right)\right\rangle}{\left\langle \hat{N}_2\left(t+\tau\right)\right\rangle\left\langle \hat{N}_2\left(t\right)\right\rangle}$$

$$=\frac{\left\langle \hat{\sigma}_{+}\left(t\right)\hat{N}_2\left(t+\tau\right)\hat{\sigma}_{-}\left(t\right)\right\rangle}{\left\langle \hat{N}_2\right\rangle^2_{\substack{steady\\state}}}$$

Single-Photon Sources and Photon Anti-Bunching



$$-\frac{d\hat{N}_1(t)}{dt}=\frac{d\hat{N}_2(t)}{dt}=\frac{\hat{N}_1(t)}{T_{pump}}-\frac{\hat{N}_2(t)}{T_{sp}}$$

$$-\frac{i}{\sqrt{V}}\sum_{j,\vec{k}}\left[\Pi_j\left(\vec{k}\right)\hat{\sigma}_+(t)\,\hat{a}_j\left(\vec{k}\right)e^{-i\omega_k t}-\Pi_j^*\left(\vec{k}\right)\hat{a}_j^+\left(\vec{k}\right)\hat{\sigma}_-(t)\,e^{i\omega_k t}\right]$$

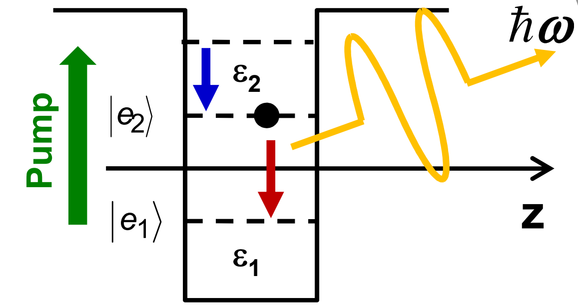
$$+\hat{D}_{pump}(t)$$

$$\Rightarrow\frac{d\left\langle\hat{N}_2(t)\right\rangle}{dt}=\frac{\left\langle\hat{N}_1(t)\right\rangle}{T_{pump}}-\frac{\left\langle\hat{N}_2(t)\right\rangle}{T_{sp}}$$

$$\left\langle\hat{N}_2\right\rangle_{\substack{steady\\state}}=\frac{T_{sp}}{T_{sp}+T_{pump}}$$

$$\left\langle\hat{N}_1\right\rangle_{\substack{steady\\state}}=\frac{T_{pump}}{T_{sp}+T_{pump}}$$

Single-Photon Sources and Photon Anti-Bunching



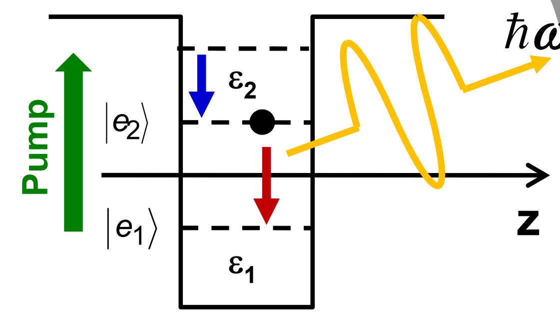
$$\frac{d\hat{N}_d(t+\tau)}{d\tau} = -\hat{N}_d(t+\tau)\left[\frac{1}{T_{pump}}+\frac{1}{T_{sp}}\right]+\left[\frac{1}{T_{pump}}-\frac{1}{T_{sp}}\right]$$
$$-\frac{2i}{\sqrt{V}}\sum_{j,\vec{k}}\left[\Pi_j(\vec{k})\hat{\sigma}_+(t+\tau)\hat{a}_j(\vec{k})e^{-i\omega_k(t+\tau)}-\Pi_j^*(\vec{k})\,\hat{a}_j^+(\vec{k})\hat{\sigma}_-(t+\tau)e^{i\omega_k(t+\tau)}\right]$$
$$+2\hat{D}_{pump}(t+\tau)$$

$$\frac{d\left\langle\hat{\sigma}_+(t)\hat{N}_d(t+\tau)\hat{\sigma}_-(t)\right\rangle}{d\tau} = -\left\langle\hat{\sigma}_+(t)\hat{N}_d(t+\tau)\hat{\sigma}_-(t)\right\rangle\left[\frac{1}{T_{pump}}+\frac{1}{T_{sp}}\right]+\left[\frac{1}{T_{pump}}-\frac{1}{T_{sp}}\right]\left\langle\hat{\sigma}_+(t)\hat{\sigma}_-(t)\right\rangle$$
$$-\frac{2i}{\sqrt{V}}\sum_{j,\vec{k}}\left[\cancelto{0}{\Pi_j(\vec{k})\left\langle\hat{\sigma}_+(t)\hat{\sigma}_+(t+\tau)\hat{a}_j(\vec{k})\hat{\sigma}_-(t)\right\rangle e^{-i\omega_k(t+\tau)}}-\cancelto{0}{\Pi_j^*(\vec{k})\left\langle\hat{\sigma}_+(t)\hat{a}_j^+(\vec{k})\hat{\sigma}_-(t+\tau)\hat{\sigma}_-(t)\right\rangle e^{i\omega_k(t+\tau)}}\right]$$
$$+\cancelto{0}{\left\langle\hat{\sigma}_+(t)2\hat{D}_{pump}(t+\tau)\hat{\sigma}_-(t)\right\rangle}$$

$$\frac{d\left\langle\hat{\sigma}_+(t)\hat{N}_d(t+\tau)\hat{\sigma}_-(t)\right\rangle}{d\tau} = -\left\langle\hat{\sigma}_+(t)\hat{N}_d(t+\tau)\hat{\sigma}_-(t)\right\rangle\left[\frac{1}{T_{pump}}+\frac{1}{T_{sp}}\right]+\left[\frac{1}{T_{pump}}-\frac{1}{T_{sp}}\right]\left\langle\hat{N}_2\right\rangle_{\substack{steady\\state}}$$

Single-Photon Sources and Photon Anti-Bunching

$$\frac{d\left\langle \hat{\sigma}_+(t)\hat{N}_d(t+\tau)\hat{\sigma}_-(t)\right\rangle}{d\tau} = -\left\langle \hat{\sigma}_+(t)\hat{N}_d(t+\tau)\hat{\sigma}_-(t)\right\rangle\left[\frac{1}{T_{pump}}+\frac{1}{T_{sp}}\right] + \left[\frac{1}{T_{pump}}-\frac{1}{T_{sp}}\right]\left\langle \hat{N}_2\right\rangle_{\substack{steady\\state}}$$



Boundary conditions:

$$\left\langle \hat{\sigma}_+(t)\hat{N}_d(t+\tau)\hat{\sigma}_-(t)\right\rangle_{\tau\to 0} = \left\langle \hat{\sigma}_+(t)\hat{N}_d(t)\hat{\sigma}_-(t)\right\rangle = -\left\langle \hat{N}_2\right\rangle_{\substack{steady\\state}}$$

$$\left\langle \hat{\sigma}_+(t)\hat{N}_d(t+\tau)\hat{\sigma}_-(t)\right\rangle_{\tau\to \infty} = \left\langle \hat{N}_2\right\rangle_{\substack{steady\\state}}\frac{\left[T_{sp}-T_{pump}\right]}{\left[T_{sp}+T_{pump}\right]}$$

Solution is:

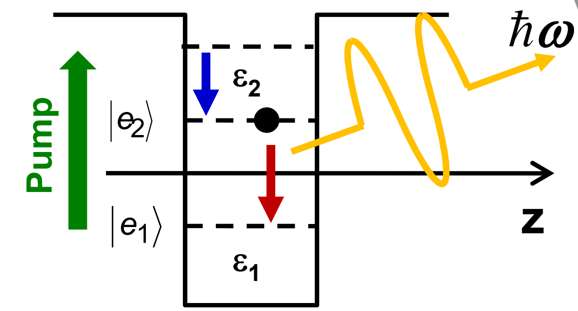
$$\Rightarrow \left\langle \hat{\sigma}_+(t)\hat{N}_d(t+\tau)\hat{\sigma}_-(t)\right\rangle = -\left\langle \hat{N}_2\right\rangle_{\substack{steady\\state}} e^{-\left(\frac{1}{T_{sp}}+\frac{1}{T_{pump}}\right)|\tau|} + \left\langle \hat{N}_2\right\rangle_{\substack{steady\\state}}\frac{\left[T_{sp}-T_{pump}\right]}{\left[T_{sp}+T_{pump}\right]}\left(1-e^{-\left(\frac{1}{T_{sp}}+\frac{1}{T_{pump}}\right)|\tau|}\right)$$

$$\Rightarrow 2\left\langle \hat{\sigma}_+(t)\hat{N}_2(t+\tau)\hat{\sigma}_-(t)\right\rangle = +\left\langle \hat{N}_2\right\rangle_{\substack{steady\\state}}\left(1-e^{-\left(\frac{1}{T_{sp}}+\frac{1}{T_{pump}}\right)|\tau|}\right) + \left\langle \hat{N}_2\right\rangle_{\substack{steady\\state}}\frac{\left[T_{sp}-T_{pump}\right]}{\left[T_{sp}+T_{pump}\right]}\left(1-e^{-\left(\frac{1}{T_{sp}}+\frac{1}{T_{pump}}\right)|\tau|}\right)$$

$$\Rightarrow \left\langle \hat{\sigma}_+(t)\hat{N}_2(t+\tau)\hat{\sigma}_-(t)\right\rangle = \left\langle \hat{N}_2\right\rangle^2_{\substack{steady\\state}}\left(1-e^{-\left(\frac{1}{T_{sp}}+\frac{1}{T_{pump}}\right)|\tau|}\right)$$

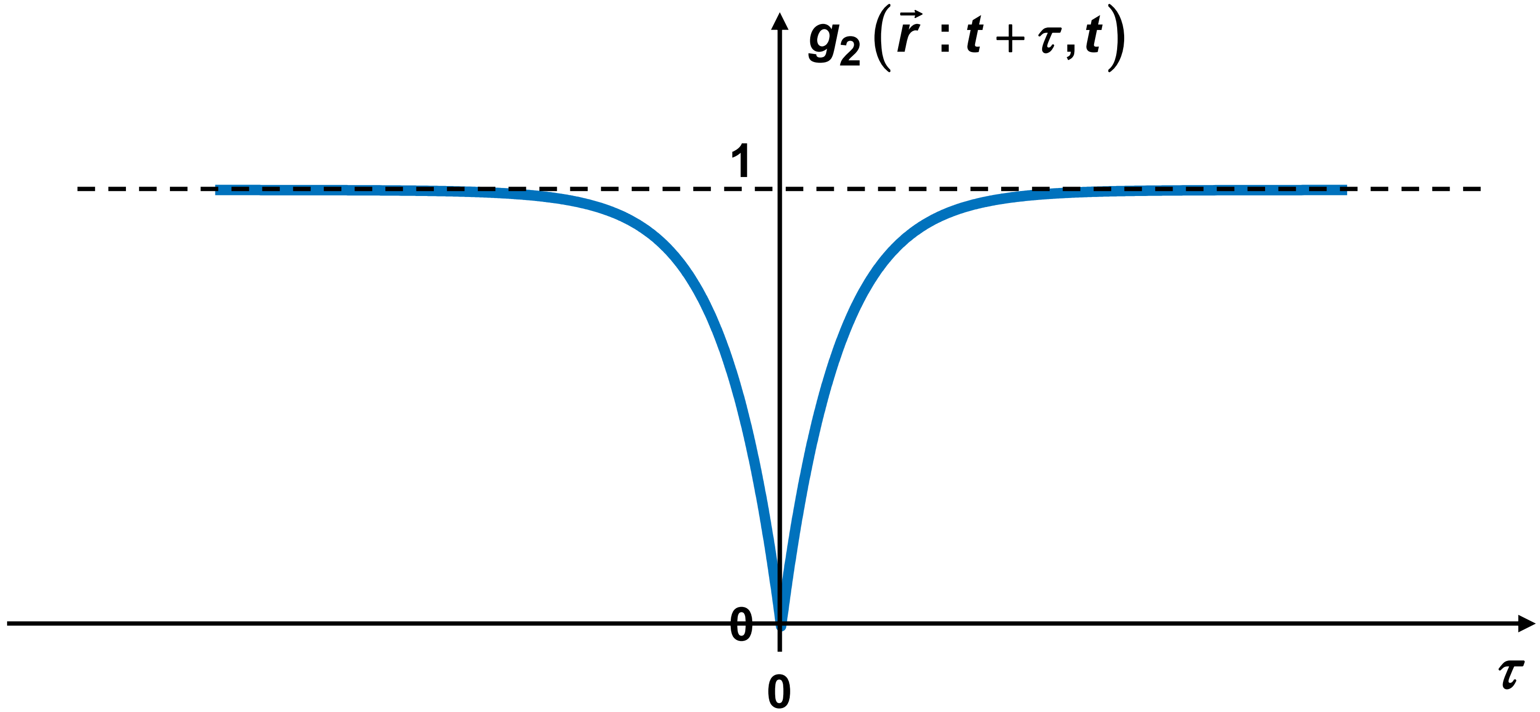
Single-Photon Sources and Photon Anti-Bunching

$$\left\langle \hat{\sigma}_+(t)\hat{N}_2(t+\tau)\hat{\sigma}_-(t)\right\rangle = \left\langle \hat{N}_2\right\rangle^2_{\substack{steady\\state}}\left(1-e^{-\left(\frac{1}{T_{sp}}+\frac{1}{T_{pump}}\right)|\tau|}\right)$$



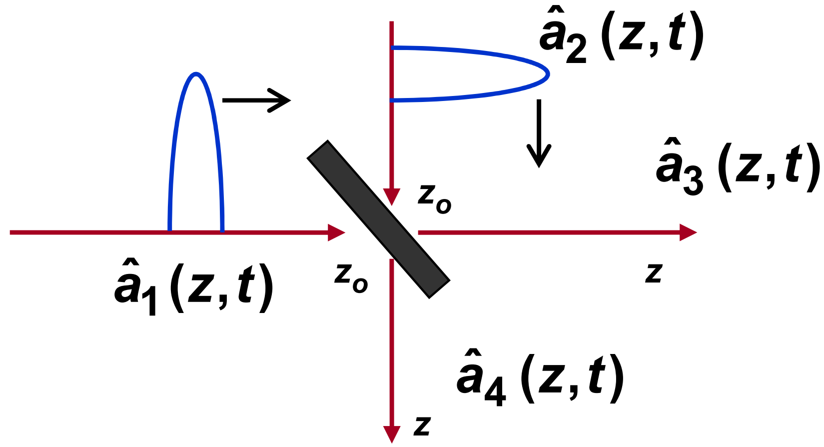
Finally:

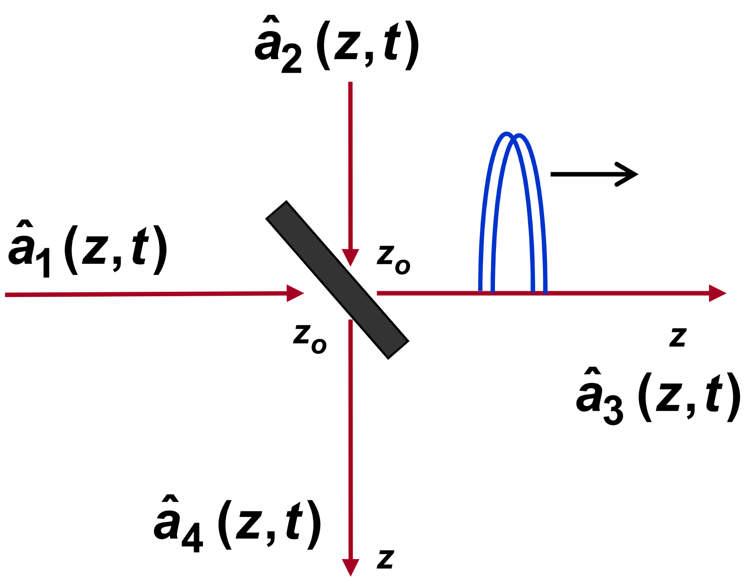
$$g_2(\vec{r}:t+\tau,t) = \frac{\left\langle \hat{\sigma}_+(t)\hat{N}_2(t+\tau)\hat{\sigma}_-(t)\right\rangle}{\left\langle \hat{N}_2\right\rangle^2_{\substack{steady\\state}}} = 1-e^{-\left(\frac{1}{T_{sp}}+\frac{1}{T_{pump}}\right)|\tau|}$$

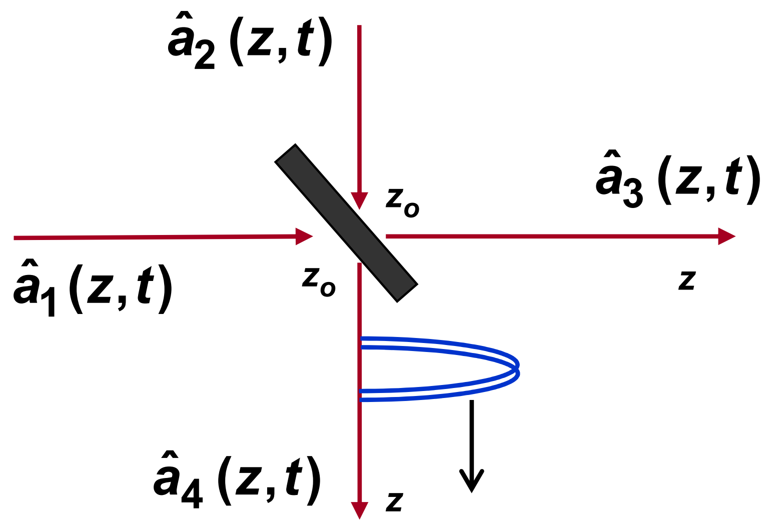


Photon Bunching: The Hong-Ou-Mandel Effect

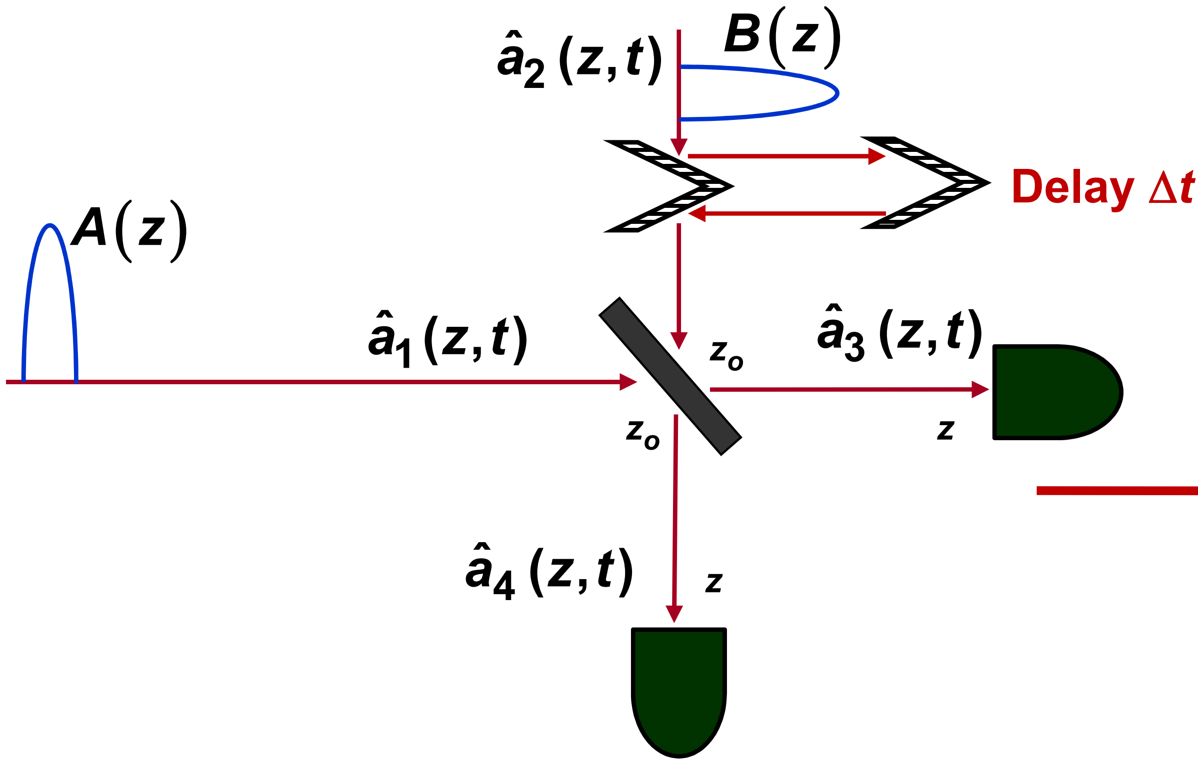
Recall from the previous chapter:

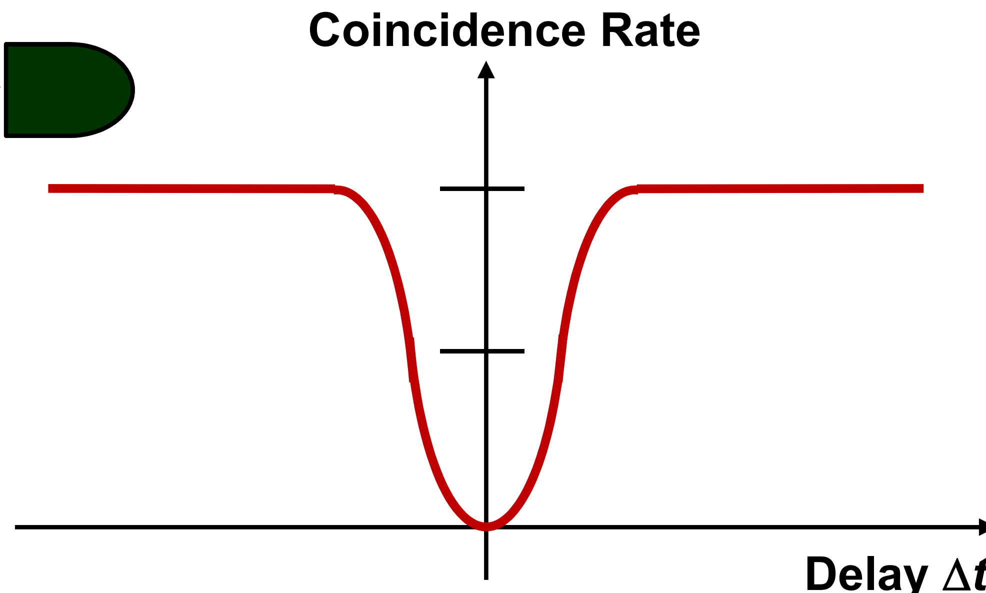






The Hong-Ou-Mandel Effect and the HOM Interferometer for Identical Photons

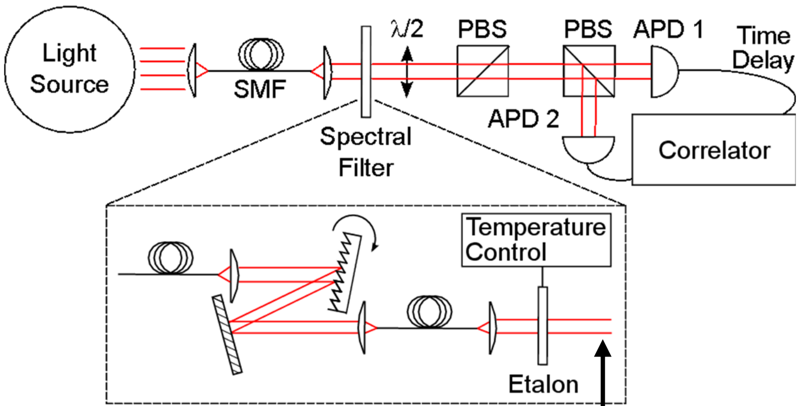




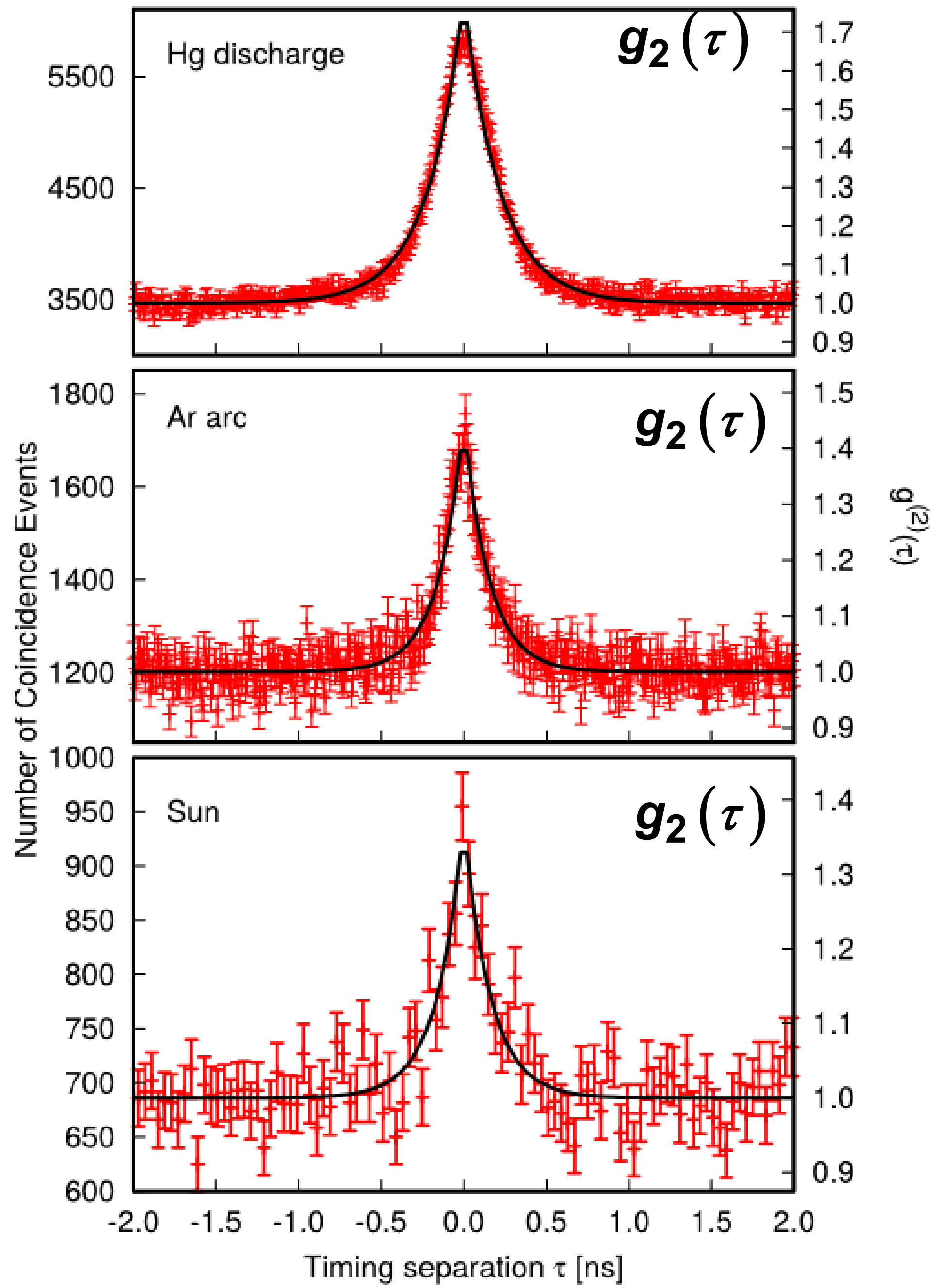
If the two photons are not "identical", the probability of finding a photon in each output port goes up and the coincidence rate recorded by the detectors goes up

Identical photons are needed in many photon based quantum computation protocols

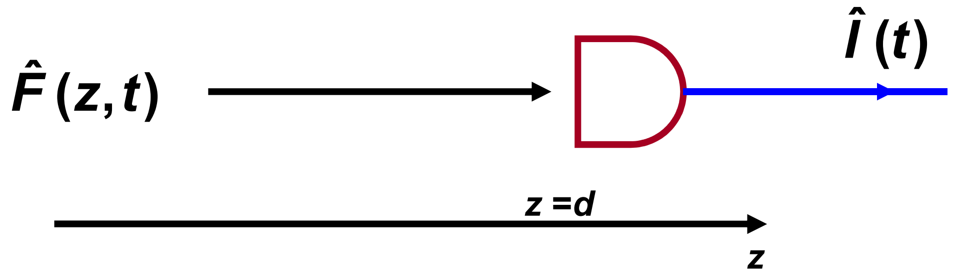
Photon Bunching: Experiments



~2 GHz optical bandwidth after filtering



Photon Counting Statistics



Suppose:

$$\left\langle \hat{F}(z_d, t_1) \right\rangle = F_o \longrightarrow \left\langle \hat{I}(t) \right\rangle = qF_o$$

We have:

$$\left\langle \hat{F}(z_d, t+\tau)\, \hat{F}(z_d, t) \right\rangle = F_o\, \delta(\tau) + F_o^2\, g_2(z_d : \tau)$$

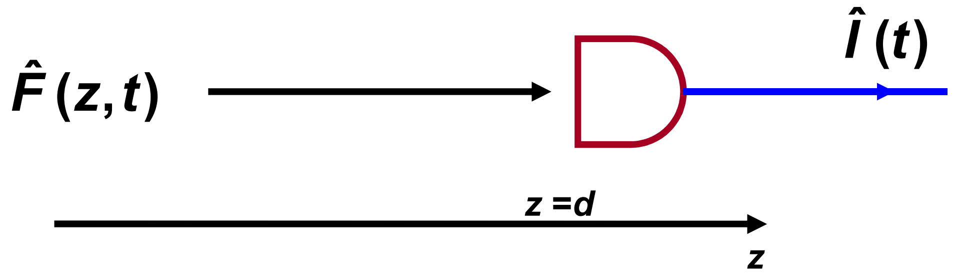
The average number of photons counted in time interval *T* is:

Use:

$$\hat{N}(z_d) = \int_{-T/2}^{T/2} \hat{F}(z_d, t)\, dt$$

$$\left\langle \hat{N}(z_d) \right\rangle = \int_{-T/2}^{T/2} \left\langle \hat{F}(z_d, t) \right\rangle dt = F_o T$$

Photon Counting Statistics



The variance in the number of photons counted is:

$$\left\langle \Delta\hat{N}^2(z_d)\right\rangle = \left\langle \hat{N}^2(z_d)\right\rangle - \left\langle \hat{N}(z_d)\right\rangle^2$$

$$= \int_{-T/2}^{T/2} dt_1 \int_{-T/2}^{T/2} dt_2 \ \left\langle \Delta\hat{F}(z_d,t_1)\ \Delta\hat{F}(z_d,t_2)\right\rangle dt$$

$$= \int_{-T}^{T} d\tau \ \left(T-|\tau|\right)\left\{F_o\ \delta(\tau) + F_o^2\left[g_2(z_d:\tau)-1\right]\right\}$$

$$= \left\langle \hat{N}(z_d)\right\rangle + F_o^2 \int_{-T}^{T} d\tau \ \left(T-|\tau|\right)\left[g_2(z_d:\tau)-1\right]$$

If T is large, the relative variance is:

$$\frac{\left\langle \Delta\hat{N}^2(z_d)\right\rangle}{\left\langle \hat{N}(z_d)\right\rangle} = 1 + F_o \int_{-\infty}^{\infty} d\tau \quad \left[g_2(z_d:\tau)-1\right]$$

Photon Counting Statistics

Relative variance is:

$$\frac{\left\langle \Delta \hat{N}^2(z_d) \right\rangle}{\left\langle \hat{N}(z_d) \right\rangle} = 1 + F_o \int_{-\infty}^{\infty} d\tau \quad [g_2(z_d : \tau) - 1]$$

1) Poissonian Statistics:

$$\left\langle \Delta \hat{N}^2(z_d) \right\rangle = \left\langle \hat{N}(z_d) \right\rangle$$

Example, when: $g_2(z : \tau) = 1$

Example: A CW coherent state

2) Sub-Poissonian Statistics:

$$\left\langle \Delta \hat{N}^2(z_d) \right\rangle \leq \left\langle \hat{N}(z_d) \right\rangle$$

Example, when: $g_2(z : \tau) \leq 1$

Example: A CW amplitude squeezed state

2) Super-Poissonian Statistics:

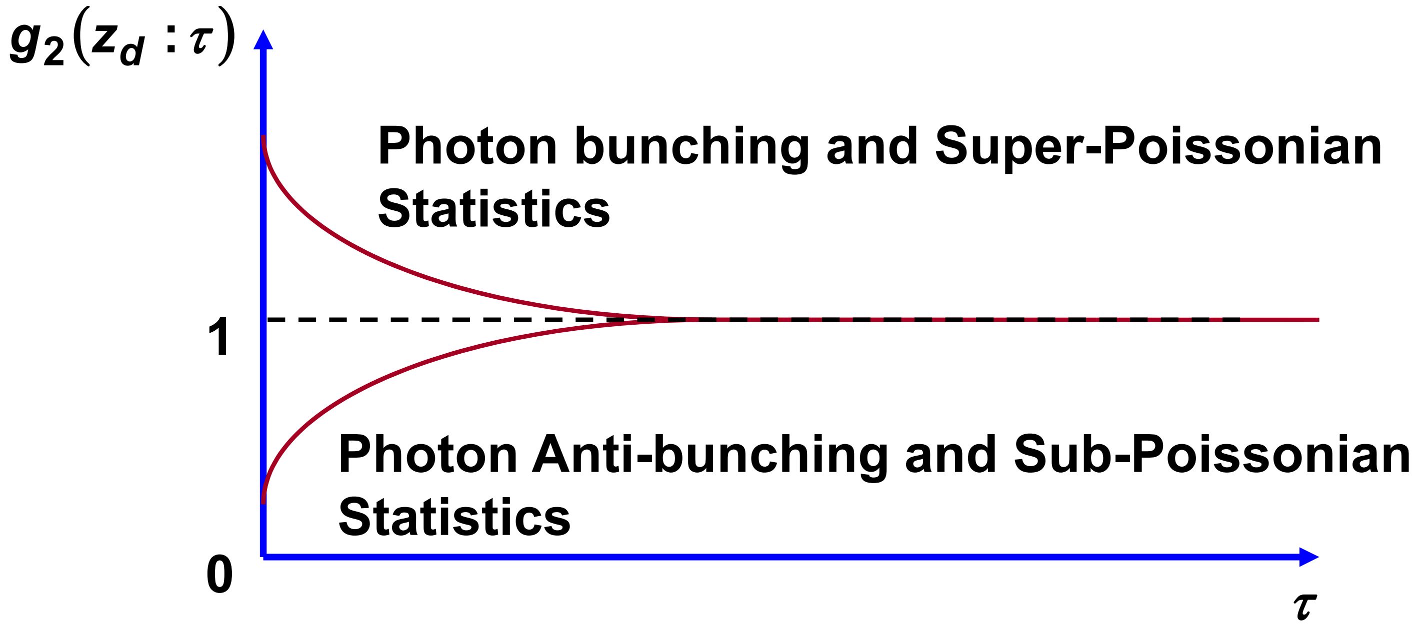
$$\left\langle \Delta \hat{N}^2(z_d) \right\rangle \geq \left\langle \hat{N}(z_d) \right\rangle$$

Example, when: $g_2(z : \tau) \geq 1$

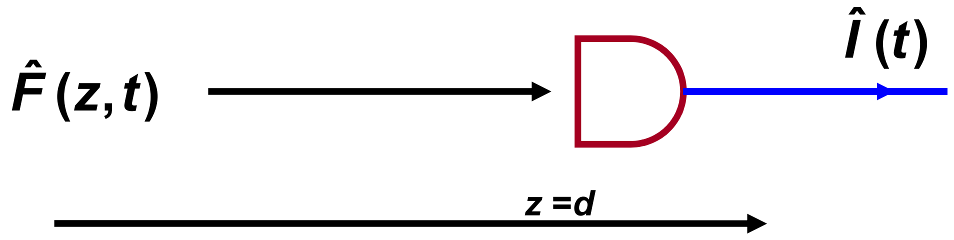
Example: Thermal radiation state

Photon Counting Statistics

$$\frac{\left\langle \Delta\hat{N}^2(z_d)\right\rangle}{\left\langle \hat{N}(z_d)\right\rangle} = 1 + F_o \int_{-\infty}^{\infty} d\tau \quad [g_2(z_d:\tau) - 1]$$



Photon Counting Distribution



Experiments often count the number of photons received/detected in a given time interval *T* and use this information to generate histograms

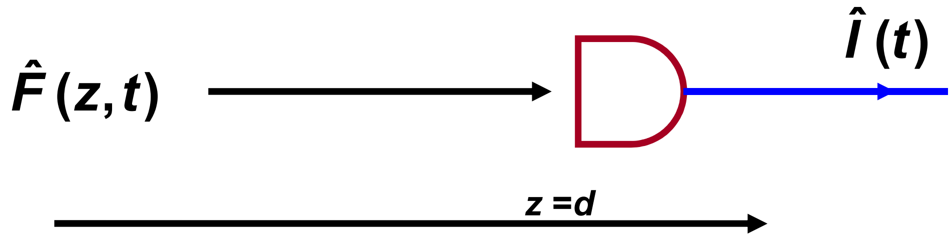
So its is useful to have an expression for the probability of counting *m* photons in a time interval *T* using a photon detector in the configuration shown above for different quantum optical states going into the detector assumed to be of unity quantum efficiency

We divide the time interval *T* into *N* small time intervals of duration Δt each

Suppose the incoming quantum state of light is $|\psi\rangle$

The quantity $\hat{a}(z_d, t_j)|\psi\rangle$ **will be nonzero only if the detector detects a photon at time** t_j

Photon Counting Distribution



We can expect the norm:

$$\langle\psi|\hat{a}^{\dagger}(z_d,t_j)\hat{a}(z_d,t_j)|\psi\rangle$$

to be proportional to the probability of detecting a photon at time t_j, but the units are not quite right. The actual probability is given by:

$$v_g\Delta t\langle\psi|\hat{a}^{\dagger}(z_d,t_j)\hat{a}(z_d,t_j)|\psi\rangle = v_g\Delta t\left\langle :\hat{a}^{\dagger}(z_d,t_j)\hat{a}(z_d,t_j):\right\rangle$$

More generally, this probability can be written as,

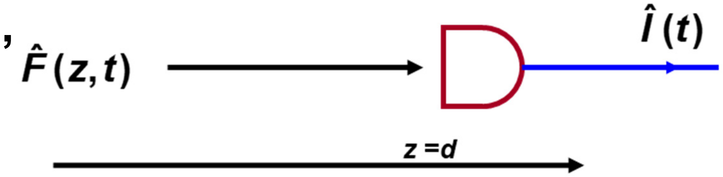
$$v_g\Delta t\left\langle :\hat{a}^{\dagger}(z_d,t_j)\hat{a}(z_d,t_j):\right\rangle$$

The colon represents normal ordering of the operators

Next, we calculate the probability of detecting one, and ONLY one, photon in the time interval T

Photon Counting Distribution

We now calculate the probability *P*(1,*T*) of detecting one, and ONLY one, photon in the time interval *T*



This is given by the probability of detecting one photon in some time interval, times the probability of detecting no photon in any other time interval:

$$P(1,T) = \underset{\Delta t \to 0}{Lt} \sum_{j=1}^{N} \left\langle : v_g \Delta t \hat{a}^{\dagger}(z_d, t_j)\hat{a}(z_d, t_j) \prod_{k \neq j} \left[1 - v_g \Delta t \hat{a}^{\dagger}(z_d, t_k)\hat{a}(z_d, t_k) \right] : \right\rangle$$

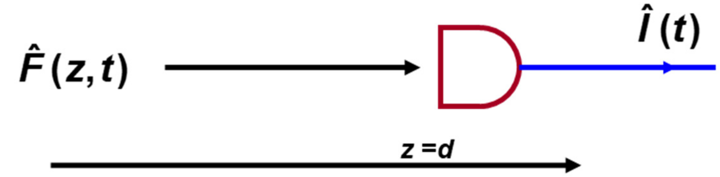
$$= \underset{\Delta t \to 0}{Lt} \sum_{j=1}^{N} \left\langle \begin{matrix} : v_g \Delta t \hat{a}^{\dagger}(z_d, t_j)\hat{a}(z_d, t_j) \left[1 + v_g \Delta t \hat{a}^{\dagger}(z_d, t_j)\hat{a}(z_d, t_j) \right] \\ \times \prod_{k=1}^{N} \left[1 - v_g \Delta t \hat{a}^{\dagger}(z_d, t_k)\hat{a}(z_d, t_k) \right] : \end{matrix} \right\rangle$$

$$= \underset{\Delta t \to 0}{Lt} \sum_{j=1}^{N} \left\langle : v_g \Delta t \hat{a}^{\dagger}(z_d, t_j)\hat{a}(z_d, t_j) \prod_{k=1}^{N} \left[1 - v_g \Delta t \hat{a}^{\dagger}(z_d, t_k)\hat{a}(z_d, t_k) \right] : \right\rangle$$

$$= \left\langle : v_g \int_0^T dt\; \hat{a}^{\dagger}(z_d, t)\hat{a}(z_d, t)\, e^{-v_g \int_0^T dt\; \hat{a}^{\dagger}(z_d, t)\hat{a}(z_d, t)} : \right\rangle$$

Photon Counting Distribution

Next, we calculate the probability $P(2,T)$ of detecting two, and ONLY two, photons in the time interval T



This is given by the probability of detecting one photon in some time interval, another photon in some other time interval, times the probability of detecting no photon in any other time interval:

$$P(2,T) = \underset{\Delta t \to 0}{Lt} \sum_{\substack{j,p=1 \\ j \neq p}}^{N} \left\langle \begin{array}{l} :\left[v_g \Delta t \hat{a}^\dagger(z_d,t_j)\hat{a}(z_d,t_j)\right]\left[v_g \Delta t \hat{a}^\dagger(z_d,t_p)\hat{a}(z_d,t_p)\right] \\ \times \prod\limits_{k \neq j,p} \left[1 - v_g \Delta t \hat{a}^\dagger(z_d,t_k)\hat{a}(z_d,t_k)\right]: \end{array} \right\rangle$$

$$= \underset{\Delta t \to 0}{Lt} \sum_{\substack{j,p=1 \\ j \neq p}}^{N} \left\langle \begin{array}{l} :\left[v_g \Delta t \hat{a}^\dagger(z_d,t_j)\hat{a}(z_d,t_j)\right]\left[v_g \Delta t \hat{a}^\dagger(z_d,t_p)\hat{a}(z_d,t_p)\right] \\ \times \prod\limits_{k=1}^{N} \left[1 - v_g \Delta t \hat{a}^\dagger(z_d,t_k)\hat{a}(z_d,t_k)\right]: \end{array} \right\rangle$$

$$= \left\langle : \frac{\left[v_g \int_0^T dt\, \hat{a}^\dagger(z_d,t)\hat{a}(z_d,t)\right]^2}{2!} e^{-v_g \int_0^T dt\, \hat{a}^\dagger(z_d,t)\hat{a}(z_d,t)} : \right\rangle$$

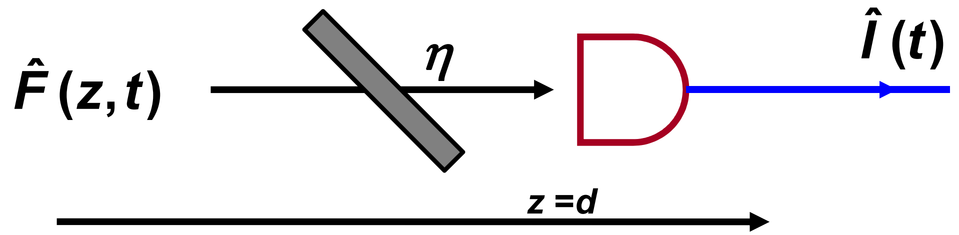
Photon Counting Distribution

Finally, we can generalize and write down the probability $P(m,T)$ of detecting m, and ONLY m, photons in the time interval T

$$P(m,T) = \left\langle : \frac{\left[v_g \int_0^T dt\, \hat{a}^\dagger(z_d,t)\,\hat{a}(z_d,t) \right]^m}{m!} e^{-v_g \int_0^T dt\, \hat{a}^\dagger(z_d,t)\hat{a}(z_d,t)} : \right\rangle$$

What if the quantum efficiency of the detector is η and not unity?

We can replace the detector with a beam-splitter/detector combination, as shown:



Since the vacuum introduced by the beam splitter does not contribute in the above expression for $P(m,T)$ because of normal ordering, we get:

$$P(m,T) = \left\langle : \frac{\left[v_g \eta \int_0^T dt\, \hat{a}^\dagger(z_d,t)\,\hat{a}(z_d,t) \right]^m}{m!} e^{-v_g \eta \int_0^T dt\, \hat{a}^\dagger(z_d,t)\hat{a}(z_d,t)} : \right\rangle$$

Photon Counting Distribution

Example 1: Suppose the input state is a CW coherent state with:

$$\alpha = \sqrt{\frac{P_o}{v_g \hbar\omega_o}} e^{i\phi} \qquad \Rightarrow \left\langle \hat{F}(z,t) \right\rangle = \frac{P_o}{\hbar\omega_o}$$

Then:

$$P(m,T) = \frac{1}{m!}\left(\frac{P_o T}{\hbar\omega_o}\right)^m e^{-\frac{P_o T}{\hbar\omega_o}} \longrightarrow$$

Poisson counting statistics!

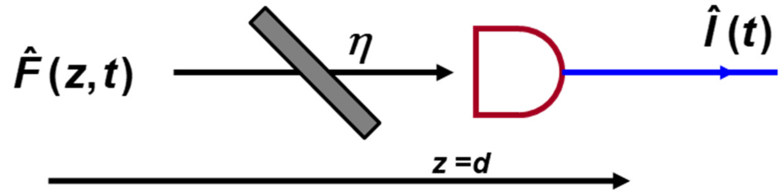
Example 2: Suppose the input state is a number state packet with "*N*" photons

$$\left|\psi\right\rangle = \frac{\left(\int_{-\infty}^{\infty} A(z')\,\hat{a}^{+}(z',0)\,dz'\right)^N}{\sqrt{N!}}\left|0\right\rangle$$

And we assume that the detection time is long enough that the entire packet has gone into the detector. Then obviously we get,

$$P(m,T) = \delta_{m,N}$$

Photon Counting Distribution



Example 3: Suppose the input state is a number state packet with "*N*" photons

$$|\psi\rangle = \frac{\left(\int_{-\infty}^{\infty} A(z')\,\hat{a}^{+}(z',0)\,dz'\right)^{N}}{\sqrt{N!}}|0\rangle$$

And we assume that the detection time is long enough that the entire packet has gone into the detector. But now the detector quantum efficiency is η :

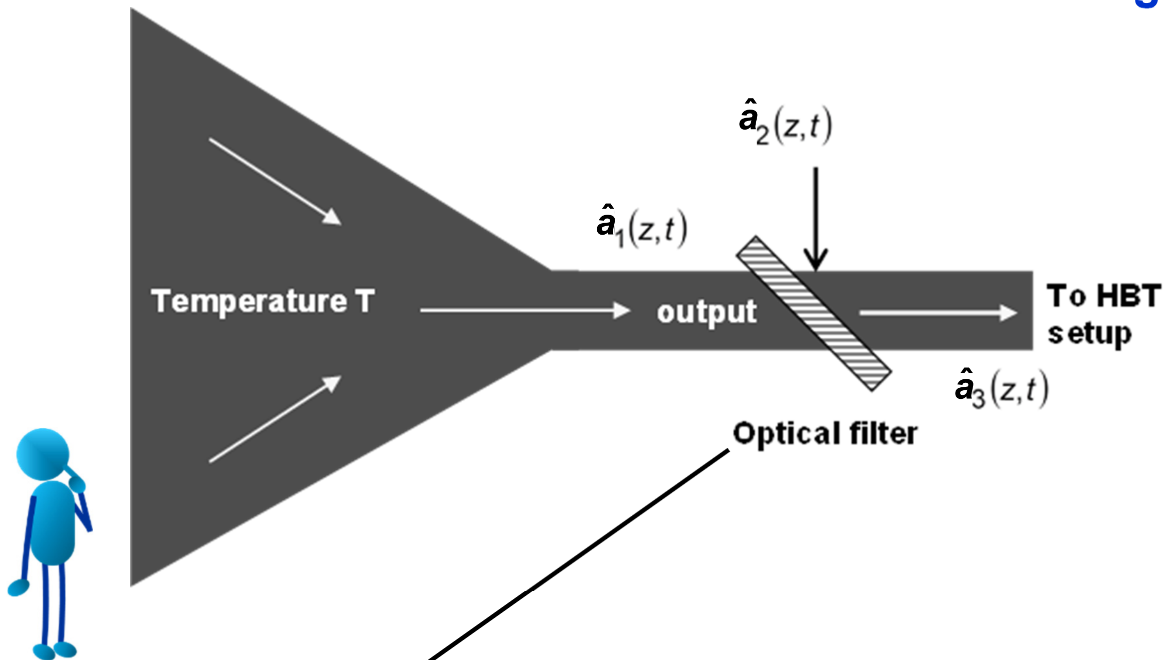
$$P(m,T) = \left\langle :\frac{\left[v_g\eta\int_0^T dt\;\hat{a}^{\dagger}(z_d,t)\,\hat{a}(z_d,t)\right]^m}{m!}e^{-v_g\eta\int_0^T dt\;\hat{a}^{\dagger}(z_d,t)\hat{a}(z_d,t)}:\right\rangle$$

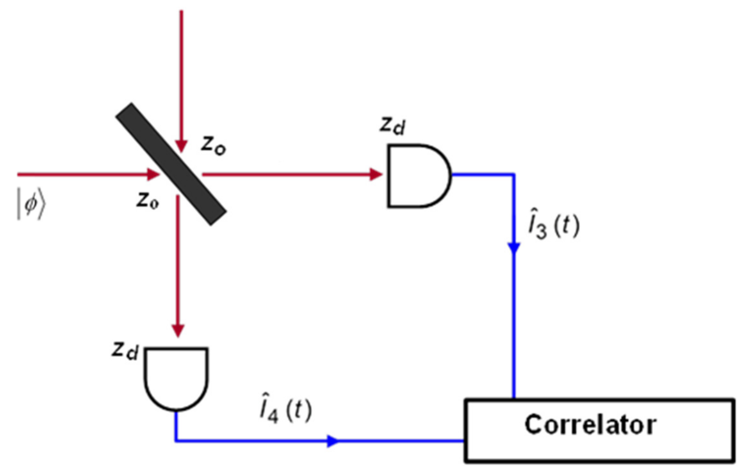
$$= \frac{N!}{m!(N-m)!}\eta^m(1-\eta)^{N-m} \qquad \{ \text{ provided } m \le N$$

Note the binomial distribution resulting from photon loss at the beam splitter!

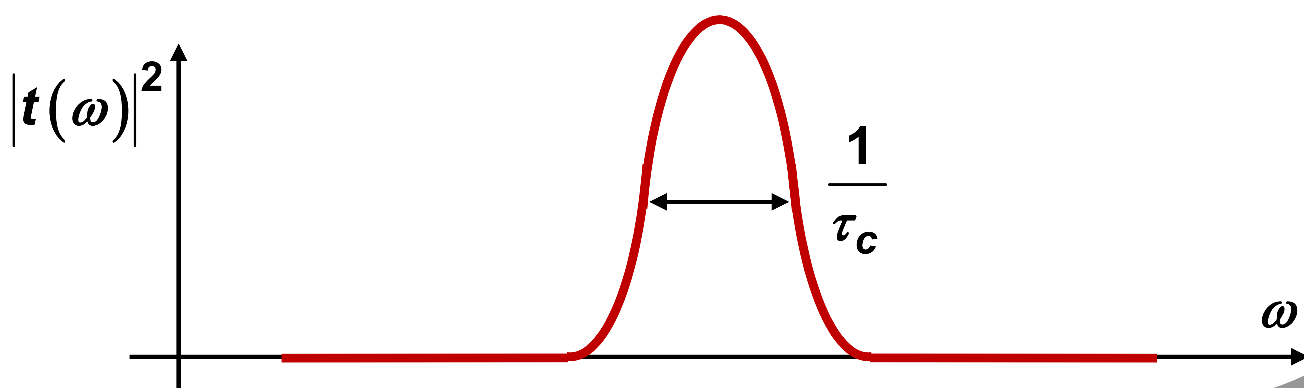
Filtered Thermal Radiation and HBT Setup

Consider filtered radiation from a thermal source going into a HBT setup:

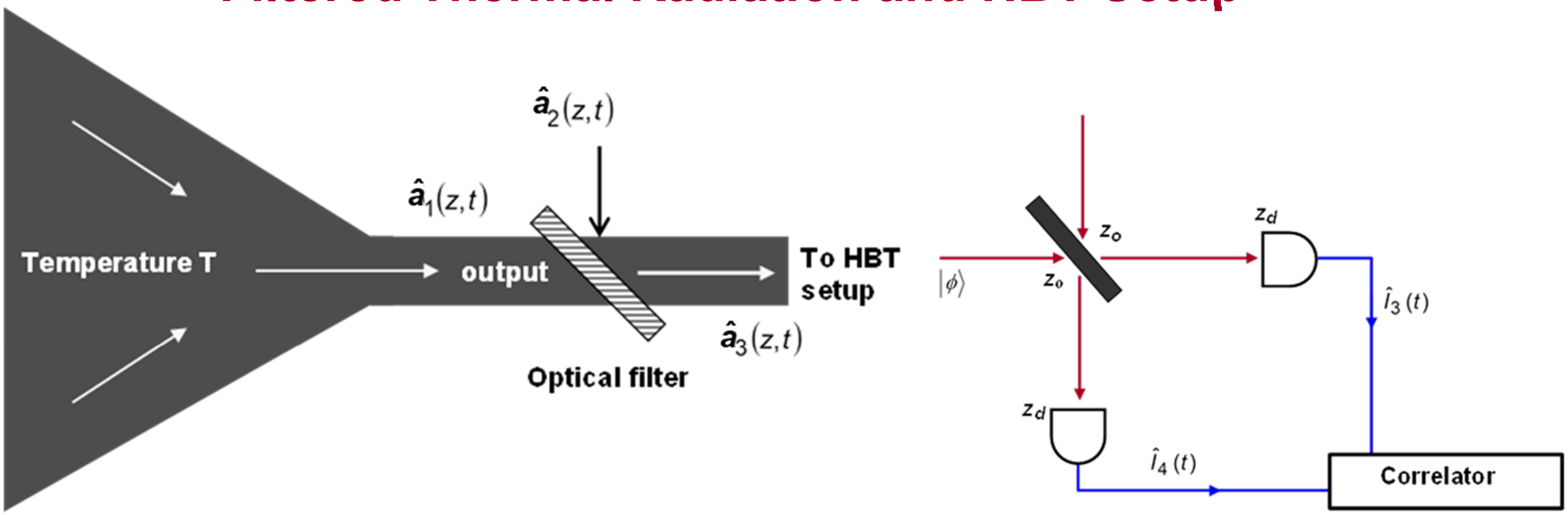




$$|t(\beta)|^2 = e^{-\frac{(\beta-\beta_o)^2 L_c^2}{2}} = e^{-\frac{(\omega(\beta)-\omega_o)^2 L_c^2}{2 v_g^2}} = |t(\omega)|^2 = e^{-\frac{(\omega(\beta)-\omega_o)^2 \tau_c^2}{2}}$$



Filtered Thermal Radiation and HBT Setup



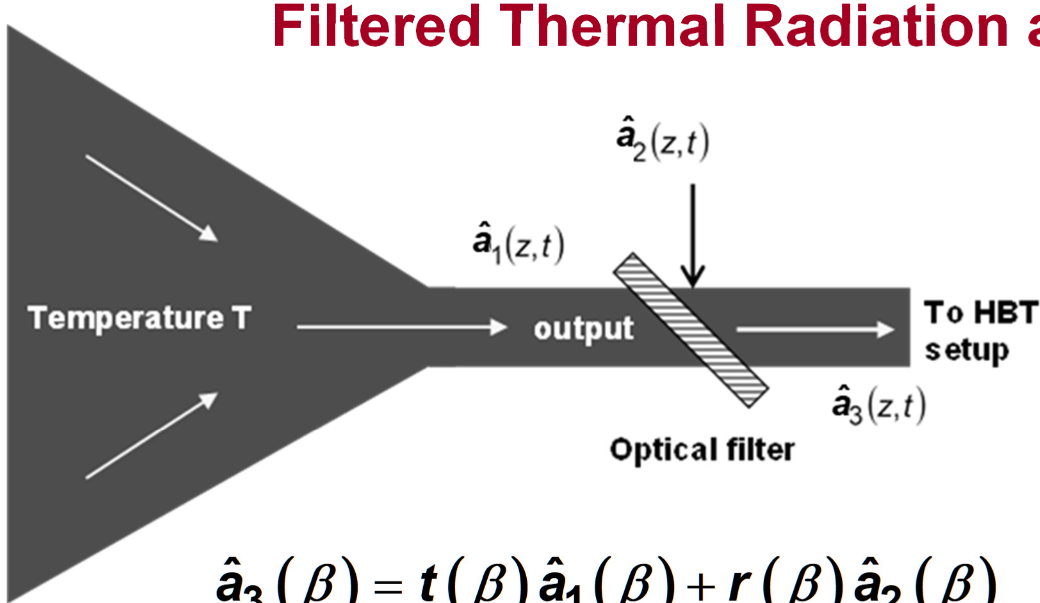
$$\hat{a}_1(z,t) = L \int_{\beta_o-\Delta\beta/2}^{\beta_o+\Delta\beta/2} \frac{d\beta}{2\pi}\ \hat{a}_1(\beta,t)\, \frac{\exp\left[i(\beta-\beta_o)\,z\right]}{\sqrt{L}}$$

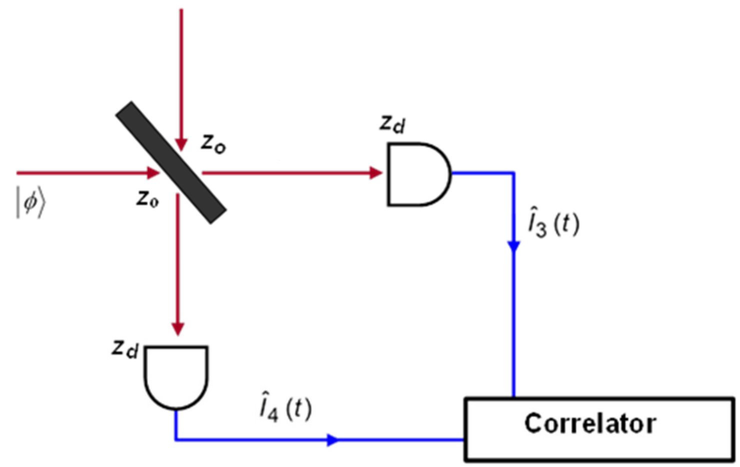
Characteristics of thermal radiation are as follows:

$$\left\langle \hat{a}_1^+(\beta)\,\hat{a}_1(\beta')\right\rangle = n_{th}(\omega(\beta))\,\delta_{\beta\beta'} = n_{th}(\omega_o)\,\delta_{\beta\beta'}$$

$$\left\langle \hat{a}_1^+(\beta)\,\hat{a}_1^+(\beta')\,\hat{a}_1(\beta'')\,\hat{a}_1(\beta''')\right\rangle = \left(n_{th}(\omega_o)\right)^2 \left(\delta_{\beta\beta'''}\delta_{\beta'\beta''} + \delta_{\beta\beta''}\delta_{\beta'\beta'''}\right)$$

Filtered Thermal Radiation and HBT Setup





$$\hat{a}_3(\beta) = t(\beta)\,\hat{a}_1(\beta) + r(\beta)\,\hat{a}_2(\beta)$$

Average flux:

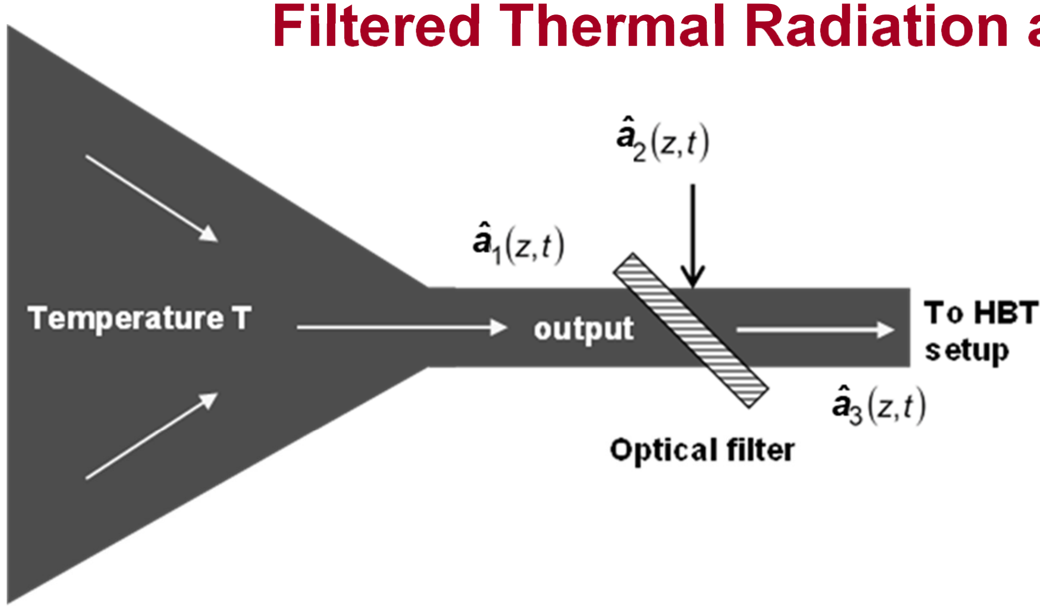
$$\left\langle \hat{F}_3(z,t) \right\rangle = v_g \left\langle \hat{a}_3^{+}(z,t)\,\hat{a}_3(z,t) \right\rangle$$

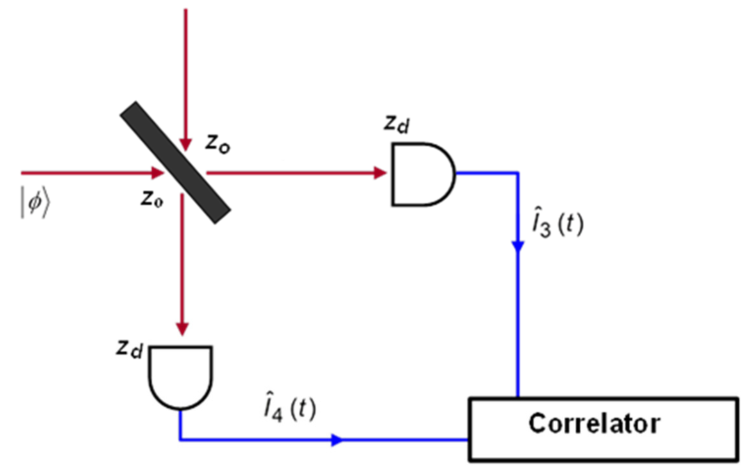
$$= v_g n_{th}(\omega_o) \int_{\beta_o+\Delta\beta/2}^{\beta_o+\Delta\beta/2} \frac{d\beta}{2\pi} \left| t(\beta)^2 \right|$$

$$= \frac{v_g}{\sqrt{2\pi}\;L_c} n_{th}(\omega_o)$$

$$= \frac{n_{th}(\omega_o)}{\sqrt{2\pi}\;\tau_c} = F_o$$

Filtered Thermal Radiation and HBT Setup





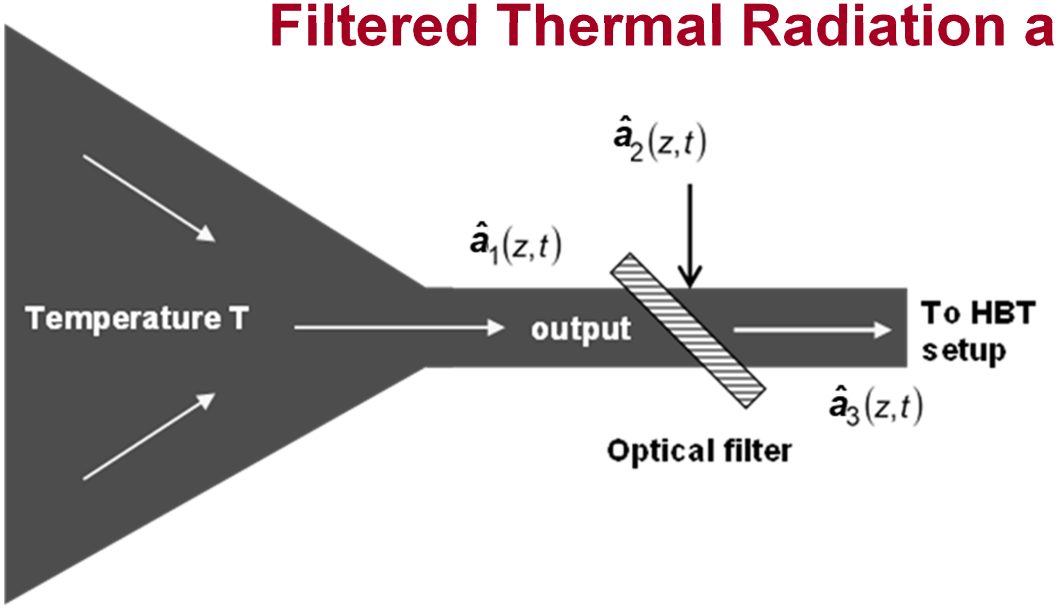
First Order Coherence:

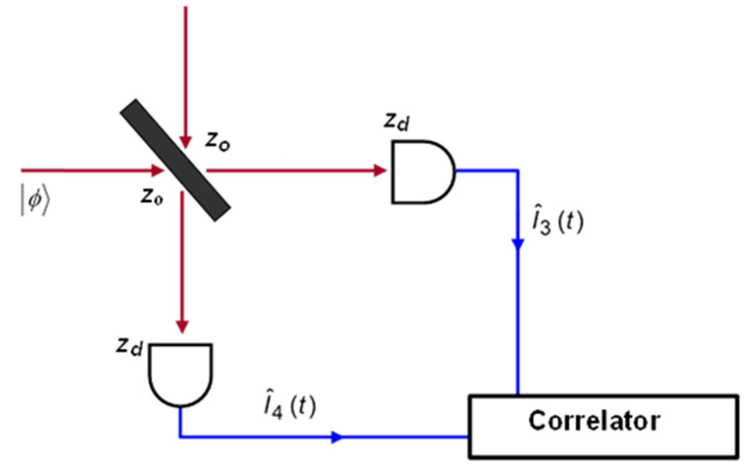
$$g_1(z_o:t_1,t_2)=\frac{\left\langle \hat{a}_3^{+}(z_o,t_1)\,\hat{a}_3(z_o,t_2)\right\rangle}{\sqrt{\left\langle \hat{a}_3^{+}(z_o,t_1)\,\hat{a}_3(z_o,t_1)\right\rangle\left\langle \hat{a}_3^{+}(z_o,t_2)\,\hat{a}_3(z_o,t_2)\right\rangle}}$$

$$=\frac{L\int_{\beta_o-\Delta\beta/2}^{\beta_o+\Delta\beta/2}\frac{d\beta'}{2\pi}\,L\int_{\beta_o-\Delta\beta/2}^{\beta_o+\Delta\beta/2}\frac{d\beta}{2\pi}\,t^{*}(\beta')\,t(\beta)\left\langle \hat{a}_1^{+}(\beta')\,\hat{a}_1(\beta)\right\rangle\frac{e^{i(\beta-\beta')z}}{L}\,e^{-i\left(\omega(\beta)t_2-\omega(\beta')t_1\right)}}{F_o}$$

$$g_1(\tau)=e^{-(\tau/\tau_c)^2/2}\,e^{-i\omega_o\tau}$$

Filtered Thermal Radiation and HBT Setup





Second Order Coherence:

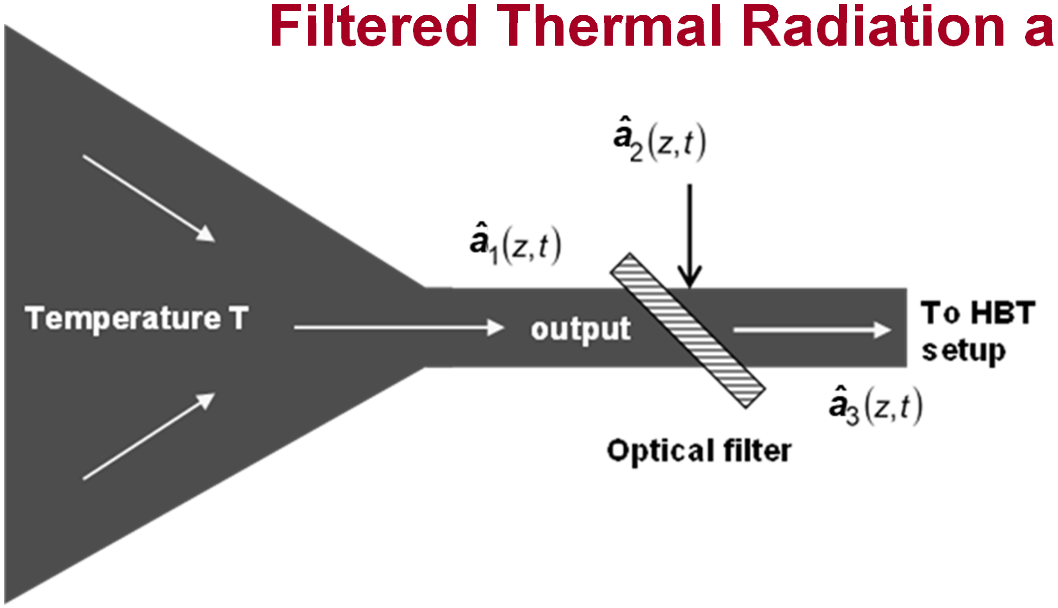
$$g_2(z_o : t_1, t_2) = \frac{\left\langle \hat{a}_3^+(z_o,t_2)\hat{a}_3^+(z_o,t_1)\ \hat{a}_3(z_o,t_1)\ \hat{a}_3(z_o,t_2)\right\rangle}{\left\langle \hat{a}_3^+(z_o,t_1)\hat{a}_3(z_o,t_1)\right\rangle\left\langle \hat{a}_3^+(z_o,t_2)\hat{a}_3(z_o,t_2)\right\rangle}$$

$$= \frac{\left\langle \hat{a}_3^+(z_o,t_2)\hat{a}_3^+(z_o,t_1)\ \hat{a}_3(z_o,t_1)\ \hat{a}_3(z_o,t_2)\right\rangle}{F_o^2}$$

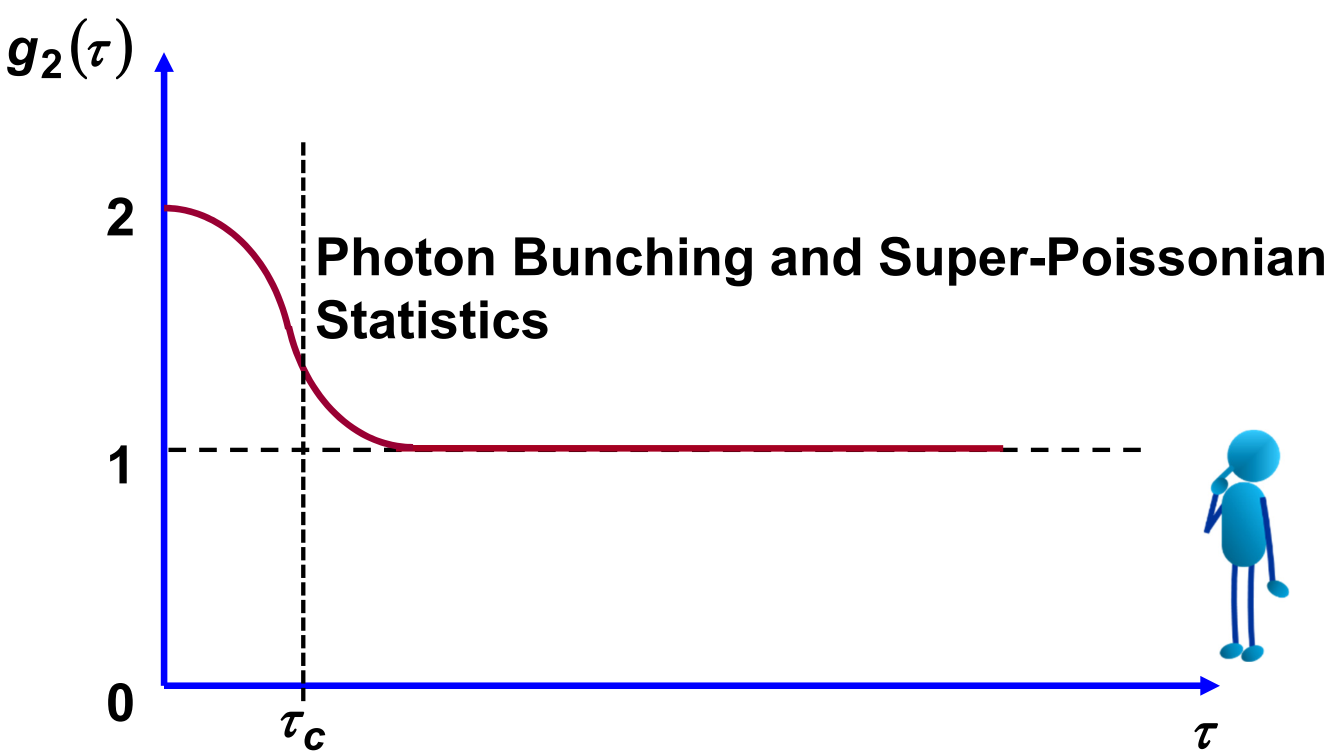
$$g_2(\tau) = 1 + e^{-(\tau/\tau_c)^2} = 1 + \left|g_1(\tau)\right|^2$$

A general feature of light with Gaussian statistics!

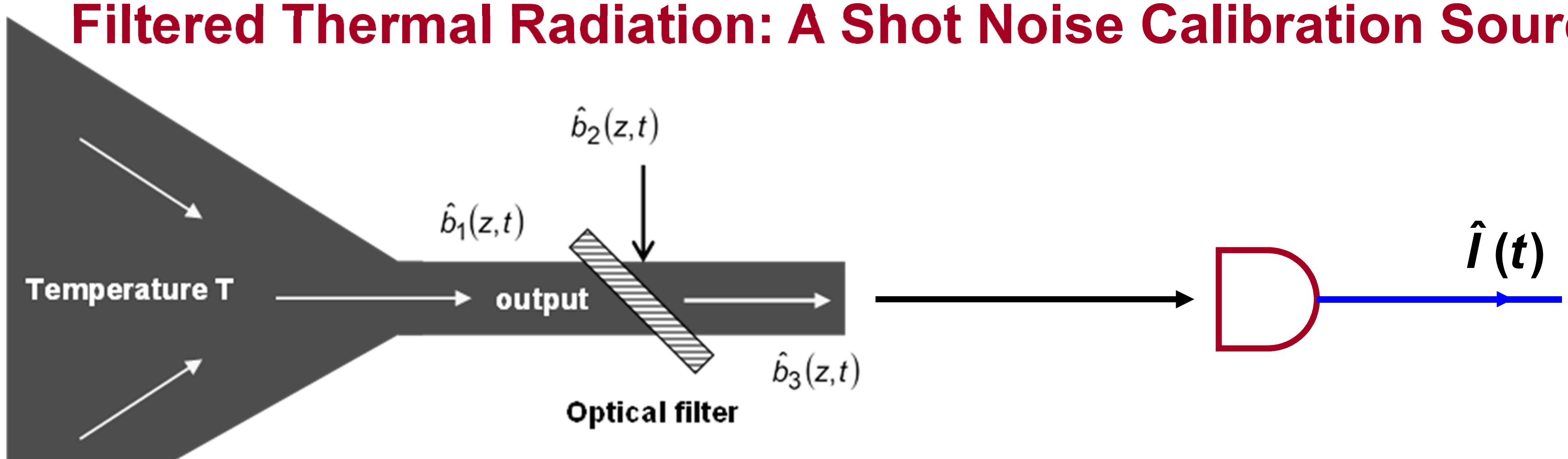
Filtered Thermal Radiation and HBT Setup



$$g_2(\tau) = 1 + e^{-(\tau/\tau_c)^2} = 1 + |g_1(\tau)|^2$$



Filtered Thermal Radiation: A Shot Noise Calibration Source

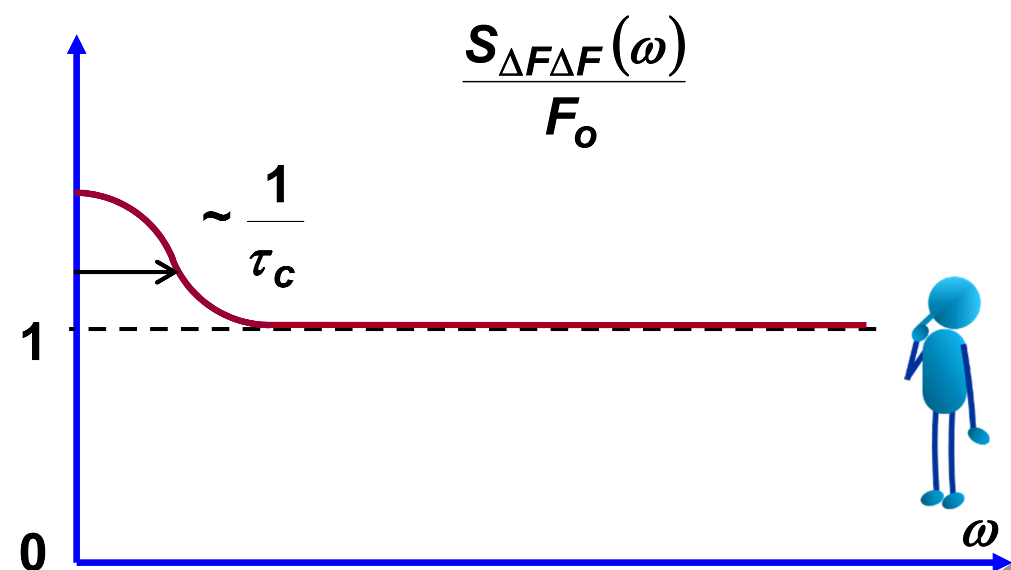


$$\left\langle \Delta\hat{F}(z_o, t+\tau)\, \Delta\hat{F}(z_o, t) \right\rangle = F_o\, \delta(\tau) + F_o^2\, [g_2(\tau) - 1] = F_o\, \delta(\tau) + F_o^2\, |g_1(\tau)|^2$$

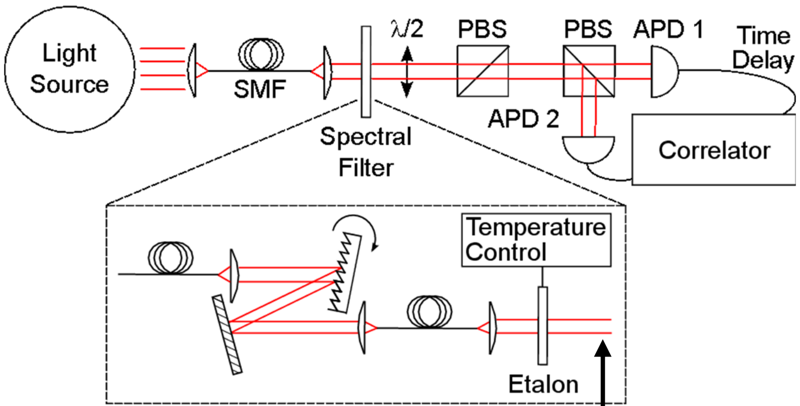
$$\Rightarrow S_{\Delta F \Delta F}(\omega) = F_o + F_o^2 \sqrt{\pi}\tau_c e^{-\frac{\omega^2 \tau_c^2}{4}}$$

$$\Rightarrow \frac{S_{\Delta F \Delta F}(\omega)}{F_o} = 1 + F_o \sqrt{\pi}\tau_c e^{-\frac{\omega^2 \tau_c^2}{4}}$$

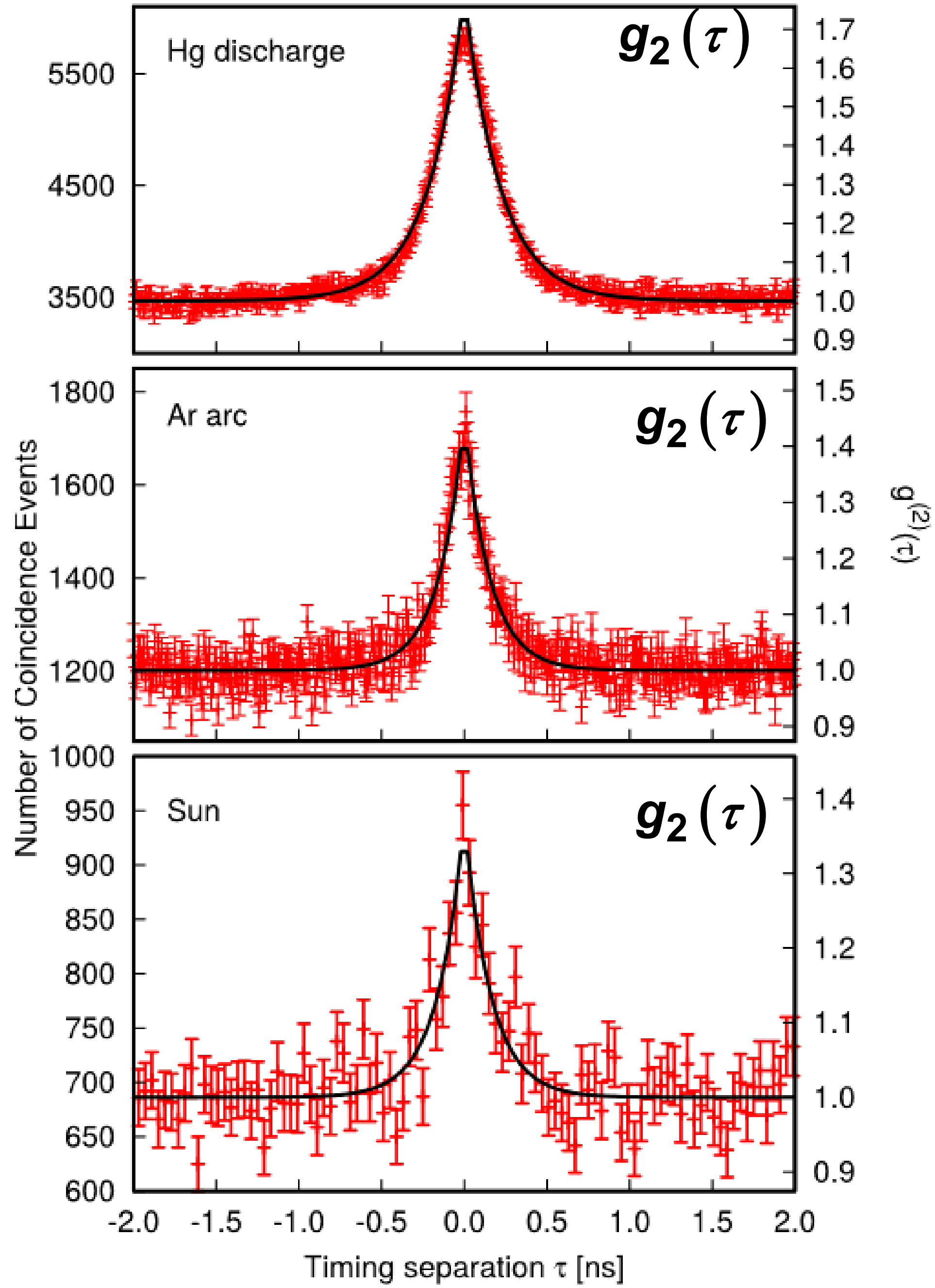
$$= 1 + \frac{n_{th}(\omega_o)}{\sqrt{2}} e^{-\frac{\omega^2 \tau_c^2}{4}}$$



Photon Bunching: Experiments



~2 GHz optical bandwidth after filtering



Photon Bunching: Classical Gaussian Light

The relation:

$$g_2(\tau) = 1 + |g_1(\tau)|^2$$

is a general feature of classical Gaussian light – defined as one for which all higher order correlation functions can be written as sums of all non-zero two-time correlation functions.

Suppose for a Gaussian light:

$$\langle E^*(t)E(t)\rangle = |E_o|^2$$

And:

$$g_1(\tau) = \frac{\langle E^*(t)E(t+\tau)\rangle}{|E_o|^2}$$

Photon Bunching: Classical Gaussian Light

For Gaussian light:

$$g_2(\tau) = \frac{\left\langle E^*(t)E^*(t+\tau)E(t+\tau)E(t)\right\rangle}{\left|E_o\right|^4}$$

$$= \left\langle E^*(t)E(t)\right\rangle\left\langle E^*(t+\tau)E(t+\tau)\right\rangle + \left\langle E^*(t)E(t+\tau)\right\rangle\left\langle E^*(t+\tau)E(t)\right\rangle$$

$$= 1 + \left|g_1(\tau)\right|^2$$

Photon Bunching: Classical Chaotic Light

The relation:

$$g_2(\tau) = 1 + \left|g_1(\tau)\right|^2$$

is a general feature of classical chaotic light – defined as made up of a superposition of randomly phased fields (from different atoms, for example):

$$E(t) = \sum_{i=1}^{N} E_i(t)$$

For chaotic light:

$$\boxed{\left\langle E^*(t)E(t)\right\rangle = \left\langle \sum_{i=1}^{N} E_i^*(t) \sum_{j=1}^{N} E_j(t)\right\rangle = \sum_{j=1}^{N}\left\langle E_j^*(t)E_j(t)\right\rangle = N\left|E_o\right|^2}$$

$$\Rightarrow g_1(\tau) = \frac{\left\langle E^*(t)E(t+\tau)\right\rangle}{N\left|E_o\right|^2} = \frac{\left\langle \sum_{i=1}^{N} E_i^*(t) \sum_{j=1}^{N} E_j(t+\tau)\right\rangle}{N\left|E_o\right|^2} = \frac{\sum_{j=1}^{N}\left\langle E_j^*(t)E_j(t+\tau)\right\rangle}{N\left|E_o\right|^2}$$

Photon Bunching: Classical Chaotic Light

For chaotic light:

$$g_2(\tau) = \frac{\left\langle E^*(t)E^*(t+\tau)E(t+\tau)E(t)\right\rangle}{N^2|E_o|^4} = \sum_{i,j,k,\ell=1}^{N} \frac{\left\langle E_i^*(t)E_j^*(t+\tau)E_k(t+\tau)E_\ell(t)\right\rangle}{N^2|E_o|^4}$$

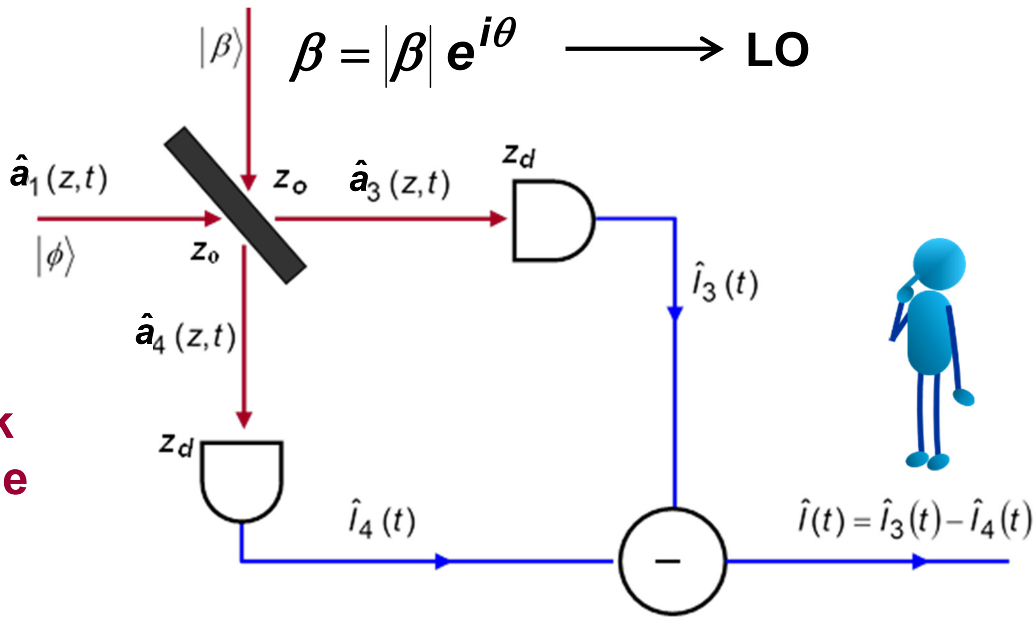
$$= \sum_{i=1}^{N} \frac{\left\langle E_i^*(t)E_i^*(t+\tau)E_i(t+\tau)E_i(t)\right\rangle}{N^2|E_o|^4} + \sum_{\substack{i,j\\ i\neq j}}^{N} \frac{\left\langle E_i^*(t)E_j^*(t+\tau)E_j(t+\tau)E_i(t)\right\rangle}{N^2|E_o|^4} +$$

$$+ \sum_{\substack{i,j\\ i\neq j}}^{N} \frac{\left\langle E_i^*(t)E_j^*(t+\tau)E_i(t+\tau)E_j(t)\right\rangle}{N^2|E_o|^4}$$

$$= \frac{1}{N} + \frac{N(N-1)}{N^2} + \frac{N(N-1)}{N^2}|g_1(\tau)|^2$$

$$= 1 + \left(1 - \frac{1}{N}\right)|g_1(\tau)|^2 \approx 1 + |g_1(\tau)|^2 \qquad (N \to \infty)$$

Balanced Homodyne Detection



Consider the following input state:

$$|\psi(t=0)\rangle = |\phi\rangle_1 \otimes |\beta\rangle_2$$

What happens when one mixes a weak signal with a strong signal (at the same frequency) in photodetection?

$$\left| |\beta| e^{i\theta - i\omega_o t} \pm a(t) e^{-i\omega_o t} \right|^2 \approx |\beta|^2 \pm 2|\beta| \left(\frac{a(t)e^{-i\theta} + a^*(t)e^{i\theta}}{2} \right) \longrightarrow \text{Quadrature!}$$

We have a 50-50 beam splitter:

$$\begin{bmatrix} \hat{a}_3(z_o,t)\, e^{i\beta_o z_o} \\ \hat{a}_4(z_o,t)\, e^{i\beta_o z_o} \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} \hat{a}_1(z_o,t)\, e^{i\beta_o z_o} \\ \hat{a}_2(z_o,t)\, e^{i\beta_o z_o} \end{bmatrix}$$

$$\Rightarrow \hat{a}_3(z_o,t) = \frac{1}{\sqrt{2}} \left[\hat{a}_1(z_o,t) + \hat{a}_2(z_o,t) \right]$$

$$\Rightarrow \hat{a}_4(z_o,t) = \frac{1}{\sqrt{2}} \left[-\hat{a}_1(z_o,t) + \hat{a}_2(z_o,t) \right]$$

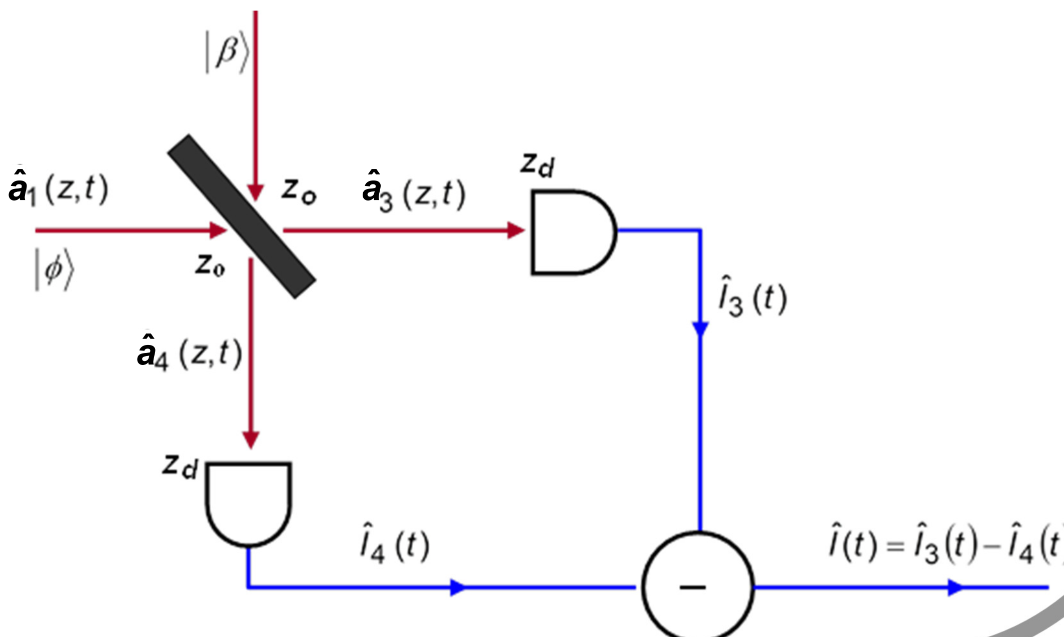
Balanced Homodyne Detection

Current Operator:

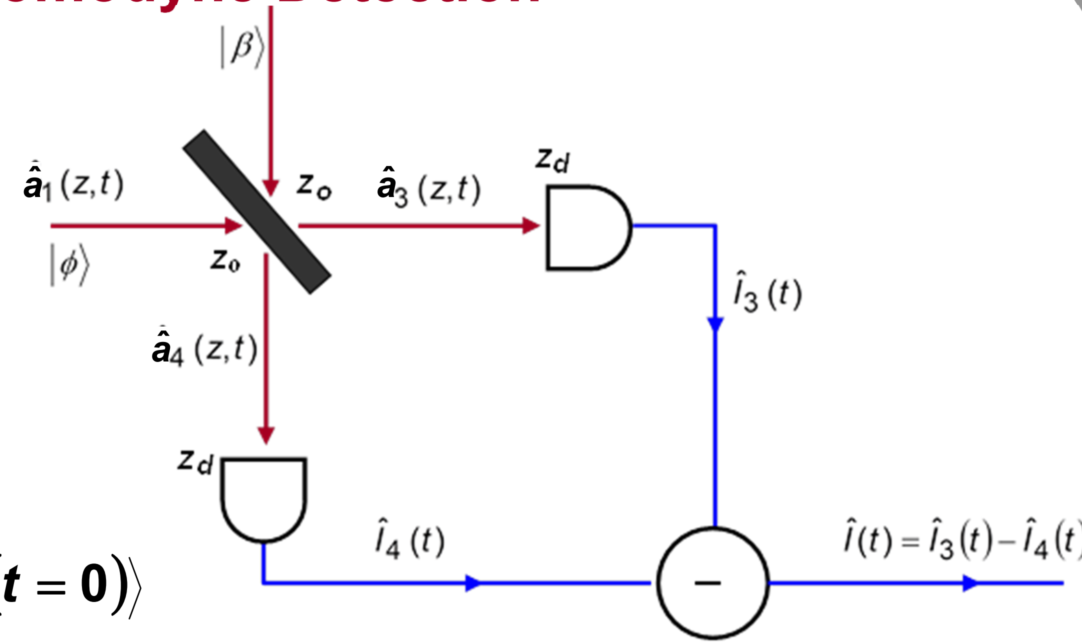
$$\hat{i}(t) = \hat{i}_3(t) - \hat{i}_4(t)$$
$$= q v_g \left[\hat{a}_3^+(z_d,t)\, \hat{a}_3(z_d,t) - \hat{a}_4^+(z_d,t)\, \hat{a}_4(z_d,t) \right]$$
$$= q v_g \left[\hat{a}_2^+(z_d - v_g t,0)\, \hat{a}_1(z_d - v_g t,0) + \hat{a}_1^+(z_d - v_g t,0)\, \hat{a}_2(z_d - v_g t,0) \right]$$

Average Current:

$$\left\langle \hat{i}(t) \right\rangle = \left\langle \psi(t=0) \right| \hat{i}(t) \left| \psi(t=0) \right\rangle$$
$$= q v_g \left\langle \psi(t=0) \right| \hat{a}_2^+(z_d - v_g t,0)\, \hat{a}_1(z_d - v_g t,0) + \hat{a}_1^+(z_d - v_g t,0)\, \hat{a}_2(z_d - v_g t,0) \left| \psi(t=0) \right\rangle$$
$$= q v_g \left|\beta\right|\, {}_1\!\left\langle \varphi \right| e^{-i\theta}\, \hat{a}_1(z_d - v_g t,0) + e^{i\theta}\, \hat{a}_1^+(z_d - v_g t,0) \left| \varphi \right\rangle_1$$
$$= 2 q v_g \left|\beta\right|\, {}_1\!\left\langle \varphi \right| \hat{x}_\theta(z_d - v_g t,0) \left| \varphi \right\rangle_1$$



Balanced Homodyne Detection



Current Correlation:

$$\left\langle \hat{I}(t_1)\,\hat{I}(t_2)\right\rangle = \langle\psi(t=0)|\,\hat{I}(t_1)\,\hat{I}(t_2)\,|\psi(t=0)\rangle$$
$$\approx \left(2qv_g\,|\beta|\right)^2\;{}_1\langle\phi|\,\hat{x}_\theta\left(z_d - v_g t_1, 0\right)\hat{x}_\theta\left(z_d - v_g t_2, 0\right)|\phi\rangle_1$$

(Ignoring terms that are not proportional to $|\beta|^2$)

Current Noise Correlation:

$$\left\langle \Delta\hat{I}(t_1)\,\Delta\hat{I}(t_2)\right\rangle = \langle\psi(t=0)|\,\hat{I}(t_1)\,\hat{I}(t_2)\,|\psi(t=0)\rangle$$
$$\approx \left(2qv_g\,|\beta|\right)^2\;{}_1\langle\phi|\,\Delta\hat{x}_\theta\left(z_d - v_g t_1, 0\right)\Delta\hat{x}_\theta\left(z_d - v_g t_2, 0\right)|\phi\rangle_1$$

Quadratures and quadrature noise can be measured using Homodyne detection!

Squeezed State Generation and Detection

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Generation of Squeezed States by Parametric Down Conversion

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(Received 11 September 1986)

Squeezed states of the electromagnetic field are generated by degenerate parametric down conversion in an optical cavity. Noise reductions greater than 50% relative to the vacuum noise level are observed in a balanced homodyne detector. A quantitative comparison with theory suggests that the observed squeezing results from a field that in the absence of linear attenuation would be squeezed by greater then tenfold.

PACS numbers: 42.50.Dv, 03.65−w, 42.65.Ky

Squeezed State Generation and Detection

$$\left\langle \Delta\hat{i}\left(t_1\right)\Delta\hat{i}\left(t_2\right)\right\rangle \approx \left(2qv_g\left|\beta\right|\right)^2\ {}_1\!\left\langle\varphi\right|\Delta\hat{x}_\theta\left(z_d-v_gt_1,0\right)\Delta\hat{x}_\theta\left(z_d-v_gt_2,0\right)\left|\varphi\right\rangle_1$$

$$S_{\Delta I\Delta I}\left(\omega\right)=\left(2qv_g\left|\beta\right|\right)^2 S_{\theta\theta}\left(\omega\right)$$

$$V\left(\theta\right)=\int\frac{d\omega}{2\pi}S_{\Delta I\Delta I}\left(\omega\right)R_{input}^2$$

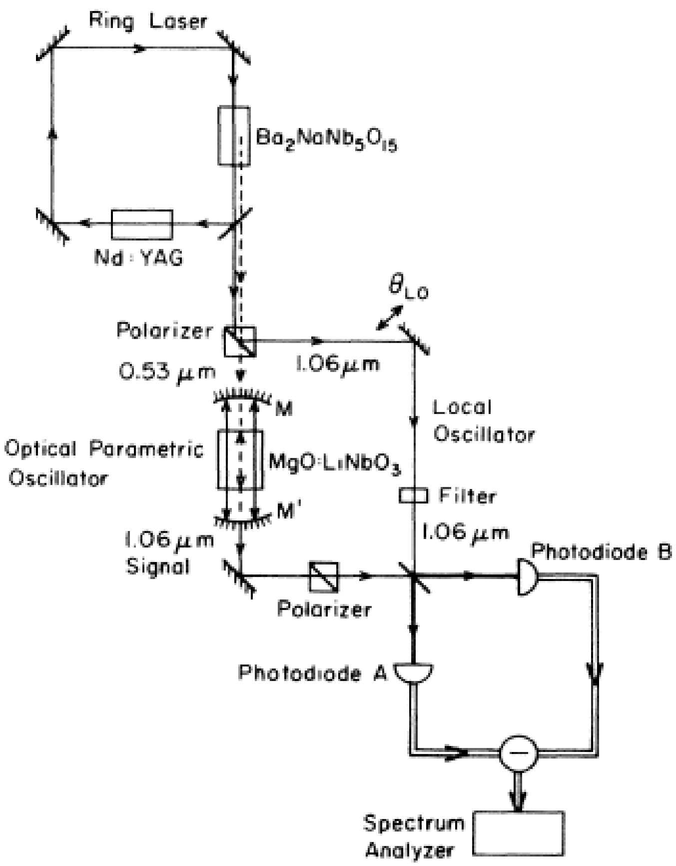
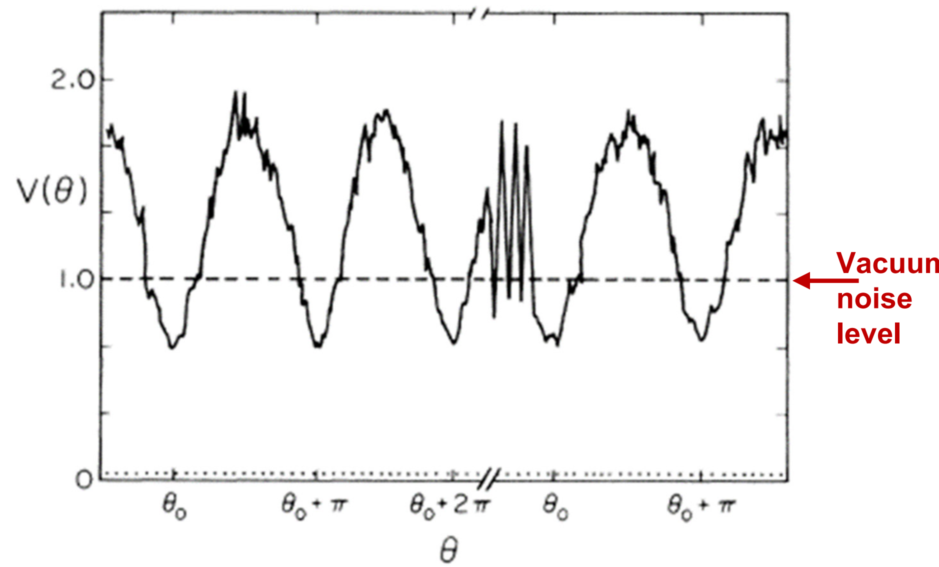


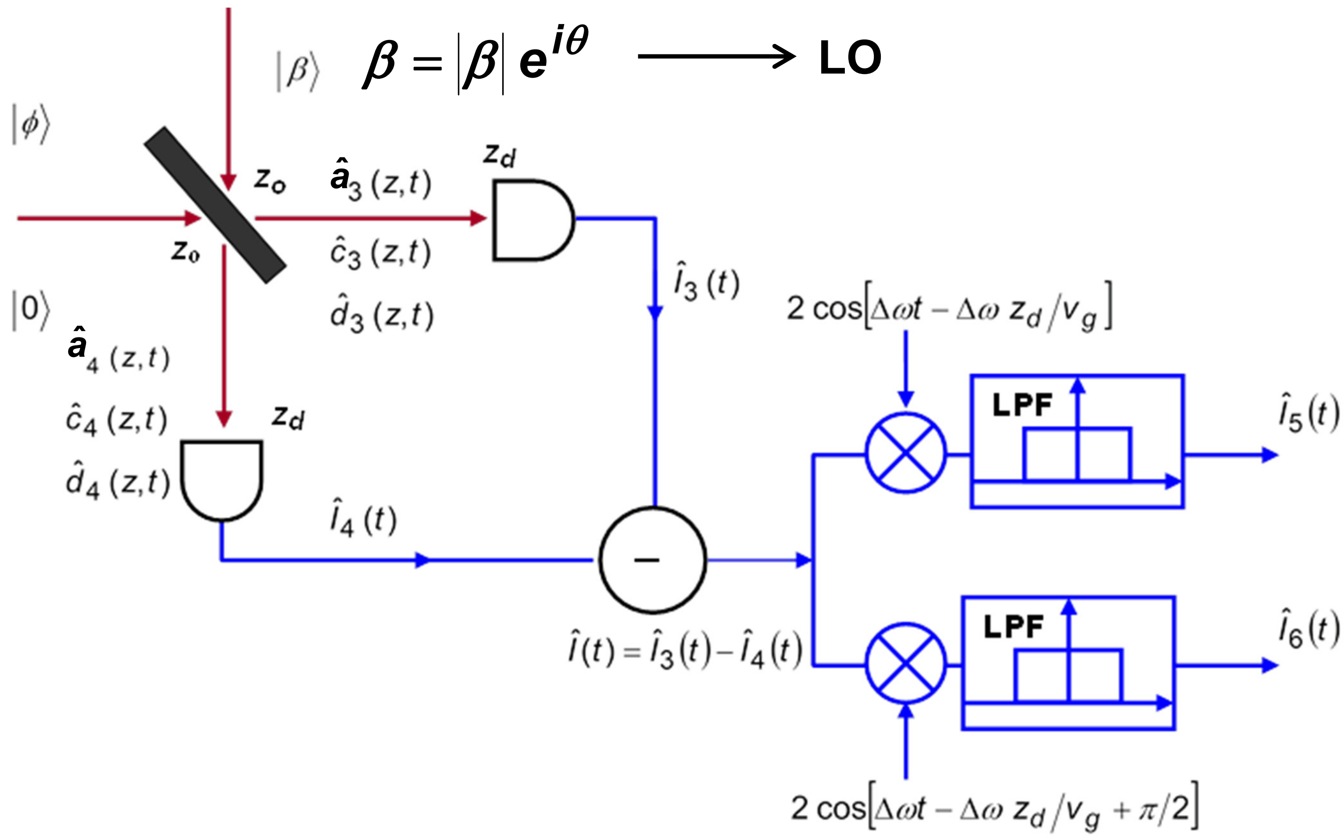
FIG. 2. Diagram of the principal elements of the apparatus for squeezed-state generation by degenerate parametric down conversion.



Measurement of the phase dependence of the quantum fluctuations in a squeezed state produced by degenerate parametric down conversion.

the phase dependence of the rms noise voltage $V(\theta)$ from a balanced homodyne detector is displayed as a function of local oscillator phase θ at fixed analysis frequency (1.8 MHz) and bandwidth (100 kHz) in the spectral distribution of photocurrent fluctuations. With the OPO input blocked, the vacuum field entering the signal port of the detector produces the noise voltage given by the dashed line with no sensitivity on θ. With the OPO input present, the dips below the vacuum level represent a 50% reduction in noise power relative to the vacuum noise level. Note that the ordinate is a linear scale in noise voltage. The dotted line is the amplifier noise level.

Balanced Heterodyne Detection



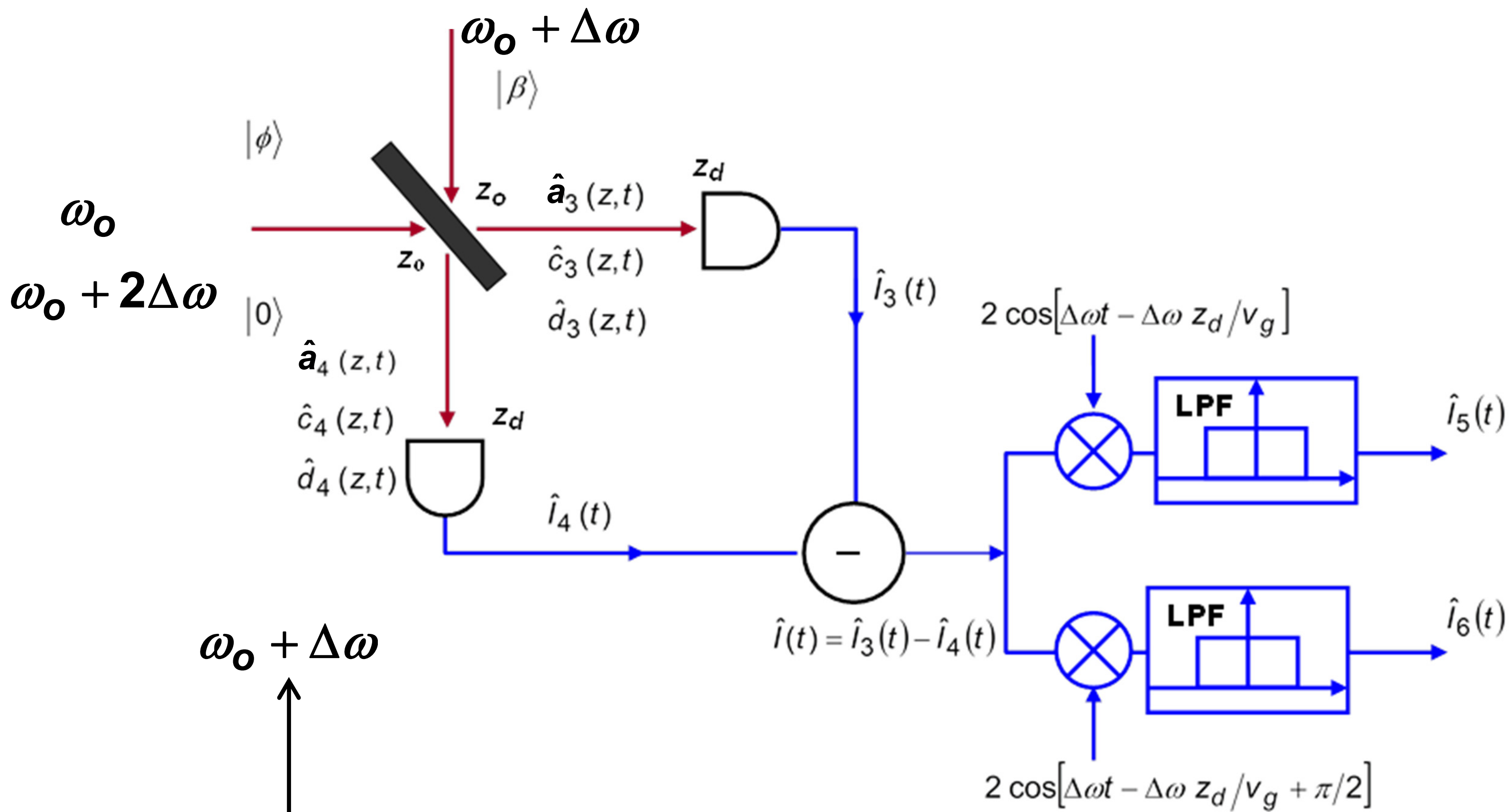
$$\left| |\beta| e^{i\theta - i(\omega_o + \Delta\omega)t} \pm a(t) e^{-i\omega_o t} \right|^2 \approx |\beta|^2 \pm 2|\beta| \left(\frac{a(t)e^{-i\theta}e^{i\Delta\omega t} + a^*(t)e^{i\theta}e^{-i\Delta\omega t}}{2} \right)$$

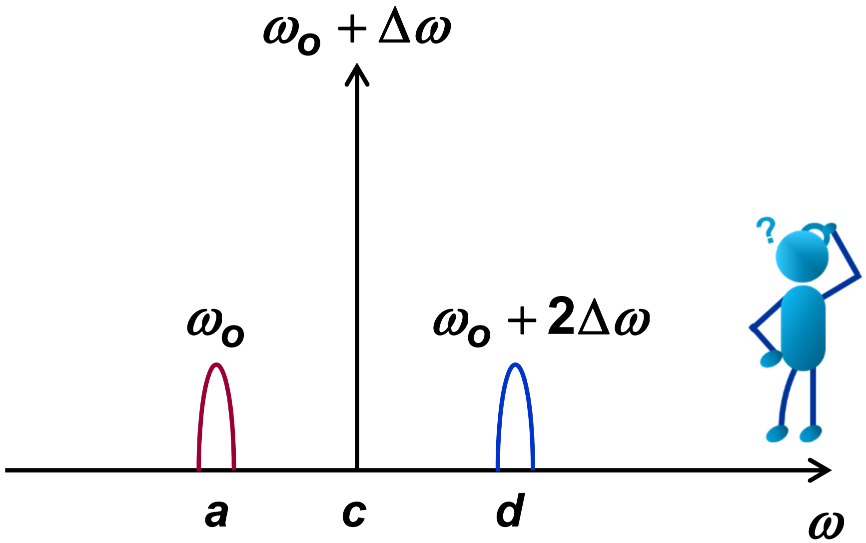
$\otimes$ $\cos(\Delta\omega t)$ $\qquad\qquad$ $\otimes$ $\sin(\Delta\omega t)$

$$\propto |\beta| \left(\frac{a(t)e^{-i\theta} + a^*(t)e^{i\theta}}{2} \right) \qquad\qquad \propto |\beta| \left(\frac{a(t)e^{-i\theta} - a^*(t)e^{i\theta}}{2i} \right)$$

Balanced Heterodyne Detection

What kind of signal frequencies can show up in the currents I_5 and I_6?





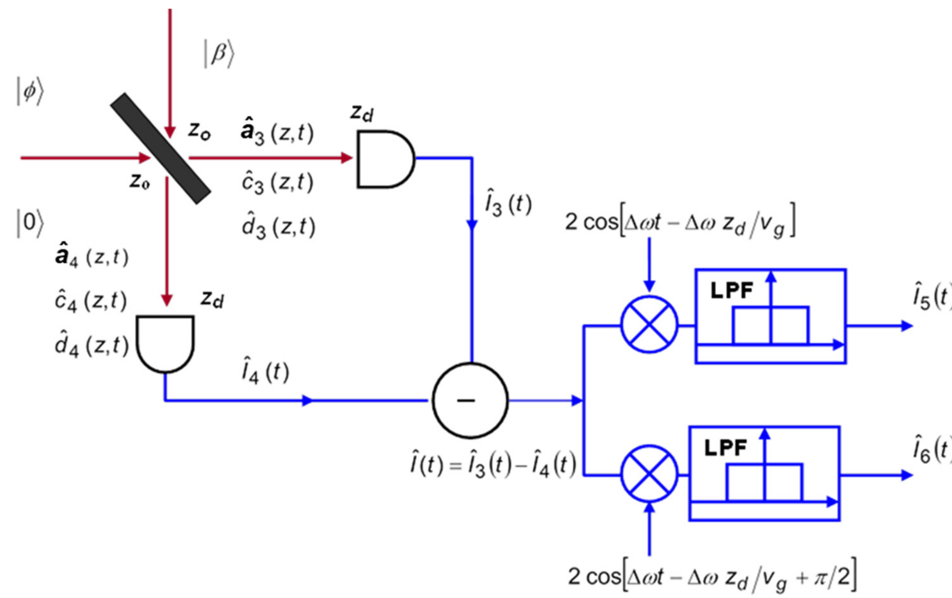
Balanced Heterodyne Detection

The operators for the field in each channel must include contributions from each frequency:

$$\hat{E}_k(z,t) \propto \hat{a}_k(z,t)e^{i\beta_o z} + \hat{c}_k(z,t)e^{i\left(\beta_o + \frac{\Delta\omega}{v_g}\right)z} + \hat{d}_k(z,t)e^{i\left(\beta_o + \frac{2\Delta\omega}{v_g}\right)z} + h.c.$$

Input state:

$$|\psi(t=0)\rangle = |\phi\rangle_1^{\omega_o} \otimes |0\rangle_1^{\omega_o+2\Delta\omega} \otimes |\beta\rangle_2^{\omega_o+\Delta\omega}$$



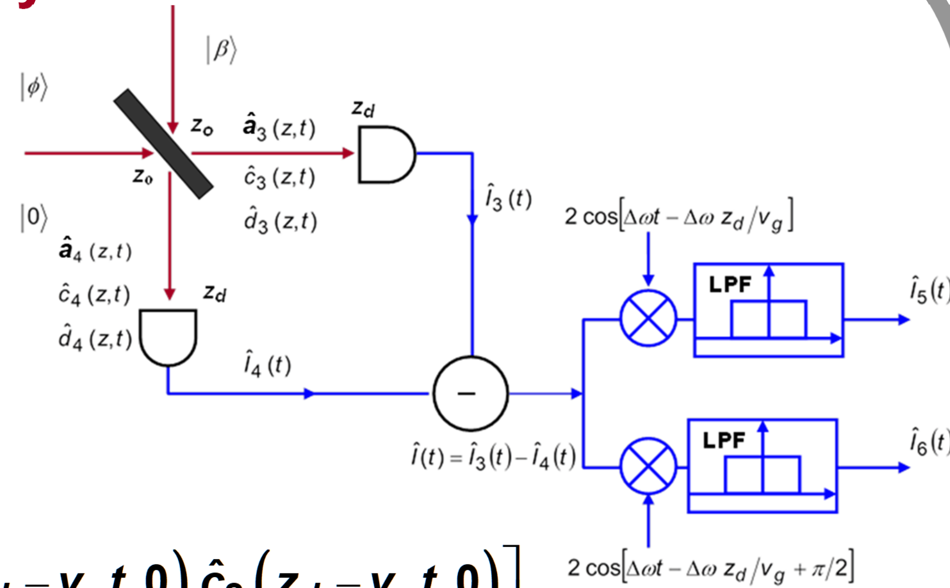
Beam splitter relations:

$$\begin{bmatrix} \hat{a}_3(z_o,t) \\ \hat{a}_4(z_o,t) \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} \hat{a}_1(z_o,t) \\ \hat{a}_2(z_o,t) \end{bmatrix}$$

$$\begin{bmatrix} \hat{c}_3(z_o,t) \\ \hat{c}_4(z_o,t) \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} \hat{c}_1(z_o,t) \\ \hat{c}_2(z_o,t) \end{bmatrix}$$

$$\begin{bmatrix} \hat{d}_3(z_o,t) \\ \hat{d}_4(z_o,t) \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} \hat{d}_1(z_o,t) \\ \hat{d}_2(z_o,t) \end{bmatrix}$$

Balanced Heterodyne Detection



Lets first look at the current I_5:

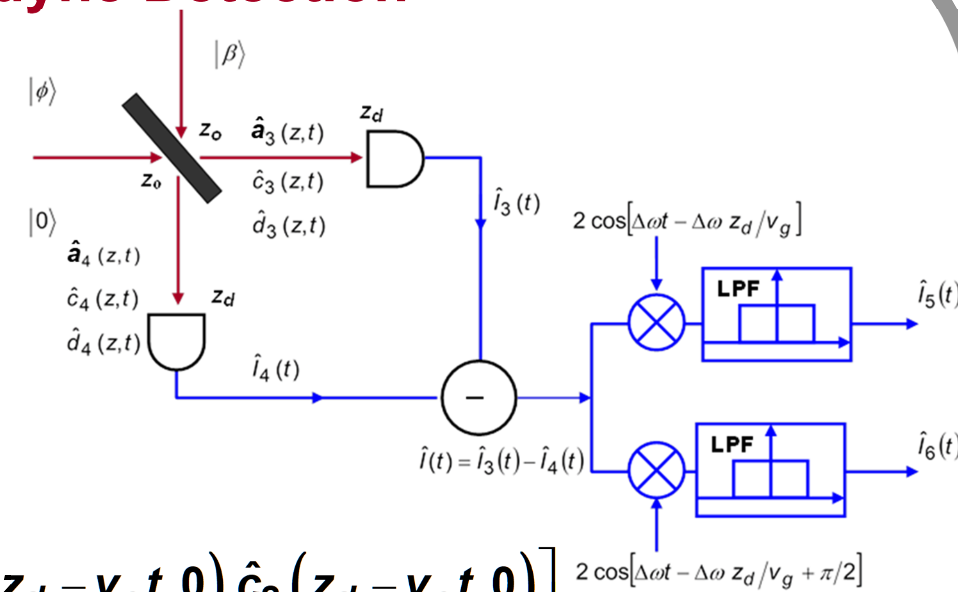
Current Operator:

$$\hat{i}_5(t) = qv_g\left[\hat{c}_2^{+}\left(z_d - v_g t,0\right)\hat{a}_1\left(z_d - v_g t,0\right) + \hat{a}_1^{+}\left(z_d - v_g t,0\right)\hat{c}_2\left(z_d - v_g t,0\right)\right]$$
$$+ qv_g\left[\hat{c}_2^{+}\left(z_d - v_g t,0\right)\hat{d}_1\left(z_d - v_g t,0\right) + \hat{d}_1^{+}\left(z_d - v_g t,0\right)\hat{c}_2\left(z_d - v_g t,0\right)\right]$$

Average Current:

$$\left\langle\hat{i}_5(t)\right\rangle = \left\langle\psi(t=0)\right|\hat{i}_5(t)\left|\psi(t=0)\right\rangle$$
$$= qv_g\left\langle\psi(t=0)\right|\hat{c}_2^{+}\left(z_d - v_g t,0\right)\hat{a}_1\left(z_d - v_g t,0\right) + \hat{a}_1^{+}\left(z_d - v_g t,0\right)\hat{c}_2\left(z_d - v_g t,0\right)\left|\psi(t=0)\right\rangle$$
$$= qv_g\,|\beta|\;{}_1^{\omega_o}\langle\varphi|\, e^{-i\theta}\,\hat{a}_1\left(z_d - v_g t,0\right) + e^{i\theta}\,\hat{a}_1^{+}\left(z_d - v_g t,0\right)|\varphi\rangle_1^{\omega_o}$$
$$= 2qv_g\,|\beta|\;{}_1^{\omega_o}\langle\varphi|\,\hat{x}_\theta\left(z_d - v_g t,0\right)|\varphi\rangle_1^{\omega_o}$$

Balanced Heterodyne Detection



Now look at the current I_6:

Current Operator:

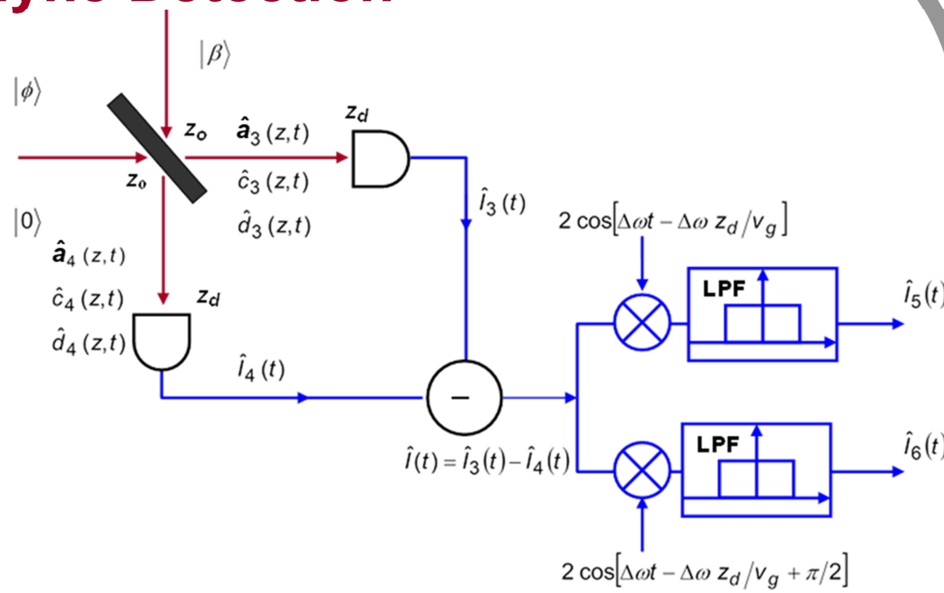
$$\hat{I}_6(t)=-iqv_g\left[\hat{c}_2^{+}\left(z_d-v_gt,0\right)\hat{a}_1\left(z_d-v_gt,0\right)-\hat{a}_1^{+}\left(z_d-v_gt,0\right)\hat{c}_2\left(z_d-v_gt,0\right)\right]$$
$$+iqv_g\left[\hat{c}_2^{+}\left(z_d-v_gt,0\right)\hat{d}_1\left(z_d-v_gt,0\right)-\hat{d}_1^{+}\left(z_d-v_gt,0\right)\hat{c}_2\left(z_d-v_gt,0\right)\right]$$

Average Current:

$$\left\langle\hat{I}_6(t)\right\rangle=\langle\psi(t=0)|\,\hat{I}_6(t)\,|\psi(t=0)\rangle$$
$$=2qv_g\,|\beta|\;{}_1^{\omega_o}\langle\phi|\,\hat{x}_{\theta+\pi/2}\left(z_d-v_gt,0\right)|\phi\rangle_1^{\omega_o}$$

Both quadratures of the input radiation can be measured simultaneously!!

Balanced Heterodyne Detection



Current Noise Correlation:

$$\left\langle \Delta\hat{I}_5(t_1)\,\Delta\hat{I}_5(t_2)\right\rangle = \langle\psi(t=0)|\,\Delta\hat{I}_5(t_1)\,\Delta\hat{I}_5(t_2)\,|\psi(t=0)\rangle$$

$$= \left(2qv_g\,|\beta|\right)^2\ {}_1^{\omega_o}\langle\phi|\,\Delta\hat{x}_\theta\left(z_d - v_g t_1, 0\right)\Delta\hat{x}_\theta\left(z_d - v_g t_2, 0\right)|\phi\rangle_1^{\omega_o}$$

$$+ \left(2qv_g\,|\beta|\right)^2\ {}_1^{\omega_o+2\Delta\omega}\langle\phi|\,\Delta\hat{x}_\theta\left(z_d - v_g t_1, 0\right)\Delta\hat{x}_\theta\left(z_d - v_g t_2, 0\right)|\phi\rangle_1^{\omega_o+2\Delta\omega}$$

$$\left\langle \Delta\hat{I}_5(t_1)\,\Delta\hat{I}_5(t_2)\right\rangle = \left(2qv_g\,|\beta|\right)^2\left[\ {}_1^{\omega_o}\langle\phi|\,\Delta\hat{x}_\theta\left(z_d - v_g t_1, 0\right)\Delta\hat{x}_\theta\left(z_d - v_g t_2, 0\right)|\phi\rangle_1^{\omega_o} + \underbrace{\frac{\delta(t_1 - t_2)}{4v_g}}_{\text{Noise penalty!}}\right]$$

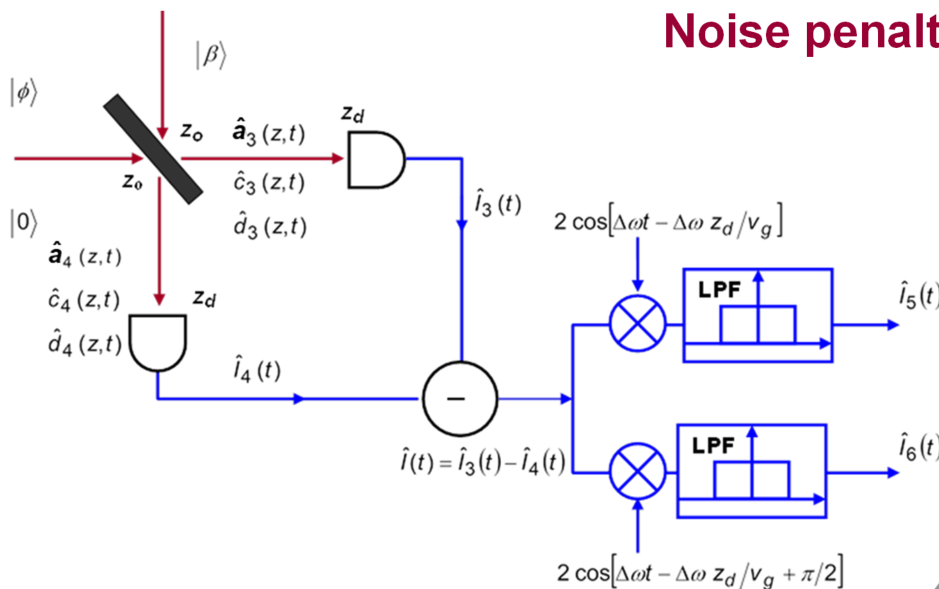
Balanced Heterodyne Detection

Current Noise Correlation:

$$\left\langle \Delta \hat{I}_5(t_1)\,\Delta \hat{I}_5(t_2)\right\rangle = \left(2qv_g\,|\beta|\right)^2 \left[{}_1^{\omega_o}\langle\phi|\,\Delta\hat{x}_\theta\left(z_d - v_g t_1,0\right)\Delta\hat{x}_\theta\left(z_d - v_g t_2,0\right)|\phi\rangle_1^{\omega_o} + \underbrace{\frac{\delta(t_1-t_2)}{4v_g}}_{\text{Noise penalty!}} \right]$$

$$\left\langle \Delta \hat{I}_6(t_1)\,\Delta \hat{I}_6(t_2)\right\rangle = \left(2qv_g\,|\beta|\right)^2 \left[{}_1^{\omega_o}\langle\phi|\,\Delta\hat{x}_{\theta+\pi/2}\left(z_d - v_g t_1,0\right)\Delta\hat{x}_{\theta+\pi/2}\left(z_d - v_g t_2,0\right)|\phi\rangle_1^{\omega_o} + \underbrace{\frac{\delta(t_1-t_2)}{4v_g}}_{\text{Noise penalty!}} \right]$$

The second term represents the extra noise introduced due to the fact that the two non-commuting quadratures are being measured simultaneously!



Coherent Optical Communication Systems: The New Revolution Happening Before Your Eyes with > 200 Gb/s/Channel Data Rates

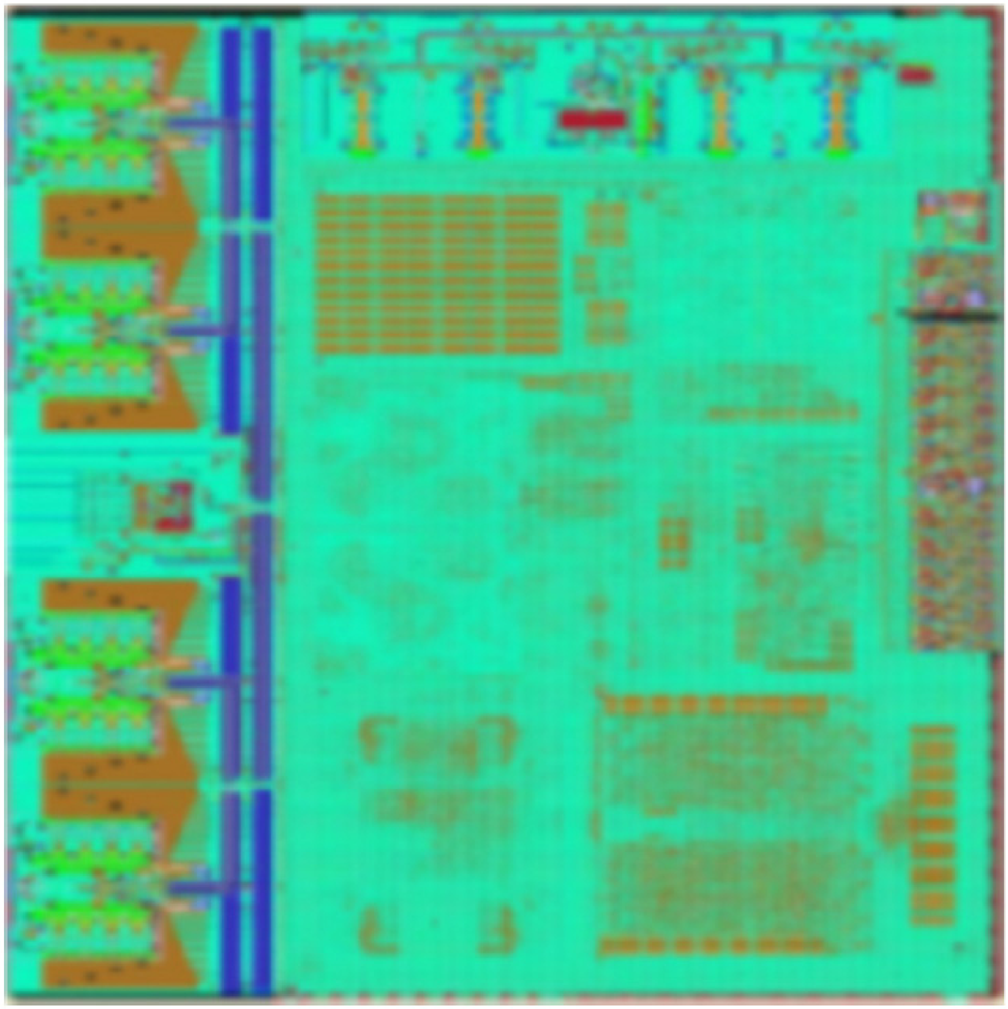


Figure 8. Example of transmitter and receiver functions combined into a single ASIC layout in 28 nm CMOS for metro optimized coherent applications.

Simultaneous Measurement of Two Non-Commuting Quantities

Suppose the commutator of two quantities is as follows:

$$\left[\hat{X}_1, \hat{X}_2\right] = iC$$

$$\Rightarrow \left\langle \Delta\hat{X}_1^2 \right\rangle \left\langle \Delta\hat{X}_1^2 \right\rangle \geq \frac{C^2}{4}$$

Min. uncertainty states would have:

$$\left\langle \Delta\hat{X}_1^2 \right\rangle = \left\langle \Delta\hat{X}_2^2 \right\rangle = \frac{C}{2}$$

Suppose we want to measure both these quantities SIMULTANEOUSLY

We make up a scheme such that we measure two derived quantities A and B and we are able to measure these derived quantities simultaneously:

$$\left.\begin{aligned} \hat{A} &= \alpha \hat{X}_1 \\ \hat{B} &= \alpha \hat{X}_2 \\ \left[\hat{A}, \hat{B}\right] &= 0 \end{aligned}\right\}$$

The relation between the derived and the actual quantities must be linear (why?)

Clearly, $\left[\hat{A}, \hat{B}\right] = 0$ **does not hold!**

Can we fix this ………..?

Simultaneous Measurement of Two Non-Commuting Quantities

We must have:

$$\hat{A} = \alpha \hat{X}_1 + \hat{F}_1$$
$$\hat{B} = \alpha \hat{X}_2 + \hat{F}_2$$
$$\left[\hat{A}, \hat{B}\right] = 0$$

$\Rightarrow$

$$\hat{Y}_1 = \frac{\hat{A}}{\alpha} = \hat{X}_1 + \frac{\hat{F}_1}{\alpha}$$
$$\hat{Y}_2 = \frac{\hat{B}}{\alpha} = \hat{X}_2 + \frac{\hat{F}_2}{\alpha}$$
$$\left[\hat{Y}_1, \hat{Y}_2\right] = 0$$

Where $\hat{F}_1$ and $\hat{F}_2$ are zero mean noise sources that must be present!
We assume that these noise sources commute with all other quantities.

If we require $\left[\hat{A}, \hat{B}\right] = 0$ then we must have: $\left[\hat{F}_1, \hat{F}_2\right] = -i\alpha^2 C$

$$\Rightarrow \left\langle \Delta \hat{F}_1^2 \right\rangle \left\langle \Delta \hat{F}_2^2 \right\rangle \geq \alpha^4 \frac{C^2}{4}$$

Min. uncertainty states would have: $\left\langle \Delta \hat{F}_1^2 \right\rangle = \left\langle \Delta \hat{F}_2^2 \right\rangle = \alpha^2 \frac{C}{2}$

The uncertainty in the values of the desired quantities would then be:

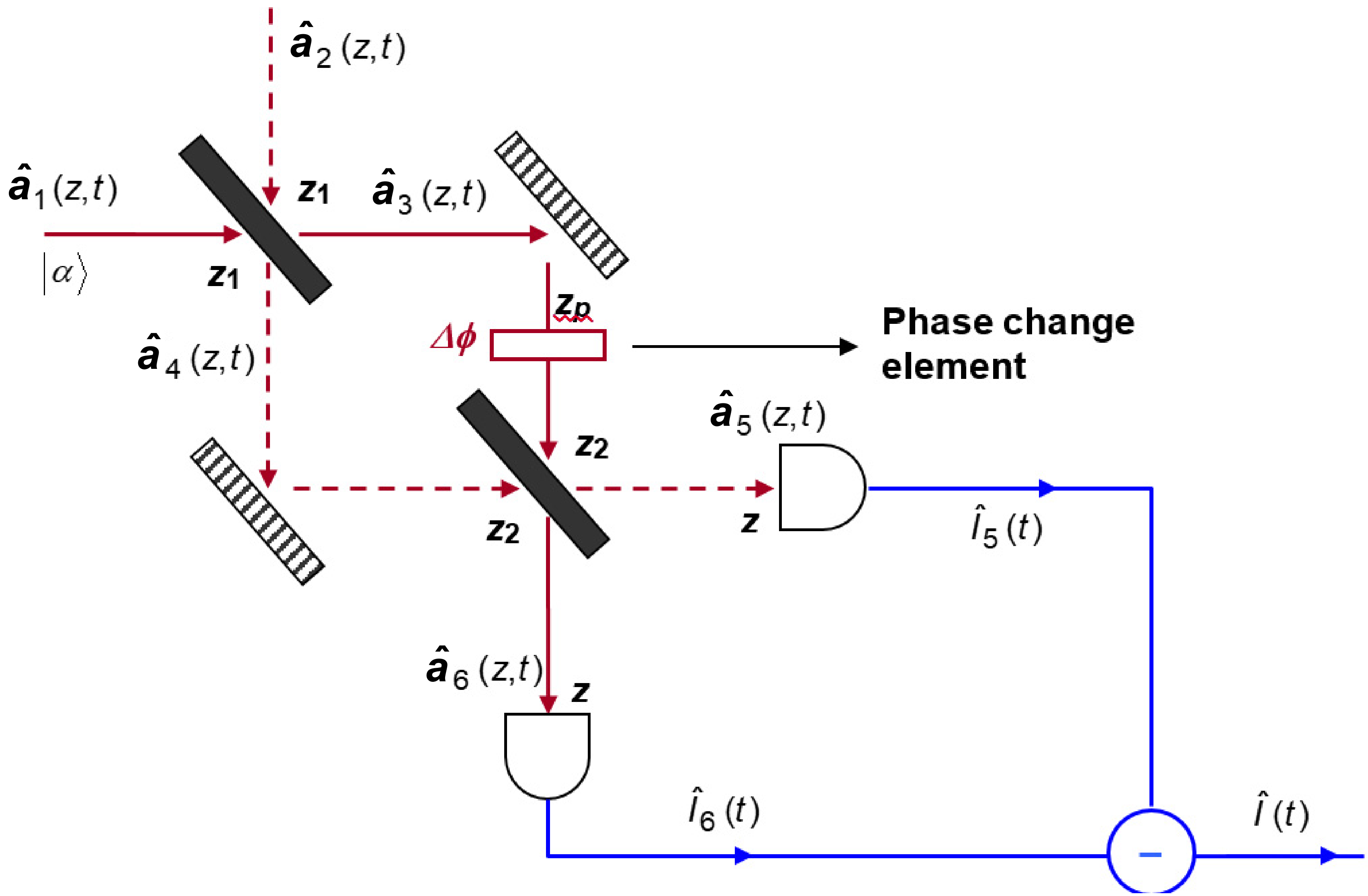
$$\left\langle \Delta \hat{Y}_1^2 \right\rangle = \left\langle \Delta \hat{X}_1^2 \right\rangle + \frac{1}{\alpha^2} \left\langle \Delta \hat{F}_1^2 \right\rangle$$
$$\left\langle \Delta \hat{Y}_2^2 \right\rangle = \left\langle \Delta \hat{X}_2^2 \right\rangle + \frac{1}{\alpha^2} \left\langle \Delta \hat{F}_2^2 \right\rangle$$

Min. uncertainty states would have:

$$\left\langle \Delta \hat{Y}_1^2 \right\rangle = C$$
$$\left\langle \Delta \hat{Y}_1^2 \right\rangle = C$$

The min noise encountered in making simultaneous measurements is twice the Heisenberg value

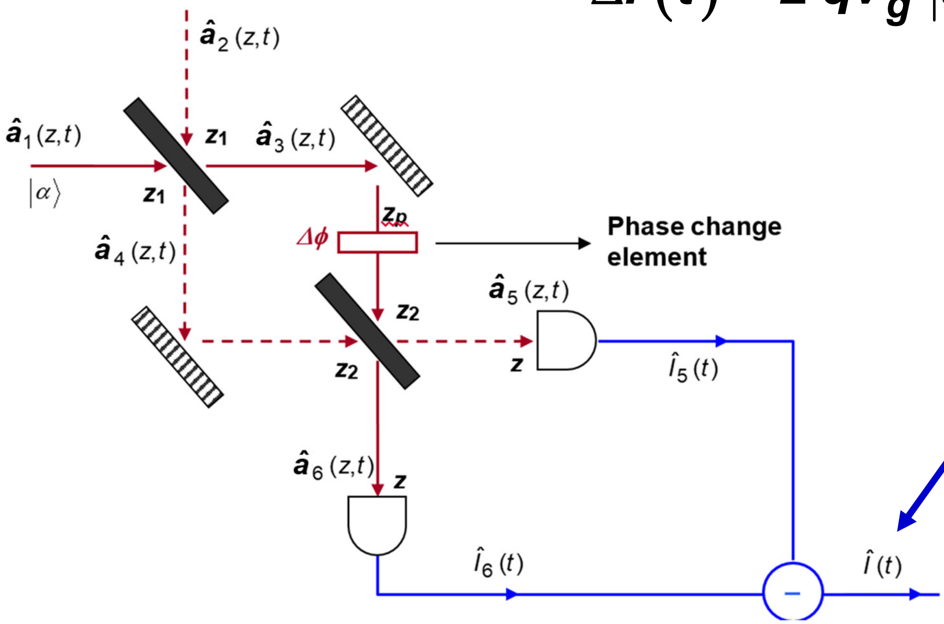
Optical Phase Detection and Standard Quantum Limit (SQL)



$$\langle \hat{i}(t) \rangle = q v_g |\alpha|^2 \Delta\phi$$

Optical Phase Detection and the Standard Quantum Limit

$$\Delta\hat{I}(t) = 2\,q v_g\,|\alpha|\left[\frac{\hat{a}_2\left(z - v_g t, 0\right)e^{-i\theta} - \hat{a}_2^{+}\left(z - v_g t, 0\right)e^{i\theta}}{2i}\right]$$



The current also has a current noise!

The current noise is related to the quadrature of the vacuum (entering from the free port of the first beam splitter) that effects the phase of the coherent state

$$\left\langle \Delta\hat{I}^2(t)\right\rangle = \int_{-\pi/T}^{\pi/T}\frac{d\omega}{2\pi}\,S_{\Delta I\,\Delta I}(\omega)$$ (integrate over the measurement bandwidth $2\pi/T$)

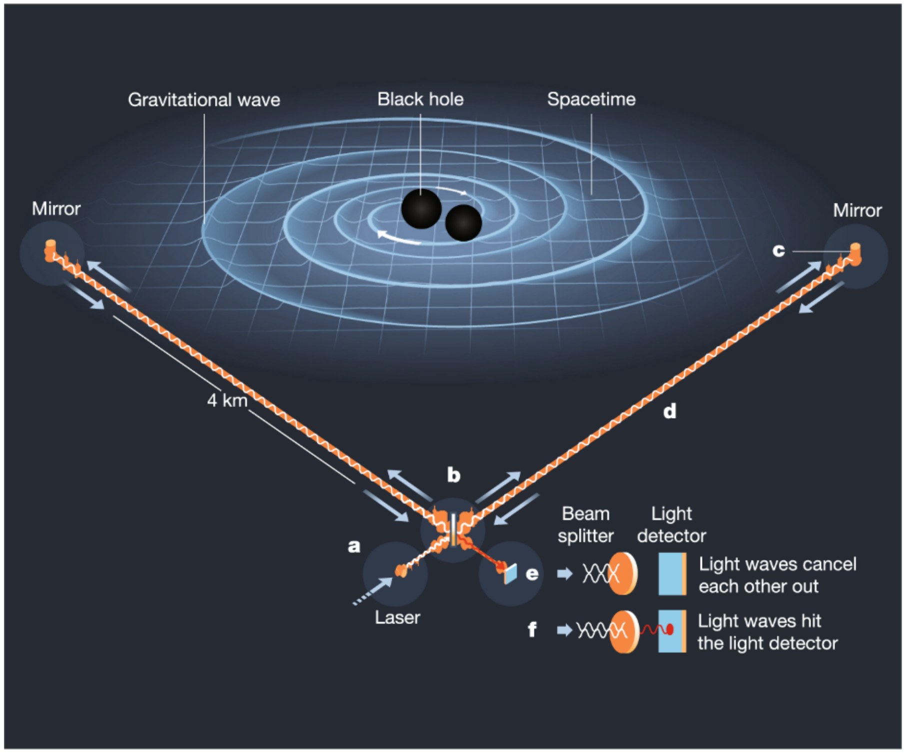
$$SNR = \frac{\left\langle \hat{I}(t)\right\rangle}{\sqrt{\left\langle \Delta\hat{I}^2(t)\right\rangle}} = \sqrt{N}\,\Delta\phi$$ $\longrightarrow$ **Equal to the square root of the photons used (or detected) in making the phase measurement**

$$\Rightarrow \Delta\phi = \frac{1}{\sqrt{N}}$$ (for $SNR = 1$)

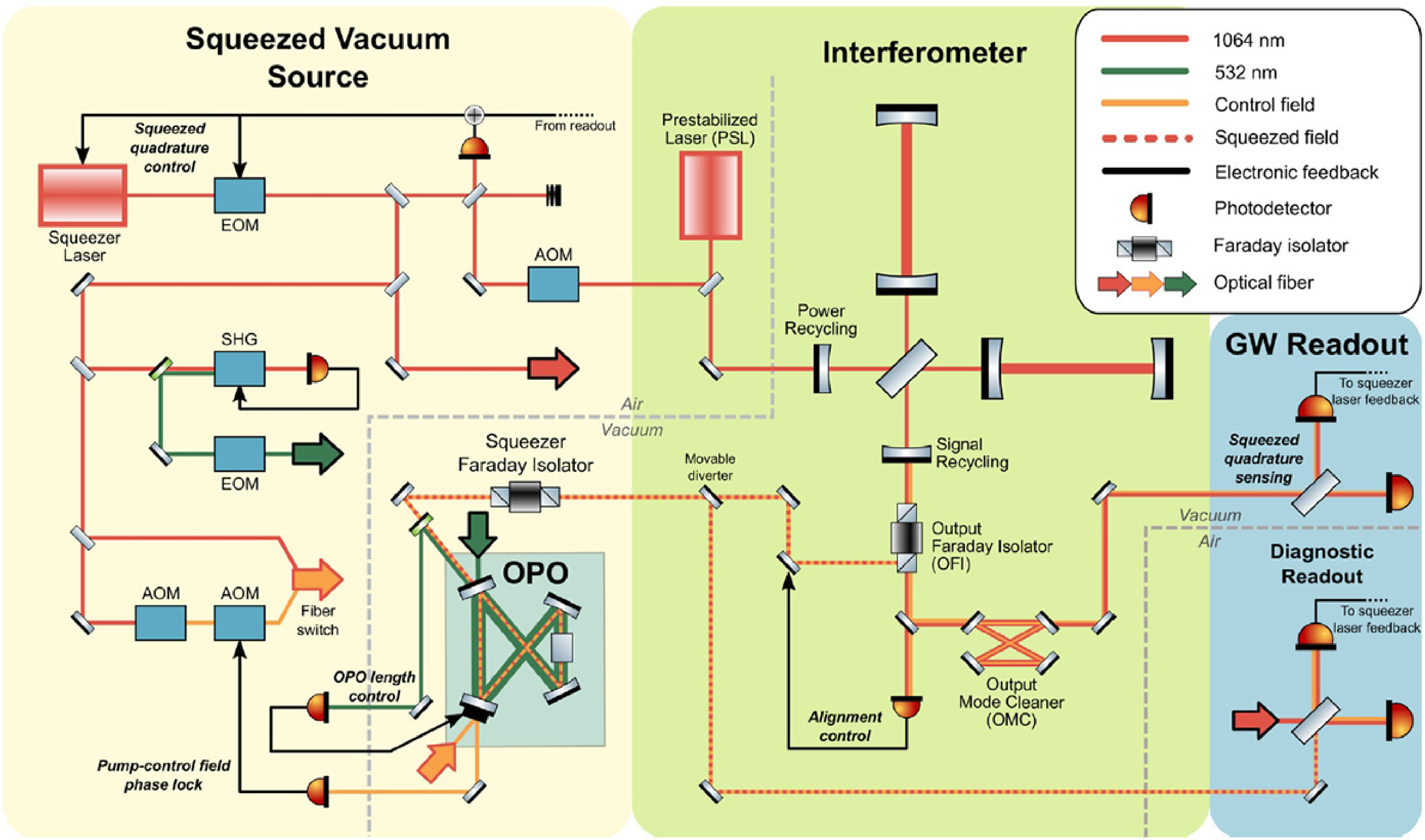
How can one beat the SQL on min phase detection?

Laser Interferometer Gravitational-Wave Observatory (LIGO)

Nobel Prize 2017

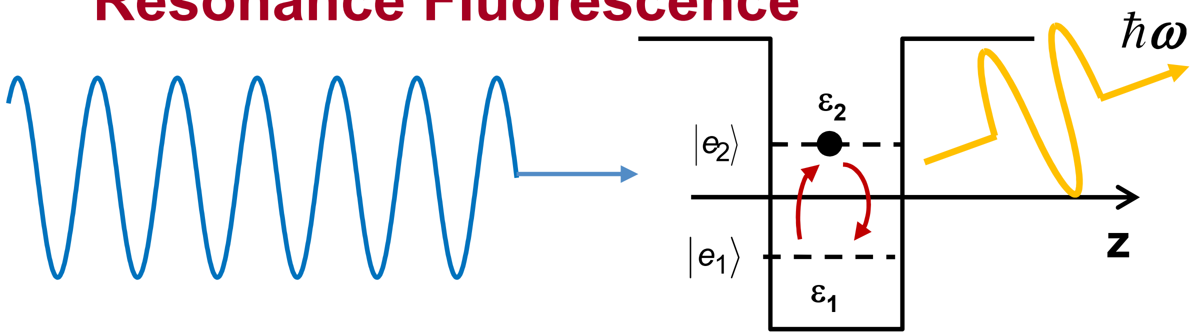


Squeezed State Interferometer at LIGO



Resonance Fluorescence

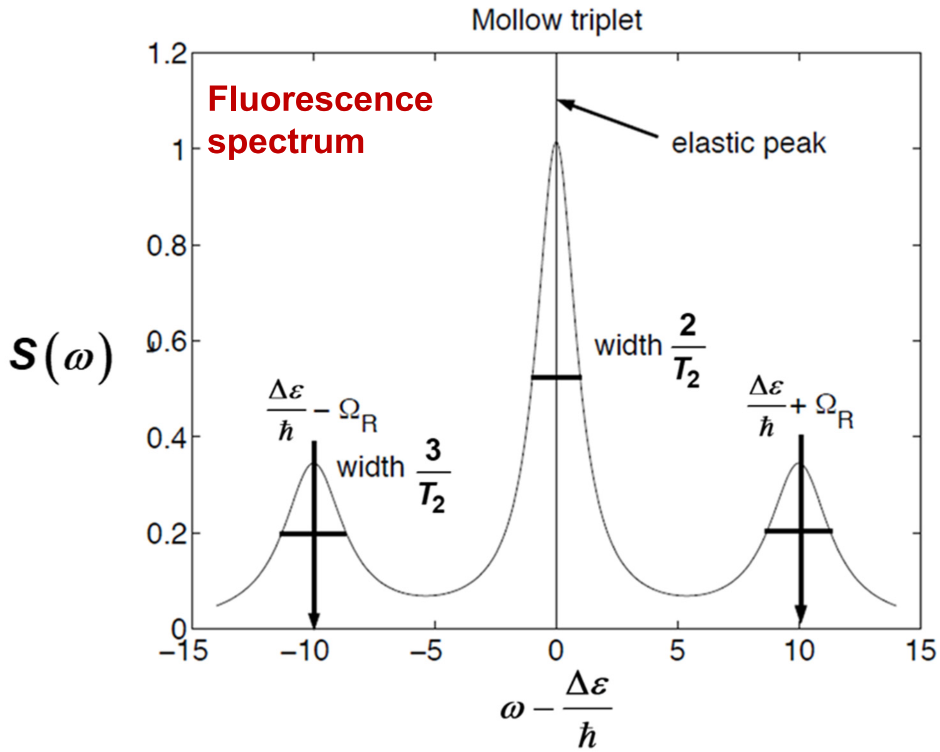
Strong driving laser field on resonance



Questions:

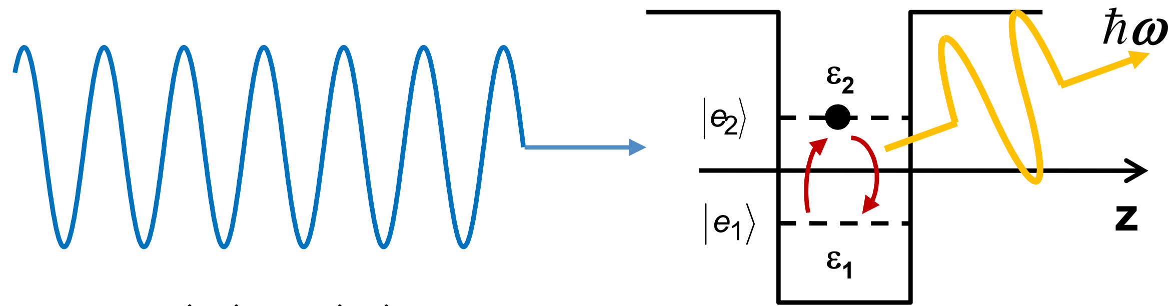
1) **What is the frequency of the spontaneously emitted photon? Or what is the spectrum of the spontaneously emitted light?**

2) **What characteristics does the photon flux exhibit? Are the photons bunched or anti-bunched?**



Resonance Fluorescence

Strong driving field at frequency ω



The Hamiltonian is:

$$\hat{H} \approx \varepsilon_1 \hat{N}_1 + \varepsilon_2 \hat{N}_2 + V\int \frac{d^3\vec{k}}{(2\pi)^3} \sum_j \hbar\omega_k \, \hat{a}_j^+(\vec{k})\hat{a}_j(\vec{k})$$

$$+\sum_{j,\vec{k}} \frac{\hbar}{\sqrt{V}} \left[\Pi_j(\vec{k})\hat{\sigma}_+ \hat{a}_j(\vec{k}) + \Pi_j^*(\vec{k})\hat{a}_j^+(\vec{k})\,\hat{\sigma}_- \right]$$

→ Interaction with all the modes

$$+\frac{1}{2}\left[\hbar\Omega_R \, \hat{\sigma}_+ e^{-i\omega t} + \hbar\Omega_R^* \, \hat{\sigma}_- e^{+i\omega t} \right]$$

→ Interaction with the driving laser field

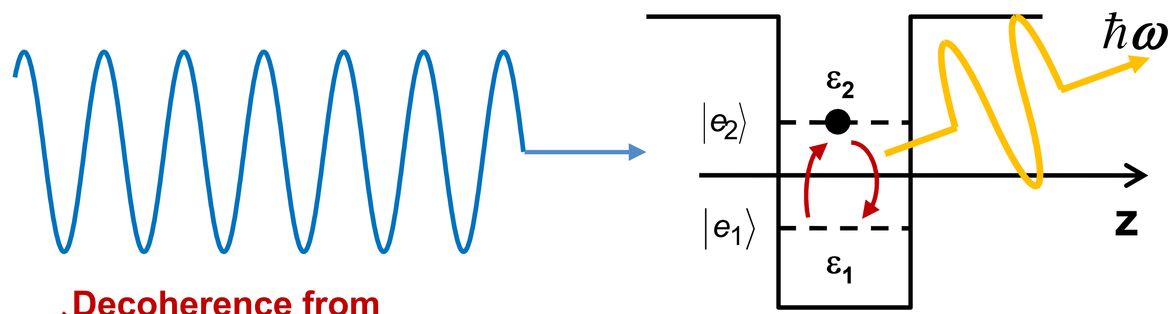
$$\hat{\vec{E}}(\vec{r},t) = V\int \frac{d^3\vec{k}}{(2\pi)^3} \sum_j i\sqrt{\frac{\hbar\omega_k}{2\varepsilon_o}} \left[\hat{a}_j(\vec{k},t) - \hat{a}_j^+(-\vec{k},t) \right] \frac{e^{i\vec{k}.\vec{r}}}{\sqrt{V}} \hat{\varepsilon}_j(\vec{k})$$

The coupling parameter is:

$$\Pi_j(\vec{k}) = -i\sqrt{\frac{\omega_k}{2\hbar\varepsilon_o}}\, q\vec{d}.\hat{\varepsilon}_j(\vec{k})$$

Resonance Fluorescence

Strong driving field at frequency ω



Decoherence from spontaneous emission into all radiation modes

$$\frac{d\hat{\sigma}_-(t)}{dt} = -i\frac{\Delta\varepsilon}{\hbar}\hat{\sigma}_-(t) - \frac{\hat{\sigma}_-(t)}{T_2} + \frac{i}{2}\Omega_R\hat{N}_d(t)e^{-i\omega t} + \hat{G}_-(t)$$

$$\frac{d\hat{\sigma}_+(t)}{dt} = i\frac{\Delta\varepsilon}{\hbar}\hat{\sigma}_+(t) - \frac{\hat{\sigma}_+(t)}{T_2} - \frac{i}{2}\Omega_R^*\hat{N}_d(t)e^{+i\omega t} + \hat{G}_+(t)$$

Langevin noise sources representing incomimg vacuum in all the radiation modes

$$\frac{d\hat{N}_d(t)}{dt} = -\frac{\left[\hat{N}_d(t)+1\right]}{T_1} - i\left[\Omega_R\,\hat{\sigma}_+(t)e^{-i\omega t} - \Omega_R^*\,\hat{\sigma}_-(t)e^{+i\omega t}\right] + +\hat{G}_d(t)$$

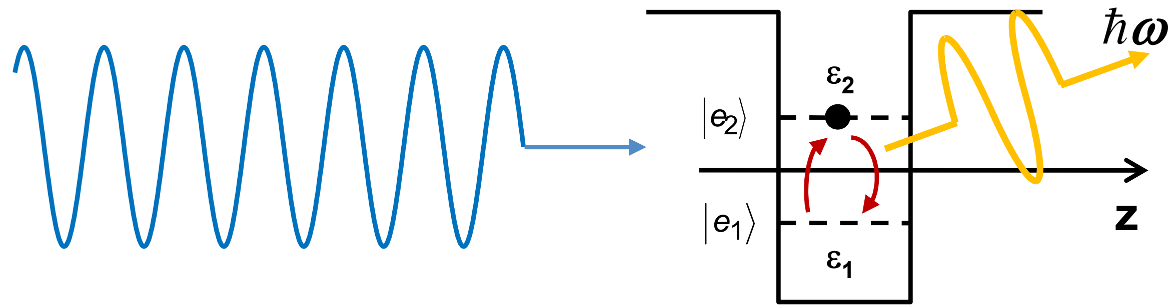
Population relaxation from spontaneous emission

From an earlier chapter, emitted radiation in the far-field is:

$$\hat{\vec{E}}_-(\vec{r},t) = \hat{\vec{E}}_-^{o}(\vec{r},t) + \frac{\omega_o^2}{4\pi\varepsilon_o c^2 r}\left[1 - \hat{r}\otimes\hat{r}\right].\left(q\vec{d}\right)\hat{\sigma}_-\left(t - \frac{r}{c}\right)$$

Resonance Fluorescence

Strong driving field at frequency ω



$$g_1(\vec{r}:t,t+\tau)=\frac{\langle \hat{E}_-(\vec{r},t)\hat{E}_+(\vec{r},t+\tau)\rangle}{\sqrt{\langle \hat{E}_-(\vec{r},t+\tau)\hat{E}_+(\vec{r},t+\tau)\rangle\langle \hat{E}_-(\vec{r},t)\hat{E}_+(\vec{r},t)\rangle}}$$

$$=\frac{\langle \hat{\sigma}_+(\vec{r},t)\hat{\sigma}_-(\vec{r},t+\tau)\rangle}{\sqrt{\langle \hat{\sigma}_+(\vec{r},t+\tau)\hat{\sigma}_-(\vec{r},t+\tau)\rangle\langle \hat{\sigma}_+(\vec{r},t)\hat{\sigma}_-(\vec{r},t)\rangle}}$$

Resonance Fluorescence

Let:

$$\hat{\sigma}_{\pm}(t) = \hat{s}_{\pm}(t)\, e^{\pm i\omega t}$$

$$\frac{d}{dt}\begin{bmatrix} \hat{s}_{-}(t) \\ \hat{s}_{+}(t) \\ \hat{N}_d(t) \end{bmatrix} = \begin{bmatrix} \left(-i\frac{\Delta}{\hbar} - \frac{1}{T_2}\right) & 0 & +\frac{i}{2}\Omega_R \\ 0 & \left(i\frac{\Delta}{\hbar} - \frac{1}{T_2}\right) & -\frac{i}{2}\Omega_R^* \\ i\Omega_R^* & -i\Omega_R & -\frac{1}{T_1} \end{bmatrix} \begin{bmatrix} \hat{s}_{-}(t) \\ \hat{s}_{+}(t) \\ \hat{N}_d(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ -\frac{1}{T_1} \end{bmatrix} + \begin{bmatrix} \hat{F}_{-}(t) \\ \hat{F}_{+}(t) \\ \hat{F}_d(t) \end{bmatrix}$$

$$\frac{d}{dt}\begin{bmatrix} \langle \hat{s}_{-}(t) \rangle \\ \langle \hat{s}_{+}(t) \rangle \\ \langle \hat{N}_d(t) \rangle \end{bmatrix} = \underbrace{\begin{bmatrix} \left(-i\frac{\Delta}{\hbar} - \frac{1}{T_2}\right) & 0 & +\frac{i}{2}\Omega_R \\ 0 & \left(i\frac{\Delta}{\hbar} - \frac{1}{T_2}\right) & -\frac{i}{2}\Omega_R^* \\ i\Omega_R^* & -i\Omega_R & -\frac{1}{T_1} \end{bmatrix}}_{\overline{\overline{A}}} \begin{bmatrix} \langle \hat{s}_{-}(t) \rangle \\ \langle \hat{s}_{+}(t) \rangle \\ \langle \hat{N}_d(t) \rangle \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ -\frac{1}{T_1} \end{bmatrix}$$

Resonance Fluorescence: Steady State

In steady state:

$$0 = \frac{d}{dt}\begin{bmatrix} \langle \hat{s}_-(t) \rangle \\ \langle \hat{s}_+(t) \rangle \\ \langle \hat{N}_d(t) \rangle \end{bmatrix} = \bar{\bar{A}}\begin{bmatrix} \langle \hat{s}_-(t) \rangle \\ \langle \hat{s}_+(t) \rangle \\ \langle \hat{N}_d(t) \rangle \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ -\frac{1}{T_1} \end{bmatrix}$$

Assuming zero detuning and $T_2 = 2T_1$:

$$\begin{bmatrix} \langle \hat{s}_-(t) \rangle \\ \langle \hat{s}_+(t) \rangle \\ \langle \hat{N}_d(t) \rangle \end{bmatrix}_{t\to\infty} = \frac{1}{\Omega_R^2 + \frac{2}{T_2^2}}\begin{bmatrix} -i\frac{\Omega_R}{T_2} \\ i\frac{\Omega_R}{T_2} \\ -\frac{2}{T_2^2} \end{bmatrix}$$

Resonance Fluorescence: Correlation Functions

Start from:

$$\frac{d}{dt}\begin{bmatrix}\hat{s}_-(t)\\ \hat{s}_+(t)\\ \hat{N}_d(t)\end{bmatrix} = \bar{\bar{A}}\begin{bmatrix}\hat{s}_-(t)\\ \hat{s}_+(t)\\ \hat{N}_d(t)\end{bmatrix} + \begin{bmatrix}0\\ 0\\ -\dfrac{1}{T_1}\end{bmatrix} + \begin{bmatrix}\hat{F}_-(t)\\ \hat{F}_+(t)\\ \hat{F}_d(t)\end{bmatrix}$$

Change of variables (τ > 0):

$$\frac{d}{d\tau}\begin{bmatrix}\hat{s}_-(t+\tau)\\ \hat{s}_+(t+\tau)\\ \hat{N}_d(t+\tau)\end{bmatrix} = \bar{\bar{A}}\begin{bmatrix}\hat{s}_-(t+\tau)\\ \hat{s}_+(t+\tau)\\ \hat{N}_d(t+\tau)\end{bmatrix} + \begin{bmatrix}0\\ 0\\ -\dfrac{1}{T_1}\end{bmatrix} + \begin{bmatrix}\hat{F}_-(t+\tau)\\ \hat{F}_+(t+\tau)\\ \hat{F}_d(t+\tau)\end{bmatrix}$$

Exterior multiply from the right:

$$\frac{d}{d\tau}\begin{bmatrix}\hat{s}_-(t+\tau)\\ \hat{s}_+(t+\tau)\\ \hat{N}_d(t+\tau)\end{bmatrix}\begin{bmatrix}\hat{s}_-(t) & \hat{s}_+(t) & \hat{N}_d(t)\end{bmatrix} = \bar{\bar{A}}\begin{bmatrix}\hat{s}_-(t+\tau)\\ \hat{s}_+(t+\tau)\\ \hat{N}_d(t+\tau)\end{bmatrix}\begin{bmatrix}\hat{s}_-(t) & \hat{s}_+(t) & \hat{N}_d(t)\end{bmatrix} - \begin{bmatrix}0\\ 0\\ \dfrac{1}{T_1}\end{bmatrix}\begin{bmatrix}\hat{s}_-(t) & \hat{s}_+(t) & \hat{N}_d(t)\end{bmatrix}$$

$$+\begin{bmatrix}\hat{F}_-(t+\tau)\\ \hat{F}_+(t+\tau)\\ \hat{F}_d(t+\tau)\end{bmatrix}\begin{bmatrix}\hat{s}_-(t) & \hat{s}_+(t) & \hat{N}_d(t)\end{bmatrix}$$

Resonance Fluorescence: Correlation Functions

$$\frac{d}{d\tau}\left\langle \begin{bmatrix} \hat{s}_-(t+\tau) \\ \hat{s}_+(t+\tau) \\ \hat{N}_d(t+\tau) \end{bmatrix} \begin{bmatrix} \hat{s}_-(t) & \hat{s}_+(t) & \hat{N}_d(t) \end{bmatrix} \right\rangle = \bar{\bar{A}} \left\langle \begin{bmatrix} \hat{s}_-(t+\tau) \\ \hat{s}_+(t+\tau) \\ \hat{N}_d(t+\tau) \end{bmatrix} \begin{bmatrix} \hat{s}_-(t) & \hat{s}_+(t) & \hat{N}_d(t) \end{bmatrix} \right\rangle - \begin{bmatrix} 0 \\ 0 \\ \dfrac{1}{T_1} \end{bmatrix} \begin{bmatrix} \langle \hat{s}_-(t) \rangle & \langle \hat{s}_+(t) \rangle & \langle \hat{N}_d(t) \rangle \end{bmatrix}$$

$$\frac{d}{d\tau}\bar{\bar{G}}(\tau) = \bar{\bar{A}}\bar{\bar{G}}(\tau) - \begin{bmatrix} 0 \\ 0 \\ \dfrac{1}{T_1} \end{bmatrix} \begin{bmatrix} \langle \hat{s}_-(t) \rangle & \langle \hat{s}_+(t) \rangle & \langle \hat{N}_d(t) \rangle \end{bmatrix}$$

$$\Rightarrow s\bar{\bar{G}}(s) - \bar{\bar{G}}(\tau=0) = \bar{\bar{A}}\bar{\bar{G}}(s) - \frac{2}{sT_2} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \langle \hat{s}_-(t) \rangle & \langle \hat{s}_+(t) \rangle & \langle \hat{N}_d(t) \rangle \end{bmatrix}$$

$$\bar{\bar{G}}(\tau=0) = \begin{bmatrix} 0 & \langle \hat{N}_1(t) \rangle & \langle \hat{s}_-(t) \rangle \\ \langle \hat{N}_2(t) \rangle & 0 & -\langle \hat{s}_+(t) \rangle \\ -\langle \hat{s}_-(t) \rangle & \langle \hat{s}_+(t) \rangle & 1 \end{bmatrix}$$

$$\Rightarrow \left[s\bar{\bar{1}} - \bar{\bar{A}} \right] \bar{\bar{G}}(s) = \bar{\bar{G}}(\tau=0) - \frac{2}{sT_2} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \langle \hat{s}_-(t) \rangle & \langle \hat{s}_+(t) \rangle & \langle \hat{N}_d(t) \rangle \end{bmatrix}$$

$$\Rightarrow \bar{\bar{G}}(s) = \left[s\bar{\bar{1}} - \bar{\bar{A}} \right]^{-1} \bar{\bar{G}}(\tau=0) - \frac{2}{sT_2} \left[s\bar{\bar{1}} - \bar{\bar{A}} \right]^{-1} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \langle \hat{s}_-(t) \rangle & \langle \hat{s}_+(t) \rangle & \langle \hat{N}_d(t) \rangle \end{bmatrix}$$

$$\Rightarrow \bar{\bar{G}}(\tau) = ?$$

Resonance Fluorescence

$$g_1(\vec{r}:t,t+\tau)=\frac{\left\langle\hat{\sigma}_+(\vec{r},t)\hat{\sigma}_-(\vec{r},t+\tau)\right\rangle}{\sqrt{\left\langle\hat{\sigma}_+(\vec{r},t+\tau)\hat{\sigma}_-(\vec{r},t+\tau)\right\rangle\left\langle\hat{\sigma}_+(\vec{r},t)\hat{\sigma}_-(\vec{r},t)\right\rangle}}$$

$$=W_0e^{-i\frac{\Delta\varepsilon}{\hbar}\tau}+W_1\,e^{-i\frac{\Delta\varepsilon}{\hbar}\tau-\frac{|\tau|}{T_2}}+W_3e^{-i\frac{\Delta\varepsilon}{\hbar}\tau-\left(\frac{3}{2T_2}+\beta\right)|\tau|}+W_4e^{-i\frac{\Delta\varepsilon}{\hbar}\tau-\left(\frac{3}{2T_2}-\beta\right)|\tau|}$$

$$\beta=\sqrt{\frac{1}{4T_2^2}-\Omega_R^2}=i\sqrt{\Omega_R^2-\frac{1}{4T_2^2}}$$

$$W_0=\frac{2}{\Omega_R^2T_2^2+2}$$

$$W_1=\frac{1}{2}$$

$$W_3=\frac{1}{2}\left[\frac{2}{\Omega_R^2T_2^2+2}\left(\frac{3/2-\beta T_2}{\beta T_2}\right)-\frac{1}{\beta T_2}-\left(\frac{1/2-\beta T_2}{2\beta T_2}\right)\right]$$

$$W_4=-\frac{1}{2}\left[\frac{2}{\Omega_R^2T_2^2+2}\left(\frac{3/2+\beta T_2}{\beta T_2}\right)-\frac{1}{\beta T_2}-\left(\frac{1/2+\beta T_2}{2\beta T_2}\right)\right]$$

Resonance Fluorescence Spectrum: Mollow Triplet

For **weak driving fields**, $\Omega_R << \dfrac{1}{2T_2}$, the spectrum becomes:

$$g_1(\vec{r}:t,t+\tau) \sim e^{-i\frac{\Delta\varepsilon}{\hbar}\tau}$$

$$\Rightarrow S(\omega) \sim 2\pi\delta(\omega - \Delta\varepsilon/\hbar)$$

Elastic scattering

For **Strong driving fields**, $\Omega_R >> \dfrac{1}{2T_2}$, the spectrum becomes:

$$g_1(\vec{r}:t,t+\tau) = \frac{1}{2}e^{-i\frac{\Delta\varepsilon}{\hbar}\tau - \frac{|\tau|}{T_2}} + \frac{1}{4}e^{-i\left(\frac{\Delta\varepsilon}{\hbar} - \Omega_R\right)\tau - \left(\frac{3}{2T_2}\right)|\tau|} + \frac{1}{4}e^{-i\left(\frac{\Delta\varepsilon}{\hbar} + \Omega_R\right)\tau - \left(\frac{3}{2T_2}\right)|\tau|}$$

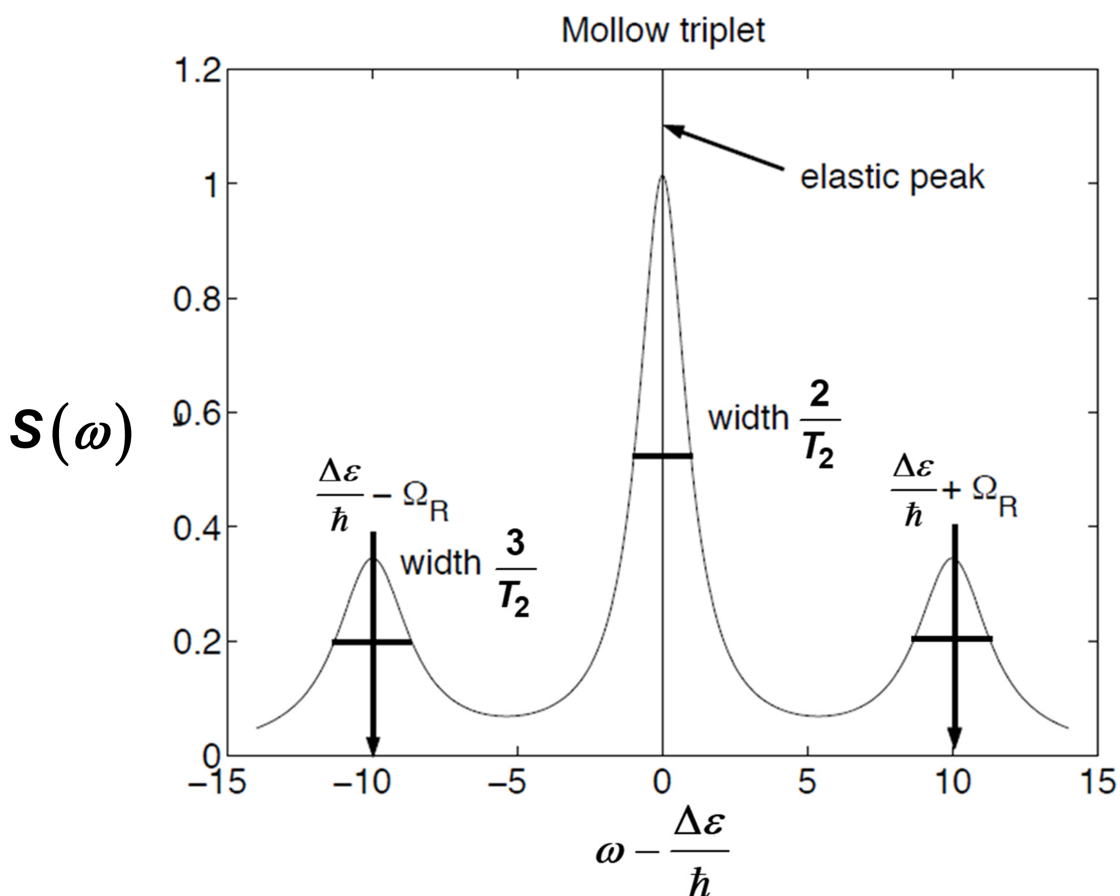
$$S(\omega) = \frac{1}{2}\frac{2/T_2}{\left[(\omega - \Delta\varepsilon/\hbar)^2 + (1/T_2)^2\right]}$$

$$+ \frac{1}{4}\frac{3/T_2}{\left[(\omega - \Delta\varepsilon/\hbar + \Omega_R)^2 + (3/2T_2)^2\right]} + \frac{1}{4}\frac{3/T_2}{\left[(\omega - \Delta\varepsilon/\hbar - \Omega_R)^2 + (3/2T_2)^2\right]}$$

Inelastic scattering !!

Resonance Fluorescence Spectrum: Mollow Triplet

$$S(\omega) = \frac{1}{2}\frac{2/T_2}{\left[(\omega - \Delta\varepsilon/\hbar)^2 + (1/T_2)^2\right]} + \frac{1}{4}\frac{3/T_2}{\left[(\omega - \Delta\varepsilon/\hbar + \Omega_R)^2 + (3/2T_2)^2\right]} + \frac{1}{4}\frac{3/T_2}{\left[(\omega - \Delta\varepsilon/\hbar - \Omega_R)^2 + (3/2T_2)^2\right]}$$



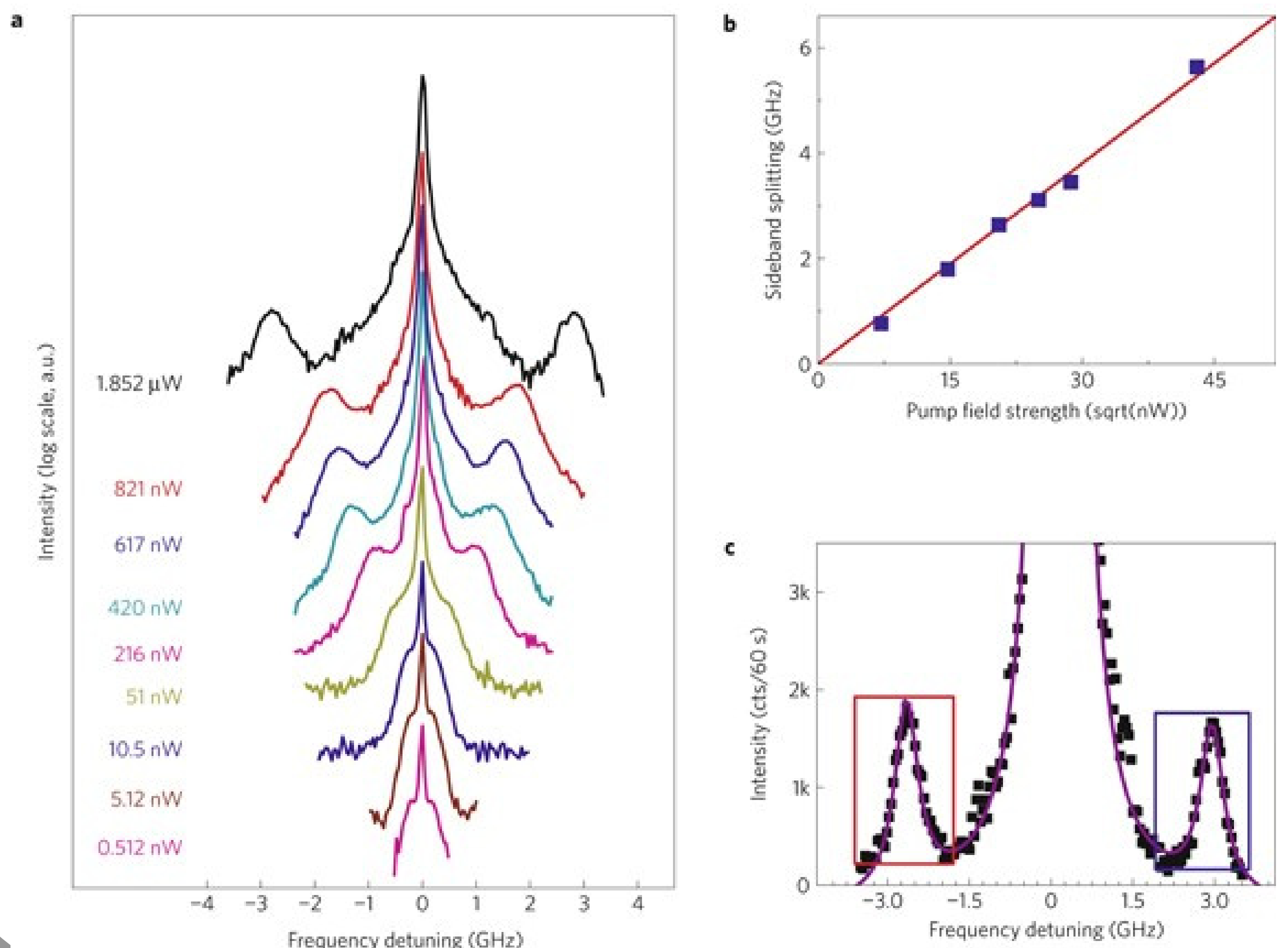
Resonance Fluorescence Spectrum: Mollow Triplet

Published: 25 January 2009

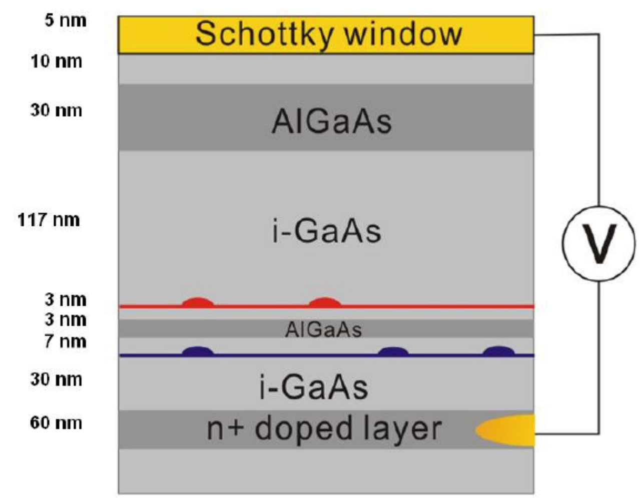
Spin-resolved quantum-dot resonance fluorescence

A. Nick Vamivakas, Yong Zhao, Chao-Yang Lu & Mete Atatüre

Nature Physics **5**, 198–202(2009) | Cite this article



InAs/GaAs quantum dots grown by molecular beam epitaxy



Resonance Fluorescence Spectrum of a Driven Superconducting Qubit

Resonance Fluorescence of a Single Artificial Atom

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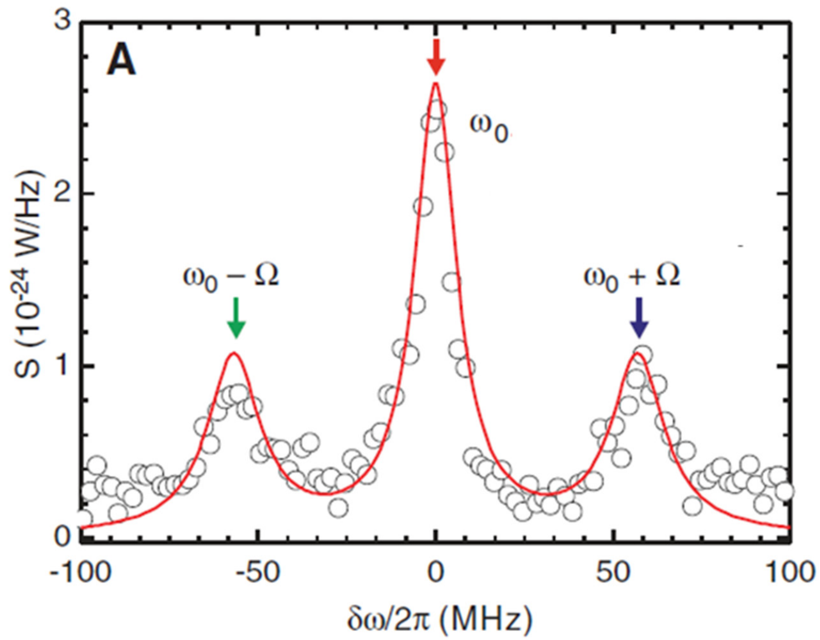
An atom in open space can be detected by means of resonant absorption and reemission of electromagnetic waves, known as resonance fluorescence, which is a fundamental phenomenon of quantum optics. We report on the observation of scattering of propagating waves by a single artificial atom. The behavior of the artificial atom, a superconducting macroscopic two-level system, is in a quantitative agreement with the predictions of quantum optics for a pointlike scatterer interacting with the electromagnetic field in one-dimensional open space. The strong atom-field interaction as revealed in a high degree of extinction of propagating waves will allow applications of controllable artificial atoms in quantum optics and photonics.

12 FEBRUARY 2010 VOL 327 SCIENCE www.sciencemag.org



Resonance Fluorescence Spectrum: Mollow Triplet

Resonance Fluorescence from a Coherently Driven Semiconductor Quantum Dot in a Cavity

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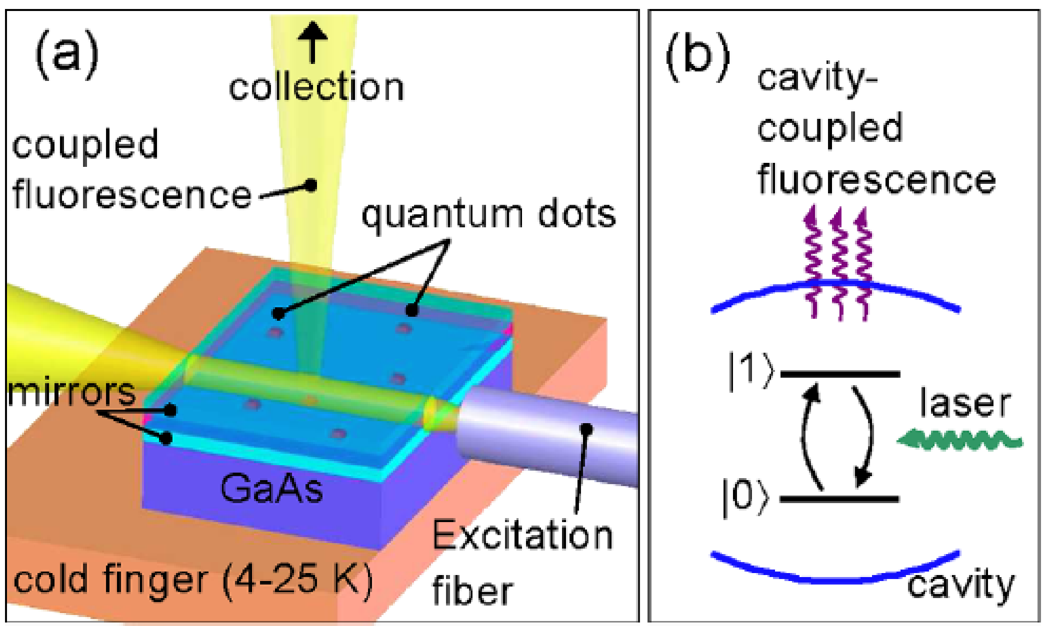
We show that resonance fluorescence, i.e., the *resonant* emission of a *coherently* driven two-level system, can be realized with a semiconductor quantum dot. The dot is embedded in a planar optical microcavity and excited in a waveguide mode so as to discriminate its emission from residual laser scattering. The transition from the weak to the strong excitation regime is characterized by the emergence of oscillations in the first-order correlation function of the fluorescence, $g(\tau)$, as measured by interferometry. The measurements correspond to a Mollow triplet with a Rabi splitting of up to 13.3 μeV. Second-order correlation measurements further confirm nonclassical light emission.



Resonance Fluorescence Photon Correlation: Anti-bunching

$$g_2(\vec{r}:t+\tau,t) = \frac{\left\langle \hat{E}_-(\vec{r},t)\hat{E}_-(\vec{r},t+\tau)\hat{E}_+(\vec{r},t+\tau)\hat{E}_+(\vec{r},t)\right\rangle}{\left\langle \hat{E}_-(\vec{r},t+\tau)\hat{E}_+(\vec{r},t+\tau)\right\rangle\left\langle \hat{E}_-(\vec{r},t+\tau)\hat{E}_+(\vec{r},t+\tau)\right\rangle}$$

$$= \frac{\left\langle \hat{\sigma}_+(t)\hat{\sigma}_+(t+\tau)\hat{\sigma}_-(t+\tau)\hat{\sigma}_-(t)\right\rangle}{\left\langle \hat{\sigma}_+(t+\tau)\hat{\sigma}_-(t+\tau)\right\rangle\left\langle \hat{\sigma}_+(t+\tau)\hat{\sigma}_-(t+\tau)\right\rangle}$$

$$= \frac{\left\langle \hat{\sigma}_+(t)\hat{N}_2(t+\tau)\hat{\sigma}_-(t)\right\rangle}{\left\langle \hat{N}_2(t+\tau)\right\rangle\left\langle \hat{N}_2(t)\right\rangle}$$

$$= \frac{\left\langle \hat{\sigma}_+(t)\hat{N}_2(t+\tau)\hat{\sigma}_-(t)\right\rangle}{\left\langle \hat{N}_2\right\rangle^2_{\substack{steady\\state}}}$$

$$\left\langle \hat{N}_d(t\to\infty)\right\rangle = -\frac{\dfrac{2}{T_2^2}}{\Omega_R^2 + \dfrac{2}{T_2^2}}$$

$$\left\langle \hat{N}_2(t\to\infty)\right\rangle = \frac{1}{2}\frac{\Omega_R^2}{\Omega_R^2 + \dfrac{2}{T_2^2}}$$

Resonance Fluorescence Photon Correlation: Anti-bunching

$$\frac{d}{d\tau}\left\langle \hat{s}_+(t)\begin{bmatrix}\hat{s}_-(t+\tau)\\ \hat{s}_+(t+\tau)\\ \hat{N}_d(t+\tau)\end{bmatrix}\begin{bmatrix}\hat{s}_-(t) & \hat{s}_+(t) & \hat{N}_d(t)\end{bmatrix}\right\rangle = \bar{\bar{A}}\left\langle \hat{s}_+(t)\begin{bmatrix}\hat{s}_-(t+\tau)\\ \hat{s}_+(t+\tau)\\ \hat{N}_d(t+\tau)\end{bmatrix}\begin{bmatrix}\hat{s}_-(t) & \hat{s}_+(t) & \hat{N}_d(t)\end{bmatrix}\right\rangle - \begin{bmatrix}0\\0\\ \frac{1}{T_1}\end{bmatrix}\begin{bmatrix}\left\langle \hat{N}_2(t)\right\rangle & 0 & -\left\langle \hat{s}_+(t)\right\rangle\end{bmatrix}$$

$$\frac{d}{d\tau}\bar{\bar{G}}(\tau) = \bar{\bar{A}}\bar{\bar{G}}(\tau) - \begin{bmatrix}0\\0\\ \frac{1}{T_1}\end{bmatrix}\begin{bmatrix}\left\langle \hat{N}_2(t)\right\rangle & 0 & -\left\langle \hat{s}_+(t)\right\rangle\end{bmatrix}$$

$$\Rightarrow s\bar{\bar{G}}(s) - \bar{\bar{G}}(\tau=0) = \bar{\bar{A}}\bar{\bar{G}}(s) - \frac{2}{sT_2}\begin{bmatrix}0 & 0 & 0\\ 0 & 0 & 0\\ \left\langle \hat{N}_2(t)\right\rangle & 0 & -\left\langle \hat{s}_+(t)\right\rangle\end{bmatrix}$$

$$\Rightarrow \left[s\bar{\bar{1}} - \bar{\bar{A}}\right]\bar{\bar{G}}(s) = \bar{\bar{G}}(\tau=0) - \frac{2}{sT_2}\begin{bmatrix}0 & 0 & 0\\ 0 & 0 & 0\\ \left\langle \hat{N}_2(t)\right\rangle & 0 & -\left\langle \hat{s}_+(t)\right\rangle\end{bmatrix}$$

$$\Rightarrow \bar{\bar{G}}(s) = \left[s\bar{\bar{1}} - \bar{\bar{A}}\right]^{-1}\bar{\bar{G}}(\tau=0) - \frac{2}{sT_2}\left[s\bar{\bar{1}} - \bar{\bar{A}}\right]^{-1}\begin{bmatrix}0 & 0 & 0\\ 0 & 0 & 0\\ \left\langle \hat{N}_2(t)\right\rangle & 0 & -\left\langle \hat{s}_+(t)\right\rangle\end{bmatrix}$$

$$\Rightarrow \bar{\bar{G}}(\tau) = ?$$

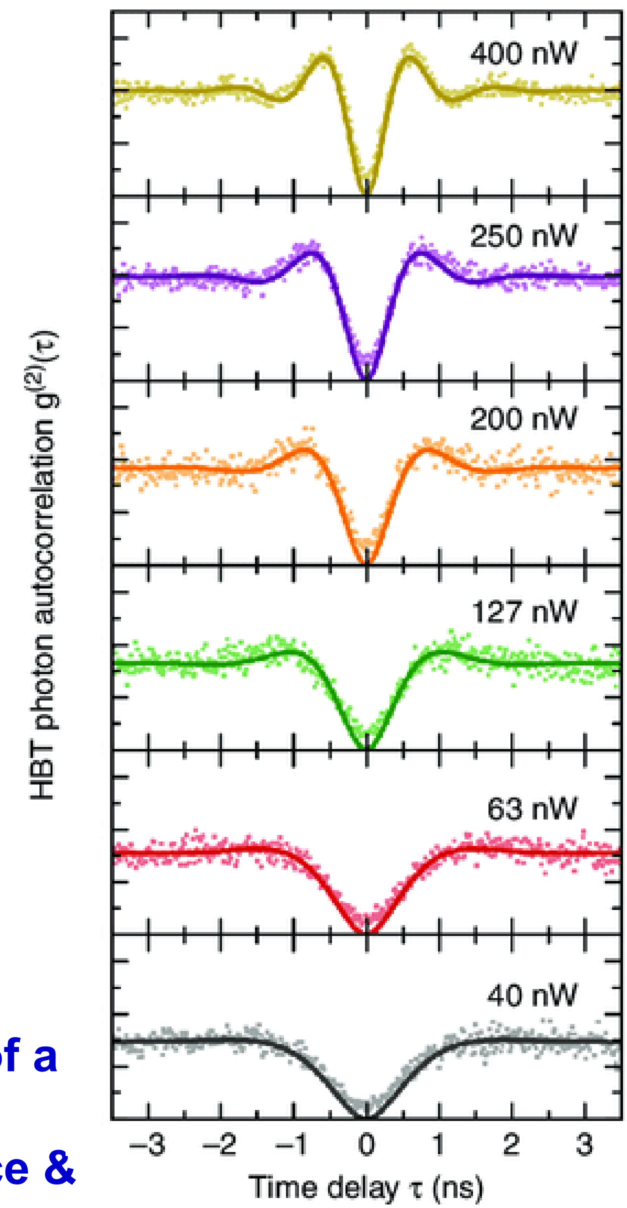
$$\bar{\bar{G}}(\tau=0) = \begin{bmatrix}0 & \left\langle \hat{s}_+(t)\right\rangle & \left\langle \hat{N}_2(t)\right\rangle\\ 0 & 0 & 0\\ -\left\langle \hat{N}_2(t)\right\rangle & 0 & \left\langle \hat{s}_+(t)\right\rangle\end{bmatrix}$$

Resonance Fluorescence Photon Correlation: Anti-bunching

$$g_2\left(\vec{r}:t+\tau,t\right)=\frac{\left\langle \hat{\sigma}_+(t)\hat{N}_2(t+\tau)\hat{\sigma}_-(t)\right\rangle}{\left\langle \hat{N}_2\right\rangle^2_{\substack{steady\\state}}}$$

$$=1-e^{-\frac{3}{2T_2}|\tau|}\left[\cosh\beta|\tau|+\frac{3}{2T_2\beta}\sinh\beta|\tau|\right]$$

$$\beta=\sqrt{\frac{1}{4T_2^2}-\Omega_R^2}=i\sqrt{\Omega_R^2-\frac{1}{4T_2^2}}$$



Resonance Fluorescence g_2 of a semiconductor quantum dot (Kreinberg et al. Light: Science & Applications (2018) 7:41)

Proof of the Behavior of Correlation Functions under Time Reversal

Consider the Heisenberg operator: $\hat{O}(t) = e^{+\frac{i}{\hbar}\hat{H}t}\hat{O}e^{-\frac{i}{\hbar}\hat{H}t}$

Suppose $\hat{T}$ is the anti-linear operator for time-reversal

And suppose that the Hamiltonian has time-reversal symmetry: $\hat{T}\hat{H}\hat{T}^{+} = \hat{H}$

Then:

$$\hat{T}\hat{O}(t)\hat{T}^{+} = \hat{T}e^{+\frac{i}{\hbar}\hat{H}t}\hat{O}e^{-\frac{i}{\hbar}\hat{H}t}\hat{T}^{+} = e^{-\frac{i}{\hbar}\hat{H}t}\hat{T}\hat{O}\hat{T}^{+}e^{+\frac{i}{\hbar}\hat{H}t} = \eta_O e^{-\frac{i}{\hbar}\hat{H}t}\hat{O}e^{+\frac{i}{\hbar}\hat{H}t} = \eta_O\hat{O}(-t)$$

The anti-linearity of the time-reversal operator implies the following two identities:

a) $\left(\langle\phi|\hat{T}\right)|\psi\rangle = \left[\langle\phi|\left(\hat{T}|\psi\rangle\right)\right]^{*}$

b) $\langle\phi|\left(\hat{T}^{+}|\psi\rangle\right) = \left[\left(\langle\psi|\hat{T}\right)|\phi\rangle\right]^{*} = \langle\psi|\left(\hat{T}|\phi\rangle\right)$

Proof of the Behavior of Correlation Functions under Time Reversal

Now consider the steady state or equilibrium correlation function:

$$\left\langle \hat{A}(t)\hat{B}(0)\right\rangle = \mathrm{Tr}\left[\hat{\rho}\hat{A}(t)\hat{B}(0)\right] = \sum_n \left\langle \phi_n \middle| \hat{\rho}\hat{A}(t)\hat{B}(0) \middle| \phi_n \right\rangle$$

Under time-reversal:

$$\hat{T}\hat{\rho}\hat{T}^+ = \hat{\rho} \qquad \hat{T}\hat{A}(t)\hat{T}^+ = \eta_A\hat{A}(-t) \qquad \hat{T}\hat{B}(0)\hat{T}^+ = \eta_B\hat{B}(0)$$

Therefore:

$$\begin{aligned}
&\left\langle \hat{A}(t)\hat{B}(0)\right\rangle \\
&= \sum_n \left\langle \phi_n \middle| \hat{\rho}\hat{A}(t)\hat{B}(0) \middle| \phi_n \right\rangle \\
&= \sum_n \left\langle \phi_n \middle| \hat{T}^+\hat{T}\hat{\rho}\hat{T}^+\hat{T}\hat{A}(t)\hat{T}^+\hat{T}\hat{B}(0)\hat{T}^+\hat{T} \middle| \phi_n \right\rangle \\
&= \eta_A\eta_B\sum_n \left\langle \phi_n \middle| \hat{T}^+\rho\hat{A}(-t)\hat{B}(0)\hat{T} \middle| \phi_n \right\rangle \\
&= \eta_A\eta_B\sum_n \left\langle \phi_n \middle| \hat{T}^+\rho\hat{A}(-t)\hat{B}(0) \middle| \hat{T}\phi_n \right\rangle \\
&= \eta_A\eta_B\sum_n \left\langle \hat{T}\phi_n \middle| \hat{B}^+(0)\hat{A}^+(-t)\rho \middle| \hat{T}\phi_n \right\rangle \\
&= \eta_A\eta_B\left\langle \hat{B}^+(0)\hat{A}^+(-t)\right\rangle
\end{aligned}$$

$$\begin{aligned}
&\longrightarrow = \eta_A\eta_B\sum_n \left\langle \phi_n \middle| \hat{T}^+\rho\hat{A}(-t)\hat{B}(0) \middle| \hat{T}\phi_n \right\rangle \\
&= \eta_A\eta_B\sum_n \left\langle \hat{T}\phi_n \middle| \rho\hat{A}(-t)\hat{B}(0) \middle| \hat{T}\phi_n \right\rangle^* \\
&= \eta_A\eta_B\left\langle \hat{A}(-t)\hat{B}(0)\right\rangle^*
\end{aligned}$$

Proof of Quantum Regression Theorem

Suppose for all operators operator $\hat{A}_j$:

$$\left\langle \hat{A}_\ell(t)\right\rangle = \mathrm{Tr}\left\{\hat{A}_\ell \hat{\rho}(t)\right\} = \mathrm{Tr}\left\{\hat{A}_\ell \hat{U}(t,0)\,\hat{\rho}(0)\,\hat{U}^{+}(t,0)\right\}$$

$$= \mathrm{Tr}\left\{\hat{U}^{+}(t,0)\,\hat{A}_\ell \hat{U}(t,0)\,\hat{\rho}(0)\right\} = \mathrm{Tr}\left\{\hat{A}_\ell(t)\,\hat{\rho}(0)\right\}$$

And suppose that for all choices of the initial density operator:

$$\frac{d\left\langle \hat{A}_\ell(t)\right\rangle}{dt} = \mathrm{Tr}\left\{\hat{A}_\ell \frac{d\hat{\rho}(t)}{dt}\right\} = \sum_k G_{\ell k}\left\langle \hat{A}_k(t)\right\rangle$$

We will make the additional assumption that the initial density matrix of the environment and the system is separable:

$$\hat{\rho}(0) = \hat{\rho}_{sys}(0)\otimes\hat{\rho}_{env}(0)$$

This implies:

$$\left\langle \hat{A}_\ell(t)\right\rangle = \mathrm{Tr}\left\{\hat{A}_\ell \hat{U}(t,0)\,\hat{\rho}(0)\,\hat{U}^{+}(t,0)\right\} = \mathrm{Tr}\left\{\hat{A}_\ell \hat{U}(t,0)\,\hat{\rho}_{sys}(0)\otimes\hat{\rho}_{env}(0)\,\hat{U}^{+}(t,0)\right\}$$

$$= \mathrm{Tr}_{sys}\left\{\underset{\substack{\uparrow\\ \text{Average of this operator}}}{\hat{A}_\ell}\ \mathrm{Tr}_{env}\left\{\hat{U}(t,0)\underbrace{\left[\hat{\rho}_{sys}(0)\right]}_{\substack{\uparrow\\ \text{With this initial density operator}}}\otimes\hat{\rho}_{env}(0)\,\hat{U}^{+}(t,0)\right\}\right\}$$

Proof of Quantum Regression Theorem

Now look at the correlation function:

$$\left\langle \hat{A}_\ell(t+\tau)\hat{A}_j(t)\right\rangle = \mathrm{Tr}\left\{\hat{U}^\dagger(t+\tau,0)\hat{A}_\ell\hat{U}(t+\tau,0)\hat{U}^\dagger(t,0)\hat{A}_j\hat{U}(t,0)\hat{\rho}(0)\right\}$$

$$= \mathrm{Tr}\left\{\hat{U}^\dagger(t+\tau,0)\hat{A}_\ell\hat{U}(t+\tau,0)\hat{U}^\dagger(t,0)\hat{A}_j\hat{U}(t,0)\hat{\rho}(0)\hat{U}^\dagger(t,0)\hat{U}(t,0)\right\}$$

$$= \mathrm{Tr}\left\{\hat{U}(t,0)\hat{U}^\dagger(t+\tau,0)\hat{A}_\ell\hat{U}(t+\tau,0)\hat{U}^\dagger(t,0)\hat{A}_j\hat{U}(t,0)\hat{\rho}(0)\hat{U}^\dagger(t,0)\right\}$$

$$= \mathrm{Tr}\left\{\hat{A}_\ell\hat{U}(t+\tau,t)\left[\hat{A}_j\hat{\rho}(t)\right]\hat{U}^\dagger(t+\tau,t)\right\}$$

If we make the assumption that the density matrix of the environment and the system at time *t* is separable and given by:

$$\hat{\rho}(t) = \hat{\rho}_{sys}(t)\otimes\hat{\rho}_{env}(t) = \hat{\rho}_{sys}(t)\otimes\hat{\rho}_{env}(0)$$

$$\left\langle \hat{A}_\ell(t+\tau)\hat{A}_j(t)\right\rangle = \mathrm{Tr}_{sys}\left\{\underset{\substack{\uparrow\\ \text{Average of this operator}}}{\hat{A}_\ell}\ \mathrm{Tr}_{env}\left\{\hat{U}(t+\tau,t)\left[\underbrace{\hat{A}_j\hat{\rho}_{sys}(t)}_{\text{With this initial density operator}}\right]\otimes\hat{\rho}_{env}(0)\hat{U}^\dagger(t+\tau,t)\right\}\right\}$$

So comparing with the earlier result, it must be that,

$$\frac{d\left\langle \hat{A}_\ell(t+\tau)\hat{A}_j(t)\right\rangle}{d\tau} = \sum_k G_{\ell k}\left\langle \hat{A}_k(t+\tau)\hat{A}_j(t)\right\rangle$$

Proof of Quantum Regression Theorem

Now look at the correlation function:

$$\left\langle \hat{A}_k(t)\hat{A}_\ell(t+\tau)\hat{A}_j(t)\right\rangle$$

$$= \mathrm{Tr}\left\{\hat{U}^\dagger(t,0)\hat{A}_k\hat{U}(t,0)\hat{U}^\dagger(t+\tau,0)\hat{A}_\ell\hat{U}(t+\tau,0)\hat{U}^\dagger(t,0)\hat{A}_j\hat{U}(t,0)\hat{\rho}(0)\right\}$$

$$= \mathrm{Tr}\left\{\hat{U}^\dagger(t+\tau,t)\hat{A}_\ell\hat{U}(t+\tau,t)\hat{A}_j\hat{\rho}(t)\hat{A}_k\right\}$$

$$= \mathrm{Tr}\left\{\hat{A}_\ell\hat{U}(t+\tau,t)\left[\hat{A}_j\hat{\rho}(t)\hat{A}_k\right]\hat{U}^\dagger(t+\tau,t)\right\}$$

If we make the assumption that the density matrix of the environment and the system at time *t* is separable:

$$\hat{\rho}(t) = \hat{\rho}_{sys}(t)\otimes\hat{\rho}_{env}(0) \longrightarrow$$

Environment has no memory

$$\left\langle \hat{A}_k(t)\hat{A}_\ell(t+\tau)\hat{A}_j(t)\right\rangle$$

$$= \mathrm{Tr}_{sys}\left\{\hat{A}_\ell\mathrm{Tr}_{env}\left\{\hat{U}(t+\tau,t)\left[\hat{A}_j\hat{\rho}_{sys}(t)\hat{A}_k\right]\otimes\hat{\rho}_{env}(0)\hat{U}^\dagger(t+\tau,t)\right\}\right\}$$

So comparing with the earlier result, it must be that,

$$\frac{d\left\langle \hat{A}_t(t)\hat{A}_\ell(t+\tau)\hat{A}_j(t)\right\rangle}{d\tau} = \sum_k G_{\ell k}\left\langle \hat{A}_t(t)\hat{A}_k(t+\tau)\hat{A}_j(t)\right\rangle$$