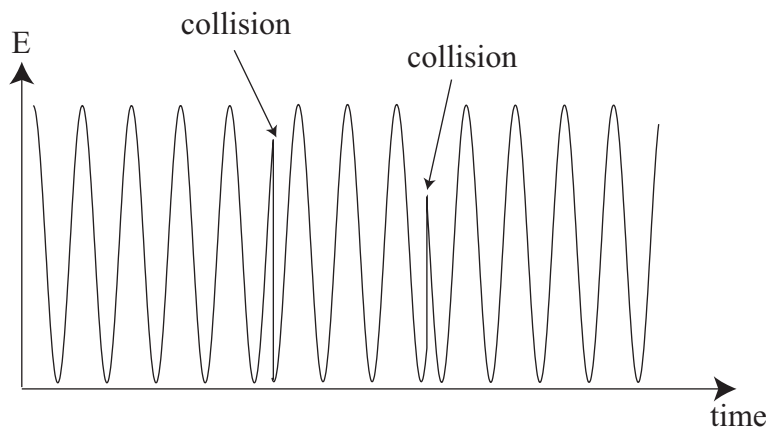


# Chapter 8

## Coherence

In Chapter 6, we discussed that interference happens when mixing two optical waves with well-defined phases between the two waves. However, this well-defined phase difference is typically only valid in small dimensions for most natural sources and hence interference generally can only be observed in small dimensions; for example, two slits separated by only a few millimeters, or thin films much less than a millimeter thick, are required to produce interference fringes in natural light. The need for small dimensions is not, however, associated with the wavelength of light. Rather, it is a consequence directly resulting from the random fluctuation and collisions among the molecules constituting the light source. When the molecule (or electric dipole) collides with other molecules, the phase of the electric field will have a sudden “jump” due to the collision, and the well-defined phase of the preceding wave will be lost, as shown in the graph below.



### 8.1 Coherence and visibility of fringes

In Chapter 6 when we discussed about interference, we assume that the electric fields have a well-defined phase that does not fluctuate in time. In that situation, we can write the total intensity of mixing two electric fields as

$$I = I_1 + I_2 + 2\sqrt{I_1 I_2} \cos \delta$$

where  $\delta$  is a fixed phase difference between the two waves. However, in reality, molecular collision inside the source will alter the phase in a random fashion. Therefore, it is more meaningful to calculate interference as a time average. In this case, the intensity of mixing two fields  $\vec{E}_1$  and  $\vec{E}_2$  can be written as

$$\begin{aligned} I &= \langle (\vec{E}_1 + \vec{E}_2) \cdot (\vec{E}_1 + \vec{E}_2)^* \rangle \\ &= \langle |\vec{E}_1|^2 \rangle + \langle |\vec{E}_2|^2 \rangle + 2 \langle \text{Re}(\vec{E}_1 \cdot \vec{E}_2^*) \rangle \end{aligned}$$

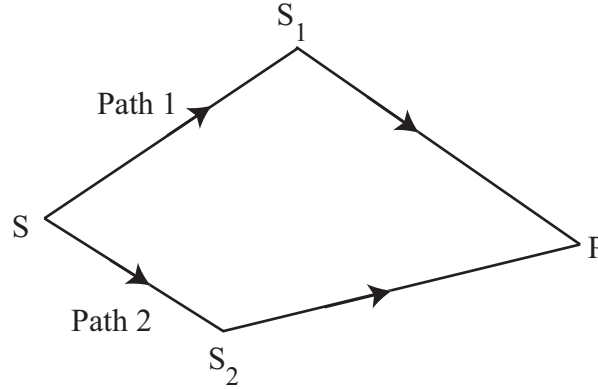
where the brackets denotes the time average:

$$\langle f \rangle = \lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T f(t) dt$$

Here, we assume that the optical fields have the same polarization; then, the total intensity can be written as

$$I = I_1 + I_2 + 2\text{Re} \langle E_1 E_2^* \rangle$$

where  $I_1 = \langle |\vec{E}_1|^2 \rangle$  and  $I_2 = \langle |\vec{E}_2|^2 \rangle$ . In a typical interference experiment the two fields  $E_1$  and  $E_2$  originate from some common source. They differ because they have different optical paths, and can be shown abstractly in the graph below:



Let us call  $t$  the time for one optical wave traveling from  $S$  to  $P$  through path 1, and  $t + \tau$  the time for the other optical wave traveling through path 2. The interference term can be written as

$$2\text{Re}\Gamma_{12}(\tau)$$

where

$$\Gamma_{12}(\tau) = \langle E_1(t) E_2^*(t + \tau) \rangle$$

The function  $\Gamma_{12}(\tau)$  is called the *mutual coherence function* or the *correlation function* of the two electric field  $E_1$  and  $E_2$ . If  $E_1$  and  $E_2$  are the same but only  $E_2$  is time delayed with  $\tau$ , then

$$\Gamma_{11}(\tau) = \langle E_1(t)E_1^*(t + \tau) \rangle$$

is known as the *autocorrelation function* or the *self-coherence function*. From the definition, we see that  $\Gamma_{11}(0) = I_1$  and  $\Gamma_{22}(0) = I_2$ . The correlation functions sometimes are written in a normalized form and is called the *degree of partial coherence function*,

$$\gamma_{12}(\tau) = \frac{\Gamma_{12}(\tau)}{\sqrt{\Gamma_{11}(0)\Gamma_{22}(0)}} = \frac{\Gamma_{12}(\tau)}{\sqrt{I_1 I_2}}$$

The total intensity of mixing two optical waves is then expressed as

$$I = I_1 + I_2 + 2\sqrt{I_1 I_2} \text{Re}[\gamma_{12}(\tau)]$$

Since the phase difference between the two waves is not completely well-defined; but instead, these frustrations are varying the phases of the electric fields statistically, and the function  $\gamma_{12}(\tau)$  correlates these statistical fluctuations in time. In general, the function  $\gamma_{12}(\tau)$  is a complex function of  $\tau$ , and  $0 < |\gamma_{12}(\tau)| < 1$ . In terms of  $|\gamma_{12}(\tau)|$ , we have the following types of coherence:

$$\begin{array}{ll} |\gamma_{12}| = 1 & \text{complete coherence} \\ 0 < |\gamma_{12}| < 1 & \text{partial coherence} \\ |\gamma_{12}| = 0 & \text{complete incoherence} \end{array}$$

In a pattern of interference fringes, the intensity varies between two limits  $I_{max}$  and  $I_{min}$ . The minimum and maximum value of the intensity is simply as

$$I_{max} = I_1 + I_2 + 2\sqrt{I_1 I_2} |\gamma_{12}|$$

$$I_{min} = I_1 + I_2 - 2\sqrt{I_1 I_2} |\gamma_{12}|$$

The *fringe visibility*  $\mathcal{V}$  is defined as the ratio

$$\begin{aligned} \mathcal{V} &= \frac{I_{max} - I_{min}}{I_{max} + I_{min}} \\ &= \frac{2\sqrt{I_1 I_2} |\gamma_{12}|}{I_1 + I_2} \end{aligned}$$

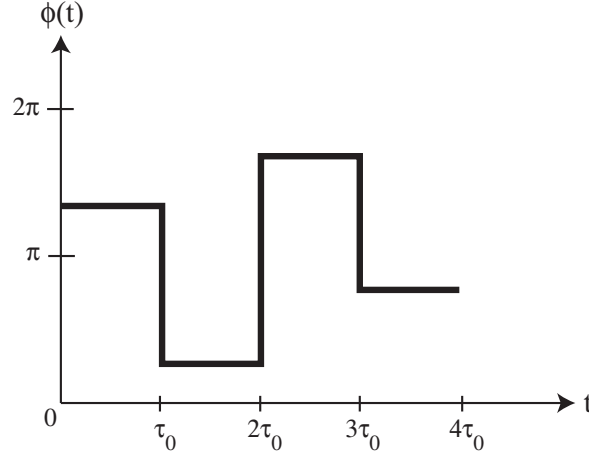
In particular, if  $I_1 = I_2$ , then

$$\mathcal{V} = |\gamma_{12}|$$

that is, the fringe visibility is equal to the modulus of the degree of partial coherence. In the case of complete coherence ( $|\gamma_{12}| = 1$ ) the interference fringes have the maximum contrast of unity, whereas for complete incoherence ( $|\gamma_{12}| = 0$ ) the contrast is zero; that is, there are no interference fringes at all.

## 8.2 Coherence time and coherence length

In order to see how the degree of partial coherence is related to the characteristics of the source, let us consider the case of a hypothetical “quasi-monochromatic” source having the following property: The oscillation and the subsequent field vary sinusoidally for a certain time  $\tau_0$  and then change phase abruptly. This sequence keeps repeating indefinitely.  $\tau_0$  is called the *coherence time*, and the phase change that occurs after each coherence time is considered to be randomly distributed between 0 and  $2\pi$ .



The electric field of this quasi-monochromatic wave can be expressed as

$$E(t) = E_0 e^{-i\omega t} e^{i\phi(t)}$$

where the phase  $\phi(t)$  is randomly frustrating with the coherence time  $\tau_0$ . The coherence time here can be think as the mean collision time between molecules inside the light source.

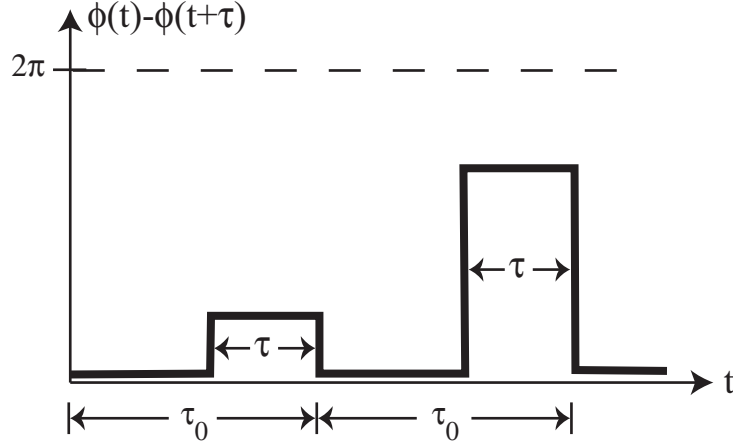
Suppose that this light wave is sent to a Michelson interferometer and observe the interference pattern by a photo-diode. The degree of partial coherence can be evaluated by assuming that the beam splitter of the Michelson interferometer is 50/50, i.e.  $|E_1| = |E_2| = |E|$ . Since it is the autocorrelation is of our concern, we delete the subscripts of  $\gamma$  and write

$$\gamma(\tau) = \frac{\langle E(t)E^*(t+\tau) \rangle}{\langle |E|^2 \rangle}$$

Substituting by the electric field, the self-coherence function now becomes

$$\begin{aligned} \gamma(\tau) &= \left\langle \frac{E_0 e^{-i\omega t} e^{i\phi(t)} E_0 e^{i\omega(t+\tau)} e^{-i\phi(t+\tau)}}{E_0^2} \right\rangle \\ &= \left\langle e^{i\omega\tau} e^{i[\phi(t)-\phi(t+\tau)]} \right\rangle \\ &= e^{i\omega\tau} \lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T e^{i[\phi(t)-\phi(t+\tau)]} dt \end{aligned}$$

Plotting the quantity  $\phi(t) - \phi(t+\tau)$  using the value from the previous graph,



Now for the first coherence time interval,  $0 < t < \tau_0$ , we observe that  $\phi(t) - \phi(t+\tau) = 0$  for  $0 < t < \tau_0 - \tau$ . On the other hand for  $\tau_0 - \tau < t < \tau_0$ , it assumes some random value between 0 and  $2\pi$ . the same is true for each succeeding coherence time interval  $\tau_0$ . The integral of  $\gamma(\tau)$  can be evaluated as the following:

$$\begin{aligned} \frac{1}{\tau_0} \int_0^{\tau_0} e^{i[\phi(t) - \phi(t+\tau)]} dt &= \frac{1}{\tau_0} \int_0^{\tau_0 - \tau} dt + \frac{1}{\tau_0} \int_{\tau_0 - \tau}^{\tau_0} e^{i\Delta} dt \\ &= \frac{\tau_0 - \tau}{\tau_0} + \frac{\tau}{\tau_0} e^{i\Delta} \end{aligned}$$

where  $\Delta$  is the random phase difference.

The same result is obtained for all subsequent intervals, except that  $\Delta$  is different for each interval. Since  $\Delta$  is random, the term involving  $e^{i\Delta}$  will average to zero. The other term,  $(\tau_0 - \tau)/\tau_0$ , is the same for all intervals, hence it is equal to average value of the integral. Of course, if  $\tau > \tau_0$ , the phase difference  $\phi(t) - \phi(t + \tau)$  is always random and, consequently, the whole integral averages to zero. From the above above result, we find that the normalized autocorrelation function for a quasi-monochromatic source is given by

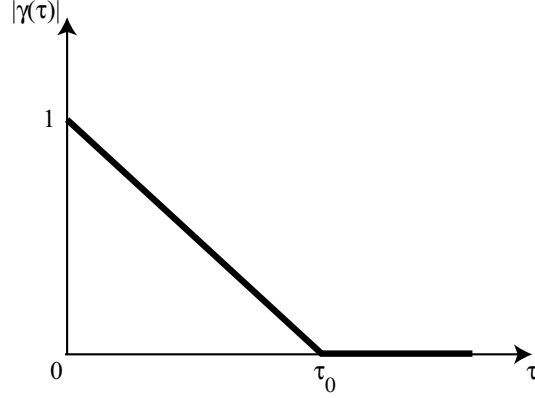
$$\begin{aligned} \gamma(\tau) &= \left(1 - \frac{\tau}{\tau_0}\right) e^{i\omega\tau} & \tau < \tau_0 \\ &= 0 & \tau \geq \tau_0 \end{aligned}$$

$$\begin{aligned} |\gamma(\tau)| &= 1 - \frac{\tau}{\tau_0} & \tau < \tau_0 \\ &= 0 & \tau \geq \tau_0 \end{aligned}$$

A graph of  $|\gamma|$  is shown below and from the previous section, the autocorrelation function is equal to the fringe visibility  $\mathcal{V}$  for the case of equal amplitude in a two-beam interference arrangement. Evidently the fringe visibility drops to zero if  $\tau$  exceeds the coherence time  $\tau_0$ . This means that the path difference between the two beam must not exceed the value

$$c\tau_0 = l_c$$

in order to obtain interference fringes. The quantity  $l_c$  is called the *coherence length*. It is essentially the length of an uninterrupted wave train.



### 8.3 Coherence for a finite wave train

In practice, no source of light is even strictly monochromatic. Even in the best so-called monochromatic sources there is always some finite spread of frequency centered about some mean frequency. We now proceed to investigate the relationship between the frequency spread, or line width, and the coherence of a light source. To do this, we make use of the *Fourier integral theorem*. According to the theorem, stated here without proof, a function  $f(t)$  can be expressed as an integral over the variable  $\omega$  in the following way:

$$\begin{aligned} f(t) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g(\omega) e^{-i\omega t} d\omega \\ g(\omega) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) e^{i\omega t} dt \end{aligned}$$

The functions  $f(t)$  and  $g(\omega)$  are called *Fourier transforms* of each other and are said to constitute a Fourier transform pair. In our present application, the variables  $t$  and  $\omega$  are time and frequency, respectively. The function  $g(\omega)$  then constitutes a frequency resolution of the time dependent function  $f(t)$  or stated in another way,  $g(\omega)$  represents the function in the frequency domain.

Let us consider now the particular case in which the function  $f(t)$  represents a single wave train of finite duration  $\tau_0$ . The time variation of this wave train is given by the function

$$\begin{aligned} f(t) &= e^{-i\omega_0 t} & \text{for } -\tau_0/2 < t < \tau_0/2 \\ f(t) &= 0 & \text{otherwise} \end{aligned}$$

Taking the Fourier transform, we have

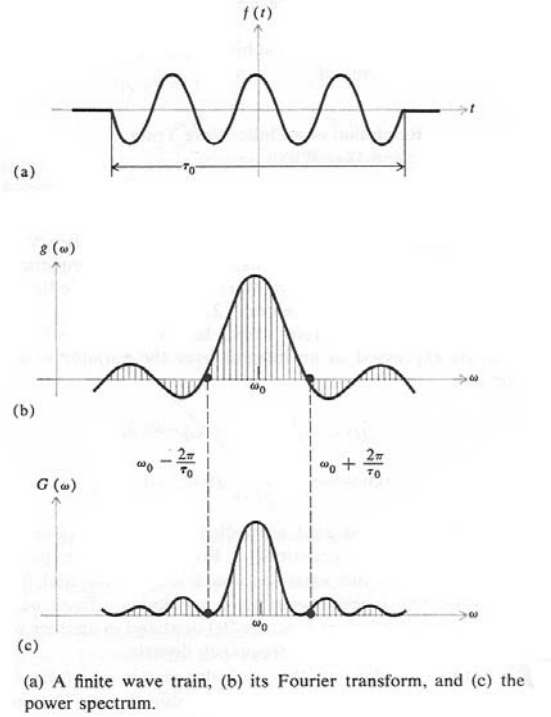
$$\begin{aligned} g(\omega) &= \frac{1}{\sqrt{2\pi}} \int_{-\tau_0/2}^{\tau_0/2} e^{i(\omega - \omega_0)t} dt \\ &= \sqrt{\frac{2}{\pi}} \frac{\sin[(\omega - \omega_0)\tau_0/2]}{\omega - \omega_0} \end{aligned}$$

The power spectrum is defined as the square of the  $g(\omega)$ ,

$$G(\omega) = |g(\omega)|^2$$

The function, in the case of a finite wave train, is given by

$$G(\omega) = \frac{2 \sin^2[(\omega - \omega_0)\tau_0/2]}{\pi (\omega - \omega_0)^2}$$



We see that the spectral distribution is maximum for  $\omega = \omega_0$  and drops to zero for  $\omega = \omega_0 \pm 2\pi/\tau_0$ . Most of the energy is contained in the region between the first two minima on either side of the central maximum at  $\omega_0$ . The spectral width  $\Delta\omega$  of the frequency distribution is therefore given by

$$\Delta\omega = \frac{2\pi}{\tau_0}$$

or

$$\Delta\nu = \frac{1}{\tau_0}$$

Now if we have a sequence of wave trains, each lasting for a time  $\tau_0$  but occurring at random times, then the power spectrum is exactly the same as that of the single pulse given above. On the other hand, if the pulses are not all of the same duration, that is, if  $\tau_0$  varies from pulse to pulse, then we can think of an average time  $\langle \tau_0 \rangle$ . The precise form of the spectral distribution is different from that of the single pulse,

but the width of the corresponding frequency spectrum is approximately  $1/\langle \tau_0 \rangle$ . Suppose we now take the reverse reasoning, namely, that if a spectral source has a line width  $\Delta\nu$ , then the corresponding coherence time  $\langle \tau_0 \rangle$  is given by

$$\langle \tau_0 \rangle = \frac{1}{\Delta\nu}$$

and the coherence length  $l_c$  is

$$l_c = c \langle \tau_0 \rangle = \frac{c}{\Delta\nu}$$

We can also express the coherence length in terms of wavelength. Using the fact that  $\Delta\nu/\nu = |\Delta\lambda|/\lambda$ , we obtain

$$l_c = \frac{\lambda^2}{\Delta\lambda}$$

where  $\Delta\lambda$  is the width of the spectrum line on the wavelength scale.

As a specific example, ordinary spectral source, such as discharge tubes, have line width of  $\sim 5000$ . The corresponding coherence length is of the order of 5000 wavelengths, or about 2mm. Comparing to the line width of a typical gas laser, it is typically as narrow as  $10^3$ Hz or less. This corresponds to a coherence length of  $\nu/\Delta\nu \simeq 10^{14}/10^3 = 10^{11}$  wavelengths, which is the order of 50km.