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REVIEW

The Dicke model in quantum optics: Dicke model revisited

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A short review of recent developments of the Dicke model in quantum optics is presented. The focus is on the model in a cavity at zero temperature and in the rotating wave approximation. Topics discussed include spectroscopic structures, the giant quantum oscillator, entanglement and phase transitions.

Keywords: quantum optics; Dicke model; Tavis–Cummings model; spectroscopy; entanglement; phase transition

1. Introduction

The Dicke model [1] is simple to describe, but hard to realize in a physical system. The system is composed of N atoms cooperatively interacting with a single mode of the radiation field. The essence of *cooperative* behaviour is that, for atomic systems, the atomic dipoles interact *coherently* with the privileged radiation mode. In recent years, the Dicke model has had renewed interest because of its phase transitions and because it is a simple model system in which one can find multi-partite entanglement. Above all, the Dicke model had renewed interest because it can be realized in systems more widely than in the original cavity QED case. Much of the original interest in the model included the phase transitions and related to it being an exactly solvable quantum system. Further, it was an early investigation of something similar to a laser-like behaviour, super-radiance, from a time well before the laser.

For N atoms to cooperatively interact with a single privileged mode of the radiation field, we require a special environment: one mode, or frequency, must dominate while other frequencies are suppressed. We will take the basic Hamiltonian for the Dicke model in the rotating wave approximation (RWA) as

$$H_0 = \omega_c a^\dagger a + \omega_0 S_z + g(a^\dagger S_- + a S_+), \quad (1.1)$$

where ω_c is the frequency of the cavity mode, ω_0 the resonant frequency of one of the atoms, and g is the single-atom cavity–atom coupling constant. (We take $\hbar = 1$.) We will neglect the possible space dependence of g ; this is discussed further

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One contribution of 13 to a Theme Issue ‘Nonlinear phenomena, optical and quantum solitons’.

in §3. The operators a and a^\dagger are the usual annihilation and creation operators for the quantized cavity field, and the collective atomic operators involved are $S_{\pm,z}$

$$S_{\pm} = \sum_{i=1}^N S_{\pm}^{(i)} \quad \text{and} \quad S_z = \sum_{i=1}^N S_z^{(i)}. \quad (1.2)$$

In this case, as is quite typical, underlying the collective operators (1.2) is a set of two-level atoms, where the spin-1/2 operators $S_{\pm,z}^{(i)}$ are given by

$$S_-^{(i)} = |g_i\rangle\langle e_i|, \quad S_+^{(i)} = |e_i\rangle\langle g_i| \quad \text{and} \quad S_z^{(i)} = \frac{1}{2}(|e_i\rangle\langle e_i| - |g_i\rangle\langle g_i|), \quad (1.3)$$

where $|g_i\rangle$ and $|e_i\rangle$ are the ground and excited states for the i th atom. It is straightforward to confirm, using equations (1.2) and (1.3), that the usual angular momentum commutation relations are satisfied, i.e. $[S_+, S_-] = 2S_z$ and $[S_z, S_{\pm}] = S_{\pm}$.

If we had no coupling g , and considered just a Hamiltonian $\omega_0 S_z$ for the atoms, we could already observe several features. Firstly, the ground state energy, in this basis, is $-(S/2)\omega_0$, where S is the spin of the system. Secondly, there are $2S + 1$ spin states, but, if we examine the number of states possible with combinations of e and g , there should be 2^N with many degenerate states. The simple explanation of the difference is that the N atoms can, of course, be represented by a spin state with spin $S/2, S/2 - 1, S/2 - 2, \dots$. The simplest example of this is the two-atom system which can have total spin 1 (termed the super-radiant state), or total spin 0 (often called a sub-radiant state). In a spin basis $|S, m_s\rangle$, the connections are: $|S = 1, m_s = 1\rangle \equiv |e_1, e_2\rangle$, $|S = 1, m_s = 0\rangle \equiv (|e_1, g_2\rangle + |g_1, e_2\rangle)/\sqrt{2}$, $|S = 1, m_s = -1\rangle \equiv |g_1, g_2\rangle$ and $|S = 0, m_s = 0\rangle \equiv (|e_1, g_2\rangle - |g_1, e_2\rangle)/\sqrt{2}$. The super-radiant and sub-radiant terminology refers to the radiation rate of the Dicke system if it is coupled to an environment. The additional values of S are not enough to make up the 2^N states, and Nussenzveig [2], for example, points out that the states for which $S < N/2$ are, in general, degenerate. By working from an extreme value of S_z eigenvalue, one can easily show that the degeneracy is $(2S + 1)N! / [(N/2 + S + 1)!(N/2 - S)!]$, which will show for $N > 2$ and $S < N/2$. In what follows, unless otherwise stated, we will assume that the maximal spin system is considered, i.e. $S = N/2$. We should again emphasize that a system of N atoms could have smaller values of S depending on the initial state. However, since $S^2 = S_x^2 + S_y^2 + S_z^2$ commutes with the Hamiltonian (1.1), the value of this spin will be conserved.

The basic Hamiltonian (1.1) is also known as the Tavis–Cummings model [3,4] in the N -atom case (and is known as the Jaynes–Cummings model in the single-atom case). Although we will refer to the Hamiltonian (1.1) as the Dicke Hamiltonian, in fact the original paper of Dicke [1] treated the electromagnetic field semi-classically rather than using a quantum oscillator. The system described above has no dissipation, i.e. it is not an open quantum system. There are two principal ways in which decoherence can manifest itself: as collective damping in which the atoms act together, and as non-collective damping in which the atoms tend to decay independently.

In the following, we will discuss the energy levels and spectroscopy of the Dicke system, with aspects of the decay and probing in §2. In §3 we look at some potential candidates for realizing the Dicke system in the context of quantum optics. Section 4 explores developments in Dicke model entanglement, and the decay of that entanglement, with emphasis on the two-atom system. Recently, the Dicke model phase transition has been a topic of discussion [5] and experimental observation [6]; we will discuss the phase transitions further in §5. And, finally, a brief conclusion is presented in §6.

2. Eigenstructure of the Dicke model

(a) Two-atom Dicke model

The simple two-atom model system serves as a useful introduction to the energy level structure of the Dicke model. To find the eigenstates, we note that the excitation number $a^\dagger a + S_z$ commutes with H_0 , and try a linear combination of the $S = 1$ Dicke states with appropriate field number states $|n\rangle$ to keep the excitation number as a constant with value n (not to be confused with a photon number). In the case of resonance, we then find that the eigenstates are

$$\begin{aligned} |n, 0\rangle &= \sqrt{\frac{n+1}{2n+1}} |S=1, m_s=1\rangle |n-1\rangle - \sqrt{\frac{n}{2n+1}} |S=1, m_s=-1\rangle |n+1\rangle, \\ |n, \pm 1\rangle &= \sqrt{\frac{n}{2(2n+1)}} |S=1, m_s=1\rangle |n-1\rangle + \sqrt{\frac{n+1}{2(2n+1)}} |S=1, m_s=-1\rangle |n+1\rangle \\ &\quad \pm \frac{1}{\sqrt{2}} |S=1, m_s=0\rangle |n\rangle, \end{aligned} \quad (2.1)$$

with eigenvalues

$$n\omega_c, \quad n\omega_c \pm 2g\sqrt{n + \frac{1}{2}}. \quad (2.2)$$

The eigenvalue spectrum is a succession of triplets except for $n=1$ (a doublet) and $n=0$ (which is a singlet). Although the triplets have an equal spacing between the two pairs, this changes if you go off-resonance. The triplet spacing is equal to the doublet spacing on-resonance.

(b) $N > 2$ atoms and approximations to the highly excited energy levels

As we consider more atoms and allow any amount of excitation, the eigenstates of the Dicke Hamiltonian have a structure illustrated in figure 1. The ground state of the system is the simple state $|g_1 g_2 \cdots g_N\rangle |0\rangle$, where $|0\rangle$ represents the state of the field. The next pair of states contains combinations of states with a single excitation, then there are three states and an increasing number of levels in each manifold until we reach $N+1$ levels; thereafter, the number of levels remains constant although the energy-level spacing changes [7].

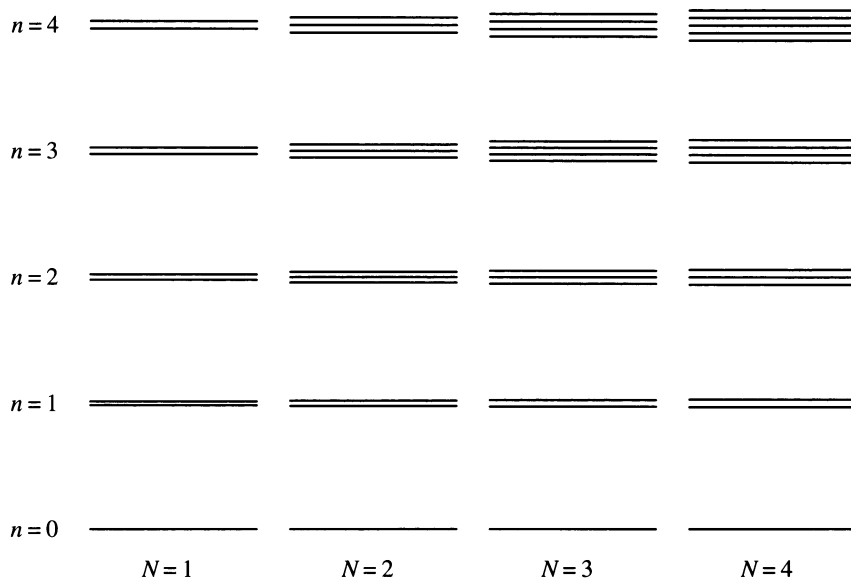


Figure 1. Illustration of low-lying energy levels of the Dicke Hamiltonian (1.1) on resonance (i.e. $\omega_c = \omega_0$). The energy levels are shown for different numbers N of atoms and excitations n . The atoms are maximally coupled so that, for spin S , $N = 2S$. The spin-0 case has been omitted from the figure.

If we are sufficiently high up the ladder of states seen in figure 1 that there are $N + 1$ states in a multiplet, the energy levels can be approximately written as [7]

$$E(n, j) \sim n\omega_c + j\sqrt{4g^2n + \Delta^2} + E^{(1)}(n, j), \quad (2.3)$$

where the detuning $\Delta = \omega_0 - \omega_c$ and $-N/2 \leq j \leq N/2$. The energy of the multiplet is first given by n and then the members are identified by j (with slightly different energies). The corresponding states are

$$|n, j\rangle = \sum_{m_S=-S}^S \langle S, m_S | \exp(-i\theta S_y) | S, j\rangle | S, m_S\rangle | n - m_S\rangle, \quad (2.4)$$

where $|S, j\rangle$ and $|S, m_S\rangle$ are angular momentum states (belonging to spin S) and the state $|n - m_S\rangle$ is an eigenstate of $a^\dagger a$.

In equation (2.3), $E^{(1)}(n, j)$ is a small correction to the energy-level structure, which is found from perturbation theory,

$$E^{(1)}(n, j) = \frac{g}{2\sqrt{n}} \left\{ \left[\left(\frac{N}{2} \right) \left(\frac{N}{2} + 1 \right) - 3j^2 \right] \sin \theta \cos \theta + j \sin \theta \right\}, \quad (2.5)$$

where $\tan \theta = 2g\sqrt{n/\Delta}$. In situations where n is very large, it can be appropriate to replace n by an average, i.e. let $n \approx n + 1 \rightarrow \bar{n}$.

(c) *Low-lying states and the giant quantum oscillator*

For the lower lying states (where $n < N$) we have, from equation (1.1), the approximate eigenenergies

$$E(n, j) \sim \left(n - \frac{N}{2}\right) \omega_0 - \frac{n}{2} \Delta + j \sqrt{4g^2 N + \Delta^2}, \quad (2.6)$$

where $-n/2 \leq j \leq n/2$ and $0 \leq n \leq N$, i.e. the number of states in a multiplet n is now $n + 1$ rather than $N + 1$. The corresponding states are

$$|n, j\rangle \sim \sum_{\ell=-L}^L \langle L, \ell | \exp(i\theta L_y) | L, j \rangle |S, L - S - \ell\rangle |L + \ell\rangle, \quad (2.7)$$

where $|L, j\rangle$ and $|S, L - S - \ell\rangle$ are angular momentum states (belonging to spin $L = n/2$ and spin S , respectively), and the state $|L + \ell\rangle$ is an eigenstate of $a^\dagger a$. Note that the ground state of the system is $|g_1 g_2 g_3 \cdots g_N\rangle |0\rangle$, which has an energy of $-(N/2)\omega_0$ ($n=0, j=0$ in equation (2.6)) because of the choice of form for equation (1.1) involving the collective S_z operator.

As pointed out by Bullough [8] and Bullough *et al.* [9], we can use the Holstein–Primakoff transformation [9,10] to represent the collective Dicke operators as

$$S_+ = b^\dagger (N - b^\dagger b)^{1/2}, \quad S_- = (N - b^\dagger b)^{1/2} b \quad \text{and} \quad S_z = b^\dagger b - \frac{N}{2}. \quad (2.8)$$

This approach has also recently been taken by Emary & Brandes [11,12]. Then the angular momentum commutators are obeyed when the operators b and b^\dagger obey the usual Bose commutation relations. Equation (2.8) suggests that, for very large N , the uncoupled Dicke spin behaves like a harmonic oscillator system, i.e. the *giant quantum oscillator* [8]. If we expand the operators S_\pm in $b^\dagger b$, the Hamiltonian (1.1) becomes approximately

$$H_0 \sim \omega_c a^\dagger a + \omega_0 \left(-\frac{N}{2} + b^\dagger b\right) + g\sqrt{N}(a^\dagger b + ab^\dagger), \quad (2.9)$$

if we neglect the smaller terms $(b^\dagger b/N)$ compared with unity. This is an approximate system of two coupled oscillators, one associated with the Dicke spin and the other with the electromagnetic field. The Hamiltonian is easily diagonalized to yield the energy levels (2.6). If the number of excitations n is given by $a^\dagger a + b^\dagger b$, it is clear that the spectrum of energy levels for $n \leq N$ in figure 1 consists of multiplets of n levels (split by the interaction $g\sqrt{N}$) at a basic energy (on resonance) of $(n - N/2)\omega_0$.

(d) *Thermal equilibrium and the giant quantum oscillator*

For the case of thermal equilibrium, rather than having any coherent process, the properties of the system can be found from the partition function

$$Z \sim \sum_{n=0}^{\infty} \sum_{j=-n/2}^{n/2} \exp(-\beta E(n, j)). \quad (2.10)$$

Using equation (2.6) and the eigenstates (2.7), the number of excited atoms, $\bar{N}_+ = \langle S_z + N/2 \rangle$, is found to be

$$\bar{N}_+ = \frac{(1 + \cos \theta)/2}{\exp(\beta\omega_+) - 1} + \frac{(1 - \cos \theta)/2}{\exp(\beta\omega_-) - 1}, \quad (2.11)$$

where $\omega_{\pm} = \bar{\omega} \pm \Omega/2$, $\bar{\omega} = (\omega_0 + \omega_c)/2$ and $\cos \theta = \Delta/\Omega$. The mean excitation of the electromagnetic field is given by the same expression, but with $\theta \rightarrow -\theta$. Expression (2.11) has a nice interpretation of a weighted average of two thermally excited oscillators with frequencies ω_{\pm} . These are the oscillators found by diagonalizing equation (2.9). For large detunings (or small couplings), and bearing in mind that Ω depends on detuning, the frequency ω_+ tends to the value of the larger of ω_0 or ω_c , and ω_- tends to the smaller of the two. As a result, the two terms approach the thermal excitation of a single mode and the thermal excitation of the uncoupled spin. However, because of the weights involving θ , the cavity term is eliminated, leaving $\bar{N}_+ \sim 1/(\exp(\beta\omega_0) - 1)$ in this limit (as expected for weak coupling).

It would appear that a discontinuity in the second term of equation (2.11) occurs when $\omega_- = 0$, i.e. when $\omega_0\omega_c = Ng^2$. This corresponds, in part, to the thermodynamic phase transition of Wang & Hioe [13] and Hepp & Lieb [14,15]. However, note that we already took the large N limit by considering only the $n < N$ low- n eigenvalues, or, in other words, the sum over n in equation (2.10) was extended to infinity. Furthermore, the coupling strength is so strong that the RWA made in equation (1.1) should be suspect and equation (2.11) should not be valid. This phase transition is further discussed briefly in §5a.

3. Practical considerations and realizations

(a) *The Dicke model, super-radiance, super-fluorescence and amplified spontaneous emission*

There can be confusion concerning the relationship between the Dicke model and super-radiance, super-fluorescence and collective resonance fluorescence. These issues, in particular, involve the environment of the collective atomic system, and there are two key concerns: the spatial size of the region occupied by the atoms, and the presence, or not, of a cavity. If there is no cavity then collective spontaneous emission requires the atoms to be close together, on a scale much shorter than a wavelength. Otherwise, the emission from an initially excited system is the amplified spontaneous emission [16]. Practically speaking, there must always be some separation between radiating atoms, although the Dicke model is often seen to apply to atoms on a single site. However, if the atoms are too close then dipole-dipole interactions become important (and these act to break the symmetry of the Dicke model). If the atoms are set apart, then spatial effects can occur: a photon radiated by one atom can be absorbed by another and cooperative emission is reduced. Indeed, in free space, the shape of the distribution of atoms will heavily influence the direction of collective emission, or super-radiance. We will not discuss this further here, but refer the interested reader to an earlier review [17].

In the presence of a cavity, the electromagnetic field is dominated by a single wavelength and direction and the Dicke model can potentially be realized. Atoms can be more easily placed at the nodes of a standing electromagnetic wave, either together at one node (in which case dipole–dipole interactions may again have important effects) or apart at different nodes. We will discuss these arrangements further in the next sections.

(b) *The micromaser*

The earliest likely candidate to realize a Dicke model system was composed of Rydberg atoms in a microwave cavity [18]. Rydberg atoms have long lifetimes and, being highly excited, the transitions involve long wavelengths. They also couple strongly to electromagnetic fields. Microwave cavity resonators can have long lifetimes (approaching a second for superconducting cavities with small entrance holes). They can have dimensions of the order of 10^{-2} m, and can be more or less completely enclosing. All these qualities make them ideal candidates for demonstrating collective and single-atom cavity-field quantum effects as oscillations of energy between the atom and cavity field can take place before decay becomes significant. The relatively large physical size (compared with the atomic beam used for the atoms) ensures that many atoms can easily be accommodated inside. Enclosure reduces the access of the environment outside the cavity to the atoms inside. Many important experiments focused on the single-atom regime; however, collective effects were also observed [19,20]. It is important to note, though, that in these experiments there was a constant flux of atoms through the cavity. Since the atoms typically enter in the excited state and pump energy into the cavity system, this leads to the idea of the micromaser and, in particular, the single-atom micromaser [21]. However, this is far removed from the simple model system of equation (1.1), which implies a fixed number of atoms. In addition, the timing of the atoms entering the cavity is uncertain and, as a result, if there are already N atoms in the cavity, and in the middle of an interaction with that cavity, then at the moment when one atom is gained or lost the change in cooperativity may not exactly match the change in atom number. Furthermore, the low (microwave) energy of Rydberg atom transitions means that thermal excitation is an issue, although low temperatures can be achieved with cryogenic cooling of the cavity.

(c) *The optical cavity*

In the early 1980s, Rydberg atoms and microwave transitions were the only possibility for the experimental realization of cooperative Dicke effects in cavities. However, as the technology for making high-Q optical cavities improved, it became possible to reach a strong-coupling regime in the optical domain with high-quality, small-volume, optical cavities. Reducing the volume enhances the coupling strength. This, in turn, reduces the time scale for a Rabi oscillation and, taken together with improvements in cavity decay time, we enter a strong-coupling, low-decay regime, where equation (1.1) has some validity.

The vacuum-field Rabi splitting, or normal-mode splitting, was observed in an optical cavity as a cooperative effect [22,23]. (Later it was observed in Rydberg atom experiments also [24].) If we examine the energy-level spectrum

shown in figure 1, we will note that the first set of states above the ground state is always a pair. However, the energy separation of that pair increases with the square-root of the atom number when on resonance (see equation (2.6) with $n = 1$ so that $j = \pm 1/2$). Measurements of this vacuum-field Rabi splitting typically use a weak field to probe transitions between the ground state and the first excited state pair. The frequency is scanned and the N -dependent splitting can then be observed. The theory of such atom-cavity spectroscopy was reviewed by Agarwal [25]. The theoretical spectra can be quite complex and this reflects the many different possible transitions between the members of multiplets (as in figure 1). In general, a single photon transition might occur with a frequency determined from equation (2.6) for $n \ll N$,

$$\begin{aligned}\Delta E(j, j') &= E(n+1, j) - E(n, j') \\ &= \frac{(\omega_0 + \omega_c)}{2} + (j - j')\sqrt{4g^2N + \Delta^2}.\end{aligned}\quad (3.1)$$

For $n \gg N$, we may use equation (2.3) to find [25]

$$\begin{aligned}\Delta E(n, j, j') &= E(n+1, j) - E(n, j') \\ &= \omega_c + j\sqrt{4g^2(n+1) + \Delta^2} - j'\sqrt{4g^2n + \Delta^2}.\end{aligned}\quad (3.2)$$

In the highly excited case of equation (3.2), the spectra would be observed in emission rather than in absorption. Some of the fine details would be hard to observe: for example, the corrections found in equation (2.5) introduce an n -dependence into equation (3.2), which is very small as the transitions are between $n+1$ and n . In addition, not all transitions are equally favoured as, for $n \gg N$, emission is dominated by $j - j' = 0, \pm 1$.

Even if the Dicke system is more or less in the ground state, a cavity experiment would not show all the possible structures unless a high resolution is available; typically there is an average atom number ($N \rightarrow \bar{N}$) owing to fluctuations in numbers of atoms crossing the cavity and the Rabi splitting, or normal mode splitting seen owing to probe absorption [23] consists of two peaks with a separation $g\sqrt{\bar{N}}$ and a width determined by experimental factors such as the cavity loss rate. These observations clearly indicate a cooperative effect, although other factors such as the spatial dependence of g and the atom number fluctuations play a role.

(d) Recent developments: ion-trap cavity QED

The experiments described in §3*b,c* still had a flux of atoms, which meant that the atom number was not constant, but with the development of combined ion-trap cavity systems this has changed [26]. Because ion traps can maintain the position of ions in a cavity, the possibility exists for realizing the Dicke model with a fixed number of ions. However, one difficulty is that cavities for ion traps tend to be rather open. One reason is that access is needed for laser cooling of the motional state of the ions. As a result of having a cavity that is not completely enclosed, spontaneous emission from individual ions is possible. The situation is illustrated for two ions in figure 2. Here, the decay

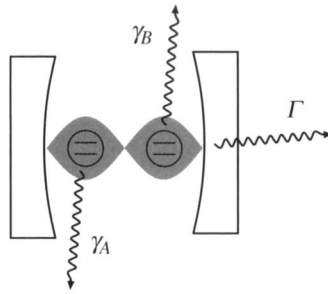


Figure 2. Schematic for two two-level atoms, or ions, which are stationary inside an optical cavity. The ions are arranged to be at the anti-nodes of the cavity field so that the coupling g is the same and maximal. The cavity field leaks at a rate Γ . The atoms can also access their environment through the sides of the cavity and they decay independently of each other at rates $\gamma_{A,B}$. Reproduced with permission from Mazzola *et al.* [27], in which the effect of the two types of decay channel on entanglement was explored. In this paper, we will take $\gamma_A = \gamma_B = \gamma$. © Springer Science + Business Media. (Online version in colour.)

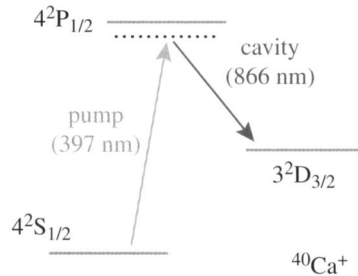


Figure 3. Energy-level scheme for three-level ions in a cavity. The classical coherent field ensures a quantum coupling between the two lower states. By detuning the two-photon resonance, spontaneous emission from the upper state is reduced and the population oscillates between the two lower states if the cavity decay is low. Reproduced with permission from Macmillan Publishers Ltd from the study of Keller *et al.* [28], in which the classical field is pulsed to form a photon ‘pistol’, or train of single photons from a single trapped ion. (Online version in colour.)

rate Γ indicates the loss through the cavity mirrors. The decay rates γ_A and γ_B represent the losses from individual ions, sideways, to the environment. These decay processes break the cooperativity of the system, while the cavity loss Γ does not.

Spontaneous emission can be reduced by detuning the cavity field from the excited state, except that this would adversely affect the dynamics by taking the system off-resonance. In recent years, ions have been trapped in cavities with three-level schemes such as the one in figure 3 [28]. By combining the quantum field of the cavity with the coherent excitation of a laser, it is possible to bypass the excited state by ensuring a two-photon resonance between different ground states (or a ground and long-lived state). The excited state ($4^2P_{1/2}$ in figure 3) is adiabatically eliminated and, as it is not populated, spontaneous emission is reduced. The theory for two such ions in a cavity field was developed recently in the study of Härkönen *et al.* [29] and shows that the two-atom Dicke model can be realized this way and used for entanglement creation.

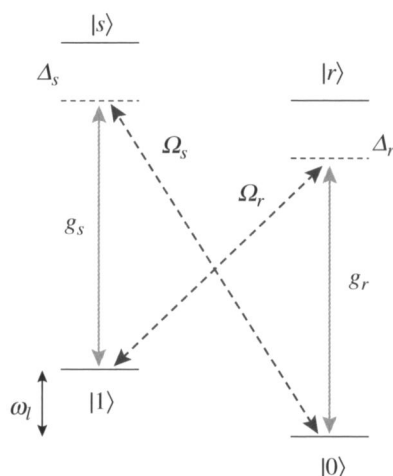


Figure 4. Single-atom energy levels for a Dicke model with two Raman transitions involving a cavity mode and two classical fields. Reproduced with permission from Dimer *et al.* [30]. Copyright (2007) by the American Physical Society.

Recently, Dimer *et al.* [30] proposed a related method to realize the Dicke model with strong coupling and low energy differences between the two involved levels. The scheme uses two Raman resonances, as seen in figure 4, which allows the inclusion of counter-rotating Hamiltonian terms in the model as a kind of simulation of the ideal Dicke model without the RWA. This would permit an experimental investigation of the phase transition of Wang & Hioe [13] and Hepp & Lieb [14,15] using cavity QED (see §5). Although the proposal in the paper involves a ring cavity and neutral atoms, the scheme would be well suited to an ion trap system with two cavity mode polarizations. Recently, a variant of this scheme involving distinct atomic and momentum states was used to observe the phase transition in an ultra-cold atomic gas [6].

(e) Recent developments: circuit QED

Circuit QED (CQED) [31–33] has now realized the Tavis–Cummings model for a small number of qubits (corresponding to $N = 3$ [34]). In this approach, the cavity mode is replaced by a resonant waveguide mode at microwave frequencies and the atoms are replaced by superconducting qubits. A compact flat geometry can be adopted, as seen in figure 5.

CQED systems have demonstrated strong coupling [31] and recently very strong coupling where the coupling constant g reached 12 per cent of the cavity frequency [35]. This is confirmed by performing a spectroscopy of the eigenvalue spectrum by adjusting the detuning of the qubits from the resonator frequency and mapping out the transmission of a very weak electromagnetic resonator field. The Jaynes–Cummings spectroscopy has been performed in this way, even reaching states with $n > 1$ [35,36], and collective effects have been observed as mentioned above [34]. The method in effect maps out the eigenvalues for low n given in §3c, where we see that the splitting of the $n = 1$ doublet (figure 1) is $2g\sqrt{N}$. Fink *et al.* [34] show this splitting for $N = 1$ –3.

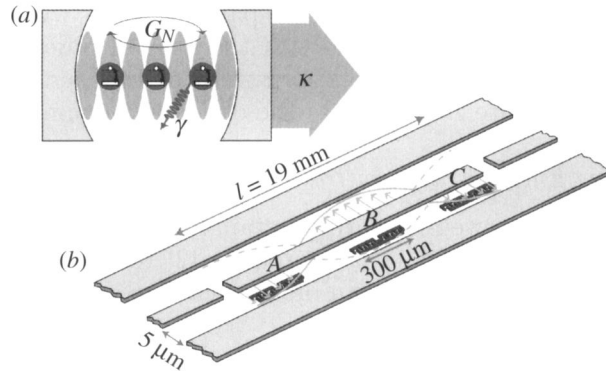


Figure 5. (a) Dicke system of three atoms in a cavity. The atoms are coupled with strength g (shown as G_N) and decay at rate γ . (b) CQED realization of the Dicke system shown in (a). The three atoms are replaced by superconducting qubits at A , B and C . These locations correspond to anti-nodes of the electromagnetic field on the waveguide resonator. Reproduced with permission from Fink *et al.* [34]. Copyright (2009), with permission from Elsevier. (Online version in colour.)

The clear advantages of CQED are that precise engineering determines where the qubits interact with the field standing wave and that the individual qubit frequencies are very controllable in terms of adjusting their frequency. In addition, the coupling strength is potentially very *strong*. However, for large N there could be an issue of the physical scale of the device and, also because of operating at gigahertz frequency, it is hard to have an efficient quantum detector for the electromagnetic field (an issue also very apparent for the original micromaser format; §3*b*). Nevertheless, the CQED approach holds much promise for moderate N systems that could realize the Dicke model in the future. It will also be a candidate for quantum information processing [32], for example.

4. Entanglement and the Dicke model

Entanglement has been important in quantum theory since the Einstein–Podolsky–Rosen paradox and since playing a central role in Bell’s inequalities. Recently, entanglement has been seen as a resource for some parts of quantum information processing and issues of quantifying entanglement have arisen. The dynamics of entanglement has become important, for example in *entanglement sudden death* [37], and the decay of entanglement is of interest. In this context the Dicke model, at least for $N = 2$, is one paradigm for an entangled system, and for $N > 2$ it becomes an example of a multi-partite entangled system. Although the focus is on two-atom systems below, there has been some notable work on entanglement with $N > 2$ (e.g. [38,39]).

The entanglement of two qubits, i.e. of two two-level systems, is most easily measured in terms of the concurrence [40]. One has to note that the highly entangled two-qubit states $(|e_1 g_2\rangle \pm |g_1 e_2\rangle)/\sqrt{2}$ are simply connected to the Dicke spin states $|S = 1, m_S = 0\rangle$ and $|S = 0, m_S = 0\rangle$, respectively, and this motivates a study of cooperative effects in two-atom systems. Once again we need to decide whether cooperative effects are appropriate when decoherence is taken into

account, and this has led to recent studies of pairs of atoms with common (Dicke-like) reservoirs that can be contrasted with the case of independent reservoirs. In figure 2, decay at a rate γ sideways represents the independent case while the decay of the mode at a rate Γ represents the effect of a common reservoir.

(a) *Entanglement in the two-atom Dicke model*

Ficek & Tanas [41] reviewed two-atom entanglement and decay in which they used a conventional master equation approach [42] to develop the dynamics of continuous excitation and loss. An example of such a cooperative master equation at zero temperature is

$$\frac{d}{dt}\rho = -i\Omega[S_+ + S_-, \rho] + i(\omega_L - \omega_0)[S_z, \rho] - \frac{\gamma_0}{2}(S_+S_-\rho + \rho S_+S_- - 2S_-\rho S_+), \quad (4.1)$$

although Ficek & Tanas also included the dipole-dipole interactions, which are neglected here. Equation (4.1) would also apply to more than two atoms as derived originally by means of the RWA and Born and Markov assumptions [42]. In equation (4.1), the system is driven with a Rabi frequency Ω at a frequency ω_L ; the phase of this coherent driving is assumed to be the same at the site of all the atoms. The decay rate γ_0 is not the same quantity as the sideways emission at the rate γ , as shown in figure 2. The difference is that the γ_0 case is cooperative while the sideways rate γ is not. Because of the Born and Markov approximations, equation (4.1) represents a low-Q cavity limit. As was shown in the study of Bullough [8] (following an argument in the study of [43,44]), by starting with an ad hoc master equation

$$\frac{d}{dt}\rho = -i\Omega[H_0, \rho] - \mathcal{E}[a - a^\dagger, \rho] + i(\omega_L - \omega_0)[S_z, \rho] - \frac{\Gamma}{2}(a^\dagger a \rho + \rho a^\dagger a - 2a \rho a^\dagger), \quad (4.2)$$

the cavity mode can be adiabatically eliminated to obtain a master equation similar to equation (4.1) with $\gamma_0 = g^2\Gamma/(\Gamma^2 + (\omega_c - \omega_L)^2)$. The adiabatic elimination requires $g\sqrt{N} \ll \Gamma$. In equation (4.2), the Dicke system H_0 (equation (1.1)) is driven by a cavity field, represented by the scaled electric field \mathcal{E} which is itself damped, as the presence of the a and a^\dagger operators in the Liouvillian indicate.

Recently, a more direct approach has been taken in a number of works that examine the decay of entanglement for two qubits which are *strongly coupled* to a cavity mode which decays as a common reservoir for the qubits [27,45–51]. In these cases we can couple the atoms directly to a bath represented by a reservoir structure, or density of states. Then solving for the atom, or qubit system, we find the evolution of the system-reduced density matrix and, to quantify the entanglement, we can calculate the concurrence of the two qubits. In one case, the quantum ‘discord’ is calculated [51], which is the difference between quantum and ‘classical’-like information.

The starting point can be illustrated with a single excitation and a single two-level qubit system. For this problem, the Hamiltonian for the model system can be written as

$$H = \sum_{\lambda} \omega_{\lambda} a_{\lambda}^{\dagger} a_{\lambda} + \sum_i \omega_i |i\rangle \langle i| + \sum_{i,\lambda} g_{\lambda}^{(i)} (a_{\lambda}^{\dagger} |0\rangle \langle i| + a_{\lambda} |i\rangle \langle 0|), \quad (4.3)$$

where $g_\lambda^{(i)}$ is the direct frequency-dependent coupling between the transition $i - 0$ and the reservoir mode labelled λ . Here, for two atoms, i will take the value 1, or 2, and the states $|1\rangle$ and $|2\rangle$ correspond to $|e_1 g_2\rangle$ and $|g_1 e_2\rangle$, respectively. We also have written $|0\rangle \equiv |g_1 g_2\rangle$. The mode λ belongs to a continuum in the (zero temperature) reservoir, and the structure of the coupling $g_\lambda^{(i)}$ determines the physical nature of the decay: a Lorentzian defined through

$$\rho_\lambda (g_\lambda^{(i)})^2 = \frac{g_i^2}{2\pi} \frac{\Gamma}{(\omega_k - \omega_0)^2 + (\Gamma/2)^2} \quad (4.4)$$

is found to lead to a leaky cavity mode. We take an Ansatz for the single excitation state,

$$\Psi(t) = c_0|000\rangle + c_1(t)|100\rangle + c_2(t)|010\rangle + \sum_\lambda c_\lambda|001_k\rangle, \quad (4.5)$$

and, by substituting it into the Schrödinger equation, we can solve for the system dynamics [52,53]. We can also show that the solution leads to an *exact* master equation, in the form of equation (4.2) with $\mathcal{E} \rightarrow 0$. In doing this, a ‘fictitious’ mode, the *pseudomode*, is introduced [53,54]. In constructing the master equation Born and Markov approximations are *not* made now and non-Markovian behaviour, owing to the cavity interaction, is seen in the dynamics.

Among the studies of the decay of two-qubit concurrence, several works [27,48–51] also include initial states with two excitations. As the reservoir may still be strongly coupled, the analysis is now non-trivial and relies on the extension of the decay of a single excitation in a reservoir to two (or more) excitations. The Fano technique can be used for this and was originally applied to multi-level systems [55] and explicitly three-level systems, which can be mapped onto the Dicke spin-1 system [56,57].

Using the pseudomode approach, Härkönen *et al.* [29] have recently examined in detail how two ions with a level structure as in figure 3 can be described as a two-atom Dicke model in cavity. They examined the concurrence for realistic parameters and showed how entanglement can be produced by decay in such a system. Essentially, because of unequal coupling, both a Dicke spin 1 and Dicke spin 0 (i.e. sub-radiant state) are produced, in general, by the initial conditions. However, the spin-1 state decays, while the sub-radiant state does not. In the end, the entanglement is created by the presence of the sub-radiant state, with a slight contamination by the decayed spin-1 state $|g_1 g_2\rangle$.

5. Dicke model phase transitions

(a) Finite temperature phase transition

Here, we note the thermodynamic phase transition of Wang & Hioe [13] and Hepp & Lieb [14,15], in which consideration was given to the thermal equilibrium of the Dicke Hamiltonian (1.1) in the limit $N \rightarrow \infty$. It was found that there was a phase transition for very strong couplings ($g\sqrt{N} \sim \sqrt{\omega_0\omega_c}/2$). The couplings are so strong that the RWA cannot be made [58], and this is the reason for a factor

of two difference with §2*d* taken in the large coupling limit. From a practical point of view, the two-level approximation is suspect, too. However, this phase transition was recently observed in an ultra-cold atomic gas [6]. The key was to use a level scheme analogous to that described in §3*d* and figure 4. However, the different atomic transitions are replaced by transitions between momentum states. The phase transition involves the fully quantum model, equation (1.1), and differs from the zero temperature phase transition described in §5*b*.

In terms of the original Dicke model, one has to treat carefully the question of this phase transition for a closed system, and not just because the two-level approximation may be violated. In the case of dipole radiation a no-go theorem stimulated much discussion [59–61]. In the case of CQED, for example, the no-go theorem does not apply [5]. It also does not apply in the case of the phase transition discussed in §5*b*.

(b) Zero temperature phase transition

The zero temperature ‘phase transition’ is seen in the coherently and collectively driven, and collectively damped, Dicke model. The phase transition appears as the coherent driving is changed through a critical value. The master equation that includes this behaviour is precisely the low-Q cavity master equation (4.1). The study of the phase transition is facilitated by an exact steady-state solution to the collective resonance fluorescence master equation which was found in the study of Lawande *et al.* [62]. The resonant case had previously been studied in Puri *et al.* [63]. The exact solution takes the form of a sum over all combinations of raising and lowering operators, i.e. [62]

$$\rho_{SS} = \frac{1}{D} \sum_{m=0}^N \sum_{n=0}^N a_{m,n} \left(-\frac{i\Omega}{\gamma_0}\right)^{-m} \left(\frac{i\Omega}{\gamma_0}\right)^{-n} S_-^m S_+^n, \quad (5.1)$$

with a normalization

$$D = \sum_{m=0}^N \left(\frac{\Omega}{\gamma_0}\right)^{-2m} \frac{(N+m+1)!}{(N-m)!(2m+1)!} a_{m,m}, \quad (5.2)$$

and coefficients

$$a_{m,n} = \frac{\Gamma(m - i\Delta/\gamma_0 + 1)\Gamma(n + i\Delta/\gamma_0 + 1)}{m!n!\Gamma(1 + i\Delta/\gamma_0)\Gamma(1 - i\Delta/\gamma_0)}. \quad (5.3)$$

In the case of resonance, the coefficients $a_{m,n}$ are simply all unity.

The phase transition has already been found by Drummond & Carmichael [64] and it appears when $\Omega = \gamma_0 N/2$. Using the exact solution (5.1), it is straightforward to calculate expectation values for operators numerically. Figure 6 shows the appearance of this phase transition in the value of $\langle S_z \rangle$ as the number of atoms $N \rightarrow \infty$. The effects of this phase transition were seen in other observables, such as $g^{(2)}$ [66], but it does not survive if the driving field is detuned from the atoms [62].

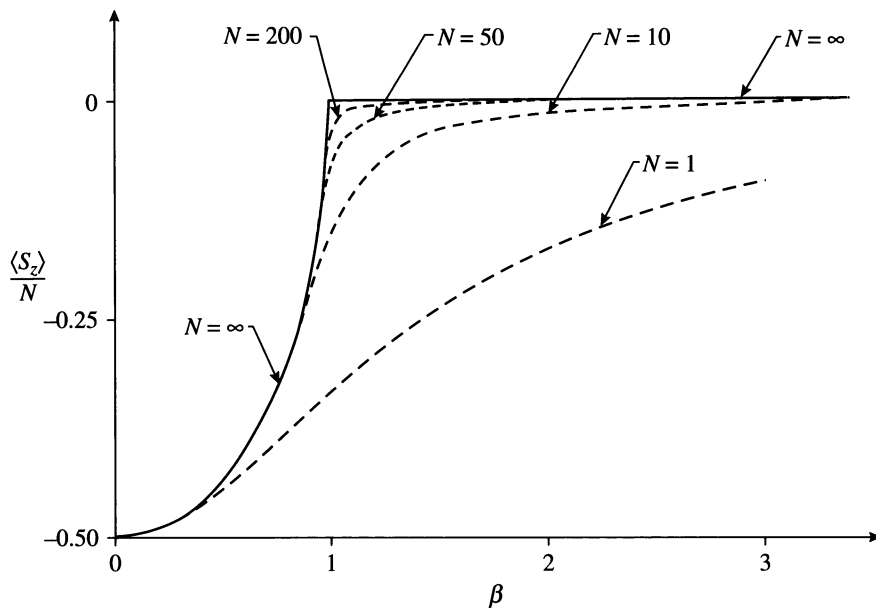


Figure 6. Dicke model phase transition as seen in the observable S_z , which is plotted as a function of a scaled driving field amplitude $\beta = 2\Omega/(N\Gamma)$ (using the notation of the present paper). Reprinted with permission from Elsevier [65].

6. Conclusion

Collective effects on atoms in cavities have been observed experimentally for over 20 years, but it has not been possible to demonstrate all the details of the Dicke model because of technological limitations and the use of atoms flying through cavities. There is interest in this model because of the transition to the large N thermodynamic limit, because of the quantum–classical boundary appearing, and because it is a system with potentially large amounts of entanglement.

In recent experiments with cold quantum gases, it has been possible to couple a Bose–Einstein condensate (BEC) in a single motional state to a cavity mode [67,68]. These experiments have demonstrated the cooperative effect on the coupling, although fluctuations in atom number, and overlap of the electromagnetic mode with the BEC, is an issue. A BEC in a cavity is also the subject of several related theoretical proposals [69,70]. However, the Dicke model phase transition (§5*a*), has now been observed experimentally with a cold quantum gas [6]. This experiment used a cavity, but the atomic states were replaced by momentum states.

Ion traps combined with high- Q cavities may offer new ways to realize the Dicke model and observe *all* the phase transitions and demonstrate entanglement. However, it may be possible to realize an ion trap Dicke model without using a cavity at all. In the study of Schneider & Milburn [39], a collective coupling of the ions by using the centre-of-mass vibrational mode of the ions is proposed. By using sympathetic cooling on one of the ions, this collective mode can be ‘reservoir-engineered’ and kept at a finite temperature. Coherent classical excitation of the Dicke system is proposed using identical coupling of a laser to each ion.

The CQED systems discussed in §3e have demonstrated strong coupling effects for low excitations and small N . In these cases, the atoms of the Dicke model are replaced by superconducting qubits while the electromagnetic field remains. They appear to be quite flexible systems (each qubit could be tuned) and the number of qubits could be increased.

The zero temperature phase transition of §5b has not apparently been observed. The future for this, and other interests such as entanglement, may lie in CQED [32], where the system is relatively easy to control. However, there may also be new ways of realizing the Dicke model system, perhaps with arrays of coupled cavities [71] suitably arranged for a non-local Dicke-like coupling. Alternatively, the model may be realized with the ion traps as discussed above, or with condensed matter systems such as excitonic excitations in a quantum well [72].

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