My wish: to develop a complete set of differential equation for an operator at a given order of cumulant expansion

Lindblad master equation for an operator

$$\dot{O} = \frac{i}{\hbar}[H, O] + \sum_i \gamma_i \left(L_i^\dagger O L_i - \frac{1}{2} \left\{ L_i^\dagger L_i, O \right\} \right)$$

Hamiltonian (Rotating wave approximation)

$$H = \hbar \Delta \hat{n} + \hbar G \left(\hat{s_+} \hat{a} + \hat{s_-} \hat{a}^\dagger \right)$$

Lindblad terms

$$\begin{cases} L_1, \gamma_1 = \hat{a}, \kappa & \text{Cavity decay (loss of photons)} \\ L_2, \gamma_2 = \hat{s_+}, \gamma & \text{Incoherent repump} \\ L_3, \gamma_3 = \hat{s_-}, \nu & \text{Decay into free space} \end{cases}$$

Example: development of the **creation operator** at **order 1** (= mean field)

$$\dot{O} = \frac{i}{\hbar}[H, O] + \sum_i \gamma_i \left(L_i^\dagger O L_i - \frac{1}{2}\left\{ L_i^\dagger L_i, O \right\} \right)$$

$$H = \underbrace{\hbar\Delta\hat{n}}_{} + \hbar G\left(\hat{s_+}\hat{a} + \hat{s_-}\hat{a}^\dagger \right)$$

Here is an example of how the algorithme develop the equations

$$\begin{aligned}
[H_0, \hat{a^\dagger}] &= \hbar\Delta\hat{n}\hat{a}^\dagger - \hbar\Delta\hat{a}^\dagger\hat{n} \\
&= \hbar\Delta\hat{a}^\dagger\hat{a}\hat{a}^\dagger - \hbar\Delta\hat{a}^\dagger\hat{a}^\dagger\hat{a} \\
&= \hbar\Delta\hat{a}^\dagger\left(1 + \hat{a}^\dagger\hat{a}\right) - \hbar\Delta\hat{a}^\dagger\hat{a}^\dagger\hat{a} \\
&= \hbar\Delta\hat{a}^\dagger + \hbar\Delta\hat{a}^\dagger\hat{a}^\dagger\hat{a} - \hbar\Delta\hat{a}^\dagger\hat{a}^\dagger\hat{a} \\
&= \hbar\Delta\hat{a}^\dagger
\end{aligned}$$

Algorithme rule = commute all the operators until we reach a full (a’, a) and (sz, s-, s+) in this order respectively using those relations:

$$\hat{a}\hat{a}^\dagger = 1 + \hat{a}^\dagger\hat{a} \qquad \hat{s_-}\hat{s_z} = \hbar\hat{s_-} + \hat{s_z}\hat{s_-}$$

$$\hat{s_+}\hat{s_z} = \hat{s_z}\hat{s_+} - \hbar\hat{s_+} \qquad \hat{s_+}\hat{s_-} = 2\hbar\hat{s_z} + \hat{s_-}\hat{s_+}$$

$$\dot{\hat{a}}^\dagger = i\Delta\hat{a}^\dagger + iG\hat{s}_+ - \frac{\kappa}{2}\hat{a}^\dagger$$

s+ is missing! Let’s develop it…

$$\dot{\hat{s}}_+ = -2iG\hbar\hat{s}_+\hat{a}^\dagger + -\gamma\hbar\hat{s}_z\hat{s}_+ + \nu\hbar\hat{s}_z\hat{s}_+ - \nu\hbar^2\hat{s}_+$$

Operators missing again? No! Because we asked order 1: we don’t need more than one operator!

Now we can get the expectation values “<A>” from the equations, where “A” is an operator (<.> is a linear operation by the way)

$$\Rightarrow \quad < \overset{\circ}{\hat{a}^{\dagger}} >= i\Delta < \hat{a}^{\dagger} > +iG < \hat{s_{+}} > -\frac{\kappa}{2} < \hat{a}^{\dagger} >$$

$$\Rightarrow \quad < \overset{\circ}{\hat{s_{+}}} >= -2iG\hbar < \hat{s_{+}}\hat{a}^{\dagger} > + -\gamma\hbar < \hat{s_{z}}\hat{s_{+}} > +\nu\hbar < \hat{s_{z}}\hat{s_{+}} > -\nu\hbar^{2} < \hat{s_{+}} >$$

But we still don’t have these equations… So here comes the trick: cumulant expansion!

Cumulant expansion at order 2 (“2” stands for two operators): $< \hat{A}\hat{B} >=< \hat{A} >< \hat{B} >$

Now we have: $<\dot{\hat{s}}_+> = -2iG\hbar <\hat{s}_+><\hat{a}^\dagger> -\gamma\hbar <\hat{s}_z><\hat{s}_+> +\nu\hbar <\hat{s}_z><\hat{s}_+> -\nu\hbar^2 <\hat{s}_+>$

sz is missing!

The idea is to develop again a set of equation for “sz” without more than 1 operator. Then applying the cumulant expansion. From that we can find the missing operators and find a set of equation for each missing operators.

Complete set of equation

$$
\left\{
\begin{aligned}
&<\dot{\hat{a}}^\dagger> = i\Delta <\hat{a}^\dagger> +iG <\hat{s}_+> -\frac{\kappa}{2} <\hat{a}^\dagger> \\
&<\dot{\hat{s}}_+> = -2iG\hbar <\hat{s}_+><\hat{a}^\dagger> -\gamma\hbar <\hat{s}_z><\hat{s}_+> +\nu\hbar <\hat{s}_z><\hat{s}_+> -\nu\hbar^2 <\hat{s}_+> \\
&<\dot{\hat{s}}_z> = -iG\hbar <\hat{s}_+><\hat{a}> +iG\hbar <\hat{s}_-><\hat{a}^\dagger> +\gamma\hbar <\hat{s}_-><\hat{s}_+> -\nu\hbar <\hat{s}_-><\hat{s}_+> -2\nu\hbar^2 <\hat{s}_z> \\
&<\dot{\hat{s}}_-> = iG\hbar <\hat{s}_z><\hat{a}> -\gamma\hbar <\hat{s}_z><\hat{s}_-> +-\gamma\hbar^2 <\hat{s}_-> +\nu\hbar <\hat{s}_z><\hat{s}_-> \\
&<\dot{\hat{a}}> = -i\Delta <\hat{a}> -iG <\hat{s}_-> -\frac{\kappa}{2} <\hat{a}>
\end{aligned}
\right.
$$

Last improvement: we can remove the redundant equations. For example, knowing that the creation operator is the hermitian of the destructor operator, we can replace <a> by the conjugate of <a’> which leads us to…

$$\left\{ \begin{array}{l} \dot{\overbrace{< \hat{a}^\dagger >}} = i\Delta < \hat{a}^\dagger > + iG < \hat{s_+} > - \dfrac{\kappa}{2} < \hat{a}^\dagger > \\[2ex] \dot{\overbrace{< \hat{s_+} >}} = -2iG\hbar < \hat{s_+} >< \hat{a}^\dagger > - \gamma\hbar < \hat{s_z} >< \hat{s_+} > + \nu\hbar < \hat{s_z} >< \hat{s_+} > - \nu\hbar^2 < \hat{s_+} > \\[2ex] \dot{\overbrace{< \hat{s_z} >}} = -iG\hbar < \hat{s_+} > \underbrace{< \hat{a}^\dagger >^*}_{<a>} + iG\hbar \underbrace{< \hat{s_+} >^*}_{<s->} < \hat{a}^\dagger > + \gamma\hbar < \hat{s_+} >^* < \hat{s_+} > - \nu\hbar < \hat{s_+} >^* < \hat{s_+} > - 2\nu\hbar^2 < \hat{s_z} > \end{array} \right.$$

This set is easily integrable with a RK4 method. That’s all!

… wait wait wait! Where are the initial values (in order to solve the system)?

Well, now this set of equation is fixed, i.e we would develop the same set for **EVERY** quantum system with the **same hamiltonian**, **lindblad terms** and of course for **order 1** (For higher order, i.e with more than one operator inside <.>, we would have more equations). Then, the only thing which can differentiate, let's say, a superradiant system from a normal laser, is the initial density operator ρ_0 and the constants G, $\boldsymbol{\Delta}$, $\boldsymbol{\kappa}$, etc

Then we find the initial values of the equation as follow: $< \hat{a}^\dagger > (t = 0) = tr\left\{\rho_0 \hat{a}_0^\dagger\right\}$

Now we can really compute the evolution of the expectation values!

What is happening at orders higher than one for the cumulant expansion?

Going back to our previous example: $$< \mathring{\hat{s_+}} >= -2iG\hbar \underbrace{< \hat{s_+}\hat{a}^\dagger >}_{\boxed{\text{Now we also develop this couple of operator}}} +-\gamma\hbar < \hat{s_z}\hat{s_+} > +\nu\hbar < \hat{s_z}\hat{s_+} > -\nu\hbar^2 < \hat{s_+} >$$

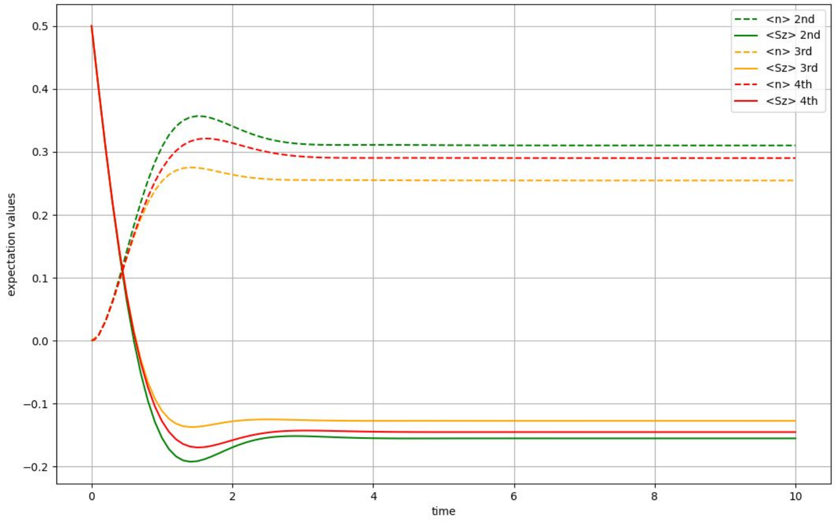
General expression: $$\langle X_1 X_2 \dots X_n \rangle = \sum_{p \in P(\mathcal{I}) \setminus \mathcal{I}} (|p|-1)!(-1)^{|p|} \prod_{B \in p} \langle \prod_{i \in B} X_i \rangle$$

If we want order **2 (maximum two operators side by side in <.>)**, we need to apply a cumulant expansion on the third order, i.e on $<X_1X_2X_3>$:

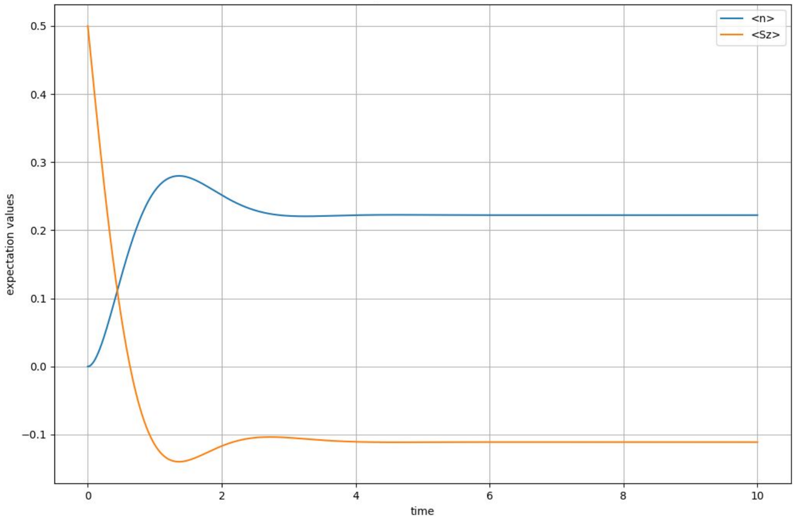
$$\langle X_1 X_2 X_3 \rangle = \langle X_1 X_2 \rangle \langle X_3 \rangle + \langle X_1 X_3 \rangle \langle X_2 \rangle + \langle X_1 \rangle \langle X_2 X_3 \rangle - 2\langle X_1 \rangle \langle X_2 \rangle \langle X_3 \rangle$$

Then we check if we have an equation for $<X_1X_2>$, $<X_1X_3>$, $<X_2X_3>$, etc… if not we develop a set, apply cumulant expansion and so on…

Start with one excited atom, zero photon inside the cavity
$G = \kappa = \gamma = \nu = 1$

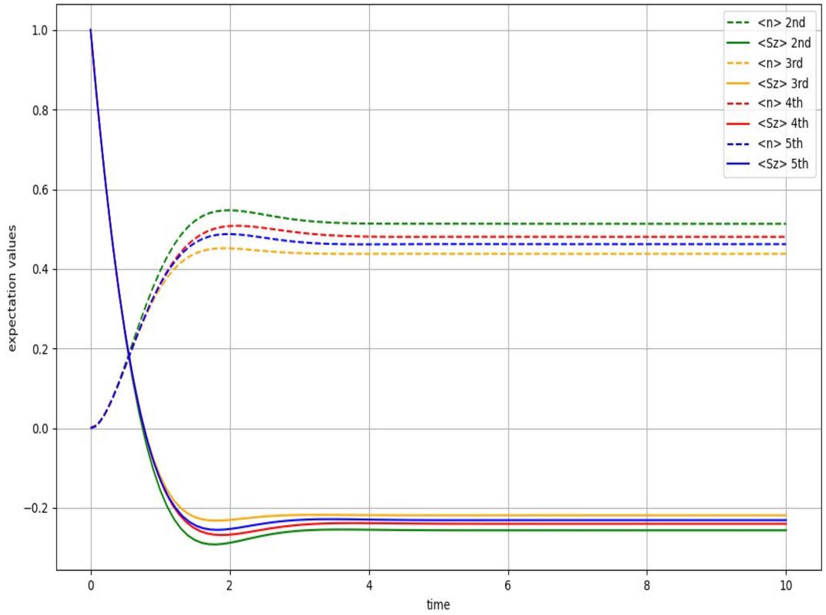


Cumulant expansion

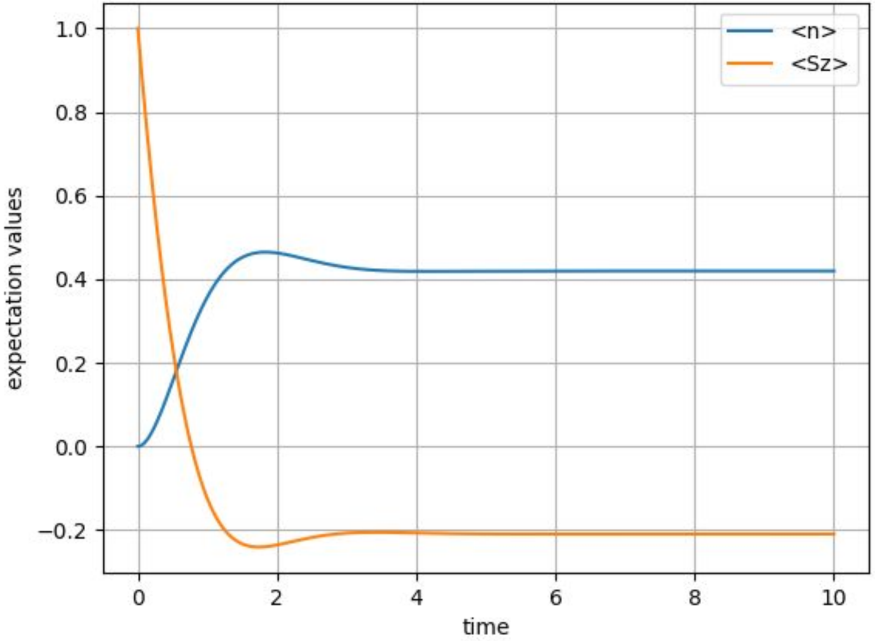


“Exact” integration from the master equation

Start with two excited atom, zero photon inside the cavity
$G = \kappa = \gamma = \nu = 1$



Cumulant expansion

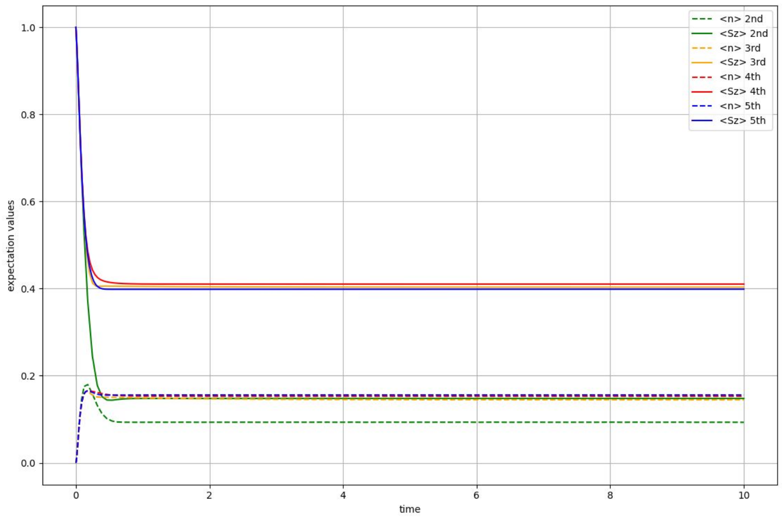


“Exact” integration from the master equation

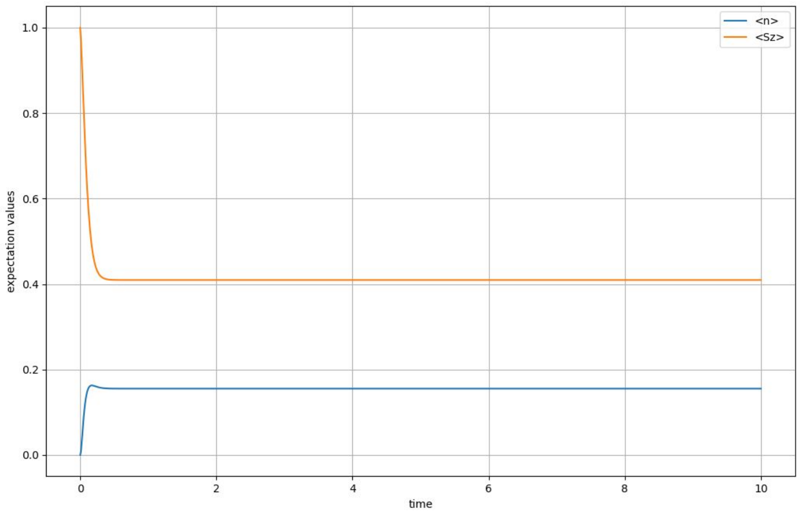
Start with two excited atom, zero photon inside the cavity

$G = 10$, $\kappa = 40$, $\gamma = 9$, $\nu = 1$

(Superradiant burst configuration from the Julia package)



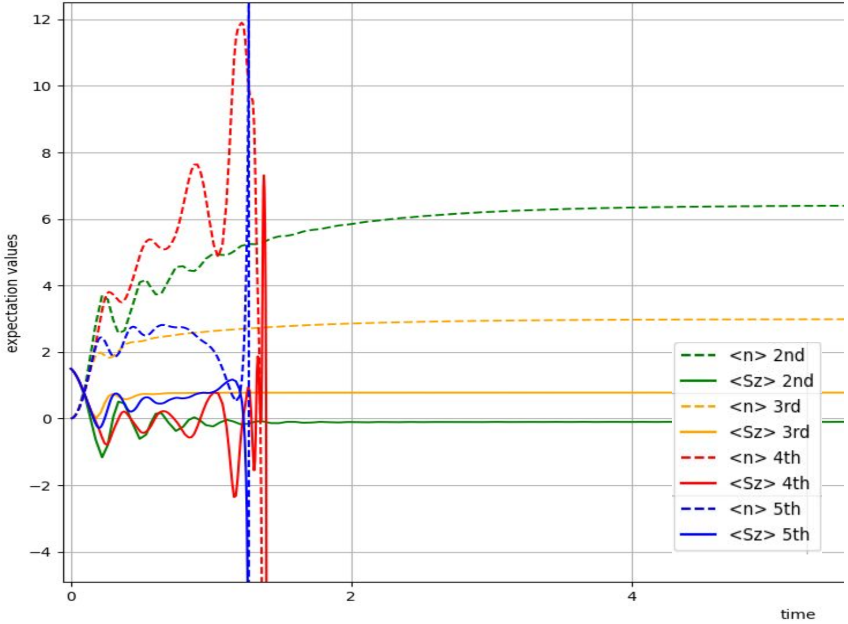
Cumulant expansion

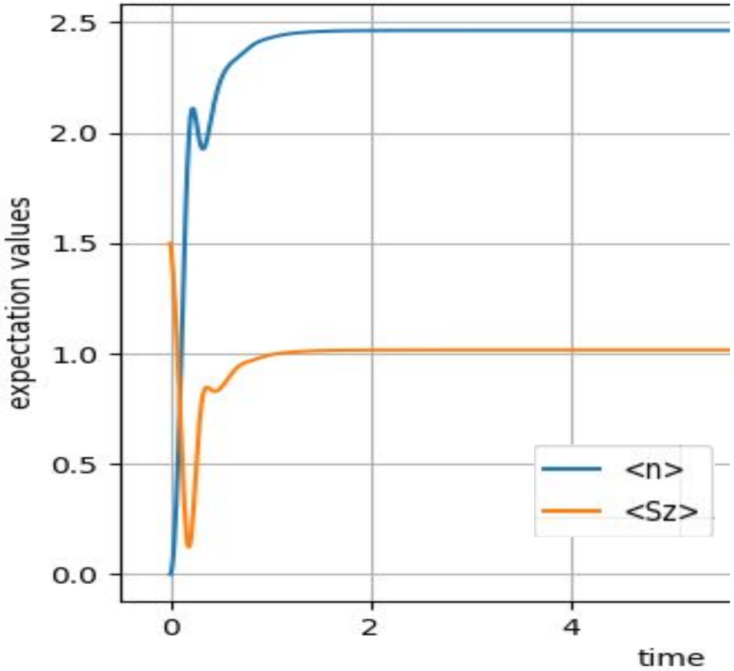


“Exact” integration from the master equation

The small gap between the “exact” results and the cumulant expansion one certainly come from the integration of the system and its precision (error tolerance in RK4)

Fancy results with three excited atom, zero photon inside the cavity
$G = 10$, $\kappa = 1$, $\gamma = 5$, $\nu = 1$





Same result at 4th order with my own implementation of RK4.
Then, from my experience, this behavior comes from an average of two divergents function: one goes to +∞ and the other to -∞.
As we get a steady-state after t=2s (approximately), the system of equation tries to get a rate from those two divergent functions.
However, our computer have a finite precision and has difficulties to get a finite result of “+∞/-∞”. After getting a “NaN” value for one of the two functions, the curve drops to an “invalid” result, here -∞.