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EQUIVALENT HIGHER ORDER LINEAR AND NON-LINEAR SYSTEMS

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The object of this study is to put forth the concept of equivalence of classes of linear and non-linear systems of higher order. In particular, the equivalence of classes of non-linear (non-autonomous/autonomous) systems of order n described by partial/ordinary differential equations with corresponding classes of linear systems of order $(n+1)$ is established through a differential transformation of the dependent variable. In view of the fact that the resulting linear systems are amenable to existing state-space techniques, this approach can be expected to be of value in the study of many non-linear problems arising in a variety of disciplines. The possible applications of the technique are illustrated with an example.

1. INTRODUCTION

Most of the studies [1–10] concerning equivalence of linear and non-linear systems (and related topics in non-linear analysis) are limited to systems of lower order (first, second and rarely third [9]). Also, very few of these studies [6, 10] relate to systems described by partial differential equations.

The object of this study is to put forth the concept of equivalence of linear and non-linear systems of higher order (say n). In particular, the equivalence of a class of non-linear (non-autonomous/autonomous) systems of order n described by partial/ordinary differential equations with corresponding linear systems of order $(n+1)$ is established through a differential transformation technique. The results of this analysis for lower order systems ($n = 1, 2$) correspond to those of the earlier studies [7, 10] on Ricatti and other first- and second-order non-linear systems.

2. ANALYSIS

First, a theorem, stating the equivalence of a class of non-linear systems of order n and a class of linear systems of order $(n+1)$, is presented followed by the proof. The possible applications of the technique are illustrated with an example.

2.1. THEOREM

Any non-linear (non-autonomous/autonomous) system of order n described by a partial/ordinary differential equation of the form

$$\mathbf{C}^{\mathrm{T}}(\mathbf{t})\,\mathbf{D}\{f(x,\mathbf{t})\} = F(\mathbf{t}) \tag{1}$$

under a differential transformation law of the type

$$f(x,\mathbf{t})\,u = pu \tag{2}$$

is equivalent to a linear (non-autonomous/autonomous) system of order $(n+1)$ described by a partial/ordinary differential equation of the form

$$\mathbf{C}^{\mathrm{T}}(\mathbf{t})\,\mathbf{P}u = F(\mathbf{t})\,u, \tag{3}$$

where $\mathbf{P} = \{P_i : i = (1, n+1)\}$ is a *linear* differential operator vector with

$$P_i = p^{(i)} = \left[\mathbf{a}^{\mathrm{T}}(\mathbf{t})\frac{\mathrm{d}}{\mathrm{d}\mathbf{t}}\right]^{(i)}; \tag{4}$$

$\mathbf{D} = \{D_j : j = (0, n)\}$ is a *non-linear* differential operator vector with

$$D_j = [p + f(x,\mathbf{t})]^{(j)} = \left[\mathbf{a}^{\mathrm{T}}(\mathbf{t})\frac{\mathrm{d}}{\mathrm{d}\mathbf{t}} + f(x,\mathbf{t})\right]^{(j)} \tag{5}$$

(superscripts i and j denote the order of differentiation); $\mathbf{t}$ is an m vector ($m = 1$ reduces the system to the domain of ordinary differential equations); $\mathbf{C}(\mathbf{t})$ is a $(n+1)$ (coefficient) vector, $\mathbf{a}(\mathbf{t})$ is an m (coefficient) vector.

2.2. PROOF

Now, the governing equation of the non-linear system of order n can be written as

$$\mathbf{C}^{\mathrm{T}}(\mathbf{t})\,\mathbf{D}\{f(x,\mathbf{t})\} = \sum_{j=0}^{n} C_j(\mathbf{t})\, D_j\{f(x,\mathbf{t})\},$$

[using equation (5)]

$$= \sum_{j=0}^{n} C_j(\mathbf{t})\,[p + f(x,\mathbf{t})]^{(j)} f(x,\mathbf{t}),$$

[using equation (2)]

$$= \sum_{j=0}^{n} C_j(\mathbf{t})\left[p + \frac{1}{u}pu\right]^{(j)}\left\{\frac{1}{u}pu\right\}. \tag{6}$$

Here,

$$\begin{aligned}
\left(p + \frac{1}{u}pu\right)^{(j)}\frac{1}{u}pu &= \left(p + \frac{1}{u}pu\right)^{(j-1)}\left(p + \frac{1}{u}pu\right)\frac{1}{u}pu \\
&= \left(p + \frac{1}{u}pu\right)^{(j-1)}\frac{1}{u}p^{(2)}u \\
&= \ldots\ldots\ldots\ldots \\
&= \ldots\ldots\ldots\ldots \\
&= \ldots\ldots\ldots\ldots \\
&= \frac{1}{u}p^{(j+1)}u.
\end{aligned} \tag{7}$$

Use of equations (7) and (1) in equation (6) gives

$$\frac{1}{u}\sum_{j=0}^{n} C_j(t)\,p^{(j+1)}u = \frac{1}{u}\sum_{i=1}^{n+1} C_{i-1}\,p^{(i)}u = F(\mathbf{t}) \tag{8}$$

(changing $j + 1 = i$). Hence,

$$\mathbf{C}^{\mathrm{T}} Pu = F(\mathbf{t})\,u \tag{9}$$

represents the equivalent linear system of order $(n+1)$.

2.3. EXAMPLE

Consider a class of non-linear, non-autonomous system of second order (representing the forced vibrations of a single-degree-of-freedom system with time-dependent parameters and non-linear damping and stiffness elements) given by the equation,

$$\ddot{x} + [3\alpha x + \beta(t)]\,\dot{x} + \gamma(t)\,x + \alpha\beta(t)\,x^2 + \alpha^2 x^3 = F(t). \tag{10}$$

Equation (10) can be rewritten in the form of equation (1) with

$$f(x,t)=x, \qquad m=1, \qquad C_1(t)=\beta(t), \qquad a(t)=1/\alpha,$$
$$n=2, \qquad C_2(t)=1, \qquad C_0(t)=\gamma(t),$$

which immediately leads to the equivalent linear third-order system given by

$$\dddot{u}+\beta(t)\ddot{u}+\gamma(t)\dot{u}=F(t)u \tag{11}$$

through a transformation of the type

$$\alpha x=\dot{u}/u. \tag{12}$$

Equation (11) can be rewritten in the form

$$\dot{\mathbf{U}}(t)=\mathbf{A}(t)\mathbf{U}(t), \tag{13}$$

where

$$\mathbf{U}(t)=\begin{bmatrix}U_1\\U_2\\U_3\end{bmatrix}=\begin{bmatrix}u\\\dot{u}\\\ddot{u}\end{bmatrix}$$

and

$$\mathbf{A}(t)=\begin{bmatrix}0 & 1 & 0\\0 & 0 & 1\\F(t) & -\gamma(t) & -\beta(t)\end{bmatrix},$$

whose solution may be obtained by state-space techniques (when $\gamma(\mathrm{t})$, $\beta(\mathrm{t})$ and $F(\mathrm{t})$ are specified).

Then,

$$x(t)=(1/\alpha)(U_2/U_1) \tag{14}$$

represents the response of the equivalent non-linear system given by equation (10). This will be particularly simple to carry out when β, γ and $F(t)$ are constants as the state transition matrix $\boldsymbol{\phi}(t)=\mathrm{e}^{\mathbf{A}(t)}$ can be obtained analytically. Also, when β, γ and $F(t)$ are functions of time, such that [11] the matrices $\mathbf{A}(t)$ and $\int_{t_0}^{t_1}\mathbf{A}(\tau)\mathrm{d}\tau$ commute, then the state transition matrix, $\boldsymbol{\phi}(t,t_0)=\exp\{\int_{t_0}^{t_1}\mathbf{A}(\tau)\mathrm{d}\tau\}$, can again be analytically determined. However, when $\mathbf{A}(t)$ and $\int_{t_0}^{t_1}(\mathbf{A}\tau)\mathrm{d}\tau$ do not commute, a perturbation method [12] can be developed.

3. DISCUSSIONS AND CONCLUSIONS

The first-order case of this study (i.e. $n=1$) with $f(x,\mathrm{t})=x$ corresponds to the system represented by a Ricatti partial differential equation [10]. Further, if $m=1$, i.e. $\mathbf{t}$ is a scalar, the system reduces to the domain of a Ricatti ordinary differential equation. The corresponding second-order linear equivalent obtained by this analysis agrees with the known results [10].

The second-order case (i.e. $n=2$) with $f(x,\mathrm{t})=x$, $m=1$ and $F(\mathrm{t})=0$ corresponds to the case mentioned by McLachlan [5]. A further generalization applicable to second-order systems only is reported in an earlier study [7] wherein the case $b=0$ reduces to the second-order case obtained here. A possible extension of this work is therefore to use a differential transformation technique of the type used in the earlier study [7] to formulate higher order equivalent linear and non-linear systems.

Thus the foregoing analysis introduces the concept of equivalence of higher order non-linear and linear systems and establishes the equivalence of a fairly wide class of non-linear systems of order n represented by ordinary/partial differential equations with a linear system or order $(n+1)$ through a differential transformation law. As mentioned before, more general classes of non-linear systems with equivalent linear systems can be construed by considering more general forms of transformation techniques. This study seems to be one of the first of its kind in attempting some general results of this nature for higher order non-linear systems and hence is expected to be a stimulus for further work in this direction.

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