

Title

Neven Gentil

May 2024

Abstract

My very nice abstract

Chapter 1

First chapter

1.1 Optical Cavity

We firstly introduce one of the most common object of quantum optic, namely, the *optical cavity*. It is mainly constituted of two mirrors mounted perpendicularly to the optical axis at a certain distance from each other. The idea behind this simple setup is to select a discret range of wavelengths defined by the space between the mirrors.

Indeed, one can modelize the cavity as an one-dimensional box, then apply the wave equation $\square \vec{E} = 0$ where \square is the d'Alembertian operator and \vec{E} the electric field's amplitude, together with the Maxwell equation $\nabla \cdot \vec{E} = \partial_x E = 0$. As the wave must vanish at the boundaries of the box (meaning that the energy cannot propagate outside of the box's region) one would find that the solutions are linear combinations of sinuzoidale functions, each of them having a spatial frequency multiple of $\frac{\pi}{L}$ where L denotes the length of the box/cavity. We shall consider that our model lies into a vacuum chamber, leading to the refractive index $n = 1$ for the void and the well known constant c for the speed of light. Then, thanks to the relation $\omega = ck$ we see that the angular frequency is also quantized.¹ It is important to note that the argument is valid for other refractive index and higher dimensions (i.e in 2D, 3D, ...).

With a more pragmatic point of view, one can think of this wavelength selection as an interferometer. In fact, let's imagine for a while that we emplace a cavity in front of the sun, fast enough to capture a beam of sunlight. Then we stop the clock before further propagation. Mathematically, the waves which are differents from the selective frequency vansih instantaneously, because of the presence of the cavity. However, physically, the full spectrum of the sun still resides between the two mirrors. Then if we run the clock again, the light is reflected from the back mirror. As we captured a continous stream, the π -dephased reflected light encouters the incoming light: interference happens and after a couple of round-trips, only the waves generating an in-phase match at each round-trip (i.e with a space-frequency or wave-number $\frac{n\pi}{L}, n \in \llbracket 0, +\infty \rrbracket$) remains by constructive interference, creating standing waves of the same frequency. On the other hand, all the other frequencies produce at reflection, in average on some round-trips, a *random-like* phase difference: the sum shall give a null coefficient in term of amplitude for those given frequencies. Although this is a simple explanation, two things are important to note:

- For the constructive interference case, we say that we need to be in-phase after one round-trip, meaning that after two reflections, which happens after hitting both back and front mirrors of the cavity, the wave must replicate the same amplitude at any position. However, the electric field is not static but moves in space. Then, it becomes more evident that twice the length of the cavity must be a multiple of the frequency in order to give the exact same phase, even with a space-motion of the wave.

¹R.Loudon derives rigorous equations in sections 1.1 and 1.2 of *Quantum Theory of Light*.

- For all the other frequencies, the phase difference is not random but actually well behaved. Nevertheless, After a couple of round-trip, the set of generated phase difference is uniformly distributed among $[-\pi, \pi]$ reproducing a sort of randomness.

Thereby, that is why the device created by C.Fabry and A.Pérot in 1899 is called an *interferometer* as, in their case, the light hit the mirrors with an angle, creating the famous pattern of fringes. Equivalently, this category of instrument is often named *resonator* as the amplitude of the electric field is enhanced in the case of resonance with the cavity or decreases in other cases: by conception, it can be modeled as a classical harmonic oscillator or with the quantum equivalent.²

In real experiments, the device is static and one prefers to let the light enter partially, to feed the cavity, and exit partially to collect the selected waves. In that way, we use *partially reflective* mirrors parametrized by an electric field reflection coefficient r and an electric field transmission coefficient t obeying the relation $r^2 + t^2 = 1$ (the squared terms are the respective coefficient for the intensity of the field), meaning that all the absorbed energy must be re-emitted by the mirror, which is of course not fully true as some amount of the energy is transformed into heat. However, current manufacturers creates mirrors with less than 0.01% of loss and the above formula is sufficient for most of the applications.

The most commonly used approach to express the cavity response is the Airy distribution which represents the intensity ratio between the internal electric field and the *launching* electric field (i.e the amount of *feeding*-field succeeding to transmit through the first mirror and penetrate the cavity), in function of the phase accumulated by round-trip:

$$\text{Airy distribution : } A(\phi) \stackrel{\text{def}}{=} \frac{I_{\text{circ}}}{I_{\text{launch}}} \quad (1.1)$$

In other words, it just makes the link between the enhancement factor of the cavity field and the constructive-destructive interference phase pattern. In order to derive it, we shall use the circulating field approach³, represented in Fig (add figure!) and constituted of:

- A launching electric: E_{launch}
- A round-trip electric field which is the field after one complete round-trip: one reflection on the back mirror plus another on the front mirror. We call it E_{RT}
- A circulating electric field which is the sum of the launched field and the round-trip one: $E_{\text{circ}} = E_{\text{launch}} + E_{RT}$. It is by definition the internal electric field.

Here, we use the phasor representation where the electric field (a real analytic signal) is splitted into two conjugate complex numbers:

$$E(x, t) = \overline{E}(x, t) + \overline{E}^*(x, t) \quad \text{s.t} \quad E(x, t) \in \mathbb{R}, \quad \overline{E}(x, t) \in \mathbb{C} \quad (1.2)$$

$$= 2\Re\{\overline{E}(x, t)\} \quad (1.3)$$

Then, we consider fields evolving in time and in space along the optical axis x . Moreover, one considers only the amplitude of the fields and deals with scalar equation. However one can assume linearly polarized input field to get the same mathematical treatment but the Airy distribution is valid for any kind of polarization and can be generalized for vectorial equations instead (i.e at higher dimension). Note that (1.2) is not a definition sign " $\stackrel{\text{def}}{=}$ " as an analytic signal can be written in that form.⁴

Rewriting the equation for the circular field gives:

$$2\Re\{\overline{E}_{\text{circ}}\} = 2\Re\{\overline{E}_{\text{launch}}\} + 2\Re\{\overline{E}_{RT}\} \quad (1.4)$$

²As we will see later, each resonant frequency owns a linewidth as a classical oscillator, in contrary to the quantum equivalent where the energy ladders (i.e the frequencies, in virtue of the relation $\hbar\omega$) are well defined.

³A. E. Siegman, section 11.3, *Lasers*

⁴More explanations about the hypothesis and what an analytic signal implies in section ??

which, by linearity and analyticity of the signal, is equivalent to:

$$\overline{E}_{circ} = \overline{E}_{launch} + \overline{E}_{RT} \quad (1.5)$$

Now, one can express the round-trip field from the circular one, dephased by a certain angular amount. Indeed, one knows that each mirror add a phase of π to the original field and the amplitude is modulated by the reflection factor R . As a field of space-frequency k and propagating over a distance l is shifted by:

$$\omega \times \text{"propagation time"} = \omega \times \frac{l}{c} \quad (1.6)$$

Then, after a complete round-trip, one obtains for the "RT" field a total angular shift of:

$$2\phi \stackrel{\text{def}}{=} \frac{2L}{c}\omega + 2\pi \quad (1.7)$$

Thereby:

$$\overline{E}_{RT} = r_1 r_2 e^{-i2\phi} \overline{E}_{circ} \quad (1.8)$$

where $r_{1,2}$ are the reflection coefficients of front and back mirror respectively. In other words, instead of propagating the field from its original value \overline{E}_{circ} through time or space to get a phase, we compute the relative phase ϕ from the length of the cavity L with the field's frequency ω .

One can highlight that for a characteristic space-frequency of the cavity $k = \frac{n\pi}{L}$, we have:

$$2\phi = \frac{2L}{c} \times \frac{cn\pi}{L} + 2\pi = 2\pi(n+1) \equiv 0 \quad [2\pi] \quad (1.9)$$

meaning that those frequencies does not generate any phase shift after one complete round-trip, involving constructive interferences then standing waves.

Putting (1.8) in (1.5) leads to:

$$\frac{\overline{E}_{circ}}{\overline{E}_{launch}} = \frac{1}{1 - r_1 r_2 e^{-i2\phi}} \quad (1.10)$$

Then, one uses the relation $I \simeq |\overline{E}|^2$, which is valid for an average of the intensity over some periods of the \overline{E} -field ⁵ and by the definition (1.1), one obtains:

$$A(\phi) = \frac{1}{(1 - r_1 r_2)^2 + 4r_1 r_2 \sin^2(\phi)} \quad (1.11)$$

Moreover, to get the actual distribution that we observe through the cavity, that is to say, the enhancement factor at the output of the cavity, we set:

$$A'(\phi) = (1 - r_1^2)(1 - r_2^2) \times A(\phi) \quad (1.12)$$

where one uses the intensity transmission coefficients $t_{1,2}^2 = 1 - r_{1,2}^2$. Now, in order to sketch the function, independent of the size of the cavity L , we shall use the third term of the equality (1.9) to reparametrize the round-trip dephasing variable ϕ :

$$\phi = \pi(x+1) \equiv \pi x \quad [\pi] \quad (1.13)$$

where the integer variable n has been replaced by the continuous one $x \in \mathbb{R}$ so that one can also investigate the behavior of the enhancement factor $A'(\phi) \sim A'(x)$ away from the resonant frequencies. Fig (??) is a normalized drawing⁶ of the airy distribution for a short amount of negative and positive resonant frequencies through a couple of reflexion coefficient $r_{1,2}$.

⁵See section ??

⁶One divides the whole graph by $A'(0)$

(add figure!)

To clarify, the abscisse coordinate is the factor quantity between two characteristic frequencies $\frac{n\pi}{L}$, which can be equivalently spatial or temporal as we get ride of the constant c in our calculation: then x is a frequency divided by the free spectral range (FSR) In other words, for $x = \frac{1}{2}$, one gets the enhancement factor A' for exactly a frequency halfway between two resonances, whereas $x = 1$ gives the A' factor for the first resonant frequency of the cavity.⁷

From those previous results, we can now define the *full width at half maximum* (FWHM) which approximately represents how broad is the range around a resonant frequency where the field is modulated by more than one-half. Thanks to (1.11), one obtains $A(0)/2$ when the denominator respects:

$$(1 - r_1 r_2)^2 = 4r_1 r_2 \sin^2(\Delta\phi) \Rightarrow \Delta\phi = \arcsin\left(\frac{1 - r_1 r_2}{2\sqrt{r_1 r_2}}\right) \quad (1.14)$$

with, in virtue of (1.13):

$$\Delta\phi = \frac{\pi}{2} \Delta\nu \quad (1.15)$$

where $\text{FWHM} \stackrel{\text{def}}{=} \Delta\nu$.

Finally, one can sketch the FWHM in fonction of the mirror reflectivities $r_{1,2}$ and obtains Fig ??

(add figure)

Two important points are noticeable:

- The linewidth of a cavity is invertly dependent on the mirrors reflectivity. In other words, more transmissive are the mirrors, broader is the frequency range around resonances, giving a *less selective* cavity, and reciprocally. Then, two perfectly reflective mirrors allow the FWHM to tend to zero: the cavity selects only and exactly the resonant frequencies $\frac{n\pi}{L}$. Formally, the *selectivity* of a cavity is defined by its *fineness* which reads $F \stackrel{\text{def}}{=} \frac{\Delta\nu}{\text{FSR}}$: one normalizes the linewidth by the distance (in term of frequency) separating two resonances. On Fig ??, as FSR is unity then the ordinate axis is also the finesse by definition.
- The function modelizing the FWHM tends to infinity below a certain value which can be evaluated to approximately $r_1 r_2 \approx 0.17$. That is due to the behavior of the arcsin which is not defined beyond $x = 1$ for $x \in \mathbb{R}^+$.

Nevertheless, it is always possible to build an experiment involving mirrors with high-losses (i.e very low reflection or very high transmission), without the output frequency going crazy. Thereby, what happens? Well, this is a limitation case of this *Airy*-model and in the common scientific description of a cavity, one utilizes the Lorentzian approach where one defines a coherent time of the light from the reflection coefficient and the cavity length, then gets a differential rate equation of the amount of photons (i.e an exponential decay of the electric field) which, by virtue of the Fourier transform, gives a Lorentzian as a spectrum around one resonant frequency. Then, to obtain the full cavity spectrum, one has to place each Lorentzian of a given resonance on top of each other on the same graph.⁸

The Lorentzian model is highly faithful to the real world experiment for all the possible reflective coefficients and tends to give the same characteristic values (linewidth, finesse, ...)

⁷One could say that the first resonant frequency is $x = 0$ according to the figure or even the formula (1.11) itself. However, $x = 0$ also means $n = 0$: the launched field would have null frequency, which is not a physical solution.

⁸Note that when treating the spontaneous emission of an atom (or more formally, a two-level atomic system), one also deals with differential rate equations, which represents the exponential lifetime decay of the excitation, giving a Lorentzian as a spectrum: longer it takes to the system to decay, broader is the bandwidth. However, the analogy stops here: for a cavity, the whole electric field spectrum is affected by the linewidth, whereas, for a two-level atom, the spectrum represents the frequency uncertainty as the atom can spontaneously emit a photon with frequency slightly different from what the atom has been excited with.

than the Airy-model from half to full reflective mirrors. However, it may seem to be less intuitive even if the derivation is straightforward and, after all, the goal here was to show the principle of interference inside a cavity due to the round-trip dephasing term.

Finally, the next step for an optical cavity is, as many of physical systems, to interact with its environment. In other words, what happens if one adds an element, let's say atoms, inside of it? Well, this is basically the principle of lots of current experiments and advanced optical tools, namely the *LASER* among others, and the next section is dedicated to the mathematical formulation of one first model for a cavity filled with a gain medium.

1.2 Jaynes-Cummings model

As an introduction, it is important to remind that the Jaynes-Cummings (JC) approach is no more no less than a model, which means that it is based on some hypothesis and has limitations which leads to refinements for more complex physical systems. In the current section, we shall present as wide as possible the principles of this model which will be the foundations of all the next derivated/evolved models⁹, even if the section is not supposed to be exhaustive but only a reference-like for further theoretical developments.

First of all, our system is composed of:

- An electric field inside a cavity: this E-field is quantized and can only be a superposition of resonances.
- A single two-level system: this system has a structure of ground-excited state and can represent a transition of an atom for example.

When quantizing the the E-field inside of a box of volume $V = L^3$ (i.e the cavity in three dimensions), one finds that the corresponding hamiltonian operator is [??Loudon p143]:

$$\hat{H}_{cav} = \int_{cavity} \left[\epsilon_0 \hat{E}(\vec{r}, t) \cdot \hat{E}(\vec{r}, t) + \mu_0^{-1} \hat{B}(\vec{r}, t) \cdot \hat{B}(\vec{r}, t) \right] dV \quad (1.16)$$

$$= \sum_{\vec{k}} \sum_{\lambda} \hbar \omega_{c, \vec{k}} \left(\hat{a}_{\lambda \vec{k}}^\dagger \hat{a}_{\lambda \vec{k}} + \frac{1}{2} \right) \quad (1.17)$$

where $\hat{E}(\vec{r}, t)$ and $\hat{B}(\vec{r}, t)$ are respectively the electric and magnetic operators; ϵ_0 and μ_0 the electric and magnetic susceptibility of the vacuum respectively, $\omega_{c, \vec{k}}$ the frequency mode; $\lambda = \pm 1$ the polarization; $\hat{a}_{\lambda \vec{k}}^\dagger$ with $\hat{a}_{\lambda \vec{k}}$ the ladders operator of the field.

Now, we define the electric field as exactly a unique mode with a unique polarization. Thereby:

$$\hat{H}_{cav} = \hbar \omega_c \left(\hat{a}^\dagger \hat{a} + \frac{1}{2} \right) \quad (1.18)$$

where we dropped the subscripts \vec{k} and λ . In other words, we represent the electric wave as a perfectly coherent field which fits a resonance of the cavity at frequency ω_c . As we can see, this one-mode cavity system is treated as a quantum harmonic oscillator.

Then comes the choice of the model for the ground-excited system. Sometimes, in the litterature [??MandelWolf], one utilizes another quantum harmonic oscillator, namely:

$$\hat{H}_{atom} = \hbar \omega_a \left(\hat{b}^\dagger \hat{b} + \frac{1}{2} \right) \quad (1.19)$$

where \hat{b} is the bosonic operator of the two-level system. However, this model allows mutiple excited levels and the energy of the atom-system is artificially restrained to the first two levels.

⁹As written, the JC-model may seem to be a pre-model or even a *toy tool* in view of introducing other legitimate models. Nevertheless, the JC-model is an actual physical approach, currently used for real use-cases in Quantum Optics, especially whith single atom interactions.

In fact, an advantageous choice is to use the spin-half quantum system:

$$\hat{H}_{atom} = \hbar\omega_a\hat{\sigma}_z \quad (1.20)$$

where $\hat{\sigma}_z$ is the third Pauli matrix. This model allows us to use the spin-down $|\downarrow\rangle$ as the ground state with an energy:

$$E_g \stackrel{\text{def}}{=} \langle\downarrow|\hat{H}_{atom}|\downarrow\rangle = -\hbar\omega_a \quad (1.21)$$

and $|\uparrow\rangle$ as the excited state with an energy:

$$E_e \stackrel{\text{def}}{=} \langle\uparrow|\hat{H}_{atom}|\uparrow\rangle = \hbar\omega_a \quad (1.22)$$

Moreover, the Pauli matrices can exponentiates to the Lie algebra of SU(2) then follows similar commutator relations, namely:

$$[\hat{\sigma}_j, \hat{\sigma}_k] = 2i\varepsilon_{jkl} \hat{\sigma}_l, \quad (1.23)$$

where ε_{jkl} is the Levi-Civita symbol. For the ladder operators $\hat{\sigma}_\pm = \hat{\sigma}_x \pm i\hat{\sigma}_y$, one can easily prove the following projections:

$$\hat{\sigma}_+|\downarrow\rangle = |\uparrow\rangle \quad \hat{\sigma}_-|\downarrow\rangle = 0 \quad \hat{\sigma}_-|\uparrow\rangle = |\downarrow\rangle \quad \hat{\sigma}_+|\uparrow\rangle = 0 \quad (1.24)$$

which means that one can excite a ground state or damp an excited state but one cannot excite an already excited state or damp a ground state: the spin-1/2 model is restrained to a pair of states by definition.¹⁰

Now that we got the hamiltonian operators for the cavity field and the two-level system, we use the tensor product in order to create a quantum system all-encompassing those two systems:

$$\hat{H}_{JC} = \hat{H}_{atom} \otimes \mathbb{1}_{cav,2} + \mathbb{1}_{atom,n} \otimes \hat{H}_{cav} + \hat{H}_{int} \quad (1.25)$$

where $\mathbb{1}_{cav,2}$ and $\mathbb{1}_{atom,n}$ are the identity operators for the Hilbert-space of the cavity and the atom respectively. Naturally, by their definition the Hilbert-space of the atom is of dimension 2 (i.e two states form the basis) whereas the Hilbert-space of the cavity is of dimension n as the quantum harmonic oscillator is associated to a Fock-space of the same dimension (i.e one has n possible different levels of energy for the electric field). In other words, we build another Hilbert-space of dimension $2n$ from two separated system.¹¹ We also defined \hat{H}_{int} as an additional hamiltonian operator belonging to an Hilbert-space which is of dimension $2n$ by construction: this new term arises from the physical interaction of the two systems but not by the *mixture* of the cavity and atom systems (i.e there is nothing related with the tensor product). This interaction term can be mathematically formulated with the electric field's acting on the dipole constituted by the atom:

$$\hat{H}_{int} \stackrel{\text{def}}{=} -\hat{d} \cdot \hat{E}(\vec{r}, t) \quad (1.26)$$

where \hat{d} is the dipole operator in term spin-half operators (using Pauli matrices) and \hat{E} the quantized electric field of the cavity. After centering the frame on the atom with $\vec{r} = 0$, the interaction hamiltonian transforms as:

$$\hat{H}_{int} = \hbar g \left[(\hat{\sigma}_+ - \hat{\sigma}_-) \otimes (\hat{a}^\dagger - \hat{a}) \right] \quad (1.27)$$

¹⁰From an autor to another, the eigenvalues may differ from a constant (in particular by a factor of 2). Keep in mind that the Pauli matrix operators are taken from $\hat{\sigma}_i = \frac{2}{\hbar} \hat{S}_i$ where \hat{S}_i is the spin operator.

¹¹ \hat{H}_{atom} and \hat{H}_{cav} are often refered to the energy of the atom and the cavity field respectively, which is a bit tendencious as this is only true when one takes those two systems separately, but falls down as soon as we mix them and add the interaction term: the eigenvalues (i.e energies) with an interaction are very different from the eigenvalues of the separated systems. Moreover, a hamiltonian operator corresponds to the energy if and only if it is *explicitly* independent of time (which is the case here but it is always nice to precise the reason).

where g is the coupling constant arising from multiple constant terms belonging to the dipole and electric operators: g defines *how strong/often* the cavity field and the atom will exchange energy. Finally, one obtains the complete hamiltonian operator for the Jaynes-Cummings model:

$$\hat{H}_{JC} \stackrel{\text{def}}{=} \hat{H}_0 + \hat{H}_{int} \quad (1.28)$$

$$= \hbar\omega_c \left(\hat{a}^\dagger \hat{a} + \frac{1}{2} \right) \otimes \mathbb{1}_{cav,2} + \mathbb{1}_{atom,n} \otimes \hbar\omega_a \hat{\sigma}_z + \hbar g \left[(\hat{\sigma}_+ - \hat{\sigma}_-) \otimes (\hat{a}^\dagger - \hat{a}) \right] \quad (1.29)$$

$$= \hbar\omega_c \hat{a}^\dagger \hat{a} + \hbar\omega_a \hat{\sigma}_z + \hbar g \left[(\hat{\sigma}_+ - \hat{\sigma}_-) (\hat{a}^\dagger - \hat{a}) \right] \quad (1.30)$$

where we omit the tensor product on the last line and remove the $\frac{1}{2}\hbar\omega_c$, which is only a choice of reference for the vacuum energy (i.e here we set zero). In the future, the *atomic* operators will lie on the right of the *field* operators in order to remind the implicit tensorial structure of the system.¹²

Then, we shall change the reference frame of the hamiltonian in order to deal only with the frequency difference $\Delta\omega \stackrel{\text{def}}{=} \omega_c - \omega_a$, called *detuning*. It will be advantageous for the simulation part and also allow us to apply a very common approximation. In this way, we use a U(1) symmetry for the JC-hamiltonian in this manner:

$$\hat{H}_{JC} \rightarrow U \hat{H}_{JC} U^{-1} + i\hbar \dot{U} U^{-1} \quad (1.31)$$

$$|\psi\rangle \rightarrow U|\psi\rangle \quad (1.32)$$

where $|\psi\rangle$ is a solution of the Schrödinger equation for the JC-hamiltonian; U is a unitary operator with its inverse U^{-1} belonging to its dual space, involving $U^{-1} = U^\dagger$; \dot{U} is the one-time derivative of U . It can be proven that the transformed hamiltonian and $U|\psi\rangle$ vector plugged into the Schrödinger equation, give the same equation as the JC-hamiltonian and $|\psi\rangle$. Thereby, if one finds a solution $|\phi\rangle$ of the transformed system, then $U^\dagger|\phi\rangle$ gives $|\psi\rangle$ which is a solution of the original system.

In our case, we choose the unitary transformation as:

$$U \stackrel{\text{def}}{=} e^{\frac{i\omega_a t}{\hbar} \hat{H}_0} = e^{i\omega_a t (\hat{n} + \hat{\sigma}_z)} \quad (1.33)$$

where we utilize the common photon number operator of the field, $\hat{n} = \hat{a}^\dagger \hat{a}$. As the U(1) symmetry is essentially a rotation along the complex unitary circle, we explicitly choose here a rotation of $\omega_a t$ which means that the transformed hamiltonian will be placed in the same frame as the atom: one calls this frame the *rotating frame*.

From Annexe (??) one finally obtains the following parts for the transformed hamiltonian:

$$i\hbar \dot{U} U^\dagger = -\hbar\omega_a (\hat{n} + \hat{\sigma}_z) \quad (1.34)$$

$$U \hbar\omega_c \hat{n} U^\dagger = \hbar\omega_c \hat{n} \quad (1.35)$$

$$U \hbar\omega_a \hat{\sigma}_z U^\dagger = \hbar\omega_a \hat{\sigma}_z \quad (1.36)$$

$$U \hbar g \hat{\sigma}_+ \hat{a} U^\dagger = \hbar g \hat{\sigma}_+ \hat{a} \quad (1.37)$$

$$U \hbar g \hat{\sigma}_- \hat{a}^\dagger U^\dagger = \hbar g \hat{\sigma}_- \hat{a}^\dagger \quad (1.38)$$

$$U \hbar g \hat{\sigma}_+ \hat{a}^\dagger U^\dagger = e^{2i\omega_a t} \hbar g \hat{\sigma}_+ \hat{a}^\dagger \quad (1.39)$$

$$U \hbar g \hat{\sigma}_- \hat{a} U^\dagger = e^{-2i\omega_a t} \hbar g \hat{\sigma}_- \hat{a} \quad (1.40)$$

As we shall use the detuning $\Delta\omega$ for the transformed hamiltonian instead of the respective cavity and atom frequency, it makes sense to write down a strong hypothesis on the system: $\omega_a \sim \omega_c$ which means that one assumes that the cavity and atom frequencies have the same order of magnitude. Thereby, the terms (1.39) and (1.40) are rotating extremely fast

¹²Note that a tensor product is a commutative relation and this artificial sort is only present to clarify the notation and equations.

compared to our reference frame, about twice the cavity or atom frequency, and are qualified of *anti-resonant*. Then, those terms average to zero over some cycles and can be discarded. This approximation is called the *rotating wave approximation* (RWA).¹³

At the end of the day, one finds the transformed hamiltonian:

$$\hat{H}_{JC} \rightarrow \hbar\Delta\omega \hat{n} + \hbar g \left(\hat{\sigma}_+ \hat{a} + \hat{\sigma}_- \hat{a}^\dagger \right) \quad (1.41)$$

that we rename by convenience and without ambiguity \hat{H}_{JC} . One may first notice that the atomic hamiltonian term vanished and only the field operator remains of \hat{H}_0 : that is due to the new frame which *follows* the atomic frequency, then the transformed \hat{H}_{cav} is proportional to the detuning. It is also interesting to see how a *generic* JC-state (i.e $|\uparrow\rangle \otimes |n\rangle$ or $|\downarrow\rangle \otimes |n\rangle$ which are also called *bare states*) is influenced by the transformed interaction hamiltonian:

$$\hat{H}_{int} \cdot |\uparrow\rangle \otimes |n\rangle = \hbar g \left[(\hat{\sigma}_+ \otimes \hat{a}) \cdot (|\uparrow\rangle \otimes |n\rangle) + (\hat{\sigma}_- \otimes \hat{a}^\dagger) \cdot (|\uparrow\rangle \otimes |n\rangle) \right] \quad (1.42)$$

$$= \hbar g \left[\hat{\sigma}_+ |\uparrow\rangle \otimes \hat{a} |n\rangle + \hat{\sigma}_- |\uparrow\rangle \otimes \hat{a}^\dagger |n\rangle \right] \quad (1.43)$$

$$= \hbar g \left[0 + |\downarrow\rangle \otimes \sqrt{n+1} |n+1\rangle \right] \quad (1.44)$$

$$= \hbar g |\downarrow\rangle \otimes \sqrt{n+1} |n+1\rangle \stackrel{\text{def}}{=} \hbar g \sqrt{n+1} |\downarrow, n+1\rangle \quad (1.45)$$

where we used (1.24). Equivalently:

$$\hat{H}_{int} \cdot |\downarrow, n\rangle = \hbar g \left[\hat{\sigma}_+ \hat{a} |\downarrow, n\rangle + \hat{\sigma}_- \hat{a}^\dagger |\downarrow, n\rangle \right] \quad (1.46)$$

$$= \hbar g \sqrt{n} |\downarrow, n-1\rangle \quad (1.47)$$

In other words, in the first case, an excited two-level system decays into its ground state, releasing a photon into the field (i.e the total energy is conserved), whereas in the second case, an atom *at rest* absorbs a photon of the field and ends up in its excited state: this is intuitively how one may imagine the interaction between the atom and the cavity field system.

In order to solve the system evolution, one wants to diagonalize the hamiltonian by finding the eigenstates and the eigenvalues. It turns out that \hat{H}_{JC} can be written by 2×2 blocks in the JC-states:¹⁴

$$\hat{H}_{JC} = \bigotimes_n \begin{bmatrix} \hbar\Delta\omega (n-1) & 2\hbar g \sqrt{n} \\ 2\hbar g \sqrt{n} & \hbar\Delta\omega n \end{bmatrix} \quad (1.48)$$

where n under the tensor product is the fock space dimension associated to the cavity system.

After diagonalizing each block of (1.48) (Annexe ??), one finds the following eigenvalues and eigenstates:

$$E_{n,\pm} = \hbar\Delta\omega \left(n - \frac{1}{2} \right) \pm \frac{\hbar}{2} \sqrt{4g^2 n + \Delta\omega^2} \quad (1.49)$$

$$|n, +\rangle = \cos\left(\frac{\theta_n}{2}\right) |\uparrow, n-1\rangle + \sin\left(\frac{\theta_n}{2}\right) |\downarrow, n\rangle \quad (1.50)$$

$$|n, -\rangle = \cos\left(\frac{\theta_n}{2}\right) |\downarrow, n\rangle - \sin\left(\frac{\theta_n}{2}\right) |\uparrow, n-1\rangle \quad (1.51)$$

with:

$$\tan(\theta_n) = -\frac{4\sqrt{ng}}{\Delta\omega} \quad (1.52)$$

¹³Here, we firstly change the reference frame in order to introduce the RWA. However, this is not necessary. Indeed, the RWA only influence the interaction hamiltonian and one can apply this approximation directly in (1.30) through the interaction picture evolution.

¹⁴The reason comes from the bare states being degenerated with eigenvalue $n+1$ when applied on the non-transformed base hamiltonian \hat{H}_0 .

To finish with the JC-model, we shall demonstrate one relevant case: the *Rabi flopping* (or more explicitly the Rabi oscillations) when the detuning is null or can be assimilated as vanishing (i.e $\Delta\omega = 0$). Thereby, one starts with the two-level system as excited with an *empty* cavity field:

$$|\psi(0)\rangle = |\uparrow, n=0\rangle = \cos\left(\frac{\theta_1}{2}\right) |1, +\rangle - \sin\left(\frac{\theta_1}{2}\right) |1, -\rangle \quad (1.53)$$

where we rewrite the initial state in terms of eigenstates.

The, applying the spatial evolution operator, one finds (see Annexe ??)

$$|\psi(t)\rangle = e^{-\frac{i}{\hbar}E_1, +t} \cos\left(\frac{\theta_1}{2}\right) |1, +\rangle - e^{-\frac{i}{\hbar}E_1, -t} \sin\left(\frac{\theta_1}{2}\right) |1, -\rangle \quad (1.54)$$

$$= \cos(\Omega t) |\uparrow, 0\rangle - i \sin(\Omega t) |\downarrow, 1\rangle \quad (1.55)$$

where we defined the Rabi frequency $\Omega = 2g$. Making the projection (i.e a measurement) along $|\uparrow, 0\rangle$ or $|\downarrow, 1\rangle$ and one ends up with the Rabi oscillations sketched on Fig ??.

(add figure!!)

As we can see on Fig ??, the probabilities of finding one photon with the atom in its ground state, while no photon with the atom in the excited state, oscillate in opposite phase.

Numerically, if one puts a small detuning (i.e ω_a is still about the same order of magnitude of ω_{ac}), one can numerically sketch on Fig ?? where we see that the energy transfer is progressively discarded as we increase the detuning: the probabilities of finding an inversed state decreases.