Plesio-Geostrophy and Data Assimilation: Formulations

Missing ingredients and new recipes

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1 Vorticity equation

In this section I derive an alternative form of the vorticity equation, which is the starting point of some eigenvalue problems. Starting from the dimensionless form of the vorticity equation, we have

$$-2\nabla_e^2 \frac{\partial \psi}{\partial t} = \frac{dH}{ds}\left(\frac{4}{sH}\frac{\partial \psi}{\partial \phi} - \frac{2}{H}\frac{\partial}{\partial s}\frac{\partial \psi}{\partial t} - \frac{1}{sH}\frac{\partial^2}{\partial \phi^2}\frac{\partial \psi}{\partial t}\right) - \frac{dH}{ds}\left(2f_\phi^e + \frac{1}{s}\frac{\partial \widetilde{f_z}}{\partial \phi}\right) + \hat{\mathbf{z}}\cdot\nabla\times\overline{\mathbf{f}_e}. \tag{1}$$

The superscript e means the field is evaluated on the equatorial plane. Here we used the dimensionless form as in Jackson and Maffei (2020), where the characteristic time scale is chosen to be the rotation time scale Ω^{-1} (the "inertial time scale"), instead of the Alfvén time scale L/V_A, as in Holdenried-Chernoff (2021). The force $\mathbf{f}$ contains all the external forces on the right-hand-side of the Navier-Stokes equation, e.g. Lorentz force, viscous force, buoyancy, etc. For the eigenvalue problem, it is convenient to move the terms involving all the time derivatives to one side,

$$\left[-2\nabla_e^2 + \frac{dH}{ds}\left(\frac{2}{H}\frac{\partial}{\partial s} + \frac{1}{sH}\frac{\partial^2}{\partial \phi^2}\right)\right]\frac{\partial \psi}{\partial t} = \frac{4}{sH}\frac{dH}{ds}\frac{\partial \psi}{\partial \phi} - \frac{dH}{ds}\left(2f_\phi^e + \frac{1}{s}\frac{\partial \widetilde{f_z}}{\partial \phi}\right) + \hat{\mathbf{z}}\cdot\nabla\times\overline{\mathbf{f}_e}$$

$$\left[-\frac{1}{s}\frac{\partial}{\partial s}\left(s\frac{\partial}{\partial s}\right) + \frac{1}{H}\frac{dH}{ds}\frac{\partial}{\partial s} + \left(\frac{1}{2sH}\frac{dH}{ds} - \frac{1}{s^2}\right)\frac{\partial^2}{\partial \phi^2}\right]\frac{\partial \psi}{\partial t} = \frac{2}{sH}\frac{dH}{ds}\frac{\partial \psi}{\partial \phi} - \frac{dH}{ds}\left(f_\phi^e + \frac{1}{2s}\frac{\partial \widetilde{f_z}}{\partial \phi}\right) + \frac{\hat{\mathbf{z}}}{2}\cdot\nabla\times\overline{\mathbf{f}_e}$$

In cases where different azimuthal wavenumber separates (e.g. when the system has rotational invariance with respect to ϕ), this equation will be readily converted to an ordinary differential equation (ODE) in s. In this case, it would be desirable to write the differential operators concerning s in the self-adjoint form $\frac{d}{ds}(p(s)\frac{d}{ds})$, to form a standard Sturm-Liouville problem,

$$-\frac{1}{s}\frac{\partial}{\partial s}\left(s\frac{\partial}{\partial s}\right) + \frac{1}{H}\frac{dH}{ds}\frac{\partial}{\partial s} = -\frac{\partial^2}{\partial s^2} - \left(\frac{1}{s} - \frac{1}{H}\frac{dH}{ds}\right)\frac{\partial}{\partial s}$$

and we can deduce the term $p(x)$ using the relation

$$\frac{1}{p(s)}\frac{dp(s)}{ds} = \frac{1}{s} - \frac{1}{H}\frac{dH}{ds} \quad\Longrightarrow\quad d\ln p = d\ln s - d\ln H = d\ln\frac{s}{H} \quad\Longrightarrow\quad p = \frac{s}{H}.$$

And the original equation can be rewritten as

$$\left[\frac{\partial}{\partial s}\left(\frac{s}{H}\frac{\partial}{\partial s}\right) + \left(\frac{1}{sH} - \frac{1}{2H^2}\frac{dH}{ds}\right)\frac{\partial^2}{\partial \phi^2}\right]\frac{\partial \psi}{\partial t} = -\frac{2}{H^2}\frac{dH}{ds}\frac{\partial \psi}{\partial \phi} + \frac{dH}{ds}\left(\frac{s}{H}f_\phi^e + \frac{1}{2H}\frac{\partial \widetilde{f_z}}{\partial \phi}\right) - \frac{s}{2H}\hat{\mathbf{z}}\cdot\nabla\times\overline{\mathbf{f}_e} \tag{2}$$

2 Spherical-Cylindrical transformation

Spherical coordinates and cylindrical coordinates can be transformed to one another via

$$\begin{pmatrix} s \\ \phi \\ z \end{pmatrix} = \begin{pmatrix} r\sin\theta \\ \phi \\ r\cos\theta \end{pmatrix}, \quad \begin{pmatrix} r \\ \theta \\ \phi \end{pmatrix} = \begin{pmatrix} \sqrt{s^2+z^2} \\ \arccos\frac{z}{\sqrt{s^2+z^2}} \\ \phi \end{pmatrix} \tag{3}$$

The vector components are converted using the rotation matrix

$$\begin{pmatrix} A_s \\ A_\phi \\ A_z \end{pmatrix} = \mathbf{R} \begin{pmatrix} A_r \\ A_\theta \\ A_\phi \end{pmatrix}, \quad \begin{pmatrix} A_r \\ A_\theta \\ A_\phi \end{pmatrix} = \mathbf{R}^T \begin{pmatrix} A_s \\ A_\phi \\ A_z \end{pmatrix} \tag{4}$$

where the rotation matrix is an orthogonal matrix

$$\mathbf{R} = \begin{pmatrix} \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \\ \cos\theta & -\sin\theta & 0 \end{pmatrix} = \begin{pmatrix} \frac{s}{\sqrt{s^2+z^2}} & \frac{z}{\sqrt{s^2+z^2}} & 0 \\ 0 & 0 & 1 \\ \frac{z}{\sqrt{s^2+z^2}} & -\frac{s}{\sqrt{s^2+z^2}} & 0 \end{pmatrix}. \tag{5}$$

The Jacobian from spherical to cylindrical coordinates and its inverse, i.e. the Jacobian from cylindrical to spherical coordinates are given by

$$\begin{aligned} \mathbf{J} &= \frac{\partial(s,\phi,z)}{\partial(r,\theta,\phi)} = \begin{pmatrix} \sin\theta & r\cos\theta & 0 \\ 0 & 0 & 1 \\ \cos\theta & -r\sin\theta & 0 \end{pmatrix} = \begin{pmatrix} \frac{s}{\sqrt{s^2+z^2}} & z & 0 \\ 0 & 0 & 1 \\ \frac{z}{\sqrt{s^2+z^2}} & -s & 0 \end{pmatrix}, \\ \mathbf{J}^{-1} &= \frac{\partial(r,\theta,\phi)}{\partial(s,\phi,z)} = \begin{pmatrix} \sin\theta & 0 & \cos\theta \\ r^{-1}\cos\theta & 0 & -r^{-1}\sin\theta \\ 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} \frac{s}{\sqrt{s^2+z^2}} & 0 & \frac{z}{\sqrt{s^2+z^2}} \\ \frac{z}{s^2+z^2} & 0 & -\frac{s}{s^2+z^2} \\ 0 & 1 & 0 \end{pmatrix}. \end{aligned} \tag{6}$$

The derivatives in spherical harmonics are transformed into derivatives in cylindrical coordinates via

$$\begin{pmatrix} \partial_r \\ \partial_\theta \\ \partial_\phi \end{pmatrix} = \frac{\partial(s,\phi,z)}{\partial(r,\theta,\phi)}^T \begin{pmatrix} \partial_s \\ \partial_\phi \\ \partial_z \end{pmatrix} = \begin{pmatrix} \sin\theta & 0 & \cos\theta \\ r\cos\theta & 0 & -r\sin\theta \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} \partial_s \\ \partial_\phi \\ \partial_z \end{pmatrix} = \begin{pmatrix} \frac{s}{\sqrt{s^2+z^2}} & 0 & \frac{z}{\sqrt{s^2+z^2}} \\ z & 0 & -s \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} \partial_s \\ \partial_\phi \\ \partial_z \end{pmatrix} \tag{7}$$

where the matrix elements are already changed to cylindrical coordinates. Inversely, we have

$$\begin{pmatrix} \partial_s \\ \partial_\phi \\ \partial_z \end{pmatrix} = \frac{\partial(r,\theta,\phi)}{\partial(s,\phi,z)}^T \begin{pmatrix} \partial_r \\ \partial_\theta \\ \partial_\phi \end{pmatrix} = \begin{pmatrix} \sin\theta & \frac{1}{r}\cos\theta & 0 \\ 0 & 0 & 1 \\ \cos\theta & -\frac{1}{r}\sin\theta & 0 \end{pmatrix} \begin{pmatrix} \partial_r \\ \partial_\theta \\ \partial_\phi \end{pmatrix} = \begin{pmatrix} \frac{s}{\sqrt{s^2+z^2}} & \frac{z}{s^2+z^2} & 0 \\ 0 & 0 & 1 \\ \frac{z}{\sqrt{s^2+z^2}} & -\frac{s}{s^2+z^2} & 0 \end{pmatrix} \begin{pmatrix} \partial_r \\ \partial_\theta \\ \partial_\phi \end{pmatrix} \tag{8}$$

where the matrix elements in spherical coordinates are also shown.

3 Induction equation at the boundary

Induction equation for the radial component at the boundary

$$\frac{\partial B_r}{\partial t} = -\nabla_H \cdot (\mathbf{u}_H B_r) \tag{9}$$

which can be expanded in spherical coordinates

$$\frac{\partial B_r}{\partial t} = -\frac{1}{\sin\theta}\left(\frac{\partial}{\partial\theta}\left(\sin\theta\, u_\theta B_r\right) + \frac{\partial}{\partial\phi}\left(u_\phi B_r\right)\right).$$

Alternatively, we can also use the induction equation in cylindrical coordinates at the boundary. These quantities are

$$\begin{aligned}
\frac{\partial B_s}{\partial t} &= (\mathbf{B}\cdot\nabla\mathbf{u})_s - (\mathbf{u}\cdot\nabla\mathbf{B})_s = B_s\frac{\partial u_s}{\partial s} + \frac{B_\phi}{s}\frac{\partial u_s}{\partial\phi} + B_z\frac{\partial u_s}{\partial z} - u_s\frac{\partial B_s}{\partial s} - \frac{u_\phi}{s}\frac{\partial B_s}{\partial\phi} - u_z\frac{\partial B_s}{\partial z}\\
\frac{\partial B_\phi}{\partial t} &= (\mathbf{B}\cdot\nabla\mathbf{u})_\phi - (\mathbf{u}\cdot\nabla\mathbf{B})_\phi = B_s\frac{\partial u_\phi}{\partial s} + \frac{B_\phi}{s}\frac{\partial u_\phi}{\partial\phi} + B_z\frac{\partial u_\phi}{\partial z} - u_s\frac{\partial B_\phi}{\partial s} - \frac{u_\phi}{s}\frac{\partial B_\phi}{\partial\phi} - u_z\frac{\partial B_\phi}{\partial z} + \frac{B_\phi u_s - u_\phi B_s}{s}\\
\frac{\partial B_z}{\partial t} &= (\mathbf{B}\cdot\nabla\mathbf{u})_z - (\mathbf{u}\cdot\nabla\mathbf{B})_z = B_s\frac{\partial u_z}{\partial s} + \frac{B_\phi}{s}\frac{\partial u_z}{\partial\phi} + B_z\frac{\partial u_z}{\partial z} - u_s\frac{\partial B_z}{\partial s} - \frac{u_\phi}{s}\frac{\partial B_z}{\partial\phi} - u_z\frac{\partial B_z}{\partial z}
\end{aligned} \tag{10}$$

At the boundary, the induction equation takes the form

$$\begin{aligned}
\frac{\partial B_s^\pm}{\partial t} &= B_s^\pm\left.\frac{\partial u_s}{\partial s}\right|_{\pm H} + \frac{B_\phi^\pm}{s}\frac{\partial u_s^\pm}{\partial\phi} + B_z^\pm\left.\frac{\partial u_s}{\partial z}\right|_{\pm H} - u_s^\pm\left.\frac{\partial B_s}{\partial s}\right|_{\pm H} - \frac{u_\phi^\pm}{s}\frac{\partial B_s^\pm}{\partial\phi} - u_z^\pm\left.\frac{\partial B_s}{\partial z}\right|_{\pm H}\\
\frac{\partial B_\phi^\pm}{\partial t} &= B_s^\pm\left.\frac{\partial u_\phi}{\partial s}\right|_{\pm H} + \frac{B_\phi^\pm}{s}\frac{\partial u_\phi^\pm}{\partial\phi} + B_z^\pm\left.\frac{\partial u_\phi}{\partial z}\right|_{\pm H} - u_s^\pm\left.\frac{\partial B_\phi}{\partial s}\right|_{\pm H} - \frac{u_\phi^\pm}{s}\frac{\partial B_\phi^\pm}{\partial\phi} - u_z^\pm\left.\frac{\partial B_\phi}{\partial z}\right|_{\pm H} + \frac{B_\phi^\pm u_s^\pm - u_\phi^\pm B_s^\pm}{s}\\
\frac{\partial B_z^\pm}{\partial t} &= B_s^\pm\left.\frac{\partial u_z}{\partial s}\right|_{\pm H} + \frac{B_\phi^\pm}{s}\frac{\partial u_z^\pm}{\partial\phi} + B_z^\pm\left.\frac{\partial u_z}{\partial z}\right|_{\pm H} - u_s^\pm\left.\frac{\partial B_z}{\partial s}\right|_{\pm H} - \frac{u_\phi^\pm}{s}\frac{\partial B_z^\pm}{\partial\phi} - u_z^\pm\left.\frac{\partial B_z}{\partial z}\right|_{\pm H}
\end{aligned}$$

where the $\pm$ superscript shows that the quantity is evaluated at the boundary $z = \pm H$. Terms in the form $\frac{\partial}{\partial s}|_{\pm H}$ means the field has to be differentiated first and evaluated at the boundary later. Hence, we see that these evolution equations are not closed in themselves, in the sense that the derivatives $\partial_s\mathbf{B}$ and $\partial_z\mathbf{B}$ cannot be evaluated or represented unless the field $\mathbf{B}$ can be evaulated or represented in the entire volume in the parameterization. This is not the case with the PG model, where the parameterization of the magnetic quantities only involves the integrated moments, the boundary field, and the equatorial field. Therefore, these equations cannot be used for time stepping or simulation. The only way to move forward seems to use eq.(9). This equation only involves surface operators (owing to the non-penetration condition $u_r = \hat{\mathbf{n}}\cdot u = 0$), and is closed on the surface of the sphere.

Nevertheless, using the induction equation of the cylindrical components can be very useful in solving eigenvalue problems where the background velocity field is zero (it is almost the case with the eigenvalue problems of interest). In these problems, the linearized version of the equation will only involve cross terms of the background magnetic field, whose derivatives are known everywhere in space, and the perturbed velocity field. If we keep the notation u for perturbational velocity, and introduce notation b for perturbational magnetic field, the linearized induction equation takes the form

$$\begin{aligned}
\frac{\partial b_s^\pm}{\partial t} &= B_s^{0\pm}\left.\frac{\partial u_s}{\partial s}\right|_{\pm H} + \frac{B_\phi^{0\pm}}{s}\frac{\partial u_s^\pm}{\partial\phi} + B_z^{0\pm}\left.\frac{\partial u_s}{\partial z}\right|_{\pm H} - u_s^\pm\left.\frac{\partial B_s^0}{\partial s}\right|_{\pm H} - \frac{u_\phi^\pm}{s}\frac{\partial B_s^{0\pm}}{\partial\phi} - u_z^\pm\left.\frac{\partial B_s^0}{\partial z}\right|_{\pm H}\\
\frac{\partial b_\phi^\pm}{\partial t} &= B_s^{0\pm}\left.\frac{\partial u_\phi}{\partial s}\right|_{\pm H} + \frac{B_\phi^{0\pm}}{s}\frac{\partial u_\phi^\pm}{\partial\phi} + B_z^{0\pm}\left.\frac{\partial u_\phi}{\partial z}\right|_{\pm H} - u_s^\pm\left.\frac{\partial B_\phi^0}{\partial s}\right|_{\pm H} - \frac{u_\phi^\pm}{s}\frac{\partial B_\phi^{0\pm}}{\partial\phi} - u_z^\pm\left.\frac{\partial B_\phi^0}{\partial z}\right|_{\pm H} + \frac{B_\phi^{0\pm}u_s^\pm - u_\phi^\pm B_s^{0\pm}}{s}\\
\frac{\partial b_z^\pm}{\partial t} &= B_s^{0\pm}\left.\frac{\partial u_z}{\partial s}\right|_{\pm H} + \frac{B_\phi^{0\pm}}{s}\frac{\partial u_z^\pm}{\partial\phi} + B_z^{0\pm}\left.\frac{\partial u_z}{\partial z}\right|_{\pm H} - u_s^\pm\left.\frac{\partial B_z^0}{\partial s}\right|_{\pm H} - \frac{u_\phi^\pm}{s}\frac{\partial B_z^{0\pm}}{\partial\phi} - u_z^\pm\left.\frac{\partial B_z^0}{\partial z}\right|_{\pm H}
\end{aligned}$$

Recall that in the plesio-geostrophic ansatz for the velocity field, $\mathbf{u}_e = \frac{1}{H}\nabla\times\psi\hat{\mathbf{z}}$, $u_z = \frac{z}{H}\frac{dH}{ds}u_s$ and the stream function ψ is z-invariant. Therefore, the equations can be simplified as

$$\begin{aligned}
\frac{\partial b_s^\pm}{\partial t} &= B_s^{0\pm} \frac{\partial}{\partial s}\left(\frac{1}{sH}\frac{\partial \psi}{\partial \phi}\right) + \frac{B_\phi^{0\pm}}{s^2 H}\frac{\partial^2 \psi}{\partial \phi^2} - \frac{1}{sH}\frac{\partial \psi}{\partial \phi}\frac{\partial B_s^0}{\partial s}\bigg|_{\pm H} + \frac{1}{sH}\frac{\partial \psi}{\partial s}\frac{\partial B_s^{0\pm}}{\partial \phi} \mp \frac{1}{sH}\frac{dH}{ds}\frac{\partial \psi}{\partial \phi}\frac{\partial B_s^0}{\partial z}\bigg|_{\pm H}, \\
\frac{\partial b_\phi^\pm}{\partial t} &= -B_s^{0\pm}\frac{\partial}{\partial s}\left(\frac{1}{H}\frac{\partial \psi}{\partial s}\right) - \frac{B_\phi^{0\pm}}{sH}\frac{\partial^2 \psi}{\partial s \partial \phi} - \frac{1}{sH}\frac{\partial \psi}{\partial \phi}\frac{\partial B_\phi^0}{\partial s}\bigg|_{\pm H} + \frac{1}{sH}\frac{\partial \psi}{\partial s}\frac{\partial B_\phi^{0\pm}}{\partial \phi} \mp \frac{1}{sH}\frac{dH}{ds}\frac{\partial \psi}{\partial \phi}\frac{\partial B_\phi^0}{\partial z}\bigg|_{\pm H} \\
&\quad + \frac{1}{s}\left(\frac{B_\phi^{0\pm}}{sH}\frac{\partial \psi}{\partial \phi} + \frac{B_s^{0\pm}}{H}\frac{\partial \psi}{\partial s}\right), \qquad (11) \\
\frac{\partial b_z^\pm}{\partial t} &= \pm H B_s^{0\pm}\frac{\partial}{\partial s}\left(\frac{1}{sH^2}\frac{dH}{ds}\frac{\partial \psi}{\partial \phi}\right) \pm \frac{B_\phi^{0\pm}}{s^2 H}\frac{dH}{ds}\frac{\partial^2 \psi}{\partial \phi^2} + \frac{B_z^{0\pm}}{sH^2}\frac{dH}{ds}\frac{\partial \psi}{\partial \phi} - \frac{1}{sH}\frac{\partial \psi}{\partial \phi}\frac{\partial B_z^0}{\partial s}\bigg|_{\pm H} \\
&\quad + \frac{1}{sH}\frac{\partial \psi}{\partial s}\frac{\partial B_z^{0\pm}}{\partial \phi} \mp \frac{1}{sH}\frac{dH}{ds}\frac{\partial \psi}{\partial \phi}\frac{\partial B_z^0}{\partial z}\bigg|_{\pm H}.
\end{aligned}$$

If we look at the right-hand-side of these induction equations, we see that the right-hand-side is free of perturbed magnetic fields, but only involves background magnetic fields and perturbed velocity field. Therefore, when given the background field, the boundary terms can be written as

$$\frac{\partial b_a^\pm}{\partial t} = \mathcal{L}_i^\pm \psi \quad \Longrightarrow \quad b_a^\pm = \frac{1}{i\omega}\mathcal{L}_i^\pm \psi$$

where $\mathcal{L}_i^\pm$ are some linear operators. In fact, this is a feature that applies to all induction equation, summarized in the following statement.

Proposition 3.1 *The ideal induction equations of the boundary magnetic field or the integrated magnetic moments, when linearized around a background field with zero velocity, involves only the background magnetic field / moment and the perturbed velocity. In other words, all of them can be written as*

$$\frac{\partial b_a}{\partial t} = \mathcal{L}_a \psi,$$

or in the frequency domain

$$i\omega b_a = \mathcal{L}_a \psi,$$

where $b_a \in \{\overline{m_{ss}}, \overline{m_{\phi\phi}}, \overline{m_{s\phi}}, \widetilde{m_{sz}}, \widetilde{m_{\phi z}}, \widetilde{zm_{ss}}, \widetilde{zm_{\phi\phi}}, \widetilde{zm_{s\phi}}, b_{es}, b_{e\phi}, b_{ez}, b_{es,z}, b_{e\phi,z}, b_s^\pm, b_\phi^\pm, b_z^\pm\}$.

This proposition leads to the following statement.

Corollary 3.2 *When linearized around a background field with zero velocity, the complete PG system with diffusionless vorticity and induction equations and boundary terms can always be reduced to a single equation*

$$\left[\frac{\partial}{\partial s}\left(\frac{s}{H}\frac{\partial}{\partial s}\right) + \left(\frac{1}{sH} - \frac{1}{2H^2}\frac{dH}{ds}\right)\frac{\partial^2}{\partial \phi^2}\right]\frac{\partial^2 \psi}{\partial t^2} = -\frac{2}{H^2}\frac{dH}{ds}\frac{\partial}{\partial \phi}\frac{\partial \psi}{\partial t} + \mathcal{L}_{\mathrm{tot}}\psi$$

where $\mathcal{L}_{\mathrm{tot}}$ is the combined linear operator that gives the Lorentz force. Furthermore, considering the forms of the induction equations and vorticity equation, $\mathcal{L}_{\mathrm{tot}}$ is at most 3rd order in (s, ϕ, z). In the frequency domain, it is written as

$$-\omega^2\left[\frac{\partial}{\partial s}\left(\frac{s}{H}\frac{\partial}{\partial s}\right) + \left(\frac{1}{sH} - \frac{1}{2H^2}\frac{dH}{ds}\right)\frac{\partial^2}{\partial \phi^2}\right]\psi = -i\omega\frac{2}{H^2}\frac{dH}{ds}\frac{\partial \psi}{\partial \phi} + \mathcal{L}_{\mathrm{tot}}\psi$$

This gives a further dilemma: the eigenvalue problem will be closed in the vorticity itself, regardless of the boundary condition. In other words, changing the boundary condition does not even change the eigenvalue problem. How is that possible? Does that mean the eigenmode is not even affected by the choice of boundary conditions? Will the boundary condition be automatically satisfied by the perturbed magnetic field? For instance, will $b_s^\pm$, $b_\phi^\pm$ and $b_z^\pm$ solved in this way automatically match an insulating boundary condition, and if not, when will it or is it necessary?

Corollary 3.2 is useful conceptually, but cannot be directly implemented as an eigenvalue problem, since the right-hand-sides contain both first derivative and stream function itself. We must instead flatten out the second order derivative, and consider the augmented system. One way to achieve this is to write

$$\begin{aligned}\left[\frac{\partial}{\partial s}\left(\frac{s}{H}\frac{\partial}{\partial s}\right)+\left(\frac{1}{sH}-\frac{1}{2H^2}\frac{dH}{ds}\right)\frac{\partial^2}{\partial\phi^2}\right]\frac{\partial\psi}{\partial t} &= -\frac{2}{H^2}\frac{dH}{ds}\frac{\partial\psi}{\partial\phi}+F \\ \frac{\partial F}{\partial t} &= \mathcal{L}_{\mathrm{tot}}\psi\end{aligned}$$

or in matrix form of the eigenvalue problem

$$i\omega\begin{pmatrix}\frac{\partial}{\partial s}\left(\frac{s}{H}\frac{\partial}{\partial s}\right)-\frac{m^2}{sH}+\frac{m^2}{2H^2}\frac{dH}{ds} & 0 \\ 0 & 1\end{pmatrix}\begin{pmatrix}\psi^m \\ F^m\end{pmatrix}=\begin{pmatrix}-\frac{2im}{H^2}\frac{dH}{ds} & 1 \\ \mathcal{L}_{\mathrm{tot}} & 0\end{pmatrix}\begin{pmatrix}\psi^m \\ F^m\end{pmatrix}$$

This is similar to the velocity-stress formulation, often used in seismological simulations. Interestingly, since we know $\{\psi^{mn}(s)=s^{|m|}H^3P_n^{(\frac{3}{2},|m|)}(2s^2-1)\}$ are the eigenfunctions for the Sturm-Liouville problem (first equation, without F contribution), we can conclude these $\{\psi^{mn}(s)\}$ form a complete orthogonal basis with respect to weight $\frac{2ism}{H^3}=-\frac{2im}{H^2}\frac{dH}{ds}$. In other words, we should expect that the appropriate expansion for F that can be the solution to the eigenvalue problem should take the form

$$F^{mn}(s)=\frac{s}{H^3}\psi^{mn}(s)=s^{|m|+1}P_n^{(\frac{3}{2},|m|)}(2s^2-1).$$

As I have mentioned, the boundary induction equation in cylindrical coordinates cannot be used in a time-stepping solver with PG formulations. The only equation that seems to be closed in itself is eq.(9). This, however, involves one complication and one limitation. First, as the equation is not in cylindrical coordinates, while all other equations are, we need an explicit spherical-cylindrical transform. Among other complications, this means the sparsity of the matrix or orthogonality of the basis might be partially destroyed. Second, noting that the Lorentz force involves only the s, z and ϕ components of the boundary magnetic fields, we need to link the radial field to the three components. This can be easily done with an insulating boundary condition, where the magnetic field external to the sphere is harmonic. However, once this assumption is dropped, it will be much more challenging to derive a general link.

4 Conversion between the two dimensionless forms

$$L=r_0,\quad \tau\sim\frac{L}{U}$$

Holdenried-Chernoff (2021) uses the Alfven wave velocity

$$U=V_A=\frac{\mathscr{B}}{\sqrt{\rho_0\mu_0}}$$

as the characteristic velocity scale. This scale is ultimately determined by the characteristic magnetic field strength $\mathscr{B}$. The relative strength of Lorentz force to Coriolis is given by the Lehnert number,

$$\mathrm{Le}=\frac{\mathscr{B}}{\sqrt{\rho_0\mu_0}\Omega L}=\frac{V_A}{V_\Omega}$$

where $V_\Omega=\Omega L$ is the rotation speed. In other words, the characteristic scales for the magnetic field and the spin rate are linked via this dimensionless number. The dimensionless Navier-Stokes equation takes the form

$$\frac{\partial\mathbf{u}}{\partial t}+\mathbf{u}\cdot\nabla\mathbf{u}+\frac{2}{\mathrm{Le}}\hat{\mathbf{z}}\times\mathbf{u}=-\nabla p+(\nabla\times\mathbf{B})\times\mathbf{B}$$

and the vorticity equation is

$$-2\nabla_e^2 \frac{\partial \psi}{\partial t} = \frac{dH}{ds}\left(\mathrm{Le}^{-1}\frac{4}{sH}\frac{\partial \psi}{\partial \phi} - \frac{2}{H}\frac{\partial}{\partial s}\frac{\partial \psi}{\partial t} - \frac{1}{sH}\frac{\partial^2}{\partial \phi^2}\frac{\partial \psi}{\partial t}\right) - \frac{dH}{ds}\left(2 f_{e\phi} + \frac{1}{s}\frac{\partial \widetilde{f_\phi}}{\partial \phi}\right) + \hat{\mathbf{z}}\cdot\nabla\times\overline{\mathbf{f}_e}$$

$$\left[\frac{\partial}{\partial s}\left(\frac{s}{H}\frac{\partial}{\partial s}\right) + \left(\frac{1}{sH} - \frac{1}{2H^2}\frac{dH}{ds}\right)\frac{\partial^2}{\partial \phi^2}\right]\frac{\partial \psi}{\partial t} = -\mathrm{Le}^{-1}\frac{2}{H^2}\frac{dH}{ds}\frac{\partial \psi}{\partial \phi} + \frac{dH}{ds}\left(\frac{s}{H} f_\phi^e + \frac{1}{2H}\frac{\partial \widetilde{f_z}}{\partial \phi}\right) - \frac{s}{2H}\hat{\mathbf{z}}\cdot\nabla\times\overline{\mathbf{f}_e}$$

In constrast, Jackson and Maffei (2020) uses

$$U = \Omega L$$

as the characteristic velocity, which is particularly useful when the magnetic field is absent. Now it is necessary to nondimensionalize the Lorentz force. While Jackson and Maffei (2020) uses $\mathscr{B} = \sqrt{\rho_0 \mu_0}\Omega L$, meaning $\mathrm{Le} = 1$ in the paper, it lacks flexibility. Instead, we can still use Lehnert number, in which case the Navier-Stokes equation takes the form

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u}\cdot\nabla\mathbf{u} + 2\hat{\mathbf{z}}\times\mathbf{u} = -\nabla p + \mathrm{Le}^2(\nabla\times\mathbf{B})\times\mathbf{B}$$

and the vorticity equation is

$$-2\nabla_e^2 \frac{\partial \psi}{\partial t} = \frac{dH}{ds}\left(\frac{4}{sH}\frac{\partial \psi}{\partial \phi} - \frac{2}{H}\frac{\partial}{\partial s}\frac{\partial \psi}{\partial t} - \frac{1}{sH}\frac{\partial^2}{\partial \phi^2}\frac{\partial \psi}{\partial t}\right) - \mathrm{Le}^2\left[\frac{dH}{ds}\left(2 f_{e\phi} + \frac{1}{s}\frac{\partial \widetilde{f_\phi}}{\partial \phi}\right) + \hat{\mathbf{z}}\cdot\nabla\times\overline{\mathbf{f}_e}\right]$$

$$\left[\frac{\partial}{\partial s}\left(\frac{s}{H}\frac{\partial}{\partial s}\right) + \left(\frac{1}{sH} - \frac{1}{2H^2}\frac{dH}{ds}\right)\frac{\partial^2}{\partial \phi^2}\right]\frac{\partial \psi}{\partial t} = -\frac{2}{H^2}\frac{dH}{ds}\frac{\partial \psi}{\partial \phi} + \mathrm{Le}^2\left[\frac{dH}{ds}\left(\frac{s}{H} f_\phi^e + \frac{1}{2H}\frac{\partial \widetilde{f_z}}{\partial \phi}\right) - \frac{s}{2H}\hat{\mathbf{z}}\cdot\nabla\times\overline{\mathbf{f}_e}\right]$$

The variables solved in two dimensionless forms can be easily converted to one another,

$$\mathbf{u}_\Omega = \frac{\mathbf{u}_A}{\Omega L}\frac{\mathscr{B}}{\sqrt{\rho_0\mu_0}} = \mathrm{Le}\,\mathbf{u}_A, \qquad t_\Omega = \Omega\frac{\sqrt{\rho_0\mu_0}L}{\mathscr{B}}t_A = \frac{t_A}{\mathrm{Le}}, \qquad \mathbf{B}_\Omega = \mathbf{B}_A.$$

Here the A and Ω subscripts indicate dimensionless fields in the equations nondimensionalized using Alfven wave velocity and rotation velocity, respectively. Finally, for the eigenvalue problem, the eigenvalues solved follow the following relation, inverse to t:

$$\omega_\Omega = \frac{\mathscr{B}}{\sqrt{\rho_0\mu_0}\Omega L}\omega_A = \mathrm{Le}\,\omega_A.$$

5 Regularity conditions on rank-2 tensor in cylindrical coordinates

In this section, I derive the regularity conditions for general rank-2 tensors in cylindrical coordinates. This approach has been more elaborately exploited for arbitrary ranks in *Regularity conditions for the Fourier coefficients of tensors in polar coordinates*. The excerpt here offers a more self-contained, explicit and easy-to-comprehend explanation.

Consider a rank-2 tensor field in 2-D space, denoted as $\mathbf{A} \in \mathbb{C}^{2\times 2}$. The tensor can be expressed in any locally orthogonal frame as

$$A_{ij} = \hat{\mathbf{e}}_i \cdot \mathbf{A} \cdot \hat{\mathbf{e}}_j.$$

Its components can be expressed in Cartesian coordinates as well as cylindrical coordinates using matrices, which are related via transform

$$\begin{pmatrix} A_{xx} & A_{xy} \\ A_{yx} & A_{yy} \end{pmatrix} = \begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix}\begin{pmatrix} A_{ss} & A_{s\phi} \\ A_{\phi s} & A_{\phi\phi} \end{pmatrix}\begin{pmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{pmatrix}$$

The elements in Cartesian coordinates are thus related to the elements in the cylindrical coordinates via

$$
\begin{aligned}
A_{xx} &= \cos^2\phi A_{ss} - \cos\phi\sin\phi\left(A_{s\phi} + A_{\phi s}\right) + \sin^2\phi A_{\phi\phi}, \\
A_{yy} &= \sin^2\phi A_{ss} + \cos\phi\sin\phi\left(A_{s\phi} + A_{\phi s}\right) + \cos^2\phi A_{\phi\phi}, \\
A_{xy} &= \cos\phi\sin\phi\left(A_{ss} - A_{\phi\phi}\right) + \cos^2\phi A_{s\phi} - \sin^2 A_{\phi s}, \\
A_{yx} &= \cos\phi\sin\phi\left(A_{ss} - A_{\phi\phi}\right) + \cos^2\phi A_{\phi s} - \sin^2 A_{s\phi}.
\end{aligned}
$$

Components of $\mathbf{A}$ are regular in cylindrical coordinates, which can be expanded in Fourier series of azimuthal wavenumber. For instance, the A_{ss} component can be expressed as

$$
A_{ss} = \sum_{m=-\infty}^{+\infty} A_{ss}^m(s) e^{im\phi}
$$

where A_{ss}^m is the Fourier coefficient for azimuthal wavenumber m. Expansions of other components naturally follow. Expressing the cosines and sines also in Fourier basis

$$
\cos\phi = \frac{e^{i\phi} + e^{-i\phi}}{2}, \quad \sin\phi = \frac{e^{i\phi} - e^{-i\phi}}{2i},
$$

$$
\cos^2\phi = \frac{e^{i2\phi} + e^{-i2\phi} + 2}{4}, \quad \sin^2\phi = -\frac{e^{i2\phi} + e^{-i2\phi} - 2}{4}, \quad \cos\phi\sin\phi = \frac{e^{i2\phi} - e^{-i2\phi}}{4i}
$$

We see that the tensor elements in Cartesian coordinates have the Fourier expansion

$$
\begin{aligned}
A_{xx} &= \sum_m \frac{e^{im\phi}}{4}\left\{2\left(A_{ss}^m + A_{\phi\phi}^m\right) + \left[A_{ss}^m - A_{\phi\phi}^m - i\left(A_{s\phi}^m + A_{\phi s}^m\right)\right]e^{-i2\phi} + \left[A_{ss}^m - A_{\phi\phi}^m + i\left(A_{s\phi}^m + A_{\phi s}^m\right)\right]e^{i2\phi}\right\} \\
A_{yy} &= \sum_m \frac{e^{im\phi}}{4}\left\{2\left(A_{ss}^m + A_{\phi\phi}^m\right) - \left[A_{ss}^m - A_{\phi\phi}^m - i\left(A_{s\phi}^m + A_{\phi s}^m\right)\right]e^{-i2\phi} - \left[A_{ss}^m - A_{\phi\phi}^m + i\left(A_{s\phi}^m + A_{\phi s}^m\right)\right]e^{i2\phi}\right\} \\
A_{xy} &= \sum_m \frac{e^{im\phi}}{4}\left\{2\left(A_{s\phi}^m - A_{\phi s}^m\right) + \left[A_{s\phi}^m + A_{\phi s}^m + i\left(A_{ss}^m - A_{\phi\phi}^m\right)\right]e^{-i2\phi} + \left[A_{s\phi}^m + A_{\phi s}^m - i\left(A_{ss}^m - A_{\phi\phi}^m\right)\right]e^{i2\phi}\right\} \\
A_{yx} &= \sum_m \frac{e^{im\phi}}{4}\left\{2\left(A_{\phi s}^m - A_{s\phi}^m\right) + \left[A_{s\phi}^m + A_{\phi s}^m + i\left(A_{ss}^m - A_{\phi\phi}^m\right)\right]e^{-i2\phi} + \left[A_{s\phi}^m + A_{\phi s}^m - i\left(A_{ss}^m - A_{\phi\phi}^m\right)\right]e^{i2\phi}\right\}
\end{aligned}
$$

Using these relations, we can deduce from the regularity of A_{xx}, A_{yy}, A_{xy} and A_{yx} that the following fields must also be regular

$$
\begin{aligned}
A_{xx} + A_{yy} &= \sum_m \left(A_{ss}^m + A_{\phi\phi}^m\right)e^{im\phi} \\
A_{xy} - A_{yx} &= \sum_m \left(A_{s\phi}^m - A_{\phi s}^m\right)e^{im\phi} \\
\left(A_{xx} - A_{yy}\right) + i\left(A_{xy} + A_{yx}\right) &= \sum_m \left[A_{ss}^m - A_{\phi\phi}^m + i\left(A_{s\phi}^m + A_{\phi s}^m\right)\right]e^{i(m+2)\phi} \\
\left(A_{xx} - A_{yy}\right) - i\left(A_{xy} + A_{yx}\right) &= \sum_m \left[A_{ss}^m - A_{\phi\phi}^m - i\left(A_{s\phi}^m + A_{\phi s}^m\right)\right]e^{i(m-2)\phi}
\end{aligned}
$$

Plugging in these relations back into the expansion of Cartesian components, we see that these are both necessary AND sufficient conditions for the regularity of the tensor elements under Cartesian coordinates. We can then safely further simplify the relations from here, feeling safe that no information is lost during the process. This procedure is, unfortunately, missing in Lewis and Bellan (1990). Only the terms of A_x are derived before the authors concluded that the respective terms must be regular. In fact, counterinstances are easy to find that does NOT fulfill the regularity constraints BUT yields regular

A_x, say $A_s = \frac{1}{s}(1-\cos 2\phi)$ and $A_\phi = \frac{1}{s}\sin 2\phi$. It is the extra constraints from A_y that jointly pose the constraints. As in Lewis and Bellan (1990), the exponentials can be written as

$$e^{im\phi} = \frac{(x+iy)^{|m|}}{s^{|m|}}.$$

This allows us to pose constraints on the Fourier coefficients $A_{ij}^m(s)$ as functions of cylindrical radius s. The four relations are equivalent to the following four regularity constraints:

$$\begin{aligned} A_{ss}^m + A_{\phi\phi}^m &= s^{|m|}C(s^2) \\ A_{s\phi}^m - A_{\phi s}^m &= s^{|m|}C(s^2) \\ A_{ss}^m - A_{\phi\phi}^m + i\left(A_{s\phi}^m + A_{\phi s}^m\right) &= s^{|m+2|}C(s^2) \\ A_{ss}^m - A_{\phi\phi}^m - i\left(A_{s\phi}^m + A_{\phi s}^m\right) &= s^{|m-2|}C(s^2) \end{aligned} \tag{12}$$

where we already used the symmetry or anti-symmetry in s for Cartesian tensor components. Notation $C(s^2)$ denotes a function of s^2 that is regular at $s=0$, which can be expanded into Taylor series. Now it is time to split the domain of k, $\mathbb{Z}$, into intervals, so as to simplify the relations. We see that the absolute value functions can be completely removed in each scenario if we split the domain into $m \leq -2$, $m=-1$, $m=0$, $m=1$ and $m \geq 2$. The treaments of negative and positive m are highly similar, and I shall only write out the positive branch in detail. For $m \geq 2$, we can substract the two latter equations in eq.(12) and obtain $A_{s\phi}^m + A_{\phi s}^m \sim s^{m-2}$; combining this with the second equation,

$$\begin{cases} A_{s\phi}^m + A_{\phi s}^m = s^{m-2}C(s^2) \\ A_{s\phi}^m - A_{\phi s}^m = s^m C(s^2) \end{cases} \implies \begin{cases} A_{s\phi}^m = A_{s\phi}^{m0}s^{m-2} + A_{s\phi}^{m1}s^m + s^{m+2}C(s^2) \\ A_{\phi s}^m = A_{\phi s}^{m0}s^{m-2} + A_{\phi s}^{m1}s^m + s^{m+2}C(s^2) \end{cases} \quad \text{and} \quad A_{s\phi}^{m0} = A_{\phi s}^{m0}.$$

Thus simultaneously we obtain the ansätze (this is in fact the required form for regularity) for $A_{s\phi}$ and $A_{\phi s}$, as well as a coupling condition. The second superscript on A_{ij}^{mn} gives the index for power series expansion in s. On the other hand, we can add the latter two equations of eq.(12) and combine with the first equation to similarly come up with

$$\begin{cases} A_{ss}^m + A_{\phi\phi}^m = s^m C(s^2) \\ A_{ss}^m - A_{\phi\phi}^m = s^{m-2}C(s^2) \end{cases} \implies \begin{cases} A_{ss}^m = A_{ss}^{m0}s^{m-2} + A_{ss}^{m1}s^m + s^{m+2}C(s^2) \\ A_{\phi\phi}^m = A_{\phi\phi}^{m0}s^{m-2} + A_{\phi\phi}^{m1}s^m + s^{m+2}C(s^2) \end{cases} \quad \text{and} \quad A_{ss}^{m0} = -A_{\phi\phi}^{m0}.$$

Finally, we reuse the third equation in eq.(12) to establish the relation between the coefficients for the diagonal and the off-diagonal elements. To make sure both s^{m-2} and s^m vanishes on the LHS,

$$\begin{aligned} A_{ss}^{m0} - A_{\phi\phi}^{m0} + i\left(A_{s\phi}^{m0} + A_{\phi s}^{m0}\right) = 0, \quad &\implies \quad A_{s\phi}^{m0} = iA_{ss}^{m0} \\ &A_{ss}^{m1} - A_{\phi\phi}^{m1} + i\left(A_{s\phi}^{m1} + A_{\phi s}^{m1}\right) = 0 \end{aligned}$$

These are the four regularity constraints for $m \geq 2$. With all the ansätze, it can be easily verified that as long as the coefficients fulfill these constraints, the target terms indeed satisfy eq.(12), and thus these ansätze and constraints are also sufficient conditions.

Next, we take a look at the situation where $m=1$. The latter two equations now yield

$$\begin{cases} A_{s\phi}^1 + A_{\phi s}^1 = sC(s^2) \\ A_{s\phi}^1 - A_{\phi s}^1 = sC(s^2) \end{cases} \implies \begin{cases} A_{s\phi}^1 = A_{s\phi}^{10}s + s^3C(s^2) \\ A_{\phi s}^1 = A_{\phi s}^{10}s + s^3C(s^2). \end{cases}$$

Apparently, no constraints are required; the ansatz alone suffices to enforce the correct leading power of s. This is equally true for A_{ss} and $A_{\phi\phi}$,

$$\begin{cases} A_{ss}^1 + A_{\phi\phi}^1 = s^1C(s^2) \\ A_{ss}^1 - A_{\phi\phi}^1 = s^1C(s^2) \end{cases} \implies \begin{cases} A_{ss}^1 = A_{ss}^{10}s + s^3C(s^2) \\ A_{\phi\phi}^1 = A_{\phi\phi}^{10}s + s^3C(s^2). \end{cases}$$

However, the last constraint still holds, that is we still need that the first-order term in s of $A^1_{ss} - A^1_{\phi\phi}$ and $i\left(A^1_{s\phi} + A^1_{\phi s}\right)$ cancel each other out,

$$A^{10}_{ss} - A^{10}_{\phi\phi} + i\left(A^{10}_{s\phi} + A^{10}_{\phi s}\right) = 0.$$

These constraints are absent from Holdenried-Chernoff (2021) (note here we are not yet assuming $A_{s\phi} = A_{\phi s}$).

Finally, we arrive at the $m = 0$ case.

$$\begin{cases} A^0_{s\phi} + A^0_{\phi s} = s^2 C(s^2) \\ A^0_{s\phi} - A^0_{\phi s} = C(s^2) \end{cases} \implies \begin{cases} A^0_{s\phi} = A^{00}_{s\phi} + s^2 C(s^2) \\ A^0_{\phi s} = A^{00}_{\phi s} + s^2 C(s^2) \end{cases} \text{and} \quad A^{00}_{s\phi} = -A^{00}_{\phi s}.$$

$$\begin{cases} A^0_{ss} + A^0_{\phi\phi} = C(s^2) \\ A^0_{ss} - A^0_{\phi\phi} = s^2 C(s^2) \end{cases} \implies \begin{cases} A^0_{ss} = A^{00}_{ss} + s^2 C(s^2) \\ A^0_{\phi\phi} = A^{00}_{\phi\phi} + s^2 C(s^2) \end{cases} \text{and} \quad A^{00}_{ss} = A^{00}_{\phi\phi}.$$

The third and the fourth equation in eq.(12) give the relations

$$\begin{cases} A^{00}_{ss} - A^{00}_{\phi\phi} + i\left(A^{00}_{s\phi} + A^{00}_{\phi s}\right) = 0 \\ A^{00}_{ss} - A^{00}_{\phi\phi} - i\left(A^{00}_{s\phi} + A^{00}_{\phi s}\right) = 0 \end{cases}$$

which are automatically satisfied given the previous ansätze. The negative m scenarios are also similarly derived. In the end, the required leading order and the constraints are summarized as follows

$$\begin{aligned}
m = 0: &\quad \begin{cases} A^0_{ss} = A^{00}_{ss} + s^2 C(s^2) \\ A^0_{\phi\phi} = A^{00}_{\phi\phi} + s^2 C(s^2) \\ A^0_{s\phi} = A^{00}_{s\phi} + s^2 C(s^2) \\ A^0_{\phi s} = A^{00}_{\phi s} + s^2 C(s^2) \end{cases}, \quad \begin{cases} A^{00}_{ss} = A^{00}_{\phi\phi} \\ A^{00}_{s\phi} = -A^{00}_{\phi s} \end{cases} \\
|m| = 1: &\quad \begin{cases} A^m_{ss} = A^{m0}_{ss} s + s^3 C(s^2) \\ A^m_{\phi\phi} = A^{m0}_{\phi\phi} s + s^3 C(s^2) \\ A^m_{s\phi} = A^{m0}_{s\phi} s + s^3 C(s^2) \\ A^m_{\phi s} = A^{m0}_{\phi s} s + s^3 C(s^2) \end{cases}, \quad \left\{ A^{m0}_{s\phi} + A^{m0}_{\phi s} = i\,\mathrm{sgn}(m)\left(A^{m0}_{ss} - A^{m0}_{\phi\phi}\right) \right. \\
|m| \geq 2: &\quad \begin{cases} A^m_{ss} = A^{m0}_{ss} s^{|m|-2} + A^{m1}_{ss} s^{|m|} + s^{|m|+2} C(s^2) \\ A^m_{\phi\phi} = A^{m0}_{\phi\phi} s^{|m|-2} + A^{m1}_{\phi\phi} s^{|m|} + s^{|m|+2} C(s^2) \\ A^m_{s\phi} = A^{m0}_{s\phi} s^{|m|-2} + A^{m1}_{s\phi} s^{|m|} + s^{|m|+2} C(s^2) \\ A^m_{\phi s} = A^{m0}_{\phi s} s^{|m|-2} + A^{m1}_{\phi s} s^{|m|} + s^{|m|+2} C(s^2) \end{cases}, \quad \begin{cases} A^{m0}_{ss} = -A^{m0}_{\phi\phi} \\ A^{m0}_{s\phi} = A^{m0}_{\phi s} \\ A^{m0}_{s\phi} = i\,\mathrm{sgn}(m) A^{m0}_{ss} \\ A^{m1}_{s\phi} + A^{m1}_{\phi s} = i\,\mathrm{sgn}(m)\left(A^{m1}_{ss} - A^{m1}_{\phi\phi}\right). \end{cases}
\end{aligned} \tag{13}$$

In many cases, it is further useful to assume symmetry of the tensor; this is the case with e.g. strain tensor ε, strain-rate tensor $\dot{\varepsilon}$, stress tensor $\boldsymbol{\sigma}$, and of course for our problem, Maxwell stress $\boldsymbol{\sigma}^M$. In this case $A_{s\phi} = A_{\phi s}$, and all coefficients of their power series in s should match. However, for $m = 0$ we have $A^{00}_{s\phi} = -A^{00}_{\phi s}$. The result is that $A^0_{s\phi} = A^0_{\phi s}$, when expanded in power series of s, has leading order s^2 instead of s^0. In addition, some original constraints will render redundant. In the end, the ansätze and the regularity constraints for symmetric rank-2 tensors are given by

$$\begin{aligned}
m = 0: \quad & \begin{cases} A^0_{ss} = A^{00}_{ss} + s^2 C(s^2) \\ A^0_{\phi\phi} = A^{00}_{\phi\phi} + s^2 C(s^2) \\ A^0_{s\phi} = A^{00}_{s\phi} s^2 + s^4 C(s^2) \end{cases}, \quad \begin{cases} A^{00}_{ss} = A^{00}_{\phi\phi} \end{cases} \\
|m| = 1: \quad & \begin{cases} A^m_{ss} = A^{m0}_{ss} s + s^3 C(s^2) \\ A^m_{\phi\phi} = A^{m0}_{\phi\phi} s + s^3 C(s^2) \\ A^m_{s\phi} = A^{m0}_{s\phi} s + s^3 C(s^2) \end{cases}, \quad \begin{cases} 2A^{m0}_{s\phi} = i \,\mathrm{sgn}(m) \left(A^{m0}_{ss} - A^{m0}_{\phi\phi}\right) \end{cases} \\
|m| \geq 2: \quad & \begin{cases} A^m_{ss} = A^{m0}_{ss} s^{|m|-2} + A^{m1}_{ss} s^{|m|} + s^{|m|+2} C(s^2) \\ A^m_{\phi\phi} = A^{m0}_{\phi\phi} s^{|m|-2} + A^{m1}_{\phi\phi} s^{|m|} + s^{|m|+2} C(s^2) \\ A^m_{s\phi} = A^{m0}_{s\phi} s^{|m|-2} + A^{m1}_{s\phi} s^{|m|} + s^{|m|+2} C(s^2) \end{cases}, \quad \begin{cases} A^{m0}_{ss} = -A^{m0}_{\phi\phi} \\ A^{m0}_{s\phi} = i \,\mathrm{sgn}(m) A^{m0}_{ss} \\ 2A^{m1}_{s\phi} = i \,\mathrm{sgn}(m) \left(A^{m1}_{ss} - A^{m1}_{\phi\phi}\right). \end{cases}
\end{aligned} \tag{14}$$

These ansätze are consistent with the leading order behaviour of the equatorial magnetic moments documented in Holdenried-Chernoff (2021). However, the five constraints on the equatorial magnetic moments derived here form a proper superset of the constraints in Holdenried-Chernoff (2021). Specfi-cially, two of these relations are absent in the dissertation, namely

$$\begin{aligned}
2A^{m0}_{s\phi} &= i \,\mathrm{sgn}(m) \left(A^{m0}_{ss} - A^{m0}_{\phi\phi}\right), \quad |m| = 1; \\
2A^{m1}_{s\phi} &= i \,\mathrm{sgn}(m) \left(A^{m1}_{ss} - A^{m1}_{\phi\phi}\right), \quad |m| \geq 2.
\end{aligned}$$

The first of these two has been rediscovered in the previous section by re-deriving the formulae. The second relation cannot be discovered as long as we only consider the relation between lowest order behaviours. In fact, from this we see that there are regularity constraints even on the second-order term in the Taylor expansion in s.

It should be noted that the derivations above ONLY considered regularity of the tensor fields. However, magnetic moments **BB** are formed by outer product of the magnetic field **B**. In other words, the magnetic moment tensor is the rank-1 transformation of the magnetic field

$$\begin{pmatrix} B_x^2 & B_x B_y \\ B_y B_x & B_y^2 \end{pmatrix} = \begin{pmatrix} B_x \\ B_y \end{pmatrix} \begin{pmatrix} B_x \\ B_y \end{pmatrix}^\intercal, \qquad \begin{pmatrix} B_s^2 & B_s B_\phi \\ B_\phi B_s & B_\phi^2 \end{pmatrix} = \begin{pmatrix} B_s \\ B_\phi \end{pmatrix} \begin{pmatrix} B_s \\ B_\phi \end{pmatrix}^\intercal$$

This constraints is not imposed in the derivations above, which assumes arbitrary tensor field. It thus poses a question that if we expand B_s^2, B_ϕ^2 and $B_s B_\phi$ separately, are we artificially expanding the image of field to moment mapping. Part of the space formed by the expansions might not have underlying magnetic fields (i.e. not surjective). [This problem requires further notice.]

References

Holdenried-Chernoff, Daria (2021). “The long and short timescale dynamics of planetary magnetic fields”. en. Accepted: 2021-10-15T06:43:38Z. Doctoral Thesis. ETH Zurich. DOI: 10.3929/ethz-b-000509840. URL: https://www.research-collection.ethz.ch/handle/20.500.11850/509840 (visited on 07/26/2023).

Jackson, Andrew and Stefano Maffei (Nov. 2020). “Plesio-geostrophy for Earth’s core: I. Basic equations, inertial modes and induction”. In: *Proceedings of the Royal Society A: Mathematical, Physical and Engineering Sciences* 476.2243. Publisher: Royal Society, p. 20200513. DOI: 10.1098/rspa.2020.0513. URL: https://royalsocietypublishing.org/doi/10.1098/rspa.2020.0513 (visited on 02/13/2023).

Lewis, H. Ralph and Paul M. Bellan (Nov. 1990). “Physical constraints on the coefficients of Fourier expansions in cylindrical coordinates”. In: *Journal of Mathematical Physics* 31.11. Publisher: American Institute of Physics, pp. 2592–2596. ISSN: 0022-2488. DOI: 10.1063/1.529009. URL: https://aip.scitation.org/doi/10.1063/1.529009 (visited on 02/17/2023).