Plesio-Geostrophy and Data Assimilation: Formulations

Missing ingredients and new recipes

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Chapter 1

Theory and governing equations

1.1 Spherical-Cylindrical transformation

Spherical coordinates and cylindrical coordinates can be transformed to one another via

$$\begin{pmatrix} s \\ \phi \\ z \end{pmatrix} = \begin{pmatrix} r\sin\theta \\ \phi \\ r\cos\theta \end{pmatrix}, \quad \begin{pmatrix} r \\ \theta \\ \phi \end{pmatrix} = \begin{pmatrix} \sqrt{s^2+z^2} \\ \arccos\frac{z}{\sqrt{s^2+z^2}} \\ \phi \end{pmatrix} \tag{1.1}$$

The vector components are converted using the rotation matrix

$$\begin{pmatrix} A_s \\ A_\phi \\ A_z \end{pmatrix} = \mathbf{R}\begin{pmatrix} A_r \\ A_\theta \\ A_\phi \end{pmatrix}, \quad \begin{pmatrix} A_r \\ A_\theta \\ A_\phi \end{pmatrix} = \mathbf{R}^T\begin{pmatrix} A_s \\ A_\phi \\ A_z \end{pmatrix} \tag{1.2}$$

where the rotation matrix is an orthogonal matrix

$$\mathbf{R} = \begin{pmatrix} \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \\ \cos\theta & -\sin\theta & 0 \end{pmatrix} = \begin{pmatrix} \frac{s}{\sqrt{s^2+z^2}} & \frac{z}{\sqrt{s^2+z^2}} & 0 \\ 0 & 0 & 1 \\ \frac{z}{\sqrt{s^2+z^2}} & -\frac{s}{\sqrt{s^2+z^2}} & 0 \end{pmatrix}. \tag{1.3}$$

The Jacobian from spherical to cylindrical coordinates and its inverse, i.e. the Jacobian from cylindrical to spherical coordinates are given by

$$\begin{aligned} \mathbf{J} = \frac{\partial(s,\phi,z)}{\partial(r,\theta,\phi)} &= \begin{pmatrix} \sin\theta & r\cos\theta & 0 \\ 0 & 0 & 1 \\ \cos\theta & -r\sin\theta & 0 \end{pmatrix} = \begin{pmatrix} \frac{s}{\sqrt{s^2+z^2}} & z & 0 \\ 0 & 0 & 1 \\ \frac{z}{\sqrt{s^2+z^2}} & -s & 0 \end{pmatrix}, \\ \mathbf{J}^{-1} = \frac{\partial(r,\theta,\phi)}{\partial(s,\phi,z)} &= \begin{pmatrix} \sin\theta & 0 & \cos\theta \\ r^{-1}\cos\theta & 0 & -r^{-1}\sin\theta \\ 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} \frac{s}{\sqrt{s^2+z^2}} & 0 & \frac{z}{\sqrt{s^2+z^2}} \\ \frac{z}{s^2+z^2} & 0 & -\frac{s}{s^2+z^2} \\ 0 & 1 & 0 \end{pmatrix}. \end{aligned} \tag{1.4}$$

The derivatives in spherical harmonics are transformed into derivatives in cylindrical coordinates via

$$\begin{pmatrix} \partial_r \\ \partial_\theta \\ \partial_\phi \end{pmatrix} = \frac{\partial(s,\phi,z)}{\partial(r,\theta,\phi)}^T \begin{pmatrix} \partial_s \\ \partial_\phi \\ \partial_z \end{pmatrix} = \begin{pmatrix} \sin\theta & 0 & \cos\theta \\ r\cos\theta & 0 & -r\sin\theta \\ 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} \partial_s \\ \partial_\phi \\ \partial_z \end{pmatrix} = \begin{pmatrix} \frac{s}{\sqrt{s^2+z^2}} & 0 & \frac{z}{\sqrt{s^2+z^2}} \\ z & 0 & -s \\ 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} \partial_s \\ \partial_\phi \\ \partial_z \end{pmatrix} \tag{1.5}$$

where the matrix elements are already changed to cylindrical coordinates. Inversely, we have

$$\begin{pmatrix} \partial_s \\ \partial_\phi \\ \partial_z \end{pmatrix} = \frac{\partial(r,\theta,\phi)}{\partial(s,\phi,z)}^T \begin{pmatrix} \partial_r \\ \partial_\theta \\ \partial_\phi \end{pmatrix} = \begin{pmatrix} \sin\theta & \frac{1}{r}\cos\theta & 0 \\ 0 & 0 & 1 \\ \cos\theta & -\frac{1}{r}\sin\theta & 0 \end{pmatrix}\begin{pmatrix} \partial_r \\ \partial_\theta \\ \partial_\phi \end{pmatrix} = \begin{pmatrix} \frac{s}{\sqrt{s^2+z^2}} & \frac{z}{s^2+z^2} & 0 \\ 0 & 0 & 1 \\ \frac{z}{\sqrt{s^2+z^2}} & -\frac{s}{s^2+z^2} & 0 \end{pmatrix}\begin{pmatrix} \partial_r \\ \partial_\theta \\ \partial_\phi \end{pmatrix} \tag{1.6}$$

where the matrix elements in spherical coordinates are also shown.

1.2 Vorticity equation

In this section I derive an alternative form of the vorticity equation, which is the starting point of some eigenvalue problems. Starting from the dimensionless form of the vorticity equation, we have

$$-2\nabla_e^2 \frac{\partial \psi}{\partial t} = \frac{dH}{ds}\left(\frac{4}{sH}\frac{\partial \psi}{\partial \phi} - \frac{2}{H}\frac{\partial}{\partial s}\frac{\partial \psi}{\partial t} - \frac{1}{sH}\frac{\partial^2}{\partial \phi^2}\frac{\partial \psi}{\partial t}\right) - \frac{dH}{ds}\left(2f_\phi^e + \frac{1}{s}\frac{\partial \widetilde{f_z}}{\partial \phi}\right) + \hat{\mathbf{z}}\cdot\nabla\times\overline{\mathbf{f}_e}. \tag{1.7}$$

The superscript e means the field is evaluated on the equatorial plane. Here we used the dimensionless form as in Jackson and Maffei (2020), where the characteristic time scale is chosen to be the rotation time scale Ω^{-1} (the "inertial time scale"), instead of the Alfvén time scale L/V_A, as in Holdenried-Chernoff (2021). The force $\mathbf{f}$ contains all the external forces on the right-hand-side of the Navier-Stokes equation, e.g. Lorentz force, viscous force, buoyancy, etc. For the eigenvalue problem, it is convenient to move the terms involving all the time derivatives to one side,

$$\left[-2\nabla_e^2 + \frac{dH}{ds}\left(\frac{2}{H}\frac{\partial}{\partial s} + \frac{1}{sH}\frac{\partial^2}{\partial \phi^2}\right)\right]\frac{\partial \psi}{\partial t} = \frac{4}{sH}\frac{dH}{ds}\frac{\partial \psi}{\partial \phi} - \frac{dH}{ds}\left(2f_\phi^e + \frac{1}{s}\frac{\partial \widetilde{f_z}}{\partial \phi}\right) + \hat{\mathbf{z}}\cdot\nabla\times\overline{\mathbf{f}_e}$$

$$\left[-\frac{1}{s}\frac{\partial}{\partial s}\left(s\frac{\partial}{\partial s}\right) + \frac{1}{H}\frac{dH}{ds}\frac{\partial}{\partial s} + \left(\frac{1}{2sH}\frac{dH}{ds} - \frac{1}{s^2}\right)\frac{\partial^2}{\partial \phi^2}\right]\frac{\partial \psi}{\partial t} = \frac{2}{sH}\frac{dH}{ds}\frac{\partial \psi}{\partial \phi} - \frac{dH}{ds}\left(f_\phi^e + \frac{1}{2s}\frac{\partial \widetilde{f_z}}{\partial \phi}\right) + \frac{\hat{\mathbf{z}}}{2}\cdot\nabla\times\overline{\mathbf{f}_e}$$

In cases where different azimuthal wavenumber separates (e.g. when the system has rotational invariance with respect to ϕ), this equation will be readily converted to an ordinary differential equation (ODE) in s. In this case, it would be desirable to write the differential operators concerning s in the self-adjoint form $\frac{d}{ds}(p(s)\frac{d}{ds})$, to form a standard Sturm-Liouville problem,

$$-\frac{1}{s}\frac{\partial}{\partial s}\left(s\frac{\partial}{\partial s}\right) + \frac{1}{H}\frac{dH}{ds}\frac{\partial}{\partial s} = -\frac{\partial^2}{\partial s^2} - \left(\frac{1}{s} - \frac{1}{H}\frac{dH}{ds}\right)\frac{\partial}{\partial s}$$

and we can deduce the term $p(x)$ using the relation

$$\frac{1}{p(s)}\frac{dp(s)}{ds} = \frac{1}{s} - \frac{1}{H}\frac{dH}{ds} \quad\Longrightarrow\quad d\ln p = d\ln s - d\ln H = d\ln\frac{s}{H} \quad\Longrightarrow\quad p = \frac{s}{H}.$$

And the original equation can be rewritten as

$$\left[\frac{\partial}{\partial s}\left(\frac{s}{H}\frac{\partial}{\partial s}\right) + \left(\frac{1}{sH} - \frac{1}{2H^2}\frac{dH}{ds}\right)\frac{\partial^2}{\partial \phi^2}\right]\frac{\partial \psi}{\partial t} = -\frac{2}{H^2}\frac{dH}{ds}\frac{\partial \psi}{\partial \phi} + \frac{dH}{ds}\left(\frac{s}{H}f_\phi^e + \frac{1}{2H}\frac{\partial \widetilde{f_z}}{\partial \phi}\right) - \frac{s}{2H}\hat{\mathbf{z}}\cdot\nabla\times\overline{\mathbf{f}_e} \tag{1.8}$$

1.3 Conversion between the two dimensionless forms

$$L = r_0, \quad \tau \sim \frac{L}{U}$$

Holdenried-Chernoff (2021) uses the Alfven wave velocity

$$U = V_A = \frac{\mathscr{B}}{\sqrt{\rho_0\mu_0}}$$

as the characteristic velocity scale. This scale is ultimately determined by the characteristic magnetic field strength $\mathscr{B}$. The relative strength of Lorentz force to Coriolis is given by the Lehnert number,

$$\mathrm{Le} = \frac{\mathscr{B}}{\sqrt{\rho_0\mu_0}\Omega L} = \frac{V_A}{V_\Omega}$$

where $V_\Omega = \Omega L$ is the rotation speed. In other words, the characteristic scales for the magnetic field and the spin rate are linked via this dimensionless number. The dimensionless Navier-Stokes equation takes the form

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u}\cdot\nabla\mathbf{u} + \frac{2}{\mathrm{Le}}\hat{\mathbf{z}}\times\mathbf{u} = -\nabla p + (\nabla\times\mathbf{B})\times\mathbf{B}$$

and the vorticity equation is

$$-2\nabla_e^2\frac{\partial\psi}{\partial t} = \frac{dH}{ds}\left(\mathrm{Le}^{-1}\frac{4}{sH}\frac{\partial\psi}{\partial\phi} - \frac{2}{H}\frac{\partial}{\partial s}\frac{\partial\psi}{\partial t} - \frac{1}{sH}\frac{\partial^2}{\partial\phi^2}\frac{\partial\psi}{\partial t}\right) - \frac{dH}{ds}\left(2f_{e\phi} + \frac{1}{s}\frac{\partial\widetilde{f_\phi}}{\partial\phi}\right) + \hat{\mathbf{z}}\cdot\nabla\times\overline{\mathbf{f}_e}$$

$$\left[\frac{\partial}{\partial s}\left(\frac{s}{H}\frac{\partial}{\partial s}\right) + \left(\frac{1}{sH} - \frac{1}{2H^2}\frac{dH}{ds}\right)\frac{\partial^2}{\partial\phi^2}\right]\frac{\partial\psi}{\partial t} = -\mathrm{Le}^{-1}\frac{2}{H^2}\frac{dH}{ds}\frac{\partial\psi}{\partial\phi} + \frac{dH}{ds}\left(\frac{s}{H}f_\phi^e + \frac{1}{2H}\frac{\partial\widetilde{f_z}}{\partial\phi}\right) - \frac{s}{2H}\hat{\mathbf{z}}\cdot\nabla\times\overline{\mathbf{f}_e}$$

In constrast, Jackson and Maffei (2020) uses

$$U = \Omega L$$

as the characteristic velocity, which is particularly useful when the magnetic field is absent. Now it is necessary to nondimensionalize the Lorentz force. While Jackson and Maffei (2020) uses $\mathscr{B} = \sqrt{\rho_0\mu_0}\Omega L$, meaning $\mathrm{Le} = 1$ in the paper, it lacks flexibility. Instead, we can still use Lehnert number, in which case the Navier-Stokes equation takes the form

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u}\cdot\nabla\mathbf{u} + 2\hat{\mathbf{z}}\times\mathbf{u} = -\nabla p + \mathrm{Le}^2(\nabla\times\mathbf{B})\times\mathbf{B}$$

and the vorticity equation is

$$-2\nabla_e^2\frac{\partial\psi}{\partial t} = \frac{dH}{ds}\left(\frac{4}{sH}\frac{\partial\psi}{\partial\phi} - \frac{2}{H}\frac{\partial}{\partial s}\frac{\partial\psi}{\partial t} - \frac{1}{sH}\frac{\partial^2}{\partial\phi^2}\frac{\partial\psi}{\partial t}\right) - \mathrm{Le}^2\left[\frac{dH}{ds}\left(2f_{e\phi} + \frac{1}{s}\frac{\partial\widetilde{f_\phi}}{\partial\phi}\right) + \hat{\mathbf{z}}\cdot\nabla\times\overline{\mathbf{f}_e}\right]$$

$$\left[\frac{\partial}{\partial s}\left(\frac{s}{H}\frac{\partial}{\partial s}\right) + \left(\frac{1}{sH} - \frac{1}{2H^2}\frac{dH}{ds}\right)\frac{\partial^2}{\partial\phi^2}\right]\frac{\partial\psi}{\partial t} = -\frac{2}{H^2}\frac{dH}{ds}\frac{\partial\psi}{\partial\phi} + \mathrm{Le}^2\left[\frac{dH}{ds}\left(\frac{s}{H}f_\phi^e + \frac{1}{2H}\frac{\partial\widetilde{f_z}}{\partial\phi}\right) - \frac{s}{2H}\hat{\mathbf{z}}\cdot\nabla\times\overline{\mathbf{f}_e}\right]$$

The variables solved in two dimensionless forms can be easily converted to one another,

$$\mathbf{u}_\Omega = \frac{\mathbf{u}_A}{\Omega L}\frac{\mathscr{B}}{\sqrt{\rho_0\mu_0}} = \mathrm{Le}\,\mathbf{u}_A, \qquad t_\Omega = \Omega\frac{\sqrt{\rho_0\mu_0}L}{\mathscr{B}}t_A = \frac{t_A}{\mathrm{Le}}, \qquad \mathbf{B}_\Omega = \mathbf{B}_A.$$

Here the A and Ω subscripts indicate dimensionless fields in the equations nondimensionalized using Alfven wave velocity and rotation velocity, respectively. Finally, for the eigenvalue problem, the eigenvalues solved follow the following relation, inverse to t:

$$\omega_\Omega = \frac{\mathscr{B}}{\sqrt{\rho_0\mu_0}\Omega L}\omega_A = \mathrm{Le}\,\omega_A.$$

1.4 Induction equation at the boundary

Induction equation for the radial component at the boundary

$$\frac{\partial B_r}{\partial t} = -\nabla_H\cdot(\mathbf{u}_H B_r) \tag{1.9}$$

which can be expanded in spherical coordinates

$$\frac{\partial B_r}{\partial t} = -\frac{1}{\sin\theta}\left(\frac{\partial}{\partial\theta}(\sin\theta\, u_\theta B_r) + \frac{\partial}{\partial\phi}(u_\phi B_r)\right).$$

Alternatively, we can also use the induction equation in cylindrical coordinates at the boundary. These quantities are

$$
\begin{aligned}
\frac{\partial B_s}{\partial t} &= (\mathbf{B}\cdot\nabla\mathbf{u})_s - (\mathbf{u}\cdot\nabla\mathbf{B})_s = B_s\frac{\partial u_s}{\partial s} + \frac{B_\phi}{s}\frac{\partial u_s}{\partial \phi} + B_z\frac{\partial u_s}{\partial z} - u_s\frac{\partial B_s}{\partial s} - \frac{u_\phi}{s}\frac{\partial B_s}{\partial \phi} - u_z\frac{\partial B_s}{\partial z}\\
\frac{\partial B_\phi}{\partial t} &= (\mathbf{B}\cdot\nabla\mathbf{u})_\phi - (\mathbf{u}\cdot\nabla\mathbf{B})_\phi = B_s\frac{\partial u_\phi}{\partial s} + \frac{B_\phi}{s}\frac{\partial u_\phi}{\partial \phi} + B_z\frac{\partial u_\phi}{\partial z} - u_s\frac{\partial B_\phi}{\partial s} - \frac{u_\phi}{s}\frac{\partial B_\phi}{\partial \phi} - u_z\frac{\partial B_\phi}{\partial z} + \frac{B_\phi u_s - u_\phi B_s}{s}\\
\frac{\partial B_z}{\partial t} &= (\mathbf{B}\cdot\nabla\mathbf{u})_z - (\mathbf{u}\cdot\nabla\mathbf{B})_z = B_s\frac{\partial u_z}{\partial s} + \frac{B_\phi}{s}\frac{\partial u_z}{\partial \phi} + B_z\frac{\partial u_z}{\partial z} - u_s\frac{\partial B_z}{\partial s} - \frac{u_\phi}{s}\frac{\partial B_z}{\partial \phi} - u_z\frac{\partial B_z}{\partial z}
\end{aligned}
\tag{1.10}
$$

At the boundary, the induction equation takes the form

$$
\begin{aligned}
\frac{\partial B_s^\pm}{\partial t} &= B_s^\pm\frac{\partial u_s}{\partial s}\bigg|_{\pm H} + \frac{B_\phi^\pm}{s}\frac{\partial u_s^\pm}{\partial \phi} + B_z^\pm\frac{\partial u_s}{\partial z}\bigg|_{\pm H} - u_s^\pm\frac{\partial B_s}{\partial s}\bigg|_{\pm H} - \frac{u_\phi^\pm}{s}\frac{\partial B_s^\pm}{\partial \phi} - u_z^\pm\frac{\partial B_s}{\partial z}\bigg|_{\pm H}\\
\frac{\partial B_\phi^\pm}{\partial t} &= B_s^\pm\frac{\partial u_\phi}{\partial s}\bigg|_{\pm H} + \frac{B_\phi^\pm}{s}\frac{\partial u_\phi^\pm}{\partial \phi} + B_z^\pm\frac{\partial u_\phi}{\partial z}\bigg|_{\pm H} - u_s^\pm\frac{\partial B_\phi}{\partial s}\bigg|_{\pm H} - \frac{u_\phi^\pm}{s}\frac{\partial B_\phi^\pm}{\partial \phi} - u_z^\pm\frac{\partial B_\phi}{\partial z}\bigg|_{\pm H} + \frac{B_\phi^\pm u_s^\pm - u_\phi^\pm B_s^\pm}{s}\\
\frac{\partial B_z^\pm}{\partial t} &= B_s^\pm\frac{\partial u_z}{\partial s}\bigg|_{\pm H} + \frac{B_\phi^\pm}{s}\frac{\partial u_z^\pm}{\partial \phi} + B_z^\pm\frac{\partial u_z}{\partial z}\bigg|_{\pm H} - u_s^\pm\frac{\partial B_z}{\partial s}\bigg|_{\pm H} - \frac{u_\phi^\pm}{s}\frac{\partial B_z^\pm}{\partial \phi} - u_z^\pm\frac{\partial B_z}{\partial z}\bigg|_{\pm H}
\end{aligned}
$$

where the $\pm$ superscript shows that the quantity is evaluated at the boundary $z = \pm H$. Terms in the form $\frac{\partial}{\partial s}|_{\pm H}$ means the field has to be differentiated first and evaluated at the boundary later. Hence, we see that these evolution equations are not closed in themselves, in the sense that the derivatives $\partial_s\mathbf{B}$ and $\partial_z\mathbf{B}$ cannot be evaluated or represented unless the field $\mathbf{B}$ can be evaulated or represented in the entire volume in the parameterization. This is not the case with the PG model, where the parameterization of the magnetic quantities only involves the integrated moments, the boundary field, and the equatorial field. Therefore, these equations cannot be used for time stepping or simulation. The only way to move forward seems to use eq.(1.9). This equation only involves surface operators (owing to the non-penetration condition $u_r = \hat{\mathbf{n}}\cdot u = 0$), and is closed on the surface of the sphere.

Nevertheless, using the induction equation of the cylindrical components can be very useful in solving eigenvalue problems where the background velocity field is zero (it is almost the case with the eigenvalue problems of interest). In these problems, the linearized version of the equation will only involve cross terms of the background magnetic field, whose derivatives are known everywhere in space, and the perturbed velocity field. If we keep the notation u for perturbational velocity, and introduce notation b for perturbational magnetic field, the linearized induction equation takes the form

$$
\begin{aligned}
\frac{\partial b_s^\pm}{\partial t} &= B_s^{0\pm}\frac{\partial u_s}{\partial s}\bigg|_{\pm H} + \frac{B_\phi^{0\pm}}{s}\frac{\partial u_s^\pm}{\partial \phi} + B_z^{0\pm}\frac{\partial u_s}{\partial z}\bigg|_{\pm H} - u_s^\pm\frac{\partial B_s^0}{\partial s}\bigg|_{\pm H} - \frac{u_\phi^\pm}{s}\frac{\partial B_s^{0\pm}}{\partial \phi} - u_z^\pm\frac{\partial B_s^0}{\partial z}\bigg|_{\pm H}\\
\frac{\partial b_\phi^\pm}{\partial t} &= B_s^{0\pm}\frac{\partial u_\phi}{\partial s}\bigg|_{\pm H} + \frac{B_\phi^{0\pm}}{s}\frac{\partial u_\phi^\pm}{\partial \phi} + B_z^{0\pm}\frac{\partial u_\phi}{\partial z}\bigg|_{\pm H} - u_s^\pm\frac{\partial B_\phi^0}{\partial s}\bigg|_{\pm H} - \frac{u_\phi^\pm}{s}\frac{\partial B_\phi^{0\pm}}{\partial \phi} - u_z^\pm\frac{\partial B_\phi^0}{\partial z}\bigg|_{\pm H} + \frac{B_\phi^{0\pm} u_s^\pm - u_\phi^\pm B_s^{0\pm}}{s}\\
\frac{\partial b_z^\pm}{\partial t} &= B_s^{0\pm}\frac{\partial u_z}{\partial s}\bigg|_{\pm H} + \frac{B_\phi^{0\pm}}{s}\frac{\partial u_z^\pm}{\partial \phi} + B_z^{0\pm}\frac{\partial u_z}{\partial z}\bigg|_{\pm H} - u_s^\pm\frac{\partial B_z^0}{\partial s}\bigg|_{\pm H} - \frac{u_\phi^\pm}{s}\frac{\partial B_z^{0\pm}}{\partial \phi} - u_z^\pm\frac{\partial B_z^0}{\partial z}\bigg|_{\pm H}
\end{aligned}
$$

Recall that in the plesio-geostrophic ansatz for the velocity field, $\mathbf{u}_e = \frac{1}{H}\nabla\times\psi\hat{\mathbf{z}}$, $u_z = \frac{z}{H}\frac{dH}{ds}u_s$ and the stream function ψ is z-invariant. Therefore, the equations can be simplified as

$$\begin{aligned}
\frac{\partial b_s^\pm}{\partial t} &= B_s^{0\pm}\frac{\partial}{\partial s}\left(\frac{1}{sH}\frac{\partial\psi}{\partial\phi}\right) + \frac{B_\phi^{0\pm}}{s^2H}\frac{\partial^2\psi}{\partial\phi^2} - \frac{1}{sH}\frac{\partial\psi}{\partial\phi}\frac{\partial B_s^0}{\partial s}\bigg|_{\pm H} + \frac{1}{sH}\frac{\partial\psi}{\partial s}\frac{\partial B_s^{0\pm}}{\partial\phi} \mp \frac{1}{sH}\frac{dH}{ds}\frac{\partial\psi}{\partial\phi}\frac{\partial B_s^0}{\partial z}\bigg|_{\pm H},\\
\frac{\partial b_\phi^\pm}{\partial t} &= -B_s^{0\pm}\frac{\partial}{\partial s}\left(\frac{1}{H}\frac{\partial\psi}{\partial s}\right) - \frac{B_\phi^{0\pm}}{sH}\frac{\partial^2\psi}{\partial s\partial\phi} - \frac{1}{sH}\frac{\partial\psi}{\partial\phi}\frac{\partial B_\phi^0}{\partial s}\bigg|_{\pm H} + \frac{1}{sH}\frac{\partial\psi}{\partial s}\frac{\partial B_\phi^{0\pm}}{\partial\phi} \mp \frac{1}{sH}\frac{dH}{ds}\frac{\partial\psi}{\partial\phi}\frac{\partial B_\phi^0}{\partial z}\bigg|_{\pm H}\\
&\quad + \frac{1}{s}\left(\frac{B_\phi^{0\pm}}{sH}\frac{\partial\psi}{\partial\phi} + \frac{B_s^{0\pm}}{H}\frac{\partial\psi}{\partial s}\right),\\
\frac{\partial b_z^\pm}{\partial t} &= \pm HB_s^{0\pm}\frac{\partial}{\partial s}\left(\frac{1}{sH^2}\frac{dH}{ds}\frac{\partial\psi}{\partial\phi}\right) \pm \frac{B_\phi^{0\pm}}{s^2H}\frac{dH}{ds}\frac{\partial^2\psi}{\partial\phi^2} + \frac{B_z^{0\pm}}{sH^2}\frac{dH}{ds}\frac{\partial\psi}{\partial\phi} - \frac{1}{sH}\frac{\partial\psi}{\partial\phi}\frac{\partial B_z^0}{\partial s}\bigg|_{\pm H}\\
&\quad + \frac{1}{sH}\frac{\partial\psi}{\partial s}\frac{\partial B_z^{0\pm}}{\partial\phi} \mp \frac{1}{sH}\frac{dH}{ds}\frac{\partial\psi}{\partial\phi}\frac{\partial B_z^0}{\partial z}\bigg|_{\pm H}.
\end{aligned} \tag{1.11}$$

If we look at the right-hand-side of these induction equations, we see that the right-hand-side is free of perturbed magnetic fields, but only involves background magnetic fields and perturbed velocity field. Therefore, when given the background field, the boundary terms can be written as

$$\frac{\partial b_a^\pm}{\partial t} = \mathcal{L}_i^\pm\psi \quad\Longrightarrow\quad b_a^\pm = \frac{1}{i\omega}\mathcal{L}_i^\pm\psi$$

where $\mathcal{L}_i^\pm$ are some linear operators. In fact, this is a feature that applies to all induction equation, summarized in the following statement.

Proposition 1.4.1 *The ideal induction equations of the boundary magnetic field or the integrated magnetic moments, when linearized around a background field with zero velocity, involves only the background magnetic field / moment and the perturbed velocity. In other words, all of them can be written as*

$$\frac{\partial b_a}{\partial t} = \mathcal{L}_a\psi,$$

or in the frequency domain

$$i\omega b_a = \mathcal{L}_a\psi,$$

where $b_a \in \{\overline{m_{ss}}, \overline{m_{\phi\phi}}, \overline{m_{s\phi}}, \widetilde{m_{sz}}, \widetilde{m_{\phi z}}, \widetilde{zm_{ss}}, \widetilde{zm_{\phi\phi}}, \widetilde{zm_{s\phi}}, b_{es}, b_{e\phi}, b_{ez}, b_{es,z}, b_{e\phi,z}, b_s^\pm, b_\phi^\pm, b_z^\pm\}$.

This proposition leads to the following statement.

Corollary 1.4.2 *When linearized around a background field with zero velocity, the complete PG system with diffusionless vorticity and induction equations and boundary terms can always be reduced to a single equation*

$$\left[\frac{\partial}{\partial s}\left(\frac{s}{H}\frac{\partial}{\partial s}\right) + \left(\frac{1}{sH} - \frac{1}{2H^2}\frac{dH}{ds}\right)\frac{\partial^2}{\partial\phi^2}\right]\frac{\partial^2\psi}{\partial t^2} = -\frac{2}{H^2}\frac{dH}{ds}\frac{\partial}{\partial\phi}\frac{\partial\psi}{\partial t} + \mathcal{L}_{\text{tot}}\psi$$

where $\mathcal{L}_{\text{tot}}$ *is the combined linear operator that gives the Lorentz force. Furthermore, considering the forms of the induction equations and vorticity equation,* $\mathcal{L}_{\text{tot}}$ *is at most 3rd order in* (s,ϕ,z)*. In the frequency domain, it is written as*

$$-\omega^2\left[\frac{\partial}{\partial s}\left(\frac{s}{H}\frac{\partial}{\partial s}\right) + \left(\frac{1}{sH} - \frac{1}{2H^2}\frac{dH}{ds}\right)\frac{\partial^2}{\partial\phi^2}\right]\psi = -i\omega\frac{2}{H^2}\frac{dH}{ds}\frac{\partial\psi}{\partial\phi} + \mathcal{L}_{\text{tot}}\psi$$

This gives a further dilemma: the eigenvalue problem will be closed in the vorticity itself, regardless of the boundary condition. In other words, changing the boundary condition does not even change the eigenvalue problem. How is that possible? Does that mean the eigenmode is not even affected by the choice of boundary conditions? Will the boundary condition be automatically satisfied by the perturbed magnetic field? For instance, will $b_s^\pm$, $b_\phi^\pm$ and $b_z^\pm$ solved in this way automatically match an insulating boundary condition, and if not, when will it or is it necessary?

Corollary 1.4.2 is useful conceptually, but cannot be directly implemented as an eigenvalue problem, since the right-hand-sides contain both first derivative and stream function itself. We must instead flatten out the second order derivative, and consider the augmented system. One way to achieve this is to write

$$\begin{aligned}\left[\frac{\partial}{\partial s}\left(\frac{s}{H}\frac{\partial}{\partial s}\right)+\left(\frac{1}{sH}-\frac{1}{2H^2}\frac{dH}{ds}\right)\frac{\partial^2}{\partial\phi^2}\right]\frac{\partial\psi}{\partial t} &= -\frac{2}{H^2}\frac{dH}{ds}\frac{\partial\psi}{\partial\phi}+F\\ \frac{\partial F}{\partial t} &= \mathcal{L}_{\mathrm{tot}}\psi\end{aligned}$$

or in matrix form of the eigenvalue problem

$$i\omega\begin{pmatrix}\frac{\partial}{\partial s}\left(\frac{s}{H}\frac{\partial}{\partial s}\right)-\frac{m^2}{sH}+\frac{m^2}{2H^2}\frac{dH}{ds} & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}\psi^m\\ F^m\end{pmatrix}=\begin{pmatrix}-\frac{2im}{H^2}\frac{dH}{ds} & 1\\ \mathcal{L}_{\mathrm{tot}} & 0\end{pmatrix}\begin{pmatrix}\psi^m\\ F^m\end{pmatrix}$$

This is similar to the velocity-stress formulation, often used in seismological simulations. Interestingly, since we know $\{\psi^{mn}(s)=s^{|m|}H^3P_n^{(\frac{3}{2},|m|)}(2s^2-1)\}$ are the eigenfunctions for the Sturm-Liouville problem (first equation, without F contribution), we can conclude these $\{\psi^{mn}(s)\}$ form a complete orthogonal basis with respect to weight $\frac{2ism}{H^3}=-\frac{2im}{H^2}\frac{dH}{ds}$. In other words, we should expect that the appropriate expansion for F that can be the solution to the eigenvalue problem should take the form

$$F^{mn}(s)=\frac{s}{H^3}\psi^{mn}(s)=s^{|m|+1}P_n^{(\frac{3}{2},|m|)}(2s^2-1).$$

As I have mentioned, the boundary induction equation in cylindrical coordinates cannot be used in a time-stepping solver with PG formulations. The only equation that seems to be closed in itself is eq.(1.9). This, however, involves one complication and one limitation. First, as the equation is not in cylindrical coordinates, while all other equations are, we need an explicit spherical-cylindrical transform. Among other complications, this means the sparsity of the matrix or orthogonality of the basis might be partially destroyed. Second, noting that the Lorentz force involves only the s, z and ϕ components of the boundary magnetic fields, we need to link the radial field to the three components. This can be easily done with an insulating boundary condition, where the magnetic field external to the sphere is harmonic. However, once this assumption is dropped, it will be much more challenging to derive a general link.

Chapter 2

Regularity constraints on expansions, conjugate variables

2.1 Regularity conditions on rank-2 tensor in cylindrical coordinates

In this section, I derive the regularity conditions for general rank-2 tensors in cylindrical coordinates. This approach has been more elaborately exploited for arbitrary ranks in *Regularity conditions for the Fourier coefficients of tensors in polar coordinates*. The excerpt here offers a more self-contained, explicit and easy-to-comprehend explanation.

Consider a rank-2 tensor field in 2-D space, denoted as $\mathbf{A} \in \mathbb{C}^{2\times 2}$. The tensor can be expressed in any locally orthogonal frame as

$$A_{ij} = \hat{\mathbf{e}}_i \cdot \mathbf{A} \cdot \hat{\mathbf{e}}_j.$$

Its components can be expressed in Cartesian coordinates as well as cylindrical coordinates using matrices, which are related via transform

$$\begin{pmatrix} A_{xx} & A_{xy} \\ A_{yx} & A_{yy} \end{pmatrix} = \begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix} \begin{pmatrix} A_{ss} & A_{s\phi} \\ A_{\phi s} & A_{\phi\phi} \end{pmatrix} \begin{pmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{pmatrix}$$

The elements in Cartesian coordinates are thus related to the elements in the cylindrical coordinates via

$$\begin{aligned}
A_{xx} &= \cos^2\phi A_{ss} - \cos\phi\sin\phi\left(A_{s\phi} + A_{\phi s}\right) + \sin^2\phi A_{\phi\phi}, \\
A_{yy} &= \sin^2\phi A_{ss} + \cos\phi\sin\phi\left(A_{s\phi} + A_{\phi s}\right) + \cos^2\phi A_{\phi\phi}, \\
A_{xy} &= \cos\phi\sin\phi\left(A_{ss} - A_{\phi\phi}\right) + \cos^2\phi A_{s\phi} - \sin^2 A_{\phi s}, \\
A_{yx} &= \cos\phi\sin\phi\left(A_{ss} - A_{\phi\phi}\right) + \cos^2\phi A_{\phi s} - \sin^2 A_{s\phi}.
\end{aligned}$$

Components of $\mathbf{A}$ are regular in cylindrical coordinates, which can be expanded in Fourier series of azimuthal wavenumber. For instance, the A_{ss} component can be expressed as

$$A_{ss} = \sum_{m=-\infty}^{+\infty} A_{ss}^m(s) e^{im\phi}$$

where A_{ss}^m is the Fourier coefficient for azimuthal wavenumber m. Expansions of other components naturally follow. Expressing the cosines and sines also in Fourier basis

$$\cos\phi = \frac{e^{i\phi} + e^{-i\phi}}{2}, \quad \sin\phi = \frac{e^{i\phi} - e^{-i\phi}}{2i},$$
$$\cos^2\phi = \frac{e^{i2\phi} + e^{-i2\phi} + 2}{4}, \quad \sin^2\phi = -\frac{e^{i2\phi} + e^{-i2\phi} - 2}{4}, \quad \cos\phi\sin\phi = \frac{e^{i2\phi} - e^{-i2\phi}}{4i}$$

We see that the tensor elements in Cartesian coordinates have the Fourier expansion

$$A_{xx} = \sum_m \frac{e^{im\phi}}{4} \left\{2\left(A^m_{ss} + A^m_{\phi\phi}\right) + \left[A^m_{ss} - A^m_{\phi\phi} - i\left(A^m_{s\phi} + A^m_{\phi s}\right)\right] e^{-i2\phi} + \left[A^m_{ss} - A^m_{\phi\phi} + i\left(A^m_{s\phi} + A^m_{\phi s}\right)\right] e^{i2\phi}\right\}$$
$$A_{yy} = \sum_m \frac{e^{im\phi}}{4} \left\{2\left(A^m_{ss} + A^m_{\phi\phi}\right) - \left[A^m_{ss} - A^m_{\phi\phi} - i\left(A^m_{s\phi} + A^m_{\phi s}\right)\right] e^{-i2\phi} - \left[A^m_{ss} - A^m_{\phi\phi} + i\left(A^m_{s\phi} + A^m_{\phi s}\right)\right] e^{i2\phi}\right\}$$
$$A_{xy} = \sum_m \frac{e^{im\phi}}{4} \left\{2\left(A^m_{s\phi} - A^m_{\phi s}\right) + \left[A^m_{s\phi} + A^m_{\phi s} + i\left(A^m_{ss} - A^m_{\phi\phi}\right)\right] e^{-i2\phi} + \left[A^m_{s\phi} + A^m_{\phi s} - i\left(A^m_{ss} - A^m_{\phi\phi}\right)\right] e^{i2\phi}\right\}$$
$$A_{yx} = \sum_m \frac{e^{im\phi}}{4} \left\{2\left(A^m_{\phi s} - A^m_{s\phi}\right) + \left[A^m_{s\phi} + A^m_{\phi s} + i\left(A^m_{ss} - A^m_{\phi\phi}\right)\right] e^{-i2\phi} + \left[A^m_{s\phi} + A^m_{\phi s} - i\left(A^m_{ss} - A^m_{\phi\phi}\right)\right] e^{i2\phi}\right\}$$

Using these relations, we can deduce from the regularity of A_{xx}, A_{yy}, A_{xy} and A_{yx} that the following fields must also be regular

$$\begin{aligned}
A_{xx} + A_{yy} &= \sum_m \left(A^m_{ss} + A^m_{\phi\phi}\right) e^{im\phi} \\
A_{xy} - A_{yx} &= \sum_m \left(A^m_{s\phi} - A^m_{\phi s}\right) e^{im\phi} \\
(A_{xx} - A_{yy}) + i\,(A_{xy} + A_{yx}) &= \sum_m \left[A^m_{ss} - A^m_{\phi\phi} + i\left(A^m_{s\phi} + A^m_{\phi s}\right)\right] e^{i(m+2)\phi} \\
(A_{xx} - A_{yy}) - i\,(A_{xy} + A_{yx}) &= \sum_m \left[A^m_{ss} - A^m_{\phi\phi} - i\left(A^m_{s\phi} + A^m_{\phi s}\right)\right] e^{i(m-2)\phi}
\end{aligned}$$

Plugging in these relations back into the expansion of Cartesian components, we see that these are both necessary AND sufficient conditions for the regularity of the tensor elements under Cartesian coordinates. We can then safely further simplify the relations from here, feeling safe that no information is lost during the process. This procedure is, unfortunately, missing in Lewis and Bellan (1990). Only the terms of A_x are derived before the authors concluded that the respective terms must be regular. In fact, counterinstances are easy to find that does NOT fulfill the regularity constraints BUT yields regular A_x, say $A_s = \frac{1}{s}\,(1 - \cos 2\phi)$ and $A_\phi = \frac{1}{s} \sin 2\phi$. It is the extra constraints from A_y that jointly pose the constraints. As in Lewis and Bellan (1990), the exponentials can be written as

$$e^{im\phi} = \frac{(x+iy)^{|m|}}{s^{|m|}}.$$

This allows us to pose constraints on the Fourier coefficients $A^m_{ij}(s)$ as functions of cylindrical radius s. The four relations are equivalent to the following four regularity constraints:

$$\begin{aligned}
A^m_{ss} + A^m_{\phi\phi} &= s^{|m|} C(s^2) \\
A^m_{s\phi} - A^m_{\phi s} &= s^{|m|} C(s^2) \\
A^m_{ss} - A^m_{\phi\phi} + i\left(A^m_{s\phi} + A^m_{\phi s}\right) &= s^{|m+2|} C(s^2) \\
A^m_{ss} - A^m_{\phi\phi} - i\left(A^m_{s\phi} + A^m_{\phi s}\right) &= s^{|m-2|} C(s^2)
\end{aligned} \tag{2.1}$$

where we already used the symmetry or anti-symmetry in s for Cartesian tensor components. Notation $C(s^2)$ denotes a function of s^2 that is regular at $s = 0$, which can be expanded into Taylor series. Now it is time to split the domain of k, $\mathbb{Z}$, into intervals, so as to simplify the relations. We see that the absolute value functions can be completely removed in each scenario if we split the domain into $m \leq -2$, $m = -1$, $m = 0$, $m = 1$ and $m \geq 2$. The treaments of negative and positive m are highly similar, and I shall only write out the positive branch in detail. For $m \geq 2$, we can substract the two latter equations in eq.(2.1) and obtain $A^m_{s\phi} + A^m_{\phi s} \sim s^{m-2}$; combining this with the second equation,

$$\begin{cases} A^m_{s\phi} + A^m_{\phi s} = s^{m-2}C(s^2) \\ A^m_{s\phi} - A^m_{\phi s} = s^{m}C(s^2) \end{cases} \implies \begin{cases} A^m_{s\phi} = A^{m0}_{s\phi}s^{m-2} + A^{m1}_{s\phi}s^m + s^{m+2}C(s^2) \\ A^m_{\phi s} = A^{m0}_{\phi s}s^{m-2} + A^{m1}_{\phi s}s^m + s^{m+2}C(s^2) \end{cases} \quad \text{and} \quad A^{m0}_{s\phi} = A^{m0}_{\phi s}.$$

Thus simultaneously we obtain the ansätze (this is in fact the required form for regularity) for $A_{s\phi}$ and $A_{\phi s}$, as well as a coupling condition. The second superscript on A^{mn}_{ij} gives the index for power series expansion in s. On the other hand, we can add the latter two equations of eq.(2.1) and combine with the first equation to similarly come up with

$$\begin{cases} A^m_{ss} + A^m_{\phi\phi} = s^{m}C(s^2) \\ A^m_{ss} - A^m_{\phi\phi} = s^{m-2}C(s^2) \end{cases} \implies \begin{cases} A^m_{ss} = A^{m0}_{ss}s^{m-2} + A^{m1}_{ss}s^m + s^{m+2}C(s^2) \\ A^m_{\phi\phi} = A^{m0}_{\phi\phi}s^{m-2} + A^{m1}_{\phi\phi}s^m + s^{m+2}C(s^2) \end{cases} \quad \text{and} \quad A^{m0}_{ss} = -A^{m0}_{\phi\phi}.$$

Finally, we reuse the third equation in eq.(2.1) to establish the relation between the coefficients for the diagonal and the off-diagonal elements. To make sure both s^{m-2} and s^m vanishes on the LHS,

$$A^{m0}_{ss} - A^{m0}_{\phi\phi} + i\left(A^{m0}_{s\phi} + A^{m0}_{\phi s}\right) = 0, \quad \implies \quad A^{m0}_{s\phi} = iA^{m0}_{ss}$$
$$A^{m1}_{ss} - A^{m1}_{\phi\phi} + i\left(A^{m1}_{s\phi} + A^{m1}_{\phi s}\right) = 0$$

These are the four regularity constraints for $m \geq 2$. With all the ansätze, it can be easily verified that as long as the coefficients fulfill these constraints, the target terms indeed satisfy eq.(2.1), and thus these ansätze and constraints are also sufficient conditions.

Next, we take a look at the situation where $m = 1$. The latter two equations now yield

$$\begin{cases} A^1_{s\phi} + A^1_{\phi s} = sC(s^2) \\ A^1_{s\phi} - A^1_{\phi s} = sC(s^2) \end{cases} \implies \begin{cases} A^1_{s\phi} = A^{10}_{s\phi}s + s^3C(s^2) \\ A^1_{\phi s} = A^{10}_{\phi s}s + s^3C(s^2). \end{cases}$$

Apparently, no constraints are required; the ansatz alone suffices to enforce the correct leading power of s. This is equally true for A_{ss} and $A_{\phi\phi}$,

$$\begin{cases} A^1_{ss} + A^1_{\phi\phi} = s^1C(s^2) \\ A^1_{ss} - A^1_{\phi\phi} = s^1C(s^2) \end{cases} \implies \begin{cases} A^1_{ss} = A^{10}_{ss}s + s^3C(s^2) \\ A^1_{\phi\phi} = A^{10}_{\phi\phi}s + s^3C(s^2). \end{cases}$$

However, the last constraint still holds, that is we still need that the first-order term in s of $A^1_{ss} - A^1_{\phi\phi}$ and $i\left(A^1_{s\phi} + A^1_{\phi s}\right)$ cancel each other out,

$$A^{10}_{ss} - A^{10}_{\phi\phi} + i\left(A^{10}_{s\phi} + A^{10}_{\phi s}\right) = 0.$$

These constraints are absent from Holdenried-Chernoff (2021) (note here we are not yet assuming $A_{s\phi} = A_{\phi s}$).

Finally, we arrive at the $m = 0$ case.

$$\begin{cases} A^0_{s\phi} + A^0_{\phi s} = s^2C(s^2) \\ A^0_{s\phi} - A^0_{\phi s} = C(s^2) \end{cases} \implies \begin{cases} A^0_{s\phi} = A^{00}_{s\phi} + s^2C(s^2) \\ A^0_{\phi s} = A^{00}_{\phi s} + s^2C(s^2) \end{cases} \quad \text{and} \quad A^{00}_{s\phi} = -A^{00}_{\phi s}.$$

$$\begin{cases} A^0_{ss} + A^0_{\phi\phi} = C(s^2) \\ A^0_{ss} - A^0_{\phi\phi} = s^2C(s^2) \end{cases} \implies \begin{cases} A^0_{ss} = A^{00}_{ss} + s^2C(s^2) \\ A^0_{\phi\phi} = A^{00}_{\phi\phi} + s^2C(s^2) \end{cases} \quad \text{and} \quad A^{00}_{ss} = A^{00}_{\phi\phi}.$$

The third and the fourth equation in eq.(2.1) give the relations

$$\begin{cases} A^{00}_{ss} - A^{00}_{\phi\phi} + i\left(A^{00}_{s\phi} + A^{00}_{\phi s}\right) = 0 \\ A^{00}_{ss} - A^{00}_{\phi\phi} - i\left(A^{00}_{s\phi} + A^{00}_{\phi s}\right) = 0 \end{cases}$$

which are automatically satisfied given the previous ansätze. The negative m scenarios are also similarly derived. In the end, the required leading order and the constraints are summarized as follows

$$
\begin{aligned}
m=0: &\quad \begin{cases} A^0_{ss} = A^{00}_{ss} + s^2 C(s^2) \\ A^0_{\phi\phi} = A^{00}_{\phi\phi} + s^2 C(s^2) \\ A^0_{s\phi} = A^{00}_{s\phi} + s^2 C(s^2) \\ A^0_{\phi s} = A^{00}_{\phi s} + s^2 C(s^2) \end{cases}, \quad \begin{cases} A^{00}_{ss} = A^{00}_{\phi\phi} \\ A^{00}_{s\phi} = -A^{00}_{\phi s} \end{cases} \\
|m|=1: &\quad \begin{cases} A^m_{ss} = A^{m0}_{ss} s + s^3 C(s^2) \\ A^m_{\phi\phi} = A^{m0}_{\phi\phi} s + s^3 C(s^2) \\ A^m_{s\phi} = A^{m0}_{s\phi} s + s^3 C(s^2) \\ A^m_{\phi s} = A^{m0}_{\phi s} s + s^3 C(s^2) \end{cases}, \quad \left\{ A^{m0}_{s\phi} + A^{m0}_{\phi s} = i \,\mathrm{sgn}(m) \left(A^{m0}_{ss} - A^{m0}_{\phi\phi} \right) \right. \\
|m|\geq 2: &\quad \begin{cases} A^m_{ss} = A^{m0}_{ss} s^{|m|-2} + A^{m1}_{ss} s^{|m|} + s^{|m|+2} C(s^2) \\ A^m_{\phi\phi} = A^{m0}_{\phi\phi} s^{|m|-2} + A^{m1}_{\phi\phi} s^{|m|} + s^{|m|+2} C(s^2) \\ A^m_{s\phi} = A^{m0}_{s\phi} s^{|m|-2} + A^{m1}_{s\phi} s^{|m|} + s^{|m|+2} C(s^2) \\ A^m_{\phi s} = A^{m0}_{\phi s} s^{|m|-2} + A^{m1}_{\phi s} s^{|m|} + s^{|m|+2} C(s^2) \end{cases}, \quad \begin{cases} A^{m0}_{ss} = -A^{m0}_{\phi\phi} \\ A^{m0}_{s\phi} = A^{m0}_{\phi s} \\ A^{m0}_{s\phi} = i \,\mathrm{sgn}(m) A^{m0}_{ss} \\ A^{m1}_{s\phi} + A^{m1}_{\phi s} = i \,\mathrm{sgn}(m) \left(A^{m1}_{ss} - A^{m1}_{\phi\phi} \right). \end{cases}
\end{aligned}
\tag{2.2}
$$

In many cases, it is further useful to assume symmetry of the tensor; this is the case with e.g. strain tensor ε, strain-rate tensor $\dot{\varepsilon}$, stress tensor $\boldsymbol{\sigma}$, and of course for our problem, Maxwell stress $\boldsymbol{\sigma}^M$. In this case $A_{s\phi} = A_{\phi s}$, and all coefficients of their power series in s should match. However, for $m = 0$ we have $A^{00}_{s\phi} = -A^{00}_{\phi s}$. The result is that $A^0_{s\phi} = A^0_{\phi s}$, when expanded in power series of s, has leading order s^2 instead of s^0. In addition, some original constraints will render redundant. In the end, the ansätze and the regularity constraints for symmetric rank-2 tensors are given by

$$
\begin{aligned}
m=0: &\quad \begin{cases} A^0_{ss} = A^{00}_{ss} + s^2 C(s^2) \\ A^0_{\phi\phi} = A^{00}_{\phi\phi} + s^2 C(s^2) \\ A^0_{s\phi} = A^{00}_{s\phi} s^2 + s^4 C(s^2) \end{cases}, \quad \left\{ A^{00}_{ss} = A^{00}_{\phi\phi} \right. \\
|m|=1: &\quad \begin{cases} A^m_{ss} = A^{m0}_{ss} s + s^3 C(s^2) \\ A^m_{\phi\phi} = A^{m0}_{\phi\phi} s + s^3 C(s^2) \\ A^m_{s\phi} = A^{m0}_{s\phi} s + s^3 C(s^2) \end{cases}, \quad \left\{ 2A^{m0}_{s\phi} = i \,\mathrm{sgn}(m) \left(A^{m0}_{ss} - A^{m0}_{\phi\phi} \right) \right. \\
|m|\geq 2: &\quad \begin{cases} A^m_{ss} = A^{m0}_{ss} s^{|m|-2} + A^{m1}_{ss} s^{|m|} + s^{|m|+2} C(s^2) \\ A^m_{\phi\phi} = A^{m0}_{\phi\phi} s^{|m|-2} + A^{m1}_{\phi\phi} s^{|m|} + s^{|m|+2} C(s^2) \\ A^m_{s\phi} = A^{m0}_{s\phi} s^{|m|-2} + A^{m1}_{s\phi} s^{|m|} + s^{|m|+2} C(s^2) \end{cases}, \quad \begin{cases} A^{m0}_{ss} = -A^{m0}_{\phi\phi} \\ A^{m0}_{s\phi} = i \,\mathrm{sgn}(m) A^{m0}_{ss} \\ 2A^{m1}_{s\phi} = i \,\mathrm{sgn}(m) \left(A^{m1}_{ss} - A^{m1}_{\phi\phi} \right). \end{cases}
\end{aligned}
\tag{2.3}
$$

These ansätze are consistent with the leading order behaviour of the equatorial magnetic moments documented in Holdenried-Chernoff (2021). However, the five constraints on the equatorial magnetic moments derived here form a proper superset of the constraints in Holdenried-Chernoff (2021). Specficially, two of these relations are absent in the dissertation, namely

$$
\begin{aligned}
2A^{m0}_{s\phi} &= i \,\mathrm{sgn}(m) \left(A^{m0}_{ss} - A^{m0}_{\phi\phi} \right), \quad |m| = 1; \\
2A^{m1}_{s\phi} &= i \,\mathrm{sgn}(m) \left(A^{m1}_{ss} - A^{m1}_{\phi\phi} \right), \quad |m| \geq 2.
\end{aligned}
$$

The first of these two has been rediscovered in the previous section by re-deriving the formulae. The second relation cannot be discovered as long as we only consider the relation between lowest order behaviours. In fact, from this we see that there are regularity constraints even on the second-order term in the Taylor expansion in s.

It should be noted that the derivations above ONLY considered regularity of the tensor fields. However, magnetic moments **BB** are formed by outer product of the magnetic field **B**. In other words, the magnetic moment tensor is the rank-1 transformation of the magnetic field

$$\begin{pmatrix} B_x^2 & B_x B_y \\ B_y B_x & B_y^2 \end{pmatrix} = \begin{pmatrix} B_x \\ B_y \end{pmatrix} \begin{pmatrix} B_x \\ B_y \end{pmatrix}^\mathsf{T}, \qquad \begin{pmatrix} B_s^2 & B_s B_\phi \\ B_\phi B_s & B_\phi^2 \end{pmatrix} = \begin{pmatrix} B_s \\ B_\phi \end{pmatrix} \begin{pmatrix} B_s \\ B_\phi \end{pmatrix}^\mathsf{T}$$

This constraints is not imposed in the derivations above, which assumes arbitrary tensor field. It thus poses a question that if we expand B_s^2, B_ϕ^2 and $B_s B_\phi$ separately, are we artificially expanding the image of field to moment mapping. Part of the space formed by the expansions might not have underlying magnetic fields (i.e. not surjective). [This problem requires further notice.]

2.2 Challenges in implementing the regularity conditions

In her dissertation, Daria briefly described how to implement coupling between lowest order coefficients in s, namely by expanding some quantities starting from $n = 1$, but adding manually the $n = 0$ contribution as a coupling term. This implementation (as confirmed by her Mathematica notebook) is equivalent to the following expansion:

$$\begin{aligned} B_{es} &= \sum_{n=0}^{\infty} C_{es}^{mn} \left[(1-s^2) s^{|m|-1} P_n^{(\alpha,\beta)}(2s^2-1)\right] \\ B_{e\phi} &= i\,\mathrm{sgn}(m) \sum_{n=0}^{\infty} C_{es}^{mn} \left[(1-s^2) s^{|m|-1} P_n^{(\alpha,\beta)}(2s^2-1)\right] + \sum_{n=0}^{\infty} C_{e\phi}^{mn} \left[(1-s^2) s^{|m|+1} P_n^{(\alpha',\beta')}(2s^2-1)\right] \end{aligned} \tag{2.4}$$

for $|m| \geq 1$ (for $m = 0$ there is no coupling between these coefficients). Note that the bases corresponding to coefficients $C_{e\phi}^{mn}$ has the prefactor $s^{|m|+1}$ instead of $s^{|m|-1}$. The Jacobi polynomial indices α and β can be chosen relatively freely, so as to enforce maximal sparsity on the matrices. For instance, if one uses $(1-s^2)s^{|m|-1}P_n^{(\alpha,\beta)}(2s^2-1)$ and $(1-s^2)s^{|m|+1}P_n^{(\alpha',\beta')}(2s^2-1)$ respectively as the test functions for B_{es} and $B_{e\phi}$ induction equations, a reasonable choice of the indices will be

$$\begin{cases} \alpha = 2 \\ \beta = |m| - \dfrac{3}{2} \end{cases}, \qquad \begin{cases} \alpha' = 2 \\ \beta' = |m| + \dfrac{1}{2} \end{cases}$$

This configuration will diagonalize the matrix blocks (B_{es}, C_{es}^{mn}) and $(B_{e\phi}, C_{e\phi}^{mn})$, which are the diagonal blocks in the mass matrix. By (B, C) I denote the matrix block formed by taking the inner product of the test function corresponding to field B and the bases corresponding to coefficient C. However, the coupling block, i.e. $(B_{e\phi}, C_{es}^{mn})$ will not be diagonal. This will form a dense matrix as an off-diagonal block in the mass matrix.

Although the previous expansion is only for B_{es}, $B_{e\phi}$, similar trick can also be used to implement the lowest-order coupling for $B_{es,z}$-$B_{e\phi,z}$ pair, $\widetilde{M_{sz}}$-$\widetilde{M_{\phi z}}$ pairs - basically any quantity pairs that behave like (s, ϕ) equatorial components of a vector. Daria even applied the same method to implementing the low-order coupling of the rank-2 tensors. Her implementation (see, e.g. `C1QP_reg_diff_visc_daria.nb`) is equivalent to the following expansion:

$$\begin{aligned}
\overline{M_{s\phi}}^m &= \sum_n C_{s\phi}^{mn} \left[\sqrt{1-s^2}s^{|m|-2}P_n^{(\alpha,\beta)}(2s^2-1)\right] \\
\overline{M_{\phi\phi}}^m &= i\,\mathrm{sgn}(m) \sum_n C_{s\phi}^{mn} \left[\sqrt{1-s^2}s^{|m|-2}P_n^{(\alpha,\beta)}(2s^2-1)\right] + \sum_n C_{\phi\phi}^{mn} \left[\sqrt{1-s^2}s^{|m|}P_n^{(\alpha',\beta')}(2s^2-1)\right] \\
\widetilde{zM_{s\phi}}^m &= \sum_n C_{zs\phi}^{mn} \left[(1-s^2)s^{|m|-2}P_n^{(\alpha,\beta)}(2s^2-1)\right] \\
\widetilde{zM_{\phi\phi}}^m &= i\,\mathrm{sgn}(m) \sum_n C_{zs\phi}^{mn} \left[(1-s^2)s^{|m|-2}P_n^{(\alpha,\beta)}(2s^2-1)\right] + \sum_n C_{z\phi\phi}^{mn} \left[(1-s^2)s^{|m|}P_n^{(\alpha',\beta')}(2s^2-1)\right]
\end{aligned}$$

Once again, we are looking at $m \geq 2$, as the coupling for $m = 0$ is different, and the coupling for $m = \pm 1$ is absent. This would have worked, had the coupling between tensorial elements in cylindrical coordinates only occurred in the lowest order $n = 0$. However, the previous section has already shown otherwise. In addition to the coupling in $n = 0$, we have additional three-component coupling in order $n = 1$ for $m \geq 2$, as well as a previously ignored three-component coupling in order $n = 0$ for $m = \pm 1$. Even if the same trick can be used for implementing the three-component coupling in order $n = 0$ for $m = \pm 1$:

$$2M_{s\phi}^{m0} = i\,\mathrm{sgn}(m)\left(M_{ss}^{m0} - A_{\phi\phi}^{m0}\right)$$

by taking the following expansion,

$$\begin{aligned}
\overline{M_{ss}}^m &= \sum_n C_{ss}^{mn} \left[\sqrt{1-s^2}sP_n^{(\alpha,\beta)}(2s^2-1)\right] \\
\overline{M_{\phi\phi}}^m &= \sum_n C_{\phi\phi}^{mn} \left[\sqrt{1-s^2}sP_n^{(\alpha,\beta)}(2s^2-1)\right] \\
\overline{M_{s\phi}}^m &= \frac{i\,\mathrm{sgn}(m)}{2}\left\{\sum_n C_{ss}^{mn} \left[\sqrt{1-s^2}sP_n^{(\alpha,\beta)}(2s^2-1)\right] - \sum_n C_{\phi\phi}^{mn} \left[\sqrt{1-s^2}sP_n^{(\alpha,\beta)}(2s^2-1)\right]\right\} \\
&\quad + \sum_n C_{s\phi}^{mn} \left[\sqrt{1-s^2}s^3P_n^{(\alpha',\beta')}(2s^2-1)\right],
\end{aligned}$$

the coupling in order $n = 1$ is just not feasible to be implemented in this way. Granted, it may be possible to write down the following expansion,

$$\begin{aligned}
\overline{M_{ss}}^m &= s^{|m|-2}\sqrt{1-s^2}\left\{C_0 + C_1s^2 + s^4\sum_n C_{ss}^{mn}\left[P_n^{(\alpha,\beta)}(2s^2-1)\right]\right\} \\
\overline{M_{\phi\phi}}^m &= s^{|m|-2}\sqrt{1-s^2}\left\{-C_0 + C_2s^2 + s^4\sum_n C_{\phi\phi}^{mn}\left[P_n^{(\alpha,\beta)}(2s^2-1)\right]\right\} \\
\overline{M_{s\phi}}^m &= s^{|m|-2}\sqrt{1-s^2}\left\{i\,\mathrm{sgn}(m)C_0 + \frac{i\,\mathrm{sgn}(m)}{2}(C_1-C_2)s^2 + s^4\sum_n C_{s\phi}^{mn}\left[P_n^{(\alpha,\beta)}(2s^2-1)\right]\right\},
\end{aligned}$$

but it then becomes a painstaking task to look for appropriate test functions. Note that now we have three additional bases $s^{|m|-2}\sqrt{1-s^2}$ and $s^{|m|}\sqrt{1-s^2}$ (occuring twice) in addition to the bases $s^{|m|+2}\sqrt{1-s^2}P_n^{(\alpha,\beta)}(2s^2-1)$. Therefore, a total number of $3N+3$ test functions are needed to form a linear system. Where exactly do we place the extra three test functions? It seems it doesn't make sense either way and always breaks the symmetry of the problem. This difficulty calls for a new expansion of the fields.

2.3 New expansion for coupled quantities

Is there a way to circumvent manually enforcing all of these regularity constraints by designing intricate expansions? The answer is yes, according to Matthew and Stefano. For the vector quantities in cylindrical coordinates, i.e. components A_s and A_ϕ, they suggest that instead of expanding them separately, one should rather be looking for expansions of $A_s \pm iA_\phi$. These will have the expansion

$$
\begin{aligned}
A_s + iA_\phi &= \sum_m s^{|m+1|} p(s^2) e^{im\phi} \\
A_s - iA_\phi &= \sum_m s^{|m-1|} p(s^2) e^{im\phi}
\end{aligned}
$$

These are, if I may, conjugate quantities to the original components, whose regularity is the sufficient and necessary condition that the corresponding Cartesian components are regular.

Does this trick similarly apply to rank-2 tensors? Indeed, if we take a step back from the final regularity constraints on individual matrix elements in cylindrical coordinates, we find that as an intermediate step, we have eq.(2.1), a set of regularity constraints on the Fourier coefficients that are sufficient and necessary conditions that the corresponding Cartesian components have regular Fourier coefficients. These are, after all, what gave rise to the regularity constraints on individual variables. It follows directly, that the components in the cylindrical coordinates need to and only need to have the following expansion

$$
\begin{aligned}
A_{ss} + A_{\phi\phi} &= \sum_m s^{|m|} C(s^2) e^{im\phi} \\
A_{s\phi} - A_{\phi s} &= \sum_m s^{|m|} C(s^2) e^{im\phi} \\
A_{ss} - A_{\phi\phi} + i\left(A_{s\phi} + A_{\phi s}\right) &= \sum_m s^{|m+2|} C(s^2) e^{im\phi} \\
A_{ss} - A_{\phi\phi} - i\left(A_{s\phi} + A_{\phi s}\right) &= \sum_m s^{|m-2|} C(s^2) e^{im\phi}.
\end{aligned} \tag{2.5}
$$

It is further beneficial to restrict the discussion to the relevant scenario at hand: the tensor is formed by the outer product of a vector with itself, and is of course symmetric. In this case the four relations are reduced to three relations regarding three quantities

$$
\begin{aligned}
M_{ss} + M_{\phi\phi} = M_1 &= \sum_m s^{|m|} C(s^2) e^{im\phi} \\
M_{ss} - M_{\phi\phi} + i2M_{s\phi} = M_+ &= \sum_m s^{|m+2|} C(s^2) e^{im\phi} \\
M_{ss} - M_{\phi\phi} - i2M_{s\phi} = M_- &= \sum_m s^{|m-2|} C(s^2) e^{im\phi}.
\end{aligned} \tag{2.6}
$$

Here we introduced symbols for these conjugate quantities, i.e. M_1, M_+ and M_-. Note that among these quantities, only M_1 is strictly a scalar quantity, as only M_1 exhibits true scalar regularity condition (i.e. $\sim s^{|m|}$). Indeed, a careful reader may have realized by now that M_1 is nothing but the trace of the 2-D tensor $\mathbf{M}$. That Tr$\mathbf{M}$ remains a constant during rotation of axes is a known fact, hence it is also called "the first invariant", especially in the context of strain and stress.

What about the other two quantities? In the context of an outer product tensor $\mathbf{M} = \mathbf{B}\mathbf{B}^T$, one way to look at it is that the other two conjugate quantities are the squares of the *vector* conjugate quantities:

$$
M_{ss} - M_{\phi\phi} \pm i2M_{s\phi} = \left(B_s \pm iB_\phi\right)^2
$$

There doesn't seem to be any further immediate physical meaning attached to these quantities.

It is useful to observe that these conjugate quantities are linked to the cylindrical components via an invertible linear map, that is independent of ϕ,

$$
\begin{pmatrix} M_1 \\ M_+ \\ M_- \end{pmatrix} = \begin{pmatrix} 1 & 1 & 0 \\ 1 & -1 & 2i \\ 1 & -1 & -2i \end{pmatrix} \begin{pmatrix} M_{ss} \\ M_{\phi\phi} \\ M_{s\phi} \end{pmatrix}, \qquad \begin{pmatrix} M_{ss} \\ M_{\phi\phi} \\ M_{s\phi} \end{pmatrix} = \frac{1}{4} \begin{pmatrix} 2 & 1 & 1 \\ 2 & -1 & -1 \\ 0 & -i & i \end{pmatrix} \begin{pmatrix} M_1 \\ M_+ \\ M_- \end{pmatrix}. \tag{2.7}
$$

Before we move on, we note that all of these conjugate quantities have a common status. If we first look at the transform for the vector components in cylindrical coordinates

$$\begin{pmatrix} B_x \\ B_y \end{pmatrix} = \begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix} \begin{pmatrix} B_s \\ B_\phi \end{pmatrix} = \frac{1}{2} \begin{pmatrix} e^{i\phi} + e^{-i\phi} & i\left(e^{i\phi} - e^{-i\phi}\right) \\ -i\left(e^{i\phi} - e^{-i\phi}\right) & e^{i\phi} + e^{-i\phi} \end{pmatrix} \begin{pmatrix} B_s \\ B_\phi \end{pmatrix} = \mathbf{R} \begin{pmatrix} B_s \\ B_\phi \end{pmatrix}$$

The rotation matrix has the following spectral decomposition,

$$\mathbf{R} = \begin{pmatrix} 1 & 1 \\ -i & i \end{pmatrix} \begin{pmatrix} e^{i\phi} & 0 \\ 0 & e^{-i\phi} \end{pmatrix} \begin{pmatrix} 1 & 1 \\ -i & i \end{pmatrix}^{-1} = \frac{1}{2} \begin{pmatrix} 1 & 1 \\ -i & i \end{pmatrix} \begin{pmatrix} e^{i\phi} & 0 \\ 0 & e^{-i\phi} \end{pmatrix} \begin{pmatrix} 1 & i \\ 1 & -i \end{pmatrix}$$

i.e. eigenvector $(1, \pm i)$ corresponding to eigenvalues $e^{\mp i\phi}$. This means that certain linear combinations (given by the *inverse* of the eigenvalue matrix) of the components retain their form during rotation, except for an additional phase factor:

$$\begin{aligned} B_x + iB_y &= e^{+i\phi}\left(B_s + iB_\phi\right) \\ B_x - iB_y &= e^{-i\phi}\left(B_s - iB_\phi\right) \end{aligned} \quad \Longrightarrow \quad \begin{aligned} B_s + iB_\phi &= e^{-i\phi}\left(B_x + iB_y\right) \\ B_s - iB_\phi &= e^{+i\phi}\left(B_x - iB_y\right). \end{aligned}$$

Therefore, the conjugate quantities are nothing but from the (inverse of the) eigenvectors of the rotation matrix. Moreover, one can immediately deduce the regular expansion from these relations. We know the Cartesian components behave like scalars, so the right-hand-sides have Fourier coefficients that are $\sim s^{|m|}$. Therefore, the valid expansion for these conjugate quantities would be

$$\begin{aligned} B_s + iB_\phi &= \sum_m s^{|m+1|} p(s^2) e^{im\phi} \\ B_s - iB_\phi &= \sum_m s^{|m-1|} p(s^2) e^{im\phi} \end{aligned}$$

exactly as we expected. Similarly, let us consider the transform of rank-2 tensors in the form of matrices between cylindrical and Cartesian coordinates. It can of course be done via

$$\begin{pmatrix} M_{xx} & M_{xy} \\ M_{yx} & M_{yy} \end{pmatrix} = \begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix} \begin{pmatrix} M_{ss} & M_{s\phi} \\ M_{\phi s} & M_{\phi\phi} \end{pmatrix} \begin{pmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{pmatrix}$$

but since this is after all just a linear transform, the whole operation can be written in matrix-vector multiplication if we flatten the tensor into vectors,

$$\begin{pmatrix} M_{xx} \\ M_{yy} \\ M_{xy} \end{pmatrix} = \begin{pmatrix} \cos^2\phi & \sin^2\phi & -2\sin\phi\cos\phi \\ \sin^2\phi & \cos^2\phi & +2\sin\phi\cos\phi \\ \cos\phi\sin\phi & -\cos\phi\sin\phi & \cos^2\phi - \sin^2\phi \end{pmatrix} \begin{pmatrix} M_{ss} \\ M_{\phi\phi} \\ M_{s\phi} \end{pmatrix} = \mathbf{R}_2 \begin{pmatrix} M_{ss} \\ M_{\phi\phi} \\ M_{s\phi} \end{pmatrix}.$$

Here we already assumes that the tensor is symmetric. The augmented rotation matrix for the rank-2 tensor can be shown to have eigendecomposition

$$\begin{aligned} \mathbf{R}_2 &= \begin{pmatrix} 1 & 1 & 1 \\ 1 & -1 & -1 \\ 0 & -i & i \end{pmatrix} \begin{pmatrix} 1 & & \\ & e^{i2\phi} & \\ & & e^{-i2\phi} \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 \\ 1 & -1 & -1 \\ 0 & -i & i \end{pmatrix}^{-1} \\ &= \frac{1}{4} \begin{pmatrix} 1 & 1 & 1 \\ 1 & -1 & -1 \\ 0 & -i & i \end{pmatrix} \begin{pmatrix} 1 & & \\ & e^{i2\phi} & \\ & & e^{-i2\phi} \end{pmatrix} \begin{pmatrix} 2 & 2 & 0 \\ 1 & -1 & 2i \\ 1 & -1 & -2i \end{pmatrix} \end{aligned}$$

Note in this case the rotation matrix $\mathbf{R}_2$ is not diagonalized by a Hermitian matrix, and so the inverse of the eigenvector matrix is different from the complex conjugate transpose of the original eigenvector matrix. This gives the three relations that we have already obtained before,

$$
\begin{aligned}
M_{xx} + M_{yy} &= M_{ss} + M_{s\phi} \\
M_{xx} - M_{yy} + 2iM_{xy} &= e^{i2\phi} \left(M_{ss} - M_{\phi\phi} + 2iM_{s\phi}\right) \\
M_{xx} - M_{yy} - 2iM_{xy} &= e^{-i2\phi} \left(M_{ss} - M_{\phi\phi} - 2iM_{s\phi}\right)
\end{aligned}
$$

which will give the regularity conditions given the scalar property of Cartesian components.

To summarize, all of these conjugate quantities, at least for rank-1 and rank-2 tensors, can be derived by computing the eigenvalue decomposition of the rotation matrix for the flattened component vector,

$$\mathbf{R}_k = \mathbf{V}\mathbf{\Lambda}\mathbf{V}^{-1}$$

and so the relation

$$\mathbf{y}^{\mathrm{Cart}} = \mathbf{R}_k \mathbf{y}^{\mathrm{Cyl}} \quad \Longrightarrow \quad \mathbf{V}^{-1}\mathbf{y}^{\mathrm{Cart}} = \mathbf{\Lambda}\left(\mathbf{V}^{-1}\mathbf{y}^{\mathrm{Cyl}}\right)$$

gives the conjugate quantities that happen to retain their forms during changing coordinates systems. However, it is doubted whether such expression can be found for arbitrary rank tensor. For an arbitrary rank tensor in general, it would suffice to seek the matrix decomposition of the rotation matrix in the form of

$$\mathbf{R}_k = \mathbf{V}\mathbf{\Lambda}\mathbf{U} \tag{2.8}$$

where $\mathbf{V}, \mathbf{U} \in \mathbb{C}^{2^k \times 2^k}$ are invertible matrices whose elements are constants independent of ϕ, and $\mathbf{\Lambda} = \mathrm{diag}\left(C_j e^{i n_j \phi}\right)$ is a diagonal matrix whose diagonal entries are solely an exponential function of ϕ. If such a factorization is found, the rotation transform can be rewritten as

$$\mathbf{V}^{-1}\mathbf{y}^{\mathrm{Cart}} = \mathbf{\Lambda}\left(\mathbf{U}\mathbf{y}^{\mathrm{Cyl}}\right) \tag{2.9}$$

and the elements in $\mathbf{U}\mathbf{y}^{\mathrm{Cyl}}$ would give the 2^k *conjugate* quantities whose Fourier coefficients take the form of $s^{|m+n_j|\phi}$.

2.4 Evolution equation for conjugate quantities

As previously implied, a natural way to circumvent manually enforcing all regularity constraints is to instead expand the conjugate quantities. For the 12 magnetic quantities (except for B_r, which lives on the sphere, and B_{ez}, which is a scalar) in PG variables, the corresponding conjugate quantities are

$$
\begin{aligned}
&\begin{cases} \overline{M_{ss}} \\ \overline{M_{\phi\phi}} \\ \overline{M_{s\phi}} \end{cases} \longrightarrow \begin{cases} \overline{M_1} = \overline{M_{ss}} + \overline{M_{\phi\phi}} \\ \overline{M_+} = \overline{M_{ss}} - \overline{M_{\phi\phi}} + i2\overline{M_{s\phi}} \\ \overline{M_-} = \overline{M_{ss}} - \overline{M_{\phi\phi}} - i2\overline{M_{s\phi}} \end{cases} \quad \begin{cases} \overline{M_{ss}} = \frac{1}{2}\overline{M_1} + \frac{1}{4}\overline{M_+} + \frac{1}{4}\overline{M_-} \\ \overline{M_{\phi\phi}} = \frac{1}{2}\overline{M_1} - \frac{1}{4}\overline{M_+} - \frac{1}{4}\overline{M_-} \\ \overline{M_{s\phi}} = -\frac{i}{4}\overline{M_+} + \frac{i}{4}\overline{M_-} \end{cases} \\
&\begin{cases} \widetilde{M_{sz}} \\ \widetilde{M_{\phi z}} \end{cases} \longrightarrow \begin{cases} \widetilde{M_{z+}} = \widetilde{M_{sz}} + i\widetilde{M_{\phi z}} \\ \widetilde{M_{z-}} = \widetilde{M_{sz}} - i\widetilde{M_{\phi z}} \end{cases} \quad \begin{cases} \widetilde{M_{sz}} = \frac{1}{2}\widetilde{M_{z+}} + \frac{1}{2}\widetilde{M_{z-}} \\ \widetilde{M_{\phi z}} = -\frac{i}{2}\widetilde{M_{z+}} + \frac{i}{2}\widetilde{M_{z-}} \end{cases} \\
&\begin{cases} \widetilde{zM_{ss}} \\ \widetilde{zM_{\phi\phi}} \\ \widetilde{zM_{s\phi}} \end{cases} \longrightarrow \begin{cases} \widetilde{zM_1} = \widetilde{zM_{ss}} + \widetilde{zM_{\phi\phi}} \\ \widetilde{zM_+} = \widetilde{zM_{ss}} - \widetilde{zM_{\phi\phi}} + i2\widetilde{zM_{s\phi}} \\ \widetilde{zM_-} = \widetilde{zM_{ss}} - \widetilde{zM_{\phi\phi}} - i2\widetilde{zM_{s\phi}} \end{cases} \quad \begin{cases} \widetilde{zM_{ss}} = \frac{1}{2}\widetilde{zM_1} + \frac{1}{4}\widetilde{zM_+} + \frac{1}{4}\widetilde{zM_-} \\ \widetilde{zM_{\phi\phi}} = \frac{1}{2}\widetilde{zM_1} - \frac{1}{4}\widetilde{zM_+} - \frac{1}{4}\widetilde{zM_-} \\ \widetilde{zM_{s\phi}} = -\frac{i}{4}\widetilde{zM_+} + \frac{i}{4}\widetilde{zM_-} \end{cases} \\
&\begin{cases} B_{es} \\ B_{e\phi} \end{cases} \longrightarrow \begin{cases} B_{e+} = B_{es} + iB_{e\phi} \\ B_{e-} = B_{es} - iB_{e\phi} \end{cases} \quad \begin{cases} B_{es} = \frac{1}{2}B_{e+} + \frac{1}{2}B_{e-} \\ B_{e\phi} = -\frac{i}{2}B_{e+} + \frac{i}{2}B_{e-} \end{cases} \\
&\begin{cases} B_{es,z} \\ B_{e\phi,z} \end{cases} \longrightarrow \begin{cases} B_{e+,z} = B_{es,z} + iB_{e\phi,z} \\ B_{e-,z} = B_{es,z} - iB_{e\phi,z} \end{cases} \quad \begin{cases} B_{es,z} = \frac{1}{2}B_{e+,z} + \frac{1}{2}B_{e-,z} \\ B_{e\phi,z} = -\frac{i}{2}B_{e+,z} + \frac{i}{2}B_{e-,z} \end{cases}
\end{aligned}
\tag{2.10}
$$

These conjugate quantities have very simple regularity constraints. Their regularity constraint is merely a leading order behaviour in s, determined by their scalar Cartesian counterparts, and a leading order behaviour in H from even or odd axial integration. The Fourier coefficients for these quantities are

$$
\begin{aligned}
\overline{M_1}^m &= Hs^{|m|}p(s^2) \\
\overline{M_+}^m &= Hs^{|m+2|}p(s^2) \\
\overline{M_-}^m &= Hs^{|m-2|}p(s^2) \\
\widetilde{M_{z+}}^m &= H^2 s^{|m+1|}p(s^2) \\
\widetilde{M_{z-}}^m &= H^2 s^{|m-1|}p(s^2) \\
\widetilde{zM_1}^m &= H^2 s^{|m|}p(s^2) \\
\widetilde{zM_+}^m &= H^2 s^{|m+2|}p(s^2) \\
\widetilde{zM_-}^m &= H^2 s^{|m-2|}p(s^2) \\
B_{e+}^m &= s^{|m+1|}p(s^2) \\
B_{e-}^m &= s^{|m-1|}p(s^2) \\
B_{e+,z}^m &= s^{|m+1|}p(s^2) \\
B_{e-,z}^m &= s^{|m-1|}p(s^2)
\end{aligned}
\tag{2.11}
$$

where $p(s^2)$ denotes any analytic function in s^2. When the expansion for the equatorial fields are further combined with a harmonic field contribution, the equatorial Fourier coefficients will have a further H^2 prefactor in the front. Apart from that, they are free of any coupling in their Fourier coefficients. The leading order behaviour alone guarantees regularity.

Despite all the merits with this set of expansions, it is not directly applicable to the current form of the PG equations. The reason lies in the test functions to be used to reduce the equations into linear systems. With every tensor component comprising of multiple bases, there is no straightforward and consistent way to choose a set of test functions.

One way to overcome the test function issue, and perhaps the most consistent way, is to expand the conjugate quantities in their evolution equations. In other words, the evolution equations in terms of magnetic field quantities should first be transformed into evolution equations in their conjugate quantities. I shall present here the explicit derivation of one set of these quantities, namely from $(\overline{M_{ss}}, \overline{M_{s\phi}}, \overline{M_{\phi\phi}})$ to $(\overline{M_1}, \overline{M_+}, \overline{M_-})$. Starting from the original evolution equations

$$\begin{aligned}
\frac{\partial \overline{M_{ss}}}{\partial t} &= -H(\mathbf{u}\cdot\nabla_e)\frac{\overline{M_{ss}}}{H} + 2\overline{M_{ss}}\frac{\partial u_s}{\partial s} + \frac{2}{s}\overline{M_{s\phi}}\frac{\partial u_s}{\partial \phi}\\
\frac{\partial \overline{M_{\phi\phi}}}{\partial t} &= -\frac{1}{H}(\mathbf{u}\cdot\nabla_e)\left(H\overline{M_{\phi\phi}}\right) - 2\overline{M_{\phi\phi}}\frac{\partial u_s}{\partial s} + 2s\overline{M_{s\phi}}\frac{\partial}{\partial s}\left(\frac{u_\phi}{s}\right)\\
\frac{\partial \overline{M_{s\phi}}}{\partial t} &= -(\mathbf{u}\cdot\nabla_e)\overline{M_{s\phi}} + s\overline{M_{ss}}\frac{\partial}{\partial s}\left(\frac{u_\phi}{s}\right) + \frac{1}{s}\overline{M_{\phi\phi}}\frac{\partial u_s}{\partial \phi}
\end{aligned}$$

Using the transforms (2.10), the equations can be rewritten as

$$\begin{aligned}
\frac{\partial \overline{M_{ss}}}{\partial t} &= -(\mathbf{u}\cdot\nabla_e)\overline{M_{ss}} + \frac{1}{4}\left(2\overline{M_1}+\overline{M_+}+\overline{M_-}\right)\left(2\frac{\partial u_s}{\partial s} - \frac{s u_s}{H^2}\right) - \frac{i}{2}\left(\overline{M_+}-\overline{M_-}\right)\frac{1}{s}\frac{\partial u_s}{\partial \phi}\\
\frac{\partial \overline{M_{\phi\phi}}}{\partial t} &= -(\mathbf{u}\cdot\nabla_e)\overline{M_{\phi\phi}} - \frac{1}{4}\left(2\overline{M_1}-\overline{M_+}-\overline{M_-}\right)\left(2\frac{\partial u_s}{\partial s} - \frac{s u_s}{H^2}\right) - \frac{i}{2}\left(\overline{M_+}-\overline{M_-}\right)s\frac{\partial}{\partial s}\frac{u_\phi}{s}\\
\frac{\partial \overline{M_{s\phi}}}{\partial t} &= -(\mathbf{u}\cdot\nabla_e)\overline{M_{s\phi}} + \frac{1}{4}\left(2\overline{M_1}+\overline{M_+}+\overline{M_-}\right)s\frac{\partial}{\partial s}\left(\frac{u_\phi}{s}\right) + \frac{1}{4}\left(2\overline{M_1}-\overline{M_+}-\overline{M_-}\right)\frac{1}{s}\frac{\partial u_s}{\partial \phi}
\end{aligned}$$

Re-combining these equations again using the transforms (2.10), we obtain the evolution equations for the conjugate variables

$$\begin{aligned}
\frac{\partial \overline{M_1}}{\partial t} &= -\left(\mathbf{u}\cdot\nabla_e\right)\overline{M_1} + \left(\overline{M_+}+\overline{M_-}\right)\left(\frac{\partial u_s}{\partial s} - \frac{s u_s}{2H^2}\right) - \frac{i}{2}\left(\overline{M_+}-\overline{M_-}\right)\left(s\frac{\partial}{\partial s}\frac{u_\phi}{s} + \frac{1}{s}\frac{\partial u_s}{\partial \phi}\right)\\
\frac{\partial \overline{M_+}}{\partial t} &= -\left(\mathbf{u}\cdot\nabla_e\right)\overline{M_+} + \overline{M_1}\left(2\frac{\partial u_s}{\partial s} - \frac{s u_s}{H^2}\right) + i\overline{M_1}\left(s\frac{\partial}{\partial s}\frac{u_\phi}{s} + \frac{1}{s}\frac{\partial u_s}{\partial \phi}\right) + i\overline{M_+}\left(s\frac{\partial}{\partial s}\frac{u_\phi}{s} - \frac{1}{s}\frac{\partial u_s}{\partial \phi}\right)\\
\frac{\partial \overline{M_-}}{\partial t} &= -\left(\mathbf{u}\cdot\nabla_e\right)\overline{M_-} + \overline{M_1}\left(2\frac{\partial u_s}{\partial s} - \frac{s u_s}{H^2}\right) - i\overline{M_1}\left(s\frac{\partial}{\partial s}\frac{u_\phi}{s} + \frac{1}{s}\frac{\partial u_s}{\partial \phi}\right) - i\overline{M_-}\left(s\frac{\partial}{\partial s}\frac{u_\phi}{s} - \frac{1}{s}\frac{\partial u_s}{\partial \phi}\right)
\end{aligned}$$

The complete list of evolution equations for the conjugate quantities (12×) are given by

$$\begin{aligned}
\frac{\partial \overline{M_1}}{\partial t} &= -\left(\mathbf{u}\cdot\nabla_e\right)\overline{M_1} + \left(\overline{M_+}+\overline{M_-}\right)\left(\frac{\partial u_s}{\partial s} - \frac{s u_s}{2H^2}\right) - \frac{i}{2}\left(\overline{M_+}-\overline{M_-}\right)\left(s\frac{\partial}{\partial s}\frac{u_\phi}{s} + \frac{1}{s}\frac{\partial u_s}{\partial \phi}\right)\\
\frac{\partial \overline{M_+}}{\partial t} &= -\left(\mathbf{u}\cdot\nabla_e\right)\overline{M_+} + \overline{M_1}\left(2\frac{\partial u_s}{\partial s} - \frac{s u_s}{H^2}\right) + i\overline{M_1}\left(s\frac{\partial}{\partial s}\frac{u_\phi}{s} + \frac{1}{s}\frac{\partial u_s}{\partial \phi}\right) + i\overline{M_+}\left(s\frac{\partial}{\partial s}\frac{u_\phi}{s} - \frac{1}{s}\frac{\partial u_s}{\partial \phi}\right)\\
\frac{\partial \overline{M_-}}{\partial t} &= -\left(\mathbf{u}\cdot\nabla_e\right)\overline{M_-} + \overline{M_1}\left(2\frac{\partial u_s}{\partial s} - \frac{s u_s}{H^2}\right) - i\overline{M_1}\left(s\frac{\partial}{\partial s}\frac{u_\phi}{s} + \frac{1}{s}\frac{\partial u_s}{\partial \phi}\right) - i\overline{M_-}\left(s\frac{\partial}{\partial s}\frac{u_\phi}{s} - \frac{1}{s}\frac{\partial u_s}{\partial \phi}\right)\\
\frac{\partial \widetilde{M_{z+}}}{\partial t} &= -(\mathbf{u}\cdot\nabla_e)\widetilde{M_{z+}} + \frac{1}{2}\left(3\frac{\partial u_z}{\partial z} - \frac{i}{s}\frac{\partial u_s}{\partial \phi} + is\frac{\partial}{\partial s}\frac{u_\phi}{s}\right)\widetilde{M_{z+}} + \frac{1}{2}\left(\frac{\partial u_z}{\partial z} + 2\frac{\partial u_s}{\partial s} + \frac{i}{s}\frac{\partial u_s}{\partial \phi} + is\frac{\partial}{\partial s}\frac{u_\phi}{s}\right)\widetilde{M_{z-}}\\
&\quad - \frac{1}{2}\left(\frac{\partial}{\partial s}\frac{s u_s}{H^2} + \frac{i}{H^2}\frac{\partial u_s}{\partial \phi}\right)\widetilde{zM_1} + \frac{1}{2}\left(-\frac{\partial}{\partial s}\frac{s u_s}{H^2} + \frac{i}{H^2}\frac{\partial u_s}{\partial \phi}\right)\widetilde{zM_+}\\
\frac{\partial \widetilde{M_{z-}}}{\partial t} &= -(\mathbf{u}\cdot\nabla_e)\widetilde{M_{z-}} + \frac{1}{2}\left(\frac{\partial u_z}{\partial z} + 2\frac{\partial u_s}{\partial s} - \frac{i}{s}\frac{\partial u_s}{\partial \phi} - is\frac{\partial}{\partial s}\frac{u_\phi}{s}\right)\widetilde{M_{z+}} + \frac{1}{2}\left(3\frac{\partial u_z}{\partial z} + \frac{i}{s}\frac{\partial u_s}{\partial \phi} - is\frac{\partial}{\partial s}\frac{u_\phi}{s}\right)\widetilde{M_{z-}}\\
&\quad - \frac{1}{2}\left(\frac{\partial}{\partial s}\frac{s u_s}{H^2} - \frac{i}{H^2}\frac{\partial u_s}{\partial \phi}\right)\widetilde{zM_1} + \frac{1}{2}\left(-\frac{\partial}{\partial s}\frac{s u_s}{H^2} - \frac{i}{H^2}\frac{\partial u_s}{\partial \phi}\right)\widetilde{zM_-}
\end{aligned}$$

$$\frac{\partial \widetilde{zM_1}}{\partial t} = -(\mathbf{u}\cdot\nabla_e)\widetilde{zM_1} + \frac{\partial u_z}{\partial z}\widetilde{zM_1} + \frac{1}{2}\left(2\frac{\partial u_s}{\partial s} + \frac{\partial u_z}{\partial z}\right)\left(\widetilde{zM_+} + \widetilde{zM_-}\right) - \frac{i}{2}\left(\frac{1}{s}\frac{\partial u_s}{\partial \phi} + s\frac{\partial}{\partial s}\frac{u_\phi}{s}\right)\left(\widetilde{zM_+} - \widetilde{zM_-}\right)$$

$$\frac{\partial \widetilde{zM_+}}{\partial t} = -(\mathbf{u}\cdot\nabla_e)\widetilde{zM_+} + \frac{\partial u_z}{\partial z}\widetilde{zM_+} + i\left(s\frac{\partial}{\partial s}\frac{u_\phi}{s} - \frac{1}{s}\frac{\partial u_s}{\partial \phi}\right)\widetilde{zM_+} + \left(\frac{\partial u_s}{\partial s} + \frac{1}{2}\frac{\partial u_z}{\partial z} + is\frac{\partial}{\partial s}\frac{u_\phi}{s} + \frac{i}{s}\frac{\partial u_s}{\partial \phi}\right)\widetilde{zM_1}$$

$$\frac{\partial \widetilde{zM_-}}{\partial t} = -(\mathbf{u}\cdot\nabla_e)\widetilde{zM_-} + \frac{\partial u_z}{\partial z}\widetilde{zM_-} - i\left(s\frac{\partial}{\partial s}\frac{u_\phi}{s} - \frac{1}{s}\frac{\partial u_s}{\partial \phi}\right)\widetilde{zM_-} + \left(\frac{\partial u_s}{\partial s} + \frac{1}{2}\frac{\partial u_z}{\partial z} - is\frac{\partial}{\partial s}\frac{u_\phi}{s} - \frac{i}{s}\frac{\partial u_s}{\partial \phi}\right)\widetilde{zM_1}$$

$$\begin{aligned}
\frac{\partial B_{e+}}{\partial t} = &- (\mathbf{u}_e\cdot\nabla_e)\, B_{e+} + \frac{1}{2}B_{e+}\left[\left(\frac{\partial}{\partial s} - \frac{i}{s}\frac{\partial}{\partial \phi}\right)(u_{es} + iu_{e\phi}) + \frac{1}{s}(u_{es} - iu_{e\phi})\right] \\
&+ \frac{1}{2}B_{e-}\left[\left(\frac{\partial}{\partial s} + \frac{i}{s}\frac{\partial}{\partial \phi}\right)(u_{es} + iu_{e\phi}) - \frac{1}{s}(u_{es} + iu_{e\phi})\right] \\
\frac{\partial B_{e-}}{\partial t} = &- (\mathbf{u}_e\cdot\nabla_e)\, B_{e-} + \frac{1}{2}B_{e+}\left[\left(\frac{\partial}{\partial s} - \frac{i}{s}\frac{\partial}{\partial \phi}\right)(u_{es} - iu_{e\phi}) - \frac{1}{s}(u_{es} - iu_{e\phi})\right] \\
&+ \frac{1}{2}B_{e-}\left[\left(\frac{\partial}{\partial s} + \frac{i}{s}\frac{\partial}{\partial \phi}\right)(u_{es} - iu_{e\phi}) + \frac{1}{s}(u_{es} + iu_{e\phi})\right] \\
\frac{\partial B_{e+,z}}{\partial t} = &- (\mathbf{u}_e\cdot\nabla_e)\, B_{e+,z} + \frac{1}{2}B_{e+,z}\left[\left(\frac{\partial}{\partial s} - \frac{i}{s}\frac{\partial}{\partial \phi}\right)(u_{es} + iu_{e\phi}) + \frac{1}{s}(u_{es} - iu_{e\phi})\right] \\
&- \frac{\partial u_z}{\partial z}B_{e+,z} + \frac{1}{2}B_{e-,z}\left[\left(\frac{\partial}{\partial s} + \frac{i}{s}\frac{\partial}{\partial \phi}\right)(u_{es} + iu_{e\phi}) - \frac{1}{s}(u_{es} + iu_{e\phi})\right] \\
\frac{\partial B_{e-,z}}{\partial t} = &- (\mathbf{u}_e\cdot\nabla_e)\, B_{e-,z} + \frac{1}{2}B_{e+,z}\left[\left(\frac{\partial}{\partial s} - \frac{i}{s}\frac{\partial}{\partial \phi}\right)(u_{es} - iu_{e\phi}) - \frac{1}{s}(u_{es} - iu_{e\phi})\right] \\
&- \frac{\partial u_z}{\partial z}B_{e-,z} + \frac{1}{2}B_{e-,z}\left[\left(\frac{\partial}{\partial s} + \frac{i}{s}\frac{\partial}{\partial \phi}\right)(u_{es} - iu_{e\phi}) + \frac{1}{s}(u_{es} + iu_{e\phi})\right]
\end{aligned}$$

Appendix A

Code implementation

A.1 Code design

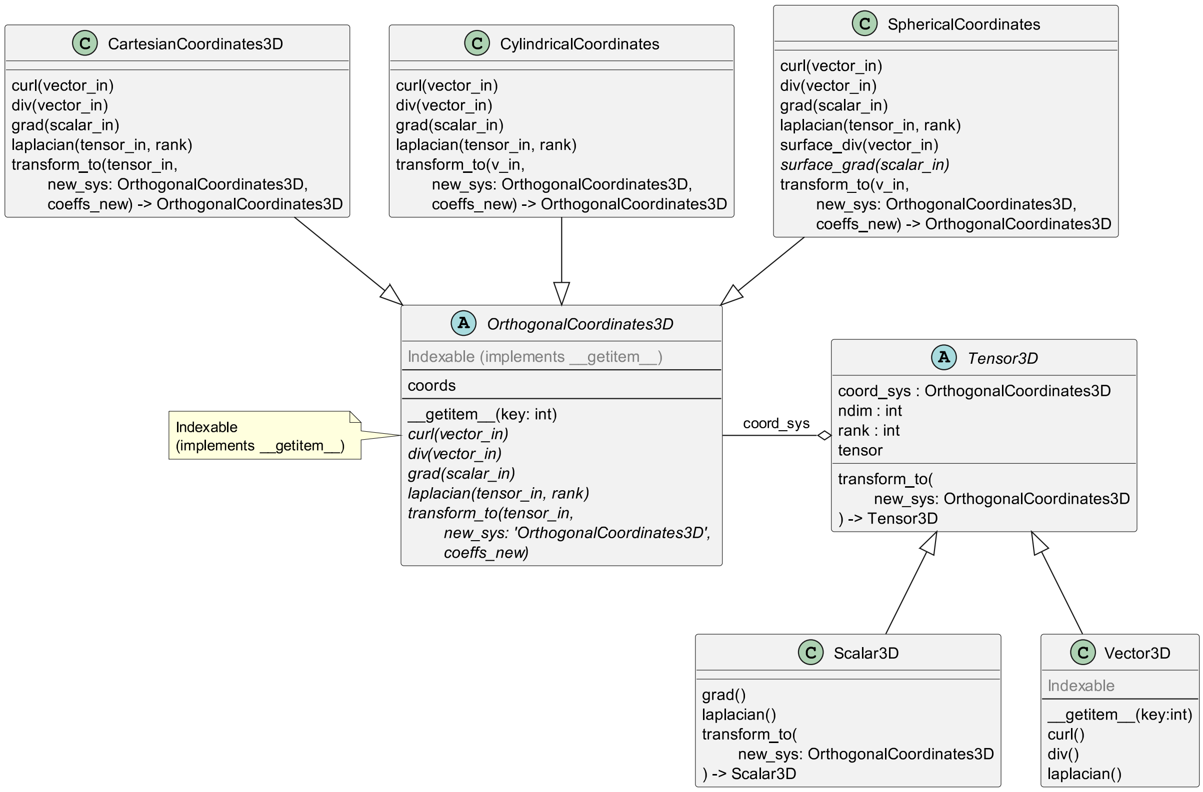

Figure A.1: Module for supplementary vector operations and calculus.

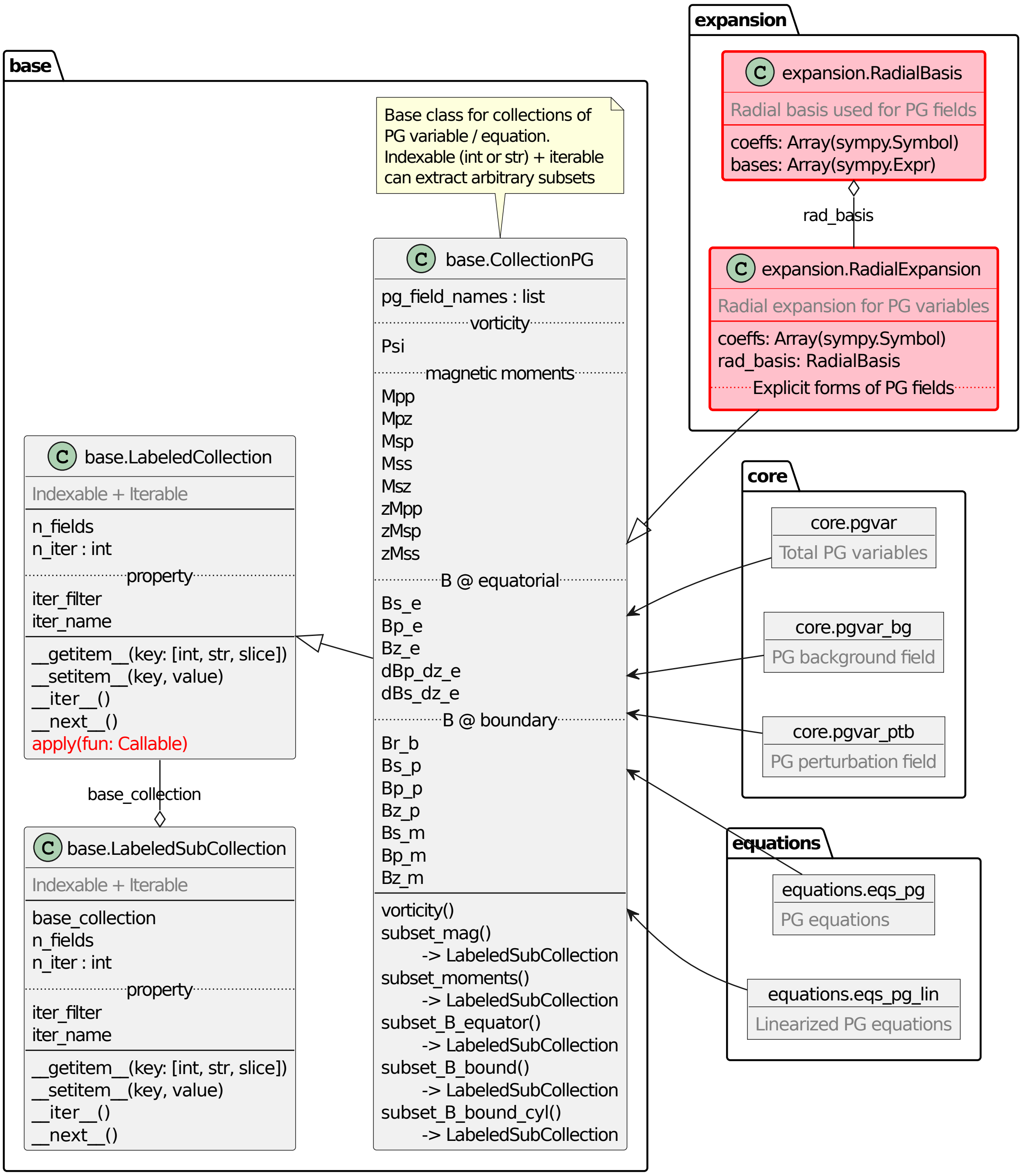

Figure A.2: Module PG model. Red items are items to be implemented

A.2 Why does the quadrature works in SciPy?

This section is regarding a technical detail on the behaviour of `eval_jacobi` in the scipy package, especially when the degree n is negative.

Why are there negative degrees in the Jacobi polynomials used in the code?

In computing the system matrices, stiffness matrix $\mathbf{K}$ in particular, it is often the case that we need to compute the inner product in the form of

$$\left\langle s^{m_1}(1-s^2)^{m_2} P_{n'}^{(\alpha',\beta')}(2s^2-1), \mathcal{L}\left(s^{m_3}(1-s^2)^{m_4} P_n^{(\alpha,\beta)}(2s^2-1)\right)\right\rangle.$$

The result of the trial function being operated on by the linear operator $\mathcal{L}$ typically involves $P_n^{(\alpha,\beta)}(2s^2-1)$, its derivative with respect to s, i.e. $\frac{d^k}{ds^k}P_n^{(\alpha,\beta)}(2s^2-1)$, or in other words, involving derivatives of the Jacobi polynomial $\frac{d^k}{d\xi^k}P_n^{(\alpha,\beta)}(\xi)|_{\xi=2s^2-1}$. This is not a problem in symbolic engines (actually, the only functioning way in `SymPy` to calculate such inner products is to keep the derivative as it is, unevaluated), but not acceptable for numerical routines. Typical numerical routines, whether using `SciPy` in Python or `MATLAB`, have no idea how to calculate the "derivative of a Jacobi polynomial". In order to use this numerical routines, the only remaining feasible way seems to be simplifying the expression into explicit polynomials at each given n and n' (using some symbolic engines), and then hand over the explicit polynomial to `SciPy`. However, this means that the most desirable feature of this numerical libraries, i.e. vectorized and parallelized operations, are out of the picture. Evaluating these inner products purely numerically thus seems to encounter a problem.

There is, however, a robust workaround: the derivatives of Jacobi polynomials can always be converted to another Jacobi polynomial, using the relation

$$\frac{d^k}{dz^k}P_n^{(\alpha,\beta)}(z) = \frac{\Gamma(\alpha+\beta+n+1+k)}{2^k\Gamma(\alpha+\beta+n+1)}P_{n-k}^{(\alpha+k,\beta+k)}(z).$$

This can be easily done as soon as `SymPy` is asked to simplify or "evaluate" the derivatives concerning the Jacobi polynomials. Now the integrand can be safely converted to a series of algebraic calculations involving only the undifferentiated Jacobi polynomials. This expression can be handed over to numerical functions, that can be evaluated at multiple n, n' as well as z in a vectorized fashion very efficiently. However, here comes another question: what is $P_{n-k}^{(\alpha+k,\beta+k)}(z)$, when $n<k$? How will the numerical routine handle this?

Is there a Jacobi polynomial with negative degree?

Strictly/semantically speaking, there is no such a thing. A polynomial is really just a polynomial, and can only have non-negative degrees. In fact, if you ask `SymPy` to evaluate a Jacobi polynomial with negative degree:

```
>>> sympy.jacobi(-1, 5/2, 4, -0.9).evalf()
...
ValueError: Cannot generate Jacobi polynomial of degree -1
```

Indeed, if we follow the definition on Wikipedia page,

$$P_n^{(\alpha,\beta)}(z) = \frac{(\alpha+1)_n}{n!}\,{}_2F_1\left(-n, 1+\alpha+\beta+n, 1+\alpha, \frac{1-z}{2}\right)$$

where ${}_2F_1$ is the hypergeometric function. The form of the prefactor is apparently only restricted to non-negative n, since factorial as well as Pochhammer's symbol usually only takes non-negative n arguments. However, both `Mathematica` and `SciPy` are okay with evaluating Jacobi polynomials with negative degrees:

```
(*Mathematica*)
In[1]=  N[JacobiP[-1, 5/2, 4, -0.9]]
Out[1]= 0.
```

```
# Python
>>> from scipy.special import eval_jacobi
>>> eval_jacobi(-1, 5/2, 4, -0.9)
0.
```

As long as... well, the polynomial with the negative degree is not evaluated at one specific point, $z = -1$:

```
(*Mathematica*)
In[1]=  N[JacobiP[-1, 5/2, 4, -1]]
   ... Power: Infinite expression 1/0^4 encountered
   ... Infinity: Indeterminate expression 0 ComplexInfinity encountered
Out[1]= Indeterminate
```

```
# Python
>>> from scipy.special import eval_jacobi
>>> eval_jacobi(-1, 5/2, 4, -1.)
nan
```

But why is this the case, if the polynomial shouldn't even have negative degree?

What is the implementation for the Jacobi polynomial with negative degree?

Now we have to understand what is happening behind the curtain: how is the Jacobi polynomial actually implemented, such that both Mathematica and SciPy allow negative degrees? Will this guarantee that the quadrature of the inner product is correct? The best way to answer these questions is to check the source code. Unfortunately, this does not work for Mathematica, a closed-source software. This can however be done for SciPy. It took me a while to find the relevant piece in the source code, as this part is in Cython:

```
cdef inline number_t eval_jacobi(double n, double alpha, double beta,
    number_t x) noexcept nogil:
     cdef double a, b, c, d
     cdef number_t g

     d = binom(n+alpha, n)
     a = -n
     b = n + alpha + beta + 1
     c = alpha + 1
     g = 0.5*(1-x)
     return d * hyp2f1(a, b, c, g)
```

It turns out that instead of using the factorial and Pochhammer's symbol, SciPy implements the following relation

$$P_n^{(\alpha,\beta)}(z) = \binom{n+\alpha}{n} {}_2F_1\left(-n, 1+\alpha+\beta+n, 1+\alpha, \frac{1-z}{2}\right)$$

where the prefactor is given by a binomial coefficient. Usually, the binomial coefficient does not make sense for negative n, but if one further looks at the source code for binom, one would realize that except for special occasions, the binomial coefficients are calculated as

$$\binom{n+\alpha}{n} = \frac{1}{(n+\alpha+1)B(1+\alpha,1+n)} = \frac{1}{n+\alpha+1}\frac{\Gamma(2+\alpha+n)}{\Gamma(1+\alpha)\Gamma(1+n)}$$

which gives 0 for any $n \in \mathbb{Z}^- \cup \{0\}$ when $\alpha + n \notin \mathbb{Z}^-$ (because $\Gamma(1+n) \to \infty$). Now, the second criterion is always fulfilled. As $P_{n-k}^{(\alpha+k,\beta+k)}$ is the ultimate function to be evaluated, our $n+\alpha$ is actually $(n-k)+(\alpha+k) = n+\alpha$. Since the original Jacobi polynomial is a legitimate polynomial, $\alpha > -1$ and

$n \geq 0$, therefore $n + \alpha \notin \mathbb{Z}^-$. In summary, `SciPy` will give 0, which is the desired outcome, when a Jacobi polynomial with negative degree is encountered.

On a related note, the Jacobi polynomial in the form of

$$P_n^{(\alpha,\beta)}(z) = \frac{1}{n+\alpha+1}\frac{\Gamma(2+\alpha+n)}{\Gamma(1+\alpha)\Gamma(1+n)}\,{}_2F_1\left(-n, 1+\alpha+\beta+n, 1+\alpha, \frac{1-z}{2}\right) \tag{A.1}$$

might be a good formula for analytic continuation in α, β, n. This formula seems to have finite value for any point, except for a zero-measure set in the 4-D space.

So why does the evaluation fail at $z = -1$ for negative n? This is the branch point for the hypergeometric function at $n \in \mathbb{Z}^-$, and evaluation is not available even for the equation above.

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