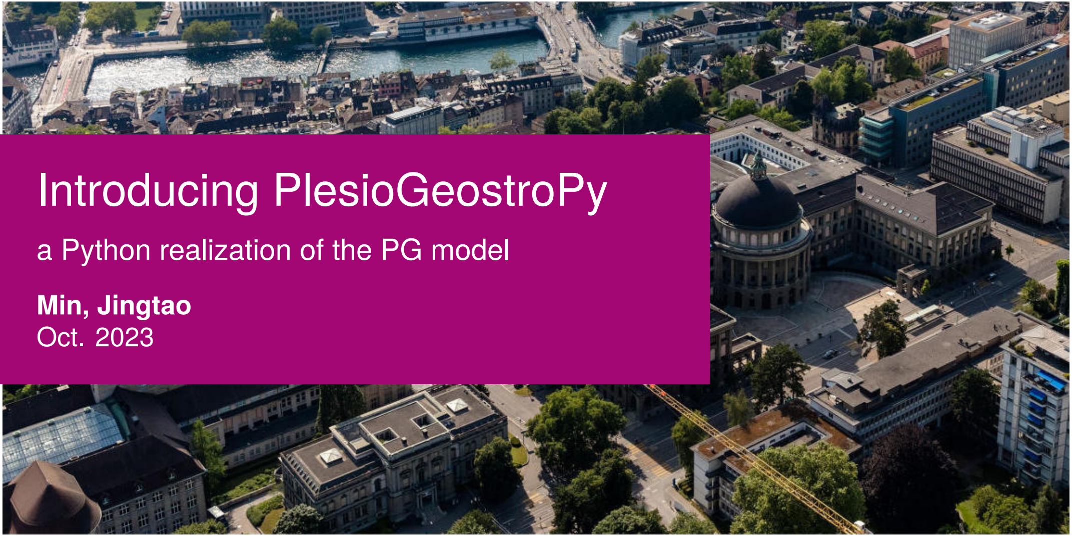

Introducing PlesioGeostroPy

a Python realization of the PG model

Min, Jingtao

Oct. 2023

Outline

1. The PG model
2. The old implementation
3. Introducing PlesioGeostroPy
4. Future works

PG, QG, and all the geostrophic models

How did I end up here?

Since the start of my doctoral studies, I

- studied surface operators and tried to derive the boundary terms in the diffusive torsional oscillation (TO) equation for 2 months $\Longrightarrow$ didn't lead anywhere;
- studied torsional Alfvén waves and calculated the 1-D eigenmodes of the torsional oscillation for 2 month $\Longrightarrow$ didn't lead anywhere;
- studied anelastic approximation for 1 month in order to work on the anelastic version of QuICC, which may help the Jupiter simulation $\Longrightarrow$ project was called off and deemed unpromising before I could start to think about the numerics;
- picked up the thread and studied the reflection of Alfvén waves at the fluid-solid interface for 2 months $\Longrightarrow$ wrote a sixty-page document and yet had more questions than before;
- have been mainly working on the PG model since Aug 2023.

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1. The PG model
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The old implementation - code *Daria*

Mathematica implementation (previous member of the group, Dr. Daria Holdenried-Chernoff)

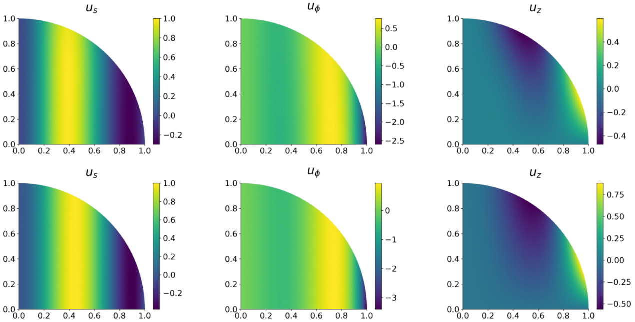

Figure: Eigenmodes calculated using 3-D code *Jiawen* (top) and PG code *Daria*.

Why reinvent the wheels when we already have a functioning implementation?

The old implementation - code *Daria*

Citizens of Gotham, riddle me this... what is this code snippet doing?

```
In[•]:= Les = 1/s D[s Bs2[s, p, t], s] + s^2/H[s] (BsH[s, p, z, t] * BsH[s, p, z, t] + BsmH[s, p, z, t] * BsmH[s, p, z, t]) + 1/s D[BsBp[s, p, t], p] - 1/s Bp2[s, p, t] + BsH[s, p, z, t] * BzH[s, p, z, t] -
    BsmH[s, p, z, t] * BzmH[s, p, z, t];

In[•]:= Lestmp =
   Les /. Bs2[s, p, t] → A0[s, p, t] + x * A[s, p, t] /. D[Bs2[s, p, t], s] → D[ A0[s, p, t] + x * A[s, p, t], s] /. D[Bs2[s, p, t], p] → D[ A0[s, p, t] + x * A[s, p, t], p] /.
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      BsBp[s, p, t] → C0[s, p, t] + x * Csp[s, p, t] /. D[BsBp[s, p, t], s] → D[ C0[s, p, t] + x * Csp[s, p, t], s] /. D[BsBp[s, p, t], p] → D[ C0[s, p, t] + x * Csp[s, p, t], p] /.
      BsmH[s, p, z, t] → amh0[s, p, z, t] + x * amh[s, p, z, t] /. D[BsmH[s, p, z, t], s] → D[amh0[s, p, z, t], s] + x * D[amh[s, p, z, t], s] /.
      D[BsmH[s, p, z, t], p] → D[amh0[s, p, z, t], p] + x * D[amh[s, p, z, t], p] /. BsH[s, p, z, t] → ah0[s, p, z, t] + x * ah[s, p, z, t] /.
      D[BsH[s, p, z, t], s] → D[ah0[s, p, z, t], s] + x * D[ah[s, p, z, t], s] /. D[BsH[s, p, z, t], p] → D[ah0[s, p, z, t], p] + x * D[ah[s, p, z, t], p] /.
      BpmH[s, p, z, t] → bmh0[s, p, z, t] + x * bmh[s, p, z, t] /. D[BpmH[s, p, z, t], s] → D[bmh0[s, p, z, t], s] + x * D[bmh[s, p, z, t] , s] /.
      D[BpmH[s, p, z, t], p] → D[bmh0[s, p, z, t], p] + x * D[bmh[s, p, z, t] , p] /. BpH[s, p, z, t] → bh0[s, p, z, t] + x * bh[s, p, z, t] /.
      D[BpH[s, p, z, t], s] → D[bh0[s, p, z, t], s] + x * D[bh[s, p, z, t] , s] /. D[BpH[s, p, z, t], p] → D[bh0[s, p, z, t], p] + x * D[bh[s, p, z, t] , p] /.
      D[BpH[s, p, z, t], z] → D[bh0[s, p, z, t], z] + x * D[bh[s, p, z, t] , z] /. BzmH[s, p, z, t] → cmh0[s, p, z, t] + x * cmh[s, p, z, t] /. BzH[s, p, z, t] → ch0[s, p, z, t] + x * ch[s, p, z, t] /.
      D[BzH[s, p, z, t], z] → D[ch0[s, p, z, t], z] + x * D[ch[s, p, z, t], z] /. D[BzH[s, p, z, t], s] → D[ch0[s, p, z, t], s] + x * D[ch[s, p, z, t], s] /.
      D[BzmH[s, p, z, t], s] → D[cmh0[s, p, z, t], s] + x * D[cmh[s, p, z, t], s];
```

The old implementation - code *Daria*

Citizens of Gotham, riddle me this... what is this code snippet doing?

```
In[•]:= Les = 1/s D[s Bs2[s, p, t], s] + s^2/H[s] (BsH[s, p, z, t] * BsH[s, p, z, t] + BsmH[s, p, z, t] * BsmH[s, p, z, t]) + 1/s D[BsBp[s, p, t], p] - 1/s Bp2[s, p, t] + BsH[s, p, z, t] * BzH[s, p, z, t] -
      BsmH[s, p, z, t] * BzmH[s, p, z, t];

In[•]:= Lestmp =
     Les /. Bs2[s, p, t] → A0[s, p, t] + x * A[s, p, t] /. D[Bs2[s, p, t], s] → D[ A0[s, p, t] + x * A[s, p, t], s] /. D[Bs2[s, p, t], p] → D[ A0[s, p, t] + x * A[s, p, t], p] /.
                            Bp2[s, p, t] → B0[s, p, t] + x * B[s, p, t] /. D[Bp2[s, p, t], s] → D[ B0[s, p, t] + x * B[s, p, t], s] /. D[Bp2[s, p, t], p] → D[ B0[s, p, t] + x * B[s, p, t], p] /.
                         BsBp[s, p, t] → C0[s, p, t] + x * Csp[s, p, t] /. D[BsBp[s, p, t], s] → D[ C0[s, p, t] + x * Csp[s, p, t], s] /. D[BsBp[s, p, t], p] → D[ C0[s, p, t] + x * Csp[s, p, t], p] /.
                      BsmH[s, p, z, t] → amh0[s, p, z, t] + x * amh[s, p, z, t] /. D[BsmH[s, p, z, t], s] →  D[amh0[s, p, z, t], s] + x * D[amh[s, p, z, t], s] /.
                    D[BsmH[s, p, z, t], p] →  D[amh0[s, p, z, t], p] + x * D[amh[s, p, z, t], p] /. BsH[s, p, z, t] → ah0[s, p, z, t] + x * ah[s, p, z, t] /.
                  D[BsH[s, p, z, t], s] →  D[ah0[s, p, z, t], s] + x * D[ah[s, p, z, t], s] /. D[BsH[s, p, z, t], p] →  D[ah0[s, p, z, t], p] + x * D[ah[s, p, z, t], p] /.
                BpmH[s, p, z, t]  → bmh0[s, p, z, t] + x * bmh[s, p, z, t]  /. D[BpmH[s, p, z, t], s] → D[bmh0[s, p, z, t], s] + x * D[bmh[s, p, z, t] , s] /.
              D[BpmH[s, p, z, t], p] → D[bmh0[s, p, z, t], p] + x * D[bmh[s, p, z, t] , p] /. BpH[s, p, z, t]  → bh0[s, p, z, t] + x * bh[s, p, z, t]  /.
            D[BpH[s, p, z, t], s] → D[bh0[s, p, z, t], s] + x * D[bh[s, p, z, t] , s] /. D[BpH[s, p, z, t], p] → D[bh0[s, p, z, t], p] + x * D[bh[s, p, z, t] , p] /.
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        D[BzH[s, p, z, t], z] → D[ch0[s, p, z, t], z] + x * D[ch[s, p, z, t], z] /. D[BzH[s, p, z, t], s] → D[ch0[s, p, z, t], s] + x * D[ch[s, p, z, t], s] /.
      D[BzmH[s, p, z, t], s] → D[cmh0[s, p, z, t], s] + x * D[cmh[s, p, z, t], s];
```

This rewrites the magnetic quantities in Lorentz force $\overline{L_\phi}$ as background + perturbation for linearization.
e.g. `Bp2[s, p, t]->B0[s, p, t] + x*B[s, p, t]` equals to $\overline{B_\phi^2} = \overline{B_\phi^2}_0 + \epsilon\overline{B_\phi^2}'$

The old implementation - code *Daria*

```
In[•]:= Les = 1/s D[s Bs2[s, p, t], s] + s^2/H[s] (BsH[s, p, z, t] * BsH[s, p, z, t] + BsmH[s, p, z, t] * BsmH[s, p, z, t]) + 1/s D[BsBp[s, p, t], p] - 1/s Bp2[s, p, t] + BsH[s, p, z, t] * BzH[s, p, z, t] -
    BsmH[s, p, z, t] * BzmH[s, p, z, t];

In[•]:= Lestmp =
   Les /. Bs2[s, p, t] → A0[s, p, t] + x * A[s, p, t] /. D[Bs2[s, p, t], s] → D[ A0[s, p, t] + x * A[s, p, t], s] /. D[Bs2[s, p, t], p] → D[ A0[s, p, t] + x * A[s, p, t], p] /.
        Bp2[s, p, t] → B0[s, p, t] + x * B[s, p, t] /. D[Bp2[s, p, t], s] → D[ B0[s, p, t] + x * B[s, p, t], s] /. D[Bp2[s, p, t], p] → D[ B0[s, p, t] + x * B[s, p, t], p] /.
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```

Why is the code undesirable for developers and users alike?

- long, repetitive operations $\Longrightarrow$ hard to debug
- namings: the background and perturbation fields are named from `A` to `H` $\Longrightarrow$ hard to debug or invoke.
- At some point I even need a correspondence table to decipher the code.

The old implementation - code *Daria*

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In[•]:= Les = 1/s D[s Bs2[s, p, t], s] + s^2/H[s] (BsH[s, p, z, t] * BsH[s, p, z, t] + BsmH[s, p, z, t] * BsmH[s, p, z, t]) + 1/s D[BsBp[s, p, t], p] - 1/s Bp2[s, p, t] - BsH[s, p, z, t] * BzH[s, p, z, t] -
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      D[BzmH[s, p, z, t], s] → D[cmh0[s, p, z, t], s] + x * D[cmh[s, p, z, t], s];
```

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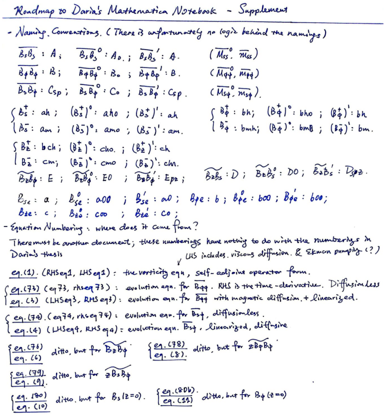

Figure: List of physical variables and their names in the code.

The old implementation - code *Daria*

```
In[]:= psi[m_, n_] := (1 - s^2)^(3/2) s^m JacobiP[n - 1, 3/2, m, 2 s^2 - 1]

In[]:= Fm[m_] := Piecewise[{{1, m == 0}, {s, m == 1}, {s^(Abs[m] - 2), m > 1}}]
       βeven[m_] := Piecewise[{{-1/2, m == 0}, {1 - 1/2, m == 1}, {m - 5/2, m > 1}}]

In[]:= BackgRegFactor = 1; (*s (-1+s^2);*)

In[]:= BsExp[m_, n_] := BackgRegFactor * (1 - s^2)^(1/2) s^(Abs[m] - 1) JacobiP[n - 1, 1, Abs[m] - 3/2, 2 s^2 - 1]
       BpExp[m_, n_] := BackgRegFactor * (1 - s^2)^(1/2) s^(Abs[m] + 1) JacobiP[n - 1, 1, Abs[m] + 1/2, 2 s^2 - 1]
       BzExp[m_, n_] := BackgRegFactor * (1 - s^2) s^Abs[m] JacobiP[n - 1, 2, Abs[m] - 1/2, 2 s^2 - 1]
```

Magnetic moments expansions for toroidal background field

```
In[]:= Bsp [m_, n_] := (1 - s^2)^(1/2) * s^Abs[m - 2] * JacobiP[n - 1, 1, Abs[m - 2] - 1/2, 2 s^2 - 1]
       Bpp [m_, n_] := (1 - s^2)^(1/2) * Fm[m] * s^2 * JacobiP[n - 1, 1, βeven[m] + 2, 2 s^2 - 1] (*Wherever Bpp
       Bpz[m_, n_] := (1 - s^2) s^(m - 1) JacobiP[n - 1, 2, m - 3/2, 2 s^2 - 1]
       Bzpp[m_, n_] := (1 - s^2) Fm[m] * s^2 * JacobiP[n - 1, 2, βeven[m] + 2, 2 s^2 - 1] (*Wherever Bzpp
       there needs to be a Bzsp contribution to ensure correct regularity. Note extra H from
       Bzsp[m_, n_] := (1 - s^2) s^Abs[m - 2] JacobiP[n - 1, 2, Abs[m - 2] - 1/2, 2 s^2 - 1]
       bs0 [m_, n_] := (1 - s^2) s^(Abs[m] - 1) JacobiP[n - 1, 2, Abs[m] - 1 - 1/2, 2 s^2 - 1]
       bp0 [m_, n_] := (1 - s^2) s^(Abs[m] + 1) JacobiP[n - 1, 2, Abs[m] + 1 - 1/2, 2 s^2 - 1] (*Factor of H ensu
```

Inflexible expansions: the expansion and the construction of matrices are hard coded in each notebook.

To change the expansion, one needs to

- produce a complete new Mathematica notebook;
- manually change the way the code collects the matrix elements.

Especially cumbersome to implement coupling, or if the bases do not coincide with the field quantities (more detail later).

The old implementation

Other problems

- The code is not efficient numerically, nor easily linked to efficient numerical libraries.
- Cryptic Mathematica syntax sugars

```
In[•]:= sol = Solve[JacobiP[colN, α, β, x] == 0 , x];
        rlocation = N[x /. # & /@ sol, prec];
```

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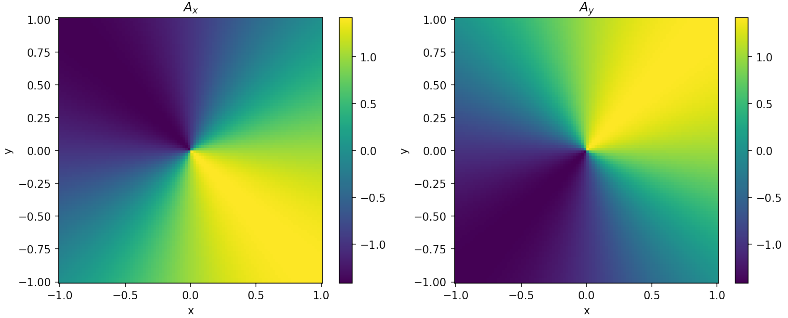
Regularity conditions for vectors

Vector $\mathbf{A} = A_s\hat{\mathbf{s}} + A_\phi\hat{\boldsymbol{\phi}} = A_x\hat{\mathbf{x}} + A_y\hat{\mathbf{y}}$
A_s and A_ϕ being regular functions of s and ϕ is not sufficient for $\mathbf{A}$ to be regular.

E.g. $A_s = A_\phi \equiv 1 \quad \Longrightarrow \quad A_x = \cos\phi - \sin\phi = \frac{x-y}{\sqrt{x^2+y^2}}, \quad A_y = \cos\phi + \sin\phi = \frac{x+y}{\sqrt{x^2+y^2}}$



Regularity conditions for vectors

However, A_x and A_y being regular functions of x and y **is sufficient** for $\mathbf{A}$ to be regular. **lewis_physical_1990** derived the regularity conditions on Fourier coefficients

- Expand A_s and A_ϕ in Fourier series;
- Convert to vector components in Cartesian coordinates,

$$\begin{pmatrix} A_x \\ A_y \end{pmatrix} = \begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix} \begin{pmatrix} A_s \\ A_\phi \end{pmatrix}$$

  and derive Fourier expansion of A_x and A_y.
- Replace

$$e^{i|m|\phi} = \frac{(x+iy)^{|m|}}{s^{|m|}}, \quad e^{i|m|\phi} = \frac{(x-iy)^{|m|}}{s^{|m|}}$$

- A_x and A_y are regular if the resulting rational terms are regular.

Regularity conditions for vectors

The required ansätze for vectors in polar coordinates

$$A_s = sg_0 + \sum_{m\neq 0} \left(\lambda_m s^{|m|-1} + g_m s^{|m|+1}\right) e^{im\phi},$$
$$A_\phi = sh_0 + \sum_{m\neq 0} \left(i\,\mathrm{sgn}(m)\lambda_m s^{|m|-1} + h_m s^{|m|+1}\right) e^{im\phi},$$

where $g_m = g_m(s^2)$, $h_m = h_m(s^2)$. The Fourier coefficients of A_s and A_ϕ are coupled at the lowest order (of Taylor series) in s.

Regularity conditions for rank-2 tensors

Similar idea, now

$$\begin{pmatrix} A_{xx} & A_{xy} \\ A_{yx} & A_{yy} \end{pmatrix} = \begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix} \begin{pmatrix} A_{ss} & A_{s\phi} \\ A_{\phi s} & A_{\phi\phi} \end{pmatrix} \begin{pmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{pmatrix}$$

Proceed similarly as follows

- Expand A_{ss}, $A_{s\phi}$, $A_{\phi s}$, $A_{\phi\phi}$ in Fourier series
- Convert to components Cartesian coordinates
- Rewrite exponentials as polynomials in Cartesian coordinates
- A_{xx}, A_{xy}, A_{yx} and A_{yy} are regular if the resulting rational terms are regular.

Regularity conditions for rank-2 tensors

Regularity conditions.

$$A^m_{ss} + A^m_{\phi\phi} = s^{|m|} C(s^2)$$

$$A^m_{s\phi} - A^m_{\phi s} = s^{|m|} C(s^2)$$

$$A^m_{ss} - A^m_{\phi\phi} + i\left(A^m_{s\phi} + A^m_{\phi s}\right) = s^{|m+2|} C(s^2)$$

$$A^m_{ss} - A^m_{\phi\phi} - i\left(A^m_{s\phi} + A^m_{\phi s}\right) = s^{|m-2|} C(s^2)$$

Assuming the Fourier coefficients

$$A_{ij} = \sum_m e^{im\phi} A^m_{ij}(s) = \sum_m e^{im\phi} s^{|m|+\Delta_m} \sum_k A^{mk}_{ij} s^{2k}$$

Regularity conditions for rank-2 tensors

The required form of the Fourier coefficients

$$m = 0: \quad \begin{cases} A^{0}_{ss} = A^{00}_{ss} + s^2 C(s^2) \\ A^{0}_{\phi\phi} = A^{00}_{\phi\phi} + s^2 C(s^2) \\ A^{0}_{s\phi} = A^{00}_{s\phi} + s^2 C(s^2) \\ A^{0}_{\phi s} = A^{00}_{\phi s} + s^2 C(s^2) \end{cases}, \quad \begin{cases} A^{00}_{ss} = A^{00}_{\phi\phi} \\ A^{00}_{s\phi} = -A^{00}_{\phi s} \end{cases}$$

$$|m| = 1: \quad \begin{cases} A^{m}_{ss} = A^{m0}_{ss} s + s^3 C(s^2) \\ A^{m}_{\phi\phi} = A^{m0}_{\phi\phi} s + s^3 C(s^2) \\ A^{m}_{s\phi} = A^{m0}_{s\phi} s + s^3 C(s^2) \\ A^{m}_{\phi s} = A^{m0}_{\phi s} s + s^3 C(s^2) \end{cases}, \quad \left\{ A^{m0}_{s\phi} + A^{m0}_{\phi s} = i \, \mathrm{sgn}(m) \left(A^{m0}_{ss} - A^{m0}_{\phi\phi} \right) \right.$$

$$|m| \geq 2: \quad \begin{cases} A^{m}_{ss} = A^{m0}_{ss} s^{|m|-2} + A^{m1}_{ss} s^{|m|} + s^{|m|+2} C(s^2) \\ A^{m}_{\phi\phi} = A^{m0}_{\phi\phi} s^{|m|-2} + A^{m1}_{\phi\phi} s^{|m|} + s^{|m|+2} C(s^2) \\ A^{m}_{s\phi} = A^{m0}_{s\phi} s^{|m|-2} + A^{m1}_{s\phi} s^{|m|} + s^{|m|+2} C(s^2) \\ A^{m}_{\phi s} = A^{m0}_{\phi s} s^{|m|-2} + A^{m1}_{\phi s} s^{|m|} + s^{|m|+2} C(s^2) \end{cases}, \quad \begin{cases} A^{m0}_{ss} = -A^{m0}_{\phi\phi} \\ A^{m0}_{s\phi} = A^{m0}_{\phi s} \\ A^{m0}_{s\phi} = i \, \mathrm{sgn}(m) A^{m0}_{ss} \\ A^{m1}_{s\phi} + A^{m1}_{\phi s} = i \, \mathrm{sgn}(m) \left(A^{m1}_{ss} - A^{m1}_{\phi\phi} \right). \end{cases}$$