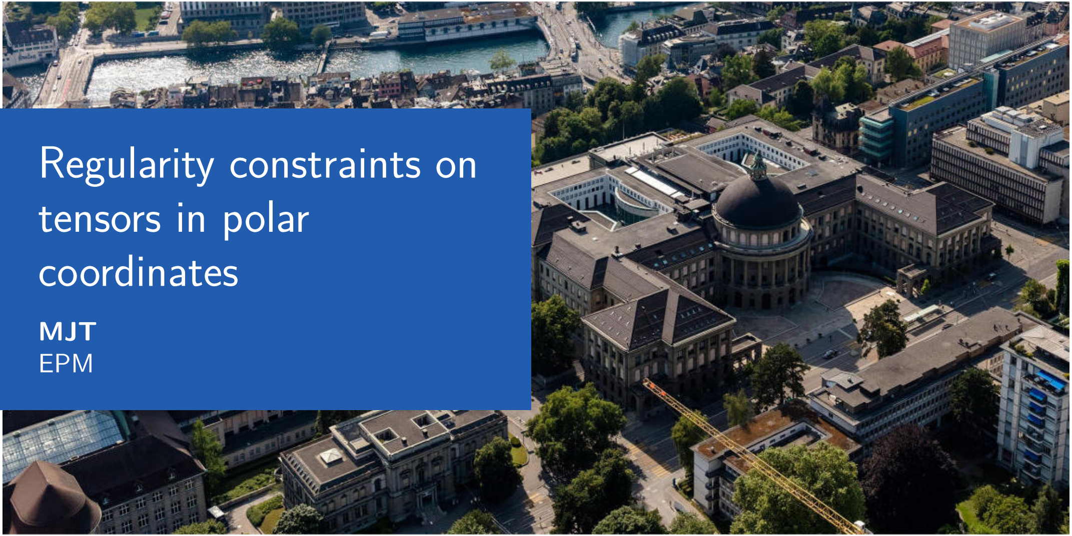

Regularity constraints on tensors in polar coordinates

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Outline

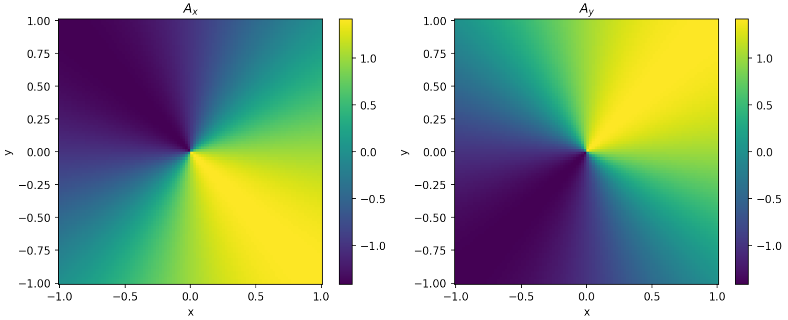
1. Vectors in polar coordinates

2. Rank-2 tensors in polar coordinates

Regularity conditions for vectors

Vector $\mathbf{A} = A_s\hat{\mathbf{s}} + A_\phi\hat{\boldsymbol{\phi}} = A_x\hat{\mathbf{x}} + A_y\hat{\mathbf{y}}$
A_s and A_ϕ being regular functions of s and ϕ is not sufficient for $\mathbf{A}$ to be regular.

E.g. $A_s = A_\phi \equiv 1 \quad \Longrightarrow \quad A_x = \cos\phi - \sin\phi = \frac{x-y}{\sqrt{x^2+y^2}}, \quad A_y = \cos\phi + \sin\phi = \frac{x+y}{\sqrt{x^2+y^2}}$



Regularity conditions for vectors

However, A_x and A_y being regular functions of x and y **is sufficient** for $\mathbf{A}$ to be regular. Lewis and Bellan (1990) derived the regularity conditions on Fourier coefficients

- Expand A_s and A_ϕ in Fourier series;
- Convert to vector components in Cartesian coordinates,

$$\begin{pmatrix} A_x \\ A_y \end{pmatrix} = \begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix} \begin{pmatrix} A_s \\ A_\phi \end{pmatrix}$$

  and derive Fourier expansion of A_x and A_y.
- Replace

$$e^{i|m|\phi} = \frac{(x+iy)^{|m|}}{s^{|m|}}, \quad e^{i|m|\phi} = \frac{(x-iy)^{|m|}}{s^{|m|}}$$

- A_x and A_y are regular if the resulting rational terms are regular.

Regularity conditions for vectors

The required ansätze for vectors in polar coordinates

$$A_s = sg_0 + \sum_{m\neq 0} \left(\lambda_m s^{|m|-1} + g_m s^{|m|+1}\right) e^{im\phi},$$

$$A_\phi = sh_0 + \sum_{m\neq 0} \left(i\,\mathrm{sgn}(m)\lambda_m s^{|m|-1} + h_m s^{|m|+1}\right) e^{im\phi},$$

where $g_m = g_m(s^2)$, $h_m = h_m(s^2)$. The Fourier coefficients of A_s and A_ϕ are coupled at the lowest order (of Taylor series) in s.

Outline

Regularity conditions for rank-2 tensors

Similar idea, now

$$\begin{pmatrix} A_{xx} & A_{xy} \\ A_{yx} & A_{yy} \end{pmatrix} = \begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix} \begin{pmatrix} A_{ss} & A_{s\phi} \\ A_{\phi s} & A_{\phi\phi} \end{pmatrix} \begin{pmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{pmatrix}$$

Proceed similarly as follows

- Expand A_{ss}, $A_{s\phi}$, $A_{\phi s}$, $A_{\phi\phi}$ in Fourier series
- Convert to components Cartesian coordinates
- Rewrite exponentials as polynomials in Cartesian coordinates
- A_{xx}, A_{xy}, A_{yx} and A_{yy} are regular if the resulting rational terms are regular.

Regularity conditions for rank-2 tensors

Regularity conditions.

$$
\begin{aligned}
A^m_{ss} + A^m_{\phi\phi} &= s^{|m|} C(s^2) \\
A^m_{s\phi} - A^m_{\phi s} &= s^{|m|} C(s^2) \\
A^m_{ss} - A^m_{\phi\phi} + i\left(A^m_{s\phi} + A^m_{\phi s}\right) &= s^{|m+2|} C(s^2) \\
A^m_{ss} - A^m_{\phi\phi} - i\left(A^m_{s\phi} + A^m_{\phi s}\right) &= s^{|m-2|} C(s^2)
\end{aligned}
$$

Assuming the Fourier coefficients

$$
A_{ij} = \sum_m e^{im\phi} A^m_{ij}(s) = \sum_m e^{im\phi} s^{|m|+\Delta_m} \sum_k A^{mk}_{ij} s^{2k}
$$

Regularity conditions for rank-2 tensors

The required form of the Fourier coefficients

$$m=0: \quad \begin{cases} A_{ss}^{0} = A_{ss}^{00} + s^2 C(s^2) \\ A_{\phi\phi}^{0} = A_{\phi\phi}^{00} + s^2 C(s^2) \\ A_{s\phi}^{0} = A_{s\phi}^{00} + s^2 C(s^2) \\ A_{\phi s}^{0} = A_{\phi s}^{00} + s^2 C(s^2) \end{cases}, \quad \begin{cases} A_{ss}^{00} = A_{\phi\phi}^{00} \\ A_{s\phi}^{00} = -A_{\phi s}^{00} \end{cases}$$

$$|m|=1: \quad \begin{cases} A_{ss}^{m} = A_{ss}^{m0} s + s^3 C(s^2) \\ A_{\phi\phi}^{m} = A_{\phi\phi}^{m0} s + s^3 C(s^2) \\ A_{s\phi}^{m} = A_{s\phi}^{m0} s + s^3 C(s^2) \\ A_{\phi s}^{m} = A_{\phi s}^{m0} s + s^3 C(s^2) \end{cases}, \quad \left\{ A_{s\phi}^{m0} + A_{\phi s}^{m0} = i\,\mathrm{sgn}(m) \left(A_{ss}^{m0} - A_{\phi\phi}^{m0} \right) \right.$$

$$|m|\geq 2: \quad \begin{cases} A_{ss}^{m} = A_{ss}^{m0} s^{|m|-2} + A_{ss}^{m1} s^{|m|} + s^{|m|+2} C(s^2) \\ A_{\phi\phi}^{m} = A_{\phi\phi}^{m0} s^{|m|-2} + A_{\phi\phi}^{m1} s^{|m|} + s^{|m|+2} C(s^2) \\ A_{s\phi}^{m} = A_{s\phi}^{m0} s^{|m|-2} + A_{s\phi}^{m1} s^{|m|} + s^{|m|+2} C(s^2) \\ A_{\phi s}^{m} = A_{\phi s}^{m0} s^{|m|-2} + A_{\phi s}^{m1} s^{|m|} + s^{|m|+2} C(s^2) \end{cases}, \quad \begin{cases} A_{ss}^{m0} = -A_{\phi\phi}^{m0} \\ A_{s\phi}^{m0} = A_{\phi s}^{m0} \\ A_{s\phi}^{m0} = i\,\mathrm{sgn}(m) A_{ss}^{m0} \\ A_{s\phi}^{m1} + A_{\phi s}^{m1} = i\,\mathrm{sgn}(m) \left(A_{ss}^{m1} - A_{\phi\phi}^{m1} \right). \end{cases}$$