

An aerial photograph of Zurich, Switzerland, showing the city's layout with a river (Limmat) winding through it. The image captures a mix of historic architecture, including a prominent domed building, and modern urban structures. A yellow construction crane is visible in the lower right quadrant. The city is surrounded by green spaces and trees.

# Introducing PlesioGeostroPy

a Python realization of the PG model

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Oct. 2023

# Outline

## 1. The PG model

## 2. The old implementation

## 3. Introducing PlesioGeostroPy

## 4. Future works

# PG, QG, and all the geostrophic models

# How did I end up here?

Since the start of my doctoral studies, I

- studied surface operators and tried to derive the boundary terms in the diffusive torsional oscillation (TO) equation for **2 months**  $\implies$  didn't lead anywhere;
- studied torsional Alfvén waves and calculated the 1-D eigenmodes of the torsional oscillation for **2 month**  $\implies$  didn't lead anywhere;
- studied anelastic approximation for **1 month** in order to work on the anelastic version of QuICC, which may help the Jupiter simulation  $\implies$  project was called off and deemed unpromising before I could start to think about the numerics;
- picked up the thread and studied the reflection of Alfvén waves at the fluid-solid interface for **2 months**  $\implies$  wrote a sixty-page document and yet had more questions than before;
- have been mainly working on the PG model since Aug 2023.

# Outline

1. The PG model
2. The old implementation
3. Introducing PlesioGeostroPy
4. Future works

# The old implementation - code *Daria*

Mathematica implementation (previous member of the group, Dr. Daria Holdenried-Chernoff)

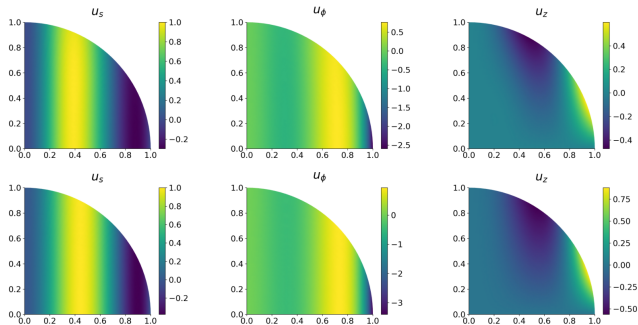


Figure: Eigenmodes calculated using 3-D code *Jiawen* (top) and PG code *Daria*.

Why reinvent the wheels when we already have a functioning implementation?

# The old implementation - code *Daria*

Citizens of Gotham, riddle me this... what is this code snippet doing?

```
in(*):= Les =  $\frac{1}{s}$  D[s Bs2[s, p, t], s] +  $\frac{s^2}{H[s]}$  (BsH[s, p, z, t] * BsH[s, p, z, t] + BsmH[s, p, z, t] * BsmH[s, p, z, t]) +  $\frac{1}{s}$  D[BsBp[s, p, t], p] -  $\frac{1}{s}$  Bp2[s, p, t] - BsH[s, p, z, t] * BzH[s, p, z, t] -  
BsmH[s, p, z, t] * BzmH[s, p, z, t];  
  
in(*):= Lestmp =  
Les /. Bs2[s, p, t] → A0[s, p, t] + x * A[s, p, t] /. D[Bs2[s, p, t], s] → D[A0[s, p, t] + x * A[s, p, t], s] /. D[Bs2[s, p, t], p] → D[A0[s, p, t] + x * A[s, p, t], p] /.  
Bp2[s, p, t] → B0[s, p, t] + x * B[s, p, t] /. D[Bp2[s, p, t], s] → D[B0[s, p, t] + x * B[s, p, t], s] /. D[Bp2[s, p, t], p] → D[B0[s, p, t] + x * B[s, p, t], p] /.  
BsBp[s, p, t] → C0[s, p, t] + x * Csp[s, p, t] /. D[BsBp[s, p, t], s] → D[C0[s, p, t] + x * Csp[s, p, t], s] /. D[BsBp[s, p, t], p] → D[C0[s, p, t] + x * Csp[s, p, t], p] /.  
BsmH[s, p, z, t] → amh0[s, p, z, t] + x * amh[s, p, z, t] /. D[BsmH[s, p, z, t], s] → D[amh0[s, p, z, t], s] + x * D[amh[s, p, z, t], s] /.  
D[BsmH[s, p, z, t], p] → D[amh0[s, p, z, t], p] + x * D[amh[s, p, z, t], p] /. BsH[s, p, z, t] → ah0[s, p, z, t] + x * ah[s, p, z, t] /.  
D[BsH[s, p, z, t], s] → D[ah0[s, p, z, t], s] + x * D[ah[s, p, z, t], s] /. D[BsH[s, p, z, t], p] → D[ah0[s, p, z, t], p] + x * D[ah[s, p, z, t], p] /.  
BpmH[s, p, z, t] → bmh0[s, p, z, t] + x * bmh[s, p, z, t] /. D[BpmH[s, p, z, t], s] → D[bmh0[s, p, z, t], s] + x * D[bmh[s, p, z, t], s] /.  
D[BpmH[s, p, z, t], p] → D[bmh0[s, p, z, t], p] + x * D[bmh[s, p, z, t], p] /. BpH[s, p, z, t] → bh0[s, p, z, t] + x * bh[s, p, z, t] /.  
D[BpH[s, p, z, t], s] → D[bh0[s, p, z, t], s] + x * D[bh[s, p, z, t], s] /. D[BpH[s, p, z, t], p] → D[bh0[s, p, z, t], p] + x * D[bh[s, p, z, t], p] /.  
D[BpH[s, p, z, t], z] → D[bh0[s, p, z, t], z] + x * D[bh[s, p, z, t], z] /. BzmH[s, p, z, t] → cmh0[s, p, z, t] + x * cmh[s, p, z, t] /. BzH[s, p, z, t] → ch0[s, p, z, t] + x * ch[s, p, z, t] /.  
D[BzH[s, p, z, t], z] → D[ch0[s, p, z, t], z] + x * D[ch[s, p, z, t], z] /. D[BzH[s, p, z, t], s] → D[ch0[s, p, z, t], s] + x * D[ch[s, p, z, t], s] /.  
D[BzmH[s, p, z, t], s] → D[cmh0[s, p, z, t], s] + x * D[cmh[s, p, z, t], s];
```

# The old implementation - code *Daria*

Citizens of Gotham, riddle me this... what is this code snippet doing?

```
in(-)= Les =  $\frac{1}{s}$  D[s Bs2[s, p, t], s] +  $\frac{s^2}{H[s]}$  (BsH[s, p, z, t] * BsH[s, p, z, t] + BsmH[s, p, z, t] * BsmH[s, p, z, t]) +  $\frac{1}{s}$  D[BsBp[s, p, t], p] -  $\frac{1}{s}$  Bp2[s, p, t] - BsH[s, p, z, t] * BzH[s, p, z, t] -
    BsmH[s, p, z, t] * BzmH[s, p, z, t];

in(-)= Lestmp =
    Les /. Bs2[s, p, t] → A0[s, p, t] + x * A[s, p, t] /. D[Bs2[s, p, t], s] → D[A0[s, p, t] + x * A[s, p, t], s] /. D[Bs2[s, p, t], p] → D[A0[s, p, t] + x * A[s, p, t], p] /.
    Bp2[s, p, t] → B0[s, p, t] + x * B[s, p, t] /. D[Bp2[s, p, t], s] → D[B0[s, p, t] + x * B[s, p, t], s] /. D[Bp2[s, p, t], p] → D[B0[s, p, t] + x * B[s, p, t], p] /.
    BsBp[s, p, t] → C0[s, p, t] + x * Csp[s, p, t] /. D[BsBp[s, p, t], s] → D[C0[s, p, t] + x * Csp[s, p, t], s] /. D[BsBp[s, p, t], p] → D[C0[s, p, t] + x * Csp[s, p, t], p] /.
    BsmH[s, p, z, t] → amh0[s, p, z, t] + x * amh[s, p, z, t] /. D[BsmH[s, p, z, t], s] → D[amh0[s, p, z, t], s] + x * D[amh[s, p, z, t], s] /.
    D[BsmH[s, p, z, t], p] → D[amh0[s, p, z, t], p] + x * D[amh[s, p, z, t], p] /. BsH[s, p, z, t] → ah0[s, p, z, t] + x * ah[s, p, z, t] /.
    D[BsH[s, p, z, t], s] → D[ah0[s, p, z, t], s] + x * D[ah[s, p, z, t], s] /. D[BsH[s, p, z, t], p] → D[ah0[s, p, z, t], p] + x * D[ah[s, p, z, t], p] /.
    BpmH[s, p, z, t] → bmh0[s, p, z, t] + x * bmh[s, p, z, t] /. D[BpmH[s, p, z, t], s] → D[bmh0[s, p, z, t], s] + x * D[bmh[s, p, z, t], s] /.
    D[BpmH[s, p, z, t], p] → D[bmh0[s, p, z, t], p] + x * D[bmh[s, p, z, t], p] /. BpH[s, p, z, t] → bh0[s, p, z, t] + x * bh[s, p, z, t] /.
    D[BpH[s, p, z, t], s] → D[bh0[s, p, z, t], s] + x * D[bh[s, p, z, t], s] /. D[BpH[s, p, z, t], p] → D[bh0[s, p, z, t], p] + x * D[bh[s, p, z, t], p] /.
    D[BpH[s, p, z, t], z] → D[bh0[s, p, z, t], z] + x * D[bh[s, p, z, t], z] /. BzmH[s, p, z, t] → cmh0[s, p, z, t] + x * cmh[s, p, z, t] /. BzH[s, p, z, t] → ch0[s, p, z, t] + x * ch[s, p, z, t] /.
    D[BzH[s, p, z, t], z] → D[ch0[s, p, z, t], z] + x * D[ch[s, p, z, t], z] /. D[BzH[s, p, z, t], s] → D[ch0[s, p, z, t], s] + x * D[ch[s, p, z, t], s] /.
    D[BzmH[s, p, z, t], s] → D[cmh0[s, p, z, t], s] + x * D[cmh[s, p, z, t], s];
```

This rewrites the magnetic quantities in Lorentz force  $\overline{L_\phi}$  as background + perturbation for linearization.

e.g.  $Bp2[s, p, t] \rightarrow B0[s, p, t] + x * B[s, p, t]$  equals to  $\overline{B_\phi^2} = \overline{B_{\phi_0}^2} + \epsilon \overline{B_\phi^2}'$



# The old implementation - code *Daria*

```
def Lee =  $\frac{1}{n} \cdot D[AB2[s, p, t], n] + \frac{n^2}{H[n]} \cdot (Bsh[s, p, z, t] + Bsh[s, p, z, t] + Bsh[s, p, z, t] + Bsh[s, p, z, t]) + \frac{1}{n} \cdot D[Bshp[s, p, t], p] - \frac{1}{n} \cdot Bp2[s, p, t] - Bsh[s, p, z, t] + Bsh[s, p, z, t] -$   
Bsh[s, p, z, t] + Bsh[s, p, z, t];  
  
def Leatmp =  
  let f, Bsh2[s, p, t] = AB[s, p, t] = x + A[s, p, t] / D[Bsh2[s, p, t], n] = D[AB[s, p, t] = x + A[s, p, t], p] = D[AB[s, p, t] = x + A[s, p, t], p] /.  
    Bp2[s, p, t] = BB[s, p, t] = x + B[s, p, t] / D[Bp2[s, p, t], n] = D[Bp2[s, p, t] = x + B[s, p, t], p] = D[Bp2[s, p, t] = x + B[s, p, t], p] /.  
    Bshp[s, p, t] = CB[s, p, t] = x + Cap[s, p, t] / D[Bshp[s, p, t], n] = D[CB[s, p, t] = x + Cap[s, p, t], p] = D[Bshp[s, p, t] = x + Cap[s, p, t], p] /.  
    Bshh[s, p, z, t] = xshh[s, p, z, t] = x + xshh[s, p, z, t] / D[Bshh[s, p, z, t], n] = D[Bshh[s, p, z, t] = x + xshh[s, p, z, t], p] = D[Bshh[s, p, z, t] = x + xshh[s, p, z, t], p] /.  
    D[Bshh[s, p, z, t], p] = D[xshh[s, p, z, t], p] = x + D[xshh[s, p, z, t], p] / Bsh[s, p, z, t] = xshh[s, p, z, t] = x + xshh[s, p, z, t] /.  
    D[Bshh[s, p, z, t], n] = D[xshh[s, p, z, t], n] = x + D[xshh[s, p, z, t], n] / D[Bshh[s, p, z, t], p] = D[xshh[s, p, z, t], p] = x + D[xshh[s, p, z, t], p] /.  
    Bshh[s, p, z, t] = xshh[s, p, z, t] = x + xshh[s, p, z, t] / D[Bshh[s, p, z, t], n] = D[Bshh[s, p, z, t] = x + xshh[s, p, z, t], p] = D[Bshh[s, p, z, t] = x + xshh[s, p, z, t], p] /.  
    D[Bshh[s, p, z, t], p] = D[xshh[s, p, z, t], p] = x + D[xshh[s, p, z, t], p] / Bshh[s, p, z, t] = xshh[s, p, z, t] = x + xshh[s, p, z, t] /.  
    D[Bshh[s, p, z, t], n] = D[xshh[s, p, z, t], n] = x + D[xshh[s, p, z, t], n] / D[Bshh[s, p, z, t], p] = D[xshh[s, p, z, t], p] = x + D[xshh[s, p, z, t], p] /.  
    D[Bshh[s, p, z, t], p] = D[xshh[s, p, z, t], p] = x + xshh[s, p, z, t] / Bshh[s, p, z, t] = xshh[s, p, z, t] = x + xshh[s, p, z, t] /.  
    D[Bshh[s, p, z, t], n] = D[xshh[s, p, z, t], n] = x + D[xshh[s, p, z, t], n] / D[Bshh[s, p, z, t], p] = D[xshh[s, p, z, t], p] = x + D[xshh[s, p, z, t], p] /.  
    D[Bshh[s, p, z, t], p] = D[xshh[s, p, z, t], p] = x + xshh[s, p, z, t] / Bshh[s, p, z, t] = xshh[s, p, z, t] = x + xshh[s, p, z, t] /.  
    D[Bshh[s, p, z, t], n] = D[xshh[s, p, z, t], n] = x + D[xshh[s, p, z, t], n] / D[Bshh[s, p, z, t], p] = D[xshh[s, p, z, t], p] = x + D[xshh[s, p, z, t], p] /.  
    D[Bshh[s, p, z, t], p] = D[xshh[s, p, z, t], p] = x + xshh[s, p, z, t] / Bshh[s, p, z, t] = xshh[s, p, z, t] = x + xshh[s, p, z, t] /.  
    D[Bshh[s, p, z, t], n] = D[xshh[s, p, z, t], n] = x + D[xshh[s, p, z, t], n] / D[Bshh[s, p, z, t], p] = D[xshh[s, p, z, t], p] = x + D[xshh[s, p, z, t], p] /.
```

Why is the code undesirable for **developers** and **users** alike?

- long, repetitive operations  $\implies$  hard to **debug**
- namings: the background and perturbation fields are named from A to H  $\implies$  hard to **debug** or **invoke**.
- At some point I even need a correspondence table to decipher the code.

## The old implementation - code *Daria*

[illegible]

Why is the code undesirable for **developers** and **users** alike?

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- At some point I even need a correspondence table to decipher the code.

Roadmap to Daria's Mathematics Notebook - Supplement

- Naming Conventions. (There is unfortunately no logic behind the namings)

$$\overline{\theta_s \theta_s} : A; \quad \overline{\theta_s \theta_s^0} : A_0; \quad \overline{\theta_s \theta_s'} : A. \quad (\overline{M_{ss}^0}, \overline{m_{ss}})$$

$$\overline{B_1 B_2} = B; \quad \overline{B_1 B_2}^0 = B_0; \quad \overline{B_1 B_2}' = B.$$

$$\overline{B_s B_f} : C_{sp}; \quad \overline{B_s B_f^0} : C_0; \quad \overline{B_s B_f^1} : C_{sp}. \quad (\overline{M_{sq}^0}, \overline{m_{sq}})$$

$$\begin{cases} \beta_s^+ : ah; & (\beta_s^+)^* : aho; & (\beta_s^+)' : ah \\ \beta_s^- : am; & (\beta_s^-)^* : amo; & (\beta_s^-)' : am. \end{cases} \quad \begin{cases} \beta_q^+ : bh; & (\beta_q^+)^* : bho; & (\beta_q^+)' : bh \\ \beta_q^- : bmi; & (\beta_q^-)^* : bmb; & (\beta_q^-)' : bm. \end{cases}$$

$$\begin{aligned} & \left\{ \begin{array}{l} B_2^+ : bck; \quad (B_2^+)^0 : cko; \quad (B_2^+)^- : ck \\ B_2^- : cm; \quad (B_2^-)^0 : cmo; \quad (B_2^-)^- : clm. \end{array} \right. \\ & \widetilde{B_2 B_2} : E; \quad \widetilde{B_2 B_2^0} : E_0; \quad \widetilde{B_2 B_2^-} : E_{-1}; \quad \widetilde{B_2 B_2^0} : D; \quad \widetilde{B_2 B_2^0} : D_0; \quad \widetilde{B_2 B_2^-} : D_{-1} \end{aligned}$$

$$B_{se} = a; \quad B_{se}^0 = a00; \quad B_{se}' = a0; \quad B_{pe} = b; \quad B_{pe}^0 = b00; \quad B_{pe}' = b00,$$

$$b_{12} = c; \quad b_{23} = c\omega; \quad b_{31} = c\omega^2;$$

- Equation Numbering: where does it come from?

There must be another document; these numberings have nothing to do with the numberings in Darwin's thesis  
(HS includes visions diffusion, & Ekman pumpkin?)

eq. (1). (RHS eq 1, LHS eq 2): five vertex eq., self-adjoint operator form

(eq. (73) (eq. 73, rhseq 73) : evolution eqn. for  $\tilde{B}_{gg}$ . RHS is the time-derivative. Diffusionless

eq. (4) (LHS eq3, RHS eq4): evolution eqn. for  $\overline{B_{eq}}$  with magnetic diffusion, + linearized

§ eq. (74). (eq 74, rhs eq 74): evaluation eq. for  $\overline{b}_{st}$ , diffusionless

eq. (4) (LHS eq. 4, RHS eq. 4) = evolution eqn. Bst, linearized, diff-sim

$\frac{\{eq. (76)}{eq. (6)}$  ditto, but for  $B_2 B_3$   $\frac{\{eq. (78)}{eq. (8)}$  ditto, but for  $z B_2 B_3$

eq. (6)

eq. (9) ditto, but for  $z \in B_1$

$\left\{ \frac{a_2}{a_1} \right\}_{190}$  ditto, but for  $B_2(z=0)$ .  $\left\{ \frac{a_2}{a_1} \right\}_{192}$  ditto, but for  $B_2(z=0)$ .

ex. (10)

100

**Figure:** List of physical variables and their names in the code.

# The old implementation - code *Daria*

```

In[ ]:= psi[m_, n_] := (1 - s^2)^(3/2) s^m JacobiP[n - 1, 3/2, m, 2 s^2 - 1]

In[ ]:= Fm[m_] := Piecewise[{{(1, m == 0), (s, m == 1), (s^Abs[m] - 2, m > 1)}}]

Beven[m_] := Piecewise[{{(-1/2, m == 0), (1 - 1/2, m == 1), (m - 5/2, m > 1)}}]

In[ ]:= BackRegFactor = 1; (* 5 (-1 + s^2) ; *)

In[ ]:= BsExp[m_, n_] := BackRegFactor * (1 - s^2)^(1/2) s^Abs[m] - 1 JacobiP[n - 1, 1, Abs[m] - 3/2, 2 s^2 - 1]

BpExp[m_, n_] := BackRegFactor * (1 - s^2)^(1/2) s^Abs[m] + 1 JacobiP[n - 1, 1, Abs[m] + 1/2, 2 s^2 - 1]

BzExp[m_, n_] := BackRegFactor * (1 - s^2) s^Abs[m] JacobiP[n - 1, 2, Abs[m] - 1/2, 2 s^2 - 1]

```

## Magnetic moments expansions for toroidal background field

```

In[ ]:= Bsp[m_, n_] := (1 - s^2)^(1/2) * s^Abs[m - 2] * JacobiP[n - 1, 1, Abs[m - 2] - 1/2, 2 s^2 - 1]

Bpp[m_, n_] := (1 - s^2)^(1/2) * Fm[m] * s^2 * JacobiP[n - 1, 1, Beven[m] + 2, 2 s^2 - 1] (*Wherever Bpp

Bpz[m_, n_] := (1 - s^2) s^(n-1) JacobiP[n - 1, 2, m - 3/2, 2 s^2 - 1]

Bzpp[m_, n_] := (1 - s^2) Fm[m] * s^2 * JacobiP[n - 1, 2, Beven[m] + 2, 2 s^2 - 1] (*Wherever Bzpp
there needs to be a Bzsp contribution to ensure correct regularity. Note extra H from

Bzsp[m_, n_] := (1 - s^2) s^Abs[m - 2] JacobiP[n - 1, 2, Abs[m - 2] - 1/2, 2 s^2 - 1]

bs0[m_, n_] := (1 - s^2) s^Abs[m] - 1 JacobiP[n - 1, 2, Abs[m] - 1/2, 2 s^2 - 1]

bp0[m_, n_] := (1 - s^2) s^Abs[m] + 1 JacobiP[n - 1, 2, Abs[m] + 1/2, 2 s^2 - 1] (*Factor of H ensu

```

Inflexible expansions: the expansion and the construction of matrices are hard coded in each notebook.

To change the expansion, one needs to

- produce a complete new Mathematica notebook;
- manually change the way the code collects the matrix elements.

Especially cumbersome to implement coupling, or if the bases do not coincide with the field quantities (more detail later).

# The old implementation

## Other problems

- The code is not efficient numerically, nor easily linked to efficient numerical libraries.
- Cryptic Mathematica syntax sugars

```
In[*]:= sol = Solve[JacobiP[colN,  $\alpha$ ,  $\beta$ , x] == 0, x];  
rlocation = N[x /. # & /@ sol, prec];
```

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4. Future works



# Outline

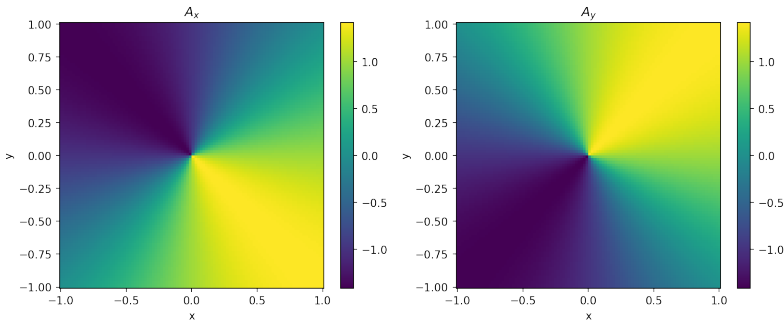
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# Regularity conditions for vectors

$$\text{Vector } \mathbf{A} = A_s \hat{s} + A_\phi \hat{\phi} = A_x \hat{x} + A_y \hat{y}$$

$A_s$  and  $A_\phi$  being regular functions of  $s$  and  $\phi$  is not sufficient for  $\mathbf{A}$  to be regular.

$$\text{E.g. } A_s = A_\phi \equiv 1 \implies A_x = \cos \phi - \sin \phi = \frac{x-y}{\sqrt{x^2+y^2}}, \quad A_y = \cos \phi + \sin \phi = \frac{x+y}{\sqrt{x^2+y^2}}$$





# Regularity conditions for vectors

However,  $A_x$  and  $A_y$  being regular functions of  $x$  and  $y$  **is sufficient** for  $\mathbf{A}$  to be regular.  
**lewis\_physical\_1990** derived the regularity conditions on Fourier coefficients

- Expand  $A_s$  and  $A_\phi$  in Fourier series;
- Convert to vector components in Cartesian coordinates,

$$\begin{pmatrix} A_x \\ A_y \end{pmatrix} = \begin{pmatrix} \cos \phi & -\sin \phi \\ \sin \phi & \cos \phi \end{pmatrix} \begin{pmatrix} A_s \\ A_\phi \end{pmatrix}$$

and derive Fourier expansion of  $A_x$  and  $A_y$ .

- Replace

$$e^{i|m|\phi} = \frac{(x + iy)^{|m|}}{s^{|m|}}, \quad e^{i|m|\phi} = \frac{(x - iy)^{|m|}}{s^{|m|}}$$

- $A_x$  and  $A_y$  are regular if the resulting rational terms are regular.

# Regularity conditions for vectors

The required ansätze for vectors in polar coordinates

$$A_s = sg_0 + \sum_{m \neq 0} (\lambda_m s^{|m|-1} + g_m s^{|m|+1}) e^{im\phi},$$
$$A_\phi = sh_0 + \sum_{m \neq 0} (i \operatorname{sgn}(m) \lambda_m s^{|m|-1} + h_m s^{|m|+1}) e^{im\phi},$$

where  $g_m = g_m(s^2)$ ,  $h_m = h_m(s^2)$ . The Fourier coefficients of  $A_s$  and  $A_\phi$  are coupled at the lowest order (of Taylor series) in  $s$ .

# Regularity conditions for rank-2 tensors

Similar idea, now

$$\begin{pmatrix} A_{xx} & A_{xy} \\ A_{yx} & A_{yy} \end{pmatrix} = \begin{pmatrix} \cos \phi & -\sin \phi \\ \sin \phi & \cos \phi \end{pmatrix} \begin{pmatrix} A_{ss} & A_{s\phi} \\ A_{\phi s} & A_{\phi\phi} \end{pmatrix} \begin{pmatrix} \cos \phi & \sin \phi \\ -\sin \phi & \cos \phi \end{pmatrix}$$

Proceed similarly as follows

- Expand  $A_{ss}$ ,  $A_{s\phi}$ ,  $A_{\phi s}$ ,  $A_{\phi\phi}$  in Fourier series
- Convert to components Cartesian coordinates
- Rewrite exponentials as polynomials in Cartesian coordinates
- $A_{xx}$ ,  $A_{xy}$ ,  $A_{yx}$  and  $A_{yy}$  are regular if the resulting rational terms are regular.

# Regularity conditions for rank-2 tensors

Regularity conditions.

$$A_{ss}^m + A_{\phi\phi}^m = s^{|m|} C(s^2)$$

$$A_{s\phi}^m - A_{\phi s}^m = s^{|m|} C(s^2)$$

$$A_{ss}^m - A_{\phi\phi}^m + i (A_{s\phi}^m + A_{\phi s}^m) = s^{|m+2|} C(s^2)$$

$$A_{ss}^m - A_{\phi\phi}^m - i (A_{s\phi}^m + A_{\phi s}^m) = s^{|m-2|} C(s^2)$$

Assuming the Fourier coefficients

$$A_{ij} = \sum_m e^{im\phi} A_{ij}^m(s) = \sum_m e^{im\phi} s^{|m|+\Delta_m} \sum_k A_{ij}^{mk} s^{2k}$$

# Regularity conditions for rank-2 tensors

The required form of the Fourier coefficients

$$\begin{aligned}
 m = 0 : & \quad \begin{cases} A_{ss}^0 = A_{ss}^{00} + s^2 C(s^2) \\ A_{\phi\phi}^0 = A_{\phi\phi}^{00} + s^2 C(s^2) \\ A_{s\phi}^0 = A_{s\phi}^{00} + s^2 C(s^2) \\ A_{\phi s}^0 = A_{\phi s}^{00} + s^2 C(s^2) \end{cases}, \quad \begin{cases} A_{ss}^{00} = A_{\phi\phi}^{00} \\ A_{s\phi}^{00} = -A_{\phi s}^{00} \end{cases} \\
 |m| = 1 : & \quad \begin{cases} A_{ss}^m = A_{ss}^{m0} s + s^3 C(s^2) \\ A_{\phi\phi}^m = A_{\phi\phi}^{m0} s + s^3 C(s^2) \\ A_{s\phi}^m = A_{s\phi}^{m0} s + s^3 C(s^2) \\ A_{\phi s}^m = A_{\phi s}^{m0} s + s^3 C(s^2) \end{cases}, \quad \{ A_{s\phi}^{m0} + A_{\phi s}^{m0} = i \operatorname{sgn}(m) (A_{ss}^{m0} - A_{\phi\phi}^{m0}) \} \\
 |m| \geq 2 : & \quad \begin{cases} A_{ss}^m = A_{ss}^{m0} s^{|m|-2} + A_{ss}^{m1} s^{|m|} + s^{|m|+2} C(s^2) \\ A_{\phi\phi}^m = A_{\phi\phi}^{m0} s^{|m|-2} + A_{\phi\phi}^{m1} s^{|m|} + s^{|m|+2} C(s^2) \\ A_{s\phi}^m = A_{s\phi}^{m0} s^{|m|-2} + A_{s\phi}^{m1} s^{|m|} + s^{|m|+2} C(s^2) \\ A_{\phi s}^m = A_{\phi s}^{m0} s^{|m|-2} + A_{\phi s}^{m1} s^{|m|} + s^{|m|+2} C(s^2) \end{cases}, \quad \begin{cases} A_{ss}^{m0} = -A_{\phi\phi}^{m0} \\ A_{s\phi}^{m0} = A_{\phi s}^{m0} \\ A_{s\phi}^{m0} = i \operatorname{sgn}(m) A_{ss}^{m0} \\ A_{s\phi}^{m1} + A_{\phi s}^{m1} = i \operatorname{sgn}(m) (A_{ss}^{m1} - A_{\phi\phi}^{m1}) \end{cases}.
 \end{aligned}$$