EE2703 : Applied Programming Lab
End-semester Examination
Magnetic Field of Loop Antenna

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Problem Statement

This problem is regarding the radiation from the loop antenna of length λ. A loop antenna is a loop or coil of wire carrying radio frequency current - predominantly receiving magnetic component of the electromagnetic wave.

Consider a long wire carrying current

$$I = \frac{4\pi}{\mu_0}\cos(\phi)\exp(j\omega t)$$

through a loop of wire. Here, ϕ is the angle in polar coordinates i.e., in (r, ϕ, z) coordinates. The wire is on the X-Y plane and centered at the origin. The radius of the loop is 10 cm and is also equal to $\frac{1}{k} = \frac{c}{\omega}$ so that the circumference of the antenna is λ

The problem is to compute and plot the magnetic field B along the z axis from 1 cm to 1000 cm, plot it and then fit the data to $B = az^b$

Through this assignment, our objective is to learn

- Vectorize loops and implement python arrays efficiently
- Calculate the magnetic field of the loop antenna of the above said current distribution along z-axis by calculating vector potential
- Trying to fit the B field in the form $B \approx az^b$ using the `lstsq` function of the `scipy.linalg` toolbox

Vector Field and Magnetic Field

Maxwell's equations in differential form can be written as

$$\nabla.\mathbf{B} = 0$$

$$\nabla.\mathbf{D} = \rho$$

$$\nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial D}{\partial t} \tag{1}$$

$$\nabla \times \mathbf{E} = -\frac{\partial B}{\partial t} \tag{2}$$

Defining Magnetic Vector Potential $\mathbf{A}$ as

$$\mathbf{B} = \nabla \times \mathbf{A} \tag{3}$$

From equation (2) and (3)

$$\nabla \times (\mathbf{E} + \dot{\mathbf{A}}) = 0 \implies \mathbf{E} + \dot{\mathbf{A}} = -\nabla . V \tag{4}$$

Applying equation (4) in (1) and defining the potential V based on **Lorentz Gauge Condition**, we obtain

$$\nabla . \mathbf{A} = -\mu_0 \epsilon_0 \dot{V} \qquad \textbf{(Lorentz Gauge Condition)}$$

$$\nabla^2 \mathbf{A} - \mu_0 \epsilon_0 \ddot{\mathbf{A}} = -\mu \mathbf{J}$$

Since the potentials vary sinusoidally, we can say $\frac{\partial}{\partial t} \sim j\omega$ and $\frac{\partial^2}{\partial t^2} \sim -\omega^2$. Thus

$$\nabla^2 \mathbf{A} - \mu_0 \epsilon_0 \ddot{\mathbf{A}} = (\nabla^2 + \mu_0 \epsilon_0 \omega^2)\mathbf{A} = (\nabla^2 + k^2)\mathbf{A} = -\mu \mathbf{J}$$

where k is the propogation constant. Solving this equation using **Green Function Technique**, we obtain

$$\mathbf{A}(\mathbf{r}) = \int_V \mu_0 \mathbf{J}(\mathbf{r}') \frac{e^{-j\beta|r-r'|}}{4\pi|r-r'|} dV$$

Effectively the above equation can be simplified into

$$\mathbf{A}(r, \phi, z) = \frac{\mu_0}{4\pi} \int \frac{\mathbf{I}(\phi)\hat{\phi} e^{-jkR} a d\phi}{R}$$

where $\vec{R} = \vec{r} - \vec{r'}$ where $\vec{r'} = a\hat{r'}$ is a point on the loop of radius $a = 10cm$. This integration can be simplified into the following summation after substituting the current distribution:

$$\mathbf{A}_{ijk} = \sum_{l=0}^{N-1} \frac{cos(\phi_l) exp(-jkR_{ijk,l}) \vec{dl}_l}{R_{ijk,l}} \tag{5}$$

where $\vec{r_l'} = acos(\phi_l)\hat{x} + asin(\phi_l)\hat{y}$ and $\vec{dl_l'} = -asin(\phi_l)\hat{x} + acos(\phi_l)\hat{y}$. From equation (3) we can obtain $\mathbf{B}$ as

$$\mathbf{B} = (\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z})\hat{x} + (\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x})\hat{y} + (\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y})\hat{z}$$

Along $\hat{z}$ axis this resolves into

$$B_z \hat{z} = \frac{A_y(\Delta x, 0, z) - A_y(-\Delta x, 0, z)}{2\Delta x} - \frac{A_x(0, \Delta y, z) - A_x(0, -\Delta y, z)}{2\Delta y} \tag{6}$$

Pseudocode

```
INIT radius = 10
INIT k = 0.1 = wavenumber
INIT Nz = 1000 = Number of points on z axis
INIT N = 100 = Number of current elements

INIT x = {-1, 0, 1}
INIT y = {-1, 0, 1}
INIT z = {1, 2, 3, 4 .... 998, 999, 1000}
INIT theta = {0, 2*pi/N, 4*pi/N......2*(N-1)*pi/N}
INIT delta_x = x[1] - x[0]
INIT delta_y = y[1] - y[0]
INIT deltaTheta = theta[1] - theta[0]

INIT MAGNITUDE_dL = deltaTheta * radius
CREATE CURRENT_DISTRIBUTION = cos(theta)*4pi/mu
CREATE CURRENT_ELEMENT_LOCATION = [cos(theta) sin(theta) 0]
CREATE CURRENT_ELEMENT_ORIENTATION = [-sin(theta) cos(theta)] * MAGNITUDE_dL
CURRENT_ELEMENT = CURRENT_DISTRIBUTION * CURRENT_ELEMENT_ORIENTATION

CREATE COORDINATE SYSTEM X,Y,Z = MESHGRID (x, y, z)

FOR EACH CURRENT_ELEMENT
    DISTANCE = (X - CURRENT_ELEMENT_LOCATION_X)^2 +
               (Y - CURRENT_ELEMENT_LOCATION_Y)^2 +
               (Z - CURRENT_ELEMENT_LOCATION_Z)^2
    Ax += mu/4pi * exponential(-j*k*DISTANCE)/DISTANCE * CURRENT_ELEMENT_X
    Ay += mu/4pi * exponential(-j*k*DISTANCE)/DISTANCE * CURRENT_ELEMENT_Y

B[z] = (Ay[1][0][z] - Ay[-1][0][z])/2delta_x -
       (Ax[0][1][z] - Ay[0][-1][z])/2delta_y

X = [1 log(z)]
logY = log(B)
FIT M,N using LEAST_SQUARE_SUM(X,logY)
N = exponential(N)
B_expected = N*z^M
```

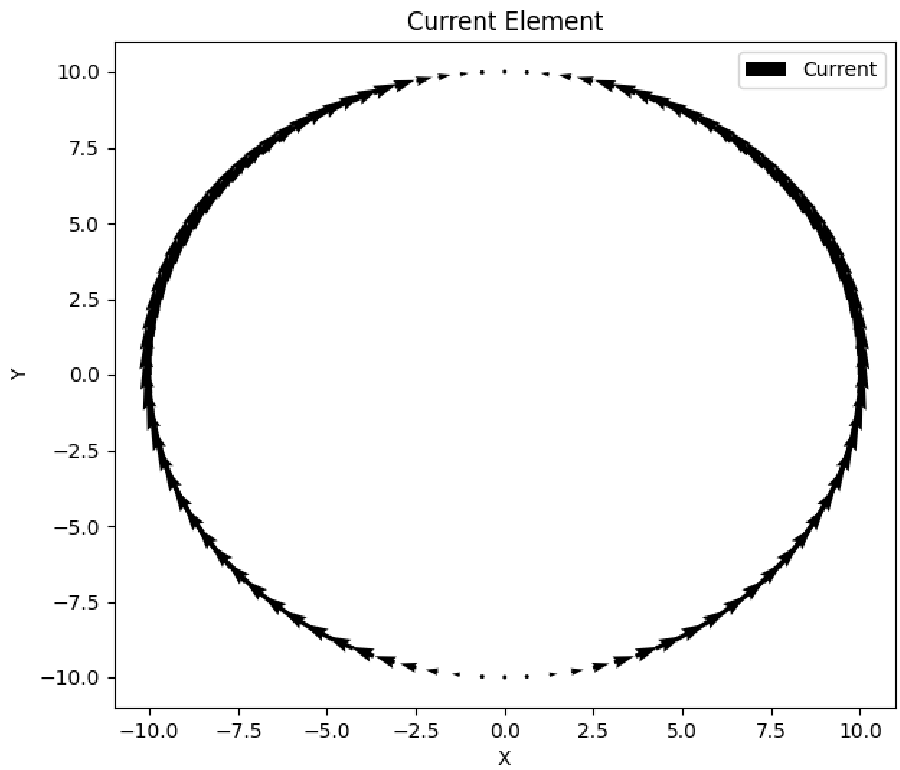
Current Distribution

The loop of wire with the following current distribution:

$$I = \frac{4\pi}{\mu_0}\cos(\phi)\exp(j\omega t)$$

is modelled numerically by breaking the loop into finite `N = 100` discrete parts. The location of each current element `rL` and its orientation `dL` is obtained following which the current distribution `IdL` is plotted

```
theta = np.linspace(0,2*np.pi,N+1,dtype = float)[:-1]
mag_dL = 2*np.pi*radius/N

dL = mag_dL* np.concatenate([[-np.sin(theta)], [np.cos(theta)]], axis = 0)
rL = radius*np.concatenate([[np.cos(theta)], [np.sin(theta)], [np.zeros(N)]], axis = 0)

I = np.cos(theta)*1e7
IdL = np.multiply(I, dL)
```



Vector Potential Calculation

The vector potential of this system is given by

$$\mathbf{A}_{ijk} = \sum_{l=0}^{N-1} \frac{cos(\phi_l)exp(-jkR_{ijk,l})d\vec{l}_l}{R_{ijk,l}}$$

where $R_{ijk,l}$ is the distance of each coordinate point (X_i, Y_j, Z_k) from the l^{th} current element. So

$$R_{ijk,l} = \sqrt{(X_i - x_l)^2 + (Y_j - y_l)^2 + (Z_k - z_l)^2}$$

Vectorially this can be reduced to

```
R = ((X - r[0])**2 + (Y - r[1])**2 + (Z - r[2])**2)**(0.5)
```

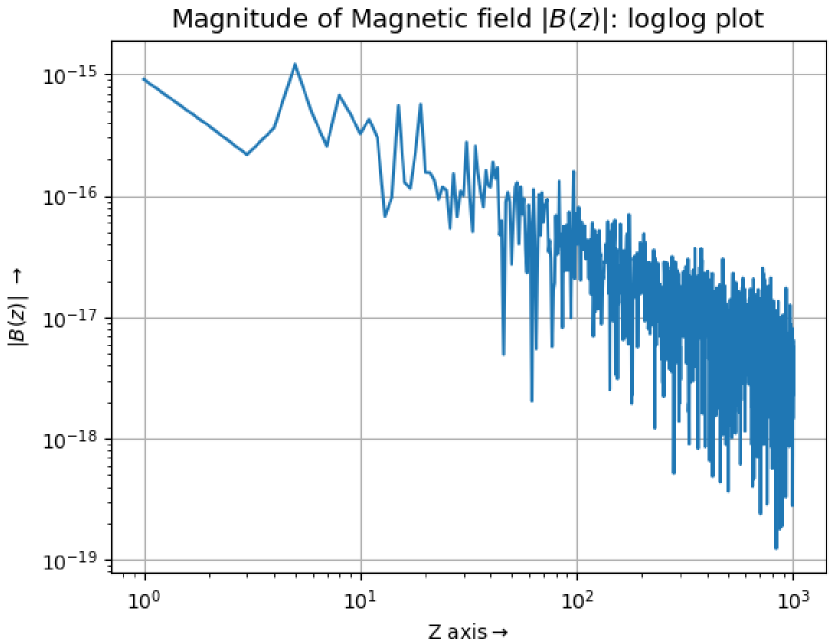
where the location of the l^{th} current element is `r = r[0]`$\hat{x}$ `+ r[1]`$\hat{y}$`+ r[2]`$\hat{z}$. Thus the vector potential `Ax, Ay` can be calculated by

```
x = np.linspace(-1,1,Nx,dtype = float);
y = np.linspace(-1,1,Ny,dtype = float);
z = np.linspace(1,Nz,Nz,dtype = float);
Y,Z,X = np.meshgrid(y,z,x);

Ax = np.zeros((Nz,Ny,Nx))
Ay = np.zeros((Nz,Ny,Nx))

def calc(X,Y,Z, r, theta, IdL):
   R = ((X - r[0])**2 + (Y - r[1])**2 + (Z - r[2])**2)**(0.5)
   Ax = 1e-7*np.multiply(np.exp(-1j*R*k)/R, IdL[0])
   Ay = 1e-7*np.multiply(np.exp(-1j*R*k)/R, IdL[1])
   return Ax, Ay

for i in range(theta.size):
   AxTemp, AyTemp = calc(X,Y,Z, rL[:,i], theta[i], IdL[:,i])
   Ax = Ax + AxTemp
   Ay = Ay + AyTemp
```

Magnetic Field Calculation

Once the vector potential is obtained the magnetic field is obtained from the equation (6) as

```
delX = x[1] - x[0]
delY = y[1] - y[0]
Xmid = int(Nx/2)
Ymid = int(Ny/2)

B = (Ay[:,Ymid,Xmid+1] - Ay[:,Ymid,Xmid-1])/(2*delX) -
    (Ax[:,Ymid+1,Xmid] - Ax[:,Ymid-1,Xmid])/(2*delY)
B = np.abs(B)
```



We can observe from the graph that the values for B are very small in the orders of 10^{-16} which is approximately zero. The expected magnetic field is also **zero** since the current distribution $I_0 cos(\phi)$ is symmetric about both X and Y axes. From the **Right Hand Rule** or **Cork Screw Rule**, the B_z components cancel out and zero effect is observed. The small errors in the orders of 10^{-16} are due to the numerical errors/approximations during the calculations

We try to fit the above mentioned graph into a loglog plot of the form:

$$B = Nz^M$$
$$\log B = \log N + M \log z$$

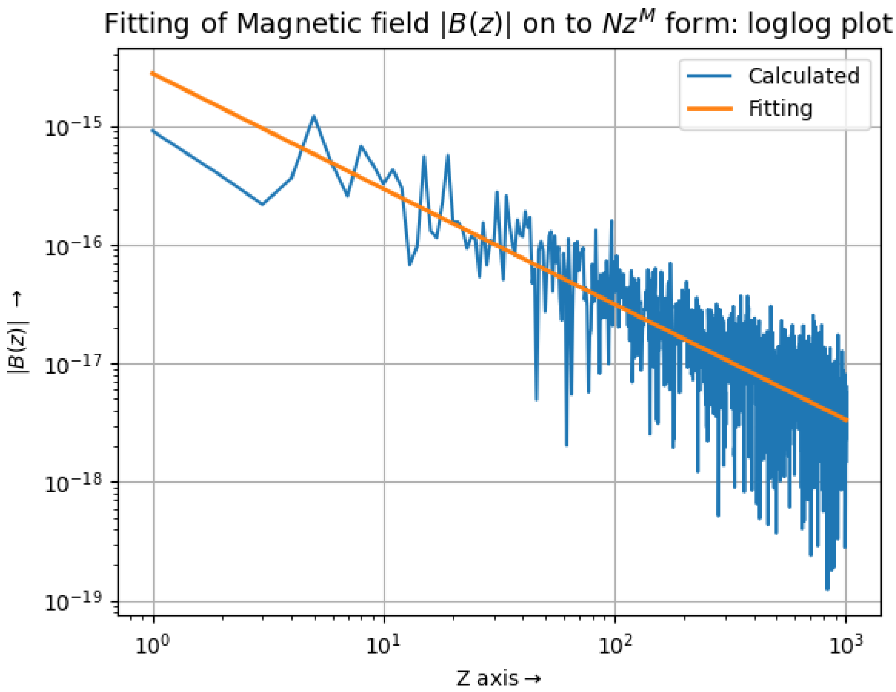
To estimate the values of A and b we use **least square method** such that the error in B is minimum by using `scipy.linalg.lstsq` function.

$$\begin{pmatrix} 1 & z_1 \\ 1 & z_2 \\ 1 & z_3 \\ .. & \\ .. & \\ 1 & z_n \end{pmatrix} \cdot \begin{pmatrix} \log(N) \\ M \end{pmatrix} = \begin{pmatrix} \log(B_1) \\ \log(B_2) \\ . \\ . \\ \log(B_n) \end{pmatrix}$$

```
X = np.vstack([log(z), np.ones(len(z))]).T
M, N = scipy.linalg.lstsq(X, log(B))[0]
N = np.exp(N)
```

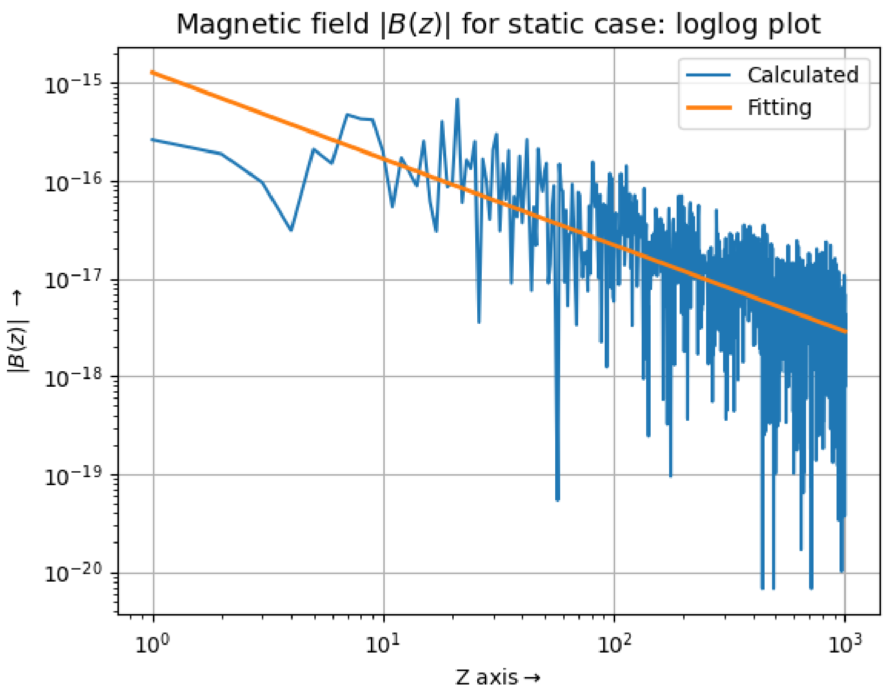
By executing the code, we obtain the fit

```
When the graph is fitted in the form of B = Nz^M, we find:
N =  2.7786522586910982e-15
M =  -0.9731108586197116
```



Since the expected B_z is zero, no definite conclusion can be observed with just the numerical errors.

In case of magnetostatics case, where the current doesn't vary with time - the expected B_z field is **zero** as before since the system is symmetric. The static case is obtained by initializing `k = 0`




```
When the graph is fitted in the form of B = Nz^M, we find:
N =  1.2849960022740135e-15
M =  -0.8828302003695143
```

As expected there is not much difference between $k = 0$ and $k \neq 0$ case.

Exploring Various Current Distributions

In the previous pages, we have considered the current distribution to be both space and time variant with $I = I_0 cos\phi exp(j\omega t)$. Now, Let us look into spatial, time invariant current distribution i.e., $I = constant$. This condition can be observed by changing the code to

```
k = 0
I = np.ones(len(theta))*1e7
```

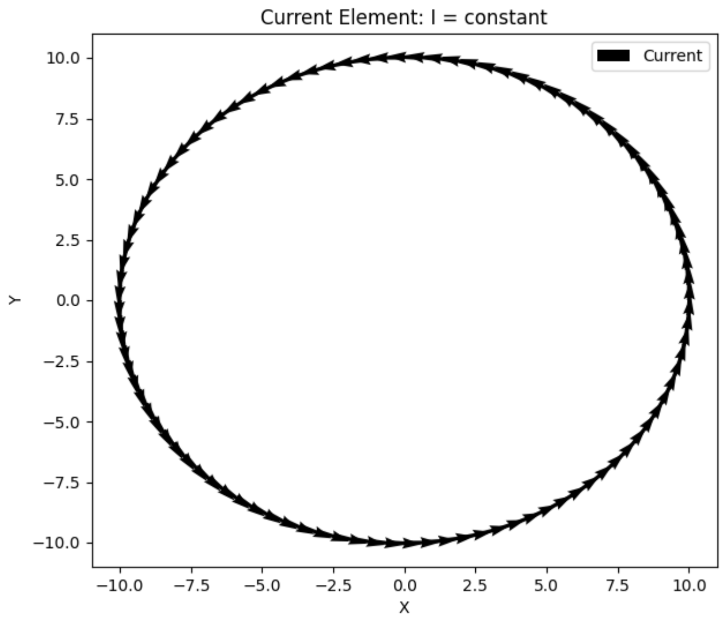
The variation of magnetic field can be easily obtained by applying **Biot-Savart's Law**

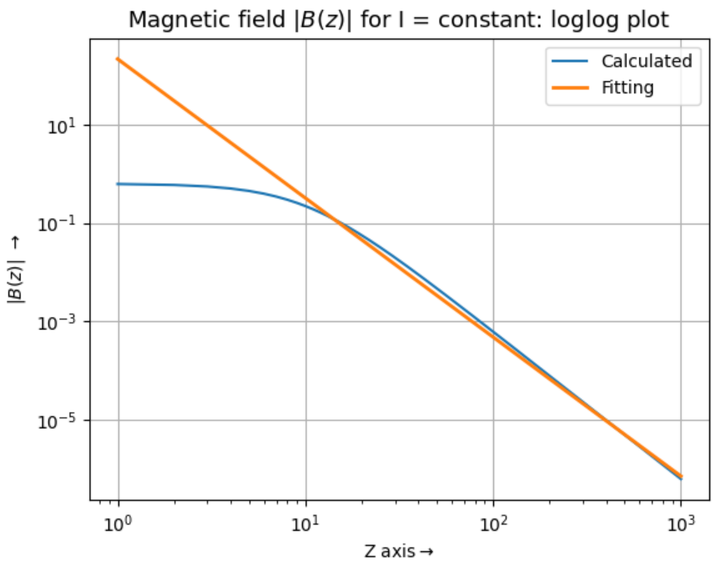
$$\mathbf{dB} = \frac{\mu_0 I d\vec{L} \times \hat{r}}{4\pi r^2}$$

where $\vec{r}$ is the vector from the current element to the position coordinate. On the z-axis, the $|\vec{r}|^2 = \sqrt{a^2 + z^2}$ where a is the radius of the loop wire.

$$dB_z = \frac{\mu_0 I dL}{4\pi} \frac{a}{(a^2+z^2)^{3/2}}$$

$$B_z = \frac{\mu_0 I_0}{2} \frac{a^2}{(a^2+z^2)^{3/2}} \propto \frac{1}{z^3}$$





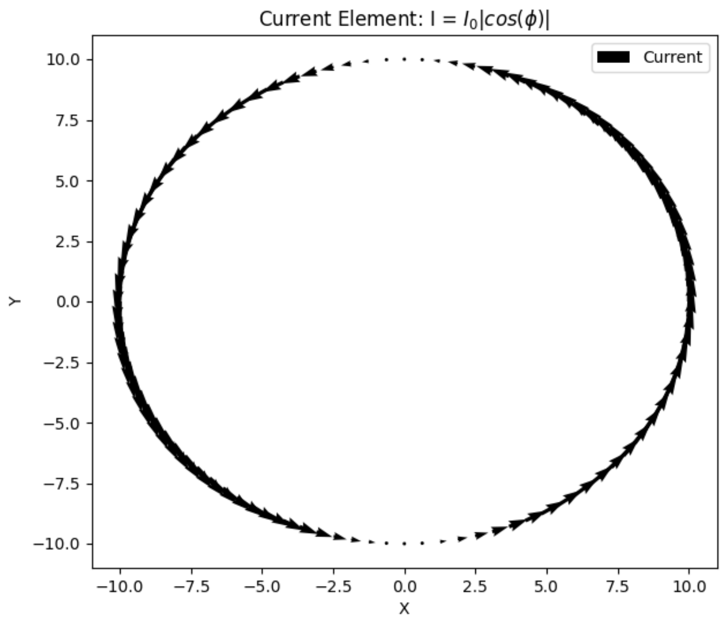

```
When the graph is fitted in the form of B = Nz^M, we find:
N =  215.85790244341976
M =  -2.826192056926644
```

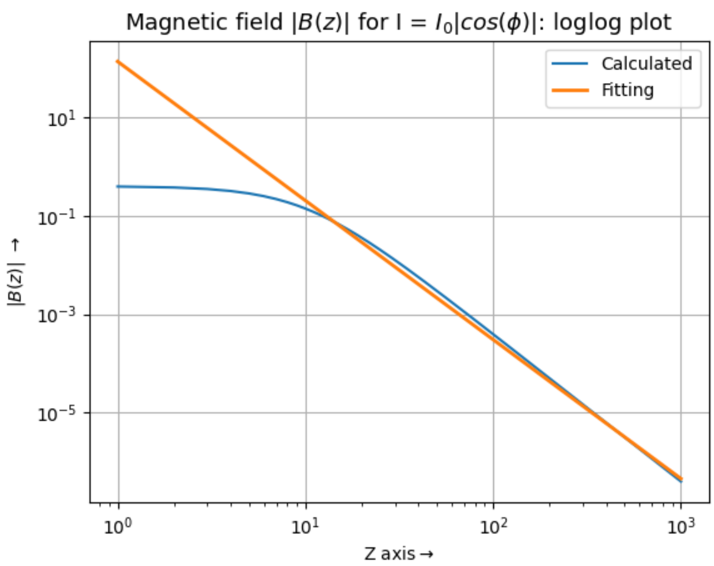
We observe from the graph that the magnetic value B falls on the power of $z^{-2.82619}$ which is close to z^{-3} as expected. Hence our simulation works well.

Let us explore one another current distribution - time invariant but spatially varying

$$I = \frac{4\pi}{\mu_0} |\cos(\phi)|$$

When the magnetic fields are plotted, we observe that





```
When the graph is fitted in the form of B = Nz^M, we find:
N =  137.36239855412188
M =  -2.8261780521825823
```

We can observe that the B falls on the power of $z^{-2.82619}$. Thus B falls off as $\frac{1}{z^3}$

Conclusion

For $I = I_0 \cos(\phi) \exp(j\omega t)$ we plotted the current distribution, calculated the vector potential and magnetic potential. We observed that the values are in the order of 10^{-15}. This is as expected since the expected magnetic field is zero.

We also tried to fit the magnetic field function in the form $B = Nz^M$ using `lstsq` function. Since the observed value are just numerical errors, no observable definite conclusions can be observed. We can observe the same for the magnetostatic time invariant case(k=0).

We also explored other current distributions like space time invariant distribution $I = I_0$. We observed that the magnetic field falls with $z^{-2.82}$ which is approximately the same as expected z^{-3}. We also plotted for the time invariant space variant distribution $I = I_0|\cos(\phi)|$ and observed that the field falls with $z^{-2.82}$.

Through this assignment, we learned on how to implement python arrays efficiently by vectorizing the loops. We also learnt how to implement Least Square Sum method using `lstsq` function