Problem 1.

$$f_x(x) = \begin{cases} cx^2, & |x| \le 1 \\ 0, & \text{otherwise} \end{cases}$$

$$F_x(x) = \int_{-1}^{x} cu^2 du = \frac{c}{3} u^3 \Big|_{-1}^{x} = \frac{c}{3}(x^3+1)$$

$$F_x(x) = \begin{cases} 0, & x < 1 \\ \frac{x^3}{2} + \frac{1}{2}, & -1 \le x \le 1 \\ 1, & x > 1 \end{cases}$$

a) $\boxed{\text{Find the constant } c.}$

$$1 = \int_{-\infty}^{\infty} f_x(x)dx = \int_{-\infty}^{-1} 0dx + \int_{-1}^{1} cx^2 dx + \int_{1}^{\infty} 0dx = \int_{-1}^{1} cx^2 dx = c\frac{x^3}{3}\Big|_{-1}^{1} = \frac{2}{3}c$$

$$\frac{2}{3}c = 1 \qquad c = \frac{3}{2}$$

b) $\boxed{\text{Find } E(x) \text{ and } Var(x)}$

$$E(x) = \int_{-\infty}^{\infty} x f_x(x) dx = \int_{-1}^{1} \frac{3}{2} x^3 dx = \frac{3}{2} \cdot \frac{x^4}{4}\Big|_{-1}^{1} = 0 \qquad E(x) = 0$$

$$E(x^2) = \int_{-\infty}^{\infty} x^2 f_x(x) dx = \int_{-1}^{1} \frac{3}{2} x^4 dx = \frac{3}{2} \cdot \frac{x^5}{5}\Big|_{-1}^{1} = \frac{3}{5}$$

$$Var(x) = E(x^2) - (E(x))^2 = \frac{3}{5} \qquad Var(x) = \frac{3}{5}$$

c) $\boxed{P(x \ge 12) - ?}$

$$P(x \ge 12) = 1 - P(x < 12) = 1 - F(12) = 1 - 1 = 0$$

$$P(x \ge 12) = \int_{12}^{\infty} f_x(x) dx = \int_{12}^{\infty} 0dx = 0 \qquad P(x \ge 12) = 0$$

Problem 2.

$$f_X(x) = \frac{1}{2} e^{-|x|} \quad , \quad \text{for all } x \in R$$

$\boxed{\text{if } Y = X^2 \text{, find the CDF of } Y.}$

$Y = X^2 \Rightarrow y \in [0; \infty)$

$$F_Y(y) = P(Y \le y) = P(X^2 \le y) = P(-\sqrt{y} \le X \le \sqrt{y}) = \int_{-\sqrt{y}}^{\sqrt{y}} \frac{1}{2} e^{-|x|} dx =$$

$$= 2 \cdot \frac{1}{2} \int_0^{\sqrt{y}} e^{-x} dx = -e^{-x} \Big/_0^{\sqrt{y}} = -e^{-\sqrt{y}} - (-e^0) = 1 - e^{-\sqrt{y}}$$

$$\boxed{F_Y(y) = \begin{cases} 1 - e^{-\sqrt{y}} & , \ y \in [0; \infty) \\ 0, & \text{otherwise} \end{cases}}$$

Problem 3.

$X \sim N(3, 9)$ $\quad \mu = 3, \quad \sigma = \sqrt{9} = 3$

a) $\boxed{\text{Find } P(X>0)}$

$P(X>0) = 1 - P(X \le 0) = 1 - \Phi\left(-\frac{\mu}{\sigma}\right) = 1 - \left(1 - \Phi\left(\frac{\mu}{\sigma}\right)\right) = \Phi\left(\frac{\mu}{\sigma}\right) =$

$= \Phi(1) = 0{,}8413$ $\qquad P(X>0) = 0{,}8413$

b) $\boxed{\text{Find } P(-3 < X < 8)}$

$P(-3<X<8) = F_X(8) - F_X(-3) = \Phi\left(\frac{8-\mu}{\sigma}\right) - \Phi\left(\frac{-3-\mu}{\sigma}\right) =$

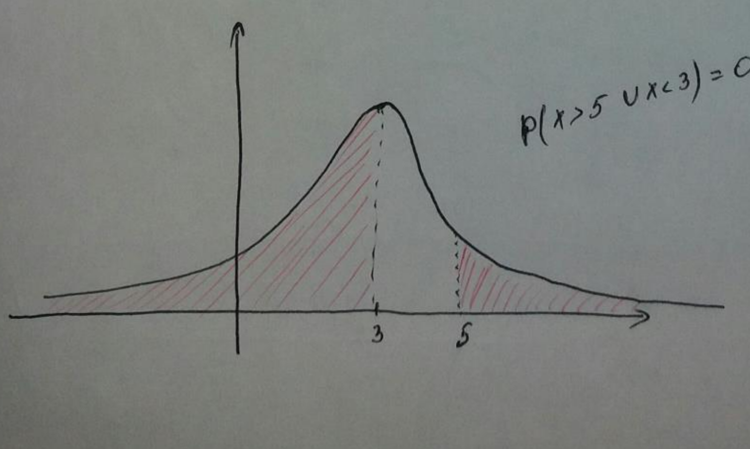
$= \Phi\left(\frac{5}{3}\right) - \Phi(-2) = \Phi(1{,}67) - (1 - \Phi(2)) = 0{,}9525 -$

$-(1 - 0{,}9772) = 0{,}9297$ $\qquad P(-3<X<8) = 0{,}9297$

c) $\boxed{P(X>5 \mid X<3) \text{ -?}}$

$P(X>5 \mid X<3) = \dfrac{P(X>5, X<3)}{P(X<3)} = \dfrac{0}{P(X<3)} = 0$

$P(X>5 \mid X<3) = 0$



Problem 4.

$$E(X) = \int_0^\infty P(X \geq x)\, dx$$

$\boxed{\text{Prove that}}$

$$P(X \geq x) = \int_x^\infty f_X(u)\, du$$

$$E(X) = \int_0^\infty \int_x^\infty f_X(u)\, du\, dx$$

$$\int_0^\infty \int_x^\infty f_X(u)\, du\, dx = \int_0^\infty \int_0^u f_X(u)\, dx\, du = \int_0^\infty f_X(u) \left(\int_0^u dx \right) du =$$

$$= \int_0^\infty u\, f_X(u)\, du = E(X)$$