Advanced methods for Information Representation

Vector spaces-I

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Group: definition

- Group (G,*)

  It is an algebraic structure where the operation "*" between the elements of G is

  1. associative, i.e. $\forall$ a,b,c $\in$ G, (a * b) * c = a * (b * c)
  2. $\exists$ a neutral element with respect to "*", i.e.

     $\forall a \in G, \exists e_* \in G, a * e_* = e_* * a = a$
  3. Every element of G has an inverse with respect to "*", i.e.

     $\forall a \in G, \exists b \in G, a * b = b * a = e_*$

- If $\forall$ a,b $\in$ G, a * b = b * a, (G,*) is said to be commutative (or abelian)

Field: definition

- $(K,+,.)$: algebraic structure over which 2 operations are defined, "+" and "." such that:
  - $(K,+)$ forms an abelian group;
  - $(K\setminus\{e_+\},.)$ forms a group with neutral element $e_.$
  - "." is distributive with respect to +, i.e. $\forall$ a,b,c $\in$ K, a.(b+c) = a.b+a.c

- $(\mathbb{C},+,.)$ represents the field of complex numbers, with "+" and "." being the addition/multiplication on complex numbers
- $(\mathbb{R},+,.)$ represents the field of real numbers, with "+" and "." being the addition/multiplication on real numbers

Vector spaces: definition

- V forms a vector space on the field of complex numbers $\mathbb{C}$ if
  1. $(V, +)$ forms a commutative group, where $+$ identifies the "sum" operation between the elements of V. Its neutral element is $\underline{0} \triangleq \underline{e}_{+}$.
  2. $\exists$ an external product "." between the elements of V and $\mathbb{C}$, for which
     a) complex number multiplication "." is interchangeable with respect to ".", i.e.

        $\forall$ a,b, $\in \mathbb{C}$, $\underline{v} \in V$, $(a.b).\underline{v} = a.(b.\underline{v})$

     b) $1 \in \mathbb{C}$ is a neutral element for ".", i.e. $\forall\ \underline{v} \in V$, $1.\underline{v} = \underline{v}$

     c) "." is distributive with respect to the sum "+" of the elements in V, i.e.

        $\forall\ a \in \mathbb{C}$, $\underline{x},\underline{y} \in V$, $a.(\underline{x}+\underline{y}) = (a.\underline{x}) + (a.\underline{y})$

     d) "." is (sort of) distributive" with respect to the sum "+" defined over $\mathbb{C}$, **i.e.**

        $\forall$ a, b $\in \mathbb{C}$, $\underline{x} \in V$, $(a+b).\underline{x} = (a.\underline{x}) + (b.\underline{y})$

- The elements of V are called "vectors".

Vector spaces: examples

- Space $\mathbb{C}^N$ / $\mathbb{R}^N$ of complex-/(real-)valued finite dimensional vectors

$$\mathbb{C}^N / \mathbb{R}^N = \{ \underline{x} = [\, x_0 \; x_1 \; \ldots \; x_{N-1} \,]^T \mid x_n \in \mathbb{C}/\mathbb{R} \,,\; n = 0, 1, \ldots, N-1 \}$$

$$\underline{x} + \underline{y} \;\triangleq\; [\, x_0+y_0 \;\; x_1+y_1 \; \ldots \; x_{N-1}+y_{N-1} \,]^T$$

$$a.\underline{x} \;\triangleq\; [\, a.x_0 \;\; a.x_1 \; \ldots \; a.x_{N-1} \,]^T$$

- Space $\mathbb{C}^{\mathbb{Z}}$ / $\mathbb{R}^{\mathbb{Z}}$ of complex-/(real-)valued infinite sequences

$$\mathbb{C}^{\mathbb{Z}} / \mathbb{R}^{\mathbb{Z}} = \{ \underline{x} = [\, \ldots \; x_{-1} \; x_0 \; x_1 \; \ldots \,]^T \mid x_n \in \mathbb{C}/\mathbb{R} \,,\; n \in \mathbb{Z} \}$$

$$\underline{x} + \underline{y} \;\triangleq\; [\, \ldots \; x_{-1}+y_{-1} \;\; x_0+y_0 \; x_1+y_1 \; \ldots \,]^T$$

$$a.\underline{x} \;\triangleq\; [\, \ldots \; a.x_{-1} \; a.x_0 \;\; a.x_1 \; \ldots \; a.x_{N-1} \,]^T$$

Vector spaces: examples

- Space $\mathbb{C}^{\mathbb{R}}$ / $\mathbb{R}^{\mathbb{R}}$ of complex-/(real-)valued functions over $\mathbb{R}$

  $\mathbb{C}^{\mathbb{R}} / \mathbb{R}^{\mathbb{R}} = \{ \underline{x} = x(t) \mid x(t) \in \mathbb{C}/\mathbb{R}, t \in \mathbb{R} \}$

  $\underline{x} + \underline{y} \quad \triangleq \quad (x+y)(t)$ sum of functions

  $a.\underline{x} \quad \triangleq \quad (a.x)(t)$ external multiplication between a scalar and a function

- Space $\mathbb{C}^{\mathbb{R}^+}$ of complex-valued functions defined over $\mathbb{R}^+$

- Space $\mathbb{C}^{[a,b]}$ of complex-valued functions defined over [a,b]

- Space of polynomial functions of order N-1: $\underline{x} = \sum_{n=0}^{N-1} \alpha_n . t^n$

Subspace: definition/examples

- A non-empty subset S of a vector space V is called a subspace of V, when it is closed with respect to the operations of vector addition "**+**" and scalar multiplication "**.** ":

  1. $\forall\ \underline{x},\underline{y} \in S,\ \underline{x} + \underline{y} \in S$
  2. $\forall\ a \in \mathbb{C},\ \underline{x} \in S,\ a.\underline{x} \in S$

  (alternatively, $\forall\ a,b \in \mathbb{C},\ \underline{x},\underline{y} \in S,\ a.\underline{x} + b.\underline{y} \in S$)

- Examples of subspaces
  - $S_1 = \{\ \underline{x} = a.\underline{x}_0 \mid \text{fixed } \underline{x}_0 \in V,\ \forall a \in \mathbb{C}\ \}$
  - $S_2 = \{\ \underline{x} \in \mathbb{C}^{\mathbb{Z}} \mid x_n = 0,\ \forall n \neq 1,2,3\ \}$, subspace of sequences having 0 value for indices $n \neq 1,2,3$
  - $S_3 = \{\ \underline{x} \in \mathbb{C}^{\mathbb{R}} \mid x(t) = -x(-t)\ \}$, subspace of odd complex-valued functions

Affine subspace: definition/examples

- A non-empty subset T of a vector space V is called an affine subspace of V, when there exists a vector $\underline{v}_0 \in V$, and a subspace S of V such that $\forall\ \underline{t} \in T,\ \exists\, \underline{s} \in S,\ \underline{t} = \underline{s} + \underline{v}_0$
- Property
  - An affine subspace is a subspace of a vector space V only if it contains $\underline{0}$
- Note: An affine subspace generalize the concept of a plane in Euclidean geometry.
- Examples
  - $T_1 = \{\ \underline{x} = a.\underline{x}_0 + \underline{y}_0 \mid \text{fixed } \underline{x}_0, \underline{y}_0 \in V,\ \forall a \in \mathbb{C}\ \}$, it is a subspace iff $\underline{y}_0 = \underline{0}$
  - $T_2 = \{\ \underline{x} \in \mathbb{C}^{\mathbb{Z}} \mid x_n = 1,\ \forall n \neq 1,2,3\ \}$, affine subspace of $\mathbb{C}^{\mathbb{Z}}$, it is not a subspace of $\mathbb{C}^{\mathbb{Z}}$ since the sequence of all "0" $\notin T_2$

Span/Linear independence: definition

- The span of a set of vectors S is the set of all finite linear combinations of vectors in S

$$S = \{\underline{\varphi}_k, k = 1,2,\ldots\}$$

$$span(S) = span\{\underline{\varphi}_k\} = \left\{ \sum_{k=1}^{N} \alpha_k . \underline{\varphi}_k \middle| \alpha_k \in C, \underline{\varphi}_k \in S, N < \infty \right\}$$

- A set of vectors $S = \{\underline{\varphi}_k$, k=1, 2, …} is said linearly independent when the system of linear equations $\sum_k \alpha_k . \underline{\varphi}_k = \underline{0}$ admits as unique solution $\alpha_k = 0 \quad \forall$ k=1,2, …

Dimension: definition

- A vector space V is said to have dimension N when it contains a linear independent set of cardinality N and any other set of higher cardinality is linearly dependent. If no finite N exists, the vector space is infinite dimensional.

- Examples
  1. $\mathbb{R}^N$ has dimension N
  2. The vector space of polynomial functions of degree N has dimensione N+1
  3. $\mathbb{C}^{\mathbb{Z}}$, $\mathbb{C}^{[a,b]}$ are infinite dimensional vector spaces

Inner product: definition

- An inner product on a vector space V over $\mathbb{C}$ (or $\mathbb{R}$) is a complex-(real-)valued function defined on VxV satisfying the following properties $\forall\, \underline{x},\underline{y},\underline{z} \in V$ and $\alpha \in \mathbb{C}$:

  a) $<\underline{x}+\underline{y},\underline{z}> = <\underline{x},\underline{z}> + <\underline{y},\underline{z}>$ linearity

  b) $<\alpha.\underline{x},\underline{y}> = \alpha.<\underline{x},\underline{y}>$ homogeneity

  [! $<\underline{x},\alpha.\underline{y}> = \alpha^*.<\underline{x},\underline{y}>$]

  c) $<\underline{x},\underline{y}> = <\underline{y},\underline{x}>^*$ [Hermitian symmetry]

  d) $<\underline{x},\underline{x}> \geq 0$ and $<\underline{x},\underline{x}> = 0$ iff $\underline{x} = \underline{0}$ [positive definiteness]

- Examples for $\forall\ (\underline{x},\underline{y}) \in \mathbb{C}^2$

  1. $<\underline{x},\underline{y}> = x_0 y_0^* + 5 x_1 y_1^*$ OK
  2. $<\underline{x},\underline{y}> = x_0^* y_0 + x_1^* y_1$ NO: violation of (c)
  3. $<\underline{x},\underline{y}> = x_0 y_0^*$ NO: violation of (d) ($<[0\ 1]^T,[0\ 1]^T> = 0$)

Inner product: standard definitions/orthogonality

- $\mathbb{C}^N$: $$\langle \underline{x},\underline{y} \rangle = \sum_{n=0}^{N-1} x_n . y_n^* \quad (\underline{x},\underline{y}) \in C^N$$
- Subspace of $\mathbb{C}^{\mathbb{Z}}$ leading to a convergent series $$\langle \underline{x},\underline{y} \rangle = \sum_{n \in Z} x_n . y_n^* \quad (\underline{x},\underline{y}) \in C^Z$$
- Subspace of $\mathbb{C}^{\mathbb{R}}$ leading to the existence of the integral $$\langle \underline{x},\underline{y} \rangle = \int_R x(t).y^*(t).dt \quad (\underline{x},\underline{y}) \in C^R$$

1. 2 vectors $\underline{x}$ and $\underline{y}$ $\in V$ are said to be orthogonal ($\underline{x} \perp \underline{y}$)

   if $<\underline{x},\underline{y}> = 0$
2. A set of vectors S is said orthogonal whenever $\underline{x} \perp \underline{y}$ $\forall \underline{x},\underline{y} \in S$ with $\underline{x} \neq \underline{y}$

Inner product: orthogonality

3. A set of vectors S is said orthonormal whenever it is orthogonal and $\forall \underline{x} \in S$, $<\underline{x},\underline{x}> = 1$
4. A vector $\underline{x}$ is said to be orthogonal to a set of vectors S ($\underline{x} \perp S$) when ($\underline{x} \perp \underline{s}$) $\forall \underline{s} \in S$
5. Two sets of vectors S_0 and S_1 are said to be orthogonal ($S_0 \perp S_1$) whenever $\underline{x}_0 \perp S_1 \; \forall \underline{x}_0 \in S_0$
6. Given a subspace S of a vector space V, the orthogonal complement of S, denoted $S^\perp$, is the set $\{\underline{x} \in V \mid \underline{x} \perp S\}$

- Properties
  - $S^\perp$ is a subspace of V
  - An orthonormal set $\{\underline{\varphi}_k\}$ is a linearly independent set

    (proof: expand $\underline{0}$ in $<\underline{0},\underline{\varphi}_i> = 0$)

Inner product examples / Inner product space

- Example of an orthonormal set

  $\varphi_0(t) = 1 \qquad t \in [-½, ½]$

  $\varphi_k(t) = 2^{½} \cos(2k\pi t) \quad t \in [-½, ½] \quad k = 1, 2\ldots$

- $\{\varphi_k(t),\ k = 0, 1, \ldots\}$ is orthogonal to the set of odd functions S_{odd}

- Definition

  A vector space equipped with an inner product is called an

  inner product space

Norm: definition

- A norm on a vector space V over $\mathbb{C}$ (or $\mathbb{R}$) is a real-valued function $||.||$ defined on V satisfying the following properties $\forall\, \underline{x}, \underline{y} \in V$ and $\alpha \in \mathbb{C}$(or $\mathbb{R}$):
  a) $||\underline{x}|| \geq 0$ and $||\underline{x}|| = 0$ iff $\underline{x} = \underline{0}$ [positive definiteness]
  b) $||\alpha.\underline{x}|| = |\alpha|.||\underline{x}||$ [homogeneity]
  c) $||\underline{x}+\underline{y}|| \leq ||\underline{x}|| + ||\underline{y}||$ [triangle inequality]
  geometric interpretation: the length of any side of a triangle $\leq$ the sum of the lengths of the other two sides
- An inner product may be used to define a norm; in such a case the norm is said to be *induced* by the inner product
- Examples for $\forall\, \underline{x} \in \mathbb{C}^2$
  1. $||\underline{x}|| = (|x_0|^2 + 5|x_1|^2)^{½}$ OK
  2. $||\underline{x}|| = |x_0| + |x_1|$ OK
  3. $||\underline{x}|| = |x_0|$ NO: violation of (a)
  4. $||\underline{x}|| = \max(|x_0|, |x_1|)$ OK

Norm: standard definitions

- Euclidean norm in $\mathbb{C}^N$: $\|\underline{x}\|_2 = \sqrt{\sum_{n=0}^{N-1} |x_n|^2} \quad \underline{x} \in C^N$

- Euclidean norm in subspace of $\mathbb{C}^{\mathbb{Z}}$ for which the inner product exists

$$\|\underline{x}\|_2 = \sqrt{\sum_{n \in Z} |x_n|^2} \quad \underline{x} \in C^Z$$

- Euclidean norm in subspace of $\mathbb{C}^{\mathbb{R}}$ for which the inner product exists

$$\|\underline{x}\|_2 = \sqrt{\int_R |x(t)|^2 . dt} \quad \underline{x} \in C^R$$

- Or more generally $\|\underline{x}\|_2^2 = \langle \underline{x}, \underline{x} \rangle$

Inner product induced norms: properties

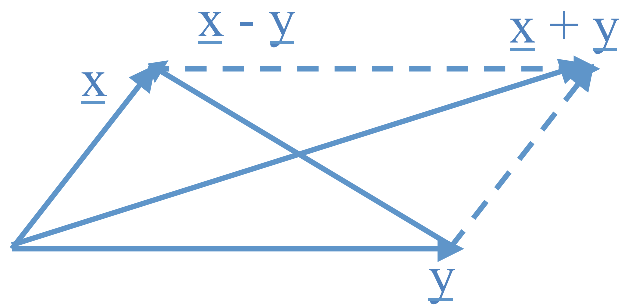
- Pythagorean theorem

  $\forall \underline{x}, \underline{y} \in V$, such that $\underline{x} \perp \underline{y}$ $\quad \|\underline{x}+\underline{y}\|^2 = \|\underline{x}\|^2 + \|\underline{y}\|^2$

  proof: inner product properties +express $\langle \underline{x}+\underline{y}, \underline{x}+\underline{y} \rangle$

  more generally

  $\{\underline{x}_k\}_{k \in K}$ being an orthogonal set $\left\|\sum_{k\in K} \underline{x}_k\right\|^2 = \sum_{k\in K} \|\underline{x}_k\|^2$



- Parallelogram law

  $\forall \underline{x}, \underline{y} \in V \quad \|\underline{x}+\underline{y}\|^2 + \|\underline{x}-\underline{y}\|^2 = 2\left(\|\underline{x}\|^2 + \|\underline{y}\|^2\right)$

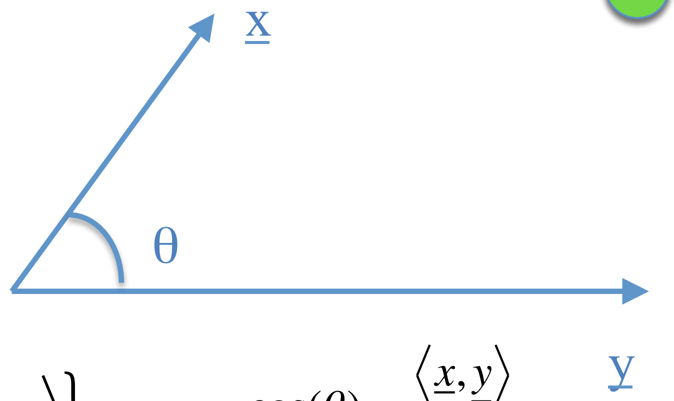
  – The parallelogram law is a necessary and sufficient condition for the norm to be induced by an inner product

Inner product induced norms: properties

- Cauchy-Schwarz inequality

$$\forall \underline{x},\underline{y} \in V, \quad \left|\left\langle \underline{x},\underline{y}\right\rangle\right| \le \|\underline{x}\|.\|\underline{y}\|$$

  - Equality holds when $\underline{x} = \alpha.\underline{y}$
  - proof: use non-negativity of $||k.\underline{x} + \underline{y}||^2$



$$\cos(\theta) = \frac{\left\langle \underline{x},\underline{y}\right\rangle}{\|\underline{x}\|\cdot\|\underline{y}\|}$$

$$0 \le \|k.\underline{x}+\underline{y}\|_2^2 = |k|^2\|\underline{x}\|_2^2 + \|\underline{y}\|_2^2 + 2.\mathrm{Re}\left\{\left\langle k.\underline{x},\underline{y}\right\rangle\right\}$$

$$= |k|^2\|\underline{x}\|_2^2 + \|\underline{y}\|_2^2 + 2.\mathrm{Re}\left\{k\left\langle \underline{x},\underline{y}\right\rangle\right\}$$

Choosing $k = -\langle\underline{x},\underline{y}\rangle^* / ||\underline{x}||^2$ leads to $$\frac{\left|\left\langle \underline{x},\underline{y}\right\rangle^*\right|^2}{\|\underline{x}\|_2^2} + \|\underline{y}\|_2^2 + 2.\mathrm{Re}\left\{-\frac{\left|\left\langle \underline{x},\underline{y}\right\rangle\right|^2}{\|\underline{x}\|_2^2}\right\} \ge 0$$

$$\frac{\left|\left\langle \underline{x},\underline{y}\right\rangle\right|^2}{\|\underline{x}\|_2^2} \le \|\underline{y}\|_2^2$$

CVD

Normed vector space

- Normed vector space: A vector space equipped with a norm is called a normed vector space

- Note: caution is necessary to limit the subspace of V for which the norm exists

Metric: definition

- In a normed vector space V over $\mathbb{C}$ (or $\mathbb{R}$), the metric or distance between 2 vectors $\underline{x}$ and $\underline{y}$ is defined as the norm of the difference vector: $d(\underline{x},\underline{y}) = ||\underline{x}-\underline{y}||$
- Given a vector space V over $\mathbb{C}$(or $\mathbb{R}$), a distance may be more generally defined even in the absence of a norm as the real-valued function defined on VxV satisfying the following properties $\forall \underline{x},\underline{y},\underline{z} \in V$
  1. $d(\underline{x},\underline{y}) \geq 0$ [positivity]
  2. $d(\underline{x},\underline{y}) = 0 \Leftrightarrow \underline{x} = \underline{y}$
  3. $d(\underline{x},\underline{y}) = d(\underline{y}, \underline{x})$ [symmetric measure]
  4. $d(\underline{x},\underline{y}) \leq d(\underline{x},\underline{z}) + d(\underline{z},\underline{y})$ [triangular inequality]
- Example of a distance not induced by a norm in $\mathbb{R}$

$$d(x,y) = |\,\mathrm{atan}(x) - \mathrm{atan}(y)\,|$$

Standard spaces

- Standard inner product spaces
  - Space of complex-valued finite dimensional vectors: $\mathbb{C}^N$
  - Space of square-summable sequences: $l^2(\mathbb{Z}) \subset \mathbb{C}^{\mathbb{Z}}$ (infinite dimensional)
  - Space of square-integrable functions: $\mathcal{L}^2(\mathbb{R}) \subset \mathbb{C}^{\mathbb{R}}$ (infinite dimensional)
  - Space of square-integrable functions over [a,b]: $\mathcal{L}^2([a,b]) \subset \mathbb{C}^{[a,b]}$
  - Space of continuous functions $C[a,b] \subset \mathbb{C}^{[a,b]}$
  - Space of continuous functions with q continuous derivatives

    $C^q[a,b] \subset C^{q-1}[a,b] \subset \ldots \subset C^0[a,b]=C[a,b]$

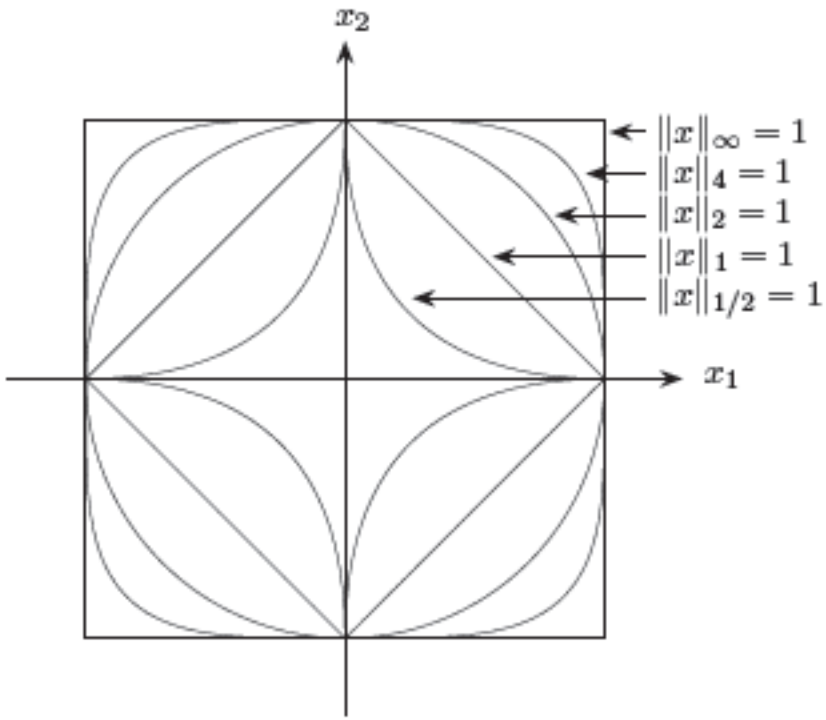
    Note: $C^q[a,b]$ is not a complete space
  - Space of polynomial functions $\subset C^\infty[a,b]$
  - Space of random variables (RVs): inner product $<\underline{X},\underline{Y}> = E[XY^*]$
    - The space of RVs with finite 2nd order moments is a normed vector space

Standard spaces

- Standard normed spaces
  - Space of complex-valued finite dimensional vectors: $\mathbb{C}^N$
    - p-norm: $$\left\|\underline{x}\right\|_p = \left(\sum_{n=0}^{N-1} |x_n|^p\right)^{1/p} \qquad \underline{x} \in C^N$$
    - p=1: Manhattan norm
    - p=2: Euclidean norm
    - p=∞: $$\left\|\underline{x}\right\|_\infty = \lim_{p\to\infty} \left(\sum_{n=0}^{N-1} |x_n|^p\right)^{1/p} = \max\left(|x_0|,|x_1|,\ldots,|x_{N-1}|\right) \qquad \underline{x} \in C^N$$
    - p∈[0,1): does not lead to a norm, but provides useful interpretation $$\left\|\begin{pmatrix}1\\0\end{pmatrix}+\begin{pmatrix}0\\1\end{pmatrix}\right\|_{1/2} = (1+1)^2 = 4 > 2 = 1+1 = \left\|\begin{pmatrix}1\\0\end{pmatrix}\right\|_{1/2} + \left\|\begin{pmatrix}0\\1\end{pmatrix}\right\|_{1/2}$$
    - p=0, $||\underline{x}||_0$ accounts for the number of non zero components in $\underline{x}$
    - Any two norms bound each other within constant factor
    - Only for p=2, the set of unit-norm vectors is invariant to a rotation of the coordinate system

l^p norms

- Set of unit-norm vectors in $\mathbb{R}^2$ for different l^p-measures



$l^p(\mathbb{Z})$ spaces

- l^p norm: $p \in [1,\infty)$

$$\|\underline{x}\|_p = \left(\sum_{n\in Z} |x_n|^p\right)^{1/p} \qquad \underline{x} \in C^Z$$

  - l^∞ norm:

$$\|\underline{x}\|_\infty = \sup_{n\in Z} |x_n| \qquad \underline{x} \in C^Z$$

  - for $p \in [1,\infty)$, the normed vector space $l^p(\mathbb{Z}) \subset \mathbb{C}^{\mathbb{Z}}$ correspond to the subspace formed by vectors in $\mathbb{C}^{\mathbb{Z}}$ with finite l^p norm

  - Property: $p<q \Rightarrow l^p(\mathbb{Z}) \subset l^q(\mathbb{Z})$
    - Corollary: If a sequence has finite l^1-norm, it has finite l^2-norm (the opposite is not necessary true)
    - Example: $x_n=1/n \quad n=1,2,\ldots$ and $x_n=0 \ n\leq 0$
      $||\underline{x}||_2 = \pi^2/6$ whereas $||\underline{x}||_1$ diverges

$\mathcal{L}^2(\mathbb{R})$ spaces

- $\mathcal{L}^p$ norm: $p \in [1,\infty)$

$$\|\underline{x}\|_p = \left(\int_R |x(t)|^p \right)^{1/p} \qquad \underline{x} \in C^R$$

  – $\mathcal{L}^\infty$ norm: $\|\underline{x}\|_\infty = \underset{t \in R}{ess\sup} |x(t)| \quad \underline{x} \in C^R$

  – for $p \in [1,\infty)$, the normed vector space $\mathcal{L}^p(\mathbb{R}) \subset \mathbb{C}^{\mathbb{R}}$ correspond to the subspace formed by vectors in $\mathbb{C}^{\mathbb{R}}$ with finite l^p norm

- It is possible to define similarly other $\mathcal{L}^p$ norm for other continuous time vector spaces such as $\mathbb{C}^{[a,b]}$