Advanced methods for Information Representation

Vector spaces-II

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Convergence: definition

- Convergent sequence of vectors

  A sequence of vectors $\underline{x}_0, \underline{x}_1, \ldots$ in a normed vector space V is said to converge to a vector $\underline{v} \in V$ when $\lim_{k\to\infty} ||\underline{v}-\underline{x}_k|| = 0$

  – In other words, given $\varepsilon>0$, $\exists K_\varepsilon$ such that $\|\underline{v} - \underline{x}_k\| < \varepsilon \quad \forall k > K_\varepsilon$

  – Note that the convergences may depend on the choice of the norm

    - Consider

$$x_k(t) = \begin{cases} 1 & t \in [0, k^{-1}] \\ 0 & otherwise \end{cases}$$

      – This sequence of vectors converges to $v(t)=0$ for all $\mathcal{L}^p$ norms with $p<\infty$

      – It does not converge for the $\mathcal{L}^\infty$ norm

Closed subspace: definition

- A subspace S of a normed vector space V is called closed when it contains all limits of sequence of vectors in S.
- Properties
  - Subspaces of all finite-dimensional normed spaces are always closed
  - Span of infinite set of vectors may not be closed
  - The closure of a set is the set of all limit points of convergent sequences in the set
  - The closure of the span of an infinite set of vectors is the set of all convergent infinite linear combination. The closure of the span of a set of vectors is always a closed subspace

$$\overline{span}\left(\{\varphi_k\} k \in K\right) = \left\{ \sum_{k \in K} \alpha_k \varphi_k \,\middle|\, \alpha_k \in C \quad \text{and the sum converges} \right\}$$

Completeness / Hilbert spaces

- Cauchy sequence of vectors

  A sequence of vectors $\underline{x}_0$, $\underline{x}_1$, … in a normed vector space *V* is called a Cauchy sequence when given $\varepsilon>0$, $\exists\, K_\varepsilon$ such that

  $$\left\|\underline{x}_k - \underline{x}_m\right\| < \varepsilon \quad \forall k,m > K_\varepsilon$$

  – The elements of a Cauchy sequence stay arbitrarily close to each other.

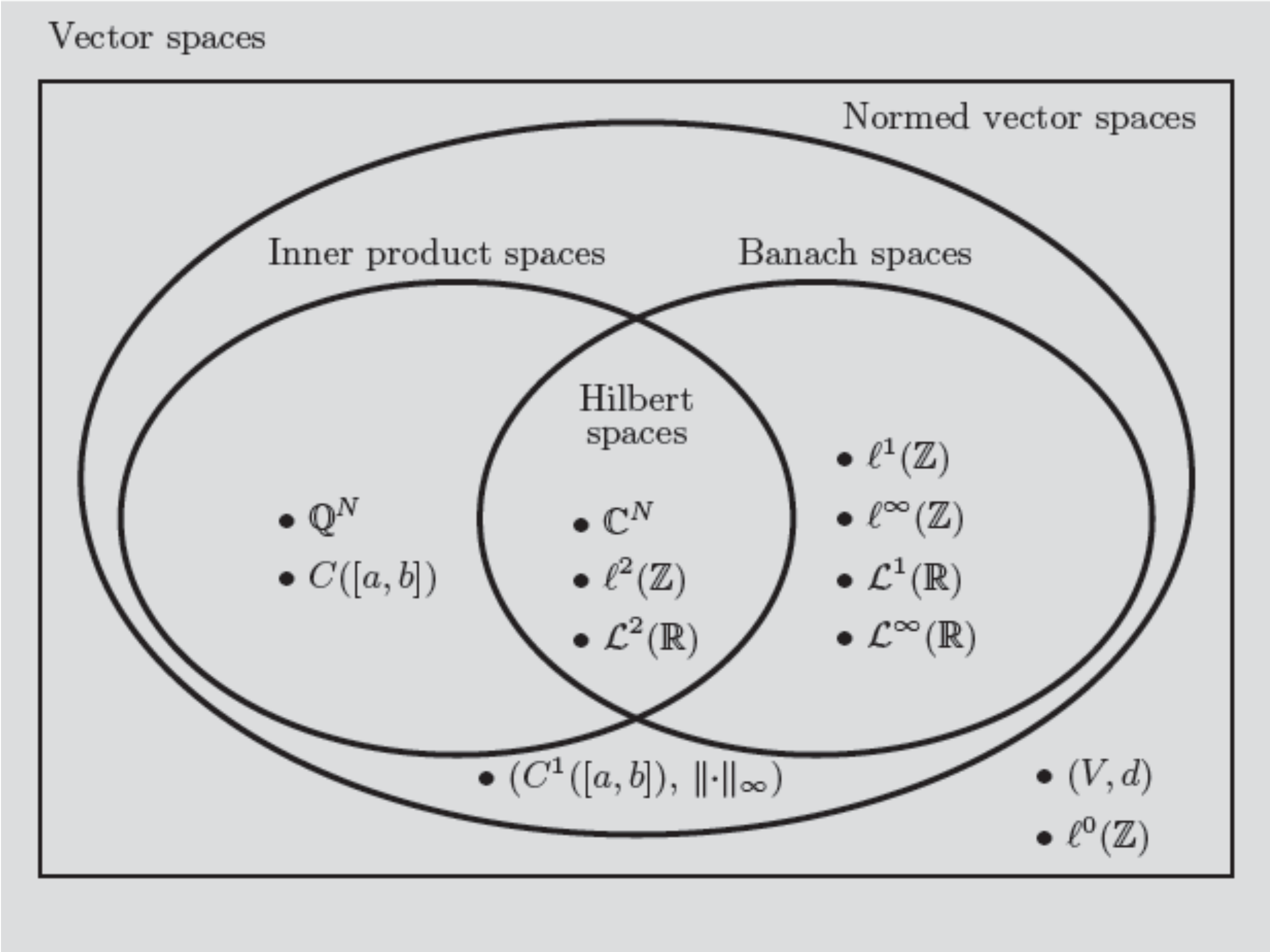
  – For real-valued sequences, it must converge (but it may not be true for all normed vector spaces)

- A normed vector space *V* is said to be complete when every Cauchy sequence in V converges to a vector in V. A complete inner product space is called a Hilbert space.

Completeness / Banach spaces

- A complete normed vector space is called a Banach space
- Properties
  - $\mathbb{Q}$ is not a complete space since there are sequences in it converging to irrational numbers
  - All finite-dimensional spaces are complete
  - All $l^p(\mathbb{Z})$ spaces are complete; in particular $l^2(\mathbb{Z})$ is a Hilbert space
  - All $\mathcal{L}^p(\mathbb{R})$ spaces are complete; in particular $\mathcal{L}^2(\mathbb{R})$ is a Hilbert space ($p<\infty$)
  - $C^q([a,b])$ are not complete under the $\mathcal{L}^p$ norm for $p \in [0,\infty)$
  - The inner product space of random variables are complete and thus Hilbert space

Complete and non complete normed spaces



Relationship between different vector spaces. (V,d) is any vector space with a metric

Separability: definition

- A space is called separable when it contains a countable dense subset
- A Hilbert space contains a countable basis if and only if it is separable
  - A closed subspace of a separable Hilbert space is separable

Linear Operators: definitions

- A function $A : H_0 \rightarrow H_1$ forms a linear operator from H_0 to H_1 when $\forall \underline{x}, \underline{y} \in H_0$, and $\forall \alpha \in \mathbb{C}$, the followings hold:
  1. Additivity: $A(\underline{x}+\underline{y}) = A\underline{x} + A\underline{y}$
  2. Homogeneity: $A(\alpha.\underline{x}) = \alpha.(A\underline{x})$

- When domain and codomain coincide ($H_0=H_1$), A forms an operator on H_0

- Notes
  - $A\underline{x}$ is a writing convention rather than $A(\underline{x})$
  - Linear operators from $\mathbb{C}^N$ to $\mathbb{C}^M$ are represented by matrices in $\mathbb{C}^M x \mathbb{C}^N$

Linear Operators: definitions

- Null space (kernel) $\mathcal{N}(A)$ of a linear operator A: subspace of H_0 which is mapped by A onto $\underline{0}$ in H_1

  $\mathcal{N}(A) = \{ \underline{x} \in H_0 \mid A\underline{x} = \underline{0} \}$

- Range $\mathcal{R}(A)$ of a linear operator A is the subspace of H_1 such that

  $\mathcal{R}(A) = \{A\underline{x} \in H_1 \mid \underline{x} \in H_0 \}$

- The operator norm of an operator A, $\|A\|$, is defined as

$$\|A\| = \sup_{\|\underline{x}\|_{H_0}=1} \|A\underline{x}\|_{H_1}$$

Linear Operators: Bounded-ness

- A linear operator is bounded when its operator norm is finite
- Property: Linear operators with finite dimensional codomains are always bounded
- Examples

1. $\underline{\underline{A}} = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}$ Hypothesis: Consider $\underline{\underline{A}}$ defined on $\mathbb{R}^2$

$$\left\|\underline{\underline{A}}\right\| = \sup_{\theta} \left\| \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} \cos(\theta) \\ \sin(\theta) \end{bmatrix} \right\| = \sup_{\theta} \left\| \begin{bmatrix} 3\cos(\theta) + \sin(\theta) \\ \cos(\theta) + 3\sin(\theta) \end{bmatrix} \right\|$$

$$\left\|\underline{\underline{A}}\right\| = \sup_{\theta} \sqrt{\left(3\cos(\theta) + \sin(\theta)\right)^2 + \left(\cos(\theta) + 3\sin(\theta)\right)^2} = \sup_{\theta} \sqrt{10 + 6\sin(2\theta)} = 2$$

$\mathcal{N}(\underline{\underline{A}}) = \{\underline{0}\}$ $\mathcal{R}(\underline{\underline{A}}) = \mathbb{R}^2$

Linear Operators: Bounded-ness

- Examples

2. Consider A defined on $\mathbb{C}^{\mathbb{Z}}$, such that $(A\underline{x})_n = |n| x_n$

   $||A||$ is infinite

   Proof: Suppose $||A|| < M$ finite; consider $\underline{x}$ with

   $x_n=0$ $n \neq M$ and $x_M=1$, then $||A\underline{x}|| = M$ (contradiction)

3. $\underline{\underline{A}} : C^3 \rightarrow C^2$

$$\underline{\underline{A}} = \begin{bmatrix} 1 & j & 0 \\ 1 & 0 & j \end{bmatrix}$$

$$\left\|\underline{\underline{A}}\right\| = \sup_{\|\underline{x}\|_2=1} \left\|\underline{\underline{A}}\underline{x}\right\|_2 = \sup_{x_i=a_i+jb_i, i\in\{1,2,3\}, \|\underline{x}\|_2=1} \left\| \begin{matrix} (a_1+jb_1)+j(a_2+jb_2) \\ (a_1+jb_1)+j(a_3+jb_3) \end{matrix} \right\|_2$$

$$\left\|\underline{\underline{A}}\right\| = \sup_{x_i=a_i+jb_i, i\in\{1,2,3\}, \|\underline{x}\|_2=1} \left[\sqrt{(a_1-b_2)^2+(a_2+b_1)^2+(a_1-b_3)^2+(a_3+b_1)^2}\right] = \sqrt{3}$$

$\mathcal{N}(\underline{\underline{A}}) = \{ [\ \alpha\ \ j\alpha\ \ j\alpha\ ]^T \}$ $\qquad$ $\mathcal{R}(\underline{\underline{A}}) = \mathbb{C}^2$

Inverse of a Bounded Linear Operator: definition

- A bounded linear operator $A : H_0 \rightarrow H_1$ is invertible if $\exists$ a bounded linear operator $B : H_1 \rightarrow H_0$ such that
  i. $\forall \underline{x} \in H_0 \quad B\, A\, \underline{x} = \underline{x}$
  ii. $\forall \underline{y} \in H_1 \quad A\, B\, \underline{y} = \underline{y}$

  If B is unique it is noted A^{-1}, and is called the inverse of A

  If only i. holds then B is called the left inverse of A

  If only ii. holds then B is called the right inverse of A

- For $\underline{\underline{A}} : \mathbb{C}^N \rightarrow \mathbb{C}^M$, basic linear algebra determines the invertibility of $\underline{\underline{A}}$

Inverse of a Bounded Linear Operator: examples

- Example

1. $\underline{\underline{A}}: C^3 \to C^2$

$$\underline{\underline{A}} = \begin{bmatrix} 1 & j & 0 \\ 1 & 0 & j \end{bmatrix}$$

$\underline{\underline{A}}$ is right invertible, i.e. $\exists\, \underline{\underline{B}}$ such that $\forall\, \underline{y} \in \mathbb{C}^2$, $\underline{\underline{A}}\,\underline{\underline{B}}\,\underline{y} = \underline{y}$

$$\begin{pmatrix} 1 & j & 0 \\ 1 & 0 & j \end{pmatrix} \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \\ b_{31} & b_{32} \end{pmatrix} \underline{y} = \underline{y}$$

$$\left(\begin{pmatrix} 1 & j & 0 \\ 1 & 0 & j \end{pmatrix} \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \\ b_{31} & b_{32} \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right) \underline{y} = \underline{0}$$

$$\begin{pmatrix} b_{11} + jb_{21} - 1 & b_{12} + jb_{22} \\ b_{11} + jb_{31} & b_{12} + jb_{22} - 1 \end{pmatrix} . \underline{y} = \underline{0}$$

$$\begin{vmatrix} b_{11} + jb_{21} - 1 & b_{12} + jb_{22} \\ b_{11} + jb_{31} & b_{12} + jb_{22} - 1 \end{vmatrix} = 0$$

$$\Leftrightarrow (b_{11} + jb_{21} - 1)(b_{12} + jb_{22} - 1) = (b_{11} + jb_{31})(b_{12} + jb_{22})$$

select any b_{ij} accordingly

Inverse of a Bounded Linear Operator: examples

- Example

  2. Sequence multiplier operator by $\underline{\alpha}$ on $l^2(\mathbb{Z})$, $\forall \underline{x} \in l^2(\mathbb{Z})$

     $(A\underline{x})_n = \alpha_n . x_n$

     A is linear (multiplier)

     A is bounded if $\underline{\alpha} \in l^\infty(\mathbb{Z})$, i.e. $||\underline{\alpha}|| = M < \infty$

     A is invertible as long as $\inf_n \alpha_n \neq 0$

     In such a case $\forall \underline{y} \in l^2(\mathbb{Z})$, $(A^{-1}\underline{y})_n = y_n / \alpha_n$

Adjoint of a Linear Operator: definition

- Generalization of the conjugate transpose of a matrix $\underline{\underline{A}}$

- The linear operator $A^* : H_1 \rightarrow H_0$ is the adjoint of the linear operator $A : H_0 \rightarrow H_1$ when $\forall \underline{x} \in H_0$, $\forall \underline{y} \in H_1$

$$<A\underline{x},\underline{y}>_{H_1} = <\underline{x},A^*\underline{y}>_{H_0}$$

- When $A^* = A$, the operator A is said self-adjoint or Hermitian

- Example: $A : H \rightarrow H$, such that $\forall \underline{x} \in H \quad A\underline{x} = \alpha\underline{x}$

$$<A\underline{x},\underline{y}> = <\alpha\underline{x},\underline{y}> = \alpha.<\underline{x},\underline{y}> = ((\alpha.<\underline{x},\underline{y}>)^*)^* = (\alpha^*.<\underline{y},\underline{x}>)^* = (<\alpha^*\underline{y},\underline{x}>)^* = <\underline{x},\alpha^*\underline{y}>$$

$$\Rightarrow A^*\underline{x} = \alpha^*\underline{x}$$

Adjoint of a Linear Operator: examples

1. $\underline{\underline{A}} : \mathbb{C}^N \to \mathbb{C}^M$, such that $\forall \underline{x} \in \mathbb{C}^N \quad \underline{\underline{A}}\underline{x} = \underline{y} \in \mathbb{C}^M$

$$\left\langle \underline{\underline{A}}\underline{x}, \underline{y} \right\rangle_{C^M} = \underline{y}^* \left(\underline{\underline{A}}\underline{x}\right) = \left(\underline{y}^* \underline{\underline{A}}\right)\underline{x} = \left(\left(\left(\underline{y}^* \underline{\underline{A}}\right)^*\right)^*\right) \underline{x} = \left(\underline{\underline{A}}^* \underline{y}\right)^* \underline{x} = \left\langle \underline{x}, \underline{\underline{A}}^* \underline{y} \right\rangle_{C^N}$$

The adjoint of $\underline{\underline{A}}$ is simply its Hermitian transposed $\underline{\underline{A}}^*$

2. Multiplier operator on $l^2(\mathbb{Z})$

$$\left\langle A\underline{x}, \underline{y} \right\rangle_{l^2(Z)} = \sum_{n \in Z} \left(\alpha_n x_n\right) y_n^* = \sum_{n \in Z} x_n \left(\alpha_n^* y_n\right)^* = \left\langle \underline{x}, A^* \underline{y} \right\rangle_{l^2(Z)} \Rightarrow \left(A^* \underline{y}\right)_n = \alpha_n^* y_n$$

- Geometric interpretation

If A has some implication, A^* preserves the geometry of such effect while acting with reversed domain and codomain

Adjoint of a Linear Operator: examples

3. Local averaging and its adjoint

Consider A: $\mathcal{L}^2(\mathbb{R}) \to l^2(\mathbb{Z})$, such that $\forall x(t) \in \mathcal{L}^2(\mathbb{R})$

$$\left(A\underline{x}\right)_n = \int_{n-1/2}^{n+1/2} x(t).dt$$

- A is clearly a linear operator by linearity of the integral
- Ax is clearly a member of $l^2(\mathbb{Z})$

$$\left\|A\underline{x}\right\|^2_{l^2(Z)} = \sum_{n\in Z}\left|\left(A\underline{x}\right)_n\right|^2 = \sum_{n\in Z}\left|\int_{n-1/2}^{n+1/2} x(t).dt\right|^2 \le \sum_{n\in Z}\int_{n-1/2}^{n+1/2}\left|x(t)\right|^2.dt = \int_{-\infty}^{\infty}\left|x(t)\right|^2.dt = \left\|\underline{x}\right\|^2_{L^2(R)}$$

- Let us find the adjoint A^* of A, which means looking for A^* such that

  $\forall \underline{x} \in \mathcal{L}^2(\mathbb{R}),\ \underline{y} \in l^2(\mathbb{Z}),\ \langle A\underline{x},\underline{y}\rangle_{l^2} = \langle \underline{x}, A^*\underline{y}\rangle_{\mathcal{L}^2(\mathbb{R})}$

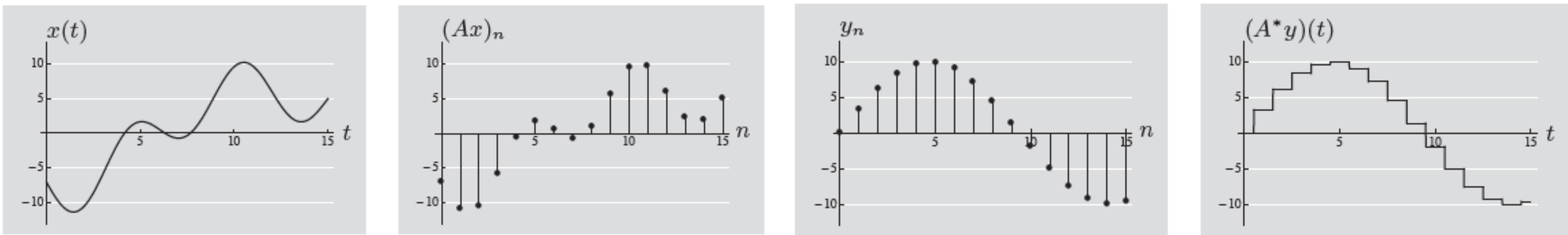
Adjoint of a Linear Operator: examples

3. Local averaging and its adjoint

$$\left\langle A\underline{x}, \underline{y} \right\rangle_{l^2(Z)} = \sum_{n \in Z} \left(A\underline{x}\right)_n y_n^* = \sum_{n \in Z} \left(\int_{n-1/2}^{n+1/2} x(t)\,dt \right) y_n^* = \sum_{n \in Z} \left(\int_{n-1/2}^{n+1/2} x(t) y_n^*\,dt \right)$$

$$\left\langle \underline{x}, A^* \underline{y} \right\rangle_{L^2(R)} = \int_{-\infty}^{\infty} x(t).\left((A^* \underline{y})(t)\right)^* dt = \sum_{n \in Z} \left(\int_{n-1/2}^{n+1/2} x(t) \underbrace{\left((A^* \underline{y})(t)\right)^*}_{\text{constant in [n-1/2,n+1/2]}} dt \right)$$

$$(A^* \underline{y})(t) = y_n \qquad t \in \left[n - \frac{1}{2}, n + \frac{1}{2} \right]$$



Adjoint of a Linear Operator: properties

- Theorem

Let $A: H_0 \rightarrow H_1$ be a bounded linear operator, then

1. Its adjoint A^* exists
2. Its adjoint A^* is unique
3. The adjoint of the adjoint $(A^*)^*$ is the original linear operator A: $(A^*)^* = A$
4. The operators (AA^*) and (A^*A) are self-adjoint
5. The operator norms of A and A^* are equal
6. If A is invertible, its adjoint is too and $(A^{-1})^* = (A^*)^{-1}$
7. Let $B: H_0 \rightarrow H_1$ be another bounded linear operator, then $(A+B)^* = A^* + B^*$
8. Let $C: H_1 \rightarrow H_2$ be another bounded linear operator, then $(CA)^* = A^*C^*$

Adjoint of a Linear Operator: properties

- Relationships between the range and null spaces of a linear operator and its adjoint
  - $\mathcal{R}(A)^{\perp} = \mathcal{N}(A^*)$ (1)
  - $\overline{\mathcal{R}(A)} = \mathcal{N}(A^*)^{\perp}$ (2) where $\overline{C}$ is the closure of C

Proof of (1)

$\mathcal{N}(A^*) \subseteq \mathcal{R}(A)^{\perp}$

Consider $\underline{y} \in \mathcal{N}(A^*)$, $\underline{y}' \in \mathcal{R}(A)$

$\underline{y}' = A\underline{x}$ for some $\underline{x}$

$<\underline{y}', \underline{y}> = <A\underline{x}, \underline{y}> = <\underline{x}, A^*\underline{y}> = <\underline{x}, \underline{0}> = 0$, thus $\underline{y} \in \mathcal{R}(A)^{\perp}$

Conversely, let us prove that $\mathcal{R}(A)^{\perp} \subseteq \mathcal{N}(A^*)$

Consider $\underline{y} \in \mathcal{R}(A)^{\perp}$, and $\forall \underline{x} \in H_0$, thus $<A\underline{x},\underline{y}> = 0 = (<\underline{x}, A^*\underline{y}>)$

By selecting $\underline{x} = A^*\underline{y}$, $A^*\underline{y} = \underline{0}$, and consequently $\underline{y} \in \mathcal{N}(A^*)$

Unitary Operators: definition

- A bounded linear operator A : $H_0 \rightarrow H_1$ is said unitary when
  i. it is invertible
  ii. it preserves inner products

$$\langle A\underline{x}, A\underline{y}\rangle_{H_1} = \langle \underline{x}, \underline{y}\rangle_{H_0} \quad \forall \underline{x}, \underline{y} \in H_0$$

- Note: preservation of inner products implies preservation of the norm, i.e.

$$\|A\underline{x}\|^2 = \langle A\underline{x}, A\underline{x}\rangle_{H_1} = \langle \underline{x}, \underline{x}\rangle_{H_0} = \|\underline{x}\|^2 \quad \forall \underline{x} \in H_0$$

- Theorem: A bounded linear operator A : $H_0 \rightarrow H_1$ is unitary iff

$$A^{-1} = A^*$$

  – Proof of A^* being the left inverse of A when A is a unitary operator

$$\langle A^* A\underline{x}, \underline{y}\rangle_{H_0} = \langle A\underline{x}, A\underline{y}\rangle_{H_1} = \langle \underline{x}, \underline{y}\rangle_{H_0} \quad \forall \underline{x}, \underline{y} \in H_0$$

Unitary Operators: Theorem

– Proof of A* being the left inverse of A implies that A is a unitary operator

$$\left\langle A\underline{x}, A\underline{y}\right\rangle_{H_1} = \left\langle \underline{x}, A^*A\underline{y}\right\rangle_{H_0} = \left\langle \underline{x}, \underline{y}\right\rangle_{H_0} \quad \forall \underline{x}, \underline{y} \in H_0$$

– Proof of A* being the right inverse of A from the fact that A is unitary

$$\left\langle AA^*\underline{x}, \underline{y}\right\rangle_{H_1} = \left\langle AA^*\underline{x}, AA^{-1}\underline{y}\right\rangle_{H_1} \underset{\text{A is unitary}}{=} \left\langle A^*\underline{x}, A^{-1}\underline{y}\right\rangle_{H_0} = \left\langle \underline{x}, AA^{-1}\underline{y}\right\rangle_{H_1} = \left\langle \underline{x}, \underline{y}\right\rangle_{H_1} \quad \forall \underline{x}, \underline{y} \in H_1$$

$$\Rightarrow AA^* = I_{H_1}$$

Eigenvector of a linear operator: Definition

- Generalization of the eigenvector of a matrix
- It applies as long as domain and codomain represent the same Hilbert space
  - If the signal is discrete-time, the eigenvector is called an eigensequence
  - If the signal is continuous time, the eigenvector is called an eigenfunction
- An eigenvector of a linear operator A: H → H is a non zero vector $\underline{v}$ such that $A\underline{v}=\lambda\underline{v}$ for some $\lambda \in \mathbb{C}$
  - λ is called an eigenvalue of A, whereas $(\lambda,\underline{v})$ is called an eigenpair of A
- Properties
  1. All eigenvalues of a self-adjoint operator A are real

$$\lambda\langle \underline{v},\underline{v}\rangle = \langle \lambda\underline{v},\underline{v}\rangle = \langle A\underline{v},\underline{v}\rangle = \langle \underline{v},A\underline{v}\rangle = \langle \underline{v},\lambda\underline{v}\rangle = \lambda^*\langle \underline{v},\underline{v}\rangle \Rightarrow \lambda \in R$$

Eigenvector of a linear operator: Properties

- Properties

  2. All eigenvectors corresponding to distinct eigenvalues are orthogonal

Proof: consider 2 eigenpairs $(\lambda_0, \underline{v}_0)$ and $(\lambda_1, \underline{v}_1)$ with $\lambda_0 \neq \lambda_1$

$$\lambda_0 \langle \underline{v}_0, \underline{v}_1 \rangle = \langle \lambda_0 \underline{v}_0, \underline{v}_1 \rangle = \langle A\underline{v}_0, \underline{v}_1 \rangle = \langle \underline{v}_0, A\underline{v}_1 \rangle = \langle \underline{v}_0, \lambda_1 \underline{v}_1 \rangle = \lambda_1^* \langle \underline{v}_0, \underline{v}_1 \rangle$$

$$\Rightarrow \langle \underline{v}_0, \underline{v}_1 \rangle = 0$$

since λ_i are real

Positive definiteness: Definition

- A self-adjoint operator A : H → H is called
  1. Positive semidefinite or nonnegative definite, written A≥0, when
     $$\langle A\underline{x},\underline{x}\rangle \geq 0 \quad \forall \underline{x} \in H$$
  2. Positive definite, written A>0, when
     $$\langle A\underline{x},\underline{x}\rangle > 0 \quad \forall \underline{x} \in H$$
  3. Negative semidefinite or nonpositive definite, when –A is positive semidefinite
  4. Negative definite, when –A is positive definite
- Positive definiteness defines a partial order on self-adjoint operators defined on the same Hilbert space
  - Given 2 self adjoint linear operators A : H → H and B : H → H

    A ≥ B means A-B ≥ 0, i.e. A-B is positive semi-definite

Positive definiteness: Definition

- Properties
  1. All eigenvalues of a positive definite operator are positive
  2. All eigenvalues of a semi-definite operator are non-negative