Wavelets, Filter Banks and Multiresolution Signal Processing

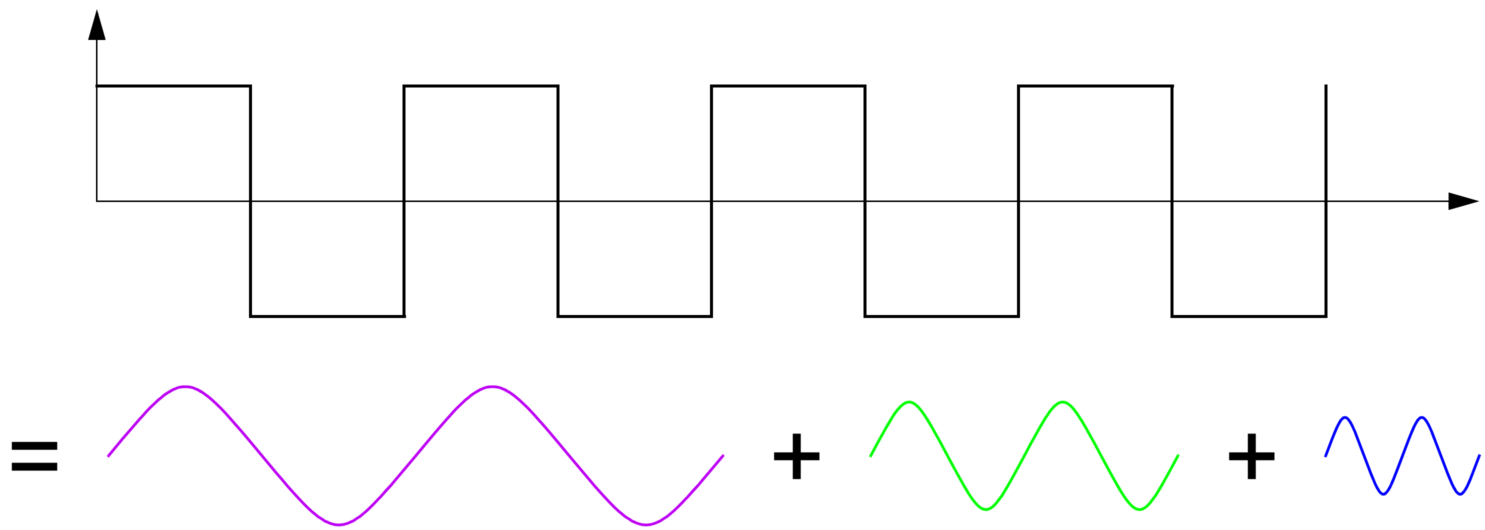
"It is with logic that one proves;
it is with intuition that one invents."

Henri Poincaré

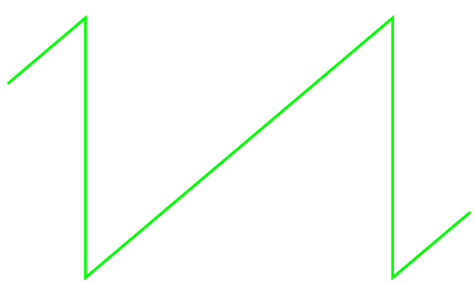
A bit of history: from Fourier to Haar to wavelets

Old topic: representations of functions

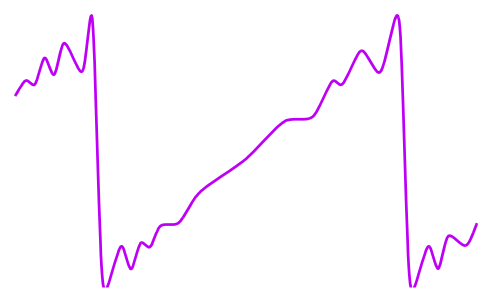
1807: Joseph Fourier upsets the French Academy



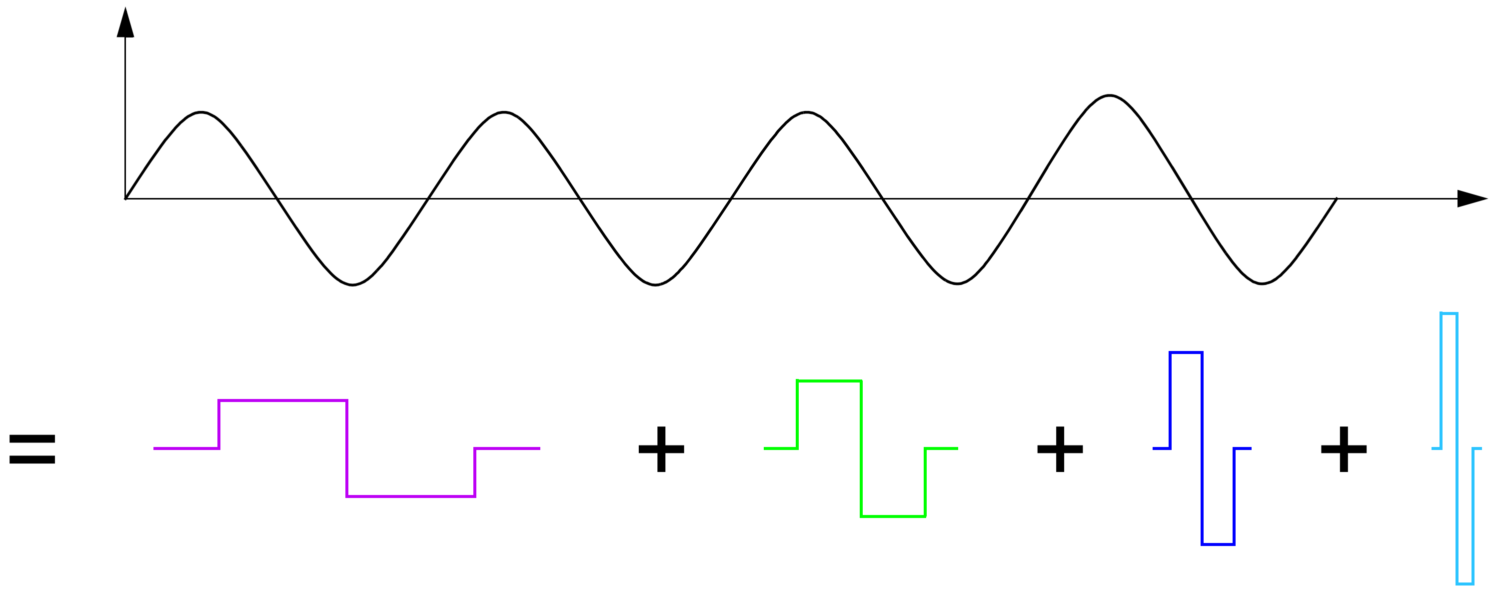
1898: Gibbs' paper



1899: Gibbs' correction



1910: Alfred Haar discovers the Haar wavelet dual to the Fourier construction



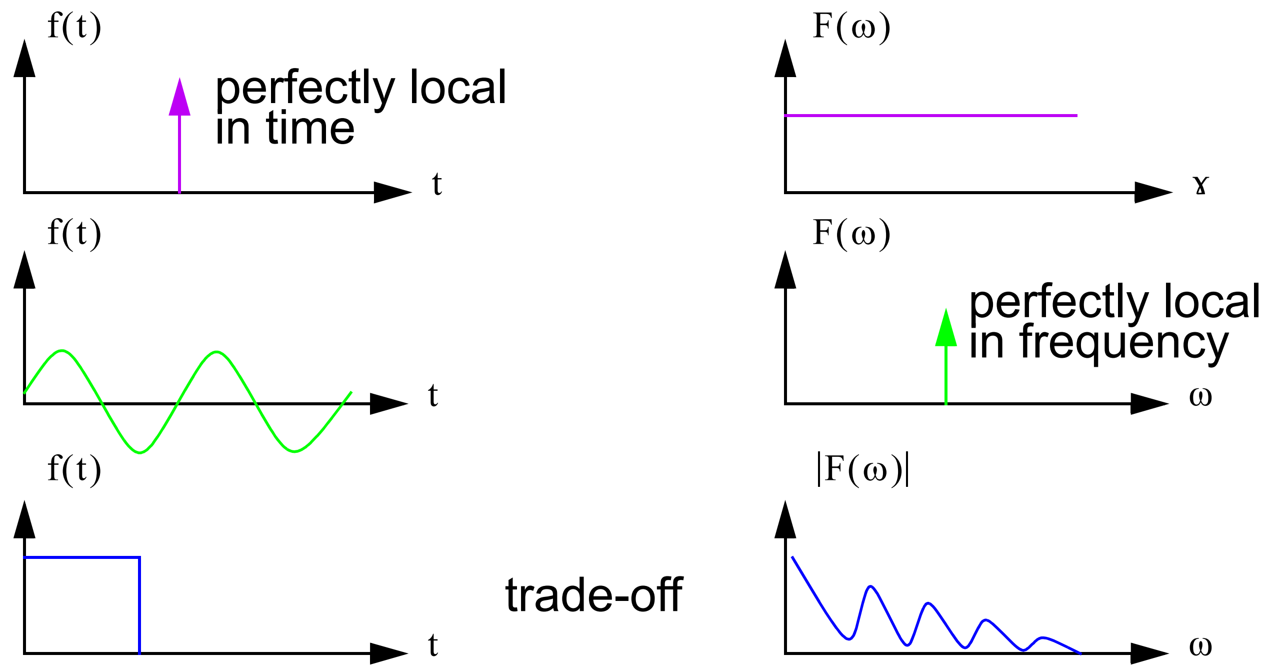
Why do this? What makes it work?

- basic atoms form an orthonormal set

Note

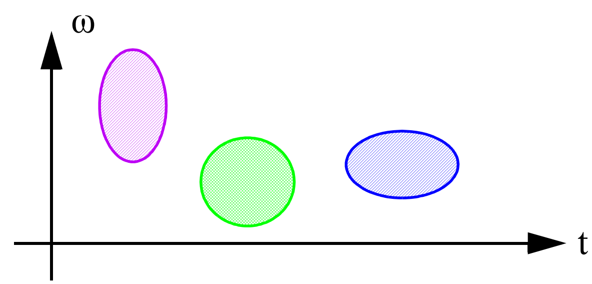
- sines/cosines and Haar functions are ON bases for $L_2(\Re)$
- both are structured orthonormal bases
- they have different time and frequency behavior

1930: Heisenberg discovers that
you cannot have your cake and eat it too!



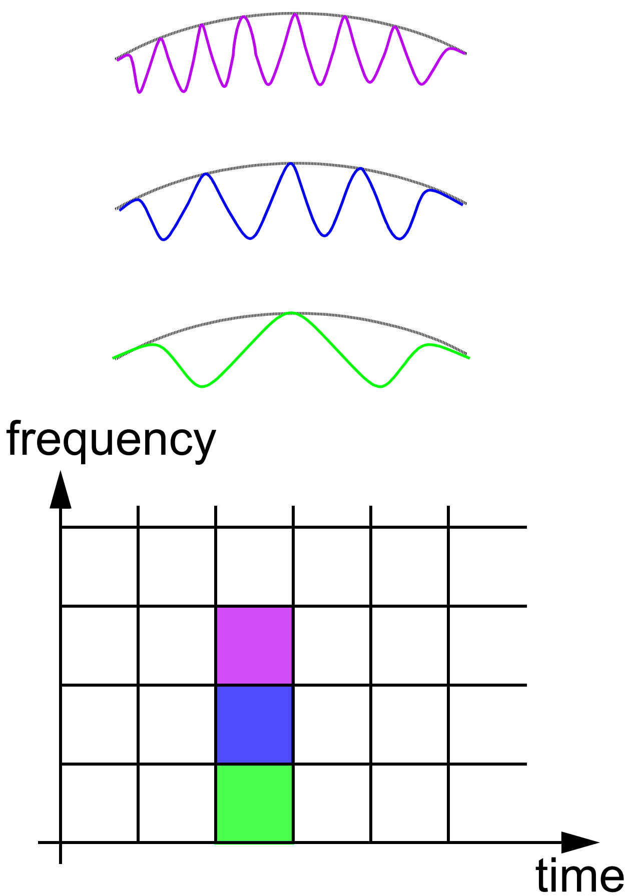
Uncertainty principle

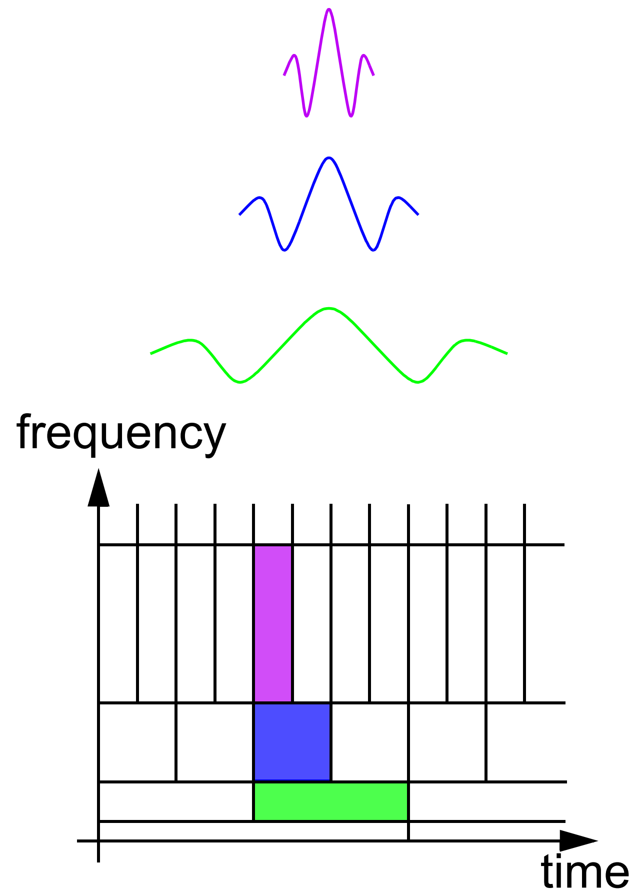
- lower bound on TF product



1945: Gabor localizes the Fourier transform $\Rightarrow$ STFT

1980: Morlet proposes the continuous wavelet transform

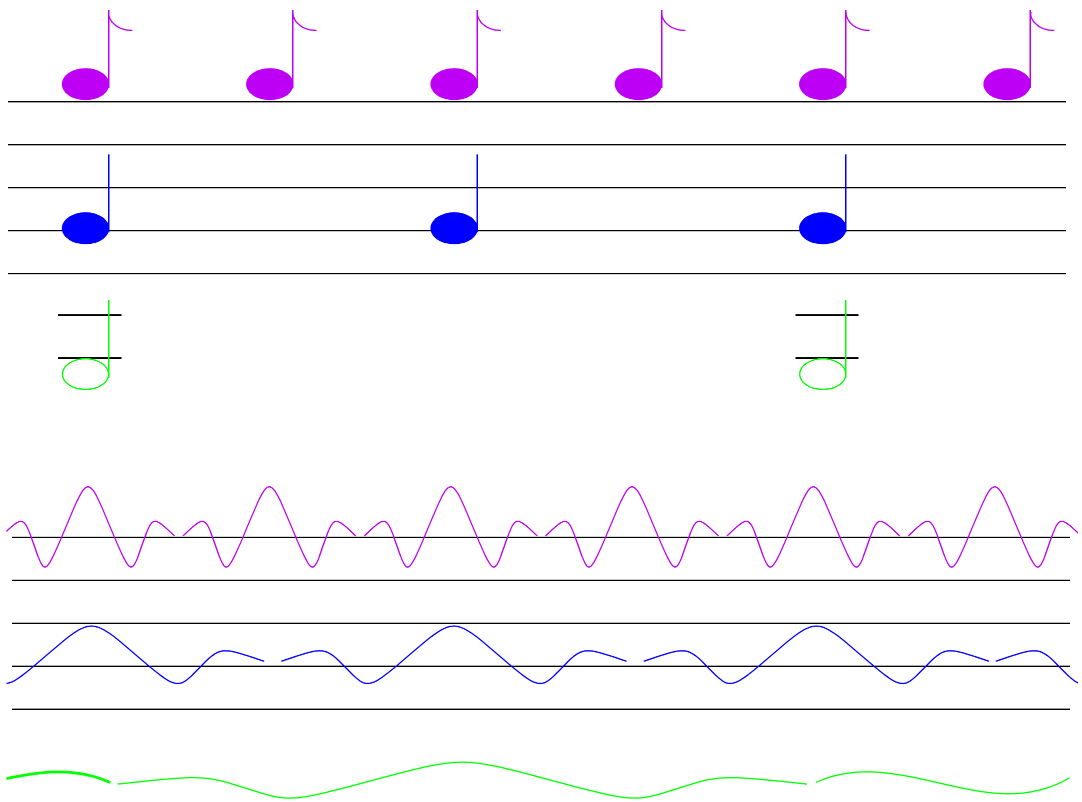




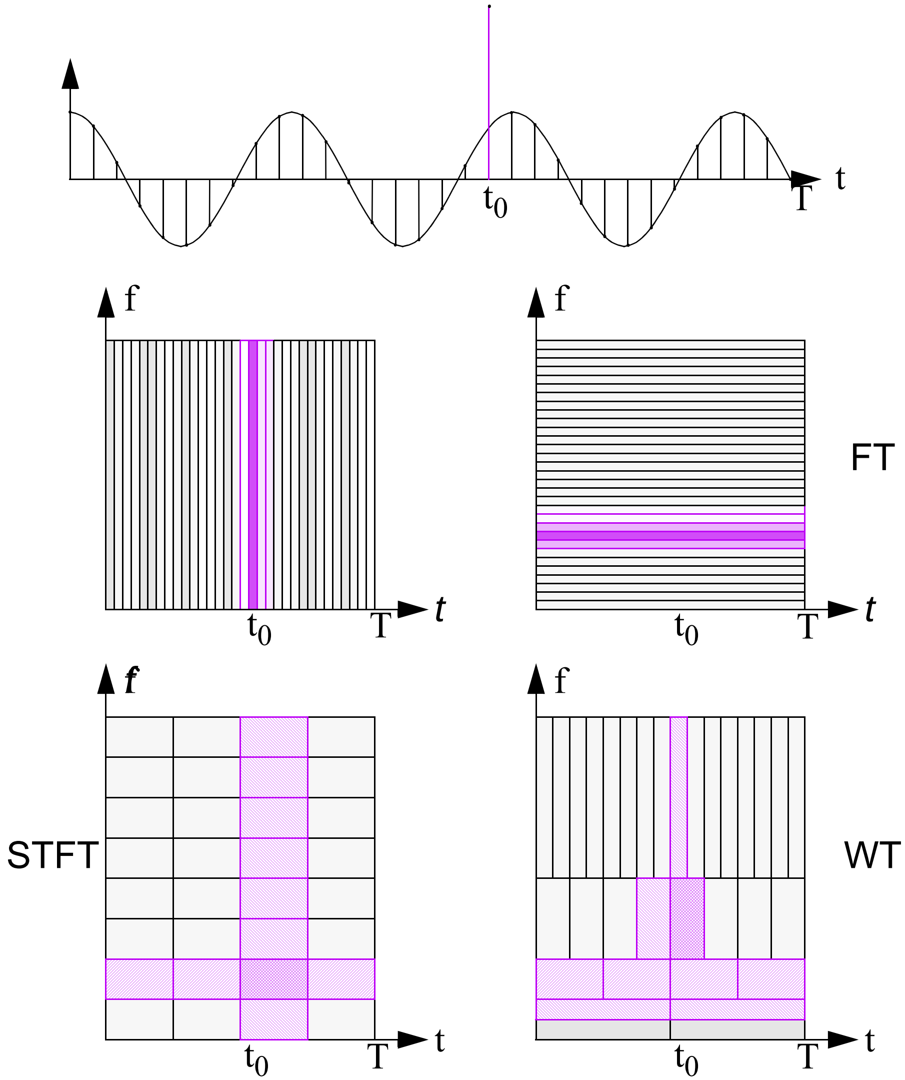
short-time Fourier transform wavelet transform

Analogy with the musical score

Bach knew about wavelets!



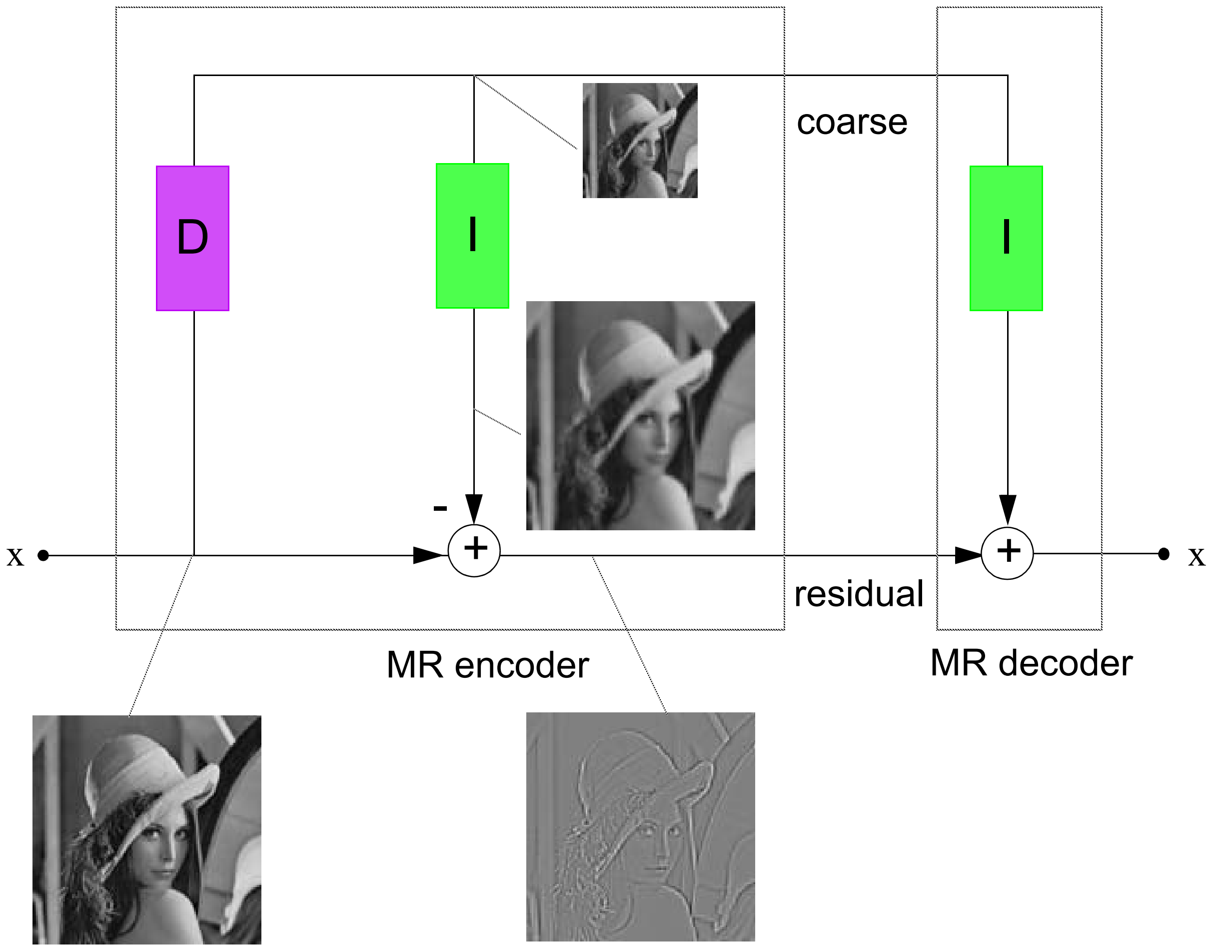
Time-frequency tiling for a sine + Delta



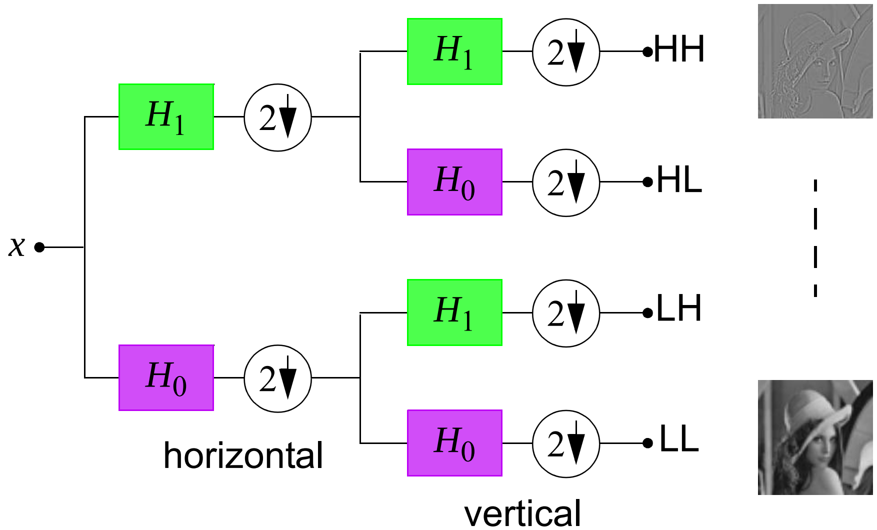
so....

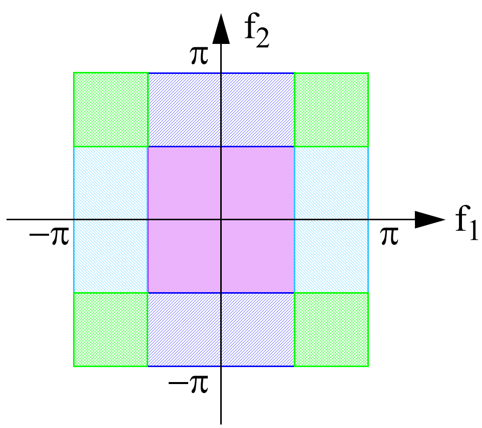
what is a good basis?

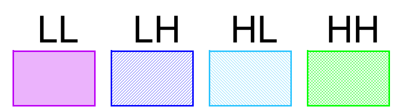
1983: Lena discovers pyramids (actually, Burt and Adelson)



1984: Lena gets critical (subband coding)





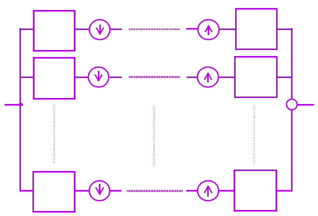


1986: Lena gets formal...
(multiresolution theory by Mallat, Meyer...)

Wavelets, filter banks and multiresolution analysis

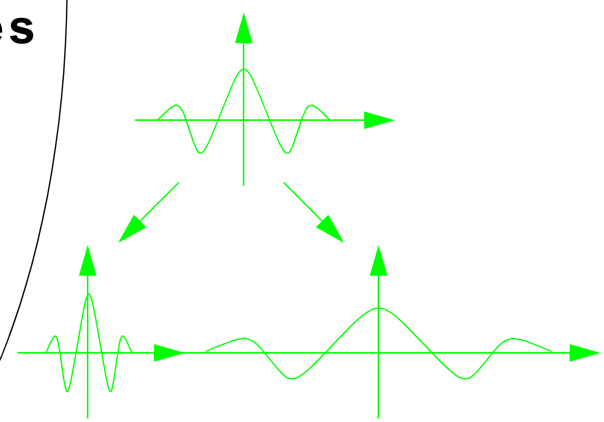
Wavelets
(applied mathematics)

Filter banks
(DSP)

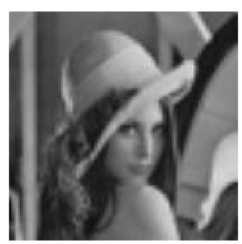


Construction of bases for signal expansions

$$x = \sum \langle \psi_i, x \rangle \psi_i$$



Multiresolution signal analysis
(computer vision)

 =  + 

Wavelets...

''All this time, the guard was looking at her,
first through a telescope,
then through a microscope,
and then through an opera glass.''

Lewis Carroll, *Through the Looking Glass*

... what are they and how to build them?

Orthonormal bases of wavelets

- Haar's construction of a basis for $L_2(\Re)$ (1910)
- Meyer, Battle-Lemarié, Stromberg (1980's)
- Mallat and Meyer's multiresolution analysis (1986)

Wavelets from iterated filter banks

- Daubechies' construction of compactly supported wavelets
- smooth wavelet bases for $L_2(\Re)$ and computational algorithms

Relation to other constructions

- successive refinements in graphics and interpolation
- multiresolution in computer vision
- multigrid methods in numerical analysis
- subband coding in speech and image processing

Goal: find $\psi(t)$ such that its scales and shifts form an orthonormal basis for $L_2(\Re)$.

Why expand signals?

Suppose

 =  + 

original = coarse + detail

signal = block 1 + block 2

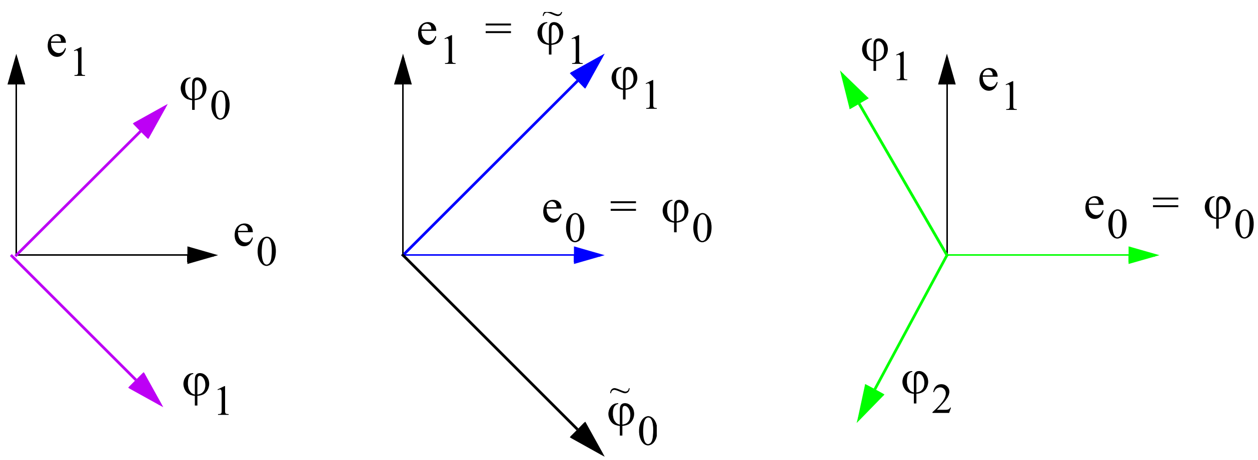
signal = $\sum$ projection x elementary signals

Advantages

- easier to analyze signal in pieces: “divide and conquer”
- extracts important features
- pieces can be treated in an independent manner

Example: Example: $\Re^2$

- orthogonal basis
- biorthogonal basis
- tight frame



Note

- orthonormal basis has successive approximation property, biorthogonal basis and frames do not
- quantization in orthogonal case is easy, unlike in the other cases

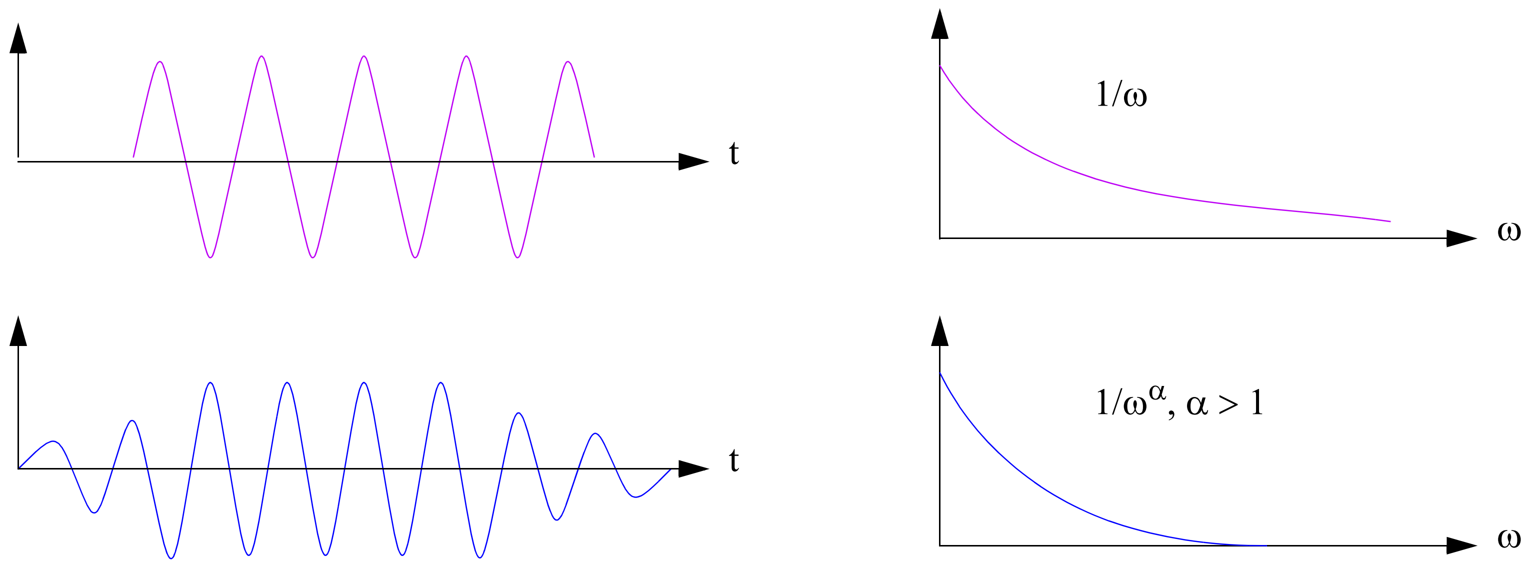
Why not use Fourier?

Block Fourier transform: bad frequency localization

Gabor transform: ill-behaved for critical sampling

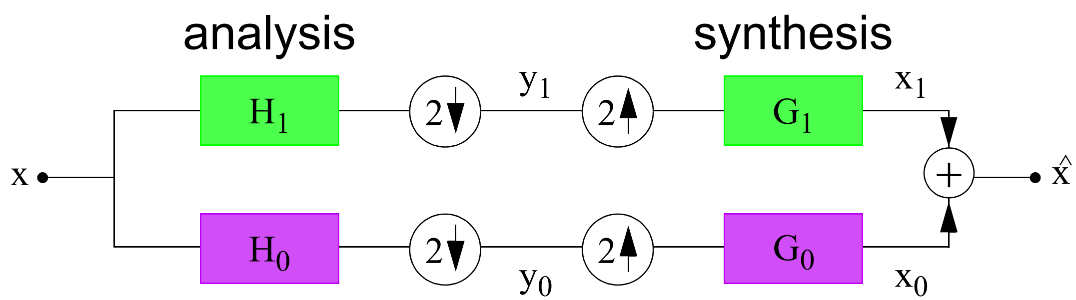
Balian-Low theorem: there is no local Fourier basis with good time and frequency localization

- however: good local cosine bases!

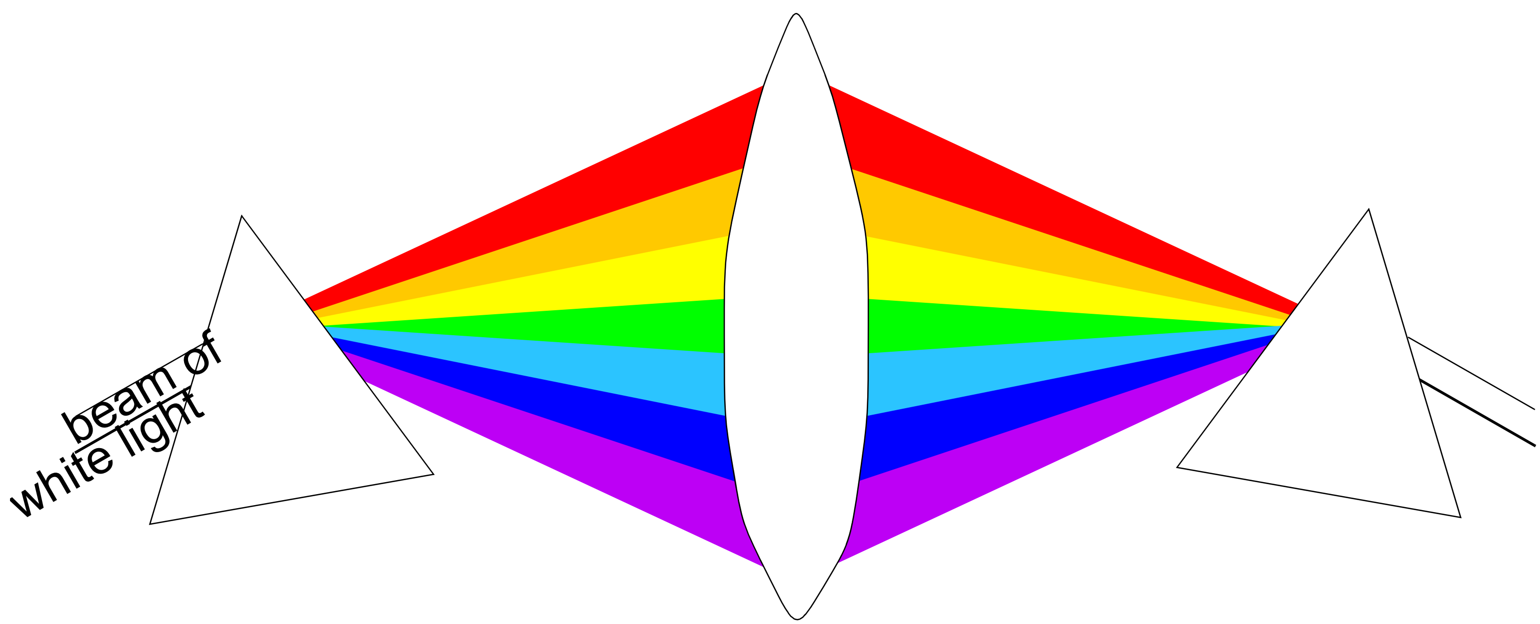


- shift and modulation

How do filter banks expand signals?

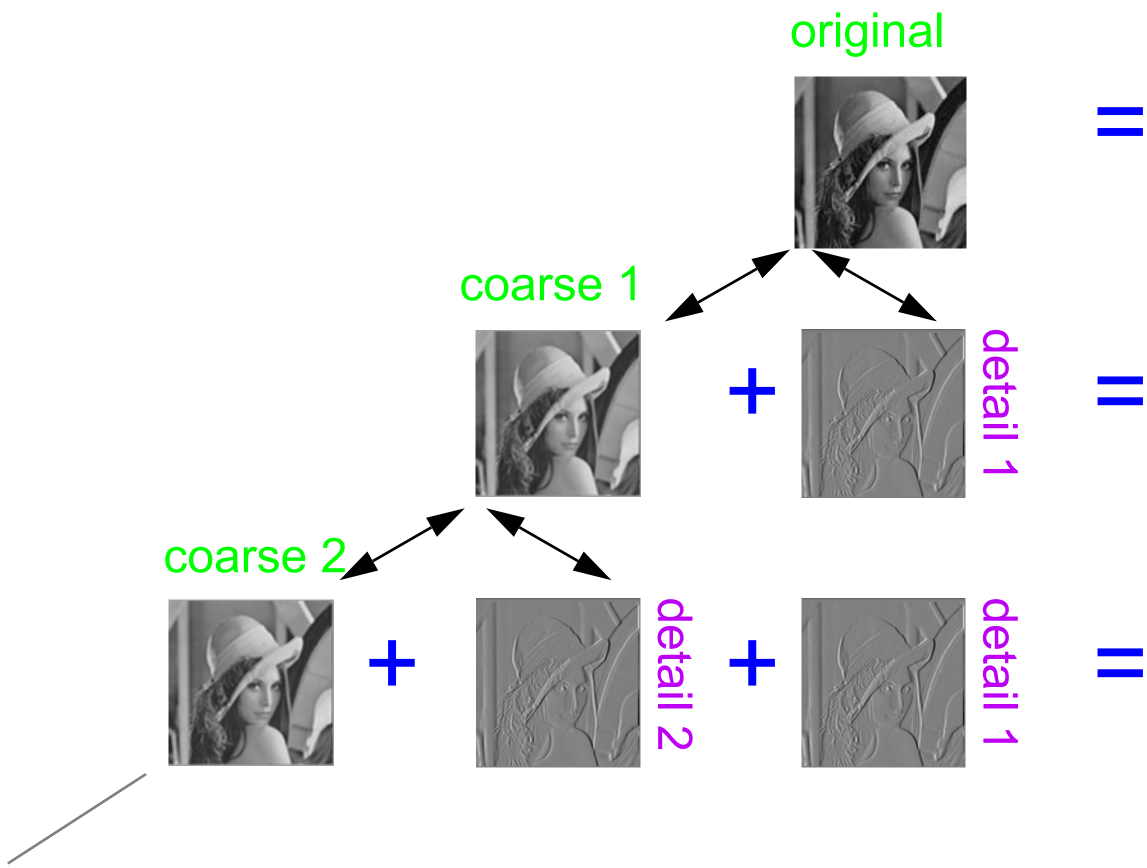


Analogy



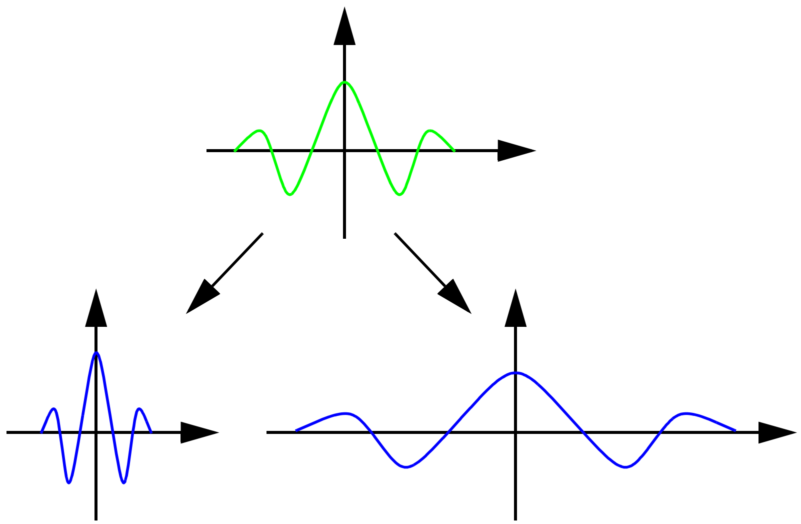
...and multiresolution analysis?

IDEA: successive approximation/refinement of the signal



... how about wavelets?

“mother” wavelet ψ



Who?

- families of functions obtained from “mother” wavelet by dilation and translation

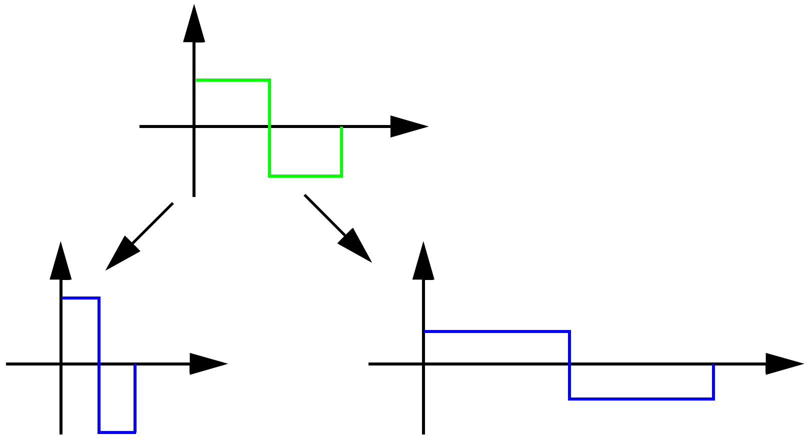
Why?

- well localized in time and frequency
- it has the ability to “zoom”

Haar system

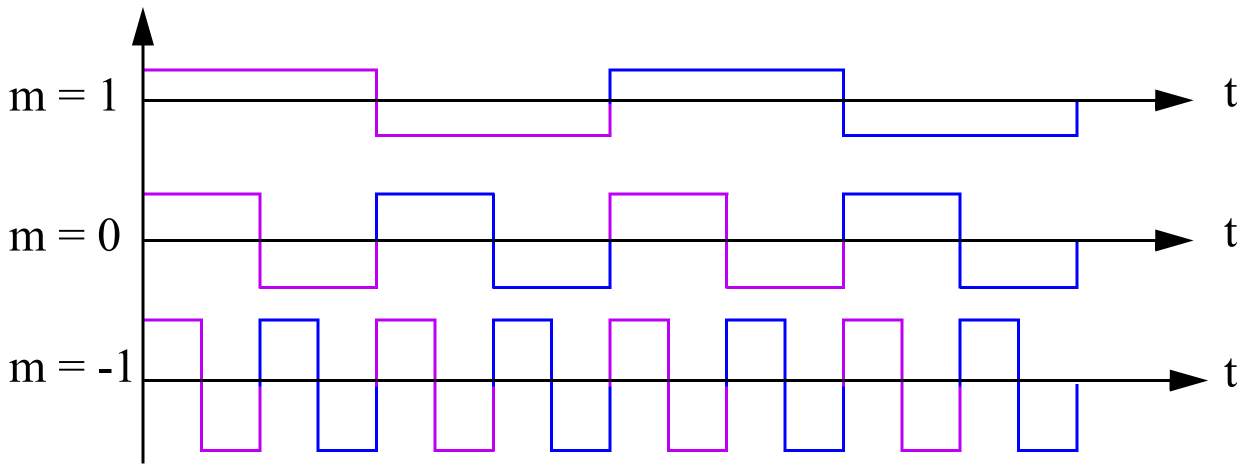
Basis functions

$$\psi(t) = \begin{cases} 1 & 0 \le t < 0.5 \\ -1 & 0.5 \le t < 1 \\ 0 & \text{else} \end{cases}$$

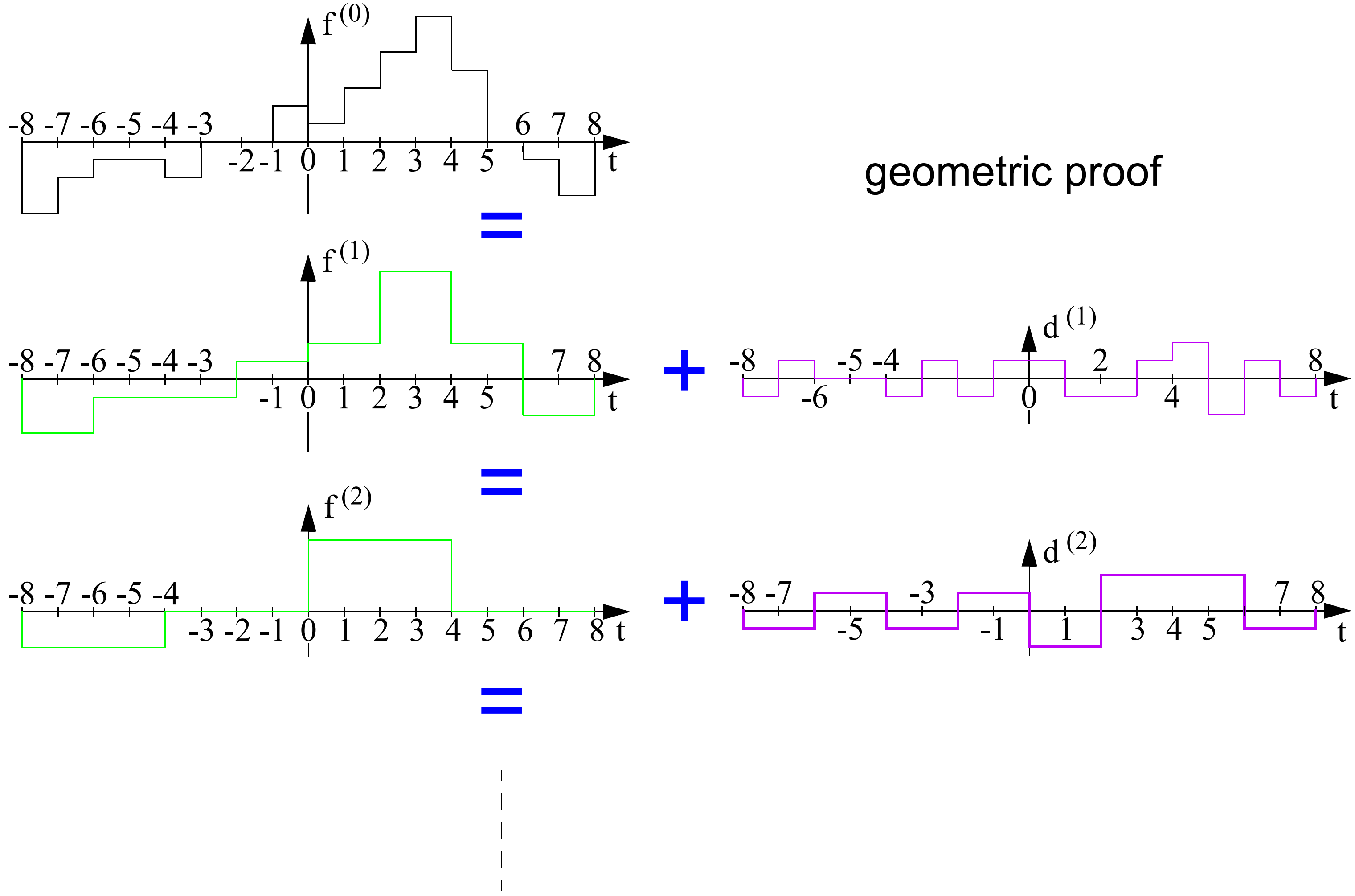


$$\psi_{m,n}(t) = 2^{-m/2}\psi(2^{-m}t - n)$$

Basis functions across scales



Haar system...
... as a basis for $L_2(\Re)$



geometric proof

Haar system...
... scaling function and wavelet

The Haar scaling function (indicator of unit interval)

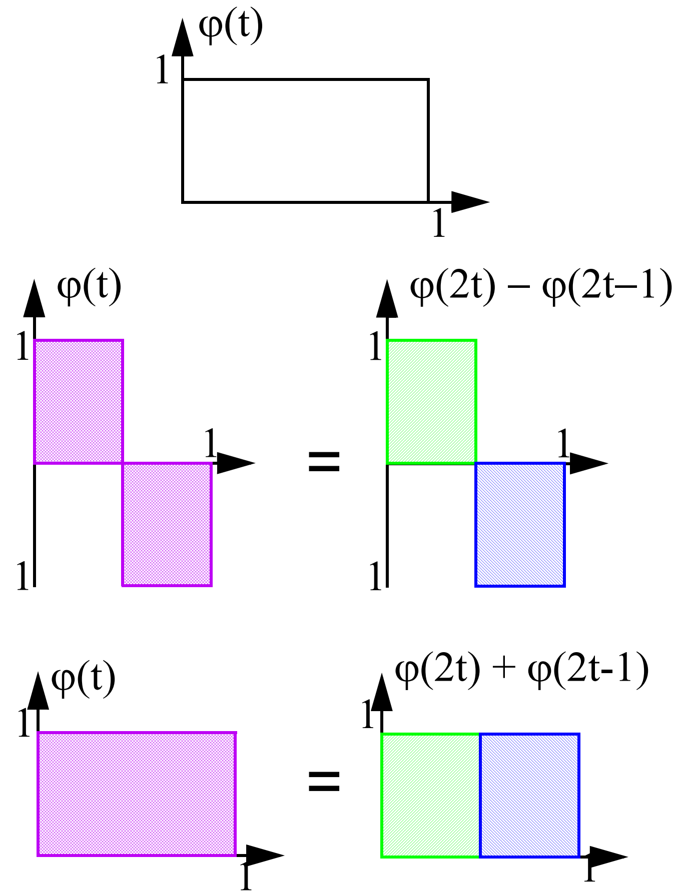
$$\varphi(t) = \begin{cases} 1 & 0 \le t < 1 \\ 0 & \text{else} \end{cases}$$

helps in the construction of the wavelet, since

$$\psi(t) = \varphi(2t) - \varphi(2t-1)$$

and satisfies a two-scale equation

$$\varphi(t) = \varphi(2t) + \varphi(2t-1)$$

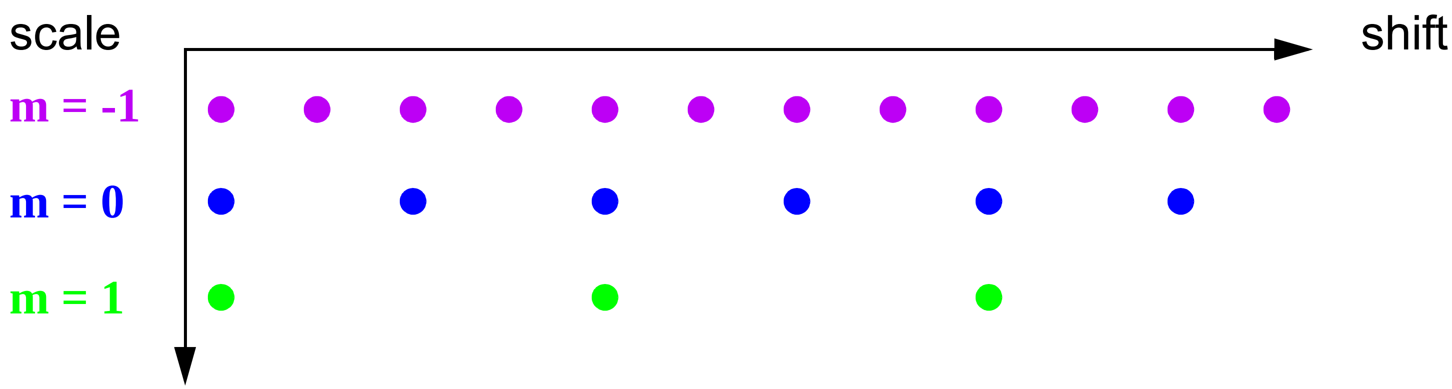


Note:

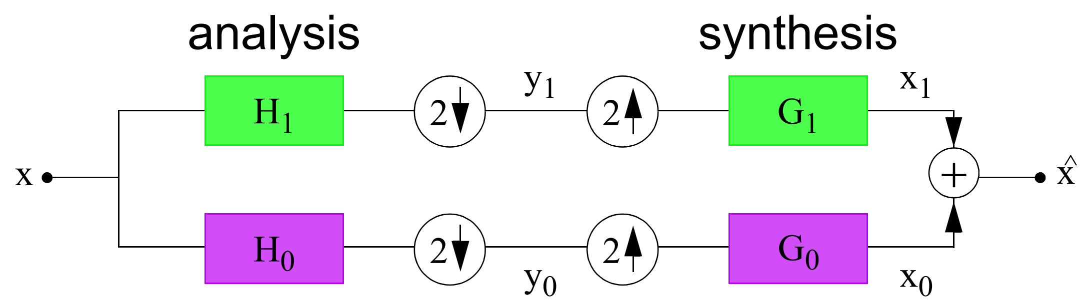
- Haar wavelet a bit too trivial to be useful...

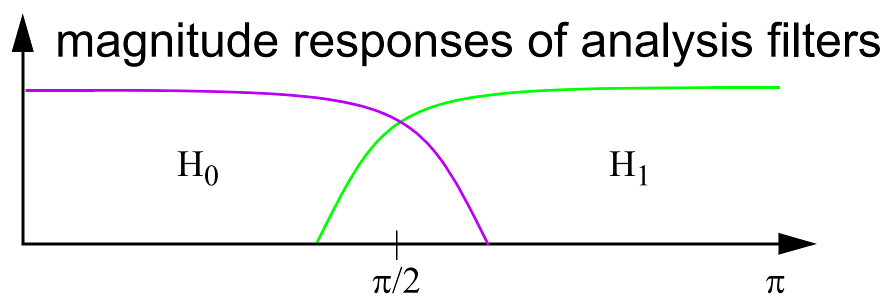
Discrete version of the wavelet transform

Compute WT on a discrete grid



Perfect reconstruction filter banks



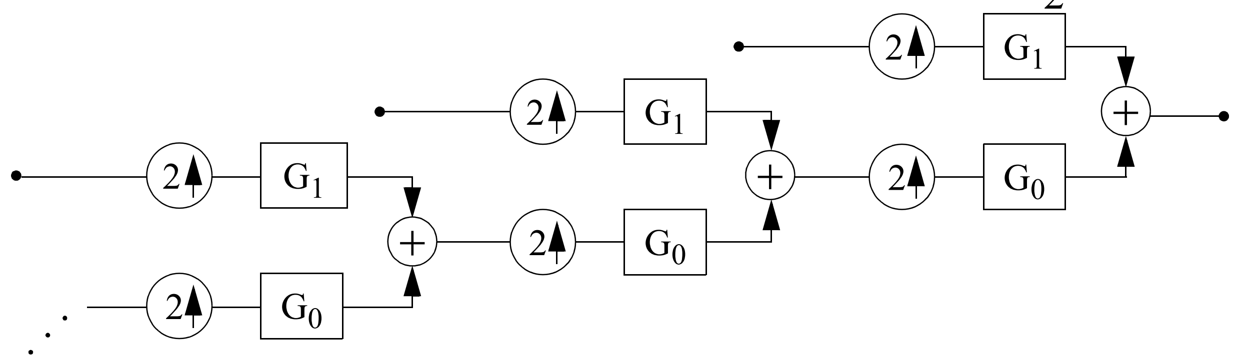


Perfect reconstruction: $G_0H_0 + G_1H_1 = I$

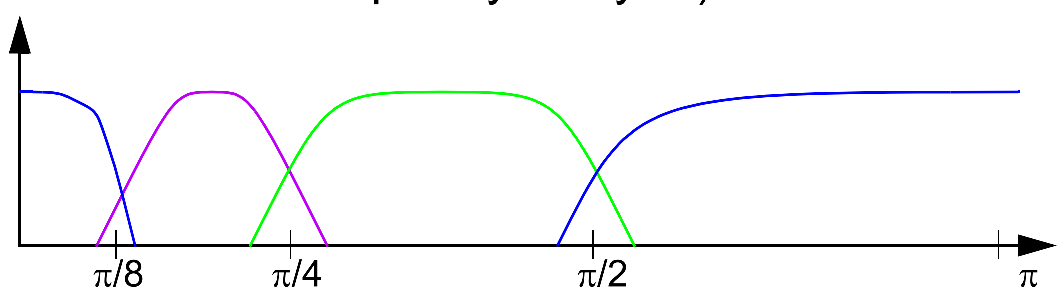
Orthogonal system: $(H_0)^*H_0 + (H_1)^*H_1 = I \quad G_0 = (H_1)^*$

Daubechies' construction...
... iterated filter banks

Iteration will generate an orthonormal basis for the space of square-summable sequences $l_2(\mathfrak{I})$



Consider equivalent basis sequences $G_0^{(i)}(z)$ and $G_1^{(i)}(z)$
(generates octave-band frequency analysis)

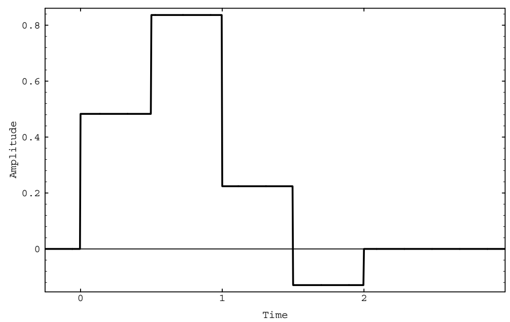


Interesting question: what happens in the limit?

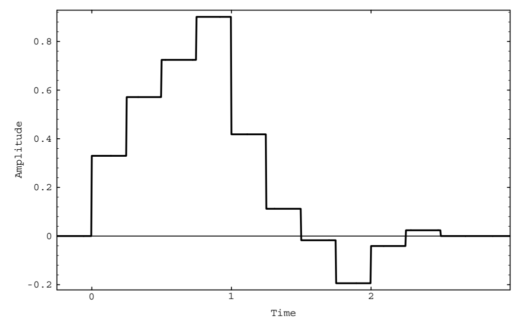
Daubechies' construction...
... iteration algorithm

At ith step associate piecewise constant approximation of length $1/2^i$ **with** $g_0^{(i)}[n]$

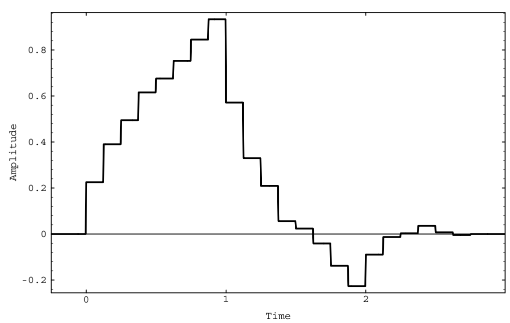
i = 1



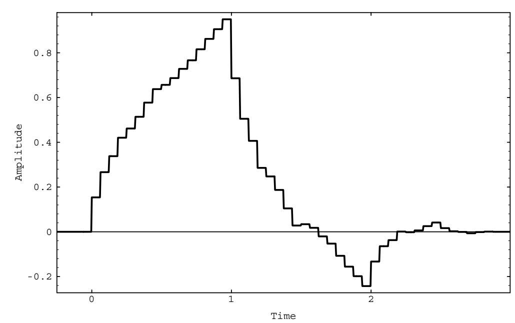
i = 2



i = 3



i = 4

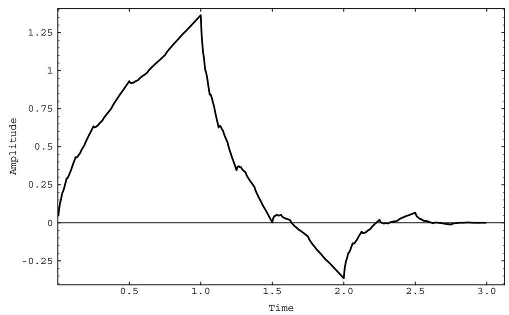


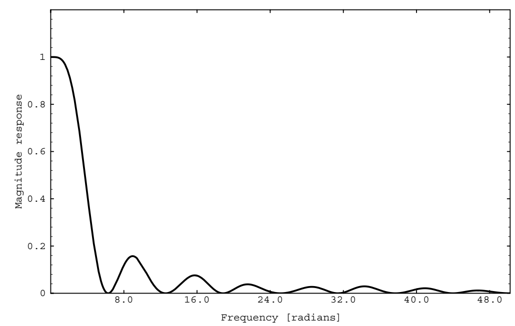
Fundamental link between discrete and continuous time!

Daubechies’ construction...
...scaling function and wavelet

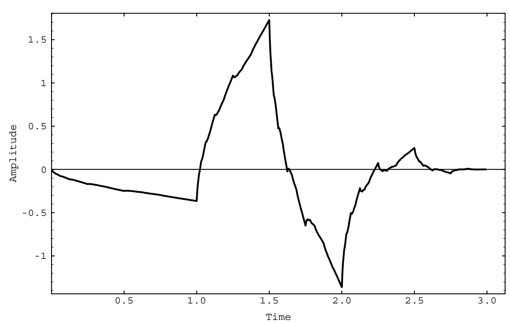
- Haar and sinc systems: either good time OR frequency localization
- Daubechies system: good time AND frequency localization

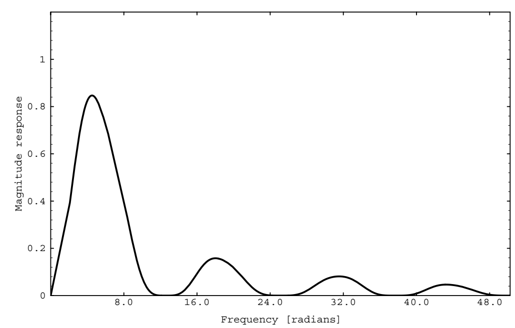
scaling function





wavelet





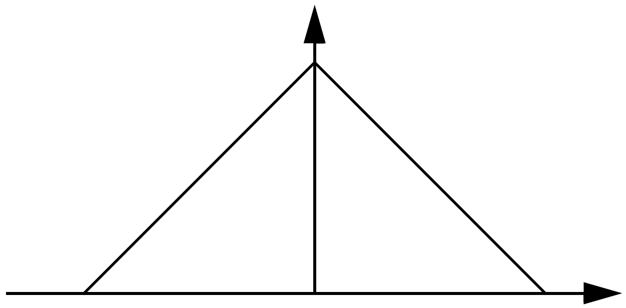
Finite length, continuous $\varphi(t)$ and $\psi(t)$, based on L=4 iterated filter

Many other constructions: biorthogonal, IIR, multidimensional...

Daubechies' construction...
... two-scale equation

$$\varphi(t) = \sum_{n} c_n \varphi(2t - n)$$

Hat function



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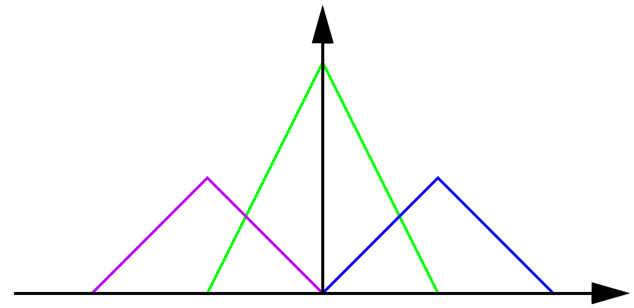
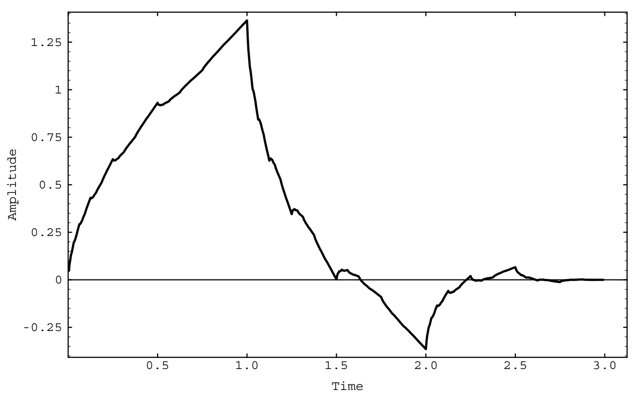


Daubechies' scaling function



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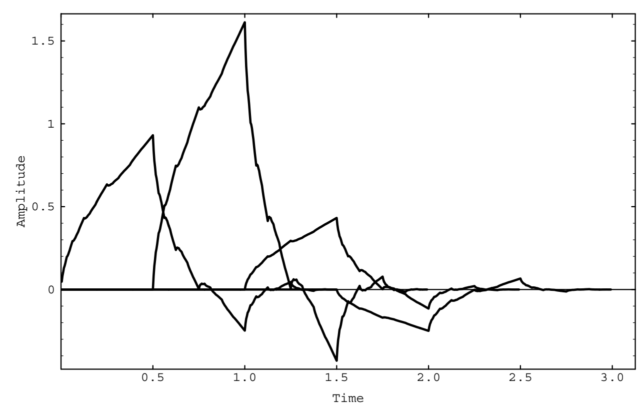
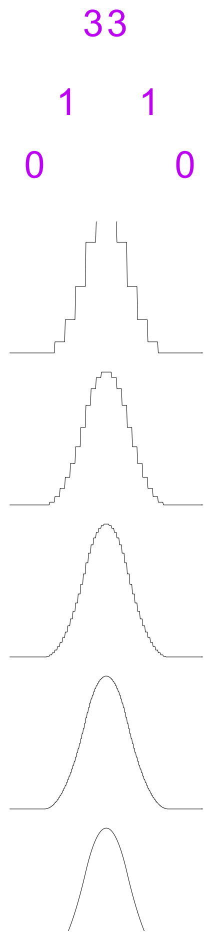
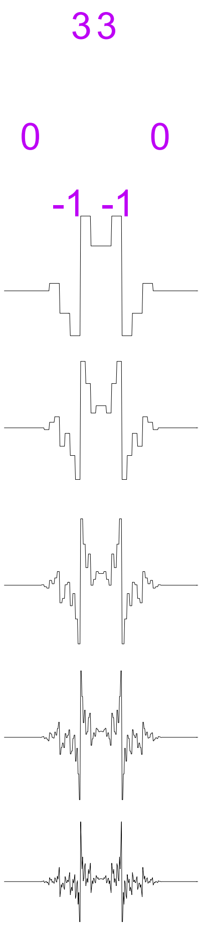


Not every discrete scheme leads to wavelets



quadratic spline

biorthogonal
filter banks



How do we know which ones will?... wait and see...

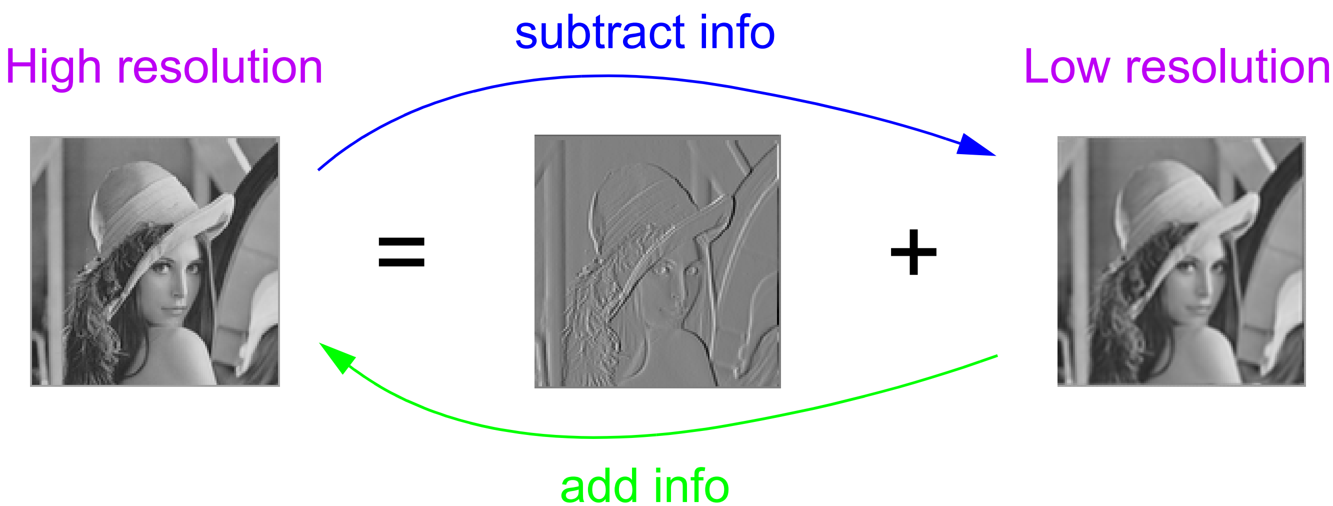
Applications

"That which shrinks must first expand."

Lao-Tzu, Tao Te Ching

Compression
Communications
Denoising
Graphics
In-painting

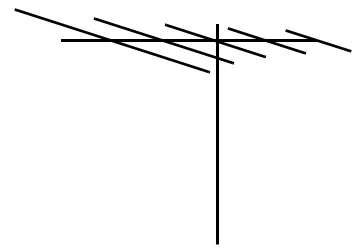
What is multiresolution?

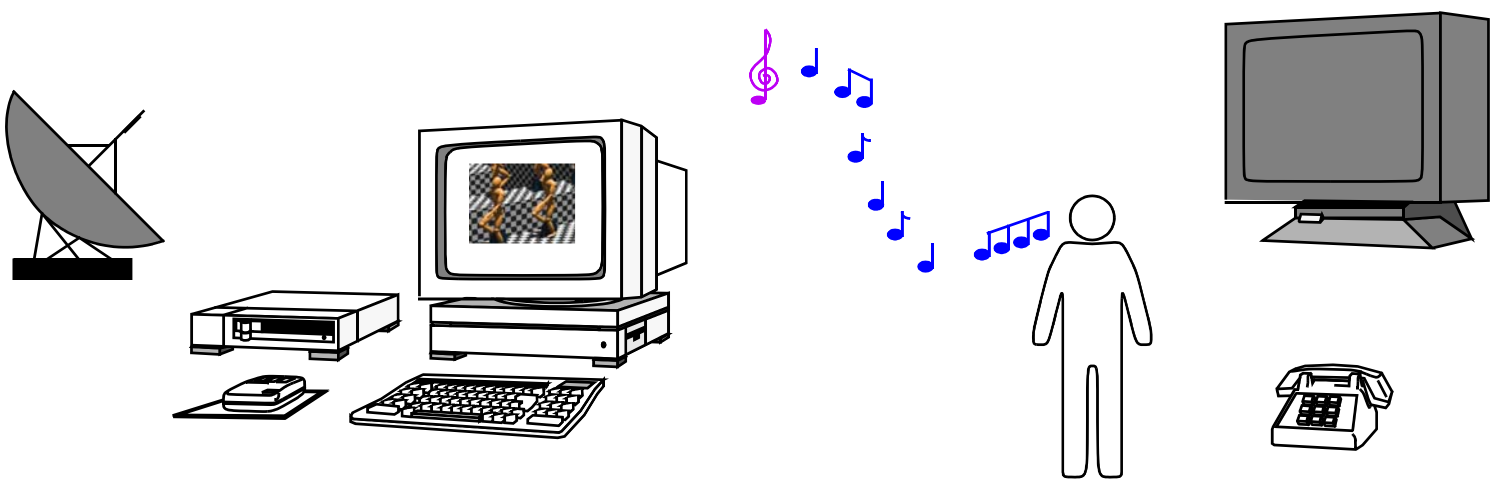


... and why use multiresolution?

A number of applications require signals to be processed and transmitted at multiple resolutions and multiple rates

- digital audio and video coding
- conversions between TV standards
- digital HDTV and audio broadcast
- remote image databases with searching
- storage media with random access
- MR coding for multicast over the Internet
- MR graphics





Compression: still a key technique in communications

Multiresolution compression...
... the DCT versus wavelet game

Question

given Lena (you have never seen before), what is the “best’’ transform to code it?

Fourier versus wavelet bases

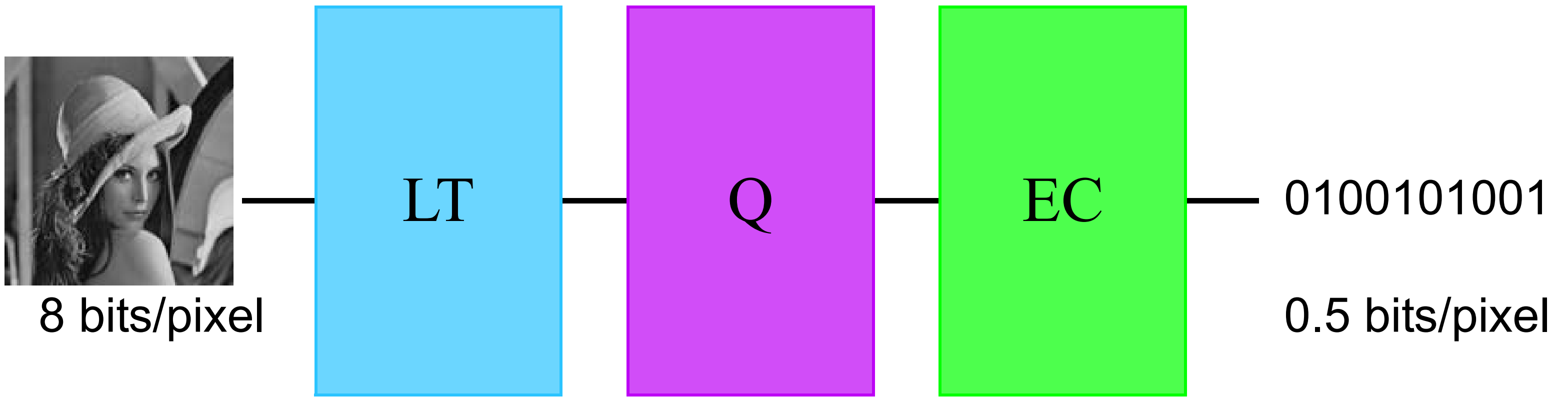
- linear versus octave-band frequency scale
- DCT versus subband coding
- JPEG versus multiresolution

Multiresolution source coding

- successive approximation
- browsing
- progressive transmission

Compression systems based on linear transforms

Goal: remove built-in redundancy, send only necessary info



- LT: linear transform (KLT, WT, SBC, DCT, STFT)
- Q: quantization
- EC: entropy coding

Gibbs phenomenon

“Blocking” effect in image compression

Wavelets

- smooth transitions
- multiscale properties
- multiresolution

A rate-distortion primer...

Compression: rate-distortion is fundamental trade-off

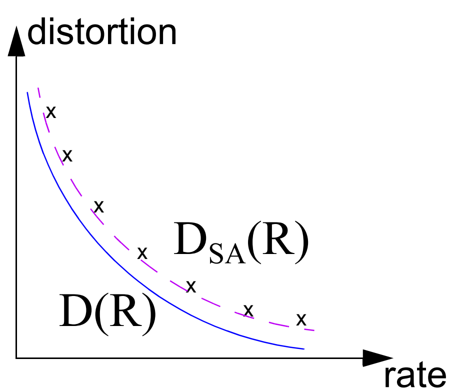
- more bitrate $\Rightarrow$ less distortion
- less bitrate $\Rightarrow$ more distortion

Standard image coder

- operates at one particular point on $D(R)$ curve

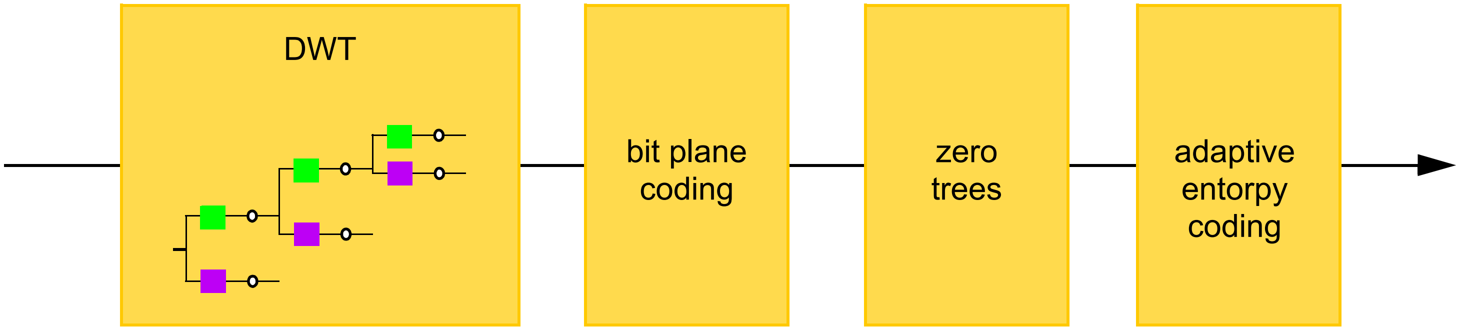
Multiresolution coder (layered, scalable)

- travels rate-distortion curve (successive approximation)
- computation scalability

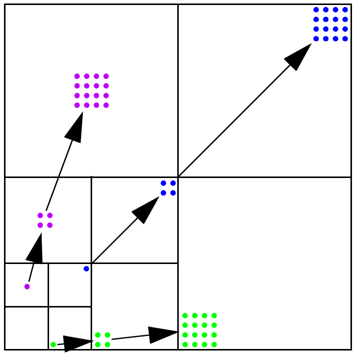


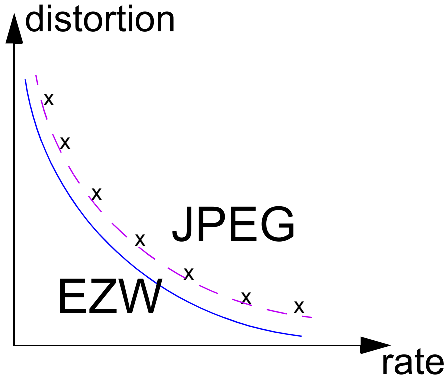
Best image coder?
... wavelet based!

Shapiro's embedded zero-tree algorithm (EZW)



- standard wavelet decomposition (biorthogonal)
- bit plane coding and zero-tree structure
- beats JPEG while achieving successive approximation





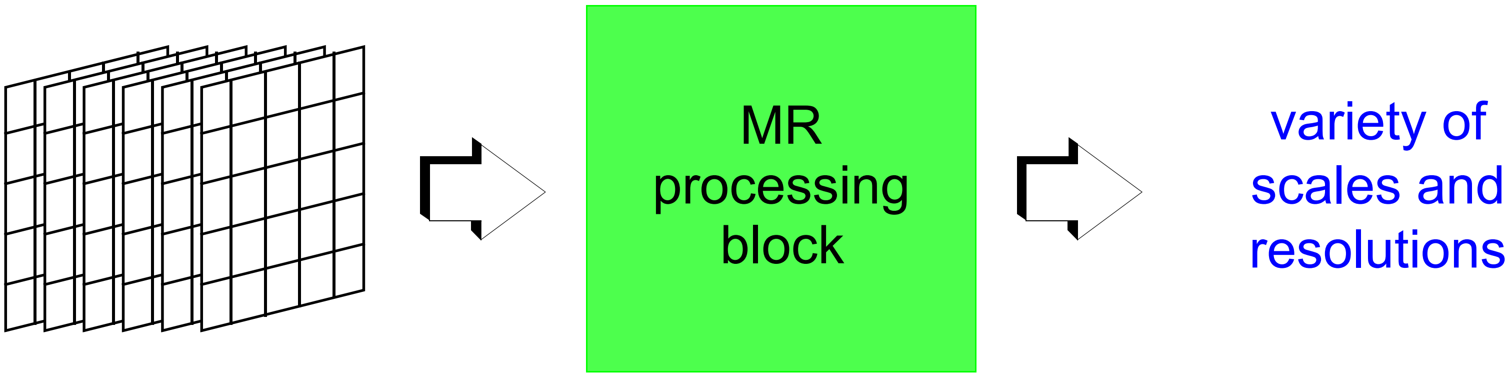
Next image coding standard... JPEG 2000

All the best coders based on wavelets

- 24 full proposals and a few partial ones
- 18 used wavelets, 4 used DCT and 5 used others
- top 75% are wavelet-based
- top 5 use advanced wavelet oriented quantization
- systems requirements ask for multiresolution

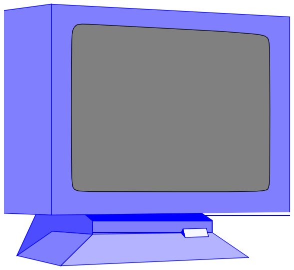
Final JPEG 2000 standard is wavelet based

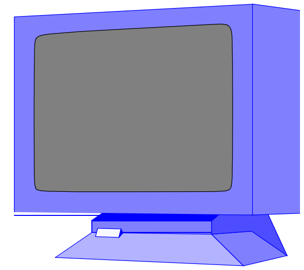
Digital video coding

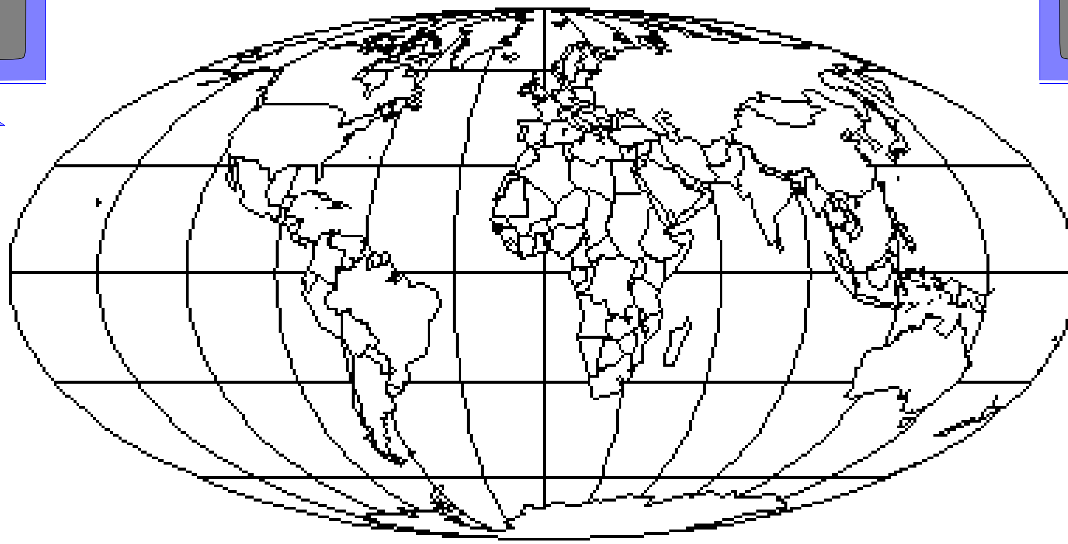


- signal decomposition for compression
- compatible subchannels
- tight control over coding error
- easy joint source/channel coding
- robustness to channel errors
- easy random access for digital storage

Conversion between TV formats

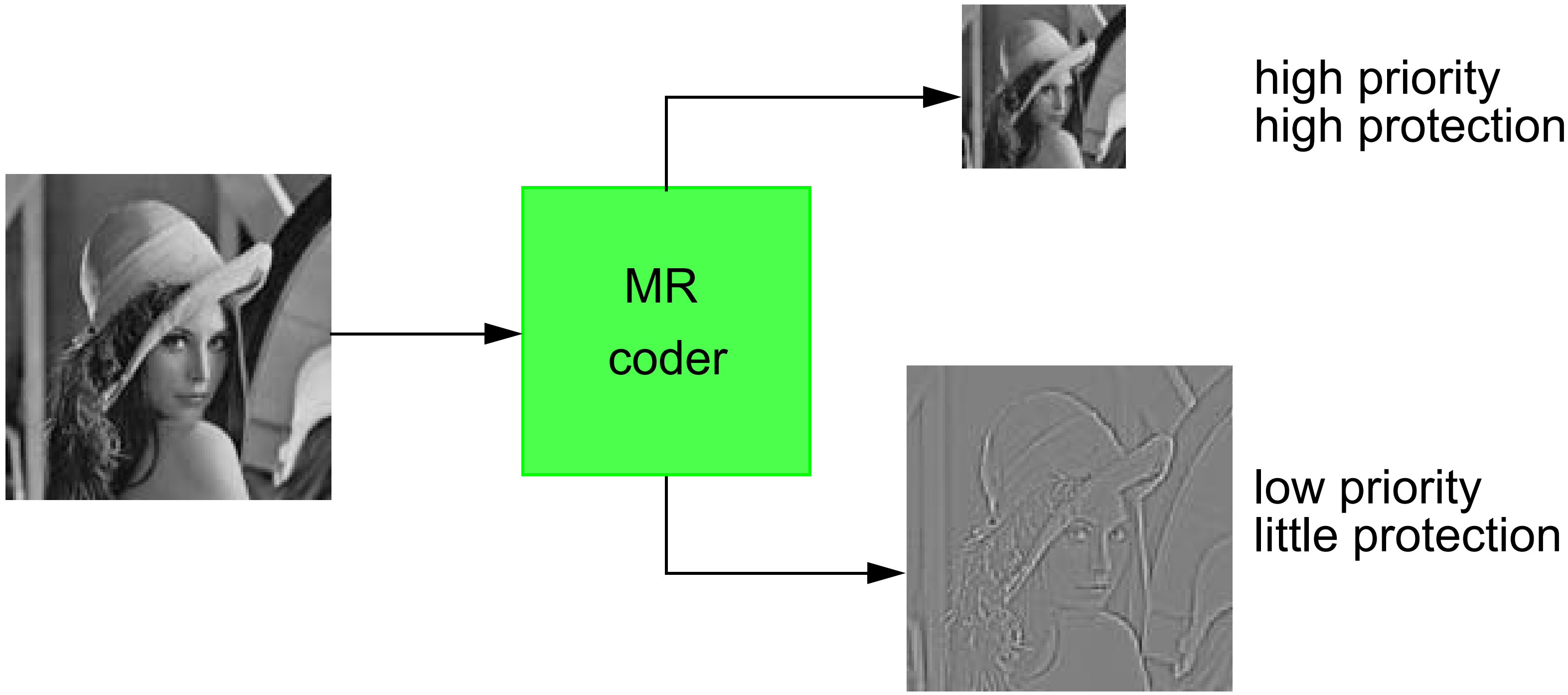






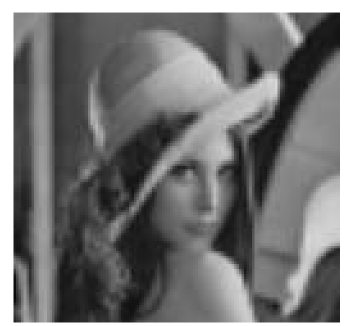
- HDTV/NTSC
- interlaced/progressive

Interaction of source and channel coding



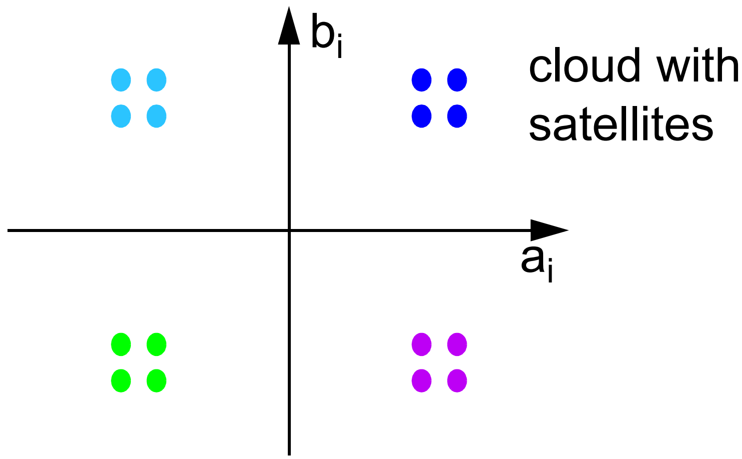
full reconstruction

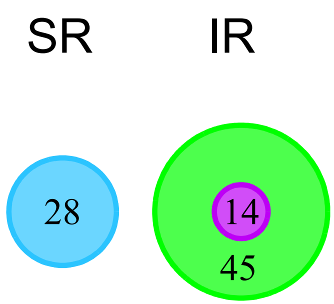
coarse reconstruction



MR transmission for digital broadcast

Embedding of coarse information within detail

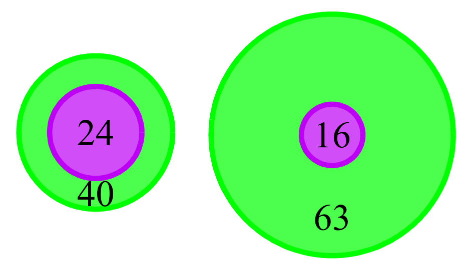




- cloud: carries coarse info
- satellite: carries detail
- blend MR transmission with MR coding

Trade-off in broad-cast ranges [miles]

MR**:** $\lambda = 0.5, 0.2$



high/low resolution

MR coding for multicast over the Internet

''I want to say a special welcome to everyone that's climbed into the Internet tonight, and has got into the MBone --- and I hope it doesn't all collapse!''

Mick Jagger, Rollings Stones on Internet,11/18/94

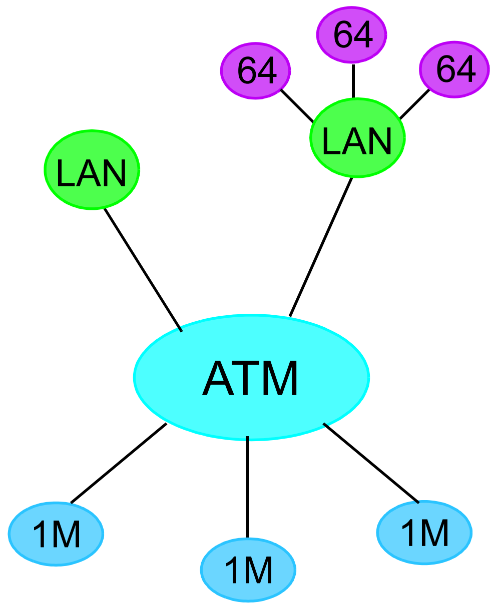
Motivation: Internet is a heterogeneous mess!

Video multicast over Mbone

- video by VIC
- software encoder/decoder
- learning experience (seminars...)

Heterogeneous user population

On-going experience



MR coding for multicast over the Internet

Fact: different users receive different bit rates

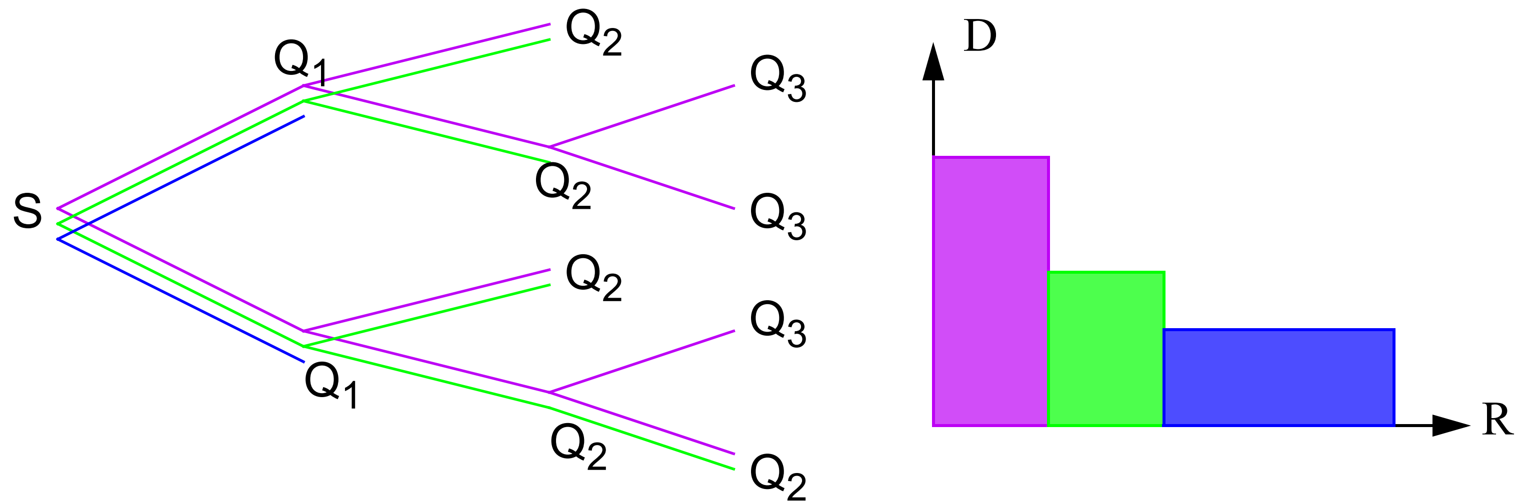
- transmission heterogeneity

Different users absorb different bit rates

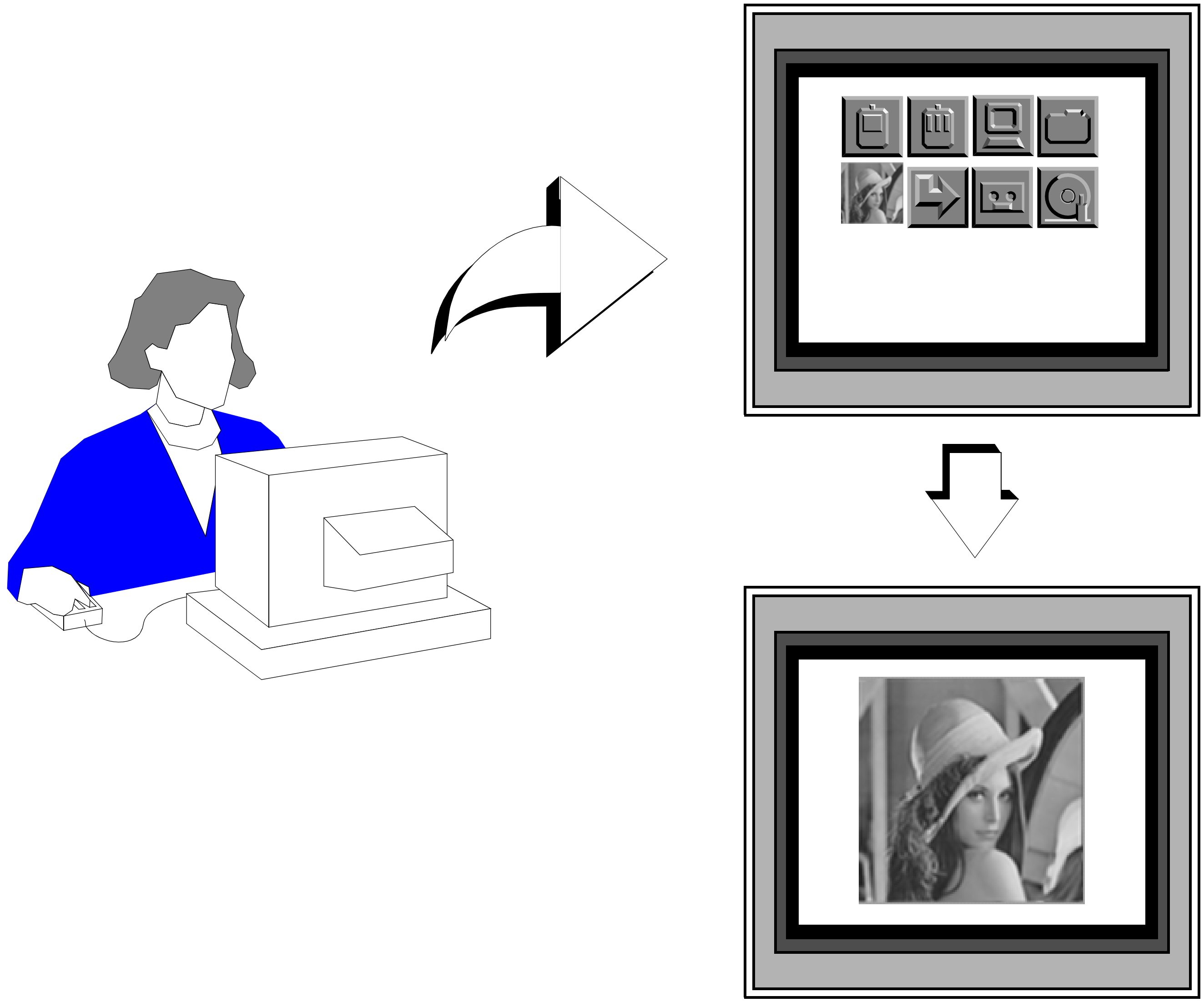
- computation heterogeneity

Solution: layered multicast trees

- different layers are transmitted over independent trees
- automatic subscribe/unsubscribe
- dynamic quality management

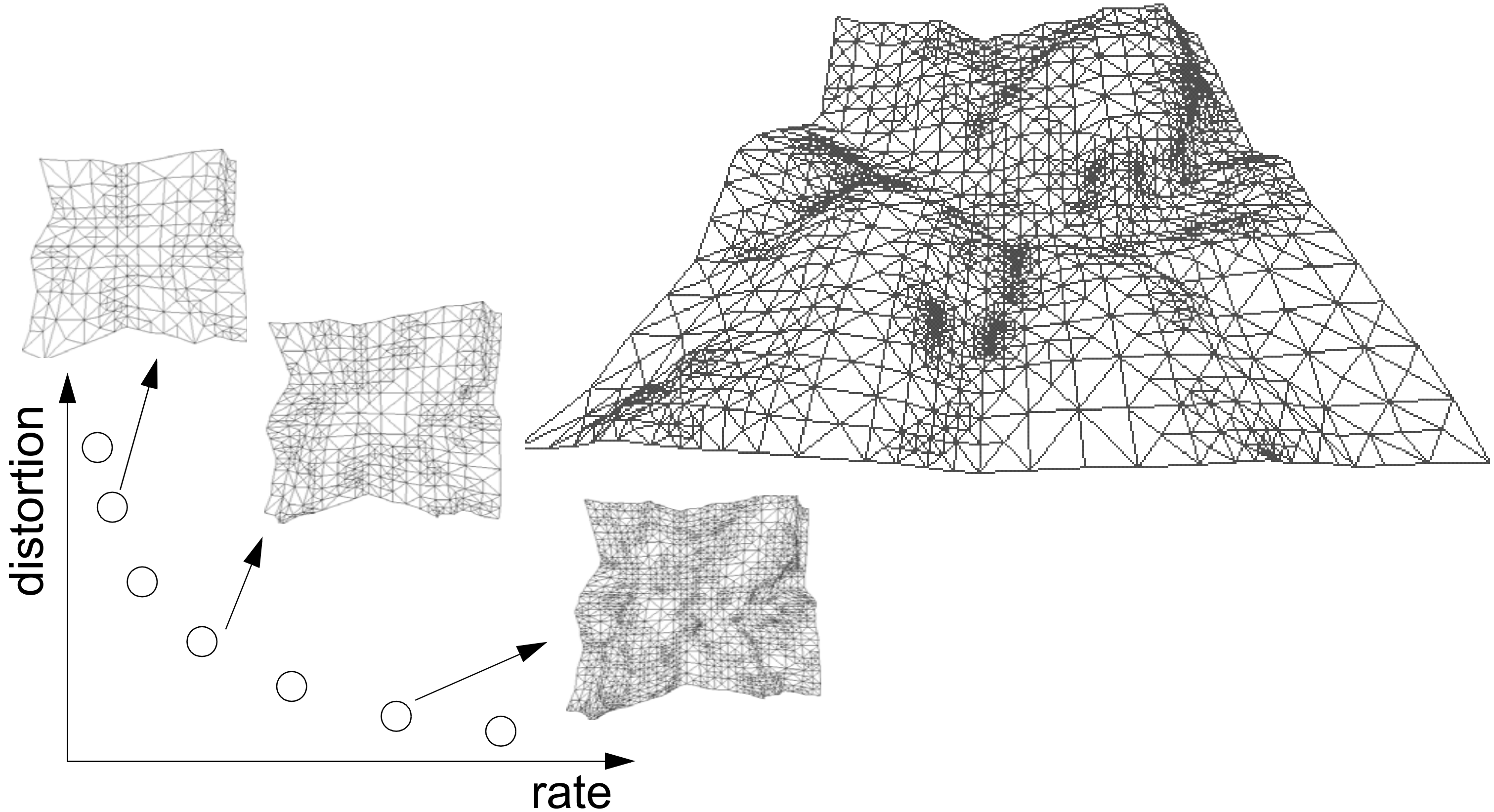


Remote image databases with browsing

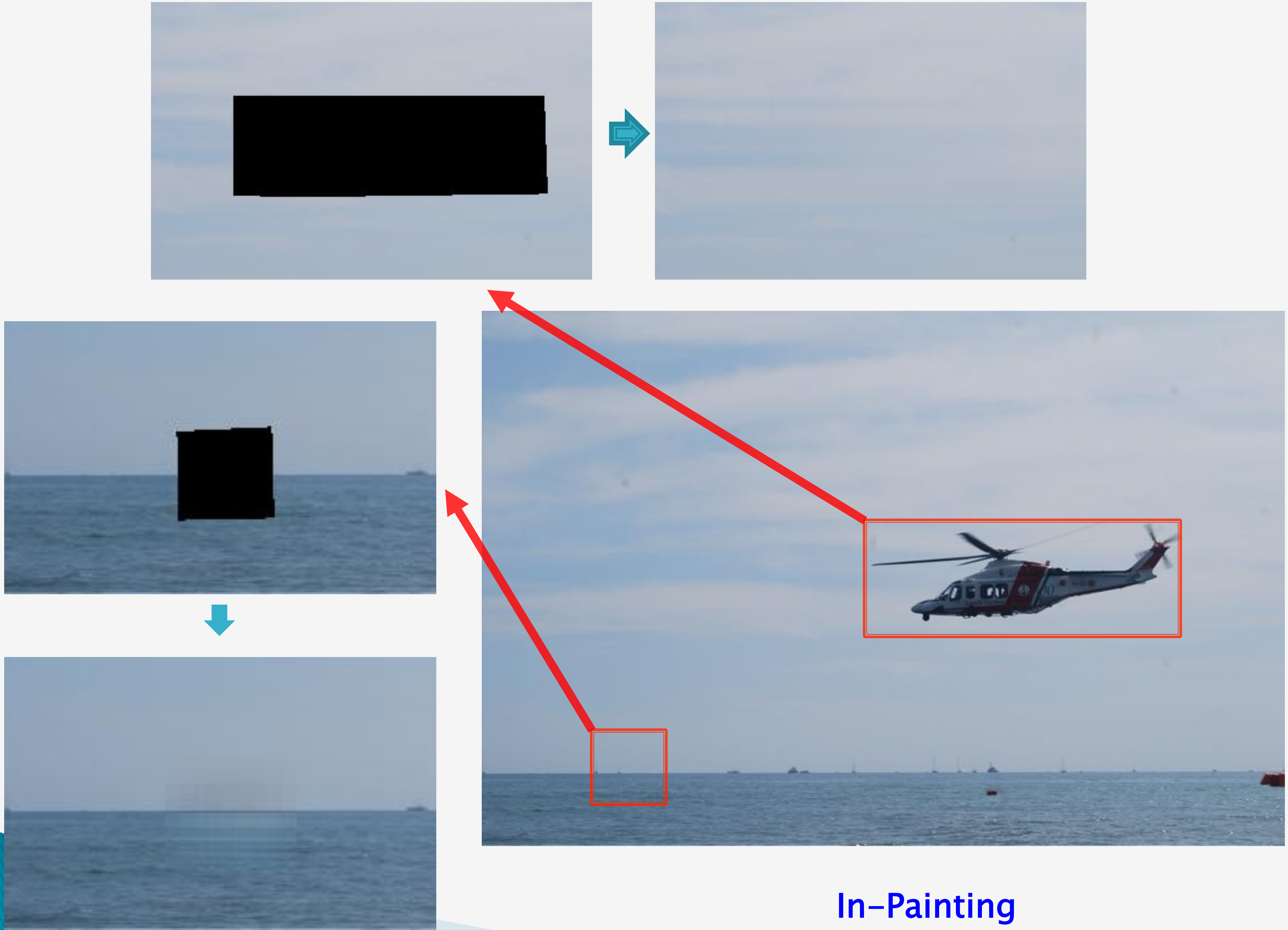


Multiresolution graphics

Example: optimize quality (distortion) for a target rate




In-Painting



In-Painting