– ADVANCED METHODS –
FOR INFORMATION REPRESENTATION

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VECTOR SPACE

A VECTOR SPACE OVER A FIELD OF SCALARS $\mathbb{C}$ (OR $\mathbb{R}$) IS A SET OF VECTORS V TOGETHER WITH OPERATIONS OF VECTOR ADDITION AND SCALAR MULTIPLICATION

- V: SET OF ELEMENTS (THE ELEMENTS ARE VECTORS)

- ADDITION BETWEEN ELEMENTS OF V $\quad +: V^2 \to V$

$$(\underline{x}, \underline{y}) \in V^2 \to \underline{x} + \underline{y} \in V$$

"+" IS COMMUTATIVE $\quad \forall \underline{x}, \underline{y} \in V \quad \underline{x} + \underline{y} = \underline{y} + \underline{x}$

"+" IS ASSOCIATIVE $\quad \forall \underline{x}, \underline{y}, \underline{z} \in V \quad (\underline{x} + \underline{y}) + \underline{z} = \underline{x} + (\underline{y} + \underline{z})$

ADDITIVE IDENTITY $\quad \exists\, \underline{0} \in V \mid \forall \underline{x} \in V \quad (\underline{x} + \underline{0}) = \underline{x}$

ADDITIVE INVERSE $\quad \exists\, \underline{x} \in V, \exists! -\underline{x} \in V \mid \underline{x} + (-\underline{x}) = \underline{0}$

- MULTIPLICATION BETWEEN A SCALAR AND AN ELEMENT OF V

$$\bullet : \mathbb{C} \times V \to V$$

$$(\alpha, \underline{x}) \in \mathbb{C} \times V \ (\text{OR } \mathbb{R} \times V) \to \alpha \circ \underline{x} \in V$$

$$(\text{OR } \bullet : \mathbb{R} \times V \to V)$$

PSEUDOASSOCIATIVITY FOR "$\circ$" $\quad \forall \alpha, \beta \in \mathbb{C}$ (OR $\mathbb{R}$), $\underline{x} \in V \quad \alpha \circ (\beta \circ \underline{x}) = (\alpha \beta) \circ \underline{x}$

MULTIPLICATION BETWEEN SCALARS AND ELEMENTS OF V

MULTIPLICATION BETWEEN COMPLEX NUMBERS

DISTRIBUTIVITY OF "$\circ$" WITH RESPECT TO "+" $\quad \forall \underline{x}, \underline{y} \in V, \alpha \in \mathbb{C}$ (OR $\mathbb{R}$) $\quad \alpha \circ (\underline{x} + \underline{y}) = \alpha \circ \underline{x} + \alpha \circ \underline{y}$

DISTRIBUTIVITY OF "$\circ$" WITH RESPECT TO ADDITION BETWEEN SCALARS

$\forall \alpha, \beta \in \mathbb{C}$ (OR $\mathbb{R}$), $\forall \underline{x} \in V \quad (\alpha + \beta) \circ \underline{x} = \alpha \circ \underline{x} + \beta \circ \underline{x}$

MULTIPLICATIVE IDENTITY $\quad \forall \underline{x} \in V \quad 1 \circ \underline{x} = \underline{x}$

$(V, +, \circ)$ FORMS A VECTOR SPACE

$\mathbb{C}^N$; SET OF VECTORS OF LENGTH N WHICH COMPONENTS ARE COMPLEX NUMBERS

$$\mathbb{C}^N = \{x = [x_0, x_1, \ldots, x_{N-1}]^T \mid x_m \in \mathbb{C}, m \in [0,1,\ldots,N-1]\}$$

EXAMPLE:
IT SATISFIES ALL THE PROPERTIES OF VECTOR SPACES, SO IT IS A VECTOR SPACE. IF WE TAKE $N=2$ THEN $\forall \underline{x}, \underline{y} \in \mathbb{C}^2$

$$\underline{x} + \underline{y} = \begin{pmatrix} x_0 + y_0 \\ x_1 + y_1 \end{pmatrix} = \begin{pmatrix} y_0 + x_0 \\ y_1 + x_1 \end{pmatrix} = \underline{y} + \underline{x}$$

$$\exists -\underline{x} = \begin{pmatrix} -x_0 \\ -x_1 \end{pmatrix} \qquad \underline{x} + (-\underline{x}) = \begin{pmatrix} x_0 + (-x_0) \\ x_1 + (-x_1) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \underline{0}$$

$\mathbb{C}^{\mathbb{Z}}$: SET OF SEQUENCES (=INFINITELY LONG VECTORS) WHICH VALUES ARE COMPLEX NUMBERS

$$\mathbb{C}^{\mathbb{Z}} = \{x = [\cdots x_{-1}\ x_0\ x_1 \cdots]^T \mid x_m \in \mathbb{C}, m \in \mathbb{Z}\}$$

$\mathbb{C}^{\mathbb{R}}$: SET OF FUNCTIONS WHICH VALUES ARE COMPLEX NUMBERS (ALSO: COMPLEX VALUED FUNCTIONS DEFINED OVER $\mathbb{R}$ OR VECTOR SPACE OF SIGNALS)

$$\mathbb{C}^{\mathbb{R}} = \{x \mid x(t) \in \mathbb{C}, t \in \mathbb{R}\}$$

$\mathbb{R}^N$; VECTOR SPACE OF POLYNOMIAL FUNCTIONS OF DEGREE AT MOST $(N-1)$ $\quad x(t) = \sum_{k=0}^{N-1} \alpha_k t^k$

NB.: $\mathbb{C}^N$ AND $\mathbb{R}^N$ ARE VECTOR SPACES OF DIMENSION N
$\mathbb{C}^{\mathbb{Z}}$ AND $\mathbb{C}^{\mathbb{R}}$ ARE VECTOR SPACES OF DIMENSION ∞

N.B.: CARTESIAN PRODUCT; IT'S A MATHEMATICAL OPERATION THAT RETURNS A SET FROM MULTIPLE SETS, THAT IS, FOR SETS A AND B, THE CARTESIAN PRODUCT $A \times B$ IS THE SET OF ALL ORDERED PAIRS (a,b) WHERE $a \in A$, $b \in B$. $A \times B = \{(a,b) \mid a \in A \wedge b \in B\}$

EXAMPLE: $A = \{0,1,2\}$, $B = \{2,3\}$, $A \times B = \{(0,2),(0,3),(1,2),(1,3)(2,2),(2,3)\}$

CARTESIAN SQUARE (OR BINARY CARTESIAN PRODUCT); IS THE PRODUCT $V \times V = V^2$

EXAMPLE;
THE TWO DIMENSIONAL PLANE $\mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$ WHERE $\mathbb{R}$ IS THE SET OF REAL NUMBERS.

SUBSPACE: A NON EMPTY SUBSET S OF A VECTOR SPACE V IS A SUBSPACE WHEN IT IS CLOSED UNDER THE OPERATIONS OF VECTOR ADDITION AND SCALAR MULTIPLICATION:

(1) $\forall \underline{x}, \underline{y} \in S \rightarrow \underline{x}+\underline{y} \in S$

(2) $\forall \alpha \in \mathbb{C}$ (OR $\mathbb{R}$), $\forall \underline{x} \in S \rightarrow \alpha \underline{x} \in S$

N.B.: A SUBSPACE S IS ITSELF A VECTOR SPACE OVER THE SAME FIELD OF SCALARS AS V AND WITH THE SAME VECTOR ADDITION AND MULTIPLICATION SCALAR OPERATIONS AS V

EXAMPLES:

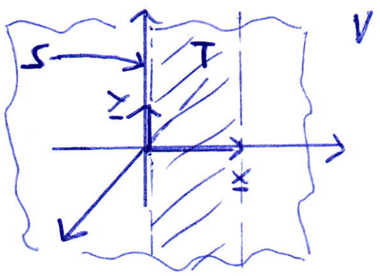
- V = VECTOR SPACE, $\underline{x} \in V$ $\alpha \underline{x}$ ($\alpha \in \mathbb{C}$) IS A SUBSPACE OF V
- $\mathbb{C}^{\mathbb{Z}}$ = VECTOR SPACE, $\underline{x} = \{2,3,4,5\}$ SEQUENCES THAT ARE $\emptyset$ OUTSIDE THIS RANGE FORM A SUBSPACE OF $\mathbb{C}^{\mathbb{Z}}$
- $\mathbb{R}^{\mathbb{R}}$ = VECTOR SPACE, FUNCTIONS THAT ARE CONSTANT ON INTERVALS $[k-\frac{1}{2}, k+\frac{1}{2})$, $k \in \mathbb{Z}$ FORM A SUBSPACE
- $\mathbb{C}^N$ IS A SUBSPACE OF $\mathbb{C}^{\mathbb{Z}}$
- $\mathbb{R}^{\mathbb{R}}$ = VECTOR SPACE, SET OF ODD/EVEN FUNCTIONS FORM A SUBSPACE

AFFINE SUBSPACE: A SUBSET T OF A VECTOR SPACE V IS AN AFFINE SUBSPACE WHEN THERE EXIST A VECTOR $\underline{x} \in V$ AND A SUBSPACE $S \subset V$ SUCH THAT ANY $\underline{t} \in T$ CAN BE WRITTEN AS $\underline{x} + \underline{s}$ FOR SOME $\underline{s} \in S$

- AN AFFINE SUBSPACE IS A SUBSPACE IF AND ONLY IF IT INCLUDES $\underline{0}$
- GENERALIZE THE CONCEPT OF A PLANE IN EUCLIDEAN GEOMETRY: SUBSPACES CORRESPOND JUST TO PLANES THAT INCLUDE THE ORIGIN
- ARE CONVEX SETS: $\lambda \underline{x} + (1-\lambda)\underline{y}$ IS IN THE SET, FOR $\lambda \in [0,1]$

EXAMPLE:

$\underline{x}, \underline{y} \in V$. THE SET OF VECTORS OF THE FORM $\underline{x} + \underbrace{\alpha \underline{y}}_{\text{SUBSPACE}}$, $\alpha \in \mathbb{C}$ IS AN AFFINE SUBSPACE



EXAMPLE:

$T = \{\underline{x} \in \mathbb{C}^{\mathbb{Z}} \mid x_m = 1, \forall m \neq 1,2,3\}$ IS AN AFFINE SUBSPACE OF $\mathbb{C}^{\mathbb{Z}}$, IT IS NOT A SUBSPACE OF $\mathbb{C}^{\mathbb{Z}}$ SINCE THE $\underline{0}$ (ALL ZERO SEQUENCE) $\notin T$.

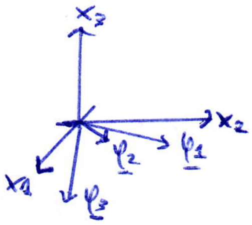
SPAN: THE SPAN OF A SET OF VECTORS S IS THE SET OF ALL LINEAR COMBINATIONS OF VECTORS IN S.

$$\mathrm{SPAN}(S) = \left\{ \sum_{k=0}^{N-1} \alpha_k \underline{\varphi_k} \;\middle|\; \alpha_k \in \mathbb{C} \,(\text{or } \mathbb{R}),\ \underline{\varphi_k} \in S,\ N \in \mathbb{N} \right\}$$

N.B.: A SPAN IS ALWAYS A SUBSPACE, THE SUM HAS A FINITE NUMBER OF TERMS EVEN IF S IS INFINITE

EXAMPLE:

$\underline{\varphi_1}, \underline{\varphi_2}, \underline{\varphi_3} \in O x_1 x_2$ PLANE



$\mathrm{SPAN}\{\underline{\varphi_1}, \underline{\varphi_2}\} = O x_1 x_2$

$\mathrm{SPAN}\{\underline{\varphi_1}, \underline{\varphi_2}, \underline{\varphi_3}\} = O x_1 x_2$

LINEARLY INDEPENDENT SET: THE SET OF VECTORS $\{\underline{\varphi_0}, \underline{\varphi_1}, \ldots, \underline{\varphi_{N-1}}\}$ IS A LINEARLY INDEPENDENT SET WHEN $\sum_{k=0}^{N-1} \alpha_k \underline{\varphi_k} = \underline{0}$ IS TRUE ONLY FOR $\alpha_k = 0, \forall k$.

OTHERWISE THE SET IS LINEARLY DEPENDENT. AN INFINITE SET OF VECTORS IS CALLED LINEARLY INDEPENDENT WHEN EVERY FINITE SUBSET IS LINEARLY INDEPENDENT.

EXAMPLE:

$$\underline{e_k} = [\underbrace{0, 0, \ldots, 0}_{(k)\ 0s}, 1, \underbrace{0, 0, \ldots, 0}_{(N-k-1)\ 0s}]^T, \quad k = 0, 1, \ldots, N-1$$

$$\sum_{k=0}^{N-1} \alpha_k \underline{e_k} = \alpha_0 \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} + \alpha_1 \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix} + \ldots + \alpha_{N-1} \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} \alpha_0 \\ \alpha_1 \\ \vdots \\ \alpha_{N-1} \end{bmatrix}$$

$\alpha_k \in \mathbb{C}\ \forall k$

THIS VECTOR IS EQUAL TO $\underline{0}$ ONLY IF $\alpha_k = 0\ \forall k$

DIMENSION OF A VECTOR SPACE: A VECTOR SPACE V IS SAID TO HAVE DIMENSION N WHEN IT CONTAINS A LINEARLY INDEPENDENT SET WITH N ELEMENTS AND EVERY SET WITH $N+1$ OR MORE ELEMENTS IS LINEARLY DEPENDENT.

IF NO SUCH FINITE N EXISTS, THE VECTOR SPACE IS ∞ DIMENSIONAL.

EXAMPLE:

$\mathbb{R}^N$ IS OF DIMENSION N.

I TAKE THE SET $\{\underline{e}_k\}_{k=0}^{N-1}$ THAT IS LINEARLY INDEPENDENT (SEE EXAMPLE OF LIN. INDEP. SET!)

I DEMONSTRATE NOW THAT EVERY SET WITH $N+1$ OR MORE ELEMENTS IS LINEARLY DEPENDENT

I TAKE $\{\underline{\varphi}_k\}_{k=0}^{M-1}$ $M>N$. IF THIS SET DOESN'T CONTAIN A SET OF LINEARLY INDEPENDENT VECTORS WE'RE DONE. IF IT CONTAINS INSTEAD, I CAN ASSUME THAT THEY ARE THE FIRST N VECTORS

$$\underline{0} = \sum_{k=0}^{M-1} \alpha_k \underline{\varphi}_k = \sum_{k=0}^{N-1} \alpha_k \underline{\varphi}_k + \sum_{k=N}^{M-1} \alpha_k \underline{\varphi}_k \quad (1)$$

BY SUBTRACTING $\sum_{k=N}^{M-1} \alpha_k \underline{\varphi}_k$ FROM BOTH SIDES I OBTAIN:

$$\sum_{k=0}^{N-1} \alpha_k \underline{\varphi}_k = -\sum_{k=N}^{M-1} \alpha_k \underline{\varphi}_k \equiv \underbrace{\begin{bmatrix} \varphi_{0,0} & \cdots & \varphi_{N-1,0} \\ \varphi_{0,1} & \ddots & \\ \vdots & & \\ \varphi_{0,N-1} & & \varphi_{N-1,N-1} \end{bmatrix}}_{\phi} \underbrace{\begin{bmatrix} \alpha_0 \\ \vdots \\ \alpha_{N-1} \end{bmatrix}}_{\alpha} = \underbrace{\begin{bmatrix} \varphi_{N,0} & \cdots & \varphi_{M-1,0} \\ \varphi_{N,1} & \cdots & \varphi_{M-1,1} \\ \vdots & & \vdots \\ \varphi_{N,N-1} & \cdots & \varphi_{M-1,N-1} \end{bmatrix}}_{\bar{\Phi}} \underbrace{\begin{bmatrix} -\alpha_N \\ \vdots \\ -\alpha_{M-1} \end{bmatrix}}_{-\bar{\alpha}}$$

ϕ IS INVERTIBLE BECAUSE THE FIRST N VECTORS ARE LINEARLY INDEPENDENT:

$$\alpha = -\phi^{-1} \bar{\Phi} \bar{\alpha}$$

WE CAN CHOOSE $\bar{\alpha}$ NON ZERO, SO (1) HAS SOLUTION WITH SOME NONZERO α_k

THUS THE SET IS LINEARLY DEPENDENT.

INNER PRODUCT

AN INNER PRODUCT ON A VECTOR SPACE V OVER $\mathbb{C}$ (OR $\mathbb{R}$) IS A COMPLEX-VALUED (OR REAL-VALUED) FUNCTION $\langle \cdot , \cdot \rangle$ DEFINED ON $V \times V$ WITH THE FOLLOWING PROPERTIES FOR ANY $\underline{x}, \underline{y}, \underline{z} \in V$ AND $\alpha \in \mathbb{C}$ (OR $\mathbb{R}$): $(V \times V \to \mathbb{C} \text{ (OR } \mathbb{R}))$

(1) DISTRIBUTIVITY $\langle \underline{x} + \underline{y}, \underline{z} \rangle = \langle \underline{x}, \underline{z} \rangle + \langle \underline{y}, \underline{z} \rangle$

(2) LINEARITY IN THE FIRST ARGUMENT $\langle \alpha \underline{x}, \underline{y} \rangle = \alpha \langle \underline{x}, \underline{y} \rangle$

(3) HERMITIAN SYMMETRY: $\langle \underline{x}, \underline{y} \rangle^* = \langle \underline{y}, \underline{x} \rangle$

(4) POSITIVE DEFINITENESS: $\langle \underline{x}, \underline{x} \rangle \geq 0$ AND $\langle \underline{x}, \underline{x} \rangle = 0$ IF AND ONLY IF $\underline{x} = \underline{0}$

N.B.: GIVES A MEASURE OF SIMILARITY BETWEEN TWO VECTORS. IT'S LIKE A NORM OF AN ORTHOGONAL PROJECTION OF ONE VECTOR ONTO A SUBSPACE SPANNED BY ANOTHER

EXAMPLES: CONSIDER VECTOR SPACE $\mathbb{C}^2$

$\langle \underline{x}, \underline{y} \rangle = x_0 y_0^* + 5 x_1 y_1^*$ IS A VALID INNER PRODUCT: IT SATISFIES ALL THE CONDITIONS

$\langle \underline{x}, \underline{y} \rangle = x_0^* y_0 + x_1^* y_1$ IS NOT A VALID INNER PRODUCT: IF $\underline{x} = \underline{y} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, $\alpha = j$

$\langle \alpha \underline{x}, \underline{y} \rangle = 0 \cdot 0 + (-j) \cdot 1 = -j$

$\alpha \langle \underline{x}, \underline{y} \rangle = j(0 \cdot 0 + 1 \cdot 1) = j$

$\langle \underline{x}, \underline{y} \rangle = x_0 y_0^*$ IS NOT A VALID INNER PRODUCT; IF $\underline{x} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ $\langle \underline{x}, \underline{x} \rangle = 0$ BUT $\underline{x} \neq \underline{0}$

N.B.: EVERY SPACE MAY SEE A DIFFERENT INNER PRODUCT DEFINITION

$\mathbb{C}^N$: $\boxed{\langle \underline{x}, \underline{y} \rangle = \sum_{n=0}^{N-1} x_n y_n^* = \underline{y}^* \underline{x}}$

$*$ = HERMITIAN TRANSPOSE

$$(y_0^* \; y_1^* \; \dots \; y_{N-1}^*) \begin{pmatrix} x_0 \\ x_1 \\ \vdots \\ x_{N-1} \end{pmatrix}$$

$\mathbb{C}^{\mathbb{Z}}$: $\boxed{\langle \underline{x}, \underline{y} \rangle = \sum_{n \in \mathbb{Z}} x_n y_n^* = \underline{y}^* \underline{x}}$

$\mathbb{C}^{\mathbb{R}}$: $\boxed{\langle \underline{x}, \underline{y} \rangle = \int_{-\infty}^{+\infty} x(t) \, y^*(t) \, dt}$

ORTHOGONALITY

(1) $\underline{x}$ AND $\underline{y}$ ARE SAID TO BE ORTHOGONAL IF $\langle \underline{x}, \underline{y} \rangle = 0$. $(\underline{x} \perp \underline{y})$

(2) A SET OF VECTORS S IS CALLED ORTHOGONAL WHEN $\underline{x} \perp \underline{y}$ $\forall \underline{x}, \underline{y} \in S$, $\underline{x} \neq \underline{y}$

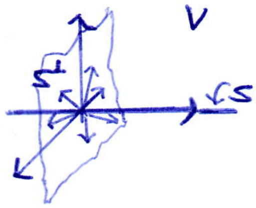
(3) A SET OF VECTORS S IS CALLED ORTHONORMAL WHEN IT'S ORTHOGONAL AND $\langle \underline{x}, \underline{x} \rangle = 1$ $\forall \underline{x} \in S$

(4) $\underline{x}$ IS ORTHOGONAL TO A SET OF VECTORS S WHEN $\underline{x} \perp \underline{y}$ $\forall \underline{y} \in S$ $(\underline{x} \perp S)$

(5) TWO SETS S_0 AND S_1 ARE SAID TO BE ORTHOGONAL WHEN $\forall \underline{s}_0 \in S_0$ IS ORTHOGONAL TO THE SET S_1 $(S_0 \perp S_1)$

(6) GIVEN A SUBSPACE S OF A VECTOR SPACE V, THE ORTHOGONAL COMPLEMENT OF S IS THE SET:

$$S^{\perp} = \{ \underline{x} \in V \mid \underline{x} \perp S \}$$



N.B.: $S^{\perp}$ IS A SUBSPACE AS WELL $\quad S^{\perp} \oplus S = V$

TH: VECTORS IN AN ORTHONORMAL SET ARE LINEARLY INDEPENDENT

$$0 = \langle \underline{0}, \underline{\varphi}_i \rangle = \left\langle \sum_{k \in K} \alpha_k \underline{\varphi}_k, \underline{\varphi}_i \right\rangle = \sum_{k \in K} \alpha_k \langle \underline{\varphi}_k, \underline{\varphi}_i \rangle = \sum_{k \in K} \alpha_k \delta_{i-k} = \alpha_i$$

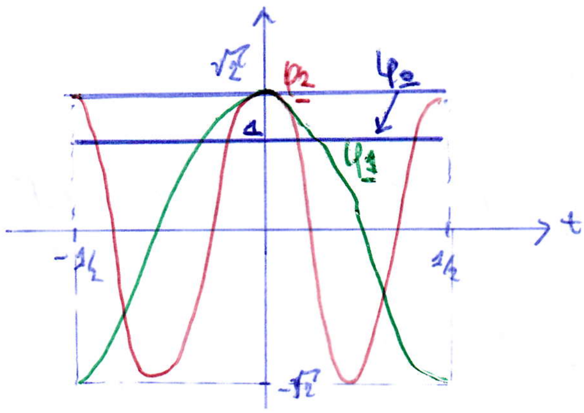
EXAMPLE:

$$\Phi = \{ \underline{\varphi}_k \}_{k \in \mathbb{N}} \subset \mathbb{C}^{[-1/2, 1/2]} \qquad \underline{\varphi}_0 = 1$$
$$\underline{\varphi}_k = \sqrt{2} \cos(2\pi k t) \qquad k = 1, 2, \ldots$$

$\varphi_0 = 1$

$\varphi_1 = \sqrt{2} \cos(2\pi t)$

$\varphi_2 = \sqrt{2} \cos(4\pi t)$



WE HAVE:

$\forall k, m \in \mathbb{Z}^+$, $k \neq m$, $\underline{\varphi}_k$ AND $\underline{\varphi}_m$ ARE ORTHOGONAL

$$\langle \underline{\varphi}_k, \underline{\varphi}_m \rangle = 2 \int_{-1/2}^{1/2} \cos(2\pi k t) \cos(2\pi m t)\, dt = \int_{-1/2}^{1/2} \left[\cos(2\pi(k+m)t) + \cos(2\pi(k-m)t) \right] dt =$$

$$= \frac{1}{2\pi} \left\{ \left[\frac{1}{k+m} \sin(2\pi(k+m)t) \right]_{-1/2}^{1/2} + \left[\frac{1}{k-m} \sin(2\pi(k-m)t) \right]_{-1/2}^{1/2} \right\} = 0$$

SO THE WHOLE SET Φ IS ORTHOGONAL: $\underline{\varphi}_k \perp \underline{\varphi}_m$ $\forall \underline{\varphi}_k, \underline{\varphi}_m \in \Phi$ SUCH THAT $\underline{\varphi}_k \neq \underline{\varphi}_m$

ALSO ϕ IS A ORTHONORMAL SET;

$$\langle \varphi_k, \varphi_k \rangle = 2\int_{-1/2}^{1/2} (\cos(2\pi k t))^2 dt = 2\int_{-1/2}^{1/2} \frac{1+\cos(4\pi k t)}{2} dt = \left[t \Big|_{-1/2}^{1/2} + \frac{1}{4\pi k} \sin(4\pi k t)\Big|_{-1/2}^{1/2} \right] = 1$$

THEN $\forall k \in N$ THE VECTOR φ_k IS ORTHOGONAL TO THE SET OF FUNCTIONS ODD (S_{ODD}): $\forall \underline{s} \in S_{ODD}$:

$$\langle \varphi_k, \underline{s} \rangle = \int_{-1/2}^{1/2} \sqrt{2}\cos(2\pi k t) \cdot s(t)\, dt = \sqrt{2}\int_{-1/2}^{0} \cos(2\pi k t) s(t)\, dt + \sqrt{2}\int_{0}^{1/2} \cos(2\pi k t) s(t)\, dt =$$

$$= -\sqrt{2}\int_{-1/2}^{0} \cos(2\pi k t)\, s(-t)\, dt + \sqrt{2}\int_{0}^{1/2} \cos(2\pi k t) s(t)\, dt =$$

$-t = \tau$

COSINE IS EVEN!

$$= -\sqrt{2}\int_{0}^{1/2} \cos(2\pi k t) s(t)\, dt + \sqrt{2}\int_{0}^{1/2} \cos(2\pi k t) s(t)\, dt = 0$$

NOTE:

SETS

NATURAL NUMBERS: $N = \{0, 1, 2, \dots\}$

$N^+ = N \setminus \{0\} = \{n \in N \mid n > 0\}$

INTEGER NUMBERS: $Z = \{0, \pm 1, \pm 2, \dots\}$

$Z^+ = \{z \in Z \mid z > 0\}$

RATIONAL NUMBERS: $Q = \{r = p/q,\ p \in Z,\ q \in N^+\}$

FINITE NUMBER OF DIGITS OR PERIODICS!!!

REAL NUMBERS: $R = \{\text{RATIONAL NUMBERS} + \text{IRRATIONAL NUMBERS}\}$

IRRATIONAL NUMBERS: $e, \pi, \sqrt{2}, \dots$ = INFINITE NUMBER OF DIGITS

$$N \subset Z \subset Q \subset R$$

NORM

A NORM ON A VECTOR SPACE V OVER $\mathbb{C}$ (OR $\mathbb{R}$) IS A REAL-VALUED FUNCTION $\|\cdot\|$ DEFINED ON V WITH THE FOLLOWING PROPERTIES FOR ANY $x, y \in V$, $\alpha \in \mathbb{C}$ (OR $\mathbb{R}$):

(1) POSITIVE DEFINITENESS $\|\underline{x}\| \geq 0$ AND $\|\underline{x}\| = 0$ IF AND ONLY IF $\underline{x} = \underline{0}$

(2) POSITIVE SCALABILITY $\|\alpha \underline{x}\| = |\alpha| \|\underline{x}\|$

(3) TRIANGLE INEQUALITY $\|\underline{x} + \underline{y}\| \leq \|\underline{x}\| + \|\underline{y}\|$ (= IFF $y = \alpha x$)

EXAMPLES: CONSIDER VECTOR SPACE $\mathbb{C}^2$

$\|\underline{x}\| = |x_0|^2 + 5|x_1|^2$ IS A VALID NORM; SATISFIES ALL THE CONDITIONS

$\|\underline{x}\| = |x_0| + |x_1|$ IS A VALID NORM (BUT NOT INDUCED BY INNER PRODUCT) (*)

$\|\underline{x}\| = |x_0|$ IS NOT A VALID NORM; IF $x = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ $\|\underline{x}\| = 0$ BUT $x \neq 0$

$\mathbb{C}^N$: $\boxed{\|\underline{x}\| = \sqrt{\langle \underline{x}, \underline{x} \rangle} = \left(\sum_{n=0}^{N-1} |x_n|^2 \right)^{1/2}}$ $= \|\underline{x}\|_2 \equiv$ EUCLIDEAN NORM IN $\mathbb{C}^N$

$\mathbb{C}^{\mathbb{Z}}$: $\boxed{\|\underline{x}\| = \sqrt{\langle \underline{x}, \underline{x} \rangle} = \left(\sum_{n \in \mathbb{Z}} |x_n|^2 \right)^{1/2}}$ $= \|\underline{x}\|_2 \equiv$ EUCLIDEAN NORM IN $\mathbb{C}^{\mathbb{Z}}$ SUBSPACE WHERE THE INNER PRODUCT EXISTS

$\mathbb{C}^{\mathbb{R}}$: $\boxed{\|\underline{x}\| = \sqrt{\langle \underline{x}, \underline{x} \rangle} = \left(\int_{-\infty}^{+\infty} |x(t)|^2 dt \right)^{1/2}}$ $= \|\underline{x}\|_2 \equiv$ EUCLIDEAN NORM IN SUBSPACE OF $\mathbb{C}^{\mathbb{R}}$ FOR WHICH THE INNER PRODUCT EXISTS

N.B.: IN THESE DEFINITIONS THE NORMS ARE INDUCED BY THE INNER PRODUCT BUT NOT ALL THE NORMS ARE INDUCED BY INNER PRODUCTS!!! (*)

PROPERTIES OF NORMS INDUCED BY INNER PRODUCTS:

PYTHAGOREAN THEOREM $\boxed{\underline{x} \perp \underline{y} \text{ IMPLIES } \|\underline{x} + \underline{y}\|^2 = \|\underline{x}\|^2 + \|\underline{y}\|^2}$

PARALLELOGRAM LAW $\boxed{\|\underline{x} + \underline{y}\|^2 + \|\underline{x} - \underline{y}\|^2 = 2(\|\underline{x}\|^2 + \|\underline{y}\|^2)}$

CAUCHY SCHWARZ INEQUALITY $\boxed{|\langle \underline{x}, \underline{y} \rangle| \leq \|\underline{x}\| \cdot \|\underline{y}\|}$

METRIC OR DISTANCE: IN A NORMED VECTOR SPACE, THE METRIC OR DISTANCE BETWEEN VECTORS $\underline{x}$ AND $\underline{y}$ IS THE NORM OF THEIR DIFFERENCE $\boxed{d(\underline{x},\underline{y}) = \|\underline{x}-\underline{y}\|}$

N.B.: IF A DEFINITION OF DISTANCE SATISFIES THE FOLLOWING PROPERTIES IT IS A VALID DEFINITION OF DISTANCE (WITHOUT USING THE NORM)

(1) $d(\underline{x},\underline{y}) \geq 0$ POSITIVITY

(2) $d(\underline{x},\underline{y}) = 0 \iff \underline{x} = \underline{y}$

(3) $d(\underline{x},\underline{y}) = d(\underline{y},\underline{x})$ SYMMETRIC MEASURE

(4) $d(\underline{x},\underline{y}) \leq d(\underline{x},\underline{z}) + d(\underline{z},\underline{y})$ TRIANGULAR INEQUALITY

EXAMPLE:

WITHOUT NORM INDUCED DISTANCE IN $\mathbb{R}$ $d(x,y) = |ATAN(x) - ATAN(y)|$

– STANDARD INNER PRODUCT SPACES –

$\mathbb{C}^N$: SPACE OF COMPLEX-VALUED FINITE-DIMENSIONAL VECTORS

$$\langle \underline{x}, \underline{y} \rangle = \sum_{n=0}^{N-1} x_n y_n^* \qquad \|\underline{x}\| = \left(\sum_{n=0}^{N-1} |x_n|^2 \right)^{1/2}$$

$\ell^2(\mathbb{Z})$: SPACE OF SQUARE-SUMMABLE SEQUENCES (FINITE ENERGY SEQUENCES)

$$\langle \underline{x}, \underline{y} \rangle = \sum_{n \in \mathbb{Z}} x_n y_n^* \qquad \|\underline{x}\| = \left(\sum_{n \in \mathbb{Z}} |x_n|^2 \right)^{1/2}$$

$\mathcal{L}^2(\mathbb{R})$: SPACE OF SQUARE INTEGRABLE FUNCTIONS (FINITE ENERGY FUNCTIONS)

$$\langle \underline{x}, \underline{y} \rangle = \int_{-\infty}^{+\infty} x(t) y^*(t)\, dt \qquad \|\underline{x}\| = \left(\int_{-\infty}^{+\infty} |x(t)|^2 dt \right)^{1/2}$$

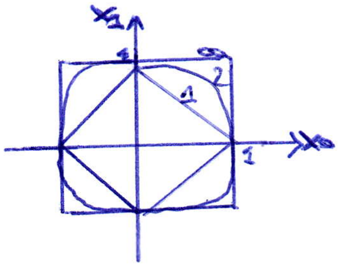
$C^q([a,b])$: SPACE OF CONTINUOUS FUNCTIONS WITH q CONTINUOUS DERIVATIVES

NOTE: THE NORMS ARE INDUCED BY THE INNER PRODUCT!!!

– STANDARD NORMED VECTOR SPACES –

$\mathbb{C}^N$ SPACES:

$$\|\underline{x}\|_p = \left(\sum_{n=0}^{N-1} |x_n|^p \right)^{1/p} \qquad p \in [1, \infty)$$



$p = 1$ TAXICAB/MANHATTAN NORM

$p = 2$ EUCLIDEAN NORM ⇐ ONLY IN THIS CASE WE HAVE A NORM INDUCED BY AN INNER PRODUCT
SQUARE

$\ell^p(\mathbb{Z})$ SPACES:

$$\underbrace{\|\underline{x}\|_p = \left(\sum_{n \in \mathbb{Z}} |x_n|^p \right)^{1/p}}_{p \in [1, \infty)} \qquad \underbrace{\|\underline{x}\|_\infty = \sup_{n \in \mathbb{Z}} |x_n|}_{\ell^\infty \text{ NORM}}$$

– $\ell^p(\mathbb{Z})$ IS THE SUBSPACE OF $\mathbb{C}^{\mathbb{Z}}$ CONSISTING OF VECTORS WITH FINITE ℓ^p NORM, $\forall p \in [1, \infty]$

– $\mathcal{L}^p(\mathbb{R})$ FOR ANY $p \in [1, \infty)$ IS THE SUBSPACE OF $\mathbb{C}^{\mathbb{R}}$ CONSISTING OF VECTORS WITH FINITE $\mathcal{L}^p$ NORM.

EXAMPLE:

NESTING OF $\ell^p(\mathbb{Z})$ SPACES. CONSIDER THE SEQUENCE $x_n = \begin{cases} 0 & \text{FOR } n \leq 0 \\ 1/n^{\alpha} & \text{FOR } n > 0 \end{cases}$

FOR SOME $\alpha \geq 0$.

WE DETERMINE WHICH OF THE SPACES $\ell^p(\mathbb{Z})$ CONTAIN x:

$$\|\underline{x}\|_p^p = \sum_{n=1}^{+\infty} \left|\frac{1}{n^{\alpha}}\right|^p = \sum_{n=1}^{+\infty} \frac{1}{n^{p\alpha}}$$

TO CONVERGE $p\alpha > 1$, SO $\underline{x} \in \ell^p(\mathbb{Z})$ FOR $p > 1/\alpha$, $\alpha > 0$

IF $p < q$ IMPLIES $\ell^p(\mathbb{Z}) \subset \ell^q(\mathbb{Z})$

EXAMPLE:

CONSIDER THE SEQUENCE $x_n = 1/n$ FOR $n \in \mathbb{Z}^+$

$$\|\underline{x}\|_2^2 = \sum_{n=1}^{+\infty} \left|\frac{1}{n}\right|^2 \quad \text{CONVERGES}$$

$$\|\underline{x}\|_2 = \sum_{n=1}^{+\infty} \left|\frac{1}{n}\right| \quad \text{DIVERGES}$$

THUS $\underline{x} \in \ell^2(\mathbb{Z})$ AND $\underline{x} \notin \ell^1(\mathbb{Z})$

- STANDARD VECTOR SPACES -

HILBERT SPACE OF SQUARE-SUMMABLE SEQUENCES	$\ell^2(\mathbb{Z})$	$\left\{x:\mathbb{Z}\to\mathbb{C} \;\middle\vert\; \sum_n \lvert x_n\rvert^2<\infty\right\}$	WITH INNER PRODUCT $\langle x,y\rangle=\sum_n x_n y_n^*$
HILBERT SPACE OF SQUARE-INTEGRABLE FUNCTIONS	$\mathcal{L}^2(\mathbb{R})$	$\left\{x:\mathbb{R}\to\mathbb{C} \;\middle\vert\; \int \lvert x(t)\rvert^2 dt<\infty\right\}$	WITH INNER PRODUCT $\langle x,y\rangle=\int x(t)\,y(t)^*\,dt$
NORMED VECTOR SPACE OF SEQUENCES WITH FINITE P NORM $1\le p<\infty$	$\ell^p(\mathbb{Z})$	$\left\{x:\mathbb{Z}\to\mathbb{C} \;\middle\vert\; \sum_n \lvert x_n\rvert^p<\infty\right\}$	WITH NORM $\lVert x\rVert_p=\left(\sum_n \lvert x_n\rvert^p\right)^{1/p}$
NORMED VECTOR SPACE OF FUNCTIONS WITH FINITE P NORM $1\le p<\infty$	$\mathcal{L}^p(\mathbb{R})$	$\left\{x:\mathbb{R}\to\mathbb{C} \;\middle\vert\; \int \lvert x(t)\rvert^p dt<\infty\right\}$	WITH NORM $\lVert x\rVert_p=\left(\int \lvert x(t)\rvert^p dt\right)^{1/p}$
NORMED VECTOR SPACE OF BOUNDED SEQUENCES WITH SUPREMUM NORM	$\ell^\infty(\mathbb{Z})$	$\left\{x:\mathbb{Z}\to\mathbb{C} \;\middle\vert\; \sup_n \lvert x_n\rvert<\infty\right\}$	WITH NORM $\lVert x\rVert_\infty=\sup_n \lvert x_n\rvert$
NORMED VECTOR SPACE OF BOUNDED FUNCTIONS WITH SUPREMUM NORM	$\mathcal{L}^\infty(\mathbb{R})$	$\left\{x:\mathbb{R}\to\mathbb{C} \;\middle\vert\; \sup_t \lvert x(t)\rvert<\infty\right\}$	WITH NORM $\lVert x\rVert_\infty=\sup_t \lvert x(t)\rvert$

VECTOR SPACE WITH INNER PRODUCT = INNER PRODUCT SPACE (PRE-HILBERT SPACE)
\+ COMPLETENESS
↳ IF EVERY CAUCHY SEQUENCE IN THE SET CONVERGES TO A VECTOR IN THE SET (*)

} (*) HILBERT SPACE = COMPLETE INNER PRODUCT SPACE

HILBERT SPACE CONTAINS A COUNTABLE ORTHONORMAL BASIS $\Leftrightarrow$ IT IS SEPARABLE

BANACH SPACE = COMPLETE NORMED VECTOR SPACE

NOTE: INNER PRODUCT SPACES ARE ALSO NORMED VECTOR SPACES AS THEY INNER PRODUCTS INDUCE THE CORRESPONDING NORMS

NORMED VECTOR SPACES: THE NORMS ARE NOT INDUCED BY AN INNER PRODUCT

(*) CONVERGENCE: A sequence of vectors $x_0, x_1, \dots, x_k$ in a normed vector space V is said to converge to $v \in V$ when $\lim_{k\to\infty} \|v - x_k\| = 0$

Equivalently: given any $\varepsilon > 0$ there exists a k_ε such that:

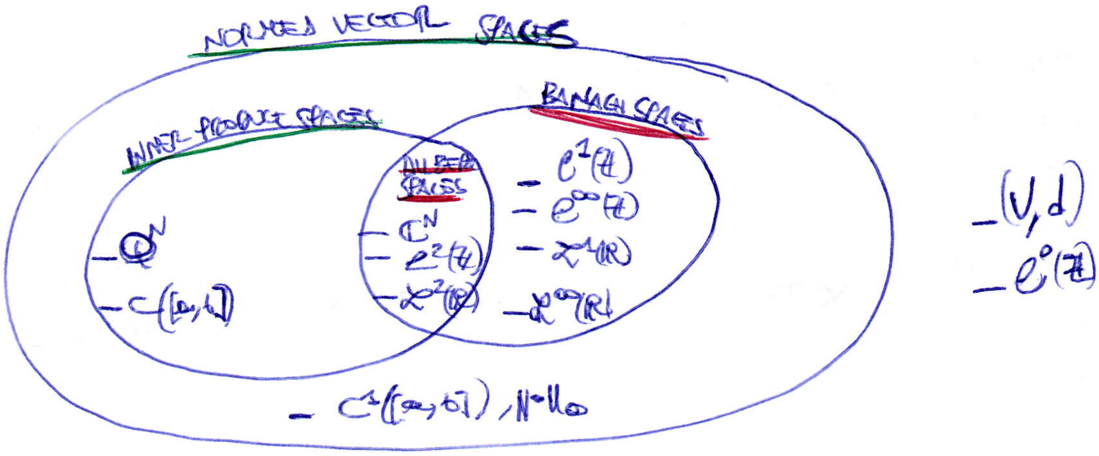
$$\|v - x_k\| < \varepsilon \quad \forall k > k_\varepsilon$$

BE CAREFUL! THE CONVERGENCE CAN DEPEND ON THE CHOICE OF NORM

(*) CAUCHY SEQUENCE: A SEQUENCE OF VECTORS $x_0, x_1, \dots$ IN A NORMED VECTOR SPACE IS CALLED A CAUCHY SEQUENCE WHEN GIVEN ANY $\varepsilon > 0$, THERE EXISTS A k_ε SUCH THAT $\|x_k - x_m\| < \varepsilon \quad \forall k, m > k_\varepsilon$

- THE ELEMENTS OF A CAUCHY SEQUENCE STAY ARBITRARILY CLOSE TO EACH OTHER
- FOR REAL-VALUED SEQUENCES IT MUST CONVERGE (IT MAY NOT BE TRUE FOR ALL NORMED VECTOR SPACES)

VECTOR SPACES



NOTES ON STANDARD SPACES:

(1) ALL FINITE-DIMENSIONAL SPACES ARE COMPLETE.
$\mathbb{C}^N$ UNDER THE 2 NORM IS A HILBERT SPACE

(2) ALL $\ell^p(\mathbb{Z})$ SPACES ARE COMPLETE. $\ell^2(\mathbb{Z})$ IS A HILBERT SPACE

(3) ALL $\mathcal{L}^p(\mathbb{R})$ SPACES ARE COMPLETE. $\mathcal{L}^2(\mathbb{R})$ IS A HILBERT SPACE

(4) $C^q([a,b])$ SPACES ARE NOT COMPLETE UNDER $\mathcal{L}^p$ NORM FOR $p \in [1, \infty)$
(EXIST CAUCHY [redacted] SEQUENCES OF CONTINUOUS FUNCTIONS WHOSE LIMITS ARE DISCONTINUOUS AND HENCE NOT IN C!)

CLOSED SUBSPACE: A SUBSPACE S OF A NORMED VECTOR SPACE V IS CALLED CLOSED WHEN IT CONTAINS ALL LIMITS OF SEQUENCES OF VECTORS IN S

N.B.: IF IT IS CLOSED ALL THE SEQUENCES OF VECTORS CONVERGE
IF IT IS COMPLETE ALL THE CAUCHY SEQUENCES CONVERGE

CLOSE $\subset$ COMPLETE

SUBSPACES OF FINITE-DIMENSIONAL NORMED VECTOR SPACES ARE ALWAYS CLOSED

SEPARABILITY: A SPACE IS CALLED SEPARABLE WHEN IT CONTAINS A COUNTABLE DENSE SUBSET.

EXAMPLE:
$\mathbb{R}$ IS SEPARABLE SINCE $\mathbb{Q}$ IS DENSE IN $\mathbb{R}$ AND IT IS COUNTABLE

Def: A GROUP CONSISTS OF A SET G AND A BINARY OPERATION "$\bullet$" DEFINED ON G, FOR WHICH THE FOLLOWING CONDITIONS ARE SATISFIED:

1) ASSOCIATIVE: $(a \cdot b) \cdot c = a \cdot (b \cdot c) \quad \forall a, b, c \in G$
2) IDENTITY: $\exists\, 1 \in G \mid a \cdot 1 = 1 \cdot a = a \quad \forall a \in G$
3) INVERSE: GIVEN $a \in G$, $\exists\, b \in G \mid a \cdot b = b \cdot a = 1$

N.B.: $(G, *)$ IS ABELIAN IF $a * b = b * a \quad \forall a, b \in G$ (COMMUTATIVE)

Def: A RING CONSISTS OF A SET R AND TWO BINARY OPERATIONS "$+$" (ADDITION) AND "$\bullet$" (MULTIPLICATION), DEFINED OVER R, FOR WHICH THE FOLLOWING CONDITIONS ARE SATISFIED:

1) ADDITIVE ASSOCIATIVE: $(a+b)+c = a+(b+c) \quad \forall a, b, c \in R$
2) ADDITIVE COMMUTATIVE: $a+b = b+a \quad \forall a, b \in R$
3) ADDITIVE IDENTITY: $\exists\, 0 \in R \mid \forall a \in R,\ 0+a = a+0 = a$
4) ADDITIVE INVERSE: $\forall a \in R, \exists\, -a \in R \mid a+(-a) = (-a)+a = 0$
5) LEFT AND RIGHT DISTRIBUTIVITY: $\forall a, b, c \in R$, $a \cdot (b+c) = a \cdot b + a \cdot c$ AND $(b+c) \cdot a = b \cdot a + c \cdot a$
6) MULTIPLICATIVE ASSOCIATIVITY: $\forall a, b, c \in R$, $(a \cdot b) \cdot c = a \cdot (b \cdot c)$

Def: A FIELD CONSISTS OF A SET F AND TWO BINARY OPERATIONS "$+$" (ADDITION) AND "$\bullet$" (MULTIPLICATION), DEFINED OVER F, FOR WHICH THE FOLLOWING CONDITIONS ARE SATISFIED:

1) $(F, +, \cdot)$ IS A RING
2) MULTIPLICATIVE COMMUTATIVE: $\forall a, b \in F$, $a \cdot b = b \cdot a$
3) MULTIPLICATIVE IDENTITY: $\exists\, 1 \in F \mid a \cdot 1 = 1 \cdot a = a \quad \forall a \in F$
4) MULTIPLICATIVE INVERSE: IF $a \in F$, $a \neq 0$, $\exists\, b \in F \mid a \cdot b = b \cdot a = 1$

EXAMPLE:

$(\mathbb{C}, +, \cdot)$ REPRESENTS THE FIELD OF COMPLEX NUMBERS, WITH "$+$" AND "$\bullet$" BEING THE ADDITION/MULTIPLICATION BETWEEN COMPLEX NUMBERS

$(\mathbb{R}, +, \cdot)$ REPRESENTS THE FIELD OF REAL NUMBERS, WITH "$+$" AND "$\bullet$" BEING THE ADDITION/MULTIPLICATION ON REAL NUMBERS

LINEAR OPERATOR

A FUNCTION $A: H_0 \to H_1$ IS CALLED A LINEAR OPERATOR FROM H_0 TO H_1 WHEN FOR ALL $\underline{x}, \underline{y} \in H_0$ AND $\alpha \in \mathbb{C}$ (OR $\mathbb{R}$):

(1) ADDITIVITY $\quad A(\underline{x}+\underline{y}) = A\underline{x} + A\underline{y}$

(2) SCALABILITY $\quad A(\alpha \underline{x}) = \alpha (A \underline{x})$

N.B.: WHEN H_0 (DOMAIN) AND H_1 (CODOMAIN) ARE THE SAME, A IS ALSO CALLED LINEAR OPERATOR ON H_0

N.B.: LINEAR OPERATORS FROM $\mathbb{C}^N$ TO $\mathbb{C}^M$ AND MATRICES IN $\mathbb{C}^{M\times N}$ ARE THE SAME THING, SO IF $H_0 \in \mathbb{C}^N$ AND $H_1 \in \mathbb{C}^M$ ANY LINEAR OPERATOR BETWEEN $\mathbb{C}^N$ AND $\mathbb{C}^M$ IS AN $M \times N$ MATRIX

$$\underline{x} \in \mathbb{C}^N, \quad \underline{y} \in \mathbb{C}^M \qquad \underset{[M\times 1]}{\underline{y}} = \underset{[M\times N]}{A}\ \underset{[N\times 1]}{\underline{x}} \qquad [M] = [M\times N][N]$$

EXAMPLE:

$H_0 = \mathcal{L}^2(\mathbb{R})$ SPACE OF FINITE ENERGY FUNCTIONS (SQUARE INTEGRABLE)

$A =$ SAMPLING OPERATOR ($T =$ SAMPLING PERIOD)

$\underline{x} \in \mathcal{L}^2(\mathbb{R}) \qquad A\underline{x} = \underline{y} \in \ell^2(\mathbb{Z})$ (SPACE OF FINITE ENERGY SEQUENCES (SQUARE SUMMABLE))

SO $H_1 = \ell^2(\mathbb{Z}) \qquad \to \qquad y_n = x(t)\big|_{t=nT}$

NULL SPACE (OR KERNEL) OF A LINEAR OPERATOR $A: H_0 \to H_1$ IS THE SUBSPACE OF H_0 THAT A MAPS TO $\underline{0}$

$$N(A) = \{\underline{x} \in H_0 \mid A\underline{x} = \underline{0}\}$$

RANGE OF A LINEAR OPERATOR $A: H_0 \to H_1$ IS A SUBSPACE OF H_1:

$$R(A) = \{A\underline{x} \in H_1 \mid \underline{x} \in H_0\}$$

NORM OF A LINEAR OPERATOR: THE OPERATOR NORM OF A, DENOTED BY $\|A\|$ IS DEFINED AS

$$\boxed{\|A\| = \sup_{\|\underline{x}\|_{H_0}=1} \|A\underline{x}\|_{H_1}}$$

N.B.: THE OPERATOR IS BOUNDED WHEN ITS OPERATOR NORM IS FINITE, BOUNDED OPERATORS ARE CONTINUOUS

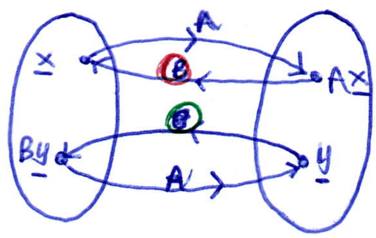
EXAMPLE:

$A: \mathbb{C}^{\mathbb{Z}} \to \mathbb{C}^{\mathbb{Z}}$ $\quad (A\underline{x})_m = x_m |m| \quad \forall m \in \mathbb{Z}$ IS UNBOUNDED. IN FACT IF $\|A\|$ IS FINITE $\exists M > \|A\|$ AND IF WE TAKE THE SEQUENCE OF ALL $\emptyset$s EXCEPT FOR 1 IN M^{TH} POSITION WE OBTAIN A CONTRADICTION: $\|A\underline{x}\| = M$ IS ABSURD: $\begin{cases} \|A\| < M \\ \|A\| \geq \|A\underline{x}\| = \|x_m |m|\| = M \end{cases}$

INVERSE: A BOUNDED LINEAR OPERATOR $A: H_0 \to H_1$ IS CALLED INVERTIBLE IF THERE EXISTS A BOUNDED LINEAR OPERATOR $B: H_1 \to H_0$ SUCH THAT:

$B A \underline{x} = \underline{x} \quad \forall \underline{x}$ IN H_0

$A B \underline{y} = \underline{y} \quad \forall \underline{y}$ IN H_1



WHEN SUCH B EXISTS IT IS UNIQUE, IS DENOTED BY A^{-1} AND IT IS CALLED THE INVERSE OF A

B IS CALLED LEFT INVERSE OF A

B IS CALLED RIGHT INVERSE OF A $\qquad B = A^{-1}$

N.B.: IN GENERAL THERE MIGHT BE INFINITE OR FINITE LEFT RIGHT INVERSE OF A MATRIX A

N.B.: WE WORK IN HILBERT SPACES SO WE HAVE THE NORM INDUCED BY THE SCALAR PRODUCT

A LINEAR OPERATOR LINKS TWO HILBERT SPACES

N.B.: LINEAR OPERATORS WITH CODOMAIN OF FINITE DIMENSIONS ARE ALWAYS LIMITED. USING BOUNDED OPERATORS WE CAN USE LINEAR ALGEBRA FINITE SPACES CONCEPTS

EXAMPLE:

$A : \mathbb{R}^2 \to \mathbb{R}^2$ $A = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}$

NORM (2 NORM FOR DOMAIN AND CODOMAIN):

$$\|A\| = \sup_{\|\underline{x}\|=1} \|A\underline{x}\| = \sup \left\| \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix} \right\| = \sup_\theta \left\| \begin{matrix} 3\cos\theta + \sin\theta \\ \cos\theta + 3\sin\theta \end{matrix} \right\| =$$

$$= \sup_\theta \sqrt{(3\cos\theta + \sin\theta)^2 + (\cos\theta + 3\sin\theta)^2} = \sup_\theta \sqrt{10\sin^2\theta + 10\cos^2\theta + 12\sin\theta\cos\theta} =$$

$$= \sup_\theta \sqrt{10 + 6\sin(2\theta)} = 4$$ (WHEN $\sin(2\theta) = 1 \to 2\theta = \frac{\pi}{2} \to \theta = \frac{\pi}{4}$)

NULL SPACE: $\underline{0}$

RANGE: $\mathbb{R}^2$

INVERSE: $A^{-1} = \frac{{}^tA_c}{|A|} = \begin{bmatrix} 3 & -1 \\ -1 & 3 \end{bmatrix}^t \cdot \frac{1}{8} = \begin{bmatrix} 3/8 & -1/8 \\ -1/8 & 3/8 \end{bmatrix}$

$BA = A^{-1}A = \begin{bmatrix} 3/8 & -1/8 \\ -1/8 & 3/8 \end{bmatrix} \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$AB = AA^{-1} = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 3/8 & -1/8 \\ -1/8 & 3/8 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

IT IS BOTH LEFT AND RIGHT INVERSE!!!

EXAMPLE:

$A : \ell^2(\mathbb{Z}) \to \ell^2(\mathbb{Z})$

$(\alpha_k)_{k\in\mathbb{Z}}$ ≡ COMPLEX VALUED SEQUENCE

$(A\underline{x})_k = \alpha_k x_k,\ k \in \mathbb{Z}$ ≡ COMPONENT WISE MULTIPLICATION

$$A\underline{x} = \begin{bmatrix} \ddots & & & 0 \\ & \alpha_{-1} & & \\ & & \alpha_0 & \\ & & & \alpha_1 \\ 0 & & & & \ddots \end{bmatrix} \begin{bmatrix} \vdots \\ x_{-1} \\ x_0 \\ x_1 \\ \vdots \end{bmatrix}$$

$\alpha_k^T = [\ldots \alpha_{-1}, \alpha_0, \alpha_1 \ldots]$

$x_k^T = [\ldots x_{-1}, x_0, x_1 \ldots]$

$A\underline{x} = [\ldots \alpha_{-1}x_{-1}, \alpha_0 x_0, \alpha_1 x_1 \ldots]^T$

IT IS BOUNDED: $\|\alpha\|_\infty = M < \infty$, $\|A\| = M$

INVERSE: IT IS INVERTIBLE WHEN $\inf_k |\alpha_k| > 0$

$$A^{-1}\underline{y} = \begin{bmatrix} \ddots 1/\alpha_{-1} & & 0 \\ & 1/\alpha_0 & \\ 0 & & 1/\alpha_1 \end{bmatrix} \begin{bmatrix} \vdots \\ y_{-1} \\ y_0 \\ y_1 \\ \vdots \end{bmatrix}$$

$(\alpha_k^T)^{-1} = [\ldots 1/\alpha_{-1}, 1/\alpha_0, 1/\alpha_1 \ldots]$

EXAMPLE:

$$A: \mathbb{C}^3 \to \mathbb{C}^2 \qquad A = \begin{bmatrix} 1 & j & 0 \\ 1 & 0 & j \end{bmatrix}$$

NORM (2 NORM FOR DOMAIN AND CODOMAIN):

$$\|A\| = \sup_{\|\underline{x}\|=1} \left\| \begin{bmatrix} 1 & j & 0 \\ 1 & 0 & j \end{bmatrix} \begin{bmatrix} a_0 + jb_0 \\ a_1 + jb_1 \\ a_2 + jb_2 \end{bmatrix} \right\| = \sup_{\|\underline{x}\|=1} \left\| \begin{matrix} a_0 + jb_0 + j(a_1 + jb_1) \\ a_0 + jb_0 + j(a_2 + jb_2) \end{matrix} \right\| =$$

$$= \sup_{\|\underline{x}\|=1} \left\| \begin{matrix} (a_0 - b_1) + j(b_0 + a_1) \\ (a_0 - b_2) + j(b_0 + a_2) \end{matrix} \right\| = \sqrt{3}$$

NULL SPACE:

$$\begin{bmatrix} 1 & j & 0 \\ 1 & 0 & j \end{bmatrix} \begin{bmatrix} a_0 + jb_0 \\ a_1 + jb_1 \\ a_2 + jb_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{cases} a_0 - b_1 + j(b_0 + a_1) = 0 \\ a_0 - b_2 + j(b_0 + a_2) = 0 \end{cases} \to \begin{cases} a_0 = b_1 \\ b_0 = -a_1 \\ a_0 = b_2 \\ b_0 = -a_2 \end{cases} \to \begin{cases} a_0 = b_1 = b_2 = \alpha \\ b_0 = -a_1 = -a_2 = \beta \end{cases} \to \begin{pmatrix} \alpha + j\beta \\ -\beta + j\alpha \\ -\beta + j\alpha \end{pmatrix} = \begin{pmatrix} z \\ jz \\ jz \end{pmatrix}$$

$$N(A) = \{ [z, jz, jz]^T \}$$

RANGE: $R(A) = \mathbb{C}^2$

INVERSE: ONLY THE RIGHT INVERSE EXISTS

$$AB\,\underline{y} = \underline{y} \qquad \forall \underline{y} \in \mathbb{C}^2$$

$$\begin{pmatrix} 1 & j & 0 \\ 1 & 0 & j \end{pmatrix} \begin{pmatrix} b_{00} & b_{01} \\ b_{10} & b_{11} \\ b_{20} & b_{21} \end{pmatrix} \underline{y} = \underline{y} \to \begin{pmatrix} b_{00} + jb_{10} & b_{01} + jb_{11} \\ b_{00} + jb_{20} & b_{01} + jb_{21} \end{pmatrix} \underline{y} = \underline{y}$$

$$\det \begin{pmatrix} b_{00} + jb_{10} - 1 & b_{01} + jb_{11} \\ b_{00} + jb_{20} & b_{01} + jb_{21} - 1 \end{pmatrix} = 0$$

ADJOINT AND SELF ADJOINT OPERATOR: THE LINEAR OPERATOR

$A^*: H_1 \to H_0$ IS CALLED ADJOINT OF THE LINEAR OPERATOR $A: H_0 \to H_1$

WHEN:

$$\boxed{\langle A\underline{x}, \underline{y}\rangle_{H_1} = \langle \underline{x}, A^*\underline{y}\rangle_{H_0}} \qquad \forall \underline{x} \in H_0,\ \underline{y} \in H_1$$

WHEN $A = A^*$ THE OPERATOR A IS CALLED SELF-ADJOINT OR HERMITIAN

EXAMPLE: (MULTIPLICATION BY A SCALAR)

HILBERT SPACE

$A: H \to H$ $\qquad A\underline{x} = \alpha\underline{x}$ for some scalar α (MULTIPLICATION BY A SCALAR)

$\forall\, \underline{x}, \underline{y} \in H$

$$\langle A\underline{x}, \underline{y}\rangle_H = \langle \alpha\underline{x}, \underline{y}\rangle_H \underset{\uparrow}{=} \alpha\langle \underline{x}, \underline{y}\rangle_H \underset{\uparrow}{=} \alpha\langle \underline{y}, \underline{x}\rangle_H^* =$$

LINEARITY OF SCALAR PRODUCT 1ST ARGUMENT — HERMITIAN SIMMETRY

$$= \left(\alpha^*\langle \underline{y}, \underline{x}\rangle_H\right)^* = \langle \alpha^*\underline{y}, \underline{x}\rangle_H^* = \langle \underline{x}, \alpha^*\underline{y}\rangle_H$$

SO $A^*\underline{y} = \alpha^*\underline{y}$ IS THE ADJOINT OF A. IN OTHER WORDS THE ADJOINT OF THE MULTIPLICATION BY A SCALAR IS THE MULTIPLICATION BY THE **CONJUGATE** (*) OF THE SCALAR.

$$\underset{[1\times 1]}{A = [\alpha]} \qquad \underset{[1\times 1]}{A^* = [\alpha^*]}$$

NOTE: IF WE WERE $A: \mathbb{R} \to \mathbb{R}$ THEN $\alpha = \alpha^*$ AND THE OPERATOR WOULD BE SELF-ADJOINT.

EXAMPLE: (MULTIPLICATION BY A MATRIX)

$A: \mathbb{C}^N \to \mathbb{C}^M$

$\langle \underline{x}, \underline{y}\rangle = \underline{y}^*\underline{x}$ = INNER PRODUCT IN $\mathbb{C}^N$ or $\mathbb{C}^M$ (* = HERMITIAN TRANSPOSE)

$\forall\, \underline{x}, \underline{y} \in \mathbb{C}^N, \mathbb{C}^M$

$$\langle A\underline{x}, \underline{y}\rangle_{\mathbb{C}^M} = \underline{y}^*(A\underline{x}) = (\underline{y}^*A)\underline{x} \overset{\text{HERMITIAN TRANSPOSE}}{=} (A^*\underline{y})^*\underline{x} = \langle \underline{x}, A^*\underline{y}\rangle_{\mathbb{C}^N}$$

SO THE ADJOINT OF MULTIPLICATION BY A MATRIX IS THE MULTIPLICATION BY THE HERMITIAN TRANSPOSE OF ~~THE~~ MATRIX

EXAMPLE:

$$(A\underline{x})_k = \alpha_k x_k, \quad k \in \mathbb{Z}$$

SCALAR PRODUCT DEFINITION IN ℓ^2

$$\langle A\underline{x}, \underline{y}\rangle_{\ell^2(\mathbb{Z})} = \sum_{k\in\mathbb{Z}} (\alpha_k x_k)\, y_k^* = \sum_{k\in\mathbb{Z}} x_k (\alpha_k^* y_k)^* = \langle \underline{x}, A^*\underline{y}\rangle$$

$$(A^*\underline{y})_k = \alpha_k^* y_k$$

EXAMPLE: LOCAL AVERAGING

$$(A\underline{x})_m = \int_{m-1/2}^{m+1/2} x(t)\,dt$$

THIS IS A SORT OF SAMPLING: THE OPERATOR IS FROM $\mathcal{L}^2(\mathbb{R})$ to $\ell^2(\mathbb{Z})$

WE NEED TO CHECK THAT THE RESULT IS IN $\ell^2(\mathbb{Z})$. IF $\underline{x} \in \mathcal{L}^2(\mathbb{R})$ COMPUTE THE ℓ^2 NORM OF $A\underline{x}$:

$$\|A\underline{x}\|_2^2 = \sum_{m\in\mathbb{Z}} |(A\underline{x})_m|^2 = \sum_{m\in\mathbb{Z}} \left| \int_{m-1/2}^{m+1/2} x(t)\,dt \right|^2 \leq \sum_{m\in\mathbb{Z}} \int_{m-1/2}^{m+1/2} |x(t)|^2 dt = \int_{-\infty}^{+\infty} |x(t)|^2 dt = \|\underline{x}\|^2_{\mathcal{L}^2}$$

$A\underline{x}$ IS IN $\ell^2(\mathbb{Z})$ SINCE ITS NORM IS BOUNDED BY $\|\underline{x}\|_{\mathcal{L}^2}$

NOW WE FIND THE ADJOINT: $\quad A^*: \ell^2(\mathbb{Z}) \to \mathcal{L}^2(\mathbb{R}) \mid \langle A\underline{x}, \underline{y}\rangle_{\ell^2} = \langle \underline{x}, A^*\underline{y}\rangle_{\mathcal{L}^2}$

$$\forall\, \underline{x} \in \mathcal{L}^2(\mathbb{R}),\ \underline{y} \in \ell^2(\mathbb{Z})$$

$$\langle A\underline{x}, \underline{y}\rangle_{\ell^2(\mathbb{R})} = \sum_{m\in\mathbb{Z}} (A\underline{x})_m y_m^* = \sum_{m\in\mathbb{Z}} \left(\int_{m-1/2}^{m+1/2} x(t)\,dt \right) y_m^* = \sum_{m\in\mathbb{Z}} \int_{m-1/2}^{m+1/2} x(t)\, y_m^*\, dt$$

$$\langle \underline{x}, A^*\underline{y}\rangle_{\mathcal{L}^2(\mathbb{R})} = \int_{-\infty}^{+\infty} x(t) \left((A^*\underline{y})(t) \right)^* dt = \sum_{m\in\mathbb{Z}} \int_{m-1/2}^{m+1/2} x(t) \left((A^*\underline{y})(t) \right)^* dt$$

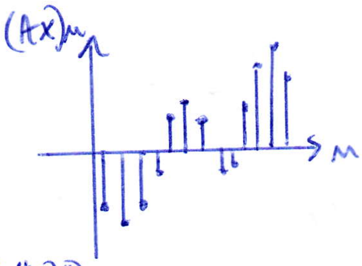
← THEY HAVE TO MATCH: WE DEFINE A PIECEWISE CONSTANT FUNCTION

$$(A^*\underline{y})(t) = y_m \quad \text{for } t \in \left[m - \tfrac{1}{2}, m + \tfrac{1}{2}\right]$$

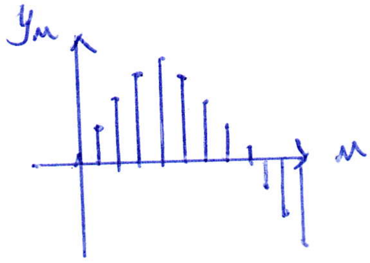
≡ CONSTANT ON THE INTERVAL

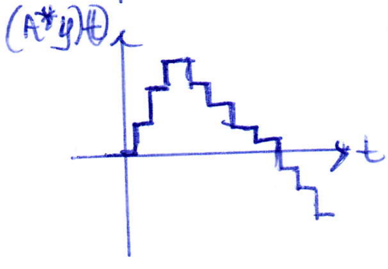
THEN THE INTEGRAL BREAKS INTO THE SUM OF INTEGRALS

$$\text{SO} \quad \langle A\underline{x}, \underline{y}\rangle_{\ell^2} = \langle \underline{x}, A^*\underline{y}\rangle_{\mathcal{L}^2}$$



LOCAL AVERAGING OPERATOR A GIVES A SEQUENCE IN $\ell^2(\mathbb{Z})$





THE ADJOINT OF A (THE ADJOINT OF LOCAL AVERAGING) IS TO FORM A PIECEWISE CONSTANT FUNCTION

ADJOINT PROPERTIES: LET $A: H_0 \to H_1$ BE A BOUNDED LINEAR OPERATOR

(1) THE ADJOINT A^* EXISTS

(2) THE ADJOINT A^* IS UNIQUE

(3) THE ADJOINT OF A^* IS EQUAL TO THE ORIGINAL OPERATOR $(A^*)^* = A$

(4) THE OPERATORS AA^* AND A^*A ARE SELF-ADJOINT

(5) THE OPERATOR NORMS OF A AND A^* ARE EQUAL $\|A^*\| = \|A\|$

(6) IF A IS INVERTIBLE, THE ADJOINT OF THE INVERSE AND THE INVERSE OF THE ADJOINT ARE EQUAL $(A^{-1})^* = (A^*)^{-1}$

(7) LET $B: H_0 \to H_1$ BE A BOUNDED LINEAR OPERATOR, THEN $(A+B)^* = A^* + B^*$

(8) LET $B: H_1 \to H_2$ BE A BOUNDED LINEAR OPERATOR, THEN $(BA)^* = A^* B^*$

RELATIONSHIP BETWEEN SUBSPACES:

$$R(A)^\perp = N(A^*)$$
$$\overline{R(A)} = N(A^*)^\perp \qquad (\overline{C} = \text{CLOSURE OF } C)$$

DIM:

(1) $N(A^*) \subseteq R(A)^\perp$

$\underline{y} \in N(A^*)$, $\underline{y}' \in R(A)$ $\qquad \underline{y}' = A\underline{x}$ for some $\underline{x}$

$$\langle \underline{y}', \underline{y} \rangle = \langle A\underline{x}, \underline{y} \rangle = \langle \underline{x}, A^*\underline{y} \rangle = \langle \underline{x}, \underline{0} \rangle = 0$$ THAT IS $\underline{y} \perp R(A)$

(2) $R(A)^\perp \subseteq N(A^*)$

$\underline{y} \in R(A)^\perp$, $\underline{x} \in H_0$

$$0 = \langle A\underline{x}, \underline{y} \rangle = \langle \underline{x}, A^*\underline{y} \rangle$$

CHOOSING $\underline{x} = A^*\underline{y} \to A^*\underline{y} = 0$ BY POSITIVE DEFINITENESS OF THE INNER PRODUCT WE MUST HAVE $\underline{y} \in N(A^*)$

UNITARY OPERATOR: A BOUNDED LINEAR OPERATOR $A: H_0 \to H_1$ IS CALLED UNITARY WHEN
(1) IT IS INVERTIBLE
(2) IT PRESERVES INNER PRODUCTS

$$\boxed{\langle A\underline{x}, A\underline{y}\rangle_{H_1} = \langle \underline{x}, \underline{y}\rangle_{H_0}} \qquad \forall \underline{x}, \underline{y} \in H_0$$

N.B.: PRESERVATION OF INNER PRODUCT LEADS TO PRESERVATION OF NORMS:

$$\|A\underline{x}\|^2 = \langle A\underline{x}, A\underline{x}\rangle_{H_1} = \langle \underline{x}, \underline{x}\rangle_{H_0} = \|\underline{x}\|^2 \qquad \forall \underline{x} \in H_0$$

TH A BOUNDED LINEAR OPERATOR $A: H_0 \to H_1$ IS UNITARY IF AND ONLY IF $A^{-1} = A^*$

Dim:

IT IS $A^*A = I$ ON H_0

A^* BEING THE LEFT INVERSE OF A WHEN A IS A UNITARY OPERATOR:

$$\langle A^*A\underline{x}, \underline{y}\rangle_{H_0} = \langle A\underline{x}, A\underline{y}\rangle_{H_1} = \langle \underline{x}, \underline{y}\rangle_{H_0} \qquad \forall \underline{x}, \underline{y} \in H_0$$

A^* BEING THE LEFT INVERSE OF A IMPLIES THAT A IS A UNITARY OPERATOR:

$$\langle A\underline{x}, A\underline{y}\rangle_{H_1} = \langle \underline{x}, A^*A\underline{y}\rangle_{H_0} = \langle \underline{x}, \underline{y}\rangle_{H_0} \qquad \forall \underline{x}, \underline{y} \in H_0$$

A^* BEING THE RIGHT INVERSE OF A FROM THE FACT THAT A IS UNITARY:

$$\langle AA^*\underline{x}, \underline{y}\rangle_{H_1} = \langle AA^*\underline{x}, AA^{-1}\underline{y}\rangle_{H_1} = \langle A^*\underline{x}, A^{-1}\underline{y}\rangle_{H_0} = \langle \underline{x}, AA^{-1}\underline{y}\rangle_{H_1} = \langle \underline{x}, \underline{y}\rangle_{H_1} \qquad \forall \underline{x}, \underline{y} \in H_1$$

$\rightarrow AA^* = I_{H_1}$

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EIGENVECTOR OF A LINEAR OPERATOR; AN EIGENVECTOR OF A LINEAR OPERATOR $A: H \to H$ IS A NONZERO VECTOR $\underline{v} \in H$ SUCH THAT $\boxed{A\underline{v} = \lambda \underline{v}}$

FOR SOME $\lambda \in \mathbb{C}$. THE CONSTANT λ IS CALLED THE CORRESPONDING EIGENVALUE AND $(\lambda, \underline{v})$ IS CALLED AN EIGENPAIR.

PROPERTIES:

(1) ALL EIGENVALUES OF A SELF-ADJOINT OPERATOR A ARE REAL

$$\lambda\langle \underline{v},\underline{v}\rangle = \langle \lambda \underline{v},\underline{v}\rangle = \langle A\underline{v},\underline{v}\rangle = \langle \underline{v}, A\underline{v}\rangle = \langle \underline{v},\lambda\underline{v}\rangle = \lambda^*\langle \underline{v},\underline{v}\rangle \Rightarrow \lambda \in \mathbb{R}$$

(2) ALL EIGENVECTORS CORRESPONDING TO DISTINCT EIGENVALUES ARE ORTHOGONAL

CONSIDER TWO EIGENPAIRS $(\lambda_0, \underline{v_0})$ AND $(\lambda_1, \underline{v_1})$ WITH $\lambda_0 \neq \lambda_1$

$$\lambda_0\langle \underline{v_0},\underline{v_1}\rangle = \langle \lambda_0\underline{v_0},\underline{v_1}\rangle = \langle A\underline{v_0},\underline{v_1}\rangle = \langle \underline{v_0}, A\underline{v_1}\rangle = \langle \underline{v_0},\lambda_1\underline{v_1}\rangle = \lambda_1^*\langle \underline{v_0},\underline{v_1}\rangle$$

$\Rightarrow \langle \underline{v_0},\underline{v_1}\rangle = 0$ SINCE λ_i ARE REAL.

N.B.: THE EIGENVECTOR OF A LINEAR OPERATOR IS THE GENERALIZATION OF THE EIGENVECTOR OF A MATRIX, IT APPLIES AS LONG AS DOMAIN AND CODOMAIN REPRESENT THE SAME HILBERT SPACE. IF THE SIGNAL IS DISCRETE-TIME THE EIGENVECTOR IS CALLED EIGENSEQUENCE IF THE SIGNAL IS CONTINUOUS TIME THE EIGENVECTOR IS CALLED **EIGENFUNCTION**.

DEFINITE LINEAR OPERATOR; A SELF ADJOINT OPERATOR $A: H \to H$ IS CALLED

(1) POSITIVE SEMIDEFINITE OR NONNEGATIVE DEFINITE $(A \geq 0)$ WHEN:

$$\langle A\underline{x},\underline{x}\rangle \geq 0 \quad \forall \underline{x} \in H$$

(2) POSITIVE DEFINITE $(A > 0)$ WHEN:

$$\langle A\underline{x},\underline{x}\rangle > 0 \quad \forall \underline{x} \in H, \ \underline{x} \neq 0$$

(3) NEGATIVE SEMIDEFINITE OR NON POSITIVE DEFINITE WHEN $-A$ IS POSITIVE SEMIDEFINITE

(4) NEGATIVE DEFINITE WHEN $-A$ IS POSITIVE DEFINITE

N.B.: POSITIVE DEFINITENESS DEFINES A PARTIAL ORDER ON SELF ADJOINT OPERATORS DEFINED ON THE SAME HILBERT SPACE.
GIVEN 2 SELF ADJOINT OPERATORS (LINEAR) $A: H \to H$ AND $B: H \to H$
$A \geq B$ MEANS $A - B \geq 0$, THAT IS $A - B$ IS A POSITIVE SEMIDEFINITE OPERATOR.

N.B.: ALL EIGENVALUES OF A POSITIVE DEFINITE OPERATOR ARE POSITIVE
ALL EIGENVALUES OF A SEMI-DEFINITE OPERATOR ARE NON-NEGATIVE.

PROJECTION THEOREM: LET S BE A CLOSED SUBSPACE OF HILBERT SPACE H AND LET $\underline{x}$ BE A VECTOR IN H

(1) EXISTENCE: THERE EXISTS $\hat{\underline{x}} \in S \mid \|\underline{x} - \hat{\underline{x}}\| \leq \|\underline{x} - \underline{s}\| \quad \forall \underline{s} \in S$

(2) ORTHOGONALITY: $\underline{x} - \hat{\underline{x}} \perp S$ IS NECESSARY AND SUFFICIENT FOR DETERMINING $\hat{\underline{x}}$

(3) UNIQUENESS: $\hat{\underline{x}}$ IS UNIQUE

(4) LINEARITY: $\hat{\underline{x}} = P\underline{x}$ WHERE P IS A LINEAR OPERATOR THAT DEPENDS ON S AND NOT ON $\underline{x}$

(5) IDEMPOTENCY: $P(P\underline{x}) = P\underline{x} \quad \forall \underline{x} \in H$

(6) SELF ADJOINTNESS: $P = P^*$

Dim (2) + (3):

WE SHOW THAT IF $\hat{\underline{x}}$ IS A MINIMIZING VECTOR, THEN THE ERROR $\underline{x} - \hat{\underline{x}}$ IS ORTHOGONAL TO S.

SUPPOSE FOR THE SAKE OF CONTRADICTION THAT $\underline{x} - \hat{\underline{x}} \not\perp S$. THEN $\exists\, \underline{\varphi} \in S, \|\underline{\varphi}\| = 1$ SUCH THAT $\langle \underline{x} - \hat{\underline{x}}, \underline{\varphi} \rangle = \alpha \neq 0$.

$$\underline{s} = \hat{\underline{x}} + \alpha\underline{\varphi} \in S: \quad \|\underline{x} - \underline{s}\|^2 = \|\underline{x} - \hat{\underline{x}} - \alpha\underline{\varphi}\|^2 = \|\underline{x} - \hat{\underline{x}}\|^2 - 2\underbrace{\langle \underline{x} - \hat{\underline{x}}, \alpha\underline{\varphi} \rangle}_{\alpha^2} + \alpha^2 \underbrace{\|\underline{\varphi}\|^2}_{1} =$$
$$= \|\underline{x} - \hat{\underline{x}}\|^2 - \alpha^2 =$$
$$< \|\underline{x} - \hat{\underline{x}}\|^2$$

THAT IS IN CONTRADICTION WITH THE HYPOTHESES THAT $\hat{\underline{x}}$ IS A MINIMIZING VECTOR

NOW WE SHOW THAT IF $\underline{x} - \hat{\underline{x}}$ IS ORTHOGONAL TO S, THEN $\hat{\underline{x}}$ IS UNIQUE: $\forall \underline{s} \in S$

$$\|\underline{x} - \underline{s}\|^2 = \|\underline{x} - \hat{\underline{x}} + \hat{\underline{x}} - \underline{s}\|^2 = \|\underline{x} - \hat{\underline{x}}\|^2 + \|\hat{\underline{x}} - \underline{s}\|^2 > \|\underline{x} - \hat{\underline{x}}\|^2 \text{ IF AND ONLY IF } \underline{s} \neq \hat{\underline{x}}$$

N.B.: $\underline{s} \in S$ AND $\underline{x} - \hat{\underline{x}} \perp S$, $\hat{\underline{x}} - \underline{s} \in S$

Dim (4):

α SCALAR, $\underline{x}_1, \underline{x}_2 \in H$ BEST APPROXIMATED BY $\hat{\underline{x}}_1, \hat{\underline{x}}_2 \in S$.

ORTHOGONALITY IMPLIES $\underline{x}_1 - \hat{\underline{x}}_1 \in S^\perp$, $\underline{x}_2 - \hat{\underline{x}}_2 \in S^\perp$

S IS A SUBSPACE SO $\hat{\underline{x}}_1 + \hat{\underline{x}}_2 \in S$

$S^\perp$ IS A SUBSPACE SO $(\underline{x}_1 - \hat{\underline{x}}_1) + (\underline{x}_2 - \hat{\underline{x}}_2) \in S^\perp$

THEN $(\underline{x}_1 + \underline{x}_2) - (\hat{\underline{x}}_1 + \hat{\underline{x}}_2) \in S^\perp$

UNIQUENESS IMPLIES THAT $\hat{\underline{x}}_1 + \hat{\underline{x}}_2$ IS THE BEST APPROXIMATION OF $\underline{x}_1 + \underline{x}_2$ (ADDITIVITY)

$\alpha\hat{\underline{x}}_1 \in S$, $\alpha(\underline{x}_1 - \hat{\underline{x}}_1) \in S^\perp$, $\alpha\underline{x}_1 - \alpha\hat{\underline{x}}_1 \in S^\perp$. $\alpha\hat{\underline{x}}_1$ IS SO THE BEST APPROXIMATION OF $\alpha\underline{x}_1$ (SCALABILITY)

Dim (5):

CHECK THAT P LEAVES $P\underline{x}$ UNCHANGED: $P\underline{x} \in S$ AND $P\underline{u} = \underline{u}$ $\forall \underline{u} \in S$

$P\underline{x} \in S \rightarrow$ DEFINITION OF $\hat{\underline{x}}$

$P\underline{u} = \underline{u} \rightarrow \|\underline{u} - \hat{\underline{u}}\| \leq \|\underline{u} - \underline{s}\|$ $\forall \underline{s} \in S$

BY UNIQUENESS PROPERTY THERE CAN BE ONLY ONE $\hat{\underline{u}}$ AND IT IS $\hat{\underline{u}} = \underline{u}$ (NON NEGATIVE NORM)

Dim (6):

$\langle P\underline{x}, \underline{y} \rangle = \langle \underline{x}, P\underline{y} \rangle$ $\forall \underline{x}, \underline{y} \in H$

$\langle P\underline{x}, \underline{y} \rangle = \langle P\underline{x}, P\underline{y} + (\underline{y} - P\underline{y}) \rangle = \langle P\underline{x}, P\underline{y} \rangle + \underbrace{\langle P\underline{x}, \underline{y} - P\underline{y} \rangle}_{P\underline{x} \in S,\ \underline{y} - P\underline{y} \in S^\perp} = \langle P\underline{x}, P\underline{y} \rangle$

$\langle \underline{x}, P\underline{y} \rangle = \langle P\underline{x} + (\underline{x} - P\underline{x}), P\underline{y} \rangle = \langle P\underline{x}, P\underline{y} \rangle + \langle \underline{x} - P\underline{x}, P\underline{y} \rangle = \langle P\underline{x}, P\underline{y} \rangle$

PROJECTION OPERATOR:

(1) AN IDEMPOTENT OPERATOR P IS AN OPERATOR SUCH THAT $P^2 = P$

(2) A PROJECTION OPERATOR IS A BOUNDED LINEAR OPERATOR THAT IS IDEMPOTENT

(3) AN ORTHOGONAL PROJECTION OPERATOR IS A PROJECTION OPERATOR THAT IS SELF ADJOINT

(4) AN OBLIQUE OPERATOR PROJECTION IS A PROJECTION OPERATOR THAT IS NOT SELFADJOINT

ORTHOGONAL PROJECTION OPERATOR: A BOUNDED LINEAR OPERATOR P ON A HILBERT SPACE H SATISFIES

$$\boxed{\langle \underline{x} - P\underline{x}, P\underline{y} \rangle = 0} \quad \forall \underline{x}, \underline{y} \in H$$

IF AND ONLY IF P IS AN ORTHOGONAL PROJECTION OPERATOR

DIM:

$$0 = \langle \underline{x} - P\underline{x}, P\underline{y} \rangle \underset{\substack{\uparrow \\ \text{ADJOINT} \\ \text{PROPERTY}}}{=} \langle P^*(\underline{x} - P\underline{x}), \underline{y} \rangle = \langle P^*(I-P)\underline{x}, \underline{y} \rangle \quad \forall \underline{x}, \underline{y} \in H$$

THEN $P^*(I-P) = 0 \rightarrow P^* = P^*P$

NOW WE SHOW THAT THIS IS EQUIVALENT TO P BEING IDEMPOTENT AND SELF ADJOINT:

$$P = (P^*)^* = (P^*P)^* = P^*P = P^* \quad \text{(SELF ADJOINT)}$$

$$P^2 = P^*P = P^* \underset{\substack{\uparrow \\ \text{SELF ADJOINT}}}{=} P \quad \text{(IDEMPOTENT)}$$

PROJECTION OPERATORS, ADJOINT AND INVERSES: LET $A: H_0 \rightarrow H_1$ AND $B: H_1 \rightarrow H_0$ BE BOUNDED LINEAR OPERATORS. IF A IS THE LEFT INVERSE OF B THEN BA IS A PROJECTION OPERATOR. IF $B = A^*$ THEN $BA = A^*A$ IS AN ORTHOGONAL PROJECTION OPERATOR.

ORTHOGONAL PROJECTION VIA PSEUDOINVERSE: LET $A: H_0 \rightarrow H_1$ BE A BOUNDED LINEAR OPERATOR

(1) IF AA^* IS INVERTIBLE THEN: $B = A^*(AA^*)^{-1}$

IS THE PSEUDOINVERSE OF A, $BA = A^*(AA^*)^{-1}A$ IS THE ORTHOGONAL PROJECTION OPERATOR OVER THE RANGE OF A^*

(2) IF A^*A IS INVERTIBLE: $B = (A^*A)^{-1}A^*$

IS THE PSEUDOINVERSE OF A, $AB = A(A^*A)^{-1}A^*$ IS THE ORTHOGONAL PROJECTION OPERATOR ONTO THE RANGE OF A

EXAMPLE:

I SUBSET OF $\mathbb{Z}$. $\quad \mathbb{1}_I : \ell^2(\mathbb{Z}) \to \ell^2(\mathbb{Z})$

$$\underline{y} = \mathbb{1}_I \, \underline{x} \qquad y_k = \begin{cases} x_k & \text{FOR } k \in I \\ 0 & \text{OTHERWISE} \end{cases}$$

$\mathbb{1}_I$ IS AN ORTHOGONAL PROJECTION OPERATOR (IDEMPOTENT + SELF ADJOINT)

$\underline{x}^T = (\dots\, x_{-2}, x_{-1}, x_0, x_1, x_2, \dots)$

$\underline{y}^T = \mathbb{1}_I \, \underline{x}^T$, $I = \{-2, -1, 0, 1, 2\}$

LINEARITY: $\underline{y_1}^T = \mathbb{1}_I \, \underline{x}^{(1)T} = (x_{-2}^{(1)}, x_{-1}^{(1)}, x_0^{(1)}, x_1^{(1)}, x_2^{(1)})$

$\underline{y_2}^T = \mathbb{1}_I \, \underline{x}^{(2)T} = (x_{-2}^{(2)}, x_{-1}^{(2)}, x_0^{(2)}, x_1^{(2)}, x_2^{(2)})$

$\underline{y_1}^T + \underline{y_2}^T = \mathbb{1}_I (\underline{x}^{(1)T} + \underline{x}^{(2)T})$

IDEMPOTENCY: $\underline{z}^T = \mathbb{1}_I \, \underline{y}^T = \mathbb{1}_I (\mathbb{1}_I \underline{x}^T) = \mathbb{1}_I^2 \underline{x}^T = \mathbb{1}_I \, \underline{x}^T$

SELF ADJOINTNESS:

$\langle \mathbb{1}_I \underline{x}, \underline{y} \rangle = \langle \underline{x}, \mathbb{1}_I \underline{y} \rangle \quad \forall \underline{x}, \underline{y} \in \ell^2(\mathbb{Z})$

$\underline{y}$ IS A SEQUENCE WITH VALUES $\neq 0$ AT LOCATIONS $\in I$

$\underline{x_1}^T = (1, 2, 3, 4, 5, 6, 7, 8, 9, 10)$

$\underline{y_1}^T = (0, 0, 0, -1, -2, -3, -4, -5, -6, -7)$

$\mathbb{1}_I$ SETS THE FIRST 3 ELEMENTS TO $\emptyset$ AND REPLICATE THE OTHERS

$\mathbb{1}_I \underline{x_1}^T = (0, 0, 0, 4, 5, 6, 7, 8, 9, 10)$

$\mathbb{1}_I \underline{y_1}^T = (0, 0, 0, -1, -2, -3, -4, -5, -6, -7)$

$\langle \mathbb{1}_I \underline{x_1}, \underline{y_1} \rangle = \langle \underline{x_1}, \mathbb{1}_I \underline{y_1} \rangle$

EXAMPLE:

$x(t) = \cos\left(\frac{3}{2}\pi t\right) \in \mathcal{L}^2([0,1])$ (HILBERT SPACE)

WE WANT TO FIND THE DEGREE-1 POLYNOMIAL CLOSEST TO $x(t)$: WE SHOULD SOLVE

$$\min_{a_0, a_1} \int_0^1 \left| \cos\left(\tfrac{3}{2}\pi t\right) - (a_0 + a_1 t) \right|^2 dt \equiv \min_{a_0, a_1} \| e(t) \|^2$$

SO I SHOULD DERIVATE: $\frac{\partial}{\partial a_0} \|e(t)\|^2 = 0$, $\frac{\partial}{\partial a_1} \|e(t)\| = 0$

IT IS LONG... I CAN USE THEN THE PROJECTION THEOREM

$$x(t) - \hat{x}(t) = \cos\left(\tfrac{3}{2}\pi t\right) - (a_0 + a_1 t)$$

THIS QUANTITY (ERROR) HAS TO BE ORTHOGONAL TO THE ENTIRE SUBSPACE OF DEGREE 1 POLYNOMIALS

(DEGREE 1 POLYNOMIALS FORM A CLOSED SUBSPACE IN THE HILBERT SPACE)

$$\text{SPAN}\{\varphi_1(t), \varphi_2(t)\} = \{a_0 + a_1 t, \; a_0, a_1 \in \mathbb{R}\} = S \qquad \begin{cases} \varphi_1(t) = 1 & t \in [0,1] \\ \varphi_2(t) = t & t \in [0,1] \end{cases}$$

$$\begin{cases} \langle x(t) - \hat{x}(t), 1 \rangle = 0 = \int_0^1 \left(\cos\left(\tfrac{3}{2}\pi t\right) - (a_0 + a_1 t)\right) \cdot 1 \, dt = -\frac{2}{3\pi} - a_0 - \frac{1}{2} a_1 \\ \langle x(t) - \hat{x}(t), t \rangle = 0 = \int_0^1 \left(\cos\left(\tfrac{3}{2}\pi t\right) - (a_0 + a_1 t)\right) \cdot t \, dt = \frac{4 + 6\pi}{9\pi^2} - \frac{1}{2} a_0 - \frac{1}{3} a_1 \end{cases}$$

BY SOLVING THE SYSTEM I OBTAIN:

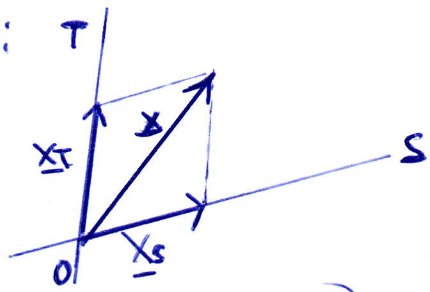
$$a_0 = \frac{8 + 4\pi}{3\pi^2}, \quad a_1 = -\frac{16 + 12\pi}{3\pi^2}$$

DIRECT SUM AND DECOMPOSITION: A VECTOR SPACE V IS A DIRECT SUM OF SUBSPACES S AND T, DENOTED $V = S \oplus T$, WHEN ANY NONZERO VECTOR $\underline{x} \in V$ CAN BE WRITTEN UNIQUELY AS $\underline{x} = \underline{x}_S + \underline{x}_T$ WHERE $\underline{x}_S \in S$, $\underline{x}_T \in T$.

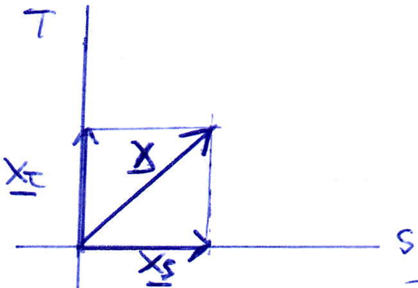
THE SUBSPACES S AND T FORM A DECOMPOSITION OF V AND THE VECTORS $\underline{x}_S$, $\underline{x}_T$ FORM A DECOMPOSITION OF $\underline{x}$.

WHEN S AND T ARE ORTHOGONAL THIS IS CALLED AN ORTHOGONAL DECOMPOSITION

EXAMPLE:



(DECOMPOSITION)



(ORTHOGONAL DECOMPOSITION)

BASES AND FRAMES

BASIS: THE SET OF VECTORS $\phi = \{\underline{\varphi}_k\}_{k \in K} \subset V$ WHERE K IS FINITE OR COUNTABLY INFINITE, IS CALLED A BASIS FOR A NORMED VECTOR SPACE V WHEN:

(1) IT IS COMPLETE IN V, THAT IS, $\forall \underline{x} \in V$ THERE IS A SEQUENCE $\alpha \in \mathbb{C}^{K}$ SUCH THAT:

$$\boxed{\underline{x} = \sum_{k \in K} \alpha_k \underline{\varphi}_k}$$

(2) $\forall \underline{x} \in V$ THE SEQUENCE α SATISFYING THE CONDITION (1) IS UNIQUE

N.B.: - WHEN K IS INFINITE THE SUM LACKS A SPECIFIC MEANING UNLESS AN ORDER OF SUMMATION IS SPECIFIED

- K MUST HAVE SINGLY-INFINITE ORDER FOR $K = \mathbb{Z}$
- THE BASIS IS CALLED UNCONDITIONAL WHEN THE SUM OF (1) CONVERGES TO $\underline{x}$ IN ONE ORDER OF SUMMATION AND IF AND ONLY IF IT CONVERGES TO $\underline{x}$ IN EVERY ORDER OF SUMMATION.
- WHEN K IS FINITE (INDEX SET) THEN $\underline{x}$ IS IN THE SPAN OF THE BASIS
- WHEN K IS INFINITE THE CLOSURE OF THE SPAN IS NEEDED TO ALLOW INFINITELY MANY TERMS ON THE LINEAR COMBINATION $V = \overline{\text{SPAN}}(\phi)$
- $(\alpha_k)_{k \in K}$ ARE CALLED THE EXPANSION COEFFICIENTS OF $\underline{x}$ WITH RESPECT TO ϕ
- THE INFINITE DIMENSIONAL HILBERT SPACES HAVE COUNTABLY-INFINITE BASES BECAUSE THEY ARE SEPARABLE
- FOR NORMED VECTOR SPACES THAT ARE NOT HILBERT SPACES, THE NORM AFFECTS WHETHER A PARTICULAR SET ϕ IS A BASIS.

RIESZ BASIS: THE SET OF VECTORS $\phi = \{\underline{\varphi}_k\}_{k \in K} \subset H$ WHERE K IS FINITE OR COUNTABLY INFINITE, IS CALLED A RIESZ BASIS WHEN, FOR HILBERT SPACE H:

(1) IT IS A BASIS FOR H

(2) $\exists$ STABILITY CONSTANTS $\lambda_{MIN}, \lambda_{MAX}$ SATISFYING $0 < \lambda_{MIN} \leq \lambda_{MAX} < \infty$ SUCH THAT, $\forall \underline{x} \in H$ THE EXPANSION OF $\underline{x}$ WITH RESPECT TO THE BASIS ϕ $\underline{x} = \sum_{k \in K} \alpha_k \underline{\varphi}_k$ SATISFIES:

$$\boxed{\lambda_{MIN} \|\underline{x}\|^2 \leq \sum_{k \in K} |\alpha_k|^2 \leq \lambda_{MAX} \|\underline{x}\|^2}$$

THE LARGEST SUCH λ_{MIN} AND THE SMALLEST SUCH λ_{MAX} ARE CALLED OPTIMAL STABILITY CONSTANTS OF ϕ.

NB.: IN AN INFINITE DIMENSIONAL VECTOR SPACE A BASIS CAN HAVE VERY LOW OR VERY HIGH COEFFICIENTS; THIS IS BAD FOR COMPUTATIONS. RIESZ BASES PUT A RESTRICTION THAT AVOID THIS PROBLEM.

BASIS SYNTHESIS OPERATOR: GIVEN A RIESZ BASIS $\{\underline{\varphi}_k\}_{k\in\mathcal{K}}$ FOR HILBERT SPACE H THE SYNTHESIS OPERATOR ASSOCIATED WITH IT IS

$$\boxed{\Phi : \ell^2(\mathcal{K}) \to H \quad \text{WITH} \quad \Phi\alpha = \sum_{k\in\mathcal{K}}{}' \alpha_k \underline{\varphi}_k}$$

N.B.: THE EXPANSION FORMULA CAN BE VIEWED AS MAPPING A COEFFICIENT SEQUENCE α TO A VECTOR $\underline{x}$. THE MAPPING IS LINEAR

N.B.: THE NORM OF A SYNTHESIS OPERATOR IS THE SUPREMUM OF THE RATIO $\|\Phi\alpha\|/\|\alpha\|$ FOR NON-ZERO $\alpha\in\ell^2(\mathcal{K})$. SO $\lambda_{min}\|\Phi\alpha\|^2 \leq \|\alpha\|^2$ AND THE NORM OF THIS LINEAR OPERATOR IS AT MOST $1/\sqrt{\lambda_{min}}$. THE NORM IS LIMITED SO THE OPERATOR Φ IS BOUNDED

N.B.: THE ADJOINT OF Φ MAPS FROM H TO A SEQUENCE IN $\ell^2(\mathcal{K})$. IT IS THE ANALYSIS OPERATOR

$\langle \Phi\alpha, \underline{y}\rangle = \langle \sum' \alpha_k \underline{\varphi}_k, \underline{y}\rangle = \sum_{k\in\mathcal{K}}{}' \alpha_k \langle \underline{\varphi}_k, \underline{y}\rangle = \sum_{k\in\mathcal{K}}{}' \alpha_k \langle \underline{y}, \underline{\varphi}_k\rangle^*$, $\alpha\in\ell^2(\mathcal{K})$, $\underline{y}\in H$

BASIS ANALYSIS OPERATOR: GIVEN A RIESZ BASIS $\{\underline{\varphi}_k\}_{k\in\mathcal{K}}$ FOR HILBERT SPACE H, THE ANALYSIS OPERATOR ASSOCIATED WITH IT IS

$$\boxed{\Phi^* : H \to \ell^2(\mathcal{K}) \quad \text{WITH} \quad (\Phi^*\underline{x})_k = \langle \underline{x}, \underline{\varphi}_k\rangle, \ k\in\mathcal{K}}$$

N.B.: $\langle \Phi\alpha, \underline{y}\rangle_H = \langle \alpha, \Phi^*\underline{y}\rangle_{\ell^2}$

N.B.: THE NORM OF THE ANALYSIS OPERATOR IS ALSO AT MOST $1/\sqrt{\lambda_{min}}$, IN FACT $\|A\| = \|A^*\|$ FOR ALL BOUNDED LINEAR OPERATORS

ORTHONORMAL BASES: THE SET OF VECTORS $\Phi = \{\underline{\varphi}_k\}_{k\in\mathcal{K}} \subset H$, WHERE $\mathcal{K}$ IS FINITE OR COUNTABLY INFINITE, IS CALLED AN ORTHONORMAL BASIS FOR THE HILBERT SPACE H WHEN

(1) IT IS A BASIS FOR H

(2) IT IS ORTHONORMAL: $\boxed{\langle \underline{\varphi}_i, \underline{\varphi}_k\rangle = \delta_{i-k} \quad \forall i,k\in\mathcal{K}}$

N.B.: ORTHONORMALITY IMPLIES UNIQUENESS OF EXPANSIONS SO WE COULD SAY THAT A SET $\Phi = \{\underline{\varphi}_k\}_{k\in\mathcal{K}} \subset H$ THAT SATISFIES (2) IS AN ORTHONORMAL BASIS WHENEVER IT IS COMPLETE THAT IS $\overline{\text{SPAN}}(\Phi) = H$.

EXAMPLE: RIESZ BASES IN $\mathbb{R}^2$

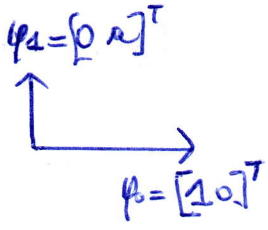
ANY TWO VECTORS φ_0, φ_1 ARE A BASIS FOR $\mathbb{R}^2$ AS LONG AS THERE IS NO SCALAR α SUCH THAT $\varphi_1 = \alpha\varphi_0$. WE FIX $\boxed{\varphi_0 = e_0 = [1,0]}$ AND WE VARY φ_1:

(1) $\boxed{\varphi_1 = a\, e_1 = [0\ a]^T}$ $a \in (0, \infty)$, THE UNIQUE EXPANSION OF $x = [x_0\ x_1]^T$ IS:

$$x = x_0 \varphi_0 + \left(\frac{x_1}{a}\right)\varphi_1 = \alpha_0 \varphi_0 + \alpha_1 \varphi_1$$

$$\lambda_{MIN} = \left(\frac{\sum_i |\alpha|^2}{\|x\|^2}\right) = \inf_{x \in \mathbb{R}^2} \frac{x_0^2 + (x_1/a)^2}{x_0^2 + x_1^2} = \begin{cases} 1 & a \in (0,1] \\ 1/a^2 & a \in (1,\infty) \end{cases}$$

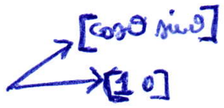
$$\lambda_{MAX} = \sup_{x \in \mathbb{R}^2} \frac{x_0^2 + (x_1/a)^2}{x_0^2 + x_1^2} = \begin{cases} 1/a^2 & a \in (0,1] \\ 1 & a \in (1,\infty) \end{cases}$$



N.B.: TO FIND THESE VALUES I PUT $\partial/\partial x_0 = 0$ AND $\partial/\partial x_1 = 0$ $\left(\frac{\partial f}{\partial g} = \frac{f'g - fg'}{|g|^2}\right)$

N.B.: BY MAKING a VERY LARGE THE BASIS BECOMES ILL-CONDITIONED THAT IS THERE ARE NONZERO VECTORS x WITH VERY SMALL EXPANSION COEFFICIENTS.
THE SAME HAPPENS FOR a VERY SMALL

(2) $\varphi_1 = [\cos\theta\ \sin\theta]^T$ $\theta \in (0, \pi/2]$



, $\varphi_0 = [1, 0]^T$

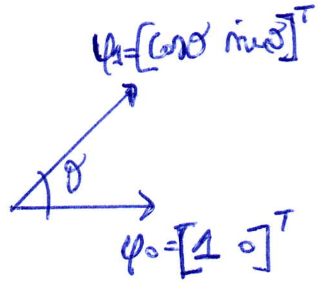
THE EXPANSION IS: $x = \alpha_0 \varphi_0 + \alpha_1 \varphi_1 = \alpha_0 [1\ 0] + \alpha_1 [\cos\theta\ \sin\theta]$

COEFFICIENTS MUST SATISFY:

$$\begin{cases} \alpha_0 + \alpha_1 \cos\theta = x_0 \\ \alpha_1 \sin\theta = x_1 \end{cases} \rightarrow \begin{cases} \alpha_1 = x_1 / \sin\theta \\ \alpha_0 = x_0 - x_1 \cos\theta / \sin\theta = x_0 - x_1 \operatorname{cotg}\theta \end{cases}$$

SO:

$$x = (x_0 - x_1 \operatorname{cotg}\theta)\varphi_0 + \left(\frac{x_1}{\sin\theta}\right)\varphi_1$$



$$\lambda_{MIN/MAX} = \inf/\sup\left(\frac{\sum_i |\alpha|^2}{\|x\|^2}\right) = \frac{(x_0 - x_1 \operatorname{cotg}\theta)^2 + (x_1/\sin\theta)^2}{(x_0)^2 + (x_1)^2}$$

I TAKE THE DERIVATIVE EQUAL TO $\emptyset$ AND I FIND:

$\lambda_{MIN} = 0$
$\lambda_{MAX} = \infty$

NUMERICAL CONDITIONING IS VERY GOOD (IDEAL) FOR $\theta = \pi/2$ (IT BECOMES A STANDARD BASIS) WHILE IT IS EXTREMELY POOR FOR SMALL θ.

ANY BASIS FOR A FINITE DIMENSIONAL HILBERT SPACE IS A RIESZ BASIS

EXAMPLE: NOT RIESZ BASIS

$\underline{\varphi_k} = \frac{1}{|k|+1} \underline{e_k}$, $k \in \mathbb{Z}$ THIS IS A SCALED VERSION OF THE STANDARD BASIS IN $\ell^2(\mathbb{Z})$

- $\|\underline{\varphi_k}\| / \|\underline{\varphi_0}\|$ IS UNBOUNDED, IT IS LIKE TO HAVE $\alpha \to 0$ OR $\lambda \to \infty$ IN THE PREVIOUS EXAMPLE
- $\Phi = \{\underline{\varphi_k}\}_{k \in \mathbb{Z}}$ IS A BASIS FOR $\ell^2(\mathbb{Z})$ BUT IT IS NOT A RIESZ BASIS

PROOF: SUPPOSE THAT $\exists$ FINITE λ_{MAX} SUCH THAT THE RIESZ BASIS DEFINITION IS SATISFIED $\forall \underline{x} \in \ell^2(\mathbb{Z})$. WE CAN TAKE $M > \sqrt{\lambda_{MAX}}$ AND $\underline{x} \in \ell^2(\mathbb{Z})$ BE THE SEQUENCE OF ALL 0s EXCEPT FOR 1 IN THE M^{TH} POSITION.

THE UNIQUE REPRESENTATION OF x USING Φ IS $\underline{x} = (|M|+1)\underline{\varphi_M}$

THEN THE COEFFICIENTS ARE $\alpha_k = \begin{cases} |M|+1 & k = M \\ 0 & \text{OTHERWISE} \end{cases}$

THIS IS IN CONTRADICTION WITH THE INEQUALITY $(|M|+1)^2 \leq \lambda_{MAX}$

ORTHONORMAL BASIS EXPANSION: LET $\Phi = \{\varphi_k\}_{k \in K}$ BE AN ORTHONORMAL BASIS FOR HILBERT SPACE H. THE UNIQUE EXPANSION WITH RESPECT TO Φ OF ANY $\underline{x}$ IN H HAS EXPANSION COEFFICIENTS

$$\boxed{\alpha_k = \langle \underline{x}, \varphi_k \rangle \quad \text{for } k \in K} \qquad \text{OR} \qquad \boxed{\underline{\alpha} = \Phi^* \underline{x}}$$

SYNTHESIS WITH THESE COEFFICIENTS YIELDS:

$$\boxed{\underline{x} = \sum_{k \in K} \langle \underline{x}, \varphi_k \rangle \varphi_k = \Phi \alpha = \Phi \Phi^* \underline{x}}$$

DIM.:

$$\langle \underline{x}, \varphi_k \rangle = \langle \sum_{i \in K} \alpha_i \varphi_i, \varphi_k \rangle = \sum_{i \in K} \alpha_i \langle \varphi_i, \varphi_k \rangle = \sum_{i \in K} \alpha_i \delta_{i-k} = \alpha_k$$

PARSEVAL'S EQUALITY: LET $\Phi = \{\varphi_k\}_{k \in K}$ BE AN ORTHONORMAL BASIS FOR HILBERT SPACE H. EXPANSION WITH COEFFICIENTS SATISFIES PARSEVAL'S EQUALITY:

$$\boxed{\|\underline{x}\|^2 = \sum_{k \in K} |\langle \underline{x}, \varphi_k \rangle|^2 = \|\Phi^* \underline{x}\|^2 = \|\underline{\alpha}\|^2}$$

AND THE GENERALIZED PARSEVAL'S EQUALITY:

$$\boxed{\langle \underline{x}, \underline{y} \rangle = \sum_{k \in K} \langle \underline{x}, \varphi_k \rangle \langle \underline{y}, \varphi_k \rangle^* = \langle \Phi^* \underline{x}, \Phi^* \underline{y} \rangle = \langle \underline{\alpha}, \underline{\beta} \rangle}$$

DIM.:

$$\langle \underline{x}, \underline{y} \rangle = \left\langle \sum_{k \in K} \langle \underline{x}, \varphi_k \rangle \varphi_k, \underline{y} \right\rangle = \sum_{k \in K} \langle \underline{x}, \varphi_k \rangle \langle \varphi_k, \underline{y} \rangle = \sum_{k \in K} \langle \underline{x}, \varphi_k \rangle \langle \underline{y}, \varphi_k \rangle^*$$

N.B.: PARSEVAL'S EQUALITY IS IMPORTANT BECAUSE IT TURNS THE ABSTRACT INNER PRODUCT COMPUTATION INTO A COMPUTATION WITH SEQUENCES.

EXAMPLE: $\underline{x} = \sum_{k \in K} \alpha_k \varphi_k \quad \underline{y} = \sum_{k \in K} \beta_k \varphi_k \quad \langle \underline{x}, \underline{y} \rangle_H = \langle \underline{\alpha}, \underline{\beta} \rangle_{\ell^2(K)} = \sum_{k \in K} \alpha_k \beta_k^*$

SO THE FINAL COMPUTATION IS AN $\ell^2(K)$ INNER PRODUCT, EVEN IF THE FIRST INNER PRODUCT IS IN AN ARBITRARY HILBERT SPACE.

ORTHOGONAL PROJECTION ONTO A SUBSPACE: GIVEN AN ORTHONORMAL SET $\Phi = \{\varphi_k\}_{k\in K} \subset H$

$$\boxed{P_I \underline{x} = \sum_{k\in I} \langle \underline{x}, \underline{\varphi_k} \rangle \underline{\varphi_k} = \Phi_I \Phi_I^* \underline{x}}$$

IS THE ORTHOGONAL PROJECTION OF $\underline{x}$ ONTO $S_I = \overline{\text{SPAN}}(\{\varphi_k\}_{k\in I})$

DIM:

P_I IS A LINEAR OPERATOR ON H. TO PROVE THAT IS AN ORTHOGONAL PROJECTION OPERATOR WE MUST SHOW THAT IT IS IDEMPOTENT AND SELF ADJOINT.

- $P_I(P_I \underline{x}) = \Phi_I \Phi_I^* (\Phi_I \Phi_I^* \underline{x}) = \Phi_I (\Phi_I^* \Phi_I) \Phi_I^* \underline{x} = \Phi_I \Phi_I^* \underline{x} = P_I \underline{x}$
- $P_I^* = (\Phi_I \Phi_I^*)^* = (\Phi_I^*)^* \Phi_I^* = \Phi_I \Phi_I^* = P_I$

ORTHOGONAL DECOMPOSITION: AN ORTHONORMAL BASIS INDUCES AN ORTHOGONAL DECOMPOSITION

$$H = \bigoplus_{k\in K} S_{\{k\}} \quad , \quad S_{\{k\}} = \text{SPAN}(\varphi_k)$$

BEST APPROXIMATION: $\hat{\underline{x}}^{(k)}$ BEST APPROXIMATION OF $\underline{x}$ IN THE SUBSPACE SPANNED BY THE ORTHONORMAL SET $\{\varphi_0, \varphi_1, \dots, \varphi_{k-1}\}$. $\hat{\underline{x}}^{(0)} = 0$ AND $\hat{\underline{x}}^{(k+1)} = \hat{\underline{x}}^{(k)} + \langle \underline{x}, \varphi_k \rangle \varphi_k \quad k = 0, 1, \dots$

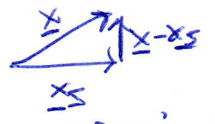
THE NEW BEST APPROXIMATION IS THE SUM OF THE PREVIOUS BEST APPROX. PLUS THE ORTHOGONAL PROJECTION ONTO THE SPAN OF THE ADDED VECTOR φ_k.

BESSEL'S INEQUALITY: GIVEN AN ORTHONORMAL SET $\Phi = \{\varphi_k\}_{k\in I}$ IN A HILBERT SPACE H, BESSEL'S INEQUALITY HOLDS:

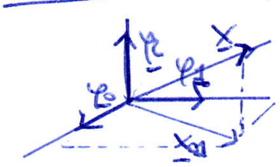
$$\boxed{\|\underline{x}\|^2 \geq \sum_{k\in I} |\langle \underline{x}, \varphi_k \rangle|^2 = \|\Phi_I^* \underline{x}\|^2}$$

THE EQUALITY FOR EVERY x IN H IMPLIES THAT THE SET Φ IS COMPLETE IN H, SO THE ORTHONORMAL SET IS AN ORTHONORMAL BASIS FOR H, SO IT'S PARSEVAL'S EQUALITY.

DIM:
$$\|\underline{x}\|^2 = \|\underline{x}_S\|^2 + \|\underline{x} - \underline{x}_S\|^2 \geq \|\underline{x}_S\|^2 = \|\Phi_I \Phi_I^* \underline{x}\|^2 = \|\Phi_I^* \underline{x}\|^2 = \sum_{k\in I} |\langle x, \varphi_k \rangle|^2$$



EXAMPLE: $\varphi_0 = [1\,0\,0]^T \quad \varphi_1 = [0\,1\,0]^T$ TWO ELEMENTS OF THE STANDARD BASIS IN $\mathbb{R}^3$, THEY ARE ORTHONORMAL



$\|\underline{x}\|^2 \geq \|\underline{x}_{01}\|^2 = |\langle x, \varphi_0 \rangle|^2 + |\langle x, \varphi_1 \rangle|^2$

ADDING $\varphi_2 = [0\,0\,1]^T$ TO THE SET GIVES AN ORTHONORMAL BASIS AND IT'S VERIFIED THE PARSEVAL'S EQUALITY.

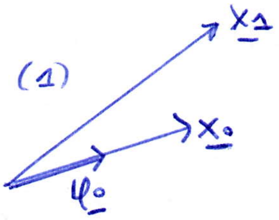
GRAM SCHMIDT ORTHOGONALIZATION:

(1) $\varphi_0 = \dfrac{x_0}{\|x_0\|}$

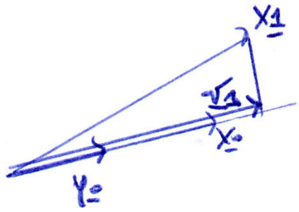
(2) $v_k = \sum_{i=0}^{k-1} \langle x_k, \varphi_i \rangle \varphi_i$

(3) $\varphi_k = \dfrac{x_k - v_k}{\|x_k - v_k\|}$

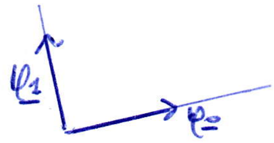
(1)



(2)



(3)



BIORTHOGONAL PAIR OF BASES: THE SETS OF VECTORS $\Phi = \{\varphi_k\}_{k \in K} \subset H$ AND $\tilde{\Phi} = \{\tilde{\varphi}_k\}_{k \in K} \subset H$, WHERE K IS FINITE OR COUNTABLY INFINITE, ARE CALLED A BIORTHOGONAL PAIR OF BASES FOR HILBERT SPACE H WHEN

(1) EACH IS A BASIS FOR H

(2) THEY ARE BIORTHOGONAL, MEANING $\langle \varphi_i, \tilde{\varphi}_k \rangle = \delta_{i-k} \quad \forall i, k \in K$

BIORTHOGONAL BASIS EXPANSION: LET $\Phi = \{\varphi_k\}_{k \in K}$ AND $\tilde{\Phi} = \{\tilde{\varphi}_k\}_{k \in K}$ BE A BIORTHOGONAL PAIR OF BASES FOR HILBERT SPACE H. THE UNIQUE EXPANSION WITH RESPECT TO THE BASIS Φ OF ANY x IN H HAS EXPANSION COEFFICIENTS.

$$\boxed{\alpha_k = \langle x, \tilde{\varphi}_k \rangle \quad \text{FOR } k \in K \quad \text{OR} \quad \alpha = \tilde{\Phi}^* x}$$

SYNTHESIS WITH THESE COEFFICIENTS YIELDS $\boxed{x = \sum_{k \in K} \langle x, \tilde{\varphi}_k \rangle \varphi_k = \Phi \alpha = \Phi \tilde{\Phi}^* x}$

Dim:

$$\langle x, \tilde{\varphi}_k \rangle = \left\langle \sum_{i \in K} \alpha_i \varphi_i, \tilde{\varphi}_k \right\rangle = \sum_{i \in K} \alpha_i \langle \varphi_i, \tilde{\varphi}_k \rangle = \sum_{i \in K} \alpha_i \delta_{i-k} = \alpha_k$$

PARSEVAL'S EQUALITIES FOR BIORTHOGONAL PAIRS OF BASES: LET $\Phi = \{\varphi_k\}_{k\in\mathcal{K}}$ AND $\tilde{\Phi} = \{\tilde{\varphi}_k\}_{k\in\mathcal{K}}$ BE A BIORTHOGONAL PAIR OF BASES FOR HILBERT SPACE H. EXPANSION WITH RESPECT TO THE BASES Φ AND $\tilde{\Phi}$ SATISFIES

$$\boxed{\|\underline{x}\|^2 = \sum_{k\in\mathcal{K}} \langle x, \varphi_k\rangle \langle \underline{x}, \tilde{\varphi}_k\rangle^* = \langle \Phi^* \underline{x}, \tilde{\Phi}^* \underline{x}\rangle = \langle \underline{\tilde{\alpha}}, \underline{\alpha}\rangle}$$

MORE GENERALLY:

$$\boxed{\langle \underline{x}, \underline{y}\rangle = \sum_{k\in\mathcal{K}} \langle x, \varphi_k\rangle \langle \underline{y}, \tilde{\varphi}_k\rangle^* = \langle \Phi^* \underline{x}, \tilde{\Phi}^* \underline{y}\rangle = \langle \underline{\tilde{\alpha}}, \underline{\beta}\rangle}$$

DIM:
$$\langle \Phi^* \underline{x}, \tilde{\Phi}^* \underline{y}\rangle = \langle \underline{x}, \Phi\tilde{\Phi}^* \underline{y}\rangle = \langle \underline{x}, \underline{y}\rangle$$

GRAM MATRIX: OFTEN ONE WANTS TO HAVE ALL EXPANSIONS TO BE WITH RESPECT TO ONE BASIS OF THE PAIR. THE OTHER BASIS OF THE PAIR HELPS COMPUTING EXPANSION COEFFICIENTS.

IF $\underline{x} = \Phi\underline{\alpha}$ AND $\underline{y} = \Phi\underline{\beta}$ THEN $\langle \underline{x}, \underline{y}\rangle = \langle \Phi\underline{\alpha}, \Phi\underline{\beta}\rangle = \langle \Phi^*\Phi \underline{\alpha}, \underline{\beta}\rangle = \langle G\underline{\alpha}, \underline{\beta}\rangle$

$G = \Phi^*\Phi \equiv$ GRAM MATRIX OR GRAMIAN

$G_{ik} = \langle \varphi_k, \varphi_i\rangle \quad \forall i, k \in \mathcal{K}$

INVERSE SYNTHESIS AND ANALYSIS: $\tilde{\Phi}^*$ IS A RIGHT INVERSE OF Φ, BUT ALSO

$\tilde{\Phi}^*\Phi = I$ ON $\ell^2(\mathcal{K})$ MAKING $\tilde{\Phi}^*$ A LEFT INVERSE OF Φ. SO $\tilde{\Phi}^*$ IS THE UNIQUE INVERSE OF Φ.

DIM:
$$\tilde{\Phi}^*\Phi\underline{\alpha} = \tilde{\Phi}^* \sum_{i\in\mathcal{K}} \alpha_i\varphi_i = \left(\langle \sum_{i\in\mathcal{K}} \alpha_i\varphi_i, \tilde{\varphi}_k\rangle\right)_{k\in\mathcal{K}} = \left(\sum_{i\in\mathcal{K}} \alpha_i \langle \varphi_i, \tilde{\varphi}_k\rangle\right)_{k\in\mathcal{K}} =$$

$$= \left(\sum_{i\in\mathcal{K}} \alpha_i \delta_{i-k}\right)_{k\in\mathcal{K}} = (\alpha_k)_{k\in\mathcal{K}} = \alpha$$

SO FOR A BIORTHOGONAL PAIR OF BASES: $\tilde{\Phi}^* = \Phi^{-1}$

AND FOR A PAIR OF RIESZ BASES: $\tilde{\lambda}_{MAX} = 1/\lambda_{MIN}$, $\tilde{\lambda}_{MIN} = 1/\lambda_{MAX}$

DUAL BASIS: LET $\Phi = \{\varphi_k\}_{k \in K}$ BE A RIESZ BASIS FOR HILBERT SPACE H AND LET $A: \ell^2(K) \to \ell^2(K)$ BE THE INVERSE OF THE GRAM MATRIX OF Φ, THAT IS, $A = (\Phi^* \Phi)^{-1}$. THEN THE SET $\tilde{\Phi} = \{\tilde{\varphi}_k\}_{k \in K}$ DEFINED VIA

$$\boxed{\tilde{\varphi}_k = \sum_{\ell \in K} a_{\ell,k} \varphi_\ell \qquad \forall k \in K}$$

TOGETHER WITH Φ FORMS A BIORTHOGONAL PAIR OF BASES FOR H. THE SYNTHESIS OPERATOR FOR THIS BASIS IS GIVEN BY $\boxed{\tilde{\Phi} = \Phi A = \Phi(\Phi^*\Phi)^{-1}}$

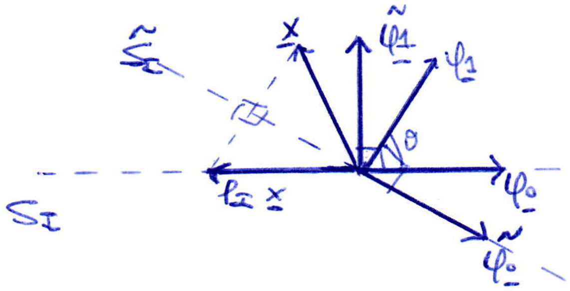
THE PSEUDOINVERSE OF Φ^*.

OBLIQUE PROJECTION: GIVEN SETS $\Phi_I = \{\varphi_k\}_{k \in I} \subset H$ AND $\tilde{\Phi}_I = \{\tilde{\varphi}_k\}_{k \in I} \subset H$ SATISFYING $\langle \varphi_n, \tilde{\varphi}_k \rangle = \delta_{n-k} \qquad \forall n,k \in I$

FOR ANY x IN H

$$\boxed{P_I x = \sum_{k \in I} \langle x, \tilde{\varphi}_k \rangle \varphi_k = \Phi_I \tilde{\Phi}_I^* x}$$

IS A PROJECTION OF x ONTO $S_I = \overline{\text{SPAN}}(\{\varphi_k\}_{k \in I})$. THE RESIDUAL SATISFIES $x - P_I x \perp \tilde{S}_I$, WHERE $\tilde{S}_I = \overline{\text{SPAN}}(\{\tilde{\varphi}_k\}_{k \in I})$



NORMAL EQUATIONS: GIVEN A VECTOR $\underline{x}$ AND A RIESZ BASIS $\{\underline{\varphi_k}\}_{k\in I}$ FOR A CLOSED SUBSPACE S IN A SEPARABLE HILBERT SPACE H, THE VECTOR CLOSEST TO $\underline{x}$ IN S IS:

$$\hat{\underline{x}} = \sum_{k\in I} \beta_k \underline{\varphi_k} = \Phi \underline{\beta}$$

WHERE $\underline{\beta}$ IS THE UNIQUE SOLUTION TO THE SYSTEM OF EQUATIONS:

$$\sum_{k\in I} \beta_k \langle \underline{\varphi_k}, \underline{\varphi_n} \rangle = \langle \underline{x}, \underline{\varphi_n} \rangle \qquad \forall n \in I$$

OR

$$\Phi^* \Phi \underline{\beta} = \Phi^* \underline{x}$$

FRAME: THE SET OF VECTORS $\phi = \{\underline{\varphi}_k\}_{k\in J} \subset H$, WHERE J IS FINITE OR COUNTABLY INFINITE, IS CALLED A FRAME FOR HILBERT SPACE H WHEN THERE EXIST POSITIVE REAL NUMBERS λ_{MAX} AND λ_{MIN} SUCH THAT

$$\boxed{\lambda_{MIN} \|\underline{x}\|^2 \leq \sum_{k\in J} |\langle \underline{x}, \underline{\varphi}_k\rangle|^2 \leq \lambda_{MAX} \|\underline{x}\|^2 \qquad \forall \underline{x} \in H}$$

THE CONSTANTS λ_{MIN} AND λ_{MAX} ARE CALLED FRAME BOUNDS. THE LARGEST SUCH λ_{MIN} AND SMALLEST SUCH λ_{MAX} ARE CALLED OPTIMAL FRAME BOUNDS OF ϕ

NOTES: - A FRAME IS ALSO CALLED RIESZ SEQUENCE
- BOTH BASES AND FRAMES CAN BE USED FOR BOTH ANALYSIS AND SYNTHESIS
- ANY RIESZ BASIS IS A FRAME

OPERATORS ASSOCIATED WITH FRAMES:

SYNTHESIS OPERATOR ASSOCIATED WITH $\{\underline{\varphi}_k\}_{k\in J}$:

$$\phi : \ell^2(J) \to H \qquad \boxed{\phi \underline{\alpha} = \sum_{k\in J} \alpha_k \underline{\varphi}_k}$$

ANALYSIS OPERATOR ASSOCIATED WITH $\{\underline{\varphi}_k\}_{k\in J}$:

$$\phi^* : H \to \ell^2(J) \qquad \boxed{(\phi^* \underline{x})_k = \langle \underline{x}, \underline{\varphi}_k\rangle \qquad k \in J}$$

THEY ARE LINEAR WITH FINITE NORM, SO THE OPERATOR IS BOUNDED, THEIR NORM IS THE SAME

$$\lambda_{MIN} I \leq \phi\phi^* \leq \lambda_{MAX} I$$

TIGHT FRAME: THE FRAME $\Phi = \{\varphi_k\}_{k \in J} \subset H$, WHERE J IS FINITE OR COUNTABLY INFINITE, IS CALLED A TIGHT FRAME, OR A λ-TIGHT FRAME, FOR HILBERT SPACE H WHEN ITS OPTIMAL FRAME BOUNDS ARE EQUAL, $\lambda_{MIN} = \lambda_{MAX} = \lambda$

IN THIS CASE $\Phi\Phi^* = \lambda I$, A TIGHT FRAME IS A COUNTERPART OF AN ORTHONORMAL BASIS

NORMALIZATION: WE CAN NORMALIZE ANY λ-TIGHT FRAME BY PULLING $1/\sqrt{\lambda}$ INTO THE SUM, TO YIELD A 1-TIGHT FRAME

$$\sum_k |\langle \underline{x}, \lambda^{-1/2} \underline{\tilde{\varphi}_k} \rangle|^2 = \sum_k |\langle \underline{x}, \underline{\tilde{\varphi}_k'} \rangle|^2 = \|\underline{x}\|^2$$

N.B.:
- THANKS TO THE NORMALIZATION WE CAN ASSOCIATE A 1-TIGHT FRAME TO ANY TIGHT FRAME
- ORTHONORMAL BASES ARE 1-TIGHT FRAMES WITH ALL UNIT-NORM VECTORS.
- IN GENERAL THE VECTORS IN A 1-TIGHT FRAME DON'T HAVE UNIT NORMS OR EVEN EQUAL NORMS

1-TIGHT FRAME EXPANSION: LET $\Phi = \{\varphi_k\}_{k \in J}$ BE A 1-TIGHT FRAME FOR HILBERT SPACE H. ANALYSIS OF ANY $\underline{x}$ IN H GIVES EXPANSION COEFFICIENTS IN $\ell^2(J)$

$$\boxed{\alpha_k = \langle \underline{x}, \varphi_k \rangle \quad \text{FOR } k \in J} \quad \text{OR} \quad \boxed{\underline{\alpha} = \Phi^* \underline{x}}$$

SYNTHESIS WITH THESE COEFFICIENTS YIELDS:

$$\boxed{\underline{x} = \sum_{k \in J} \langle \underline{x}, \varphi_k \rangle \varphi_k = \Phi \underline{\alpha} = \Phi\Phi^* \underline{x}}$$

N.B.: WITH ORTHONORMAL BASIS THE EXPANSION IS UNIQUE, IN 1-TIGHT FRAME CASE IT IS GENERALLY NOT

PARSEVAL'S EQUALITY FOR 1-TIGHT FRAMES: LET $\Phi = \{\varphi_k\}_{k \in J}$ BE A 1-TIGHT FRAME FOR HILBERT SPACE H. EXPANSION WITH COEFFICIENTS SATISFIES

$$\boxed{\|\underline{x}\|^2 = \sum_{k \in J} |\langle \underline{x}, \varphi_k \rangle|^2 = \|\Phi^* \underline{x}\|^2 = \|\underline{\alpha}\|^2}$$

MORE GENERALLY

$$\boxed{\langle \underline{x}, \underline{y} \rangle = \sum_{k \in J} \langle \underline{x}, \varphi_k \rangle \langle \underline{y}, \varphi_k \rangle^* = \langle \Phi^* \underline{x}, \Phi^* \underline{y} \rangle = \langle \underline{\alpha}, \underline{\beta} \rangle}$$

N.B.:
- THE 1-TIGHT FRAMES ARE CALLED PARSEVAL TIGHT FRAMES
- WE OFTEN WORK WITH NON TIGHT FRAMES (TIGHT FRAMES OF UNIT NORM VECTOR) THUS PRESERVATION OF THE NORM IS REPLACED BY A CONSTANT SCALING

DUAL PAIR OF FRAMES: THE SET OF VECTORS $\Phi = \{\varphi_k\}_{k\in J} \subset H$ AND $\tilde{\Phi} = \{\tilde{\varphi}_k\}_{k \in J} \subset H$, WHERE J IS FINITE OR COUNTABLY INFINITE, ARE CALLED A DUAL PAIR OF FRAMES FOR HILBERT SPACE H WHEN

(1) EACH IS A FRAME FOR H

(2) FOR ANY x IN H

$$\boxed{x = \sum_{k\in J} \langle x, \tilde{\varphi}_k \rangle \varphi_k = \Phi \tilde{\Phi}^* x}$$

INNER PRODUCT: $\Phi \equiv$ FRAME FOR H, $x = \Phi\alpha$, $y = \Phi\beta$

$$\langle x, y\rangle = \langle \Phi\alpha, \Phi\beta\rangle = \langle G\alpha, \beta\rangle = \beta^* G \alpha, \quad G = \Phi^*\Phi = \text{GRAM MATRIX}$$

IN CASE OF FRAMES THE MATRIX G IS NOT NECESSARILY INVERTIBLE, G IS A BOUNDED LINEAR OPERATOR IF AND ONLY IF THE FRAME IS A RIESZ BASIS.

INVERSE: Φ AND $\tilde{\Phi}^* \equiv$ DUAL PAIR OF FRAMES, SYNTHESIS OPERATOR Φ IS A LEFT INVERSE OF ANALYSIS OPERATOR $\tilde{\Phi}^*$. WITH 1-TIGHT FRAMES Φ IS GENERALLY NOT A RIGHT INVERSE (SO NOT AN INVERSE) OF $\tilde{\Phi}^*$ BECAUSE $\tilde{\Phi}^*\Phi \neq I$.

IN A DUAL PAIR OF FRAMES THE ROLES OF THE TWO FRAMES CAN BE REVERSED SO

$$x = \tilde{\Phi}\Phi^* x$$

$\tilde{\Phi}$ = SYNTHESIS OPERATOR IS THE LEFT INVERSE OF Φ^* (ANALYSIS OPERATOR)

OBLIQUE PROJECTION: Φ AND $\tilde{\Phi}^*$ DUAL PAIR OF FRAMES, $P = \tilde{\Phi}^*\Phi$ IS A PROJECTION OPERATOR

IDEMPOTENCY: $P^2 = (\tilde{\Phi}^*\Phi)(\tilde{\Phi}^*\Phi) = \tilde{\Phi}^*\underbrace{(\Phi\tilde{\Phi}^*)}_{\Phi \text{ IS THE LEFT INVERSE OF } \tilde{\Phi}^*}\Phi = \tilde{\Phi}^* I \Phi = \tilde{\Phi}^*\Phi = P$

CANONICAL DUAL FRAME: GIVEN ONE FRAME Φ THERE ARE ∞ MANY FRAMES $\tilde{\Phi}$ THAT COMPLETE A DUAL PAIR WITH Φ. BUT THERE IS ONLY ONE CANONICAL DUAL FRAME THAT LEADS TO AN ORTHOGONAL PROJECTION OPERATOR ON $\ell^2(J)$.

IF P IS SELF-ADJOINT IT IS AN ORTHOGONAL PROJECTION OPERATOR.

$$\tilde{\Phi} = (\Phi\Phi^*)^{-1}\Phi$$

IN FACT $P = \tilde{\Phi}^*\Phi = ((\Phi\Phi^*)^{-1}\Phi)^*\Phi = \Phi^*(\Phi\Phi^*)^{-1}\Phi$

HOW TO SELECT A GOOD BASIS/FRAME?

GOOD MEANS:

(1) THE SET OF COEFFICIENTS OF AN EXPANSION WHICH ARE ALMOST ZERO IS VERY LARGE ⇒ WE HAVE A SPARSE REPRESENTATION

(2) AN AS GOOD AS POSSIBLE REPRESENTATION OF INFORMATION JOINT IN TIME AND IN FREQUENCY

EXAMPLE:

$\varphi(t) = \sin(2\pi f_0 t)$ IS INFINITELY LONG, HAS FREQUENCY f_0 ⇒ TOTAL PRECISION ON f, ZERO PRECISION ON t $\left(\phi(f) = -\delta(f+f_0) + \delta(f-f_0) \cdot \frac{1}{2j}\right)$

$\varphi(t) = \delta(t-t_0)$ TOTAL PRECISION IN t, ZERO PRECISION IN f $\left(\phi(f) = \exp(-j2\pi f t_0)\right)$

$\varphi(t) \leftrightarrow \phi(f)$ LOCATION IS DETERMINED BY THE SUPPORT IN TIME OF $\varphi(t)$, AND **DUALLY IN FREQUENCY** OF $\phi(f)$.

$\varphi(t) = \exp(j2\pi f_0 t)$ INFINITELY PRECISE IN FREQUENCY, INFINITELY UNPRECISE IN TIME $(\phi(f) = \delta(f-f_0))$

NB: $\varphi(t) = \exp(j2\pi f_0 t) = \cos(2\pi f_0 t) + j\sin(2\pi f_0 t)$

$$\cos(2\pi f_0 t) = \frac{e^{j2\pi f_0 t} + e^{-j2\pi f_0 t}}{2} \overset{\mathcal{F}}{\Longleftrightarrow} \frac{\delta(f-f_0) + \delta(f+f_0)}{2}$$

$$\sin(2\pi f_0 t) = \frac{e^{j2\pi f_0 t} - e^{-j2\pi f_0 t}}{2j} \overset{\mathcal{F}}{\Longleftrightarrow} \frac{\delta(f-f_0) - \delta(f+f_0)}{2j}$$

$$\text{So } \exp(j2\pi f_0 t) = \frac{\delta(f-f_0) + \cancel{\delta(f+f_0)}}{2} + \cancel{j}\,\frac{\delta(f-f_0) - \cancel{\delta(f+f_0)}}{2\cancel{j}} = \delta(f-f_0).$$

OR I CAN SEE IT AS $1 \cdot \exp(j2\pi f_0 t) \overset{\mathcal{F}}{\longleftrightarrow} \delta(f-f_0)$ (SHIFT PROPERTY)

FOURIER GIVES AN EXTREMELY PRECISE REPRESENTATION/LOCALIZATION IN FREQUENCY DOMAIN, BUT IT DOESN'T GIVE SPATIAL INFORMATION.

CONVERSELY TEMPORAL REPRESENTATION IS EXTREMELY PRECISE IN TIME, BUT IT IS NOT PRECISE IN FREQUENCY.

FOURIER GIVES A GOOD DESCRIPTION FOR STATIONARY OR PSEUDO STATIONARY SIGNALS. FOR HIGHLY NON STATIONARY SIGNALS THERE ARE LIMITATIONS, THAT CAN BE OVERCOME BY THE STFT

UNTIL NOW WE CONSIDERED THE FOLLOWING REPRESENTATIONS:

CONTINUOUS LINEAR EXPANSION OF A CONTINUOUS TIME SIGNAL

$$X_f = \langle x(t), \varphi_f(t) \rangle = \int_{-\infty}^{+\infty} x(t) \cdot \varphi_f^*(t)\, dt$$

$$x(t) = \int_{-\infty}^{+\infty} X_f \cdot \varphi_f(t)\, df$$

IF $\varphi_f(t) = \exp(j2\pi ft)$ I OBTAIN THE FOURIER TRANSFORM
IT IS COMPUTED WITH A SCALAR PRODUCT

CONTINUOUS LINEAR EXPANSION OF A DISCRETE SEQUENCE

$$X_f = \langle x[n], \varphi_f[n] \rangle = \sum_{n \in \mathbb{Z}} x[n] \cdot \varphi_f^*[n]$$

$$x[n] = \int_{-\infty}^{+\infty} X_f \cdot \varphi_f[n]\, df$$

IF $\varphi_f[n] = \exp(j2\pi f n)$ I OBTAIN THE DTFT

DISCRETE EXPANSION OF A DISCRETE TIME SIGNAL

$$x[n] = \sum_{k \in \mathbb{Z}} X_k \varphi_k[n]$$

$X_k = \langle x[n], \varphi_k[n] \rangle$ IF $\varphi_k[n]$ ARE ORTHONORMAL.

IF $\varphi_k[n] = W_N^{nk}$, $W_N = \exp\left(j\frac{2\pi}{N}\right)$ I OBTAIN THE DFT

$$x[n] = \frac{1}{N} \sum_{k=0}^{N-1} X_k \cdot W_N^{nk}, \quad X_k = \sum_{n=0}^{N-1} x[n] \cdot W_N^{-nk}$$

DISCRETE EXPANSION OF A CONTINUOUS TIME SIGNAL

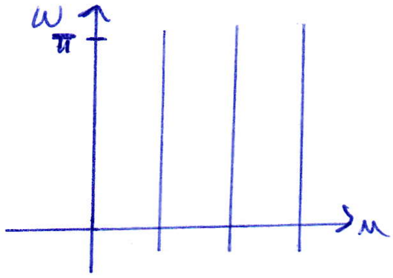
$$x(t) = \sum_{k \in \mathbb{Z}} X_k \varphi_k(t)$$

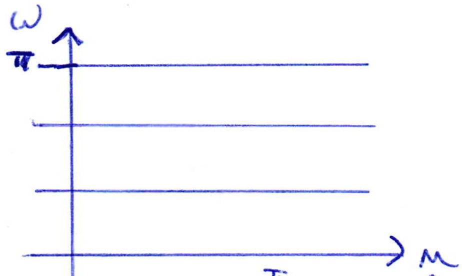
$X_k = \langle x(t), \varphi_k(t) \rangle$ IF $\varphi_k(t) = \exp\left(j\frac{2\pi nk}{T}\right)$ I OBTAIN THE FOURIER SERIES

WE WANT TO FIND A GENERAL SET OF FUNCTIONS WHICH ARE: EFFICIENT (GENERATED FROM PROTOTYPES) AND STRUCTURED (PROPERTIES ABOUT TIME/FREQUENCY LOCALIZATION)

A GENERAL ORTHONORMAL BASIS IS OF THE FORM: $b[n] = \sum_{k\in\mathbb{Z}} \langle \varphi_k[n], b[n] \rangle \cdot \varphi_k[n]$

DIRAC GIVES THE MAXIMUM RESOLUTION IN TIME DOMAIN
DTFT GIVES THE MAXIMUM RESOLUTION IN FREQUENCY DOMAIN } WE CONSIDER THEM AS THE DUAL OF THE OTHER

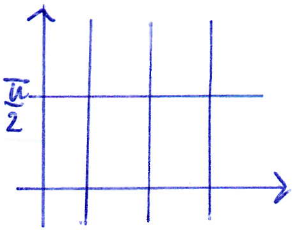




DIRAC $b[n] = \sum_{k\in\mathbb{Z}} \underbrace{x[k]}_{b[k]} \cdot \underbrace{\delta[n-k]}_{\varphi_k[n]}$

DTFT $b[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} \underbrace{X(e^{j\omega})}_{\sum b[n] e^{-j\omega n}} \underbrace{e^{j\omega n}}_{\varphi[n]} \, d\omega$

WE WANT TO FIND A REPRESENTATION "IN BETWEEN", WE START WITH DIRAC AND WE RELAX THE TEMPORAL CONDITION IN ORDER TO INCREMENT BY A FACTOR OF 2 THE FREQUENCY RESOLUTION.



$\varphi_0[n] \to n=0,\ n=1$; $\to \omega\ [0, \pi/2]$; $\to \|\varphi_0\| = 1$

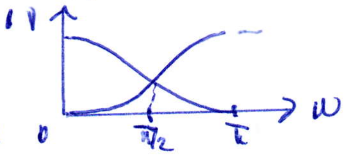
$$\begin{cases} \varphi_0[n] = \cos\theta\, \delta[n] + \sin\theta\, \delta[n-1] \\ \Phi_0(e^{j\omega}) = \cos\theta + \sin\theta\, e^{-j\omega} \Big|_{\omega=\pi} = \cos\theta - \sin\theta \end{cases}$$

$$\theta = \pi/4 + k\pi$$

I OBTAIN $\boxed{\varphi_0[n] = \frac{1}{\sqrt{2}}\delta[n] + \frac{1}{\sqrt{2}}\delta[n-1]}$. WE NEED TO FIND $\varphi_1[n]$ WITH THE SAME TEMPORAL SUPPORT, AND WITH $\langle \varphi_0, \varphi_1 \rangle = 0$

$\varphi_1[n] = \delta[n]\cos\bar\theta + \delta[n-1]\sin\bar\theta \to \langle \varphi_0, \varphi_1 \rangle = 0 \Rightarrow \frac{1}{\sqrt{2}}\cos\bar\theta + \frac{1}{\sqrt{2}}\sin\bar\theta = 0 \Rightarrow \theta = (2k-1)\frac{\pi}{2} + \frac{\pi}{4}$

$\boxed{\varphi_1[n] = \frac{1}{\sqrt{2}}\delta[n] - \frac{1}{\sqrt{2}}\delta[n-1]}$ WE OBTAIN TWO FILTERS:



TO OBTAIN OTHER FUNCTIONS, IT IS ENOUGH TO TRANSLATE φ_0 AND φ_1 BY TWO SAMPLES EACH TIME: $\varphi_{2k}[n] = \varphi_0[n-2k]$, $\varphi_{2k+1}[n] = \varphi_1[n-2k]$

WE NEED TO VERIFY COMPLETENESS OF $\Phi = \{\varphi_k\}_{k\in\mathbb{Z}}$: φ_0, φ_1 ARE ORTHONORMAL BASES FOR $\mathbb{R}^2$
JOINING THE BASES OF THE INTERVALS, I OBTAIN A BASE FOR $\ell^2(\mathbb{Z}) \Rightarrow \Phi$ IS A HAAR BASES FOR $\ell^2(\mathbb{Z})$

$$b[n] = \sum_{n\in\mathbb{Z}} \langle \varphi_k, b \rangle \varphi_k = \sum \langle \varphi_{2k}, b \rangle \varphi_{2k} + \sum \langle \varphi_{2k+1}, b \rangle \varphi_{2k+1} = \underbrace{\sum_{n\in\mathbb{Z}} \frac{1}{\sqrt{2}} (b[2k] + b[2k+1]) \varphi_{2k}[n]}_{x_V[n] \in V} + \underbrace{\sum_{n\in\mathbb{Z}} \frac{1}{\sqrt{2}} (b[2k] - b[2k+1]) \varphi_{2k+1}}_{x_W[n] \in V}$$

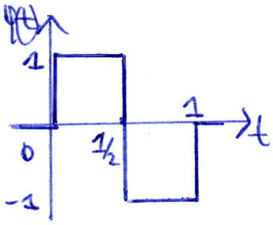
$V = \mathrm{SPAN}\{\varphi_{2k}[n]\}_{k\in\mathbb{Z}}$, $W = \mathrm{SPAN}\{\varphi_{2k}[n]\}_{k\in\mathbb{Z}}$

$\ell^2(\mathbb{Z}) = V + W$ $\quad$ $V \to$ LOW PASS FILTERING
$W \to$ HIGH PASS FILTERING

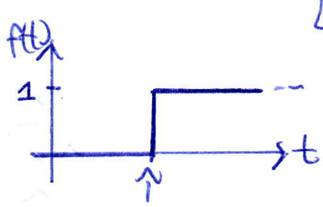
N.B.: WE CAN REPRESENT HAAR VECTORS IN A MATRIX AND TREAT IT AS A LINEAR OPERATOR

EXAMPLE:

$$\psi(t) = \begin{cases} 1 & 0 \le t < 1/2 \\ -1 & 1/2 \le t < 1 \\ 0 & \text{OTHERWISE} \end{cases}$$



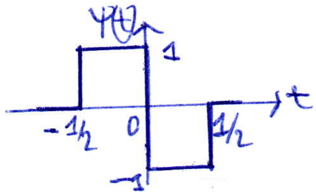
$$f(t) = \begin{cases} 0 & t < \tau \\ 1 & t \ge \tau \end{cases}$$



$$CWT_f(a,b) = \int f(t) \frac{1}{\sqrt{a}} \psi\left(\frac{t-b}{a}\right) dt$$

LET US CONSIDER A SHIFTED VERSION OF THE HAAR WAVELET:

$$\psi(t) = \begin{cases} 1 & -1/2 \le t < 0 \\ -1 & 0 \le t < 1/2 \\ 0 & \text{OTHERWISE} \end{cases}$$



INTEGRATING THE FUNCTION WE OBTAIN A TRIANGLE: $\theta(t) = \begin{cases} 1/2 - |t| & -1/2 < t < 1/2 \\ 0 & \text{OTHERWISE} \end{cases}$

WE HAVE A COMPACT SUPPORT AS $\int_{\mathbb{R}} \psi(t)\, dt = 0$

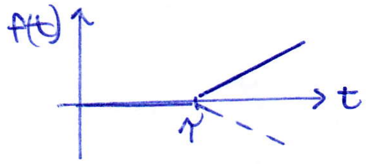
THE INTEGRAL TENDS TO: $CWT_f = \left[\sqrt{a}\, \theta\left(\frac{t-b}{a}\right) f(t)\right]_{-\infty}^{+\infty} - \int_{-\infty}^{+\infty} \sqrt{a}\, \theta\left(\frac{t-b}{a}\right) \cdot f'(t)\, dt = -a^{-1/2} \theta\left(\frac{\tau-b}{a}\right)$

CWT DECREASES AS $\frac{1}{a^{3}}$ $(b \to \tau,\ a \to 0)$

EXAMPLE:

WE WANT A WAVELET WITH AT LEAST TWO MOMENTUM NON NULL, TO ANALYZE

$$f(t) = \begin{cases} 0 & t \le \tau \\ t & t > \tau \end{cases}$$



DISCONTINUITY OF ORDER TWO; THE PRIMITIVE OF THE DERIVATIVE HAS TO HAVE COMPACT SUPPORT

WE HAVE: $CWT_f(a,b) = -\int \sqrt{a}\, \theta\left(\frac{t-b}{a}\right) f'(t)\, dt$ (FROM THE PREVIOUS EXAMPLE)

NOW: $CWT_f(a,b) = -\sqrt{a} \left[a\, \theta^{(1)}\left(\frac{t-b}{a}\right) f'(t)\right]_{-\infty}^{+\infty} + \sqrt{a} \int_{-\infty}^{+\infty} a\, \theta^{(1)}\left(\frac{t-b}{a}\right) f''(t)\, dt$

$$= a^{3/2} \cdot \theta^{(1)}\left(\frac{\tau-b}{a}\right)$$

IT DECREASES FASTER $\left(\begin{smallmatrix} b \to \tau \\ a \to 0 \end{smallmatrix}\right)$ $CWT \sim \theta^{(1)}\left(\frac{\tau-b}{a}\right)$

WE CAN USE THE CWT TO IDENTIFY THE SINGULARITIES;

THE RESPONSE TO THE DISCONTINUITY DEPENDS ON:

- THE TYPE OF WAVELET
- THE TYPE OF DISCONTINUITY

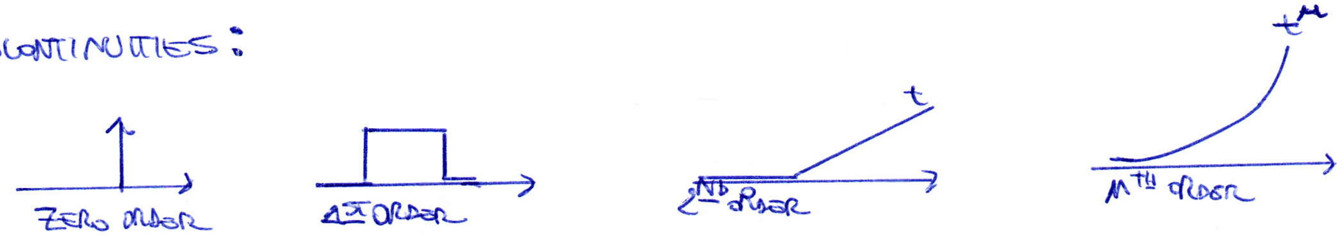
$$\psi_{a,b}(t) = \frac{1}{\sqrt{a}}\,\psi\left(\frac{t-b}{a}\right) \qquad a \in \mathbb{R}^+,\ b \in \mathbb{R}$$

TO PRESERVE $\mathcal{L}^2$ NORM

TO SIMPLIFY I CONSIDER $\|\psi_{a,b}\| = 1$

$$CWTf(a,b) = \frac{1}{\sqrt{a}}\int_{-\infty}^{+\infty} \psi\left(\frac{t-b}{a}\right) f(t)\,dt = \langle \psi_{a,b}, f\rangle$$

DISCONTINUITIES:



MOMENTUM m_M:

$$m_M = \int_{-\infty}^{+\infty} t^M f(t)\,dt \qquad M = 0, 1, \ldots$$

TH OF MOMENTUM OF FOURIER TRANSFORM

$$m_M = \frac{1}{(-j)^M} \cdot \left.\frac{d^M F(\omega)}{d\omega^M}\right|_{\omega=0} = \frac{F^{(M)}(0)}{(-j)^M}$$

IF A WAVELET HAS A NULL DERIVATIVE OF ORDER M IN $\omega = 0$ THEN IT NULLS THE POLYNOMIALS UNTIL ORDER M. NULL MOMENTUM → NULLS THE POLYNOMIAL

THE ADMISSIBLE OPTION IS $\Psi(0) = 0 \rightarrow \int_{-\infty}^{+\infty} \psi(t)\,dt = 0$ THAT MEANS THAT NEED TO HAVE AT LEAST A NULL MOMENTUM

EXAMPLE:

IF $f(t) = \delta(t-\tau)$

$$CWTf(a,b) = \frac{1}{\sqrt{a}}\int_{-\infty}^{+\infty} \psi\left(\frac{t-b}{a}\right)\delta(t-\tau)\,dt = \frac{1}{\sqrt{a}}\cdot\psi\left(\frac{\tau-b}{a}\right)$$

$a \to 0$ INFINITE HEIGHT

IN ORDER TO DETECT A DISCONTINUITY OF M-TYPE / M-ORDER ARE NEEDED AT LEAST M NULL MOMENTUM.

TH: GIVEN A WAVELET WITH COMPACT SUPPORT AND WITH N NON NULLS MOMENTUM AND A FUNCTION f WITH A SINGULARITY OF ORDER $m \leq N$.
THEN THE CWT IN THE NEIGHBORHOOD OF τ HAS THE FORM

$$CWT_f(a,b) = (-1)^m \cdot a^{m-1/2} \cdot \psi^{(m)}\left(\frac{\tau - b}{a}\right) \qquad \psi^{(m)} \triangleq m \text{ ORDER DERIVATIVE OF } \psi$$

LET'S CONSIDER A SET OF GENERIC FUNCTIONS OBTAINED FROM THE MOTHER FUNCTION $\psi(t)$.
DESCRIBE $\psi(t)$ IN THE $t-f$ PLANE MEANS DETERMINE THE SUPPORT OF THE FUNCTION IN THAT PLANE.
AT FIRST IF IT IS FINITE IN ONE DIMENSION, IT WILL BE ALWAYS INFINITE IN THE OTHER.
IN PRACTICE ONE RELAXES THE CONCEPT OF SUPPORT TO DEAL WITH FINITE SUPPORT $\psi(t)$.

WE DEFINE AN ENERGY CONSTRAINT SUPPORT = SUPPORT WITH FINITE ENERGY

(FOR TIME) I_t : $\boxed{W_\psi(I_t) = \int_{I_t} |\psi(t)|^2 dt \leq k \cdot W_\psi}$ $\quad k \in [0,1]$, IN PRACTICE $k = 90\%$

(FOR FREQUENCY) I_f : $\boxed{W_\psi(I_f) = \int_{I_f} |\Phi(f)|^2 df \leq k \cdot W_\psi}$

N.B.: I_t AND I_f CONTAIN THE 90% OF THE ENERGY OF THE TIME AND FREQUENCY DOMAIN FUNCTIONS

IN SOME SITUATIONS WE COULD HAVE MORE INTERVALS THAT SATISFY THE CONDITION

WE CONSTRUCT A FAMILY OF FUNCTIONS THAT DIFFER FROM $\psi(t)$ BY TRANSLATION IN TIME:

$$\psi_k(t) = \psi(t - t_k) \qquad k \in \mathbb{Z}$$

THE WIDTH OF THE INTERVALS DOESN'T CHANGE WITH THE TRANSLATION! BECAUSE IT IS JUST A SHIFT
SO DURATION OF $I_t(\psi_k)$ = DURATION OF $I_t(\psi)$ $\quad (I_t(\psi_k) = I_t(\psi) + t_k)$
SIMILARLY $I_f(\psi_k) = I_f(\psi)$ SINCE THE ENERGY SPECTRUM $|\Phi_k(f)|^2 = |\Phi(f)|^2$

$\forall x(t)$ APPROXIMATING $x(t)$ WITH $\{\psi_k(t), k \in \mathbb{Z}\}$ WILL BE SATISFACTORY TO MEASURE THE TIME-FREQUENCY CONTENT OF $x(t)$ AS LONG AS THERE EXISTS AN OVERLAP BETWEEN CONTIGUOUS $I_t(\psi_k)$ (BUT NOT TOO MUCH TO ENSURE TEMPORAL RESOLUTION).

DUALLY WE COULD DEFINE AN ALTERNATIVE SET OF FUNCTIONS $\psi_k(f)$ FROM THE MOTHER FUNCTION SUCH THAT $\psi_k(f) = \psi(f - f_k)$ (WE TRASLATE IN FREQUENCY).
ALSO HERE WE HAVE A GOOD DESCRIPTION DESCRIBING ANY $x(t)$ IF THERE IS NO OVERLAP AND NO FREE SPACE BETWEEN CONTIGUOUS INTERVALS.

IN GENERAL, FROM THE MOTHER FUNCTION, ONE CAN CONSTRUCT A FAMILY OF OTHER FUNCTIONS THAT TILE UP THE TIME FREQUENCY PLANE BY SHIFTING $\psi(t)$ IN TIME OR IN FREQUENCY OR BOTH SIMULTANEOUSLY.
LIMITATION: $I_t \times I_f$ IS FIXED!!! TIME-FREQUENCY RESOLUTION IS TOTALLY DETERMINED BY THE SQUARE $I_t \times I_f$ = TILE

HEISENBERG UNCERTAINTY PRINCIPLE

IN NO WAY ONE CAN FIND A $\varphi(t)$ SUCH THAT I_t AND I_f ARE AS SMALL AS WE WANT.

$\forall \varphi(t)$ ADMITTING $\phi(f)$ WITH UNIT NORM BOTH CENTERED IN TIME AND FREQUENCY

$$\int_{\mathbb{R}} t\,|\varphi(t)|^2 dt = \int_{\mathbb{R}} f\,|\phi(f)|^2 df = 0$$

THEN BY DEFINING THEIR EXTENT/DIMENSION IN TIME AND FREQUENCY AS;

$$\Delta t = \sqrt{\int_{\mathbb{R}} t^2 |\phi(t)|^2 dt} \qquad \Delta f = \sqrt{\int_{\mathbb{R}} f^2 |\phi(f)|^2 df}$$

WE HAVE THAT: $\boxed{\Delta t \cdot \Delta f \geq \frac{1}{4\pi}}$

EQUALITY $\left(\Delta t \cdot \Delta f = \frac{1}{4\pi}\right)$ HOLDS ONLY FOR $\varphi(t) = \sqrt{\frac{\alpha}{\pi}}\exp(-\alpha t^2)$

Dim.:

LET US CONSIDER A REAL FUNCTION $\varphi(t) \in \mathbb{R}$, $\frac{\partial \varphi(t)}{\partial t} = \varphi'$

$$\langle t\varphi(t), \varphi'(t)\rangle = I$$

$$|I|^2 = \left|\int_{\mathbb{R}} t\,\varphi(t)\,\varphi'(t)\,dt\right|^2 \leq \int_{\mathbb{R}} \left|t\,\varphi(t)\,\varphi'(t)\,dt\right|^2 = A$$ (CAUCHY SCHWARZ INEQUALITY)

$$A = \int_{\mathbb{R}} |t\,\varphi(t)|^2 \cdot |\varphi'(t)|^2 dt \leq \underbrace{\int_{\mathbb{R}} |t\,\varphi(t)|^2 dt}_{\Delta t^2} \cdot \underbrace{\int_{\mathbb{R}} |\varphi'(t)|^2 dt}_{\text{PARSEVAL}}$$

(N.B.: INEQUALITY HOLDS BECAUSE WE HAVE POSITIVE QUANTITIES $\sum |\alpha_i \beta_i| \leq \sum |\alpha_i| \cdot \sum |\beta_i|$)

$$\Downarrow$$

$$\int_{\mathbb{R}} |\mathcal{F}\{\varphi'(t)\}|^2 df = \int_{\mathbb{R}} |\phi(f)\, j2\pi f|^2 df = 4\pi^2 \int_{\mathbb{R}} f^2 |\phi(f)|^2 df = 4\pi^2 \Delta f^2$$

THEN: $|I|^2 \leq 4\pi^2 \Delta t^2 \Delta f^2$

$$I = \int_{\mathbb{R}} t\,\varphi(t)\,\varphi'(t)\,dt = \underbrace{\frac{t\,\varphi^2(t)}{2}\Big|_{-\infty}^{+\infty}}_{0} - \underbrace{\int_{-\infty}^{+\infty} \frac{1}{2}\varphi^2(t)\,dt}_{1/2} = -1/2$$

0 BECAUSE $\varphi(t) \in \mathcal{L}^2(\mathbb{R})$ DECREASES FASTER THAN t

1/2 BECAUSE $\varphi(t)$ HAS UNIT NORM

(N.B.: INTEGRATION BY PARTS WITH $f(t)f(t)' = \frac{1}{2}\frac{\partial f^2(t)}{\partial t}$)

$$\left(f' = \frac{\partial \varphi^2(t)}{\partial t} \quad f = \varphi^2(t) \quad g = \frac{1}{2}t \quad g' = \frac{1}{2}\right)$$

$|I|^2 = 1/4$

So: $\frac{1}{4} \leq 4\pi^2 \Delta t^2 \Delta f^2 \Rightarrow \Delta t^2 \Delta f^2 \geq \frac{1}{16\pi^2} \Rightarrow \Delta t \cdot \Delta f \geq \frac{1}{4\pi}$

TO DEMONSTRATE THE EQUALITY (CAUCHY SCHWARZ):

$|\langle t\varphi(t), \varphi(t)'\rangle| \leq \|t\varphi(t)\| \cdot \|\varphi'(t)\|$ LIMIT CONDITION IS OBTAINED FOR $kt\varphi(t) = \varphi'(t)$

(IN CAUCHY SCHWARZ IS AN EQUALITY).

SOLVING THE DIFFERENTIAL EQUATION LEADS TO A GAUSSIAN SOLUTION $\varphi(t) = \sqrt{\frac{\alpha}{\pi}}\exp(-\alpha t^2)$

IN FACT: $\varphi(t) = c\,e^{kt^2/2}$, $\|\varphi\| = 1$, $k = -2\alpha$, $c = \sqrt{\frac{\alpha}{\pi}}$.

CONSIDER A FAMILY OF FUNCTIONS SCALED AND SHIFTED:

$$\boxed{\psi_{a,b}(t) = \frac{1}{\sqrt{a}}\,\psi\!\left(\frac{t-b}{a}\right)} \in \mathcal{L}^2(\mathbb{R})$$

a: SCALE FACTOR $a \in \mathbb{R}^+$
b: SHIFT FACTOR IN TIME

WE WANT TO EVALUATE THE KIND OF TILING THAT WE OBTAIN IN THE $f-t$ PLANE. THE DIMENSION OF THE TILES IS VARIABLE, BUT THE AREA IS CONSTANT.

SCALING IMPLIES A CHANGE OF DIMENSIONS OF THE INTERVALS I_t, I_f

$$I_t(\psi_{a,b}) \text{ SUCH THAT } \int_{t=\alpha}^{\beta} |\psi_{a,b}(t)|^2\,dt = 90\%\,\|\psi_{a,b}\|^2$$

$$\psi_{a,b}(t) = \frac{1}{\sqrt{a}}\,\psi\!\left(\frac{t-b}{a}\right)$$

$$= \int_{u=\frac{\alpha-b}{a}}^{\frac{\beta-b}{a}} \frac{1}{a}\,|\psi(u)|^2\,a\,du = 90\%\,\|\psi\|^2, \quad u = \frac{t-b}{a}, \quad dt = a\,du$$

$\psi(t)$ AND $\psi_{a,b}(t)$ HAVE THEN THE SAME ENERGY $\int_{\mathbb{R}} |\psi(t)|^2 dt = \int_{\mathbb{R}} |\psi_{a,b}(t)|^2 dt$

IN TIME DOMAIN THE NEW INTERVAL BECOMES:

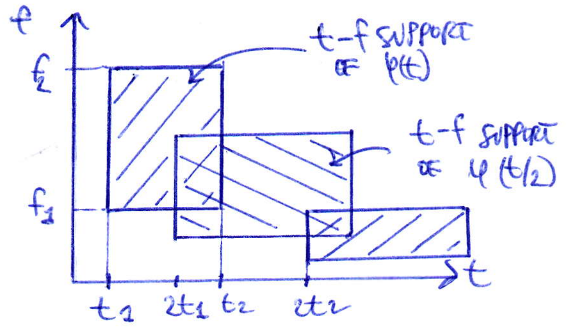
$$\boxed{I_t\!\left(\psi\!\left(\frac{t-b}{a}\right)\right) = a\,I_t(\psi) + b}$$

IN FREQUENCY WE OBTAIN:

$$\frac{1}{\sqrt{a}}\cdot\psi\!\left(\frac{t-b}{a}\right) \longleftrightarrow \phi(af)\cdot\frac{a}{\sqrt{a}}\cdot\exp(-j\,2\pi f b) \rightarrow \boxed{I_f\!\left(\psi\!\left(\frac{t-b}{a}\right)\right) = \frac{1}{a}\cdot I_f(\psi)}$$

WHEN THE INTERVAL ENLARGES IN FREQUENCY IT GETS NARROWER IN TIME, AND VICEVERSA

IF $\psi(t)$ IS SELECTED BASED PAIRS



BY DOUBLING THE SCALE AT EACH STEP WE TRADE TEMPORAL RESOLUTION FOR FREQUENCY RESOLUTION

IN ORDER TO REACH AN OPTIMAL TILING $t-f$ PLANE USE, GENERATES A FAMILY OF FUNCTIONS WHICH ARE SHIFTED AT SCALE 1 BY THE I_f EXTEND AND WHICH ARE SHIFTED AT SCALE $a = 2^m$ BY $2^m I_f$ EXTENT.

SHORT TIME FOURIER TRANSFORM (STFT) OR GABOR TRANSFORM

$\forall x \in L^2(\mathbb{R})$

$$\boxed{STFT_x(f,\tau) = \langle x(t), w(t-\tau)\exp(j2\pi f t)\rangle = \int_{\mathbb{R}} x(t)\, w^*(t-\tau)\exp(-j2\pi f t)\,dt}$$

$w(t)$ IS A FINITE SUPPORT WINDOW THAT IS SHIFTED OVER THE TIME AXIS = WINDOW FUNCTION

NB.: THE SPECTROGRAM IS THE ENERGY DISTRIBUTION ASSOCIATED WITH THE STFT
THAT IS $S(f,\tau) = |STFT(f,\tau)|^2$

NB.: WITH THE STFT WE PASS FROM A MONODIMENSIONAL REPRESENTATION TO A BIDIMENSIONAL ONE

CONTINUOUS WAVELET TRANSFORM

GIVEN A FUNCTION $\psi(t)$ SUCH THAT IT SATISFIES THE ADMISSIBILITY CONDITION THAT IS:
($\psi(t)$ IS A BAND-PASS SIGNAL) $\psi(t)$ = MOTHER WAVELET, a = SCALE FACTOR, b = SHIFT FACTOR

$$\boxed{C_\psi = \int_{\mathbb{R}} \frac{|\Psi(fa)|^2}{|f|}\,df < \infty}$$

THE ADMISSIBILITY STATES THAT IT HAS TO BE A BANDPASS SIGNAL AND THAT THE DENOMINATOR DECREASES FASTER THAN THE NUMERATOR. IT IS ALMOST ALWAYS SATISFIED, SO ADMISSIBILITY REDUCES TO $\Psi(0)=0$ (NO D.C. COMPONENT).

THEN THE CONTINUOUS WAVELET TRANSFORM (CWT) IS:

$$\boxed{CWT_x(a,b) = \langle x(t), \psi_{a,b}(t)\rangle = \int_{\mathbb{R}} x(t)\cdot \psi^*\left(\frac{t-b}{a}\right)\cdot\frac{1}{\sqrt{|a|}}\cdot dt}$$

$a \in \mathbb{R}^+$
$b \in \mathbb{R}$

INVERSION FORMULA:

$$\boxed{x(t) = \frac{1}{C_\psi}\int_{a=0^+}^{+\infty}\int_{b=0^+}^{+\infty} CWT_x(a,b)\cdot\psi\left(\frac{t-b}{a}\right)\cdot\frac{1}{\sqrt{|a|}}\cdot\frac{da\,db}{a^2}}$$

N.B. WITH CWT WE SCALE AND SHIFT THE FUNCTION IN TIME. THIS ALLOWS TO ZOOM IN THE t-f REPRESENTATION.
THE DIMENSION OF THE TILES IS NOT FIXED.

	TIME REPRESENTATION		FREQUENCY REPRESENTATION
	$x(t)$	$\xleftrightarrow{\mathcal{F}}$	$X(f)$
EQUIVALENT DISCRETE REPRESENTATION (NO ALIASING)	$\downarrow$ $x[n]$	$\xleftrightarrow{DFT}$	$X[k]$
	$x(t)$	$\longleftrightarrow$	$STFT_x(f,\tau)$
	$\downarrow$ $x[n]$	$\longleftrightarrow$	DISCRETE TIME $STFT_x$
	$x(t)$	$\longleftrightarrow$	$CWT_x(a,b)$
	$\downarrow$ $x[n]$	$\longleftrightarrow$	$DWT_x[m,n]$

Dim: INVERSION FORMULA OF CWT

USING PARSEVAL WE CAN WRITE:

$$CWT_x(a,b) = \int X(f)\, \Psi^*_{a,b}(f)\, df$$

WHERE:

$$\psi_{a,b}(t) = \frac{1}{\sqrt{|a|}}\, \psi\left(\frac{t-b}{a}\right) \overset{\mathcal{F}}{\longleftrightarrow} \Psi_{a,b}(f) = \frac{|a|}{\sqrt{|a|}} \cdot \Psi(af) \cdot \exp(-j2\pi fb)$$

SO:

$$CWT_x(a,b) = \int_{\mathbb{R}} X(f) \cdot \frac{|a|}{\sqrt{|a|}} \cdot \Psi^*(af) \cdot \exp(j2\pi fb)\, df =$$

$$= \sqrt{|a|} \int_{\mathbb{R}} \left(X(f) \cdot \Psi^*(af) \right) \cdot \exp(j2\pi fb)\, df$$

= INVERSE FOURIER TRANSFORM OF $X(f) \cdot \Psi^*(af)$ EVALUATED IN $t=b$ $= \mathcal{F}^{-1}\left\{ X(f)\Psi^*(af) \right\}\Big|_{t=b}$

NOW WE CALL:

$$\boxed{I(a) = \int_{b=-\infty}^{+\infty} CWT_x(a,b) \cdot \psi_{a,b}(t) \cdot db} = \sqrt{|a|} \int_{b=-\infty}^{+\infty} \int_{f=-\infty}^{+\infty} \left(X(f)\Psi^*(af) \exp(j2\pi fb)\, df \right) \cdot \underbrace{\psi_{a,b}(t)}_{\frac{1}{\sqrt{|a|}}\psi\left(\frac{t-b}{a}\right)} \cdot db =$$

$$= \int_{f=-\infty}^{+\infty} X(f)\Psi^*(af) \underbrace{\left[\int_{b=-\infty}^{+\infty} \psi\left(\frac{t-b}{a}\right) \exp(j2\pi fb)\, db \right]}_{\text{THIS IS THE INVERSE FOURIER TRANSFORM OF SHIFTED AND SCALED VERSION OF } \psi(t) \text{ AND MIRRORED/FLIPPED}} df =$$

$\Rightarrow \Psi(af) \exp(j2\pi ft)\, |a|$

NB: WE OBTAIN THE INVERSE TRANSFORM LIKE A DIRECT ONE BY OPERATING A CHANGE OF VARIABLES

$\psi_1(u) = \psi(-u)$

$$\int_{b=-\infty}^{+\infty} \psi_1\left(\frac{u+t}{a}\right) \exp(-j2\pi fu)\, du$$

$$= \int_{-\infty}^{+\infty} X(f)\, |\Psi(af)|^2 \exp(j2\pi ft)\, df\, |a|$$

THEN WE CALL:

$$B = \int_{a=-\infty}^{+\infty} \frac{I(a)}{a^2}\, da = \int_{a=-\infty}^{+\infty} \int_{f=-\infty}^{+\infty} X(f)\, |\Psi(af)|^2 \exp(j2\pi ft)\, df\, \frac{|a|}{a^2}\, da =$$

$$= \int_{f=-\infty}^{+\infty} \underbrace{\int_{a=-\infty}^{+\infty} \frac{|\Psi(af)|^2}{|a|}\, da}_{C\psi} \cdot X(f) \cdot \exp(j2\pi ft)\, df = X(t) \cdot C\psi$$

$$x(t) = \frac{B}{C\psi} = \frac{1}{C\psi} \int_{a=-\infty}^{+\infty} \int_{b=-\infty}^{+\infty} CWT_x(a,b) \cdot \psi_{a,b}(t) \cdot \frac{db \cdot da}{a^2} =$$

$$= \frac{1}{C\psi} \int_{a=-\infty}^{+\infty} \int_{b=-\infty}^{+\infty} CWT_x(a,b) \cdot \psi\left(\frac{t-b}{a}\right) \cdot \frac{1}{\sqrt{|a|}} \cdot \frac{da\, db}{a^2}.$$

CWT PROPERTIES

1) <u>LINEARITY</u>: SINCE $\langle , \rangle$ IS LINEAR, CWT IS LINEAR (ONLY FOR CWT !!!)

$$x_1(t) \xleftrightarrow{CWT} CWT_{x_1}(a,b)$$
$$x_2(t) \xleftrightarrow{CWT} CWT_{x_2}(a,b)$$
$$\alpha_1 x_1(t) + \alpha_2 x_2(t) \xleftrightarrow{CWT} \alpha_1 CWT_{x_1}(a,b) + \alpha_2 CWT_{x_2}(a,b)$$

2) <u>SHIFT INVARIANCE</u>:

$$x(t) \longleftrightarrow CWT_x(a,b)$$
$$x(t-t_0) \longleftrightarrow CWT_x(a, b-t_0)$$

3) <u>CHANGE OF SCALE</u>:

$$\begin{array}{ccc} x(t) & \longrightarrow & x_T(t) = \frac{1}{\sqrt{T}} x\left(\frac{t}{T}\right) \\ CWT \updownarrow & & CWT \updownarrow \\ CWT_x(a,b) & \longleftrightarrow & CWT_{x_T}\left(\frac{a}{T}, \frac{b}{T}\right) \end{array}$$

THE ENERGY REMAINS THE SAME THANKS TO THE SCALE FACTOR $1/\sqrt{T}$

<u>Dim</u>:

$$x_T(t) \leftrightarrow CWT_{x_T}(a,b) = \int_{-\infty}^{+\infty} \frac{1}{\sqrt{T}} \cdot x\left(\frac{t}{T}\right) \cdot \frac{1}{\sqrt{a}} \cdot \psi\left(\frac{t-b}{a}\right) dt =$$

$$\frac{t}{T} = u, \; t = uT, \; dt = T du$$

$$= \int_{-\infty}^{+\infty} \frac{1}{\sqrt{T}} \cdot \frac{1}{\sqrt{a}} \cdot x(u) \cdot \psi\left(\frac{uT-b}{a}\right) \cdot T \cdot du =$$

$$= \frac{1}{\sqrt{\frac{a}{T}}} \int_{-\infty}^{+\infty} x(u) \cdot \psi\left(\frac{u - b/T}{a/T}\right) \cdot du = CWT_x\left(\frac{a}{T}, \frac{b}{T}\right)$$

4) <u>ENERGY PRESERVATION</u>:

$$W_x = \int_{-\infty}^{+\infty} |x(t)|^2 dt = \frac{1}{C_\psi} \int_{\mathbb{R}^2} |CWT_x(a,b)|^2 \frac{da\,db}{a^2}$$

SIMILARLY WE HAVE THE PRESERVATION OF THE SCALAR PRODUCT

$$\langle \underline{x}, \underline{y} \rangle = \frac{1}{C_\psi} \int_{\mathbb{R}^2} CWT_x(a,b) \cdot CWT_y^*(a,b) \cdot \frac{da\,db}{a^2} = \int_{\mathbb{R}^2} x(t) y^*(t) dt$$

<u>Dim</u>:

$$I = \int_{\mathbb{R}^2} \Big| \underbrace{\int x(t) \frac{1}{\sqrt{a}} \psi^*\left(\frac{t-b}{a}\right) dt}_{CWT_x(a,b)} \Big|^2 \frac{da\,db}{a^2} \overset{(PARSEVAL)}{=} \int_{\mathbb{R}^2} \frac{1}{|a|} \Big| \int_{\mathbb{R}} \underbrace{X(f) \cdot \Psi(af)}_{P(f)} \cdot \exp(j 2\pi f b) \, df \Big|^2 da\,db =$$

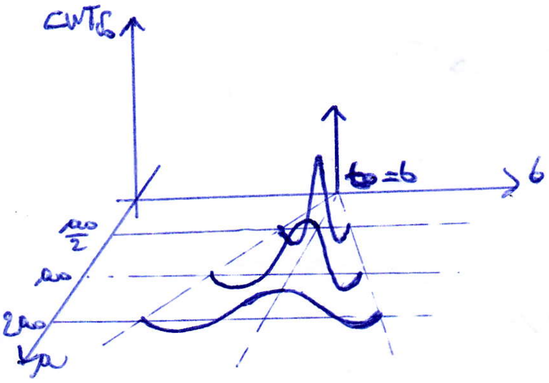
$$= \iint_{-\infty}^{+\infty} \frac{1}{|a|} |p(b)|^2 da = \int_f |X(f)|^2 \left(\int_a \frac{|\Psi(af)|^2}{|a|} da \right) df = \left(\int_f |X(f)|^2 df \right) \cdot C_\psi \Rightarrow \frac{I}{C_\psi} = \int_{\mathbb{R}} |X(f)|^2 df = W_x$$

(PARSEVAL)

5) TIME LOCATION:

$$\delta_0(t) = \delta(t - t_0)$$

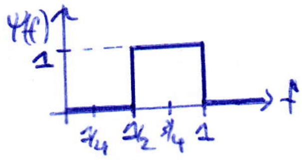
$$CWT_{\delta_0}(a, b) = \int_{-\infty}^{+\infty} \delta(t - t_0) \cdot \frac{1}{\sqrt{a}} \cdot \psi^*\left(\frac{t-b}{a}\right) dt = \frac{1}{\sqrt{a}} \cdot \psi^*\left(\frac{t_0 - b}{a}\right)$$



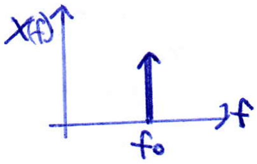
6) FREQUENCY LOCATION:

LET US ASSUME THAT $\psi(t)$ ACTS AS AN IDEAL BAND PASS FILTER BETWEEN THE FREQUENCIES $[1/2 ; 1]$

$$\psi(f) = \text{rect}\left(\frac{f - 3/4}{1/2}\right)$$



WE WANT TO FIND AT WHICH SCALES WE OBSERVE A RESPONSE FOR AN IDEAL FREQUENCY SIGNAL WITH $f = f_0$



$$x(t) = \exp(j 2\pi f_0 t) \longleftrightarrow X(f) = \delta(f - f_0)$$

$$CWT_x(a, b) = \int_{\mathbb{R}} x(t) \frac{1}{\sqrt{a}} \psi^*\left(\frac{t-b}{a}\right) dt \overset{(\text{PARSEVAL})}{=} \int_{\mathbb{R}} X(f) \left(\frac{1}{\sqrt{a}}\right)^{-1} \psi^*(af) \cdot \exp(j 2\pi f b)\, df =$$

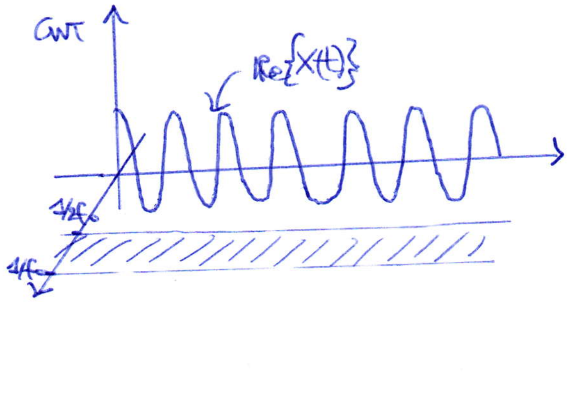
$$= \int_{\mathbb{R}} \underbrace{\delta(f - f_0)}_{X(f)} \left(\frac{1}{\sqrt{a}}\right)^{-1} \underbrace{\text{rect}\left(2af - \frac{3}{2}\right)}_{\psi(af)} \exp(j 2\pi f b)\, df =$$

$$= \left(\frac{1}{\sqrt{a}}\right)^{-1} \text{rect}\left(2af_0 - \frac{3}{2}\right) \exp(j 2\pi f_0 b)$$

WE HAVE A RESPONSE WHEN:

$$-1/2 < 2af_0 - \frac{3}{2} < 1/2$$

$$1/2f_0 < a < 1/f_0$$



SOME NOTES...

SCALE VERSUS RESOLUTION CONCEPT:

IN CONTINUOUS TIME IF A SIGNAL $x(t)$ WHICH HAS BANDWIDTH B IS SCALED BY A FACTOR a, SINCE BOTH TIME AND FREQUENCY ARE REDUCED OR AMPLIFIED BY THE SAME QUANTITY RESPECTIVELY, THE RESOLUTION REMAINS UNCHANGED.

IN THE DISCRETE TIME CASE:

$$x[n] \longrightarrow (2\downarrow) \longrightarrow y[n]$$

$x[n]$: RESOLUTION N, SCALE 1

$y[n]$: RESOLUTION HALVED, SCALE 1/2, <u>ALIASING</u>

$$x[n] \longrightarrow \boxed{LP} \longrightarrow (2\downarrow) \longrightarrow y[n]$$

$x[n]$: RESOLUTION N, SCALE 1

$y[n]$: RESOLUTION N, SCALE 1/2, <u>NO ALIASING</u>

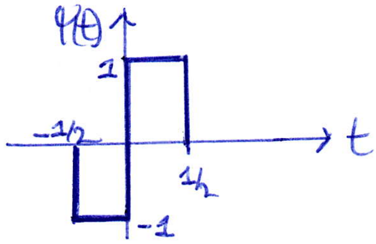
GIVEN A MOTHER FUNCTION $\psi(t)$ WHICH HAS A GIVEN TIME-FREQUENCY REPRESENTATION, SHIFTED VERSIONS OF $\psi(t)$ IN TIME AND FREQUENCY ALLOW TO COVER THE ENTIRE TIME-FREQUENCY-PLANE PROVIDED THAT THE SHIFTS ARE "DENSE" ENOUGH.

NOW BECAUSE OF THE HEISENBERG UNCERTAINTY PRINCIPLE, THE SUPPORT OF $\psi(t)$ IN ITS TIME-FREQUENCY REPRESENTATION IS BOUNDED (THE AREA OF THIS SUPPORT IS LIMITED) $\Delta t \Delta f \geq 1/4\pi$

SHORT TIME FOURIER TRANSFORM ATTEMPTS TO IMPLEMENT THIS IDEA WITH A BOUND IN PERFORMANCE DUE TO THIS INEQUALITY.

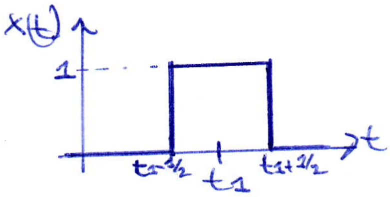
IF WE CHANGE ALSO THE SCALE WE ARE ABLE TO "ZOOM" INTO THE TIME-FREQUENCY REPRESENTATION.

EXAMPLE: BANDPASS MOTHER WAVELET



$$\psi(t) = \mathrm{rect}(2t-1) - \mathrm{rect}(2t+1)$$

THE SIGNAL IS: $x(t) = \mathrm{rect}\left(\frac{t-t_1}{T}\right)$



ITS TRANSFORM IS:

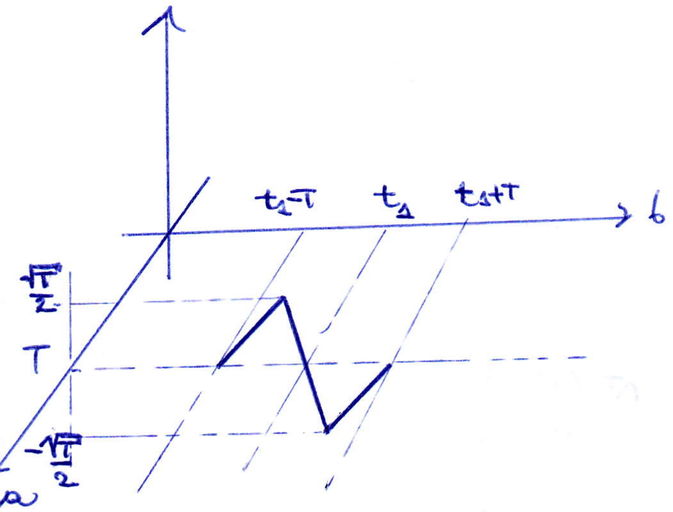
$$CWT_x(a,b) = \frac{1}{\sqrt{|a|}} \int_{-\infty}^{+\infty} x(t)\, \psi^*\left(\frac{t-b}{a}\right) dt$$

IF WE FIX, FOR EXAMPLE, $a=1$

$$CWT_x(1,b) = \int_{-\infty}^{+\infty} x(t)\, \psi^*(t-b)\, dt \equiv$$

THIS IS THE CONVOLUTION OF $x(t)$ WITH ψ THAT IS THE CORRELATION OF $x(t)$ WITH ψ FLIPPED $\varphi_{xy}(-b)$

IN OTHER WORDS THE CWT GIVES THE SIMILARITY BETWEEN THE MOTHER FUNCTION AND THE SIGNAL

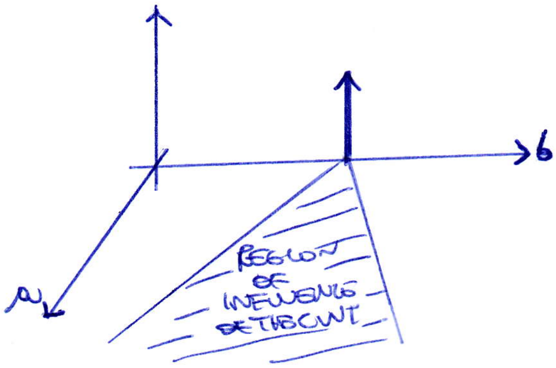


THE GRAPH REPRESENTS THE SIMILARITY WHEN b VARIES, FOR $a=T$

$$CWT_x(a,b) = \int \mathrm{rect}\left(\frac{t-t_1}{T}\right) \psi\left(\frac{t-b}{T}\right) dt$$

NOW WE FIGURE OUT HOW IT CHANGES WHEN a VARIES:

- AS a MOVES TO $\emptyset$ THE CWT ZOOMS INTO THE SIGNAL LOCATION IN TIME (WHEN a DECREASES THE WAVELET TENDS TO A DELTA)
- WHEN a INCREASES, THE PEAK DECREASES AND THE WAVELET BROADENS. SO THE AMPLITUDE DECAYS PROPORTIONALLY WITH THE SCALE.



REPRODUCING KERNEL

THE CWT IS A REDUNDANT REPRESENTATION: WE REPRESENT IN TWO DIMENSIONS A MONODIMENSIONAL SIGNAL.
SO ONLY A SMALL SUBSET OF BIDIMENSIONAL FUNCTIONS CORRESPOND TO THE WAVELET TRANSFORM, THAT IS, ONLY A SUBSPACE H OF V CORRESPONDS TO A WAVELET TRANSFORM OF A FUNCTION $\in L^2(\mathbb{R})$

$$V = \left\{ F(a,b) \;\middle|\; \int_{\mathbb{R}^2} |F(a,b)|^2 \frac{da\,db}{a^2} < \infty \right\} \qquad H \subset V$$

V: SPACE OF $F(a,b)$ FUNCTIONS WHICH ARE SQUARE INTEGRABLES
H: SUBSPACE OF ALL CONTINUOUS WAVELET ~~[illegible]~~ TRANSFORMS FOR ANY $\psi(t)$ WHICH SATISFIES THE ADMISSIBILITY CONDITION

IF ANY $F(a,b) \in H$ (F IS A CWT OF THE SIGNAL $x(t)$) I CAN RECOVER THE VALUE OF F AT ANY OTHER LOCATION (a_0, b_0) THROUGH ITS REDUNDANCY KERNEL.

TH A FUNCTION $F(a,b) \in H$, THAT IS, IT IS A WAVELET TRANSFORM OF $f(t)$ IF AND ONLY IF:

$$\boxed{F(a_0, b_0) = \frac{1}{C_\psi} \iint K(a_0, b_0, a, b)\, F(a,b) \frac{da\,db}{a^2}}$$

WHERE: $K(a_0, b_0, a, b) = \langle \underbrace{\psi_{a_0,b_0}}_{\text{WAVELET IN } a_0, b_0}, \underbrace{\psi_{a,b}}_{\text{GENERIC WAVELET}} \rangle$

IS CALLED REPRODUCING KERNEL.

THIS MEANS THAT I CAN OBTAIN THE VALUE OF F AT ANY POSITION (a_0, b_0) BY STARTING FROM ITS REPRODUCTION KERNEL $\quad x(t) \longleftrightarrow CWT_x(a,b)$

IF WE KNOW THE REPRODUCING KERNEL, IF THE WAVELET SATISFIES THE ADMISSIBILITY CONDITION, I CAN KNOW THE VALUE IN (a_0, b_0).

IT MUST BE THAT $F \in H$ AND THE MAPPING BETWEEN THE POINTS DEPENDS ON HOW THE POINTS ARE RELATED TO THE MOTHER FUNCTION.

EXAMPLE OF A MOTHER WAVELET: MORLET WAVELET

IT IS THE FIRST WAVELET! IT IS A COMPLEX EXPONENTIAL WINDOWED WITH A GAUSSIAN WINDOW

$$\boxed{\psi_M(t) = \frac{1}{\sqrt{2\pi}} \exp(-j2\pi f t) \exp\left(-\frac{t^2}{2}\right)}$$

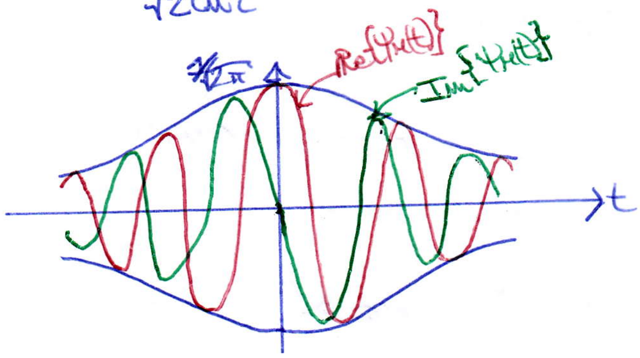
$$\mathrm{Re}\{\psi_M(t)\} = \frac{1}{\sqrt{2\pi}} \cos(2\pi f_0 t) \exp\left(-\frac{t^2}{2}\right)$$

$$\mathrm{Im}\{\psi_M(t)\} = -\frac{1}{\sqrt{2\pi}} \sin(2\pi f_0 t) \exp\left(-\frac{t^2}{2}\right)$$

$$\|\psi_M\|^2 = \int_{\mathbb{R}} |\psi_M(t)|^2 dt = \int_{\mathbb{R}} \left|\frac{1}{\sqrt{2\pi}} \exp\left(-\frac{t^2}{2}\right)\right|^2 dt = 1$$

CHOOSE f_0 SO THAT THE SECOND MAX OF $\mathrm{Re}\{\psi_M(t)\}$ IS EQUAL TO HALF OF THE VALUE IN $\emptyset$.

$$f_0 = \frac{1}{\sqrt{2\ln 2}} \simeq 0.849$$



THE ~~WAVELET~~ FOURIER TRANSFORM OF THE MOTHER FUNCTION IS:

$$\Psi(f) = k \exp\left(-\frac{(f-f_0)^2}{k}\right)^2$$

THE ADMISSIBILITY CONDITION IS:

$$C_\psi = \int_{-\infty}^{+\infty} \frac{|\Psi(f)|^2}{|f|} df$$

IT EXISTS PROVIDED THAT f_0 IS NOT TOO SMALL!

SAMPLING THE CWT

WE WANT TO UNDERSTAND IF IT IS POSSIBLE TO SAMPLE THE $CWT_x(a,b)$ OVER a AND b VALUES, SUCH THAT THE SUB-SAMPLES WILL ALLOW TO RECONSTRUCT $x(t)$.
IT IS POSSIBLE THANKS TO THE REDUNDANCY.

CWT INVERSION FORMULA:

$$x(t) = \frac{1}{C_\psi}\int_{\mathbb{R}^2} CWT_x(a,b)\,\psi_{a,b}(t)\,\frac{da\,db}{a^2}\cdot\frac{1}{\sqrt{|a|}}$$

CAN WE COMPUTE $\psi_{a,b}(t)$ AT DISCRETE LOCATIONS SO THAT THE INVERSION FORMULA HOLDS?
YES IF THE SAMPLING IS SUFFICIENTLY DENSE, IN THIS CASE WE CAN WRITE:

$$x(t) = \frac{1}{C_\psi}\int_{\mathbb{R}^2} CWT_x(a,b)\,\psi_{a,b}(t)\,\frac{1}{\sqrt{|a|}}\,\frac{da\,db}{a^2} = \sum_{m\in\mathbb{Z}}\sum_{n\in\mathbb{Z}} \langle \psi_{m,n}, x\rangle\,\tilde{\psi}_{m,n}$$

LET'S START TO REPRESENT THE FUNCTION $\psi_{1,b}(t) = \psi(t-b)$.
SINCE $\psi(t)$ SATISFIES THE ADMISSIBILITY CONDITION, $\psi(t-b)$ HAS FINITE $I_f(\psi)$ AND SO IT CAN BE SAMPLED WITH A STEP SIZE b_0 SUCH THAT:

$$1/b_0 > |I_f(\psi)|$$

N.B.: $I_f(\psi)$ IS THE FREQUENTIAL SUPPORT OF $\psi_{1,b}(t)$
AS THE WAVELETS ARE BAND PASS FILTERS IT IS POSSIBLE TO SAMPLE THEM RESPECTING SHANNON.

SO THE DISCRETE VERSION OF $\psi_{1,b}(t)$ IS: $\psi_{1,b}(t) = \psi(t - nb_0),\ n\in\mathbb{Z}$

NOW WE CONSIDER THE SCALE: $\psi_{a_0^m,b}(t) = \frac{1}{\sqrt{a_0^m}}\cdot\psi\left(\frac{t-b}{a_0^m}\right),\quad a = a_0^m$

THE SUPPORT BECOMES THEN: $I_f\left(\psi_{a_0^m,b}\right) = I_f(\psi)/a_0^m$

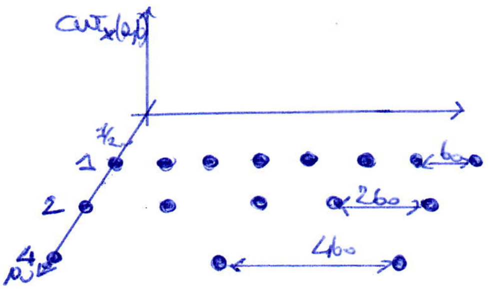
WHEREAS IN TIME: $I_t\left(\psi_{a_0^m,b}\right) = a_0^m I_t(\psi) + b$

SO A DISCRETE VERSION OF $\psi_{a_0^m,b}$ IS:

$$\psi_{m,n}(t) = \psi\left(\frac{t - n\,a_0^m b_0}{a_0^m}\right)\cdot\frac{1}{\sqrt{a_0^m}}$$

WE GENERATED A DISCRETE FAMILY OF FUNCTIONS:

$$\boxed{\Psi = \left\{\psi_{m,n}(t) = a_0^{-m/2}\cdot\psi\left(a_0^{-m}t - nb_0\right),\ m,n\in\mathbb{Z}\right\}}$$



$a_0 = 2$

$\psi(2^m t - nb_0)$

IT HAS TO BE THAT $a_0 \neq 1$ (ONLY CONSTRAINT FOR a)
WHILE WE HAVE SOME CONSTRAINTS ON THE VALUE OF b

IF Y CONSTITUTES A FRAME WE COULD GENERATE A SYNTHESIS / ANALYSIS OPERATOR SUCH THAT:

$$\forall x(t) \in \mathcal{L}^2(\mathbb{R}) \qquad \boxed{x(t) = \sum_{m \in \mathbb{Z}} \sum_{n \in \mathbb{Z}} \alpha_{m,n} \cdot a_0^{-m/2} \cdot \psi(a_0^{-m} t - n b_0)} \text{ (SYNTHESIS)}$$

IF IT IS A 1-TIGHT FRAME:

$$\boxed{\alpha_{m,n} = \langle \underline{x}, \psi_{m,n} \rangle = \int_{-\infty}^{+\infty} x(t) \cdot a_0^{-m/2} \cdot \psi^*(a_0^{-m} t - n b_0)\, dt} \text{ (ANALYSIS)}$$

TO PROVE THE FRAME DECOMPOSITION WE MUST SATISFY THE CONDITION:

$$\exists \lambda_{min} > 0, \lambda_{MAX} < \infty \mid \forall x \in \mathcal{L}^2(\mathbb{R}) \qquad \lambda_{min} \|\underline{x}\|^2 \leq \sum_{m,n \in \mathbb{Z}} |\alpha_{m,n}|^2 \leq \lambda_{MAX} \|\underline{x}\|^2$$

FROM NOW ON WE LIMIT TO THE CASE: $a_0 = 2$, $b_0 = 1$ (THAT IS THE DYADIC CASE)

THE SYNTHESIS OPERATOR BECOMES:

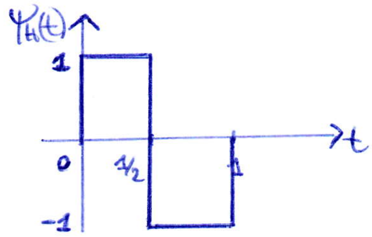
$$x(t) = \sum_{m,n \in \mathbb{Z}} \alpha_{m,n} \cdot \psi(2^{-m} t - n) \cdot 2^{-m/2}$$

WITH:

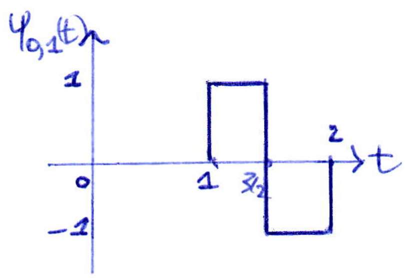
$$\alpha_{m,n} = \langle \underline{x}, \underline{\psi_{m,n}} \rangle$$

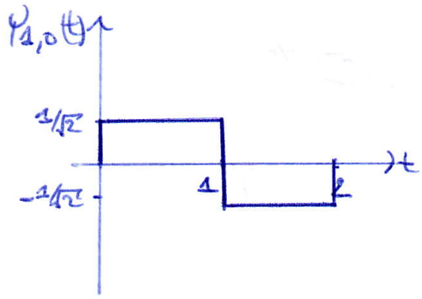
HAAR WAVELET $\equiv$ 1ST ORDER DAUBECHIES WAVELET

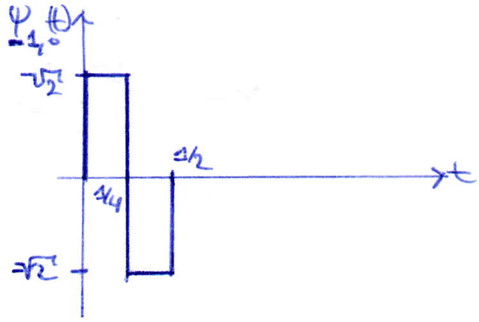
SIMPLEST WAVELET, CONSTRUCTED WITH TWO RECTANGLES, UNITARY ENERGY



$$\psi_H(t) = \begin{cases} 1 & 0 < t \leq 1/2 \\ -1 & 1/2 < t \leq 1 \\ 0 & \text{OTHERWISE} \end{cases} \qquad Y = \left\{ 2^{-m/2} \cdot \psi(2^{-m} t - n),\ m, n \in \mathbb{Z} \right\}$$







THE SCALAR PRODUCT BETWEEN TWO VERSIONS SHIFTED AND SCALED IS:

$$\langle \psi_{m,n}(t), \psi_{m',n'}(t) \rangle = \delta_{m,m'} \cdot \delta_{n,n'} = 1 \quad \text{ONLY IF } m = m' \text{ AND } n = n'$$

$$\stackrel{!}{=} \delta[m-m'] \cdot \delta[n-n'] \quad \equiv \text{ORTHOGONALITY}$$

<u>PROPOSITION</u>: Ψ IS AN ORTHONORMAL BASIS OF $\mathcal{L}^2(\mathbb{R})$

$$\forall x(t) \in \mathcal{L}^2(\mathbb{R}) \qquad x(t) = \sum_{m,n \in \mathbb{Z}} \alpha_{m,n} \cdot 2^{-m/2} \cdot \psi_{m,n}(t)$$

$$\alpha_{m,n} = \langle \underline{x}, \underline{\psi_{m,n}} \rangle = \int_{t=2^{-m}(n)}^{2^{-m}(n+1/2)} x(t)\, dt - \int_{2^{-m}(n+\frac{1}{2})}^{2^{-m}(n+1)} x(t)\, dt$$

IT MUST BE DEMONSTRATED

(1) ORTHONORMALITY: DERIVES FROM THE PROPERTIES OF $\psi_{m,n}(t)$

(2) COMPLETENESS.

WE HAVE THE SET OF FUNCTIONS $\Psi = \{\psi_{m,n}(t),\ m,n \in \mathbb{Z}\}$, $\text{SPAN}(\Psi) = \mathcal{L}^2(\mathbb{R})$

IF Ψ IS A BASE OF $\mathcal{L}^2(\mathbb{R})$ $\langle \psi_{m,n}(t), \psi^*_{m',n'}(t) \rangle = \delta[m-m']\,\delta[n-n']$

IF IT IS ORTHONORMAL $\forall x \in \mathcal{L}^2(\mathbb{R})$, $\underline{x} = \sum \langle \underline{x}, \psi_{m,n} \rangle\, \psi_{m',n'}(t)$

WE CAN CONSIDER AS MOTHER FUNCTION THE HAAR WAVELET: $\psi(t) = \text{rect}\left(\frac{t-1/4}{1/2}\right) - \text{rect}\left(\frac{t-3/4}{1/2}\right)$

$$\stackrel{!}{=} \varphi(2t) - \varphi(2t-1), \quad \varphi(t) = \text{rect}(t - 1/2)$$

THE COMPLEX PART IS TO DEMONSTRATE THAT IT IS A BASIS. WE CAN PROCEED IN THE FOLLOWING WAY:

(1) WE TAKE THE STEP FUNCTION AND WE WRITE IT AS THE SUM OF TWO FUNCTIONS.
- AVERAGE OF TWO CONSECUTIVE VALUES (COARSE)
- DIFFERENCE BETWEEN THE ORIGINAL VALUE AND THE SIGNAL THAT AVERAGES TWO ADJACENT VALUES (DETAIL).

(2) WE ITERATE THE PROCEDURE ITERATIVELY APPLYING (1) TO THE AVERAGED SIGNAL

(3) AT INFINITE WE OBTAIN A NULL SIGNAL, THAT IS DISTORTION IS ZERO. IT IS POSSIBLE TO RECONSTRUCT PROPERLY THE ORIGINAL SIGNAL THAT WAS EXPANDED OVER AN INFINITE SET OF FUNCTIONS OF DIFFERENT SCALE, AND CONSTRUCTED AS A SET OF DIFFERENCE SIGNALS.
IF THE SIGNAL IS CONTINUOUS WE REDUCE TO $\emptyset$ THE SUPPORT OF THE STEP FUNCTION.

EVERY FINITE ENERGY SIGNAL CAN BE WRITTEN AS:

$$x(t) = \sum_{m=-\infty}^{+\infty} \sum_{n=-\infty}^{+\infty} d_{m,n} \cdot 2^{-m/2} \cdot \psi(2^{-m}t - n)$$

WHERE: $\psi(t) = \varphi(2t) - \varphi(2t-1)$, $\varphi(2t) = \varphi(4t) - \varphi(4t-1)$

IF $\varphi(t)$ SATISFIES A CONDITION LIKE THE ONE SEEN, WE HAVE THE FOLLOWING PROPERTIES:

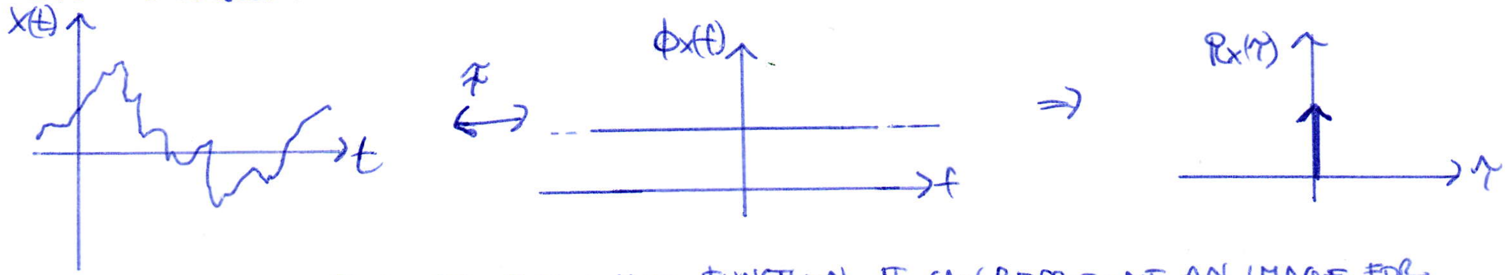
(1) $\varphi(t) = \sqrt{2} \sum_{n \in \mathbb{Z}} g_0[n]\, \varphi(2t - n)$ $\qquad g_0[n] \mid\ |G_0(f)|^2 + |G_0(f+1/2)|^2 = 2$

(2) $\psi(t) = \sqrt{2} \sum_{n \in \mathbb{Z}} g_1[n]\, \varphi(2t - n)$ $\qquad g_1[n] \mid\ G_1(f) = -\exp(-j2\pi f)\, G_0^*(f + 1/2)$

THIS MEANS THAT WE CAN PASS FROM CONTINUOUS TO DISCRETE WITHOUT LOOSING INFORMATION.

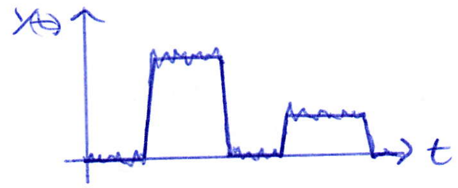
WAVELET APPLICATION: DENOISING

LET'S CONSIDER A WHITE NOISE. IT HAS CONSTANT POWER SPECTRAL DENSITY (PSD) AND IMPULSIVE AUTOCORRELATION. WHITE NOISE TRANSFORMS INTO A FULL BAND COVERAGE, NO SPARSITY CAN BE ACHIEVED



NOW CONSIDER A PIECEWISE POLYNOMIAL FUNCTION. IT CAN REPRESENT AN IMAGE FOR EXAMPLE. THE FUNCTION IS AFFECTED BY WHITE NOISE

$$y(t) = z(t) + x(t) = \text{SIGNAL} + \text{WHITE NOISE} = \text{NOISY SIGNAL}$$

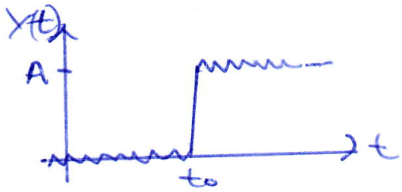


I WOULD LIKE TO OBTAIN $z(t)$ BUT I MEASURE $y(t)$. WE PERFORM A DENOISING OPERATION. WE MAKE AN ASSUMPTION ABOUT AN A PRIORI MODEL FOR $z(t)$: STEP FUNCTION (SIMPLE MODEL).

$$y(t) = A\,\varepsilon(t - t_0) + x(t)$$

$$\Big\updownarrow \mathcal{F}$$

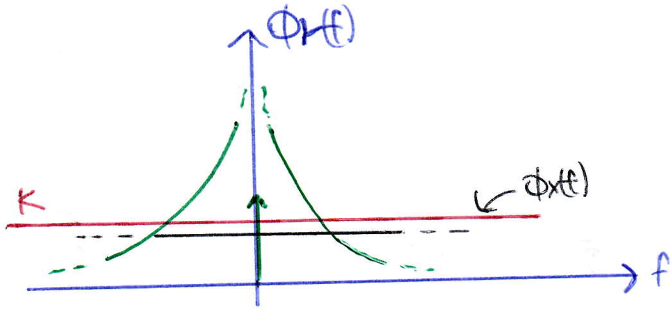
$$Y(f) = \frac{A}{2}\delta(f) + \frac{A}{j2\pi f} + X(f)$$



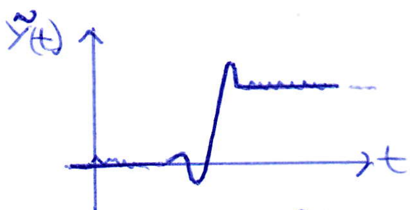
THE POWER SPECTRAL DENSITY IS:

$$\Phi_Y(f) = \mathcal{F}\{\varphi_Y(\tau)\} = |Y(f)|^2 = A^2\left(\frac{\delta(f)}{4} + \frac{1}{4\pi^2 f^2}\right) + \Phi_X(f)$$

BY SELECTING A THRESHOLD $K > \Phi_X(f)$ WE GET RID OF MOST OF THE WHITE NOISE. NOT ALL BECAUSE IT REMAINS IN THE LOW FREQUENCIES. BUT BY DOING SO I ALSO GET RID OF SOME OF THE HIGH FREQUENCIES OF $z(t)$.



$$\Phi_{\tilde{Y}}(f) = \begin{cases} \Phi_Y(f) & \text{IF } \Phi_Y(f) > K \\ 0 & \text{OTHERWISE} \end{cases}$$



RECOVERING $z(t)$ FROM $\tilde{y}(t)$ IS OK TO SOME EXTENT, EXCEPT FOR LOW PASS AND GIBBS AROUND DISCONTINUITIES.

N.B.: THE SPECTRUM OF $\varepsilon(t)$ DECREASES PROPORTIONALLY TO $1/f$

IF $Z(t)$ IS DISCONTINUOUS $Z(f)$ DECAYS AS $1/f$ (SLOW DECAY!), WITH FOURIER

IF THE SIGNAL IS DISCRETE $Z[m] = \varepsilon[m] \cdot \mathrm{rect}_N[m]$ THE DISCRETE SPECTRUM DECREASES AS $1/k$ (DFT)

$$\mathrm{rect}_N\left[m - \frac{N}{2}\right] \overset{DTFT}{\longleftrightarrow} \frac{\sin(\pi f N)}{\sin(\pi f)}$$

$\forall y \in \mathcal{L}^2(\mathbb{R})$ IT CAN BE DECOMPOSED WITH A DISCRETE WAVELET EXPANSION

$$y(t) = \sum_{m,n \in \mathbb{Z}}' \langle \underline{y}, \underline{\psi_{m,n}} \rangle \, \underline{\psi_{m,n}}(t)$$

PROVIDED $\psi_{m,n}$ ARE PART OF A MULTIRESOLUTION FRAMEWORK

IF I TAKE A WINDOWED VERSION OF A WHITE NOISE IT WILL BE FINITE ENERGY $(\in \mathcal{L}^2(\mathbb{R}))$ AND IT CAN BE DECOMPOSED USING THE DWT:

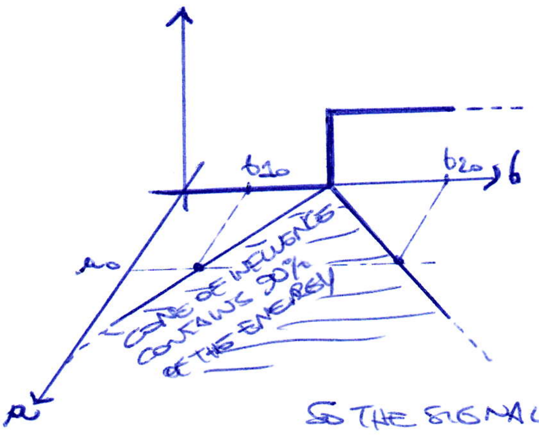
$$x(t) = \sum_{m,n \in \mathbb{Z}}' \langle \underline{x}, \underline{\psi_{m,n}} \rangle \, \psi_{m,n}(t)$$

$\langle X, \underline{\psi_{m,n}} \rangle$ IT CAPTURES ENERGY IN A GIVEN BAND

DUE TO NORMALIZATION OF THE AMPLITUDE ALL $\psi_{m,n}(t)$ HAVE THE SAME ENERGY W_ψ. THEREFORE IT MAKES SENSE TO SAY THAT THE AMOUNT OF CORRELATION OF $x(t)$ WITH RESPECT TO ANY $\psi_{m,n}(t)$ WILL REMAIN CONSTANT, SINCE THE SPECTRUM OF $X(t)$ IS CONSTANT.

SO $\langle \underline{x}, \underline{\psi_{m,n}} \rangle$ ARE THE SAME.

IF WE CONSIDER THE SIGNAL ~~[illegible]~~ $A\,\varepsilon(t-t_0)$ ~~[illegible]~~ WE HAVE THAT THE DECAY OF ITS WAVELET TRANSFORM IS FASTER RESPECT TO FOURIER



$b \in (b_{10}, b_{20})$ CONTAINS THE 90% OF THE ENERGY CONTENT OF $CWT_x(a,b)$

IF WE MOVE OVER ANOTHER INTERVAL WHAT IS THE DECAY OF $CWT_x(a_1,b)$ WITH RESPECT TO $CWT_x(a_0,b)$?

WE KNOW THAT: $\psi_{a,b}(t) = \frac{1}{\sqrt{a}} \psi\left(\frac{t-b}{a}\right)$

SO THE SIGNAL DECAYS AS $\sqrt{a_1/a_0}$.

IN GENERAL THE CWT_x DECREASES AS $\sqrt{a}$

BY SAMPLING THE $CWT_x(a,b)$ AT POWERS OF 2 ALONG THE a DIRECTION:

$\boxed{a_m = 2^{-m}}$, THE DECAY WILL BE $\boxed{2^{-m/2}}$ (SQRT $= \wedge 1/2$)

IT DECREASES EXPONENTIALLY WITH RESPECT TO m, COMPARED WITH A DECAY THAT WAS $1/k$ FOR THE DFT.

SO THE SPECTRUM OF THE DWT IS MORE COMPACT/LOCALIZED AND WHEN APPLYING A THRESHOLD WE REMOVE LESS INFORMATION THAT BELONGS TO THE SIGNAL RESPECT TO THE FOURIER CASE. (LESS RELEVANT COEFFICIENTS)

WE SAY THAT THE DENOISING CAPABILITY OF WAVELETS IS HIGHER THAN THE ONE OF FOURIER.

MULTIRESOLUTION FRAMEWORK

Def: A MULTIRESOLUTION ANALYSIS FRAME CONSISTS OF A FAMILY OF EMBEDDED CLOSED SUBSPACES

$$\dots \subset V_2 \subset V_1 \subset V_0 \subset V_{-1} \subset V_{-2} \dots \subset \mathcal{L}^2(\mathbb{R})$$

SUCH THAT:

(1) UPWARD COMPLETENESS

$$\overline{\bigcup_{m\in\mathbb{Z}}} V_m = \mathcal{L}^2(\mathbb{R})$$

(2) DOWNWARD COMPLETENESS

$$\overline{\bigcap_{m\in\mathbb{Z}}} V_m = \emptyset$$

(3) SCALE INVARIANCE

$$x(t) \in V_e \iff x(2^m t) \in V_{e-m}$$

(4) SHIFT INVARIANCE

$$x(t) \in V_e \iff x(t-2^e k) \in V_e$$

INVARIANCE FOR SHIFTS MULTIPLE OF 2^e IN FACT $x(t)$ PIECEWISE CONSTANT OVER $[2^e k, 2^e(k+1))$

(5) $\exists$ ORTHONORMAL BASIS FOR V_0

$\exists\ \varphi(t) \in V_0$ s.t. THE SET $\{\varphi(t-n),\ n\in\mathbb{Z}\}$ IS AN ORTHONORMAL BASIS FOR V_0

$\varphi(t) \to$ SCALE FUNCTION

PROPERTIES:

1. THE ORTOGONAL PROJECTION OF $x(t)$ IS SUCH THAT:

$$V_m: x_{\perp V_m}(t) = \sum_n \langle x(t), \varphi(2^m t - n)\rangle \varphi(2^m t - n)$$ (ORTHOGONAL PROJECTION OF x ON V_m)

$$\lim_{m\to\infty} V_m = \mathcal{L}^2(\mathbb{R})$$

$$\lim_{m\to\infty} x_{\perp V_m} = x(t) \text{ (UPWARD COMPLETENESS)}$$

2. CONDITION (5) WRITTEN IN FREQUENCY IS EQUIVALENT TO $\sum_{n\in\mathbb{Z}} |\Phi(f+n)|^2 = 1$

Dim:

$$\langle \varphi(t-n), \varphi(t-m)\rangle = \delta[n-m]$$

$$\int_{-\infty}^{+\infty} \varphi(t-n)\, \varphi^*(t-m)\, dt = \delta[n-m] \qquad t-n = t'$$

$$\int_{-\infty}^{+\infty} \varphi(t')\, \varphi^*(t'+n-m)\, dt' = \delta[n-m] \xrightarrow{n-m=k} = \begin{cases} 1 & k=0 \\ 0 & k\neq 0 \end{cases}$$

THE INTEGRAL TURNS INTO A CROSSCORRELATION:

$$R_\varphi(t)\Big|_{t=k} = \begin{cases} 1 & k=0 \\ 0 & k\neq 0 \end{cases}$$

WE CAN WRITE: $R_\varphi(t) \cdot \underbrace{\delta_1(t)}_{\text{TRAIN OF DELTAS}} = \delta(t)$ (ORTHOGONALITY)

ITS FOURIER TRANSFORM:

$$\mathcal{F}\{R_\varphi(t)\, \delta_1(t)\} = \mathcal{F}\{\delta(t)\}$$

$$|\Phi(f)|^2 * \delta_1(f) = 1$$

$$\Rightarrow \sum_{m\in\mathbb{Z}} |\Phi(f-m)|^2 = 1$$

3 2-SCALE EQUATION: DUE TO EMBEDDING OF SPACES IF $\varphi(t) \in V_0$ THEN $\varphi(t) \in V_{-1}$ BECAUSE $V_0 \subset V_{-1}$.

THEN $\varphi(t)$ CAN BE REPRESENTED WITH A BASIS OF V_{-1} THAT IS $\{\varphi(2t-m),\, m\in\mathbb{Z}\}$

$$(\Delta)\quad \boxed{\varphi(t) = \sqrt{2}\sum_{m\in\mathbb{Z}} g_0[m]\, \varphi(2t-m)}$$

$g_0[m] = \langle \varphi(t), \sqrt{2}\,\varphi(2t-m)\rangle$ ← TO PRESERVE THE ENERGY

$$\Phi(f) = \sqrt{2}\sum_{m\in\mathbb{Z}} g_0[m] \underbrace{\int_{-\infty}^{+\infty} \varphi(2t-m)\exp(-j2\pi f t)\,dt}_{\frac{1}{2}\Phi\left(\frac{f}{2}\right)\exp\left(-j2\pi \frac{f}{2} m\right)} =$$

$$= \Phi\left(\frac{f}{2}\right)\cdot \frac{1}{\sqrt{2}} \cdot \underbrace{\sum_{m\in\mathbb{Z}} g_0[m]\exp\left(-j2\pi \frac{f}{2} m\right)}_{G_0\left(\frac{f}{2}\right)\ (\text{DTFT})}$$

So

$$\boxed{\underbrace{\Phi(f)}_{\text{CONTINUOUS TRANSFORM}} = \frac{1}{\sqrt{2}} \cdot \underbrace{\Phi\left(\frac{f}{2}\right)}_{\text{CONTINUOUS TRANSFORM}} \cdot \underbrace{G_0\left(\frac{f}{2}\right)}_{\text{DTFT}}}$$

→ THIS IMPLIES THAT IT IS POSSIBLE TO CONSTRUCT CONTINUOUS TIME WAVELETS BASED ON A DISCRETE ITERATING FILTER.

THIS MEANS THAT A DATA WITH RESOLUTION t IS REPRESENTED WITH RESOLUTION $2t$

4) $|G_0(f)|^2 \cdot |G_0(f+1/2)|^2 = 2$

DIM: $\langle \varphi(t-m), \varphi(t-n) \rangle = \delta[n-m]$

IT IS EQUIVALENT TO:

$$\sum_{k\in\mathbb{Z}} |\Phi(f+k)|^2 = 1 \quad \text{AND WE OBTAIN} \quad \sum_{k\in\mathbb{Z}} |\Phi(2f+k)|^2 = 1 \ (*)$$

EXPLOITING THE EXPRESSION (Δ) OF THE PREVIOUS PROPERTY, (*) BECOMES:

$$\sum_{k\in\mathbb{Z}} \frac{1}{2} \left|\Phi\left(f+\frac{k}{2}\right) \cdot G_0\left(f+\frac{k}{2}\right)\right|^2 = 1$$

WE DIVIDE THE EVEN CASES $(k=2\ell)$ FROM THE ODD ONES $(k=2\ell+1)$:

$$\frac{1}{2}\Big(\underbrace{\sum_{\ell\in\mathbb{Z}}}_{k\ \text{EVEN}} \underbrace{|G_0(f+\ell)|^2}_{G_0(f)} \cdot |\Phi(f+\ell)|^2 + \underbrace{\sum_{\ell\in\mathbb{Z}}}_{k\ \text{ODD}} \left|G_0\left(f+\ell+\frac{1}{2}\right)\right|^2 \cdot \left|\Phi\left(f+\ell+\frac{1}{2}\right)\right|^2 \Big) = 1$$

AS THE DTFT IS PERIODIC OF PERIOD 1 WE OBTAIN:

$$\frac{1}{2}\Big(|G_0(f)|^2 \underbrace{\sum_{\ell\in\mathbb{Z}} |\Phi(f+\ell)|^2}_{1} + \left|G_0\left(f+\frac{1}{2}\right)\right|^2 \underbrace{\sum_{\ell\in\mathbb{Z}} \left|\Phi\left(\ell+f+\frac{1}{2}\right)\right|^2}_{f'=f+1/2 \text{ AND WE HAVE THAT } \sum|\ |^2=1} \Big) = 1$$

$$\Rightarrow \ |G_0(f)|^2 + |G_0(f+1/2)|^2 = 2.$$

TH WAVELET CONSTRUCTION

IF A MULTIRESOLUTION ANALYSIS FRAME EXISTS, THEN IT IS ALWAYS POSSIBLE TO CONSIDER A SET:
$$Y = \left\{ \psi_{m,n}(t) = 2^{-m/2}\, \psi(2^{-m}t - n),\ n \in \mathbb{Z} \right\}$$

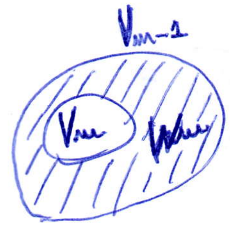
THAT IS AN ORTHONORMAL BASIS FOR W_m THAT REPRESENTS THE ORTHONORMAL COMPLEMENT OF V_m IN V_{m-1}

$$\dots \subset V_{m+1} \subset V_m \subset V_{m-1} \subset V_{m-2} \subset \dots$$

$$\boxed{V_{m-1} = V_m \oplus W_m}$$

ORTHONORMAL BASIS FOR V_m $\{\varphi(t-n),\ n \in \mathbb{Z}\}$

ORTHONORMAL BASIS $\{\psi(t-n),\ n \in \mathbb{Z}\}$



AND THANKS TO THE PROPERTIES OF UPWARD COMPLETION:
$$\overline{\bigcup_{m \in \mathbb{Z}} V_m} = \mathcal{L}^2(\mathbb{R}) \Rightarrow \bigcup_{m \in \mathbb{Z}} W_m = \mathcal{L}^2(\mathbb{R})$$

Dim.:

WE NEED TO FIND THE MOTHER WAVELET. WHAT WE WANT TO DO IS TO FIND A WAVELET $\psi(t) \in W_0$ SUCH THAT $\psi(t-n),\ n \in \mathbb{Z}$ IS AN ORTHONORMAL BASIS FOR W_0. THANKS TO SCALING PROPERTY $\psi_{m,n}(t),\ n \in \mathbb{Z}$ IS AN ORTHONORMAL BASIS FOR W_m. MOREOVER WITH THE PROPERTIES OF UPWARD/DOWNWARD COMPLETION WE HAVE THAT $Y = \{\psi_{m,n}\}$ ARE AN ORTHONORMAL BASE OF $\mathcal{L}_2(\mathbb{R})$.

WE HAVE $W_0 \subset V_{-1}$ AND THANKS TO THE SCALING PROPERTY:
$$f(t) \in W_m \Leftrightarrow f(2^m t) \in W_0$$

AS WE HAVE SEEN BEFORE WE CAN WRITE $\psi(t)$ AS A LINEAR COMBINATION OF FUNCTIONS $\varphi(2t-n)$, WITH $n \in \mathbb{Z}$

$$\psi(t) = \sqrt{2} \sum_{n \in \mathbb{Z}} g_1[n]\, \varphi(2t-n)$$
$$\Updownarrow \mathcal{F}$$
$$\Psi(f) = \frac{1}{\sqrt{2}}\, G_1\!\left(\frac{f}{2}\right) \cdot \Phi\!\left(\frac{f}{2}\right)$$

$\psi(t) \in W_0$ AND $W_0 \perp V_0$ (IT IS ORTHOGONAL TO ALL THE BASIS FUNCTIONS)

WE THEN HAVE THE FOLLOWING ORTHOGONALITY CONDITION:
$$\langle \psi(t), \varphi(t-n) \rangle = 0$$

WE CAN TRANSFORM THIS CONDITION WITH FOURIER:

$$\langle \psi(t), \varphi(t-n)\rangle = \int_{\mathbb{R}} \psi(t)\cdot \varphi^*(t-n)\,dt = R^*_{\psi\varphi}(\underbrace{t-n}_{\tau}) = R^*_{\psi\varphi}(\tau)\delta_1(\tau) = 0$$

↑ TRAIN OF DELTAS

$$\Updownarrow$$

$$\Phi^*_{\psi\varphi}(f) * \delta_1(f) = 0$$

$$\Psi(f)\cdot\Phi^*(f) * \delta_1(f) = 0$$

$$\boxed{\sum_{n\in\mathbb{Z}}{}' \Psi(f-n)\Phi^*(f-n) = 0}$$

IN THIS EXPRESSION ↑ WE SUBSTITUTE:

$$\Psi(f) = \frac{1}{\sqrt{2}} G_1\left(\frac{f}{2}\right)\Phi\left(\frac{f}{2}\right)$$

$$\Phi(f) = \frac{1}{\sqrt{2}} G_0\left(\frac{f}{2}\right)\Phi\left(\frac{f}{2}\right)$$

AND WE OBTAIN:

$$\frac{1}{2}\left\{\sum_{\ell\in\mathbb{Z}}{}' G_1\left(\frac{f-2\ell}{2}\right)G_0^*\left(\frac{f-2\ell}{2}\right)\left|\Phi\left(\frac{f-2\ell}{2}\right)\right|^2 + \sum{}' G_1\left(\frac{f-(2\ell+1)}{2}\right)G_0^*\left(\frac{f-(2\ell+1)}{2}\right)\left|\Phi\left(\frac{f-(2\ell+1)}{2}\right)\right|^2\right\} = 0$$

N.B.: WHAT WE HAVE DONE IS TO SEPARATE THE ODD ($k=2\ell+1$) AND EVEN ($k=2\ell$) COMPONENTS OF

$$\frac{1}{2}\sum_{k\in\mathbb{Z}}{}' G_1\left(\frac{f-k}{2}\right)\Phi\left(\frac{f-k}{2}\right)G_0^*\left(\frac{f-k}{2}\right)\Phi^*\left(\frac{f-k}{2}\right) = 0$$

NOW WE DEFINE $f' = f/2$

$$\frac{1}{2}\left\{\underbrace{G_1(f')\,G_0^*(f')}_{\text{PERIODIC OF PERIOD 1}}\underbrace{\sum_{\ell\in\mathbb{Z}}|\Phi(f'-\ell)|^2}_{=1} + \underbrace{G_1(f'+1/2)\,G_0^*(f'+1/2)}_{\text{IS IT PERIODIC OF PERIOD } f}\underbrace{\sum_{\ell\in\mathbb{Z}}\left|\Phi\left(f'-\ell+\frac{1}{2}\right)\right|^2}_{f''=f+1,\ \Sigma'|\,|^2=1}\right\} = 0$$

N.B.: $\sum_{k=-\infty}^{+\infty}{}' |\Phi(f\pm\ell)|^2 = 1$ IS THE POISSON FORMULA FOR THE ORTHONORMALITY OF $\{\varphi(t-n)\}, n\in\mathbb{Z}$

SO I OBTAIN:

$$\boxed{G_1(f)\,G_0^*(f) + G_1(f+1/2)\,G_0^*(f+1/2) = 0} \quad (*)$$

THIS EQUATION HIGHLIGHTS THE CONNECTION BETWEEN THE DISCRETE AND THE CONTINUOUS.

WE CANNOT HAVE SIMULTANEOUSLY THAT:

$$|G_0^*(f)| = 0 \qquad \text{AND} \qquad |G_0^*(f+1/2)| = 0$$

IT MUST BE SATISFIED THIS (SUFFICIENT) CONDITION:

$$G_1(f) = \lambda(f) \cdot G_0^*(f+1/2) \qquad \lambda(f) \text{ IS PERIODIC OF PERIOD } 1$$

WE SUBSTITUTE THIS IN THE EQUATION (*) AND WE OBTAIN:

$$\lambda(f) + \lambda(f+1/2) = 0$$

$$\boxed{\lambda(f) = -\exp(-j2\pi f)}$$

N.B.: IT IS A POSSIBLE CHOICE BUT IT IS NOT THE ONLY ONE

SO WE OBTAIN $G_1(f)$ AS A FUNCTION OF $\lambda(f)$ AND $g_0(f)$:

$$G_1(f) = -\exp(-j2\pi f)\, G_0^*(f+1/2)$$

$$\Updownarrow$$

$$g_1[n] = (-1)^n g_0[1-n]$$

AS A CONSEQUENCE, THE WAVELET IS:

$$\Psi(f) = -\frac{1}{\sqrt{2}} \exp(-j2\pi f) \cdot G_0^*(f) \cdot \Phi\left(\frac{f}{2}\right)$$

$$\Updownarrow$$

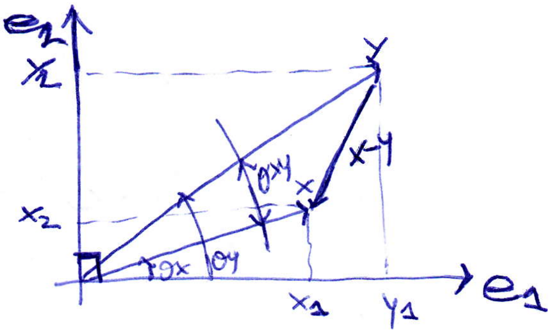
$$\psi(t) = -\sqrt{2} \sum_{n \in \mathbb{Z}} (-1)^n g_0[1-n]\, \varphi(2t-n)$$

NOW TO DEMONSTRATE THAT $\psi(t)$ AND ITS SHIFTED VERSIONS (OF INTEGER VALUES) FORM A BASE OF W_0 WE SHOULD DEMONSTRATE THE ORTHOGONALITY OF THE BASIS FUNCTIONS $\psi_{0,n}(t)$ AND COMPLETENESS.

ADVANCED METHODS FOR INFORMATION REPRESENTATION **EXERCISES (1**

REVIEW OF EUCLIDEAN VECTOR SPACES

① GEOMETRY (LENGTH, ANGLES, COORDINATES...)



$\begin{cases} \|e_1\| = \|e_2\| = 1 \quad \text{(VERSORS)} \\ e_1 \perp e_2 \end{cases}$

$\{e_1, e_2\}$ FORM AN ORTHONORMAL BASE (ONB) OF $\mathbb{R}^2$

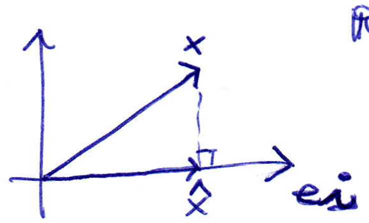
$\|x\| = \sqrt{\langle x, x\rangle}$

$\langle x,y\rangle = x_1y_1 + x_2y_2 = \|x\| \cos\theta_x \cdot \|y\| \cos\theta_y + \|x\| \sin\theta_x \cdot \|y\| \sin\theta_y =$

$= \|x\|\|y\| (\cos\theta_x \cos\theta_y + \sin\theta_x \sin\theta_y) = \|x\| \|y\| \cos(\theta_y - \theta_x) = \|x\|\|y\| \cos\theta_{xy}$

② PROJECTION

a)



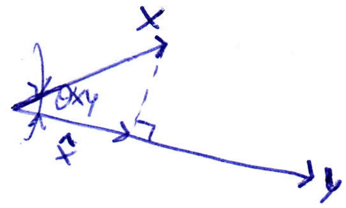
$\mathbb{R}^N$

GIVEN x THE PROJECTION $\hat{x}$ OF x ON e_i

$e_i \in \{e_j\}_{j\in[1,N]}$ ONB OF $\mathbb{R}^N$ IS

$\hat{x} = \underbrace{\langle x, e_i\rangle}_{\text{THE INNER PRODUCT IS THE LENGTH OF THE PROJECTION}} e_i = \|\hat{x}\| e_i$

b)



$\hat{x} = \langle x, y\rangle y$ IFF $\|y\| = 1$

c) IF LIKE CASE b) BUT $\|y\| \neq 1$

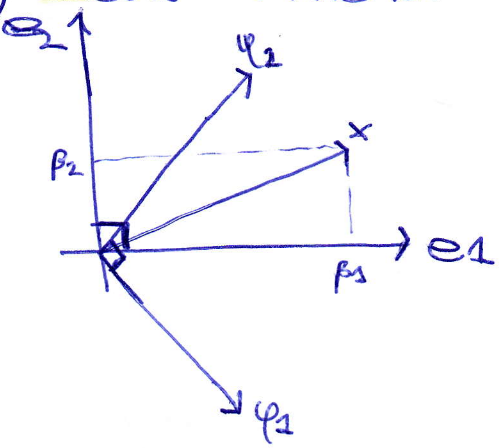
$\|\hat{x}\| = \|x\| \cos\theta_{xy}$ THE DIRECTION OF y IS e_y

$e_y = \dfrac{y}{\|y\|}$, SO $\hat{x} = \|\hat{x}\| e_y = \|x\| \cos\theta_{xy} \dfrac{y}{\|y\|} = \overbrace{\|x\| \cos\theta_{xy}}^{\langle x,y\rangle} \dfrac{y}{\|y\|^2} \|y\| =$

$\boxed{\hat{x} = \langle x, y\rangle \dfrac{y}{\|y\|^2}}$

NB I HAVE TO NORMALIZE TWICE! ONE FOR THE SCALAR PRODUCT, ONE FOR y SO IT'S $\langle\cdot\rangle / \|y\|^2$

(3) ORTHONORMAL BASIS



$\{e_1, e_2, \dots, e_N\} \equiv$ ONB

$\{\varphi_1, \varphi_2, \dots, \varphi_N\} \equiv$ ONB

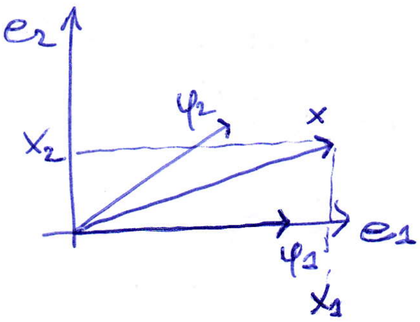
$\boxed{x =} \beta_1 e_1 + \beta_2 e_2 \quad x \in \mathbb{R}^2 \quad (\text{or } \mathbb{R}^N)$

$= \underbrace{\langle x, e_1\rangle}_{\text{ANALYSIS}} \underbrace{e_1}_{\text{SYNTHESIS}} + \langle x, e_2\rangle e_2 = \boxed{\sum_{i=1}^{N} \langle x, e_i\rangle e_i}$

$$x = \alpha_1 \varphi_1 + \alpha_2 \varphi_2 = \langle x, \varphi_1\rangle \varphi_1 + \langle x, \varphi_2\rangle \varphi_2 = \sum_{i=1}^{N} \langle x, \varphi_i\rangle \varphi_i$$

$\boxed{\|x\|^2} = \beta_1^2 + \beta_2^2 = \alpha_1^2 + \alpha_2^2 \boxed{= \sum_{i=1}^{N} |\langle x, e_i\rangle|^2}$ PARSEVAL RELATION (PARTICULAR OF BESSEL INEQUALITY)

MOST GENERAL CONDITION:



$\|\varphi_i\| \neq 1 \qquad \varphi_1 \not\perp \varphi_2$

THIS IS THE BIGGEST PROBLEM !!...

WE WANT TO USE φ_1 AND φ_2 AS A BASIS BUT φ_1, φ_2 ARE NOR UNITARY NOR ORTHOGONAL

$x = [x_1, x_2]^T$

$x = \alpha_1 \varphi_1 + \alpha_2 \varphi_2$ WE WOULD LIKE TO EXPRESS α_1, α_2 LIKE A SCALAR PRODUCT

NUMERICAL EXAMPLE:

$\varphi_1 = [1 \ 0]^T = e_1$

$\varphi_2 = [1/2 \ 1]^T$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \alpha_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \alpha_2 \begin{bmatrix} 1/2 \\ 1 \end{bmatrix} = \begin{bmatrix} \alpha_1 + 1/2\,\alpha_2 \\ \alpha_2 \end{bmatrix} \Rightarrow \begin{array}{l} \alpha_2 = x_2 \\ x_1 = \alpha_1 + \frac{1}{2}\alpha_2 = \alpha_1 + \frac{x_2}{2} \end{array}$$

$\alpha_2 = x_2 \ , \ \alpha_1 = x_1 - \frac{x_2}{2}$

$\alpha_1 = x^T \tilde{\varphi}_1 = [x_1 \ x_2] \underbrace{\begin{bmatrix} 1 \\ -1/2 \end{bmatrix}}_{\tilde{\varphi}_1}$

$\alpha_2 = [x_1 \ x_2] \underbrace{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}_{\tilde{\varphi}_2}$

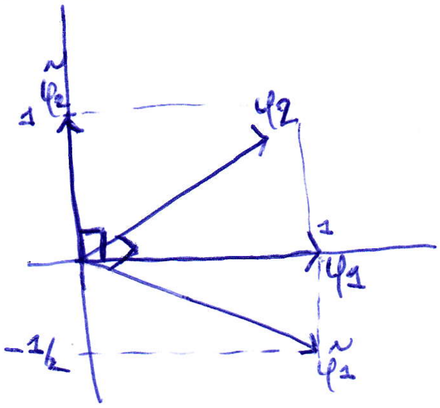
NOW I CAN REWRITE THE EQUATION:

$$x = \alpha_1\varphi_1 + \alpha_2\varphi_2 = \langle x,\tilde\varphi_1\rangle\varphi_1 + \langle x,\tilde\varphi_2\rangle\varphi_2$$

$\{\tilde\varphi_1,\tilde\varphi_2\}$ IS A DUAL BASIS

DUAL BECAUSE I CAN ALSO WRITE $x=\langle x,\varphi_1\rangle\tilde\varphi_1+\langle x,\varphi_2\rangle\tilde\varphi_2$

I REDRAW THE PROBLEM CONSIDERING $\tilde\varphi_1$ AND $\tilde\varphi_2$



$\varphi_1=[1\ \ 0]$ $\quad\tilde\varphi_1=[1\ \ -1/2]$

$\varphi_2=[1/2\ \ 1]$ $\quad\tilde\varphi_2=[0\ \ 1]$

$x\perp y$ IFF $\langle x,y\rangle=0$

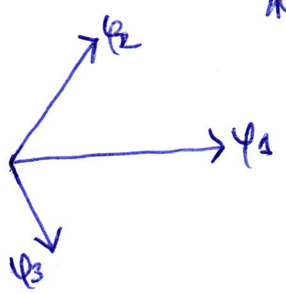
$$\boxed{\langle\varphi_i,\tilde\varphi_j\rangle = \begin{cases}1 & \text{IF } i=j\\ 0 & \text{IF } i\neq j\end{cases}}$$

BIORTHOGONALITY CONDITIONS

$\{\varphi_i\}$ IS CALLED DUAL BIORTHOGONAL TO $\{\tilde\varphi_i\}$ AND VICE VERSA

4 FRAMES

$\mathbb{R}^2$

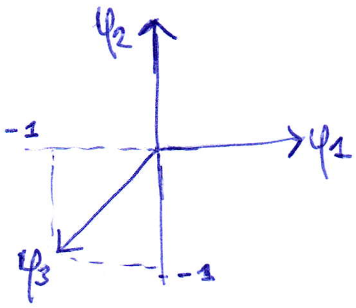


$$x=\langle x,\tilde\varphi_1\rangle\varphi_1+\langle x,\tilde\varphi_2\rangle\varphi_2+\langle x,\tilde\varphi_3\rangle\varphi_3 \text{ IN } \mathbb{R}^2$$

$$\begin{bmatrix}x_1\\x_2\end{bmatrix} = X \equiv \begin{matrix}\langle x,\tilde\varphi_1\rangle\\ \langle x,\tilde\varphi_2\rangle\\ \langle x,\tilde\varphi_3\rangle\end{matrix}$$

IF I KNOW FOR EXAMPLE THAT I HAVE LOST $\langle x,\tilde\varphi_3\rangle$ I CAN ANYWAY RECONSTRUCT x (IT'S REDUNDANT).
SO IT'S RELATED TO PRACTICAL NEEDS

$$\text{SPAN}\{\underbrace{\varphi_1,\varphi_2}^{\text{ONB}}\} = \mathbb{R}^2 = \text{SPAN}\{\varphi_1,\varphi_2,\varphi_3\}$$



$\varphi_1=[1\ \ 0]$

$\varphi_2=[0\ \ 1]$

$\varphi_3=[-1\ \ -1]$

WE ARE LOOKING FOR $\{\tilde\varphi_1,\tilde\varphi_2,\tilde\varphi_3\}$ I.E. **DUAL FRAME**

$\varphi_3=-\varphi_1-\varphi_2$

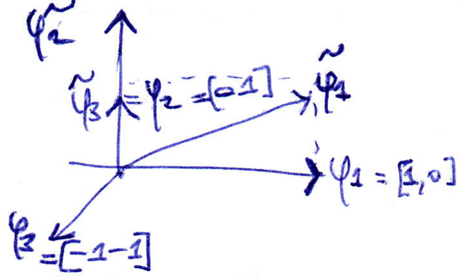
$$x=\langle x,\varphi_1\rangle\varphi_1+\langle x,\varphi_2\rangle\varphi_2+\underbrace{[\langle x,\varphi_2\rangle-\langle x,\varphi_2\rangle]\varphi_1}_{0}+\underbrace{[\langle x,\varphi_2\rangle-\langle x,\varphi_2\rangle]\varphi_2}_{0} =$$

ARBITRARY

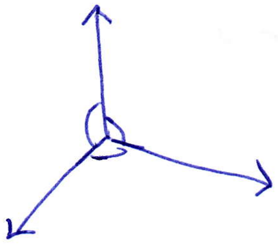
I REARRANGE A BIT

$$= \langle x, \varphi_1+\varphi_2\rangle \varphi_1 + \langle x, 2\varphi_2\rangle \varphi_2 + \langle x, \varphi_2\rangle \varphi_3 \rightarrow (-\varphi_1-\varphi_2)$$

$$\tilde{\varphi}_1 = \varphi_1+\varphi_2 \qquad \tilde{\varphi}_2 = 2\varphi_2 \qquad \tilde{\varphi}_3 = \varphi_2$$



TIGHT FRAMES (SELF-DUAL FRAMES)



$$\varphi_1 = [0\ \ 1] \qquad \|\varphi_i\| = 1$$

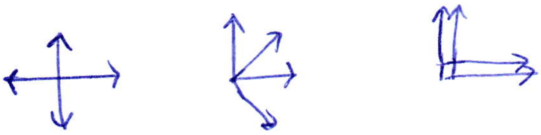
$$\varphi_2 = \left[-\frac{\sqrt{3}}{2}\ \ -\frac{1}{2}\right]$$

$$\varphi_3 = \left[\frac{\sqrt{3}}{2}\ \ -\frac{1}{2}\right]$$

$$\frac{3}{2} x = \langle x, \varphi_1\rangle \varphi_1 + \langle x, \varphi_2\rangle \varphi_2 + \langle x, \varphi_3\rangle \varphi_3$$

REDUNDANCY FACTOR

ALSO THESE:



PROJECTION INTO SUBSPACES

ASSUME S SUBSPACE OF H

A PROJECTION OPERATOR IS A P SUCH THAT $P^2 = P$ ($P \cdot P = P$)

IF WE HAVE A SET OF VECTORS $\underline{b_1}, \underline{b_2}, \dots, \underline{b_K}$ AND WE CONSIDER $B = [\underline{b_1}, \underline{b_2}, \dots, \underline{b_K}]$. WE MAY WANT TO FIND THE PROJECTION INTO THE SUBSPACE GENERATED BY THE $\underline{b_1}, \underline{b_2}, \dots, \underline{b_K}$ VECTORS THAT IS THE SET S OF VECTORS $\underline{s}$ SUCH THAT $\underline{s} = B\underline{c} = (\underline{b_1}, \underline{b_2}, \dots \underline{b_K}) \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_K \end{bmatrix} = c_1 b_1 + c_2 b_2 + \dots + c_K b_K$

AND GIVEN A $\underline{v}$ WE CAN COMPUTE A PROJECTION OF $\underline{v}$ IN S

TR: IF A IS A LEFT-INVERSE OF B THEN BA IS A PROJECTION

PROOF: $\quad B\underbrace{AB}_{I}A = BA \quad , \underset{P}{(BA)}^2 = BA$

THUS BA IS A PROJECTION INTO THE RANGE OF B

EXAMPLE: GIVEN B, FIND THE PROJECTIONS INTO THE RANGE OF B = SPAN OF THE COLUMNS OF B

$B = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{bmatrix}$ THAT IS OPERATORS P: $P^2 = P$

$P\underline{v} = \underline{v}$ FOR $\underline{v}$ IN THE RANGE OF B

WE USE THE THEOREM:

$$A = \begin{bmatrix} a_{00} & a_{01} & a_{02} \\ a_{10} & a_{11} & a_{12} \end{bmatrix}$$

NOW IF $AB = I$ THEN BA IS A PROJECTION

$$\underbrace{\begin{bmatrix} a_{00} & a_{01} & a_{02} \\ a_{10} & a_{11} & a_{12} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{bmatrix}} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} a_{00} + a_{01} & a_{01} + a_{02} \\ a_{10} + a_{11} & a_{11} + a_{12} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \Rightarrow$$

$\underset{\alpha}{a_{01}} = -a_{02} \qquad a_{00} + a_{01} = 1$

$a_{10} = -\underset{\beta}{a_{11}} \qquad a_{11} + a_{12} = 1$

$$\Rightarrow A = \begin{bmatrix} 1-\alpha & \alpha & -\alpha \\ -\beta & \beta & 1-\beta \end{bmatrix}$$

IF A IT'S LIKE THIS IT SATISFIES THE THM, SO BA IS A PROJECTION. BUT THERE IS ANOTHER CONDITION TO SATISFY:

So $BA = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1-\alpha & \alpha & -\alpha \\ -\beta & \beta & 1-\beta \end{bmatrix} = \begin{bmatrix} 1-\alpha & \alpha & -\alpha \\ 1-(\alpha+\beta) & \alpha+\beta & 1-(\alpha+\beta) \\ -\beta & \beta & 1-\beta \end{bmatrix}$

WHAT IS THE ORTHOGONAL PROJECTION?

REMEMBER THAT ORTHOGONAL PROJECTION: (1) $P^2 = P$

(2) P SELF ADJOINT $\langle P\underline{x}, \underline{y} \rangle = \langle \underline{x}, P\underline{y} \rangle \ \forall \underline{x}, \underline{y}$

SO BA IS AN ORTH PROJECTION IF:

$\alpha = \beta$

$1 - 2\alpha = \alpha \Rightarrow \alpha = 1/3 = \beta$

IN THIS CASE THE OPERATOR IS THE MATRIX $BA = P$, SO THE MATRIX MUST BE SELF-ADJOINT (OR ALSO CALLED HERMITIAN MATRIX) THAT IS:

$T = T^* \leftarrow$ TRANSPOSE CONJUGATE

$$BA = \left.\begin{bmatrix} 2/3 & 1/3 & -1/3 \\ 1/3 & 2/3 & 1/3 \\ -1/3 & 1/3 & 2/3 \end{bmatrix}\right\} \text{THIS IS THE CORRECT PROJECTION OPERATOR}$$

CHECK: WITH THE NORMAL EQUATION IF WE HAVE $\underline{v}$ AND WE COMPUTE THE ORTHOGONAL PROJECTION INTO THE RANGE OF B WE ARE COMPUTING

$$\underline{\hat{v}} = B\underline{c} = c_1 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

$\underline{c}$ ← VECTOR OF COEFFICIENTS

AND WE WANT c_1 AND c_2 BE SUCH THAT $(\underline{v} - \underline{\hat{v}}) \perp \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ Y

$(\underline{v} - \underline{\hat{v}}) \perp \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$

WE IMPOSE ORTHOGONALITY OF $\underline{v} - \underline{\hat{v}}$ TO THE RANGE OF B

B^* ← TRANSPOSE CONJUGATE

$$B^*(\underline{v} - \underline{\hat{v}}) = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} (\underline{v} - \underline{\hat{v}}) = \underline{0}$$

$\Downarrow$

$$B^*\underline{v} = B^*\underline{\hat{v}} = B^*B\,\underline{c}$$

$\underline{c}$ ← VECTOR OF UNKNOWN COEFFICIENTS

"NORMAL EQUATIONS" →

$\Rightarrow$ IF B^*B IS INVERTIBLE:

$$(B^*B)^{-1} B^* \underline{v} = \underline{c}$$

I CAN GET THE COEFFICIENTS

$$\Rightarrow \underline{\hat{v}} = B \underbrace{(B^*B)^{-1} B^*}_{A} \underline{v}$$

THE PREVIOUS A NEEDED TO SATISFY $AB = I$ AND IN FACT $\underbrace{(B^*B)^{-1} B^*}_{A} B = I$

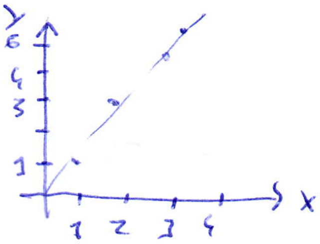
FOR THE ORTHOGONAL PROJ:

$$(B^*B)^{-1}B^* = \left(\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{bmatrix} \right)^{-1} \left(\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \right) = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}^{-1} \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 2/3 & -1/3 \\ -1/3 & 2/3 \end{pmatrix} \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 2/3 & 1/3 & -1/3 \\ -1/3 & 1/3 & 2/3 \end{pmatrix}$$

(FOR THE ORTH PROJ. $\alpha = \beta = 1/3$!) $\begin{pmatrix} 2/3 & 1/3 & -1/3 \\ -1/3 & 1/3 & 2/3 \end{pmatrix} \overbrace{\begin{pmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{pmatrix}}^{B} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

EXAMPLE OF APPLICATION OF ORTHOGONAL PROJECTIONS:

ASSUME WE HAVE A SET OF OBSERVATIONS



WE HAVE 4 OBSERVATIONS OF AN UNKNOWN FUNCTION (OR NOISY DATA)

x_i	1	2	3	4
y_i	1	3	4	6

WE WANT TO FIND THE "BEST" LINEAR FIT TO THE OBSERVED DATA

$y(x) = ax + b$ for some $a, b \in \mathbb{R}$

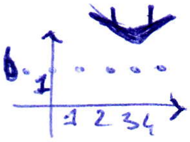
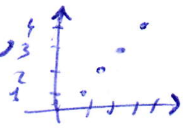
"BEST" IN THE SENSE THAT IT MINIMIZES THE SQUARE ERROR $e_i = y(x_i) - y_i$

WE WANT $\sum_{i=1}^{4} e_i^2$ TO BE AS SMALL AS POSSIBLE.

NOTE: $\underline{y} = \begin{bmatrix} 1 \\ 3 \\ 4 \\ 6 \end{bmatrix}$ IS A VECTOR IN $\mathbb{R}^4$ AND WE WANT TO FIND A FUNCTION $\underbrace{y(x)}_{=ax+b}$ GIVEN OBSERVATIONS AT POINTS $x = 1, 2, 3, 4$

THEN WE CAN WRITE: $ax + b = a \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} + b \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$

AND $\underline{e} = \begin{bmatrix} 1 \\ 3 \\ 4 \\ 6 \end{bmatrix} - \left(a \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} + b \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \right)$. SO WE WANT TO FIND AN APPROXIMATION OF $[1\ 3\ 4\ 6]^*$ IN THE SUBSPACE

$S = \text{SPAN}\left(\begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \right)$ $\approx b \cdot$  $+ a \cdot$ 

AND WE WANT $\underline{y} - \underline{y}_S$ TO HAVE MINIMUM NORM $\Rightarrow$ $\underline{y}_S$ MUST BE THE ORTHOGONAL PROJECTION

SO, USING: $B = \begin{bmatrix} 1 & 1 \\ 2 & 1 \\ 3 & 1 \\ 4 & 1 \end{bmatrix}$ $\begin{bmatrix} a \\ b \end{bmatrix}$

$\underline{y}_S = B \underbrace{\begin{bmatrix} a \\ b \end{bmatrix}}_{\underline{y}} = \begin{bmatrix} 1 & 1 \\ 2 & 1 \\ 3 & 1 \\ 4 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix}$ WE WANT $(\underline{y} - \underline{y}_S) \perp S$

TRANSPOSE CONJUGATE

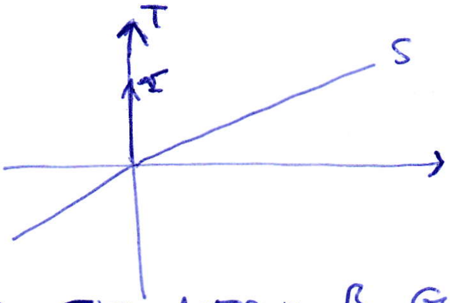
$\Rightarrow B^* \underline{y} = \underbrace{B^* \underline{y}_S}_{B^*B \begin{bmatrix} a \\ b \end{bmatrix}} \Rightarrow \begin{bmatrix} a \\ b \end{bmatrix} = \underbrace{(B^*B)^{-1} B^*}_{\text{NORMAL EQUATIONS!!}} \underline{y}$

$B^*B = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 2 & 1 \\ 3 & 1 \\ 4 & 1 \end{bmatrix} = \begin{bmatrix} 30 & 10 \\ 10 & 4 \end{bmatrix}$ $\det(B^*B) = 20 \Rightarrow (B^*B)^{-1} = \frac{1}{20} \begin{bmatrix} 4 & -10 \\ -10 & 30 \end{bmatrix}$

$B^*\underline{y} = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 3 \\ 4 \\ 6 \end{bmatrix} = \begin{bmatrix} 43 \\ 14 \end{bmatrix} \Rightarrow \begin{bmatrix} a \\ b \end{bmatrix} = \underbrace{\frac{1}{20} \begin{bmatrix} 4 & -10 \\ -10 & 30 \end{bmatrix}}_{(B^*B)^{-1}} \underbrace{\begin{bmatrix} 43 \\ 14 \end{bmatrix}}_{(B^*\underline{y})} = \begin{bmatrix} 1/5 & -1/2 \\ -1/2 & 3/2 \end{bmatrix} \begin{bmatrix} 43 \\ 14 \end{bmatrix}$

TR: GIVEN A SUBSPACE S OF H (HILBERT SPACE) AND A PROJECTION P INTO S, THEN P DEFINES A SUBSPACE T SUCH THAT:

$H = S \oplus T$



$\forall \underline{v} \in H$ THERE ARE UNIQUE $\underline{v}_S \in S$, $\underline{v}_T \in T$, $\underline{v} = \underline{v}_S + \underline{v}_T$

FOR THE ORTHOGONAL PROJECTION $T \perp S$

FOR THE MATRIX B GIVEN BEFORE AND FOR THE GENERAL PROJECTOR BA FOUND, DETERMINE T.

WE HAD $S = \text{RANGE OF } B$, $\dim(S) = 2$, $\dim(H) = 3 \Rightarrow \dim(T) = 1$

SO WE CAN DETERMINE T IF WE JUST FIND ONE VECTOR IN T.

A SIMPLE WAY TO DO THIS IS TO TAKE A $\underline{v} \notin S$, COMPUTE $\underline{v}_S$ (PROJECTION ON SUBSPACE S) AND THEN $\underline{v} - \underline{v}_S = \underline{v}_T$.

$B = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{bmatrix}$ WE CAN TAKE $\underline{v} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, THEN $\underline{v}_S = P_S \underline{v} = BA\underline{v} =$

$$\underline{v}_S = \begin{pmatrix} 1-\alpha & \alpha & -\alpha \\ 1-(\alpha+\beta) & \alpha+\beta & 1-(\alpha+\beta) \\ -\beta & \beta & 1-\beta \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1-\alpha \\ 2-(\alpha+\beta) \\ 1-\beta \end{pmatrix}$$

SO FOR FIXED α, β, WE HAVE ONE PARTICULAR PROJECTION:

$$\underline{v}_S = \begin{pmatrix} 1-\alpha \\ 2-(\alpha+\beta) \\ 1-\beta \end{pmatrix} \quad \text{AND} \quad \underline{v} - \underline{v}_S = \begin{pmatrix} \alpha \\ (\alpha+\beta)-1 \\ \beta \end{pmatrix}$$

THEN THE SUBSPACE T IS SIMPLY: $T = \left\{ c \cdot \begin{pmatrix} \alpha \\ (\alpha+\beta)-1 \\ \beta \end{pmatrix}, c \in \mathbb{R} \right\}$

CHECK: $\underline{v} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ $\quad \underline{v}_S = \begin{pmatrix} 1-2\alpha \\ 2-2(\alpha+\beta) \\ 1-2\beta \end{pmatrix}$ $\quad \underline{v} - \underline{v}_S = \begin{pmatrix} 2\alpha \\ 2(\alpha+\beta)-2 \\ 2\beta \end{pmatrix} = 2\begin{pmatrix} \alpha \\ (\alpha+\beta)-1 \\ \beta \end{pmatrix}$

WE ARE ESSENTIALLY SAYING THAT IN $\mathbb{R}^3$ WE CAN COMPLETE THE SET OF VECTORS $B = \left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right\}$ WITH THE VECTOR $\begin{pmatrix} \alpha \\ (\alpha+\beta)-1 \\ \beta \end{pmatrix}$ TO FORM A BASIS FOR $\mathbb{R}^3$.

(WE CHOSE ANOTHER $\underline{v} \notin S$ AND WE FIND A SOLUTION THAT IS LINEARLY DEPENDENT FROM THE PREVIOUS ONE!!! SO IT'S STILL A SOLUTION.)

FRAMES

LET

$$\Phi = \begin{array}{c} \begin{matrix} \varphi_1 & \varphi_2 & \varphi_3 & \varphi_4 \end{matrix} \\ \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \end{array} \qquad \Psi = \begin{array}{c} \begin{matrix} \psi_1 & \psi_2 & \psi_3 & \psi_4 \end{matrix} \\ \begin{bmatrix} 1 & 0 & 0 & -1 \\ -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{bmatrix} \end{array}$$

(1) PROVE THAT

$$\Psi = \{\underline{\psi_1}, \underline{\psi_2}, \underline{\psi_3}, \underline{\psi_4}\}$$

DO NOT FORM A BASIS FOR $\mathbb{R}^4$

(2) PROVE THAT

$$F = \{\underline{\varphi_1}, \underline{\varphi_2}, \underline{\varphi_3}, \underline{\psi_1}, \underline{\psi_2}, \underline{\psi_3}\}$$

IS A FRAME AND COMPUTE ITS BOUNDS

(3) FIND THE CANONICAL DUAL FRAME OF F

(1) THE VECTORS $\underline{\psi_1}, \underline{\psi_2}, \underline{\psi_3}, \underline{\psi_4}$ FORM A BASIS OF $\mathbb{R}^4$ IF THEY ARE LINEARLY INDEPENDENT

WE SEE THAT $\underline{\psi_4} = -\sum_{i=1}^{3} \underline{\psi_i}$ $\Rightarrow$ NOT LIN. INDEP. $\Rightarrow$ NOT A BASIS
EQUIVALENTLY THE DETERMINANT OF $\Psi = 0$
SO THE COLUMNS ARE LINEARLY DEPENDENT

(2)
$$F = \begin{pmatrix} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 & -1 \end{pmatrix}$$

WE ONLY HAVE TO CHECK THAT THE RANK OF **F** IS 4

BOUNDS OF A FRAME $F = \{f_1, f_2, \dots, f_k\}$

$$\lambda_{\min}^{f} \|\underline{x}\|^2 \leq \sum_{i=1}^{K} |\langle \underline{x}, \underline{f_i} \rangle|^2 \leq \lambda_{\max}^{f} \|\underline{x}\|^2$$

IF WE **TAKE** THE MATRIX FF^* AND COMPUTE THE EINGENVALUES

$\lambda_{\min}$ IS THE MINIMUM, $\lambda_{\max}$ IS THE MAXIMUM

$$FF^* = \begin{pmatrix} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 \end{pmatrix} = \begin{pmatrix} 2 & -1 & 0 & 0 \\ -1 & 3 & -1 & 0 \\ 0 & -1 & 3 & -1 \\ 0 & 0 & -1 & 1 \end{pmatrix}$$

$\lambda_{\min} = 3$

$\lambda_{\max} = 5$

(ACTUALLY $\lambda_{\max} = 4.3623$, $\lambda_{\min} = 0.4915$)

FIND THE CANONICAL DUAL FRAME

$$\underline{x} = F\underline{c}$$

WE WANT $\underline{c}$ TO BE A LINEAR FUNCTION OF $\underline{x}$; $\underline{c} = A\underline{x}$

$\underline{x} = FA\underline{x}$, EASY IF WE CAN FIND AN A SUCH THAT $FA = I$

CONSIDER: $A = F^*(FF^*)^{-1}$ $\rightarrow FA = FF^*(FF^*)^{-1} = I$

WE NEED FF^* TO BE INVERTIBLE. ALWAYS POSSIBLE IF F IS A FRAME BECAUSE THE COLUMNS SPAN THE WHOLE SPACE.

NOW THE MATRIX A CONTAINS THE ANALYSIS VECTORS IN ITS ROWS

$$\Rightarrow A = \tilde{\Phi}^*$$

$$\tilde{\Phi}^* = (FF^*)^{-1}F = \begin{pmatrix} 2 & -1 & 0 & 0 \\ -1 & 3 & -1 & 0 \\ 0 & -1 & 3 & -1 \\ 0 & 0 & -1 & +1 \end{pmatrix}^{-1} \begin{pmatrix} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 & -1 \end{pmatrix}$$

Ex:

$$\Phi = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix} \qquad \Psi = \begin{bmatrix} 1 & 0 & 0 & -1 \\ -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & +1 \end{bmatrix} \qquad F = [\Phi \ \Psi]$$

FIND THE BOUNDS AND THE CANONICAL DUAL FRAME

$\psi_1 - \psi_2 + \psi_3 = \psi_4 \Rightarrow \Phi$ AND Ψ NOT BASIS

$$FF^* = \begin{pmatrix} 1 & 0 & 0 & 1 & 1 & 0 & 0 & -1 \\ 1 & 1 & 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & -1 & +1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & -1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & +1 & -1 \\ -1 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 4 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 4 \end{pmatrix} = 4I$$

(IN A DIAGONAL MATRIX THE EIGENVALUES ARE EQUAL TO THE ELEMENTS ON THE DIAGONAL)

SO $\lambda_{MIN} = \lambda_{MAX} = 4 \Rightarrow$ TIGHT FRAME! THAT IS: $\lambda \|\underline{x}\|^2 = \sum_i |\langle x, f_i \rangle|^2$

DUAL FRAME:

$$\underbrace{(FF^*)^{-1}}_{\frac{1}{4}I} \cdot F$$

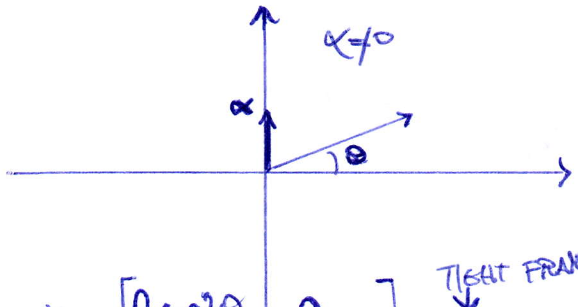
$$\frac{1}{4} I \cdot F = \frac{1}{4} F$$

NOTE THAT ~~[illegible]~~ WE HAVE A TIGHT FRAME WHEN: $FF^* = \lambda I$

THIS IS SATISFIED WHENEVER THE ROWS OF F ARE ORTHOGONAL WITH THE SAME NORM

$\lambda = 1$ IF THE ROWS OF F ARE ORTHONORMAL

IN $\mathbb{R}^2$ FIND ALL TIGHT FRAMES OF THE FORM:



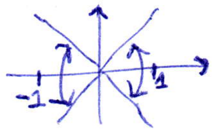
$$\phi = \begin{bmatrix} 0 & \cos\theta & -\cos\theta \\ \alpha & \sin\theta & \sin\theta \end{bmatrix}$$

$$\phi\phi^* = \begin{bmatrix} 2\cos^2\theta & 0 \\ 0 & \alpha^2 + 2\sin^2\alpha \end{bmatrix} \overset{\text{TIGHT FRAME}}{\triangleq} c \cdot I = \begin{bmatrix} c & 0 \\ 0 & c \end{bmatrix}$$

$$\begin{cases} 2\cos^2\theta = c \\ \alpha^2 + 2\sin^2\theta = c \end{cases} \Rightarrow \alpha^2 + 2\underbrace{(\cos^2\theta + \sin^2\theta)}_{1} = 2c \Rightarrow \alpha^2 + 2 = 2c = 4\cos^2\theta$$

$$\alpha^2 = 2(2\cos^2(\theta) - 1) = 2(\cos^2\theta - \sin^2\theta) = 2\cos(2\theta) \Rightarrow \alpha^2 = 2\cos(2\theta)$$

$$\Rightarrow \cos(2\theta) > 0 \qquad (\alpha^2 \geq 0,\ \alpha \neq 0)$$



BUT WE ALSO KNOW $\cos(2\theta) \leq 1 \Rightarrow \alpha^2 \leq 2 \Rightarrow 0 \leq |\alpha| < \sqrt{2}$

FROM $\cos(2\theta) > 0$ WE OBTAIN: $2k\pi - \frac{\pi}{2} < 2\theta < 2k\pi + \frac{\pi}{2} \rightarrow k\pi - \frac{\pi}{4} < \theta < k\pi + \frac{\pi}{4}$

WE FORCED THE FRAME TO BE TIGHT AND SO THE DUAL FRAME WILL BE

$$\frac{1}{c}\phi = \frac{1}{2\cos^2(\theta)} \begin{bmatrix} 0 & \cos\theta & -\cos\theta \\ \pm\sqrt{2\cos(2\theta)} & \sin\theta & \sin\theta \end{bmatrix}$$

EX:

CONSIDER $\mathcal{L}^2([0,1])$ AND THE SUBSPACE S OF AFFINE FUNCTIONS.

FIND A 4-ELEMENT $\underset{\text{TIGHT}}{\wedge}$ FRAME FOR S THAT INCLUDES THE SIGNALS

$$\varphi_0(t) = 1 \qquad \varphi_1(t) = \sqrt{3}\,t$$

FIND THE FRAME BOUND λ AND THE CANONICAL DUAL FRAME;

HINT: THE UNION OF ORTHONORMAL BASES IS ALWAYS A TIGHT FRAME

ASSUME WE HAVE A SPACE OF DIMENSION 1 AND K DIFFERENT $\underset{\text{ORTHONORMAL}}{\wedge}$ BASES FOR THE SPACE:

$$\Psi_1 = \{\psi_{11}, \ldots, \psi_{1d}\}$$

$$\Psi_K = \{\psi_{K1}, \ldots, \psi_{Kd}\}$$

$$F = \{\Psi_1, \Psi_2, \ldots, \Psi_K\} \quad \text{FRAME}$$

NOTE THAT $\forall \underline{x}$:

$$\sum_{i=1}^{K} \underbrace{\sum_{j=1}^{d} |\langle \underline{x}, \underline{\Psi}_{ij} \rangle|^2}_{\|\underline{x}\|} = K\|\underline{x}\|$$

THE INNER SUM IS THE NORM OF $\underline{x}$ COMPUTED USING THE PARSEVAL RELATION FOR THE ORTHONORMAL BASIS Ψ_i

$$\lambda_{min} \leq \sum_{i=1}^{K} \underbrace{\sum_{j=1}^{d} |\langle \underline{x}, \underline{\Psi}_{ij} \rangle|^2}_{\|\underline{x}\|^2} \leq \lambda_{max}\|\underline{x}\|^2$$

SO WE CAN BUILD 2 ORTHONORMAL BASIS FROM $\underline{\varphi}_0$ AND $\underline{\varphi}_1$ WITH $\underline{\varphi}_0$ IN THE FIRST AND $\underline{\varphi}_1$ IN THE SECOND, $\Psi_1 = \{\underset{\underset{\underline{\varphi}_0}{\shortparallel}}{\underline{\Psi}_{11}}, \underline{\Psi}_{12}\}$ $\Psi_2 = \{\underset{\underset{\underline{\varphi}_1}{\shortparallel}}{\underline{\Psi}_{21}}, \underline{\Psi}_{22}\}$

USE GRAM-SCHMIDT PROCEDURE!

START WITH $\underline{\varphi}_0$

$\Psi_{11} = \varphi_0$

$\langle \underline{\varphi}_1, \underline{\Psi}_{11} \rangle = \langle \underline{\varphi}_1, \underline{\varphi}_0 \rangle = \int_0^1 1 \cdot \sqrt{3}\, t\, dt = \frac{\sqrt{3}}{2} t^2 \Big|_0^1 = \frac{\sqrt{3}}{2}$

$$\underline{\Psi}_{12} = \frac{\boxed{\underline{\varphi}_1 - \langle \underline{\varphi}_1, \underline{\Psi}_{11} \rangle \underline{\Psi}_{11}}}{\|\underline{\varphi}_1 - \langle \underline{\varphi}_1, \underline{\Psi}_{11} \rangle \underline{\Psi}_{11}\|} \qquad \longleftrightarrow (\sqrt{3}t - \tfrac{\sqrt{3}}{2}) \qquad \rightarrow 1/2$$

$$\|\underline{\varphi}_1 - \langle \underline{\varphi}_1, \underline{\Psi}_{11} \rangle \underline{\Psi}_{11}\|^2 = \|\underline{\varphi}_1\|^2 - \|\langle \underline{\varphi}_1, \underline{\Psi}_{11} \rangle \underline{\Psi}_{11}\|^2 = 1 - 3/4 = 1/4$$

$\|\underline{\Psi}_{11}\| = 1$

$$\Psi_{12}(t) = \frac{\sqrt{3}t - \sqrt{3}/2}{1/2} = 2\sqrt{3}t - \sqrt{3}$$

STARTING WITH φ_1:

$\underline{\Psi}_{21} = \underline{\varphi}_1$

$$\underline{\Psi}_{22} = \frac{\underline{\varphi}_0 - \langle \underline{\varphi}_0, \underline{\Psi}_{21} \rangle \underline{\Psi}_{21}}{\|\underline{\varphi}_0 - \langle \underline{\varphi}_0, \underline{\Psi}_{21} \rangle \underline{\Psi}_{21}\|} = \frac{1 - \frac{\sqrt{3}}{2}\sqrt{3}t}{1/2} = 2 - 3t.$$

$$\|\underline{\varphi}_0 - \langle \underline{\varphi}_0, \underline{\Psi}_{21} \rangle \underline{\Psi}_{21}\| \rightarrow \sqrt{\|\underline{\varphi}_0\|^2 - |\langle \underline{\varphi}_0, \underline{\varphi}_1 \rangle|^2} = \sqrt{1/4} = 1/2$$

$$F = [1 \quad \sqrt{3}t \quad 2\sqrt{3}t - \sqrt{3} \quad 2 - 3t]$$

$\lambda = 2$ (TWO ORTHONORMAL BASES IN THE FRAME)

FF^*?

EXERCISE: AUDIO APPLICATION

I CAN PLAY WITH QUANTIZATION AND SAMPLING FREQUENCY. IT MAKES USE OF FRAMES. I TRY TO OBTAIN MORE BITS INCREASING THE SAMPLE FREQUENCY TO SAVE ON THE COST OF CONVERTERS. (OVERSAMPLING).

SAMPLING RATE	QUANT.		SAMPLING RATE	QUANT.
MT	q (FOR EX. 12)	$=$	$\frac{MT}{K}$	$\bar{q} + \log_2(\sqrt{K})$

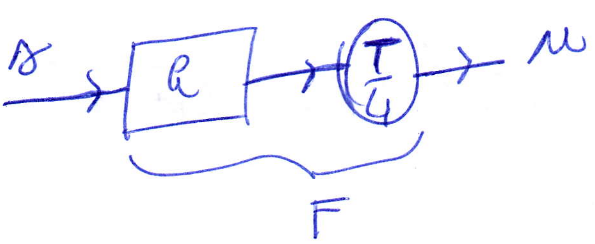
(FOR EX: $\bar{q} = 10$, $\log_2(\sqrt{K}) = 2$, $K = 16$)

(CONTINUOUS SIGNAL)

$$s(t) \in \mathcal{L}^2(\mathbb{R}) \xrightarrow{L.P.F.} \left[-\frac{1}{2T}, \frac{1}{2T}\right] \xrightarrow{\text{SAMPLING}} f_s = \frac{1}{T}$$

$\text{rect}\left(\frac{f}{1/T}\right)$

$K=4$



$$u = h * s = \left[s * \text{sinc}\left(\frac{t}{T}\right)\right]\left[\frac{kT}{4}\right]$$

$$\Updownarrow$$

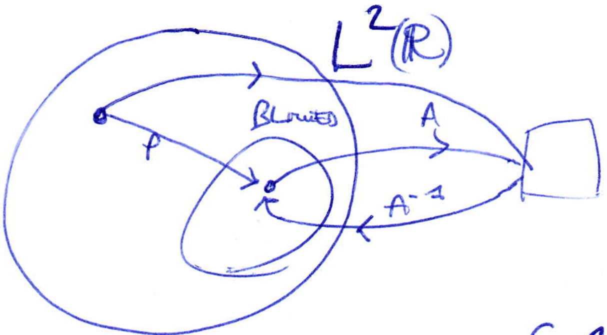
$$U = F \cdot S$$

$$u = \int_{\mathbb{R}} s(\tau)\,\text{sinc}\left(\frac{\frac{kT}{4} - \tau}{T}\right) d\tau = \underbrace{\int_{\mathbb{R}} s(\tau)\,\text{sinc}\left(\frac{\tau - \frac{kT}{4}}{T}\right) d\tau}_{\text{ANALYSIS EQUATION}} = \langle x, \sigma^{\frac{kT}{4}} \text{sinc}_T \rangle$$

DELAY OPERATOR

$$= u_k$$

$k \in \mathbb{Z}$

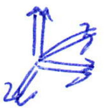


For band limited signals in $\left[-\frac{1}{2T}, \frac{1}{2T}\right]$ a base is $\{\sigma^{\ell T} \text{sinc}_T\}_{\ell \in \mathbb{Z}}$

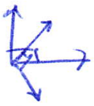
Now my operator (analysis) F is actually associated to Φ:

$$\boxed{\Phi = \left\{ \sigma^{\frac{kT}{4}} \operatorname{sinc}_T \right\}_{k\in\mathbb{Z}} \equiv \bigcup_{i=0}^{3} \left\{ \sigma^{\frac{iT}{4}} \sigma^{\ell T} \operatorname{sinc}_T \right\}_{\ell\in\mathbb{Z}}} = \text{TIGHT FRAME}$$

k

IT'S A TIGHT FRAME WITH REDUNDANCY 4 

THE REDUNDANCY IS USED TO PRODUCE HIFI MUSIC IN THIS CASE.

( REDUNDANCY FACTOR $4/2=2$

 $3/2$)

Φ IS A TIGHT FRAME $\Rightarrow \sum_{k\in\mathbb{Z}} |\langle \sigma^{\frac{kT}{4}}, x\rangle|^2 = 4\|x\|^2$ (PARSEVAL)

$\Rightarrow$ DUAL FRAME $\Psi = \left\{ \frac{1}{4} \sigma^{\frac{kT}{4}} \operatorname{sinc}_T \right\}_{k\in\mathbb{Z}}$

$\downarrow$

F^+ inverse operator $\Rightarrow F^+ F x = x = F^+ \mu_k = x$

Quantization: $\mu = \{\mu_k\}$ $\quad Q(\mu) = \hat{\mu} = \mu + \varepsilon_q \leftarrow$ QUANTIZATION ERROR

$$x \to \underbrace{\boxed{h_F} \to (\downarrow T/4)}_{F} \xrightarrow{\mu} \boxed{Q_q} \xrightarrow{\hat{\mu}} (\uparrow T/4) \to \boxed{h_{F^+}} \to \hat{x}$$

(OVERSAMPLING)

$\leftarrow$ THIS IS CHEAPER THAN THIS CAUSE USES LESS BIT

$\hat{x} = x + \varepsilon_r = F^+(\mu + \varepsilon_q) = x + F^+\varepsilon_q$

$\uparrow$ RECONSTRUCTION ERROR

$\|\varepsilon_r\|^2 = \underline{\text{ENERGY OF THE RECONSTRUCTED ERROR}} = \frac{1}{4}\|\varepsilon_q\|^2 \longrightarrow \boxed{\|\varepsilon_r\| = \frac{1}{2}\|\varepsilon_q\|}$

$$\boxed{\|\varepsilon_{q+1}\| = \frac{1}{2}\|\varepsilon_q\|}$$

$$x \to \underbrace{\boxed{h} \to (\downarrow T)}_{\text{SAMPLER}} \to \boxed{Q_{q+1}} \to \underbrace{(\uparrow T) \to \boxed{h}}_{\text{INTERPOLATOR}} \to \hat{x}$$

PROJECTION OPERATORS; WE USE THE TWO KNOWN THEOREMS

TH 1 $A: H_0 \to H_1$ AND $B: H_1 \to H_0$ ARE BOUNDED LINEAR OPERATORS

IF $AB = \mathbb{1} \Rightarrow BA$ IS THE OBLIQUE PROJECTION OPERATOR

$B^t = A^* \Rightarrow BA$ IS THE ORTHOGONAL PROJECTION OPERATOR

GIVEN B WE COMPUTE A SUCH THAT A IS THE LEFT INVERSE;

$$B = \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \Rightarrow \begin{bmatrix} a_{00} & a_{01} & a_{02} \\ a_{10} & a_{11} & a_{12} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

WE CAN FIND PARAMETRIC EXPRESSIONS OF THE ELEMENTS OF A (α, β)
WE EXPLOIT THE FACT THAT A PROJECTION OPERATOR IS IDEMPOTENT $(BA(BA) = BA)$
AND SELF ADJOINT $((BA)^* = BA)$ AND WE OBTAIN A.

WE CAN OBTAIN THE SAME RESULT IF WE KNOW THAT IT IS AN ORTHOGONAL PROJECTION OPERATOR

$$v - \hat{v} \perp R(B)$$

$$B^*(v - \hat{v}) = 0 \quad \text{(SCALAR PRODUCT)}$$

$$B^* v = B^* \hat{v} \to \hat{v} = Bc \quad \text{(IT IS A LINEAR COMBINATION OF THE VECTORS OF THE BASE } R(B))$$

$$B^* v = B^* B c$$

$$c = \underbrace{(B^*B)^{-1} B^*}_{A} v$$

$$\hat{v} = B (B^*B)^{-1} B^* v$$

WE USED THE THEOREM OF THE ORTHOGONAL PROJECTIONS BY USING THE PSEUDO INVERSE.

TH2 $A: H_0 \to H_1$ IS A BOUNDED LINEAR OPERATOR

a) IF A^*A IS INVERTIBLE, THEN $B = (A^*A)^{-1} A^*$ IS THE PSEUDO INVERSE OF A
AND $AB = A(A^*A)^{-1}A^*$ IS THE ORTHOGONAL PROJECTION OPERATOR ON THE RANGE OF A, $R(A)$.

b) IF AA^* IS INVERTIBLE, THEN $B = A^*(AA^*)^{-1}$ IS THE PSEUDO INVERSE OF A
AND $BA = A^*(AA^*)^{-1}A$ IS THE ORTHOGONAL PROJECTION OPERATOR ON THE RANGE OF A, $R(A)$.

WE CAN COMPUTE $S^\perp$, THE ORTHOGONAL COMPLEMENT OF $S = R(B)$ BY
EXPLOITING THE FOLLOWING PROPERTY $S^\perp \oplus S = H$.
WE COMPUTE THE ORTHOGONAL PROJECTION OF $v \in H$ ON $R(B)$ AND FROM
THE DIFFERENCE BETWEEN v AND $\hat{v}$ WE OBTAIN A BASE OF $B^\perp$.

LINEAR INTERPOLATION

WE CAN EXPLOIT THE ORTHOGONAL PROJECTION THEOREMS TO COMPUTE THE BEST LINEAR INTERPOLATION.

DATA: x_i, y_i

THE INTERPOLATION LINE IS OF THE FORM $y(x_i) = a x_i + b$ $\quad b = \begin{bmatrix}1\\1\\1\\1\end{bmatrix}$

WE WANT TO MINIMIZE: $e = y - y(x_i) = \begin{bmatrix}1\\3\\4\\0\end{bmatrix} - \left(a\begin{bmatrix}1\\2\\3\\4\end{bmatrix} + b\begin{bmatrix}1\\1\\1\\1\end{bmatrix}\right)$

THAT IS WE MINIMIZE $\|e\|^2$. WE LOOK FOR THE ORTHOGONAL PROJECTION ON

$\mathrm{SPAN}\left(\begin{bmatrix}1\\2\\3\\4\end{bmatrix}, \begin{bmatrix}1\\1\\1\\1\end{bmatrix}\right)$ WHICH PROJECTION OPERATOR IS $P = B(B^*B)^{-1}B^*$

FRAME

THE DUAL FRAME CAN BE COMPUTED WITH THE PSEUDO-INVERSE.

WE WANT THAT $\quad F\tilde{F}x = x \Rightarrow F\tilde{F} = I$

$$\tilde{F} = F^*(FF^*)^{-1} \Rightarrow F\tilde{F} = FF^*(FF^*)^{-1} = I$$

THE FRAME STABILITY CONSTANTS ARE THE MAX AND THE MIN EIGENVECTORS OF FF^*

IF A FRAME IS THE UNION OF TWO ORTHONORMAL BASES WE OBTAIN FOR SURE A TIGHT FRAME.

THE DUAL IS: $\quad \tilde{F} = \frac{1}{\lambda} F = \frac{1}{\lambda} \underbrace{I}_{FF^*} F$

CONSTRUCTION OF FRAMES:

THE SIMPLEST METHOD IS TO FORCE TO HAVE A TIGHT FRAME.

IF I HAVE A PARAMETRIC ESPRESSION OF ONE OF THE TWO FRAMES I IMPOSE:

$$\Phi \Phi^* \sim I$$

IF I HAVE TWO ELEMENTS THAT COMPOSE THE FRAME I CONSTRUCT ORTHONORMAL BASES STARTING FROM THOSE ELEMENTS AND I MERGE THEM.

BIORTHOGONAL BASIS

$\langle e_1, \varphi_1 \rangle = 1 \qquad \langle e_2, \varphi_1 \rangle = 0$

$\langle e_1, \varphi_2 \rangle = 0 \qquad \langle e_2, \varphi_2 \rangle = 1$

$$\begin{bmatrix} e_1\varphi_1 & e_1\varphi_2 \\ e_2\varphi_1 & e_2\varphi_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I \Rightarrow \begin{bmatrix} e_1 \\ e_2 \end{bmatrix} \cdot \begin{bmatrix} \varphi_1 & \varphi_2 \end{bmatrix} = I.$$