Planck Radiation

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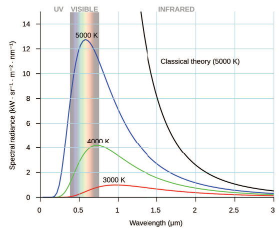
INTRODUCTION

Emission of EM Radiation

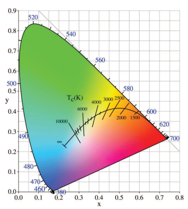
- Every physical body spontaneously and continuously emits electromagnetic radiation.
- Near thermodynamic equilibrium, the emitted radiation is nearly described by Planck's law. Because of its dependence on temperature, Planck radiation is said to be thermal.
- The higher the temperature of a body, the more radiation it emits at every wavelength. Planck radiation has a maximum intensity at a specific wavelength that depends on the temperature.



INTRODUCTION

Emission of EM Radiation

- For example, at room temperature ($300K$), a body emits thermal radiation that is mostly infrared and invisible.
  - At higher temperatures the amount of infrared radiation increases and can be felt as heat, and the body glows visibly red.
  - At even higher temperatures, a body is dazzlingly bright yellow or blue-white and emits significant amounts of short wavelength radiation, including ultraviolet and even x-rays.
  - The surface of the sun ($6000K$) emits large amounts of both infrared and ultraviolet radiation; its emission is peaked in the visible spectrum.



INTRODUCTION

Absorption and Emission of EM Radiation

- In the interior of a physical medium, radiation can be absorbed and emitted by matter.
  - This mediates transfer of energy as heat, and can change the internal energy of the matter, and the occupation numbers of the states of its molecules.
  - Planck radiation is the greatest amount of radiation that any body at thermal equilibrium can emit from its surface, whatever its chemical composition or surface structure.
  - Passage of radiation across an interface between media can be characterized by the emissivity of the interface, the radiance of the passing radiation divided by the Planck radiance.
  - It is in general dependent on chemical composition and physical structure, on temperature, on the wavelength, on the angle of passage, and on the polarization, of the radiation.

INTRODUCTION

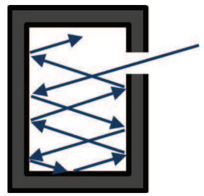
Absorption and Emission of EM Radiation

- The emissivity of an interface is also known as its transmittance or as its absorbance.
- The emissivity of a natural interface is always between zero and one. For an interface, the sum of emissivity and reflectivity is one.
- An ideally perfectly reflecting interface has emissivity zero, reflectivity one. An ideally perfectly transmitting interface has emissivity one, reflectivity zero.

BLACK BODY

Model

- A body that interfaces with another medium with emissivity one, and that absorbs all the radiation incident upon it, is said to be a black body.
  - The surface of a black body can be modeled by a small hole in the wall of a large enclosure which is maintained at a uniform temperature with rigid opaque walls that are not perfectly reflective at any wavelength.
  - At equilibrium, the radiation inside this enclosure follows Planck's law. This radiation is well sampled by the radiation that is emitted at right angles from the hole.



BLACK BODY RADIATION

Planck distribution

- Just as the Maxwell-Boltzmann distribution for thermodynamic equilibrium at a given temperature is the unique maximum entropy energy distribution for a gas of many conserved massive particles, so also is Planck's distribution for a photon gas of photons, which are not conserved and have zero rest mass.
  - By contrast to a material gas where the masses and number of particles play a role, the spectral radiance, pressure and energy density of a photon gas at equilibrium are entirely determined by the temperature.
  - If the photon gas is not initially Planckian, the second law of thermodynamics guarantees that interactions (between photons and other particles or even between the photons themselves) will cause the photon energy distribution to change and approach the Planck distribution.

BLACK BODY RADIATION

Planck distribution

- In such an approach to thermodynamic equilibrium photons are created or annihilated in the right numbers and with the right energies to fill the cavity with a Planck distribution at the eventual equilibrium temperature.
  - The quantity $B_\nu(T)$ is the spectral radiance as a function of temperature and frequency. It has units of $W sr^{-1} m^{-2} Hz^{-1}$ in the SI system.
  - An infinitesimal amount of power $B_\nu(T) cos\theta dA d\Omega d\nu$ is radiated in the direction described by the angle θ from the surface normal from infinitesimal surface area dA into infinitesimal solid angle $d\Omega$ in an infinitesimal frequency band of width $d\nu$ centered on frequency ν.
  - The total power radiated into any solid angle is the integral of $B_\nu(T)$ over those three quantities, and is given by the Stefan-Boltzmann law.
  - The spectral radiance from a black body has the same value for every direction and angle of polarization: the black body is called a Lambertian radiator.

BLACK BODY RADIATION

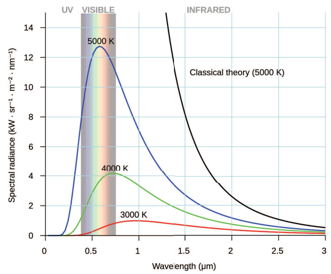
Planck's law

- *Planck's law* describes the electromagnetic radiation emitted by a *black body* in *thermal equilibrium* at a definite temperature. The law is named after Max Planck, who originally proposed it in 1900. It is a pioneer result of modern physics and quantum mechanics.
- For frequency ν, or for wavelength λ, Planckian radiation can be described thus:
  $B_\nu(T) = \frac{2h\nu^3}{c^2}\frac{1}{e^{\frac{h\nu}{k_\mathrm{B}T}}-1}$
- or
  $B_\lambda(T) = \frac{2hc^2}{\lambda^5}\frac{1}{e^{\frac{hc}{\lambda k_\mathrm{B}T}}-1}$
  where B denotes its spectral radiance, T its absolute temperature, k_B the Boltzmann constant, h the Planck's constant, and c the speed of light in the medium, whether material or vacuum.

BLACK BODY RADIATION

Planck's law

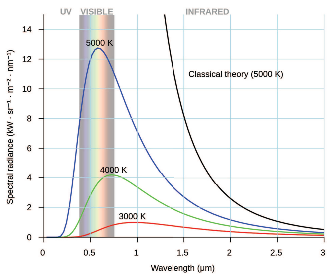
- The radiance SI units are $Wsr^{-1}m^{-2}Hz^{-1}$ for $B_\nu(T)$ and $Wsr^{-1}m^{-3}$ for $B_\lambda(T)$. The law may also be expressed in other terms, such as of the number of photons emitted at a certain wavelength, or of the energy density in a volume of radiation.
- In the limit of low frequencies (i.e. long wavelengths), Planck's law tends to the Rayleigh-Jeans law, while in the limit of high frequencies (i.e. small wavelengths) it tends to the Wien approximation.



BLACK BODY RADIATION

Planck's law

- Max Planck developed the law in 1900, originally with only empirically determined constants, and later showed that, expressed as an energy distribution, it is the unique stable distribution for radiation in thermodynamic equilibrium.
- As an energy distribution, it is one of a family of thermal equilibrium distributions which include the Bose-Einstein distribution, the Fermi-Dirac distribution and the Maxwell-Boltzmann distribution.



CORRESPONDENCE BETWEEN SPECTRAL VARIABLE FORMS

Planck's law

- Different spectral variables require different corresponding forms of expression of the law. In general, one may not convert between the various forms of Planck's law simply by substituting one variable for another, because this would not take into account that the different forms have different units.
  - Corresponding forms of expression are related because they express one and the same physical fact: for a particular physical spectral increment, a particular physical energy increment is radiated.
  - This is so whether it is expressed in terms of an increment of frequency, $d\nu$, or, correspondingly, of wavelength, $d\lambda$. Introduction of a minus sign can indicate that an increment of frequency corresponds with decrement of wavelength.

CORRESPONDENCE BETWEEN SPECTRAL VARIABLE FORMS

Planck's law

- For the above corresponding forms of expression of the spectral radiance, one may use an obvious expansion of notation, temporarily for the present calculation only. Then, for a particular spectral increment, the particular physical energy increment may be written
  $B_\lambda(\lambda,\ T)\ \mathrm{d}\lambda = -B_\nu(\nu(\lambda),\ T)\ \mathrm{d}\nu$,
- which leads to
  $B_\lambda(\lambda,\ T)\ =\ -\frac{\mathrm{d}\nu}{\mathrm{d}\lambda}B_\nu(\nu(\lambda),\ T)$.
- Also, $\nu(\lambda) = c/\lambda$, so that $d\nu/d\lambda = -c/\lambda^2$. Substitution gives the correspondence between the frequency and wavelength forms, with their different units.
- It follows that the location of the peak of the distribution for Planck's law depends on the choice of spectral variable.

SPECTRAL ENERGY DENSITY FORM

Planck's law

- Planck's law can also be written in terms of the spectral energy density u by multiplying B by $4\pi/c$:
  $u_i(T) = \frac{4\pi}{c}B_i(T)$.
- These distributions have units of energy per volume per spectral unit.

PLANCK'S LAW

Derivation

- Consider a cube of side L with conducting walls filled with electromagnetic radiation in thermal equilibrium at temperature T. If there is a small hole in one of the walls, the radiation emitted from the hole will be characteristic of a perfect *black body*.
- We will first calculate the spectral energy density within the cavity and then determine the spectral radiance of the emitted radiation.
- At the walls of the cube, the parallel component of the electric field and the orthogonal component of the magnetic field must vanish.
- Analogous to the wave function of a *particle in a box*, one finds that the fields are superpositions of periodic functions. The three wavelengths λ_1, λ_2, and λ_3, in the three directions orthogonal to the walls can be written as:
  $\lambda_i = \frac{2L}{n_i}$,
  where the n_i are integers.

PLANCK'S LAW

Derivation

- For each set of integers n_i, there are two linear independent solutions (modes). According to quantum theory, the energy levels of a mode are given by:

$$E_{n_1,n_2,n_3}(r) = \left(r + \frac{1}{2}\right)\frac{hc}{2L}\sqrt{n_1^2 + n_2^2 + n_3^2}. \tag{1}$$

- The quantum number r can be interpreted as the number of photons in the mode. The two modes for each set of n_i correspond to the two polarization states of the photon which has a spin of 1.
- Note that for $r = 0$ the energy of the mode is not zero. This vacuum energy of the electromagnetic field is responsible for the *Casimir effect*. In the following we will calculate the internal energy of the box at *absolute temperature* T.

Derivation

- According to *statistical mechanics*, the probability distribution over the energy levels of a particular mode is given by:
  $P_r = \frac{\exp(-\beta E(r))}{Z(\beta)}$.
- Here
  $\beta \stackrel{\text{def}}{=} 1/(k_B T)$.
- The denominator $Z(\beta)$, is the *partition function* of a single mode and makes P_r properly normalized:
  $Z(\beta) = \sum_{r=0}^{\infty} e^{-\beta E(r)} = \frac{e^{-\beta\varepsilon/2}}{1-e^{-\beta\varepsilon}}$.
- Here we have implicitly defined
  $\varepsilon \stackrel{\text{def}}{=} \frac{hc}{2L}\sqrt{n_1^2+n_2^2+n_3^2}$,
  which is the energy of a single photon.

Derivation

- The average energy in a mode can be expressed in terms of the partition function:
  $\langle E \rangle = -\frac{d\log(Z)}{d\beta} = \frac{\varepsilon}{2} + \frac{\varepsilon}{e^{\beta\varepsilon}-1}$.
- This formula, apart from the first vacuum energy term, is a special case of the general formula for particles obeying *Bose-Einstein statistics*. Since there is no restriction on the total number of photons, the *chemical potential* is zero.
- If we measure the energy relative to the ground state, the total energy in the box follows by summing $\langle E \rangle - \frac{\varepsilon}{2}$ over all allowed single photon states.
- This can be done exactly in the thermodynamic limit as L approaches infinity. In this limit, ϵ becomes continuous and we can then integrate $\langle E \rangle - \frac{\varepsilon}{2}$ over this parameter.

Derivation

- To calculate the energy in the box in this way, we need to evaluate how many photon states there are in a given energy range.
- If we write the total number of single photon states with energies between ϵ and $\epsilon + d\epsilon$ as $g(\epsilon)d\epsilon$, where $g(\epsilon)$ is the *density of states* (which we'll evaluate in a moment), then we can write:

$$U = \int_0^{\infty} \frac{\varepsilon}{e^{\beta\varepsilon}-1} g(\varepsilon)\, d\varepsilon. \tag{2}$$

- To calculate the density of states we rewrite equation 1 as follows:
  $\varepsilon \stackrel{\text{def}}{=} \frac{hc}{2L} n$,
- where n is the norm of the vector $\mathbf{n} = (n_1, n_2, n_3)$:
  $n = \sqrt{n_1^2+n_2^2+n_3^2}$.

Derivation

- For every vector $\mathbf{n}$ with integer components larger than or equal to zero, there are two photon states. This means that the number of photon states in a certain region of n-space is twice the volume of that region.
- An energy range of $d\epsilon$ corresponds to shell of thickness $dn = (2L/hc)d\epsilon$ in n-space. Because the components of $\mathbf{n}$ have to be positive, this shell spans an octant of a sphere.
- The number of photon states $g(\epsilon)d\epsilon$, in an energy range $d\epsilon$, is thus given by:
  $g(\varepsilon)\, d\varepsilon = 2\frac{1}{8}4\pi n^2\, dn = \frac{8\pi L^3}{h^3c^3}\varepsilon^2\, d\varepsilon$.
- Inserting this in Eq.2 gives:

$$U = L^3 \frac{8\pi}{h^3c^3} \int_0^{\infty} \frac{\varepsilon^3}{e^{\beta\varepsilon}-1}\, d\varepsilon. \tag{3}$$

Derivation

- From this equation one can derive the spectral energy density as a function of frequency $u_\nu(T)$ and as a function of wavelength $u_\lambda(T)$:
  $\frac{U}{L^3} = \int_0^\infty u_\nu(T)\, d\nu$,
  where:
  $u_\nu(T) = \frac{8\pi h\nu^3}{c^3} \frac{1}{e^{h\nu/k_\mathrm{B}T}-1}$.
  And:
  $\frac{U}{L^3} = \int_0^\infty u_\lambda(T)\, d\lambda$,
  where
  $u_\lambda(T) = \frac{8\pi hc}{\lambda^5} \frac{1}{e^{hc/\lambda k_\mathrm{B}T}-1}$.
- This is also a spectral energy density function with units of energy per unit wavelength per unit volume.
- Integrals of this type for Bose and Fermi gases can be expressed in terms of *polylogarithms*. In this case, however, it is possible to calculate the integral in closed form using only elementary functions.

Derivation

- Substituting
  $\varepsilon = k_\mathrm{B}Tx$,
  in Eq. (3), makes the integration variable dimensionless giving:
  $u(T) = \frac{8\pi (k_\mathrm{B}T)^4}{(hc)^3} J$.
- Here J is a *Bose-Einstein integral* given by:
  $J = \int_0^\infty \frac{x^3}{e^x-1}\, dx = \frac{\pi^4}{15}$.
- The total electromagnetic energy inside the box is thus given by:
  $\frac{U}{V} = \frac{8\pi^5 (k_\mathrm{B}T)^4}{15(hc)^3}$,
  where $V = L^3$ is the volume of the box.
- The combination hc/k_B has the value 14 387.770 mK.

Derivation

- This is *not* the *Stefan-Boltzmann law* (which provides the total energy radiated by a black body per unit surface area per unit time), but it can be written more compactly using the *Stefan-Boltzmann constant* σ, giving
  $\frac{U}{V} = \frac{4\sigma T^4}{c}$.
- The constant $4\sigma/c$ is sometimes called the radiation constant.
- Since the radiation is the same in all directions, and propagates at the speed of light c, the spectral radiance of radiation exiting the small hole is
  $B_\nu(T) = \frac{u_\nu(T)\, c}{4\pi}$,
  which yields
  $B_\nu(T) = \frac{2h\nu^3}{c^2} \frac{1}{e^{h\nu/k_\mathrm{B}T}-1}$.

Derivation

- It can be converted to an expression for $B_\lambda(T)$ in wavelength units by substituting ν by c/λ and evaluating $B_\lambda(T) = B_\nu(T)\left|\frac{d\nu}{d\lambda}\right|$.
- Note that dimensional analysis shows that the unit of steradians, shown in the denominator of left hand side of the equation above, is generated in and carried through the derivation but does not appear in any of the dimensions for any element on the left-hand-side of the equation.