Propagation in Matter

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Maxwell's Equations in Matter

We have seen Maxwell's equations in free space. In matter, we must change them somehow. We have

$$\nabla \cdot \mathbf{D} = \rho$$
$$\nabla \cdot \mathbf{B} = 0$$
$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$
$$\nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}$$

where

$$\mathbf{D} = \varepsilon \mathbf{E}$$
$$\mathbf{B} = \mu \mathbf{H}$$

Permittivity and Permeability

- The two coefficients introduced, ε and μ, are the electrical permittivity and the magnetic permeability of the material.
- For general materials, they are not scalar constants at all
  - For homogeneous media, they do not depend on the position
  - For isotropic media, they are scalar, while in general they are tensors
  - For dispersive media, in the sinusoidal regime, they depend on the frequency of the EM wave, while for non-dispersive media they are constant in f
  - For linear media, we can still use the Fourier decomposition and solve for the propagation of each component with its own parameters
- We will be primarily interested in linear homogeneous isotropic, possibly dispersive, media

Permittivity and Permeability

- It is usually useful to introduce relative permittivity and permeability, defined by

$$\varepsilon = \varepsilon_r \varepsilon_0$$
$$\mu = \mu_r \mu_0$$

- For a linear homogeneous and non-dispersive medium, it easily seen that the same solutions derived for vacuum still hold with minimal changes. For example, for a purely monochromatic wave

$$E_x = E_0 \cos(\omega t - kz)$$
$$B_y = \frac{E_0 \sqrt{\varepsilon_r \mu_r}}{c} \cos(\omega t - kz)$$

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Velocity of Propagation

- In matter, the velocity of the wave is changed, a fact that we want to study in terms of k and ω
- For the monochromatic wave seen before, Maxwell's equations are satisfied if
$$\frac{k}{\omega} = \frac{c}{\sqrt{\varepsilon_r \mu_r}}$$
- This is, as we have seen, the phase velocity (velocity of a monochromatic wave at that frequency)
- In almost all practical cases the medium is dispersive, that is, the values of ε_r and μ_r are not constant for varying k
- This explains why we have in general a nonlinear relation $\omega = \omega(k)$

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Other Quantities

- **Refractive index**: defined as the ratio $n = c/v$ where v is the phase velocity. We thus have
$$n = \sqrt{\varepsilon_r \mu_r}$$
- Even in matter, the solution to Maxwell's equations is a transverse wave in the far field and all properties seen for free space hold
- The irradiance is given by
$$F = \frac{S_0}{2Z}$$
where
$$Z = Z_0 \sqrt{\frac{\mu_r}{\varepsilon_r}}$$

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Attenuation

- When an EM wave propagates in matter there is usually an effect of absorption of the energy from the material
- This is primarily related to the conductance of the material
- The characteristics of the material however, including the conductance, can vary for varying wave number of the propagating EM wave
- We will see that, in a sinusoidal regime, the attenuation can be studied by properly working with the relative permittivity ε_r
- The relative permeability does not create much problem, we can almost always assume $\mu_r = 1$

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Attenuation

- If the medium has a nonzero conductance γ, in the presence of an electric field E an electric current is generated with density
$$\mathbf{J} = \gamma \mathbf{E}$$
- This current enters the equation
$$\nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}$$
and, as we see, it is added to the term
$$\frac{\partial \mathbf{D}}{\partial t} = \frac{\partial \varepsilon \mathbf{E}}{\partial t}$$

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Attenuation

- When we study Maxwell's equations in the sinusoidal regime (that is, we assume that the medium is linear and we can expand signals using the Fourier Transform), the equation

  $$\nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}$$

  becomes

  $$\nabla \times \mathbf{H} = \gamma \mathbf{E} + j\omega\varepsilon E$$

  that we can write as

  $$\nabla \times \mathbf{H} = j\omega\varepsilon_c E$$

  where we set

  $$\varepsilon_c = \varepsilon - j\frac{\gamma}{\omega}$$

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Attenuation

- This shows that we can describe linear attenuation by simply adding an imaginary part to the dielectric constant
- This imaginary part is associated to a component in the curl of **H** which is out of phase with the electric field that we would have in free space
- We often find the notation

  $$\varepsilon_c = (\varepsilon' - j\varepsilon'')\varepsilon_0$$

- It is clear that, if there is no conductance, then the dielectric constant is real
- It is an experimental fact that in all passive media the conductance is positive, and thus the imaginary part of ε_c is negative

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Attenuation

- The definition of the refractive index is extended according to the relation

  $$n^2 = \varepsilon_r$$

  In case of attenuation we thus have a complex refractive value that we write as

  $$n = m - j\kappa$$

  where

  $$\varepsilon' = m^2 - \kappa^2$$
  $$\varepsilon'' = 2m\kappa$$

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Attenuation

- Since we have only changed the dielectric constant, the solutions of Maxwell's equations that we had obtained before are still valid.
- Using the exponential notation we can write the electric field as

  $$E_x = E_0 e^{j(\omega t - kz)}$$

- Observe, however, that the condition

  $$k = \frac{\omega n}{c}$$

  now implies

  $$k = \frac{\omega}{c}(m - j\kappa)$$

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Attenuation

- This implies that for the expression of the electric field we have

$$E_x = E_0 e^{-\frac{\omega\kappa z}{c}} e^{j\left(\omega t - \frac{\omega m z}{c}\right)}$$

- That is, the amplitude of the electric field decreases exponentially in z
- The irradiance, which goes with the square, is thus

$$F = F_0 e^{-2\frac{\omega\kappa z}{c}}$$

- This leads us to define the absorption distance

$$l_a = \frac{c}{2\omega\kappa}$$