Electromagnetism

Marco Dalai (author of slides), Stefan Wabnitz (lecturer)

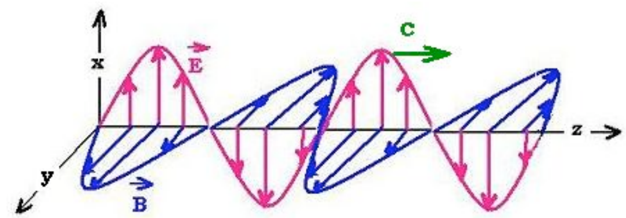
marco.dalai@unibs.it, stefan.wabnitz@unibs.it
Department of Information Engineering
University of Brescia - Italy

Academic Year 2013/2014

Electromagnetic Waves

Plane Waves

What we will now see: a plane wave in the z direction



Linearly polarized plane electromagnetic wave

Electromagnetic Waves

Maxwell's Equations in Free Space

- Maxwell's equations describe the mutual interaction between electric and magnetic fields
- Equations in free space:

$$\nabla \cdot \mathbf{E} = 0$$
$$\nabla \cdot \mathbf{B} = 0$$
$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$
$$\nabla \times \mathbf{B} = \varepsilon_0 \mu_0 \frac{\partial \mathbf{E}}{\partial t}$$

where

ε_0 is the electrical permittivity of free space
μ_0 is the magnetic permeability of free space

Electromagnetic Waves

The Wave Equation

- By applying the curl to the third equation we obtain

$$\begin{aligned}\nabla \times (\nabla \times \mathbf{E}) &= \nabla \times \left(-\frac{\partial \mathbf{B}}{\partial t}\right) \\ &= -\frac{\partial}{\partial t}(\nabla \times \mathbf{B})\end{aligned}$$

- Using the identity

$$\nabla \times (\nabla \times \mathbf{E}) = \nabla(\nabla \cdot \mathbf{E}) - \nabla^2 \mathbf{E}$$

and the fourth equation, we obtain the wave equation

$$\varepsilon_0 \mu_0 \frac{\partial^2 \mathbf{E}}{\partial t^2} = \nabla^2 \mathbf{E}$$

The Wave Equation

- Set $c = 1/\sqrt{\varepsilon_0\mu_0}$, the speed of light in vacuum!
  In cartesian coordinates the wave equation reads

$$\frac{1}{c^2}\frac{\partial^2\mathbf{E}}{\partial t^2} = \frac{\partial^2\mathbf{E}}{\partial x^2} + \frac{\partial^2\mathbf{E}}{\partial y^2} + \frac{\partial^2\mathbf{E}}{\partial z^2}$$

- In the same way we obtain

$$\frac{1}{c^2}\frac{\partial^2\mathbf{B}}{\partial t^2} = \frac{\partial^2\mathbf{B}}{\partial x^2} + \frac{\partial^2\mathbf{B}}{\partial y^2} + \frac{\partial^2\mathbf{B}}{\partial z^2}$$

- These equations relate the variation of the EM field vectors in space with their variation in time. The solutions of Maxwell's equation must thus satisfy the wave equations.

Solution of the original equations

- We obtained two separate wave equations for **E** and for **B**.
- Clearly, if we now solve those two equations independently, we do not necessarily obtain a solution to the original equations, because **E** and **B** are not related by

$$\nabla \times \mathbf{E} = -\frac{\partial\mathbf{B}}{\partial t}$$
$$\nabla \times \mathbf{B} = \varepsilon_0\mu_0\frac{\partial\mathbf{E}}{\partial t}$$

- We could thus try to first find a solution of the wave equation for **E**, and then determine **B** by using the above constraints.

Plane Waves in Free Space

- We look for a "simple" solution that gives constant fields on planes, for example over the $x-y$ plane, so that the fields only depend on the variable z

$$\mathbf{E} = \mathbf{E}(z,t) \qquad \mathbf{B} = \mathbf{B}(z,t) \tag{1}$$

- The wave equations thus become

$$\frac{1}{c^2}\frac{\partial^2\mathbf{E}}{\partial t^2} = \frac{\partial^2\mathbf{E}}{\partial z^2} \qquad \frac{1}{c^2}\frac{\partial^2\mathbf{B}}{\partial t^2} = \frac{\partial^2\mathbf{B}}{\partial z^2}$$

- Consider the components of the vectors **E** and **B**

$$\mathbf{E}(z,t) = E_x(z,t)\mathbf{i} + E_y(z,t)\mathbf{j} + E_z(z,t)\mathbf{k}$$
$$\mathbf{B}(z,t) = B_x(z,t)\mathbf{i} + B_y(z,t)\mathbf{j} + B_z(z,t)\mathbf{k}$$

Plane Waves in Free Space

- Take the z coordinate of the two equations

$$\nabla \times \mathbf{B} = \varepsilon_0\mu_0\frac{\partial\mathbf{E}}{\partial t}, \qquad \nabla \times \mathbf{E} = -\frac{\partial\mathbf{B}}{\partial t} \tag{2}$$

  that is

$$\frac{\partial B_y}{\partial x} - \frac{\partial B_x}{\partial y} = \varepsilon_0\mu_0\frac{\partial E_z}{\partial t}, \qquad \frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} = -\frac{\partial B_z}{\partial t}$$

- We obtain

$$\frac{\partial E_z}{\partial t} = 0, \qquad \frac{\partial B_z}{\partial t} = 0$$

- Hence, the components in the z direction are constant in time. For the superposition principle, it is useful to just assume them to be equal to zero.

Electromagnetic Waves

Plane Waves in Free Space

- Consider now the wave equation for $\mathbf{E}$

$$\frac{1}{c^2}\frac{\partial^2 \mathbf{E}}{\partial t^2} = \frac{\partial^2 \mathbf{E}}{\partial z^2}$$

evaluated for the x component

$$\frac{1}{c^2}\frac{\partial^2 E_x}{\partial t^2} = \frac{\partial^2 E_x}{\partial z^2}$$

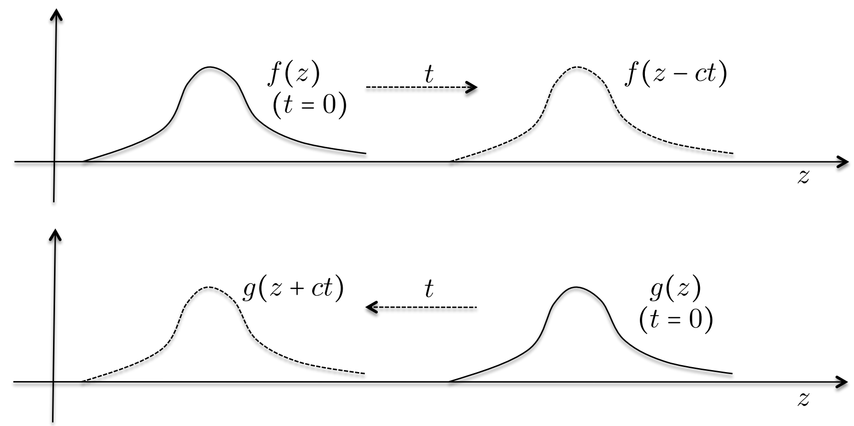
- The solution to this equation is of the type

$$E_x(z,t) = f(z-ct) + g(z+ct)$$

where f and g are arbitrary functions.

Electromagnetic Waves

Forward and Backward Waves



Electromagnetic Waves

Plane Waves in Free Space

- Can this E_x component exist alone? No ...
- From the condition

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

evaluated along the y component we have

$$-\frac{\partial E_z}{\partial x} + \frac{\partial E_x}{\partial z} = -\frac{\partial B_y}{\partial t}$$

and thus

$$\begin{aligned}\frac{\partial B_y}{\partial t} &= -\frac{\partial}{\partial z}\left(f(z-ct) + g(z+ct)\right)\\ &= \frac{1}{c}\frac{\partial f(z-ct)}{\partial t} - \frac{1}{c}\frac{\partial g(z+ct)}{\partial t}\end{aligned}$$

Electromagnetic Waves

Plane Waves in Free Space

- Integrating this in t we obtain

$$\begin{aligned}E_x(z,t) &= f(z-ct) + g(z+ct)\\ B_y(z,t) &= \frac{1}{c}f(z-ct) - \frac{1}{c}g(z+ct)\end{aligned}$$

that is, a forward and a backward wave as before.
Note the opposite sign in the backward wave for B.

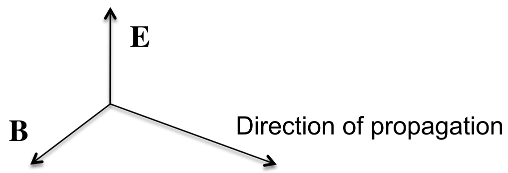
- In the same way, we can obtain a solution of the form

$$\begin{aligned}E_y(z,t) &= h(z-ct) + r(z+ct)\\ B_x(z,t) &= -\frac{1}{c}h(z-ct) + \frac{1}{c}r(z+ct)\end{aligned}$$

and all possible combinations of these two solutions.

Plane Waves in Free Space

- Consider only the forward wave and note that the vectors **E** and **B** are orthogonal to each other and to the direction of propagation, and that they form with it a right-handed basis



- For this reason we call them *transverse waves*
- For a different propagation direction we may simply change the reference system

Power Carried by a Wave

- A plane wave of this kind carries a certain amount of power
- Irradiance: the power that transverses a unit area orthogonal to the propagation direction.
- It can be expressed by using the Poynting vector

$$\mathbf{S} = \frac{1}{\mu_0}\mathbf{E} \times \mathbf{B}$$

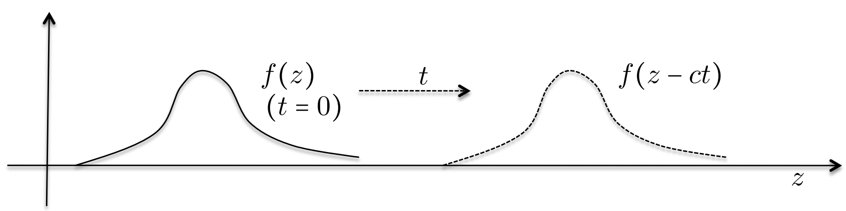
- The absolute value of this vector gives the instantaneous power that transverses the surface. By computing the mean value (in time) we can compute the mean transmitted power per unit area.

Plane Waves in Free Space

- The solution of the Maxwell's equations seen above does not describe exactly EM waves that can be generated by an antenna (we have not used any source for our EM field!)
- It can be shown, however, that all sources generate, in the far field, an EM wave that at a large distances can be locally approximated with a plane wave
- Here we thus give a characterization of EM waves by means of plane waves
- We will then consider "simple" characterizations of non-plane waves

Fourier Transform

- Consider the function f that we used as an impulse to describe our forward plane wave



- Since Maxwell's equations are linear, if we assume $f(z)$ to be the sum of different signals, we can imagine all components to propagate in the same way. (NB: in vacuum!)
- We can thus express f in terms of its frequency components by using the Fourier Transform superposition, and then study each Fourier component as an independent plane wave.

Fourier Transform

- Direct Transform

$$\hat{f}(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(z) e^{-jkz} dz$$

- Inverse Transform

$$f(z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(z) e^{jkz} dk$$

  It can be interpreted as a description of $f(z)$ as a weighted sum of harmonic "components"
- Propagation of each "component",

$$\underset{(t=0)}{e^{jkz}} \xrightarrow{t} e^{jk(z-ct)} = e^{j(kz-\omega t)} \qquad \text{where } c = \omega/k \tag{3}$$

Fourier Transform

A note on the harmonic components

$$e^{j(kz-\omega t)} = \cos(kz-\omega t) + j\sin(kz-\omega t)$$

- We can avoid complex numbers by using $\sin(\cdot)$ and $\cos(\cdot)$ functions
- If the propagation is not in vacuum, in general the speed of different components can be different.
- If the medium is *linear* and homogeneous, we can still consider harmonic components but possibly having different speeds.
- This explains why we express the harmonic components using k and ω, because in general the linear relation $\omega = ck$ does not hold and we have to use instead a more general relation $\omega = \omega(k)$

Monochromatic Plane Waves in Vacuum

- Let us consider, in light of the previous discussion, the propagation of a monochromatic plane wave in vacuum

$$\begin{aligned} E_x &= E_0\cos(\omega t - kz) & \quad B_x &= 0 \\ E_y &= 0 & B_y &= \tfrac{E_0}{c}\cos(\omega t - kz) \\ E_z &= 0 & B_z &= 0 \end{aligned}$$

- The constant c here satisfies

$$c = \frac{\omega}{k} = \frac{1}{\sqrt{\varepsilon_0\mu_0}} \approx 2,9979 \cdot 10^8 m/s$$

  which is indeed the propagation speed of light in vacuum

Monochromatic Plane Waves in Vacuum

- Parameters
  - ω: angular variation in unit time (in rad/s)
  - k: wave number, angular variation per unit length (rad/m)
- Equivalently
  - f: frequency, number of cycles per unit time

$$\omega = 2\pi f$$

  - λ: wavelength, space distance between two points in phase

$$k = \frac{2\pi}{\lambda}$$

Electromagnetic Waves

Transmitted Power

- We have introduced the irradiance, which can be expressed in terms of the Poynting vector

$$\mathbf{S} = \frac{1}{\mu_0}\mathbf{E} \times \mathbf{B}$$

- Its mean value for a sinusoidal wave as discussed above gives

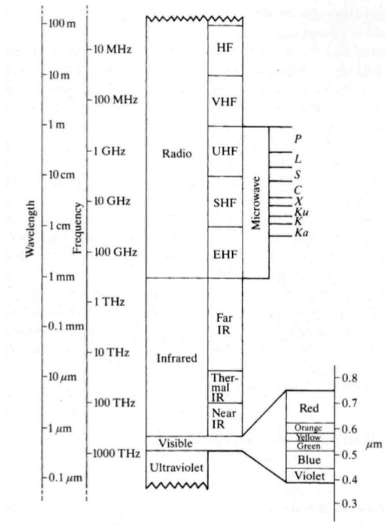
$$F = \frac{E_0^2}{2Z_0}$$

where $Z_0 = \sqrt{\frac{\mu_0}{\varepsilon_0}} \approx 377\Omega$ is the impedance of vacuum

Electromagnetic Waves

Electromagnetic Spectrum

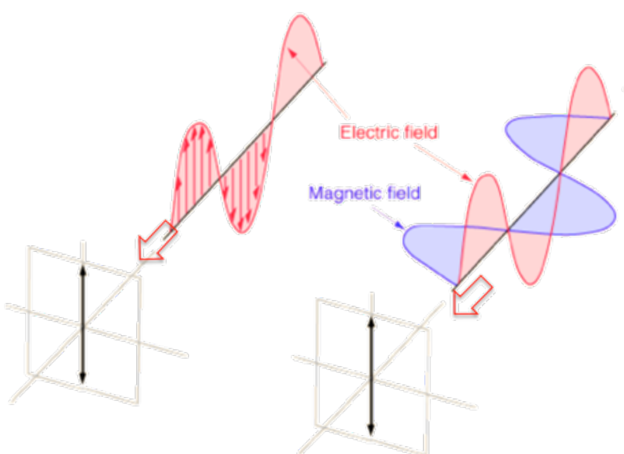
- We have considered perfect monochromatic waves
- We can study the full range of wavelengths
- This is precisely the spectrum we have discussed in previous lectures
- Each signal well always be able to interpret each signal as a "sum" (integral) of infinite purely monochromatic waves



Electromagnetic Waves

Polarization

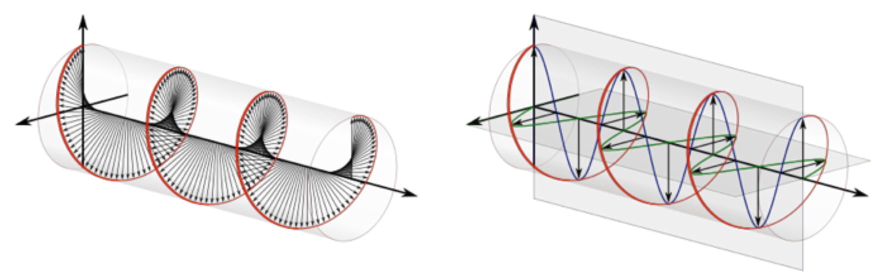
- The wave presented in the previous expression was a *linearly polarized* wave



Electromagnetic Waves

Polarization

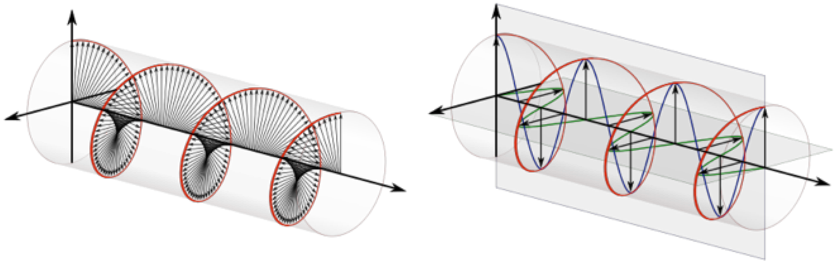
- There exist different polarization, for example circular (right-handed)



Electromagnetic Waves

Polarization

- ... (or left-handed)



Electromagnetic Waves

Polarization

- We can interpret polarization again as the effect of the combination of different waves
- Consider the sum of two waves as the previous one but with different phases

$$
\begin{aligned}
E_x &= E_{0x}\cos(\omega t - kz - \phi_x) & B_x &= 0 \\
E_y &= 0 & B_y &= \frac{E_{0x}}{c}\cos(\omega t - kz - \phi_x) \\
E_z &= 0 & B_z &= 0
\end{aligned}
$$

and

$$
\begin{aligned}
E_x &= 0 & B_x &= -\frac{E_{0y}}{c}\cos(\omega t - kz - \phi_y) \\
E_y &= E_{0y}\cos(\omega t - kz - \phi_y) & B_y &= 0 \\
E_z &= 0 & B_z &= 0
\end{aligned}
$$

Electromagnetic Waves

Polarization

- We obtain another wave of the form

$$
\begin{aligned}
E_x &= E_{0x}\cos(\omega t - kz - \phi_x) & B_x &= -\frac{E_{0y}}{c}\cos(\omega t - kz - \phi_y) \\
E_y &= E_{0y}\cos(\omega t - kz - \phi_y) & B_y &= \frac{E_{0x}}{c}\cos(\omega t - kz - \phi_x) \\
E_z &= 0 & B_z &= 0
\end{aligned}
$$

- By combining different amplitudes and phases we obtain the different polarizations
- Alert: the waves considered here are ideal, with *infinite coherence length*. True waves do not have constant phases... well see later

Electromagnetic Waves

Polarization: Examples

- Linear polarization, in those cases where

$$\phi_x - \phi_y = 0,\ \pi,\ -\pi$$

- Circular polarization

$$E_{0x} = E_{0y}$$

  - Right-handed

$$\phi_y - \phi_x = \frac{\pi}{2}$$

  - Left-handed

$$\phi_y - \phi_x = -\frac{\pi}{2}$$

Electromagnetic Waves

Non Monochromatic Waves

- As anticipated, no wave is really monochromatic
- We can focus on pass-band waves, that is, waves that can be written as

$$\mathbf{E}(z,t) = \mathbf{E}_0(z,t)e^{j(k_0 z - \omega_0 t)} + \mathbf{E}_0^*(z,t)e^{-j(k_0 z - \omega_0 t)}$$

  where $\mathbf{E}_0(z,t)$ is a slowly varying function of z and t
- Equivalently, we can assume that the quantities

$$E_{0x},\ E_{0y},\ \phi_x,\ \phi_y$$

  vary very slowly in z and t

Electromagnetic Waves

Polarization

- Stokes vector

$$\begin{aligned} S_0 &= \langle E_{0x}^2 \rangle + \langle E_{0y}^2 \rangle \\ S_1 &= \langle E_{0x}^2 \rangle - \langle E_{0y}^2 \rangle \\ S_2 &= \langle 2E_{0x}E_{0y}\cos(\phi_x - \phi_y) \rangle \\ S_3 &= \langle 2E_{0x}E_{0y}\sin(\phi_x - \phi_y) \rangle \end{aligned}$$

- Degree of polarization (DOP)

$$\sqrt{\frac{S_1^2 + S_2^2 + S_3^2}{S_0^2}}$$

  where $\mathbf{E}_0(z,t)$ is a slowly varying function of z and t
- Irradiance

$$F = \frac{S_0}{2Z_0}$$

Electromagnetic Waves

Coherence

- The fact that a quasi-monochromatic wave such as

$$\mathbf{E}(z,t) = \mathbf{E}_0(z,t)e^{j(k_0 z - \omega_0 t)} + \mathbf{E}_0^*(z,t)e^{-j(k_0 z - \omega_0 t)}$$

  can also be written in terms of slowly varying values of ϕ_x, ϕ_y leads to the definition of the concept of coherence length and time
- Coherence time: the maximum size of a a time interval over which the phases ϕ_x, ϕ_y can be considered constant (or, stated in a different way, the phase of the fields can be foreseen over time intervals of that length)
- After sufficiently high time intervals, the phase of the EM field becomes completely unpredictable

Electromagnetic Waves

Group Velocity Dispersion

- We already mentioned that if the medium is not the vacuum, in general harmonic components with different frequency will propagate at different speeds. We may use the relation $\omega = \omega(k)$.
- If the signal has a narrow band, for example

$$\mathbf{E}(z,t) = \mathbf{E}_0(z,t)e^{j(k_0 z - \omega_0 t)} + \text{c.c.}$$

  we have two important notions of velocity
  - Phase velocity:

$$v_p = \frac{\omega_0}{k_0}$$

  - Group velocity

$$v_g = \left.\frac{\partial \omega}{\partial k}\right|_{\omega = \omega_0}$$

  ... This is the true speed of the function envelope.

Dispersion

- In vacuum, $\omega = ck$, and thus phase and group velocities coincide!
- In many media (for example in the ionosphere), instead, a non-linear relation holds, and we have the *dispersion* phenomenon
  - Different Fourier components travel at different speeds
  - The envelope moves with the above mentioned group velocity
  - The shape of the envelope must change during propagation and this depends from the higher order derivatives, usually mainly on the second order term:

$$-\left.\frac{\partial^2 \omega}{\partial k^2}\right|_{\omega=\omega_0}$$

Doppler Effect

- The notion of frequency of a wave is not invariant with respect to different (moving) reference frames.
- Doppler effect: shift of the perceived frequency due to the relative motion between source and target
- A receiver that is moving, relatively to a source of frequency ω, with a speed v forming an angle θ with the segment joining it to the source will observe a wave with frequency

$$\omega' = \omega \frac{\sqrt{1 - \frac{v^2}{c^2}}}{1 - \frac{v\cos\theta}{c}}$$

Doppler Effect

- For usual speeds of vehicles on the Earth the effect is too small to be measured
- But for a satellite that is approaching at the horizon with a speed of some Km/s the effect can actually be measured.. At that speed, we can use the approximation

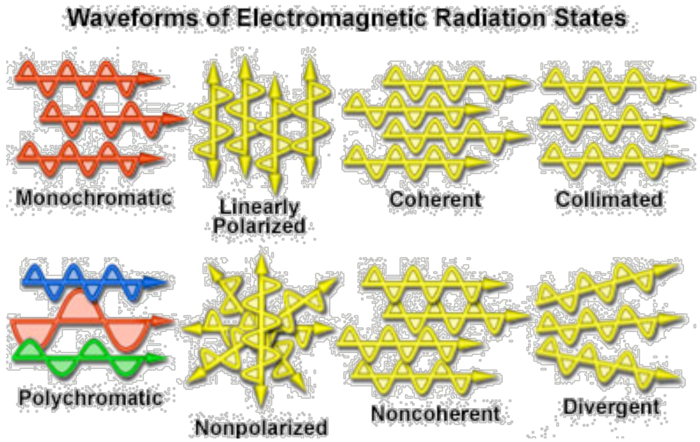
$$\omega' = \omega\left(1 + \frac{v\cos\theta}{c}\right)$$

Example: If the speed is 7 Km/s and the angle is nearly 10°, assuming $f = 5$GHz, we obtain $f = 4.999885$GHZ, a shift of nearly 115KHz.

Collimation

- True waves cannot propagate in one direction with a planar wave front
- For real sources, the wave front can never be perfectly flat
- In some cases, however, the wave can be locally approximated with a plane wave with negligible error (consider for example the sun light on the Earth's surface)
- In other cases, it is important to understand how much the rays diverge and thus give a measure of how much they are *collimated*

Summary



Radiometric Quantities

- **Radiant Energy** Q_e: the total amount of energy carried by the considered EM wave. It can be associated to the total energy irradiated by a source or the energy that trasverses a given area of interest ... [Joule]
- **Radiant flux (or radiant power)**: is the radiant energy per unit time [Watt]

$$\Phi = \frac{dQ_e}{dt}$$

The radiant flux is a fundamental quantity because it represents a limit which is independent from the directionality of the source etc..

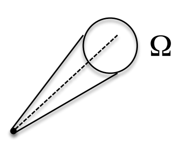
Radiometric Quantities

- **Spectral power** Φ: is the radiant power per wavelength [Wm^{-1}] (sometimes, per Hertz [WHz^{-1}]

$$\Phi(\lambda) = \frac{\partial \Phi}{\partial \lambda}$$

- **Radiant intensity** I: power radiated by a point source in a given direction per solid angle [Wsr^{-1}]

$$I = \frac{\partial \Phi}{\partial \Omega}$$



Radiometric Quantities

For extended sources

- **Exitance** M: radiant power emitted per unit source area [Wm^{-2}] (sometimes, per Hertz [WHz^{-1}]

$$M = \frac{\partial \Phi}{\partial A}$$

- **Spectral radiant exitance**:

$$M(\lambda) = \frac{\partial M}{\partial \lambda}$$

When we consider an illuminated object rather than a source

- **Irradiance** E: radiant incident power per unit area [Wm^{-2}]

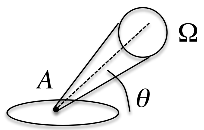
$$E = \frac{\partial \Phi}{\partial A}$$

Radiometric Quantities

For extended sources

- **Radiance** L: radiant power emitted in one direction per solid angle and unit area projected in the plane orthogonal to the considered direction [$\mathrm{Wsr^{-1}m^{-2}}$]

$$L = \frac{\partial^2 \Phi}{\partial \Omega \partial A \cos(\theta)}$$



- **Spectral radiance**: radiance per wavelength [$\mathrm{Wsr^{-1}m^{-3}}$]

$$L(\lambda) = \frac{\partial L}{\partial \lambda}$$

Quantization of the EM Field

- An electromagnetic wave cannot have an arbitrary energy, it is always composed of a number of elementary quantities called *photons*
- Every electromagnetic wave is thus a stream of a huge number of these elementary wave packets, each with energy

$$E = \frac{hc}{\lambda}$$ where $h \approx 6,626 \cdot 10^{-24}$ Js is the Plank constant

- This is useful to better understand the problem of coherence, collimation etc. It is something that has to be considered as a property of a population of photons