Diffraction

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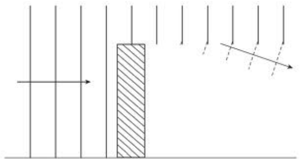
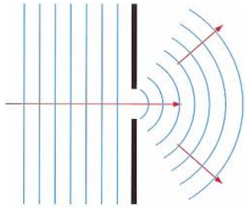
Diffraction

Intuition

- Diffraction is a phenomenon that involves the variation of the propagation direction of an electromagnetic wave caused by an obstacle on its path
- A plane electromagnetic wave in free space that hits on his path an obstacle partially blocking it, cannot propagate afterwards as a plane wave and its wavefront has to change
- The wave tends to "bend around an obstacle", or "spread out" as it goes through a gap





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What we need

- We do not consider the interaction between electromagnetic waves and matter the atomic level
- Here we only consider an effect which is intrinsically due to the wave characteristics of electromagnetic perturbations, and does not depend on the type of matter that interacts with it
- This phenomenon is important to understand the resolving power of remote sensing systems

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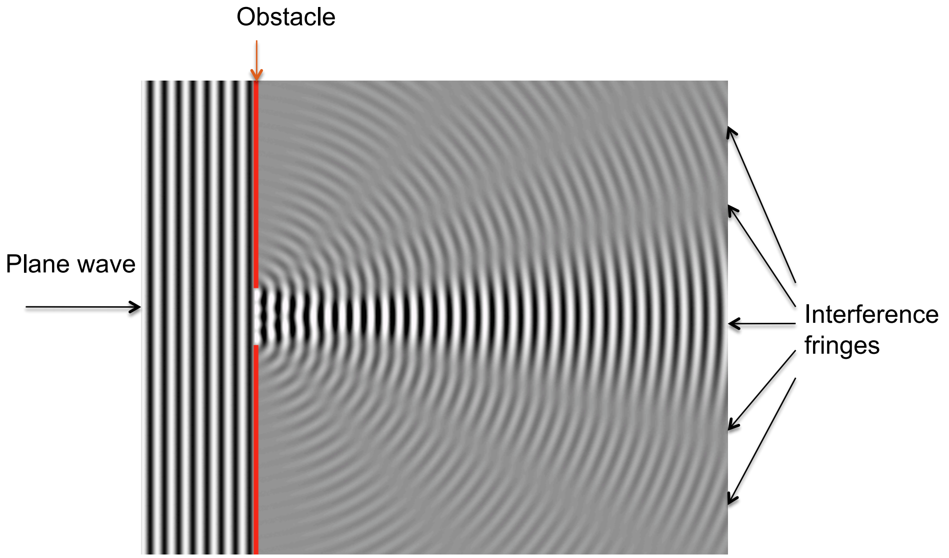
Diffraction: Interference



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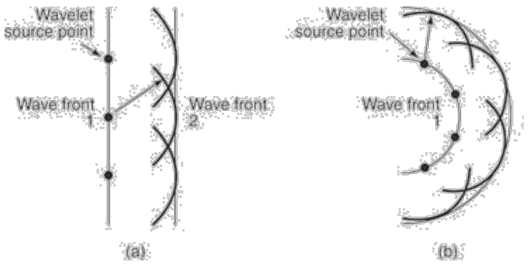
Huygens's Principle

- Huygens's principle: the propagation of a wave can be studied by assuming that each point of the wavefront is the source of a secondary spherical wave
- The wavefront propagates as the envelope of all secondary wavefronts



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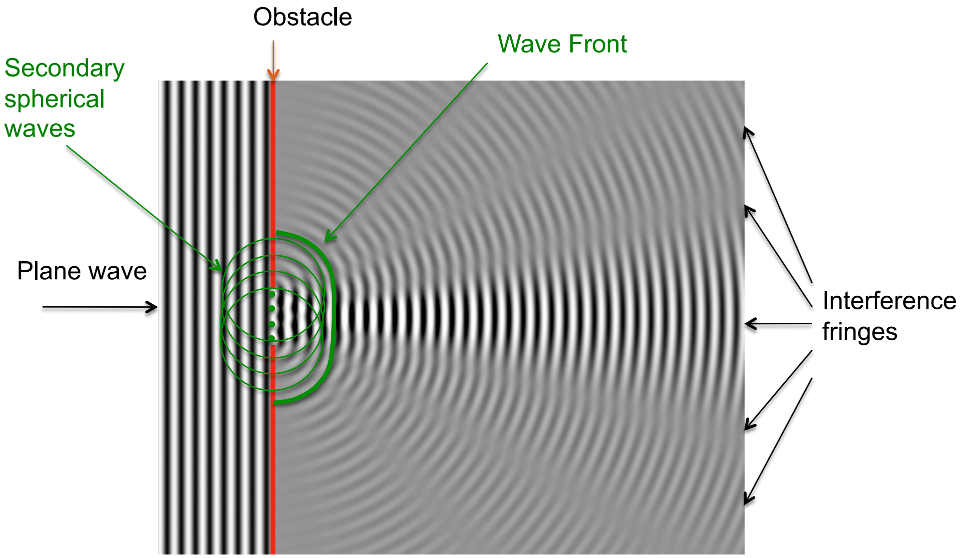
Huygens' Principle



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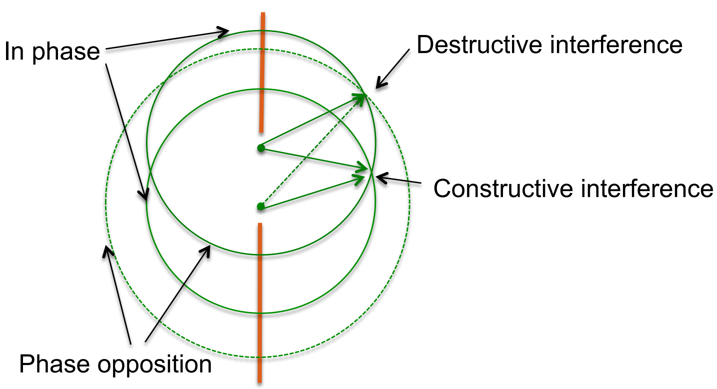
Interference

- The secondary spherical waves interact. The phase difference between any two such waves depends on the difference of the distance from the two source points



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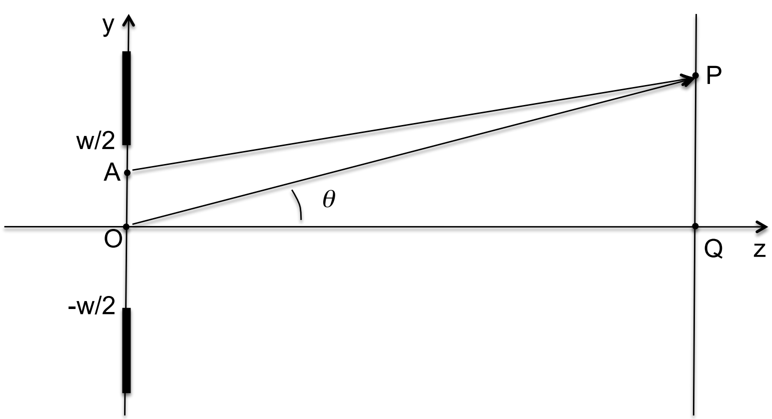
Interference: Single Slit

- Consider a slit of width w and consider a point P on a plane very "far" and parallel to the slit



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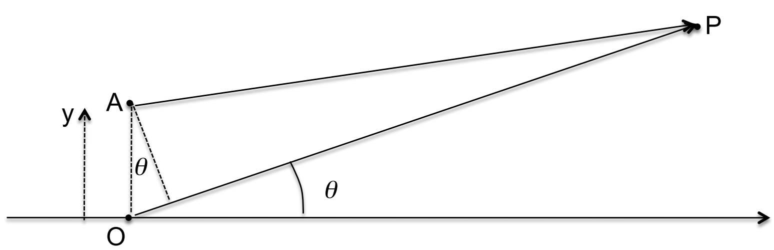
Interference: Single Slit

- Consider as a reference the phase of the wave generated by the source point O when it reaches point P
- The wave generated by the point A of coordinate y reaches P with a different phase, due to the different between the length AP and OP
- Assume P is very far, so that the segments AP and OP are almost parallel and differ by about $y \sin\theta$



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Interference: Single Slit

- If k is the wave number, the phase of the wave generated by A in P is $ky \sin\theta$
- The point A thus contributes to the wave in P with a complex contribution

$$e^{jky \sin\theta}$$

- If we integrate letting A move in the whole slit (i.e., y from $w/2$ to $w/2$) we obtain

$$a(\theta) = \int_{-w/2}^{w/2} e^{jky\sin\theta} dy$$

- Note that this looks very much like a Fourier Transform... and indeed it is

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Interference: General Transparency function

- If we assume that the slit is in reality a segment with a given transparency $f(y)$, we may consider the secondary waves to have different amplitudes, and thus obtain

$$a(\theta) = \int_{-\infty}^{\infty} f(y) e^{jky\sin\theta} dy$$

- This integral is called **Fraunhofer's diffraction integral**
- We observe that it is a Fourier Transform of the transparency function f computed in the variable

$$-k \sin\theta$$

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Interference: Single Slit

- By evaluating the transform in the "pure" ideal slit we obtain

$$a(\theta) \propto \text{sinc}\left(\frac{wk\sin\theta}{2\pi}\right)$$

  and thus the intensity of the wave is proportional to

$$I(\theta) \propto \text{sinc}^2\left(\frac{wk\sin\theta}{2\pi}\right)$$

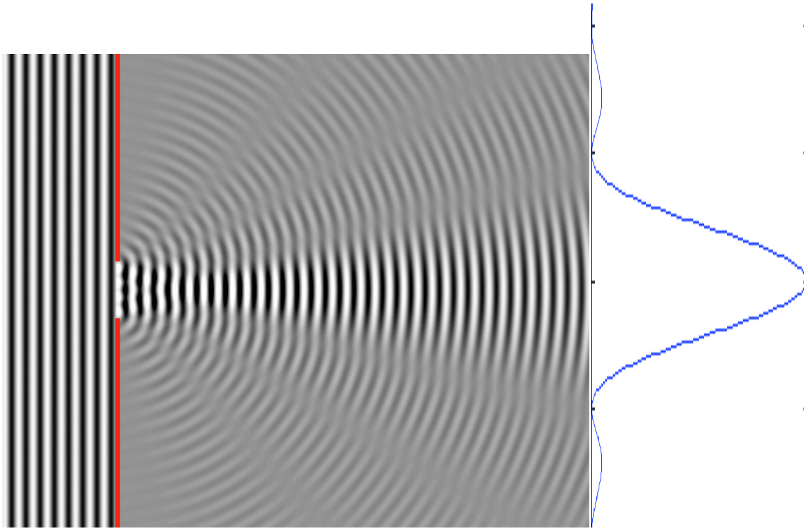
- The first zero of the sinc function gives an idea of the beam width

$$\sin\theta = \pm\frac{2\pi}{wk} = \pm\frac{\lambda}{w}$$

- If $w \gg \lambda$, then $\sin\theta \approx \theta$ and thus

$$\theta \approx \lambda/w$$

Interference: Single Slit



Far Field

- We assumed AP and OP to be almost parallel. This is reasonable only if P is sufficiently far.
- We consider the approximation valid if the difference between AQ and OQ is less than one quarter wavelength.
- In order for this to be true in the whole slit we must have

$$\frac{w^2}{8z} < \frac{\lambda}{4}$$

that is, if z is much larger than the so called **Fresnel distance**

$$z_F = \frac{w^2}{2\lambda}$$

- In this case we refer to the approximation as the **far field approximation**

Two Dimensional Aperture

- The analysis used for the one-dimensional slit contains essentially all the principles needed to solve also the two dimensional case
- Assume that we have an opaque wall with a two-dimensional region with transparency $f(x, y)$. The complex amplitude on a plane very far from the aperture will be

$$a(\theta_x, \theta_y) \propto \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} f(x, y) e^{j(kx \sin\theta_x + ky \sin\theta_y)} dx dy$$

- This corresponds to a two-dimensional transform of the function $f(x, y)$

Rectangular Aperture

- It can be solved in a separable way, that is, by multiplying the result of two one-dimensional transforms in the two orthogonal directions

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Circular Aperture

- In the case of a circular aperture with diameter D, we essentially have to compute the Fourier Transform of a circle
- The result clearly has circular symmetry and it is thus easier to express in polar coordinates
- This way, we can express the integral in terms of Bessel's functions and compute the amplitude at a given angle with respect to the aperture axis

$$a(\theta_x, \theta_y) \propto \frac{J_1\left(\frac{kD\sin\theta_r}{2}\right)}{\frac{kD\sin\theta_r}{2}}$$

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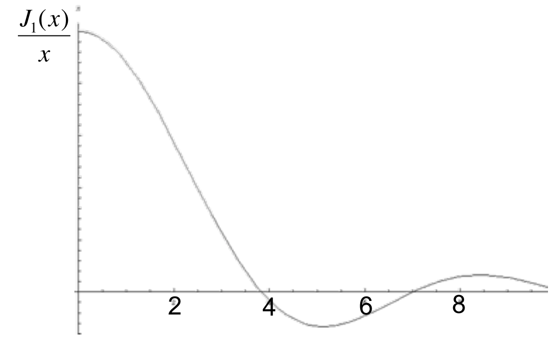
Circular Aperture



- Again, the first zero of the function gives an idea of the beam size

$$\sin\theta_r \approx 1.22\frac{\lambda}{D}$$

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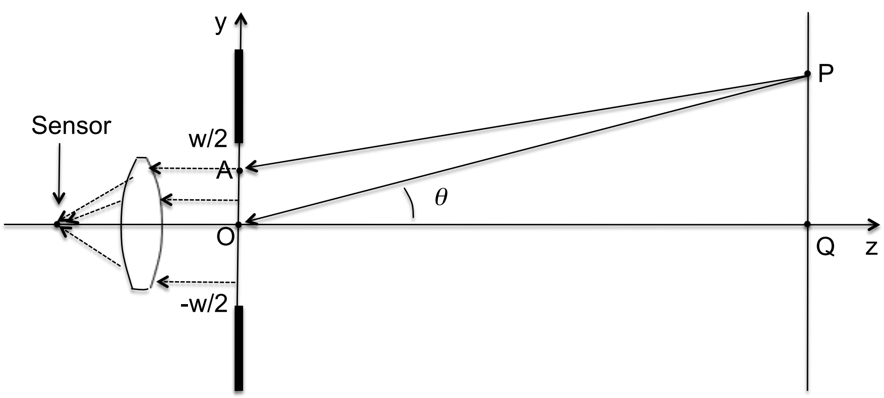
Spatial Resolution

- The diffraction phenomenon has important consequences on the resolution of acquisition systems
- Let us invert the light propagation



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Spatial Resolution

- In this case, as in the analysis of diffraction, the rays that arrive at the sensor along different paths form constructive or destructive interference depending on the path-length difference
- Hence, we deduce that those paths that led to constructive interference before will do that now as well
- Hence, the acquisition made by the sensor is not merely due to the portion of observed plane directly on the axis of the lens, but to all the lobes of the sinc or Bessel's functions seen before. We will thus have an angular resolution θ_r such that

$$\sin\theta_r \approx 1.22\frac{\lambda}{D}$$

and, for small θ_r,

$$\theta_r \approx 1.22\frac{\lambda}{D} \doteq \frac{\lambda}{D}$$

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Spatial Resolution

- Example in the region of optimal frequencies
  - Wavelength $\lambda = 0.5\mu$m
  - Lens diameter $D = 5$cm
  - Angular resolution of about $\lambda/D = 10^{-5}$ rad
  - This corresponds to a spatial resolution of 10m at a distance of 1000Km ($\gg z_F = 2.5$Km)
- Example in the microwave region (for a passive system).
  - Wavelength $\lambda = 3$cm
  - Antenna with diameter $D = 1$m
  - Angular resolution of $\lambda/D = 0.03$ rad
  - This corresponds to a spatial resolution of 30Km at a distance of 1000Km ($\gg z_F = 17$m)