BEAM PROPAGATION METHOD (BPM)

The beam propagation method is widely used for the numerical solution of the nonlinear Schrödinger equation, including wave guiding structures with weak variations along the propagation direction

A few applications of the BPM technique:

- Analysis of laser beam propagation through inhomogenous media
- Design of integrated optics devices
- Study of the evolution of short pulses in optical fibers
- Evaluation of the performance of fiber optic transmission systems

Let us start by considering the propagation of monochromatic waves in dielectric guiding structures, which we suppose for the moment to be linear

Helmholtz equation
$$\frac{\partial^2 E}{\partial z^2} + \boxed{\nabla_\perp^2 E + k^2 n_0^2 E} + \boxed{k^2 (n^2 - n_0^2) E} = 0$$

$$E = \psi \exp\left(jk_0 z \right) \quad k_0 = kn_0 = \frac{2\pi}{\lambda} n_0$$

Schrödinger equation
$$j\frac{\partial \psi}{\partial z} + \boxed{\frac{1}{2k_0} \nabla_\perp^2 \psi} + \boxed{\frac{1}{2} k_0 \left(\frac{n^2}{n_0^2} - 1 \right) \psi} = 0$$

It is clear that wave propagation is determined by two effects:

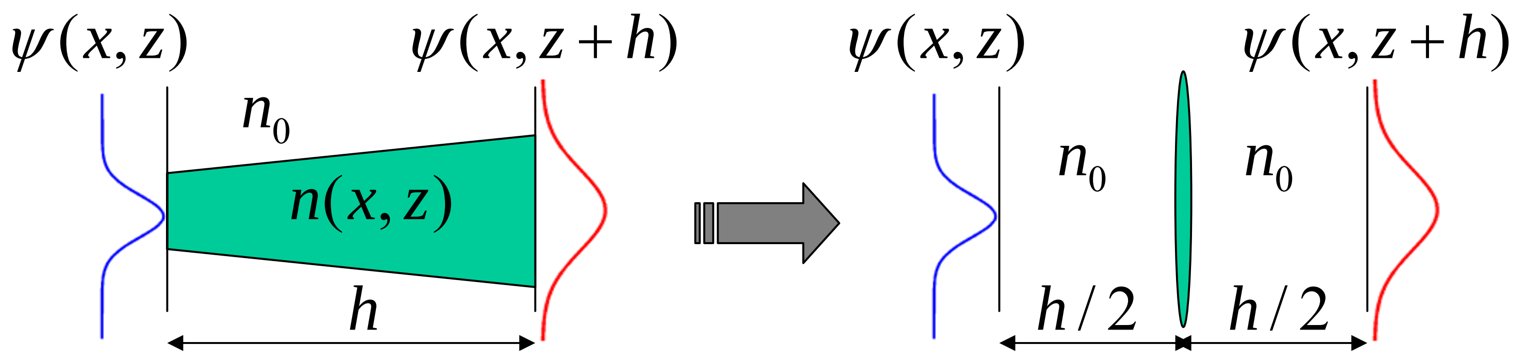
- free-propagation (diffraction) in a uniform medium with index n_0
- beam confining action of the guiding structure

The SPLIT-STEP approach

Basic approximation: although the 2 terms act simultaneously, we may suppose that, for a propagation over a very small step h, we may "split" the two effects, that is, let them act one after the other

The total propagation distance is divided in sub-intervals of length h

1. One propagates the electric field in the uniform medium over a first half step $h/2$
1. Next one adds a linear phase shift due to the presence of the guiding structure over the entire step h
2. Finally one propagates the electric field in an uniform medium over the remaining half step $h/2$



Let us represent free-space propagation by means of the U operator, and the guiding effect by the W operator

Helmholtz equation

$$\frac{\partial^2 E}{\partial z^2} = (U + W)E$$

$$\boxed{U = -\nabla_\perp^2 - k^2 n_0^2}$$

$$\boxed{W = -k^2 (n^2 - n_0^2)}$$

Schrödinger equation

$$\frac{\partial \psi}{\partial z} = (U + W)\psi$$

$$\boxed{U = j\frac{1}{2k_0}\nabla_\perp^2}$$

$$\boxed{W = j\frac{1}{2}k_0\left(\frac{n^2}{n_0^2} - 1\right)}$$

$$\frac{\partial \psi}{\partial z} = (U + W)\psi$$

Let us suppose that, over a sufficiently small step h, W is constant with z, so that the solution can be formally written as:

$$\psi(x, z + h) = \exp\big(h(U + W)\big)\,\psi(x, z)$$

It is difficult to obtain an explicit expression for the exponential of the operator U+W. Nevertheless, if the operators *U* and *W* commute, it is possible to show that the following relationship holds:

$$\Downarrow$$

$$\exp\big(h(U + W)\big) = \exp(hU)\exp(hW)$$

We have "split" the two operators!

Although the two operators U e W do not commute in general, however for small h it is possible to prove that

$$\exp(h(U+W)) = \exp(hU)\exp(hW) + O(h^2)$$

The accuracy of the splitting may be further improved if we write

$$\exp(h(U+W)) = \exp\left(\frac{h}{2}U\right)\exp(hW)\exp\left(\frac{h}{2}U\right) + O(h^3)$$

In fact, this type of factorisation is used in the split-step method

$$\boxed{\psi(x,z+h) = \exp\left(\frac{h}{2}U\right)\exp(hW)\exp\left(\frac{h}{2}U\right)\psi(x,z)}$$

DIFFRACTIVE STEP

Starting from the Schrödinger equation (with the single transverse dimension x)

$$\frac{\partial \psi}{\partial z} = j \frac{1}{2k_0} \nabla_\perp^2 \psi = j \frac{1}{2k_0} \frac{\partial^2 \psi}{\partial x^2}$$

The above equation may be easily solved in the spatial frequency domain f_x, that is by calculating the Fourier transform of the field profile

$$\Psi(f_x, z) = \int_{-\infty}^{+\infty} \psi(x, z) \exp\left(- j2\pi f_x x\right) dx = F\{\psi\}\left(f_x, z\right)$$

$$\psi(x, z) = \int_{-\infty}^{+\infty} \Psi(f_x, z) \exp\left(j2\pi f_x x\right) df_x = F^{-1}\{\Psi\}\left(x, z\right)$$

$$F\left\{\frac{\partial \psi}{\partial x}\right\} = j2\pi f_x \Psi \qquad F\left\{\frac{\partial^2 \Psi}{\partial x^2}\right\} = \left(j2\pi f_x\right)^2 \Psi$$

By calculating the Fourier transform of Schrödinger's equation, we obtain

$$\frac{\partial \Psi}{\partial z} = -j\frac{1}{2k_0}(2\pi f_x)^2 \Psi$$

In the frequency domain, the effect of diffraction on a step h is

$$\Psi(f_x, z+h) = \exp\left(-j\frac{1}{2k_0}(2\pi f_x)^2 h\right)\Psi(f_x, z)$$

Next, one may go back into the transverse spatial coordinate domain by inverse Fourier transformation

$$\psi(x, z+h) = F^{-1}\left\{\Psi(f_x, z+h)\right\}$$

When starting from Helmholtz's equation:

$$\frac{\partial^2 E}{\partial z^2} + \nabla_\perp^2 E + k^2 n_0^2 E = \frac{\partial^2 E}{\partial z^2} + \frac{\partial^2 E}{\partial x^2} + k^2 n_0^2 E$$

$$E(x,z) \overset{F}{\Rightarrow} \mathrm{E}(f_x, z)$$

$$\frac{\partial^2 \mathrm{E}}{\partial z^2} + (j2\pi f_x)^2 \mathrm{E} + k^2 n_0^2 \mathrm{E} = 0$$

$$\left[\frac{\partial \mathrm{E}}{\partial z} - j\sqrt{k^2 n_0^2 - (2\pi f_x)^2}\mathrm{E}\right]\cdot\left[\frac{\partial \mathrm{E}}{\partial z} + j\sqrt{k^2 n_0^2 - (2\pi f_x)^2}\mathrm{E}\right] = 0$$

$$\Downarrow$$

$$\frac{\partial \mathrm{E}}{\partial z} = j\sqrt{k^2 n_0^2 - (2\pi f_x)^2}\mathrm{E}$$

We chose the sign of the phase shift term which corresponds to propagation in the positive z direction

$\mathrm{E} = \Psi \exp\left(jkn_0 z \right)$ We explicitly separate the rapidly varying phase

$$\Psi(f_x , z + h) = \Psi(f_x , z) \exp\left(j\left(- kn_0 + \sqrt{k^2 n_0^2 - (2\pi f_x)^2}\right)h\right)$$

One obtains once again the result that corresponds to the solution of Schrödinger's equation

$$- kn_0 + \sqrt{k^2 n_0^2 - (2\pi f_x)^2} = kn_0 \left(-1 + \sqrt{1 - \frac{(2\pi f_x)^2}{k^2 n_0^2}} \right) \cong -\frac{1}{2} \frac{(2\pi f_x)^2}{kn_0}$$

$$\boxed{\begin{gathered} \delta\beta = -kn_0 + \sqrt{k^2 n_0^2 - (2\pi f_x)^2} \\ \Psi(f_x , z + h) = \Psi(f_x , z) \exp\left(j\delta\beta\, h \right) \end{gathered}}$$

The Fourier Transform may be numerically computed in a very efficient manner through the FFT method

L: width of calculation window in the x coordinate

N: number of steps in the discretization of the x coordinate

$$x_n = (n-1)\Delta x = (n-1)\frac{L}{N} \qquad f_l = (l-1)\Delta f = (l-1)\frac{1}{L}$$

$$\psi(x_n) \Longrightarrow \boxed{\Psi(f_l) = \sum_{n=1}^{N} \psi(x_n) \exp\left(-j2\pi \frac{(l-1)(n-1)}{N}\right)}$$

$$\Psi(f_l) \Longrightarrow \psi(x_n) = \frac{1}{N}\sum_{l=1}^{N} \Psi(f_l) \exp\left(j2\pi \frac{(l-1)(n-1)}{N}\right)$$

GUIDING STRUCTURE

$$\frac{\partial \psi}{\partial z} = W\psi$$

W is, in general, z-dependent

Solution of the differential equation

$$\psi(x, z+h) = \psi(x, z)\exp\left(\int_z^{z+h} W(x, z)\, dz\right)$$

Crank-Nicolson integration scheme

$$\int_z^{z+h} W(x, z)\, dz \cong \frac{h}{2}\big(W(x, z+h) + W(x, z)\big)$$

$$\exp(hW) = \exp\left(\frac{h}{2}\big(W(x, z+h) + W(x, z)\big)\right)$$

1. Free propagation over the first half step *h/2*

$$\psi\left(x, z+\frac{h}{2}\right)=\exp\left(\frac{hU}{2}\right)\psi(x,z)=F^{-1}\left[\exp\left(j\delta\beta\frac{h}{2}\right)F\{\psi(x,z)\}\right]$$

2. Waveguide induced phase shift over the full step *h*

$$\begin{aligned}\overline{\psi}\left(x, z+\frac{h}{2}\right) &= \exp(hW)\psi\left(x, z+\frac{h}{2}\right)\\ &= \exp\left[\frac{h}{2}\left[W(x,z)+W(x,z+h)\right]\right]\psi\left(x, z+\frac{h}{2}\right)\end{aligned}$$

3. Free propagation over the second half step *h/2*

$$\psi(x, z+h)=\exp\left(\frac{hU}{2}\right)\overline{\psi}\left(x, z+\frac{h}{2}\right)=F^{-1}\left[\exp\left(j\delta\beta\frac{h}{2}\right)F\left\{\overline{\psi}\left(x, z+\frac{h}{2}\right)\right\}\right]$$

In the presence of Kerr nonlinearity, the W operator reads as

$$\boxed{W = j\frac{1}{2}k_0\left(\frac{n^2}{n_0^2}-1\right)+jkn_2|\psi|^2}$$

The evaluation of this exponential is no longer trivial: since we do not know $\psi(x,z+h)$, we also do not know $W(x,z+h)$

$$\exp\left[\frac{h}{2}\left[W(x,z)+W(x,z+h)\right]\right]$$

However, an accurate numerical solution may still be obtained by an iterative approach:

$$W(z+h)=W(z) \Longrightarrow \psi^{(0)}(z+h)$$

$$\psi^{(0)}(z+h) \qquad W^{(0)}(z+h) \Longrightarrow \psi^{(1)}(z+h)$$

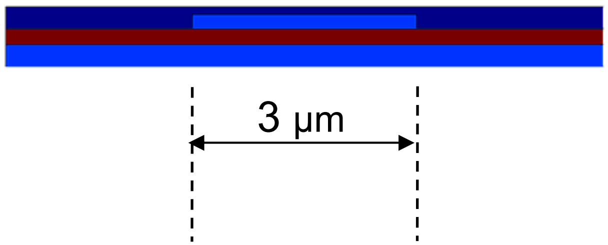
In practice, one obtains a good approximation after just one or two iterations!

Ex.: “ridge” dielectric waveguide obtained from a multi-layer structure

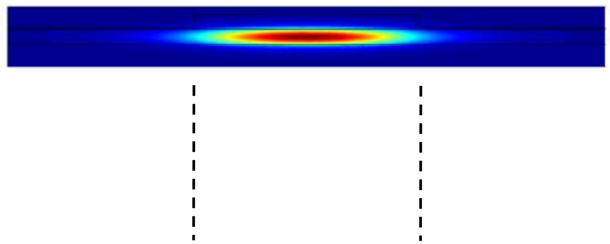
Multilayer structure:
- air
- SiO_2 with 0.2 µm width (n=1.444)
- Si with 0.2 µm width (n=3.476)
- SiO_2 substrate (n=1.444)

The “ridge” has a 3 µm width and is obtained by lithography by removing 0.2 µm of SiO_2

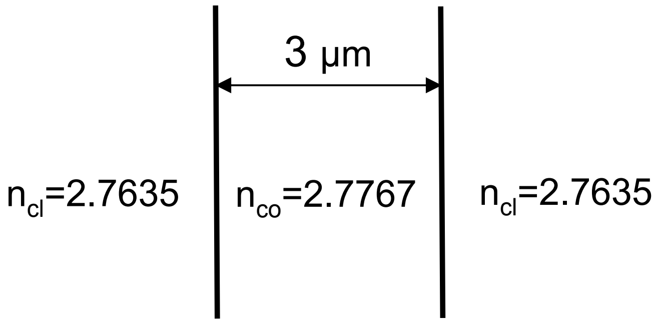
Refractive index profile of the “ridge”

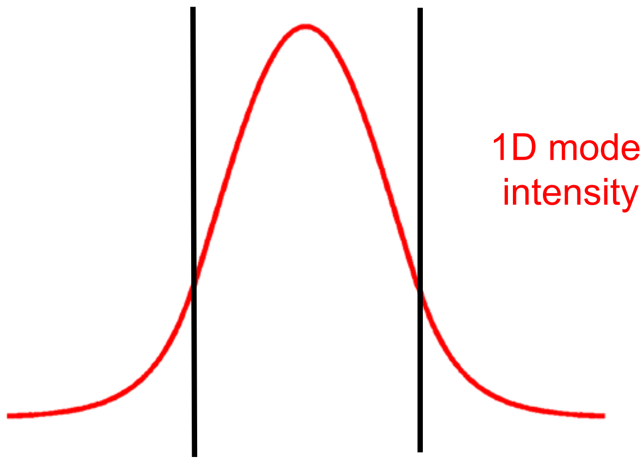


2D mode intensity (λ=1.55 µm)

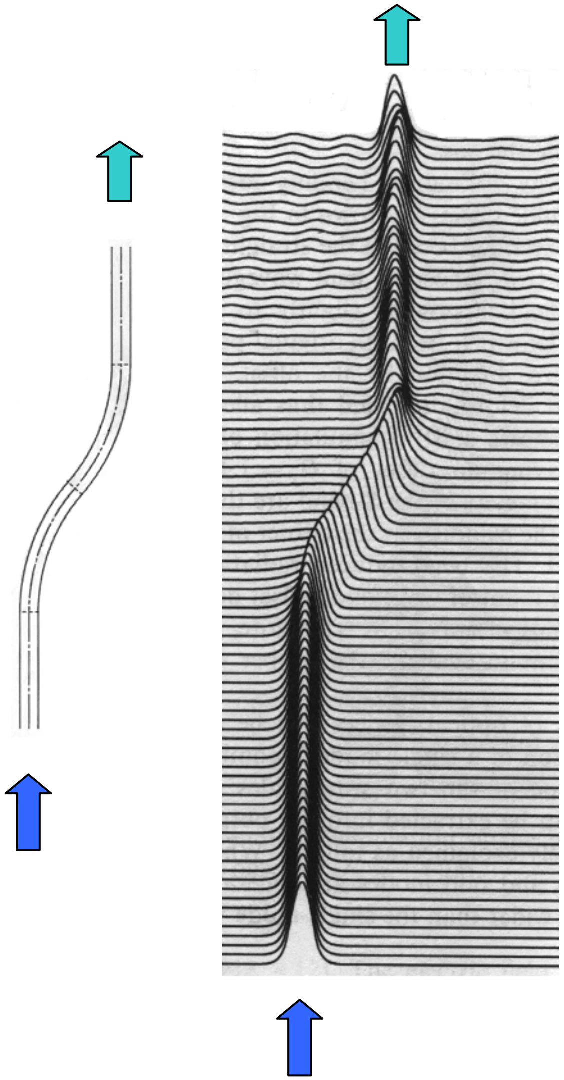


Equivalent layer obtained with the **effective index method**

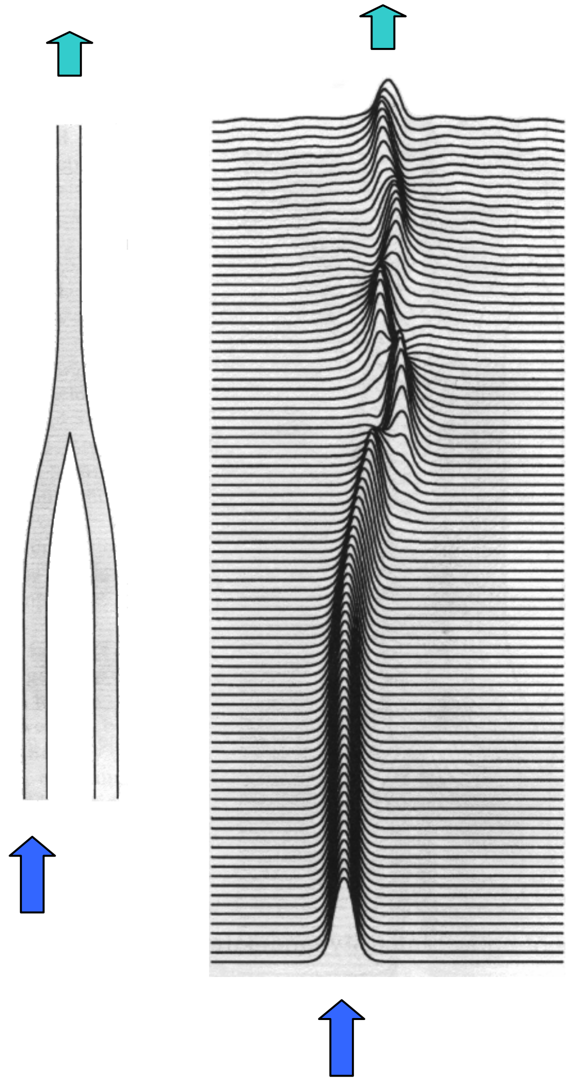




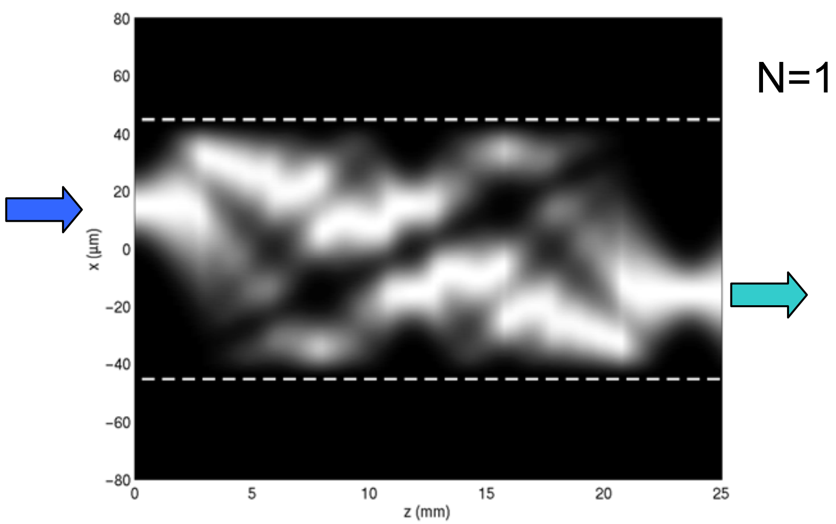
S curve

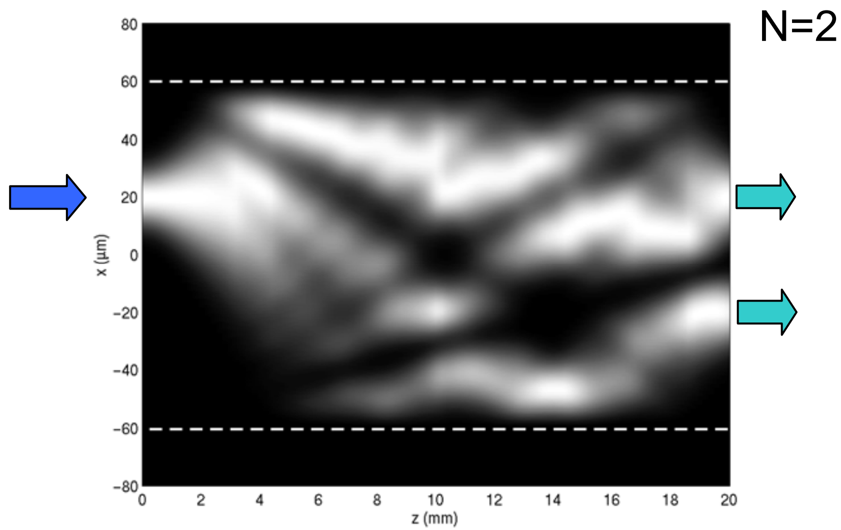


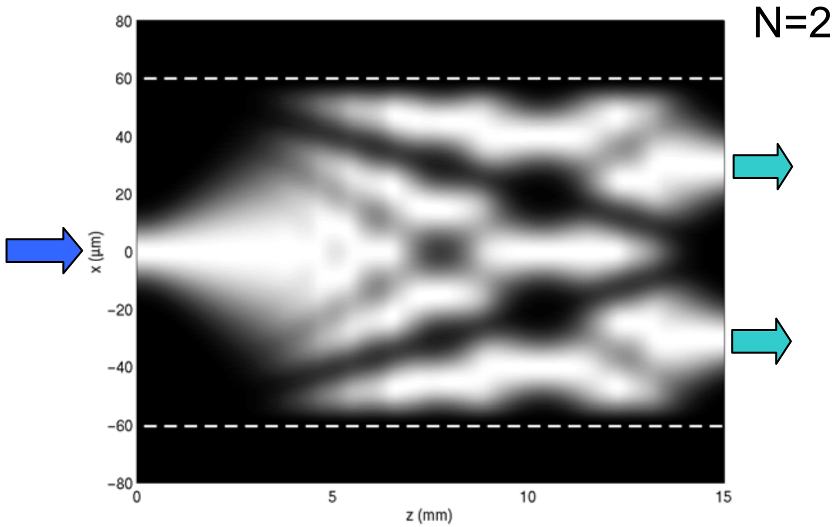
Y junction

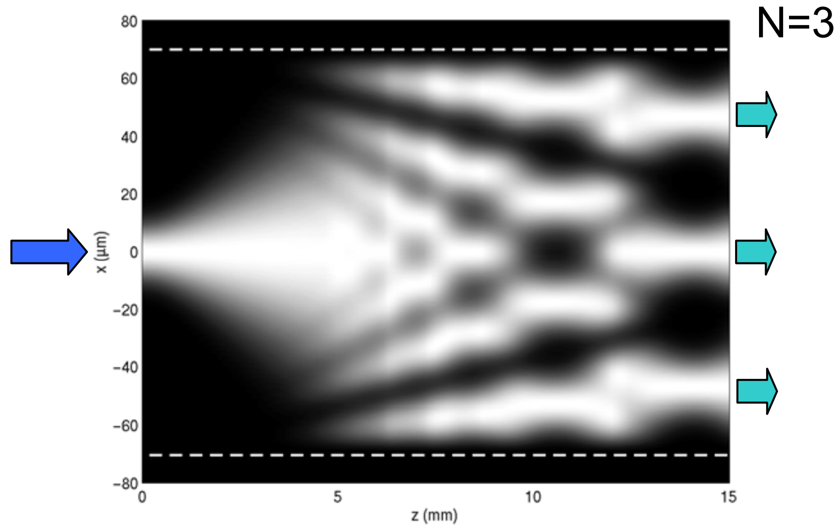


Propagation in multimode guide (1xN splitter)









Nonlinear Schrödinger eq. for fiber propagation

$$\frac{\partial A}{\partial Z} = (U + W)A$$

$$U = -j\frac{1}{2}\beta_2 \frac{\partial^2}{\partial T^2} \quad \text{dispersion}$$

$$W = j\gamma |A|^2 \quad \text{nonlinear potential}$$

$$\mathrm{A}(f_T, Z) = \int_{-\infty}^{+\infty} A(T, Z)\exp(-j2\pi f_T T)\, dT = F\{A\}(f_T, Z)$$

$$A(T, Z) = \int_{-\infty}^{+\infty} \mathrm{A}(f_T, Z)\exp(j2\pi f_T T)\, df_T = F^{-1}\{\mathrm{A}\}(T, Z)$$

1. Dispersion over the first half step *h/2*

$$A\left(T,Z+\frac{h}{2}\right)=F^{-1}\left[\exp\left(j\frac{1}{2}\beta_2(2\pi f_T)^2\frac{h}{2}\right)F\{A(T,Z)\}\right]$$

2. Phase shift due to the Kerr effect over full step *h*

$$\begin{aligned}\bar{A}\left(T,Z+\frac{h}{2}\right) &= \exp(hW)A\left(T,Z+\frac{h}{2}\right)\\ &= \exp\left[\frac{h}{2}[W(T,Z)+W(T,Z+h)]\right]A\left(T,Z+\frac{h}{2}\right)\end{aligned}$$

3. Dispersion over the second half step *h/2*

$$A(T,Z+h)=F^{-1}\left[\exp\left(j\frac{1}{2}\beta_2(2\pi f_T)^2\frac{h}{2}\right)F\left\{\bar{A}\left(T,Z+\frac{h}{2}\right)\right\}\right]$$

ALTERNATIVE TO BPM: FINITE DIFFERENCE INTEGRATION METHOD IN TIME DOMAIN

An alternative integration scheme with respect to the beam propagation method may use a discrete version of the nonlinear Schrödinger equation. Among different possible discrete versions, most stable (accurate) results obtained with Ablowitz-Ladik scheme → Obtain a set of N coupled ODEs

$$j\frac{\partial A}{\partial Z} = \frac{1}{2}\beta_2\frac{\partial^2 A}{\partial T^2} - \gamma|A|^2 A$$

$$\Downarrow$$

$$j\frac{\partial A_n}{\partial Z} = \frac{1}{2}\beta_2\Delta_2 A_n - \frac{\gamma}{2}|A_n|^2\left(A_{n+1} + A_{n-1}\right),$$

$$\Delta_2 A_n \equiv \left(A_{n+1} + A_{n-1} - 2A_n\right)/\Delta T^2$$