Maxwell equations
in <u>time domain</u>

$$\vec{\tilde{E}}(x,y,z,t) \quad \vec{\tilde{H}}(x,y,z,t)$$

$$\begin{cases} \nabla \times \vec{\tilde{E}} = -\mu_0 \dfrac{\partial \vec{\tilde{H}}}{\partial t} \\ \nabla \times \vec{\tilde{H}} = \dfrac{\partial \vec{\tilde{D}}}{\partial t} = \dfrac{\partial(\varepsilon_0 \vec{\tilde{E}} + \vec{\tilde{P}})}{\partial t} \end{cases}$$

For narrowband fields at the carrier angular frequency ω_0 ($\Delta\omega \ll \omega_0$)
It proves convenient to rewrite the equations in the frequency domain

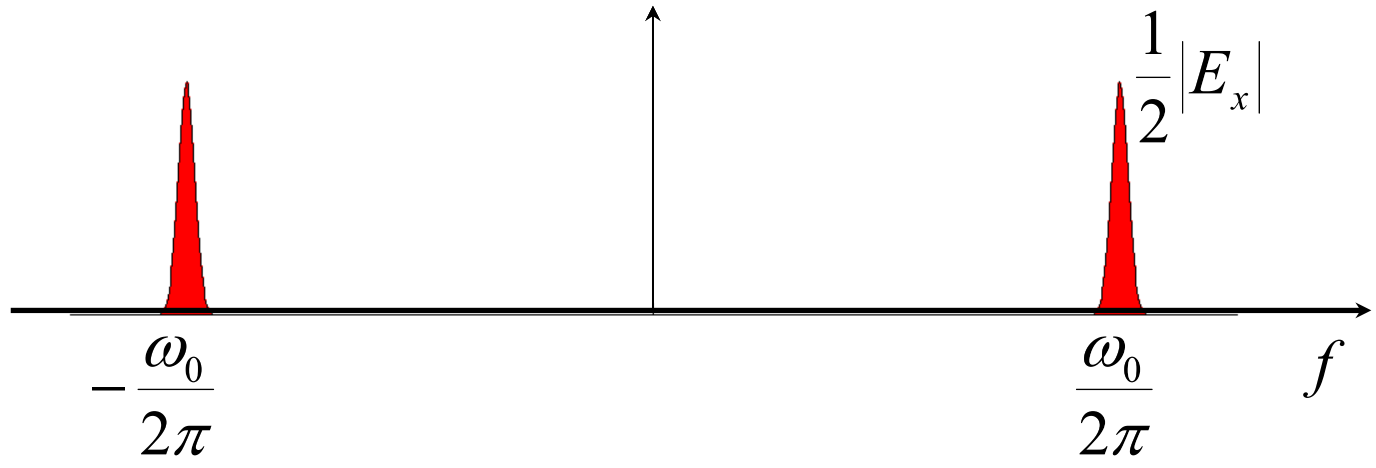
$$\vec{\tilde{E}}(x,y,z,t) = \mathrm{Re}\left\{ \vec{E}(x,y,z,t) e^{-j\omega_0 t} \right\}$$

$$\vec{\tilde{E}}(x,y,z,t) = \frac{1}{2}\left\{ \vec{E}(x,y,z,t) e^{-j\omega_0 t} + \vec{E}^*(x,y,z,t) e^{j\omega_0 t} \right\}$$

$\vec{E}(x,y,z,t)$

<u>Slowly varying envelope</u>

$$\omega_0 = 2\pi \frac{c}{\lambda} \quad \lambda = 1550 \text{ nm} \Rightarrow \omega_0 \cong 1.2 \times 10^{15} \text{ Hz}$$



Maxwell equations
in the <u>frequency</u> domain

$$\begin{cases} \nabla \times \vec{E} = j\omega_0 \mu_0 \vec{H} \\ \nabla \times \vec{H} = -j\omega_0 \vec{D} = -j\omega_0 (\varepsilon_0 \vec{E} + \vec{P}) \end{cases}$$

$\vec{E}(x,y,z) \quad \vec{H}(x,y,z)$

Let us study the propagation of a light beam in homogeneous media or in a guiding structure (possibly non-uniform)

$$\nabla \times \nabla \times \vec{E} = j\omega_0 \mu_0 \nabla \times \vec{H} = \omega_0^2 \mu_0 \varepsilon \vec{E}$$ This is rigorously valid for E

$$\nabla \times \nabla \times \vec{E} = \nabla(\nabla \cdot \vec{E}) - \nabla^2 \vec{E} \cong -\nabla^2 \vec{E}$$ $$\left(\nabla \cdot \vec{D} = 0,\ \vec{D} = \varepsilon \vec{E}\right)$$

$\nabla \cdot \vec{E} = 0$ $\Leftarrow$ homogeneous medium

$\nabla \cdot \vec{E} \cong 0$ $\Leftarrow$ slowly varying medium (weakly guiding)

the electric field components turn out to be uncoupled!

Supposing that $E(x,y,z)$ is linearly polarized, its evolution along the z direction is described by the following scalar relationship

Helmholtz equation

$$\boxed{\nabla^2 E + \omega_0^2 \mu_0 \varepsilon E = 0}$$

$\varepsilon(x,y,z)$ dielectric constant

$n(x,y,z)$ refractive index

$$\varepsilon(x,y,z)=\varepsilon_0\varepsilon_r=\varepsilon_0 n^2(x,y,z)$$

We may write the electric field E as the product of two terms:

- a <u>slowly-varying</u> (with z) envelope $\psi(x,y,z)$
- a rapidly oscillating phase term $\exp\{jk_0 z\}$

$$\boxed{E(x,y,z)=\psi(x,y,z)\,e^{jk_0 z}}$$

$$\boxed{k_0=\frac{2\pi}{\lambda}n_0}$$

ψ provides the "baseband" representation of signal E which is modulated at carrier wave number k_0

k_0 reference wave number

n_0 reference refractive index

By substitution in Helmholtz's equation

$$\frac{\partial^2\psi}{\partial z^2}+2jk_0\frac{\partial\psi}{\partial z}-k_0^2\psi+\frac{\partial^2\psi}{\partial x^2}+\frac{\partial^2\psi}{\partial y^2}+\frac{\omega_0^2}{c^2}n^2\psi=0$$

Slowly-varying envelope approx.: $\Rightarrow$ $\left|\frac{\partial^2\psi}{\partial z^2}\right| << \left|2jk_0\frac{\partial\psi}{\partial z}\right|$

$$\left(\left|\frac{\partial\psi}{\partial z}\right| << \left|\frac{\psi}{\lambda}\right| \Rightarrow \left|\frac{\partial^2\psi}{\partial z^2}\right| << \left|k_0\frac{\partial\psi}{\partial z}\right| << \left|k_0^2\psi\right|\right)$$

This equation describes the propagation in a weakly guiding waveguide in the z direction

$$\boxed{2jk_0\frac{\partial\psi}{\partial z}+\nabla_\perp^2\psi+\frac{\omega_0^2}{c^2}(n^2-n_0^2)\psi=0}$$

weak guiding hypothesis.: $\Leftrightarrow$ $\left|\frac{n^2(x,y,z)}{n_0^2}-1\right| << 1$

The above equation is only valid for describing propagation of paraxial beams (with wave vector nearly parallel to z) and cannot take into account reflections due to index discontinuities

In the frequency domain $D = \varepsilon_0 \varepsilon_r E = \varepsilon_0 (1 + \chi) E = \varepsilon_0 E + P$

Whereas for large values of the electric field, i.e., for $E \geq 10^8$ V/m

The relationship between P and E is no longer linear:

$$\boxed{D = \varepsilon_0 E + P + P_{NL}}$$

Supposing an instantaneous nonlinear response, we may write in the time domain

$$\boxed{\widetilde{P}_{NL} = \varepsilon_0 (\chi^{(2)} \widetilde{E}^2 + \chi^{(3)} \widetilde{E}^3 + \cdots)}$$

$\chi^{(2)}$ is the quadratic nonlinearity: it is absent in glass, but it is present in ferroelectric crystals ($LiNbO_3$) and in semiconductors ($GaAs$)

$\chi^{(3)}$ is the cubic nonlinearity: it cannot be neglected in glass (SiO_2) e.g., in optical fibers and in planar waveguides

$\chi^{(3)} \neq 0$ In a cubic nonlinearity medium one observes the KERR EFFECT: the refractive index has a contribution proportional to optical power

Refractive index variation:

$$\boxed{\Delta n = n_{2I} I = n_2 |E|^2} \qquad n_2 = \frac{3}{8n}\chi^{(3)} \qquad I = \frac{1}{2\eta}|E|^2,$$

$$\eta = \sqrt{\mu_0/\varepsilon} = 1/nc\varepsilon_0$$

Several materials of practical interest exhibit Kerr nonlinearity:

- Silica glass SiO_2 (optical fibers, passive planar circuits)
- Gallium arsenide $GaAs$, silicon Si (planar circuits)
- Polymers (for passive planar circuits)

material	SiO_2	$GaAs$	PTS
n (λ=1550 nm)	1.44	3.43	1.66
n_2 [m²/V²]	0.6×10^{-22}	0.65×10^{-19}	0.44×10^{-18}

In propagation equations, one obtains the square of the total refractive index:

$$n_{tot}^2 = (n + \Delta n)^2 = n^2 + 2n\,\Delta n + \Delta n^2 \cong n^2 + 2\,n\,n_2|E|^2 \cong n^2 + 2\,n_0 n_2 |E|^2$$

In nonlinear media, Helmholtz equation is modified as follows:

$$\nabla^2 E + \omega_0^2 \mu_0 \varepsilon E + \omega_0^2 \mu_0 P_{NL} = 0$$

The nonlinear polarization contribution (which represents a small perturbation with respect to the linear polarization) reads as:

$$\boxed{P_{NL} = 2\varepsilon_0 n_0\, n_2 |E|^2 E}$$

$$\Downarrow$$

$$2\,jk_0 \frac{\partial \psi}{\partial z} + \nabla_\perp^2 \psi + \frac{\omega_0^2}{c^2}(n^2 - n_0^2)\psi + \boxed{2\frac{\omega_0^2}{c^2} n\, n_2 |\psi|^2 \psi} = 0$$

$$j\frac{\partial \psi}{\partial z} + \frac{1}{2k_0}\nabla_\perp^2 \psi + \frac{1}{2}\frac{\omega_0}{c} n_0 \left(\frac{n^2}{n_0^2} - 1\right)\psi + \frac{\omega_0}{c} n_2 |\psi|^2 \psi = 0$$

Nonlinear Schrödinger equation (NLSE) in 3D

$$j\frac{\partial \psi}{\partial z}+\boxed{\frac{1}{2k_0}\nabla_\perp^2\psi}+\boxed{\frac{1}{2}k_0\left(\frac{n^2}{n_0^2}-1\right)\psi}+\boxed{\frac{2\pi}{\lambda}n_2|\psi|^2\psi}=0$$

diffraction linear guiding Kerr nonlinearity

$n = n(x, y, z)$ refractive index of the 3D waveguide

The NLSE holds under the following assumptions:

- Monochromatic or narrowband field at ω_0
- Paraxial propagation (narrow angular spectrum around k_0)
- Weak guiding
- $n(x,y,z)$ has negligible discontinuities along the z direction

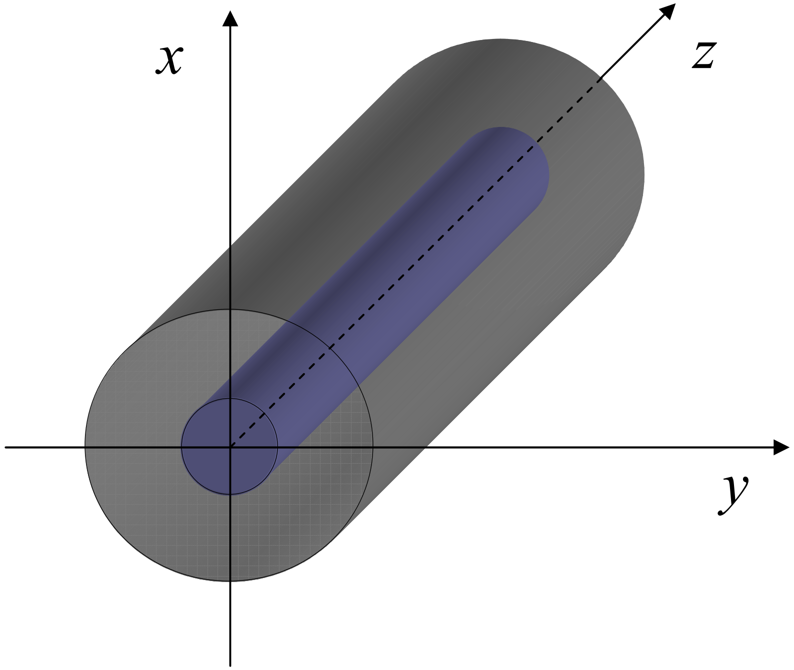
Pulse propagation in optical fibers

Core index $n_1 = 1.464$
Core radius: 4 μm

cladding index $n_2 = 1.450$

weak guiding hypothesis

$$\left(\frac{n_1}{n_2}\right)^2 - 1 << 1 \Leftrightarrow n_2 \cong n_1 \equiv n$$



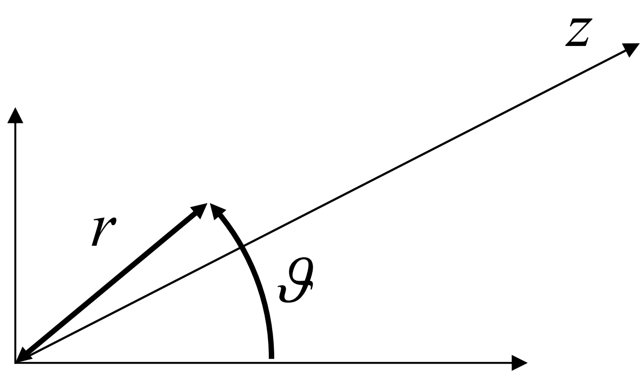
Let us suppose the fiber to be monomode the angular frequency ω_0 and that the field is linearly polarized, for example along the x direction

$$\widetilde{E} = \frac{1}{2}\left(Ee^{-j\omega_0 t} + E^* e^{j\omega_0 t}\right) = \frac{1}{2} Ee^{-j\omega_0 t} + c.c$$ $c.c.$ = complex conjugate

In cilindrical coordinates (r, ϑ, z)

Electric field envelope

$$E = E(r, \vartheta, z, t)$$



Wave equation in the time domain

$$\nabla^2 \tilde{E} = \mu_0 \varepsilon \frac{\partial^2 \tilde{E}}{\partial t^2} + \mu_0 \frac{\partial^2 \tilde{P}_{NL}}{\partial t^2}$$

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \vartheta^2} + \frac{\partial^2}{\partial z^2} = \nabla_\perp^2 + \frac{\partial^2}{\partial z^2}$$

Nonlinear polarization term due to the Kerr effect

$$\tilde{P}_{NL} = 2\varepsilon_0 n\, n_2 |E|^2 \tilde{E}$$

$$\tilde{P}_{NL} = 2\varepsilon_0 n\, n_{2I} \left(\frac{1}{2} c \varepsilon_0 n |E|^2\right) \tilde{E}$$

The electric field envelope may be decomposed as the product of 3 terms :

- Transverse field profile $R(r,\vartheta)$
- Slowly varying (longitudinally, with z) envelope $A(z,t)$
- Rapidly oscillating phase $\exp(j\beta z)$

$$\boxed{E(r,\vartheta,z,t) = R(r,\vartheta)A(z,t)e^{j\beta z}}$$

R is the transverse profile of the fundamental guided fiber mode (LP_{01})
And we suppose that its shape is not affected by nonlinearity

$$\nabla_\perp^2 R + \left(k^2 n^2 - \beta^2\right)R = 0$$ Helmholtz equation

$$\beta = \beta(\omega)$$ Mode propagation constant

field intensity $$I(r,\vartheta,z,t)=\frac{1}{2}c\varepsilon_0 n|R|^2|A|^2$$

R is normalized in such a way that the mode power P reads as:

$$P=\int_S \frac{1}{2}c\varepsilon_0 n|E|^2 ds=|A|^2\int_S \frac{1}{2}c\varepsilon_0 n|R|^2 ds=|A|^2$$

$$\boxed{P(z,t)=|A(z,t)|^2}$$

$\Uparrow$

fiber cross-section

$$\frac{1}{2}c\varepsilon_0 n|E|^2=\frac{|A|^2|R|^2}{\int_S |R|^2 ds}=\frac{|A|^2|R|^2}{\Sigma}$$

The slowly varying (in z) envelope assumption means that:

$$\left|\frac{\partial^2 A}{\partial z^2}\right| << \left|2\beta_0\frac{\partial A}{\partial z}\right|$$

By substituting in the wave equation, after some elementary steps (that we omit here) one obtains:

$$j2\beta_0 \frac{\partial A}{\partial z} + \left(\beta^2 - \beta_0^2\right)A + 2k^2 n_0 \frac{n_{2I}}{\Sigma^2} |A|^2 A \int_S |R|^4 ds = 0$$

Effective area: a_{eff}
Mode cross-section as far as the nonlinear effects are concerned

$$a_{eff} = \frac{\Sigma^2}{\int_S |R|^4 ds} = \frac{\left(\int_S |R|^2 ds\right)^2}{\int_S |R|^4 ds}$$

Given the transverse profile of the refractive index, it is possible to obtain the mode profile, hence the effective area is computed

Note: the angular frequency is present here:

$$j\frac{\partial A}{\partial z} + \frac{\beta^2(\omega) - \beta_0^2}{2\beta_0} A + \gamma |A|^2 A = 0 \qquad \gamma = \frac{2\pi n_{2I}}{\lambda a_{eff}}$$

Once the refractive index and the core radius are known,
the propagation constant may be obtained
from the solution of the Helmholtz equation

For a narrowband pulses at ω_0, we may write the mode propagation constant β by means of a Taylor expansion in ω-ω_0

$$\beta(\omega) = \beta(\omega_0) + \frac{\partial \beta}{\partial \omega}(\omega - \omega_0) + \frac{1}{2}\frac{\partial^2 \beta}{\partial \omega^2}(\omega - \omega_0)^2 + \frac{1}{6}\frac{\partial^3 \beta}{\partial \omega^3}(\omega - \omega_0)^3 + \cdots$$

$$\beta_0 = \beta(\omega_0)$$

reference propagation constant

$$\boxed{\beta_1 = \left.\frac{\partial \beta}{\partial \omega}\right|_{\omega=\omega_0} = \frac{1}{V_g}}$$

inverse of group velocity

$$\boxed{\beta_2 = \left.\frac{\partial^2 \beta}{\partial \omega^2}\right|_{\omega=\omega_0}}$$

dispersion

Let us point out the following relationships for $A(z,t)$

$$\int_{-\infty}^{+\infty} \frac{\partial A}{\partial t} \exp\big(j(\omega-\omega_0)t\big)\,dt \quad = \big(A\exp\big(j(\omega-\omega_0)t\big)\big]_{-\infty}^{+\infty}$$

$$- \int_{-\infty}^{+\infty} j(\omega-\omega_0) A \exp\big(j(\omega-\omega_0)t\big)\,dt$$

This is the well-known property of the Fourier transform of the time derivative of a function

$$= -j(\omega-\omega_0) \int_{-\infty}^{+\infty} A \exp\big(j(\omega-\omega_0)t\big)\,dt$$

With similar steps, one also obtains:

$$\int_{-\infty}^{+\infty} \frac{\partial^2 A}{\partial t^2} \exp\big(j(\omega-\omega_0)t\big)\,dt = -(\omega-\omega_0)^2 \int_{-\infty}^{+\infty} A \exp\big(j(\omega-\omega_0)t\big)\,dt$$

$$\boxed{-j(\omega-\omega_0)A \Leftrightarrow \frac{\partial A}{\partial t}}$$

$$\boxed{-(\omega-\omega_0)^2 A \Leftrightarrow \frac{\partial^2 A}{\partial t^2}}$$

$$\frac{(\beta-\beta_0)(\beta+\beta_0)}{2\beta_0} \cong (\beta(\omega)-\beta_0) \cong \beta_1(\omega-\omega_0)+\frac{1}{2}\beta_2(\omega-\omega_0)^2$$

$$j\frac{\partial A}{\partial z}+\beta_1(\omega-\omega_0)A+\frac{1}{2}\beta_2(\omega-\omega_0)^2 A+\gamma\left|A\right|^2 A=0$$

Equation that describes the evolution of the field envelope A

$$\boxed{j\frac{\partial A}{\partial z}+j\beta_1\frac{\partial A}{\partial t}-\frac{1}{2}\beta_2\frac{\partial^2 A}{\partial t^2}+\gamma\left|A\right|^2 A=0}$$

β_1, β_2, γ are constants

Let us consider the evolution of a short pulse: it is convenient to choose A reference frame moving with the pulse group velocity

v_g group velocity

$$v_g = \frac{\partial \omega}{\partial \beta} = \frac{1}{\beta_1}$$

$$\beta_2 = \frac{\partial \beta_1}{\partial \omega} = \frac{\partial}{\partial \omega}\left(\frac{1}{v_g}\right)$$

Group velocity dispersion

$$(z,t) \Leftrightarrow (Z,T)$$

$$\boxed{\begin{cases} Z = z \\ T = t - \beta_1 z \end{cases}}$$

$$\frac{\partial A}{\partial t} = \frac{\partial A}{\partial T}\frac{\partial T}{\partial t} + \frac{\partial A}{\partial Z}\frac{\partial Z}{\partial t} = \frac{\partial A}{\partial T}$$

$$\frac{\partial^2 A}{\partial t^2} = \frac{\partial^2 A}{\partial T^2}$$

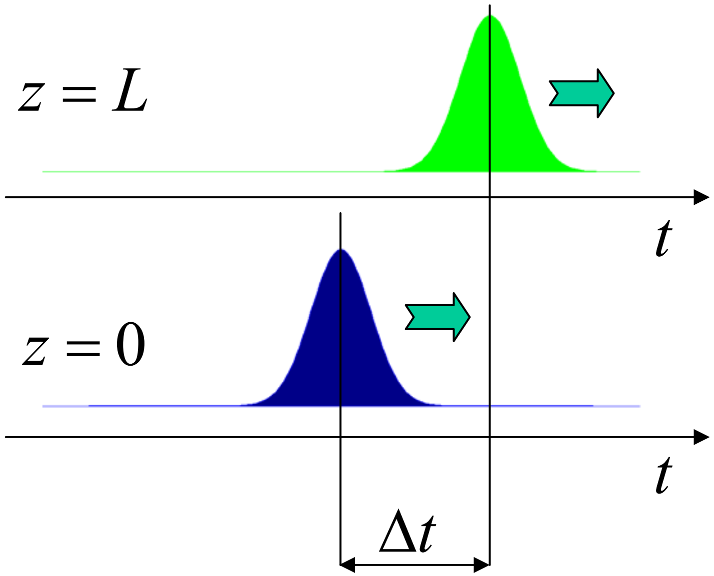
$$\frac{\partial A}{\partial z} = \frac{\partial A}{\partial T}\frac{\partial T}{\partial z} + \frac{\partial A}{\partial Z}\frac{\partial Z}{\partial z} = -\beta_1 \frac{\partial A}{\partial T} + \frac{\partial A}{\partial Z}$$

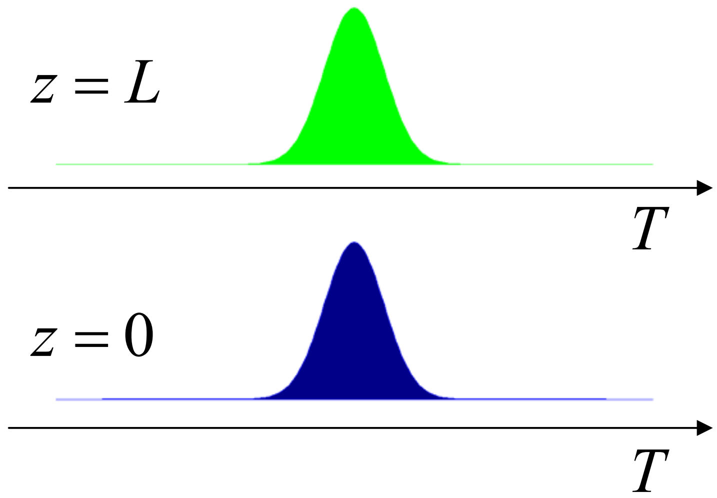
Initial reference frame

"traveling" reference frame

$$(z,t)$$

$$(Z,T) \quad \begin{cases} Z = z \\ T = t - \dfrac{z}{v_g} \end{cases}$$





The pulse "center of mass" is delayed by $\Delta t = \dfrac{L}{v_g}$

The pulse remains centered on the new time axis T

1D Nonlinear Schrödinger equation (NLSE)

$$j\frac{\partial A}{\partial Z}\boxed{-\frac{1}{2}\beta_2\frac{\partial^2 A}{\partial T^2}}+\boxed{\gamma|A|^2A}=0$$

(Z,T) time-delayed reference frame

β_2 dispersion

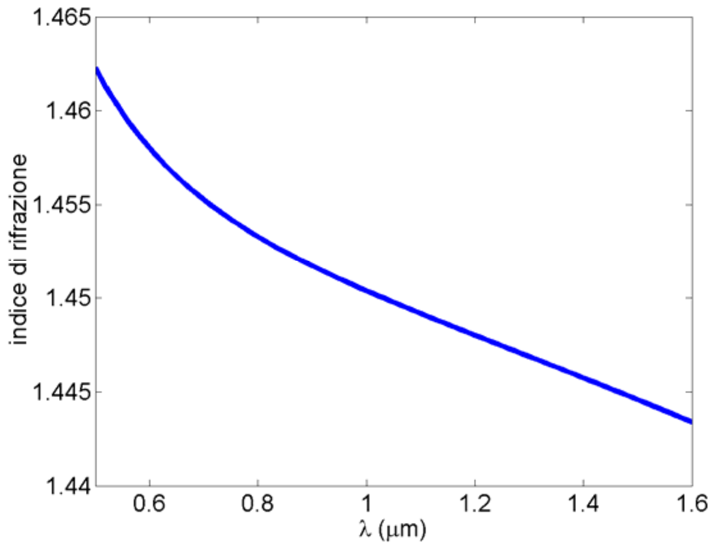
$$D=\frac{\partial}{\partial\lambda}\frac{\partial\beta}{\partial\omega}=\frac{\partial^2\beta}{\partial\omega^2}\frac{\partial\omega}{\partial\lambda}=-\frac{2\pi}{\lambda^2}c\beta_2$$

$$\gamma=\frac{2\pi}{\lambda}\frac{n_{2I}}{a_{eff}}$$

nonlinear coefficient

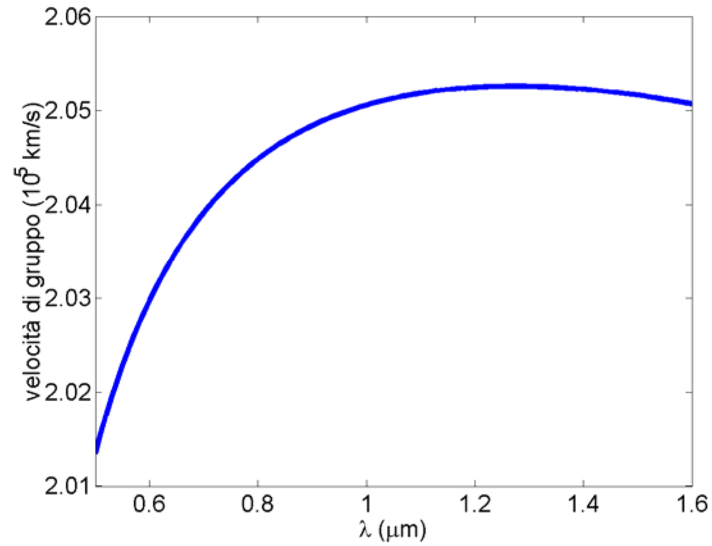
$n_{2I}=3.2\times10^{-16}\ \mathrm{cm^2/W}$ $\Rightarrow$ $\gamma=2-3\ \mathrm{W^{-1}km^{-1}}$

Material dispersion of silica glass is typically much larger than the waveguide contribution



Effective index of the guided mode

$$n_{eff} = \frac{c}{\omega}\beta$$



$$\beta_1 = \frac{1}{c}\left(n_{eff} + \omega\frac{dn_{eff}}{d\omega}\right)$$

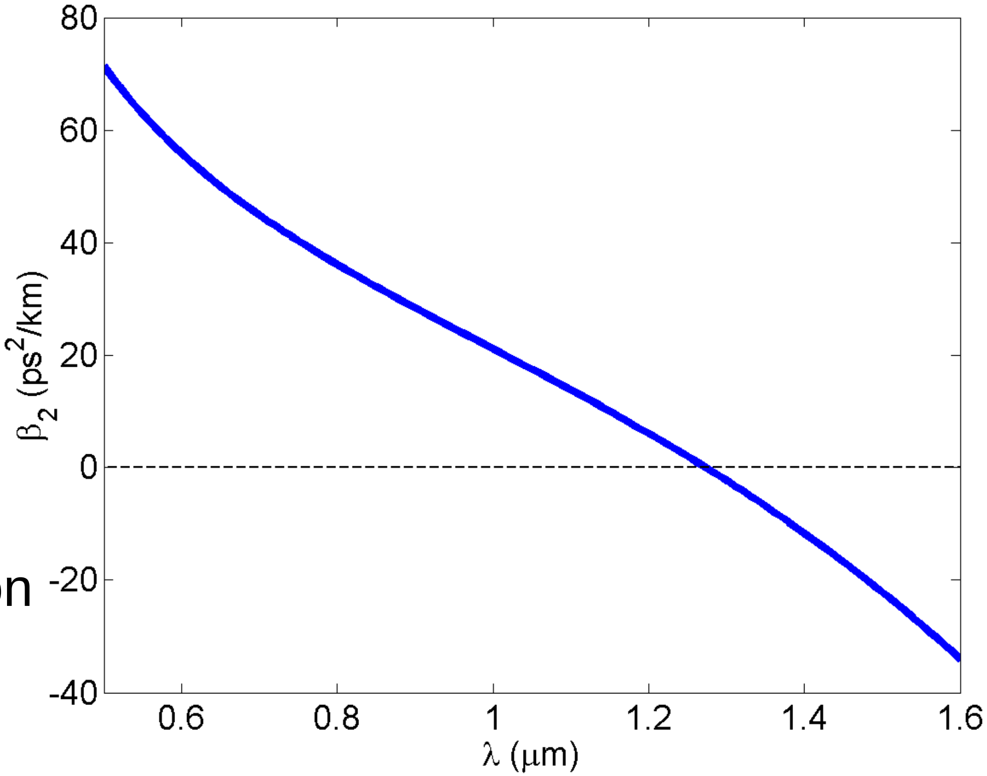
$$v_g = c \Big/ \left(n_{eff} - \lambda\frac{dn_{eff}}{d\lambda}\right)$$

$$\beta_2 = \frac{1}{c}\left(2\frac{dn_{eff}}{d\omega} + \omega \frac{d^2 n_{eff}}{d\omega^2} \right) = \frac{\lambda^3}{2\pi c^2}\frac{d^2 n_{eff}}{d\lambda^2} \qquad D = -\frac{\lambda}{c}\frac{d^2 n_{eff}}{d\lambda^2}$$

$\beta_2 > 0$ normal dispersion

$\beta_2 = 0$ @ $\lambda \approx 1300$ nm

$\beta_2 < 0$ anomalous dispersion



In standard monomode fibers, for λ close to 1550 nm

$$\beta_2 = -20\ \mathrm{ps^2/km} \quad \Rightarrow \quad D = 15.7\ \mathrm{ps/(nm \cdot km)}$$

Schrödinger equation

$$\boxed{j\hbar\frac{\partial\Psi}{\partial t} \quad \boxed{+\frac{\hbar^2}{2m}\nabla^2\Psi} \quad \boxed{-V(x,y,z,t)\Psi} \quad =0}$$

$$j\frac{\partial\psi}{\partial z} \quad +\frac{1}{2k_0}\nabla_\perp^2\psi \quad +\frac{1}{2}k_0\left(\frac{n^2}{n_0^2}-1\right)\psi+\frac{2\pi}{\lambda}n_2|\psi|^2\psi \quad =0$$

diffraction

linear potential

nonlinear potential

$$j\frac{\partial A}{\partial Z} \quad -\frac{1}{2}\beta_2\frac{\partial^2 A}{\partial T^2} \quad +\gamma|A|^2A \quad =0$$

nonlinear potential

dispersion

diffraction
dispersion

potential