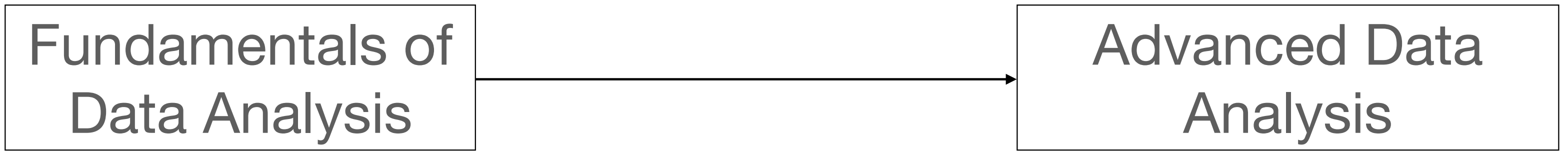
Introduction to Probability Theory 1

Fernando Racimo, 2023

Fundamentals of Data Analysis


Fundamentals of Data Analysis
Advanced Data Analysis

Fundamentals of Data Analysis → Advanced Data Analysis

- Properties of estimators
- Likelihood inference
- Bayesian thinking
- Generalized models
- Model comparison and regularization
- Resampling
- Mixed models
- Unsupervised learning

The two sides of statistics

Probability Theory

“What can we say about the data generated by a given process?”

The two sides of statistics

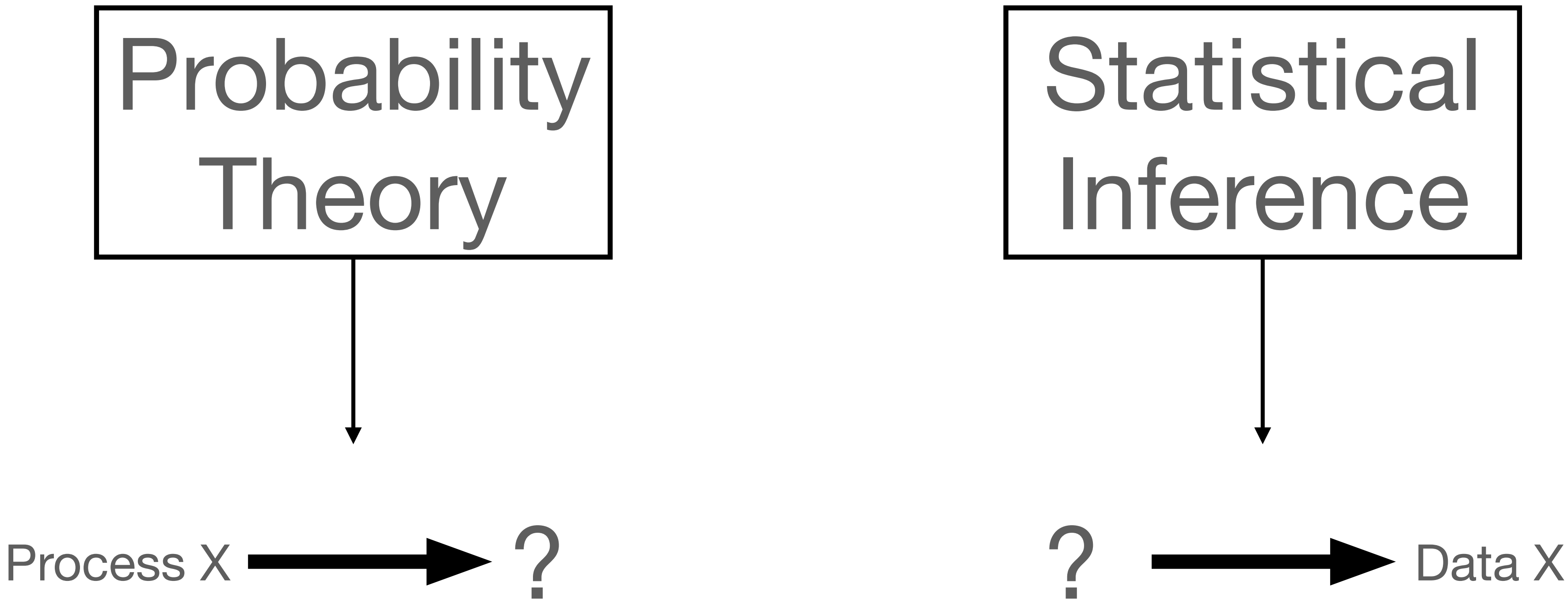
Probability Theory

"What can we say about the data generated by a given process?"

Statistical Inference

"What can we say about the process that generated a given data?"

The two sides of statistics



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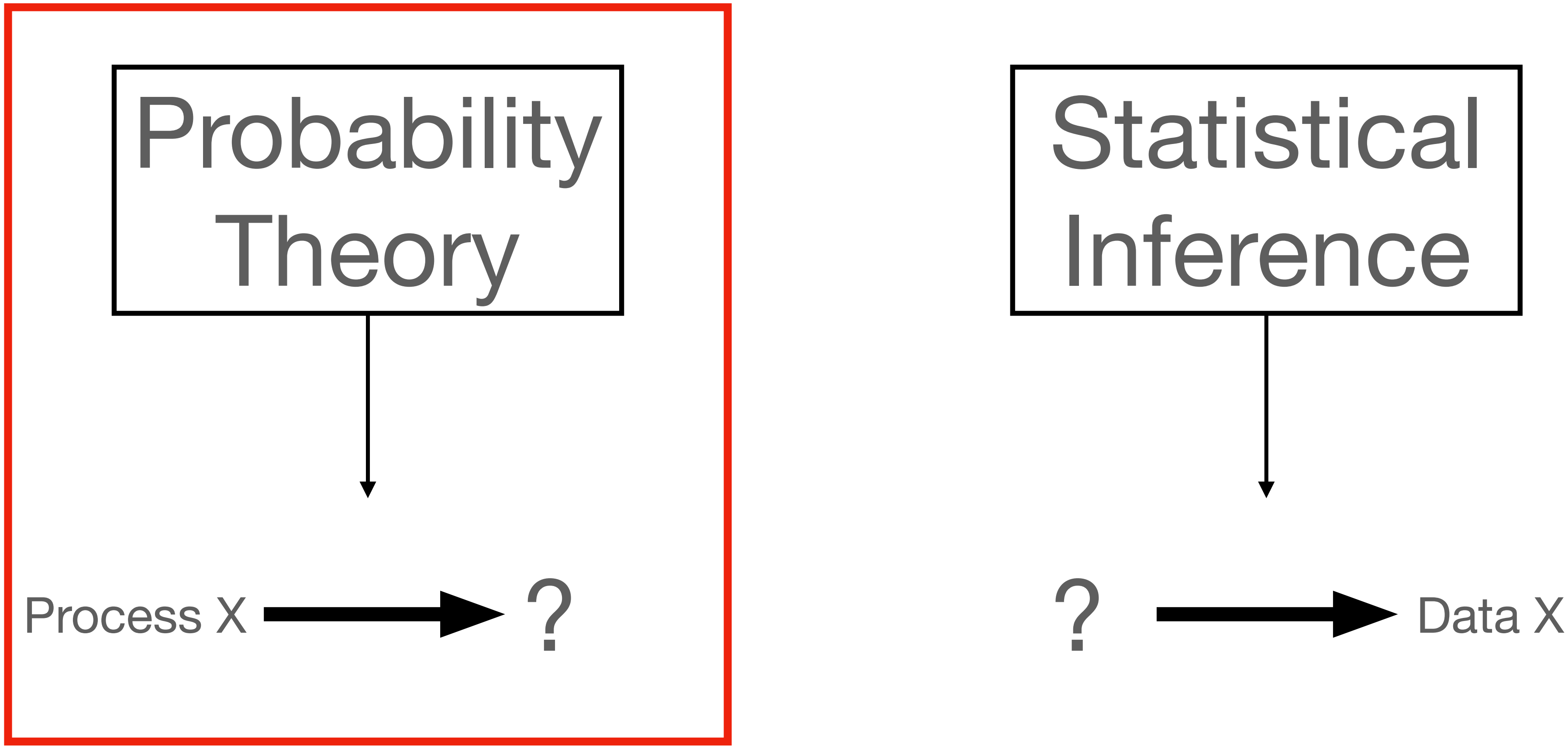

Some statistics lingo

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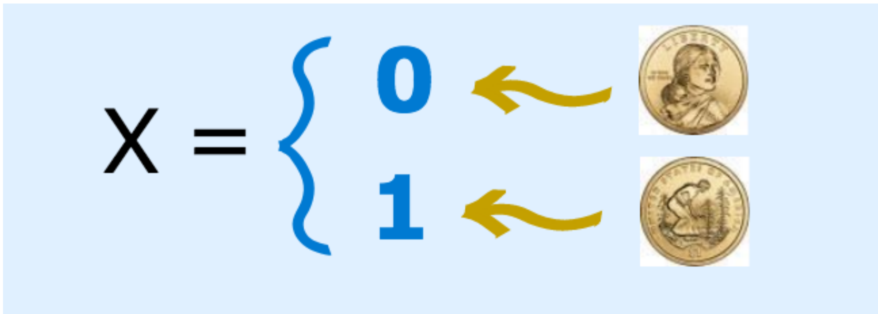

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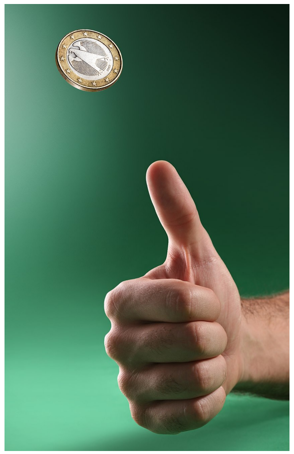

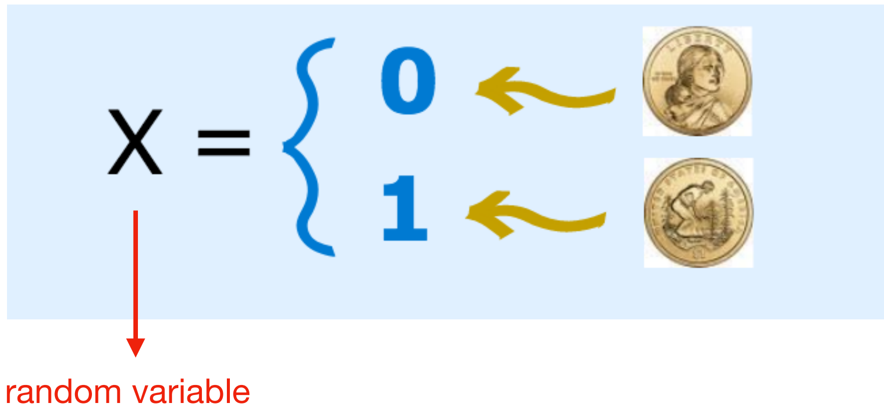

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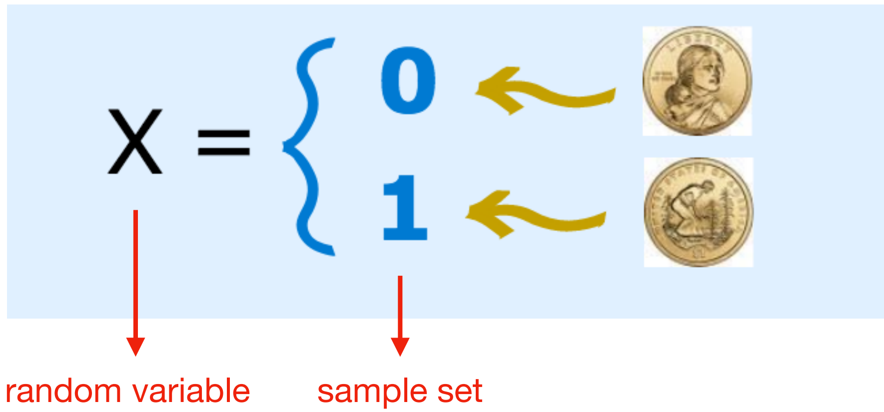

Basic rules of probability

- P[A or B] = P[A] + P[B] - P[A and B]
- In other words, the probability that A or B happens is equal to the probability that A happens plus the probability that B happens, minus the probability that both A and B happen.

Basic rules of probability

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- In other words, the probability that A or B happens is equal to the probability that A happens plus the probability that B happens, minus the probability that both A and B happen.
- <u>Exercise:</u> why do we have to subtract P[A and B] ?
- <u>Hint:</u>
  - P[A] = P[A and B] + P[A and not-B]

Rules of probability: conditioning

- P[A | B] stands for “the probability of A conditional on B having happened”

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- Example:
    - P[has covid **and** tested positive] =
        - = P[tested positive | has covid] P[has covid]
        - = P[has covid | tested positive] P[tested positive]

Rules of probability: conditioning

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Often, one of these is easier to obtain than the other

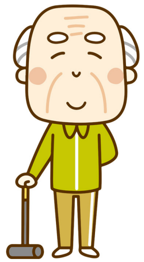
Rules of probability: independence

- A and B are said to be independent **if and only if** P[A and B] = P[A] P[B]

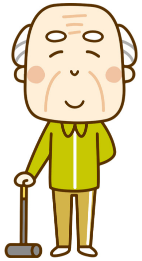
Rules of probability: independence

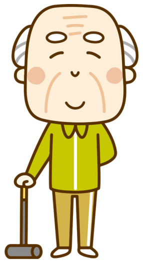
- A and B are said to be independent **if and only if** P[A and B] = P[A] P[B]
- When two events are independent, knowing that one event happens gives me no information about the other event
- If A and B are independent, then:
    - P[A | B] = P[A]
    - P[B | A] = P[B]

Rules of probability: independence

P[ | ] ≠ P[]

Rules of probability: independence

P[ | ] ≠ P[]

 and  are **not** independent

Rules of probability: independence

P[ | 

] = P[]

Rules of probability: independence

P[ | 

] = P[]

 and 

are independent

Bayes Rule

- $P[A \mid B] = \dfrac{P[B \mid A]P[A]}{P[B]}$

- <u>Exercise:</u> derive this rule.

- <u>Hint:</u> write down what P[A and B] is equal to (from previous slides)

Law of total probability

- P[A] = P[A | B] P[B] + P[A | C] P[C] + P[A | D] P[D] + …

- In other words: $$P[A] = \sum_i P[A \mid X_i] P[X_i]$$

- Why might it be useful to know this?

Probability distribution

- A **probability distribution** is a mathematical description of a random process, experiment or phenomenon (i.e. a random variable)
- Random ≠ Aleatory
- There are still **rules** that define any phenomenon, even if the phenomenon is random

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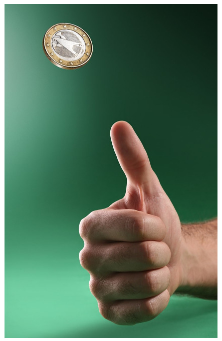

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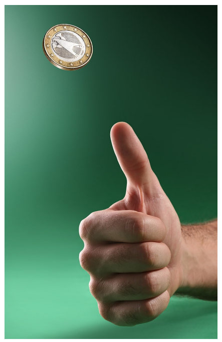

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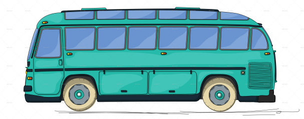

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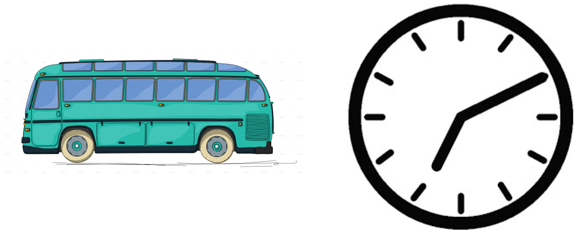

“-4”

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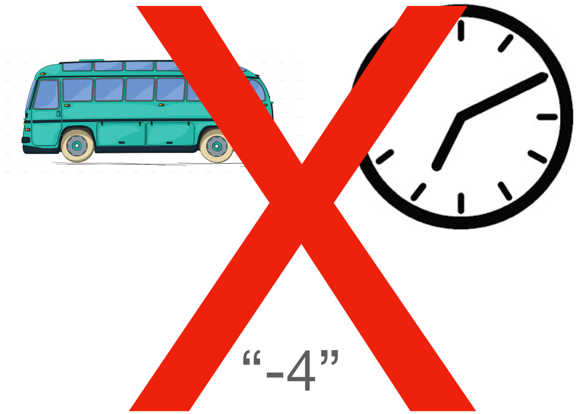

The Bernoulli distribution

$\sim$ Bernoulli (p)

- Here, p is the parameter that determines how fair the coin is, i.e. the probability of heads.
- For example:
    - **X ~ Bernoulli (0.5)** means “The random variable X can be modeled as a fair coin toss”
    - **X ~ Bernoulli (0.9)** means “The random variable X can be modeled as a biased coin toss with 90% probability of heads”

The Bernoulli distribution

This squiggly symbol means:
“can be modeled as”

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- Let a "heads" outcome be defined as X = 1 and a "tails" outcome be defined as X = 0

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~ Bernoulli (p)

- Let a “heads” outcome be defined as X = 1 and a “tails” outcome be defined as X = 0
- Assuming **X ~ Bernoulli (0.5)** then:
  - P[X = 1] = 0.5
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- Assuming **X ~ Bernoulli (0.9)** then:
  - P[X = 1] = 0.9
  - P[X = 0] = 0.1

Probability distributions: what are they good for?

- They allow us to **simulate** data
- They allow us to find good approximations to the processes that generated our data (in other words, perform **statistical inference**)
- They allow us to choose **among competing models** to explain our data

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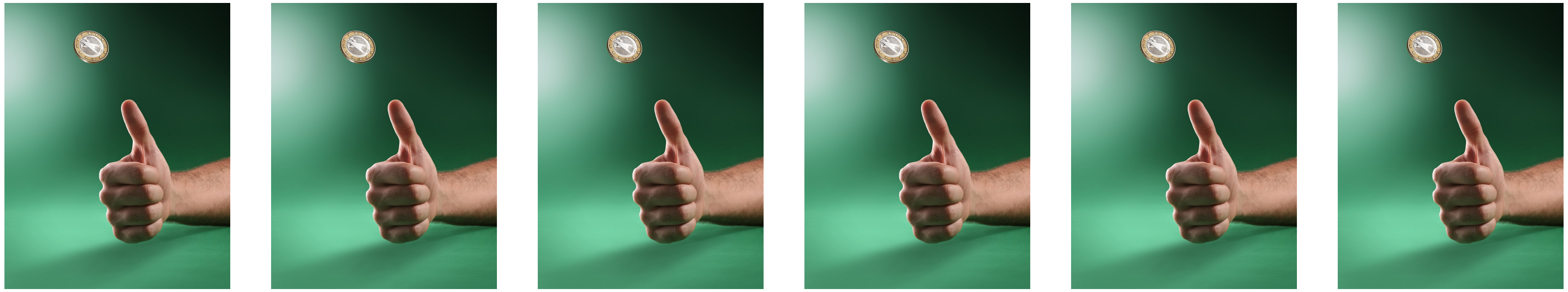
- They allow us to **simulate** data
- They allow us to find good approximations to the processes that generated our data (in other words, perform **statistical inference**)
- They allow us to choose **among competing models** to explain our data
- They allow us to better **think about** our data and the processes that generate it

The binomial distribution



$$\underbrace{0 \qquad 0 \qquad 1 \qquad 1 \qquad 0 \qquad 1}_{\text{sum} \sim \text{Binomial}(\, n,\ p\ )}$$

The binomial distribution

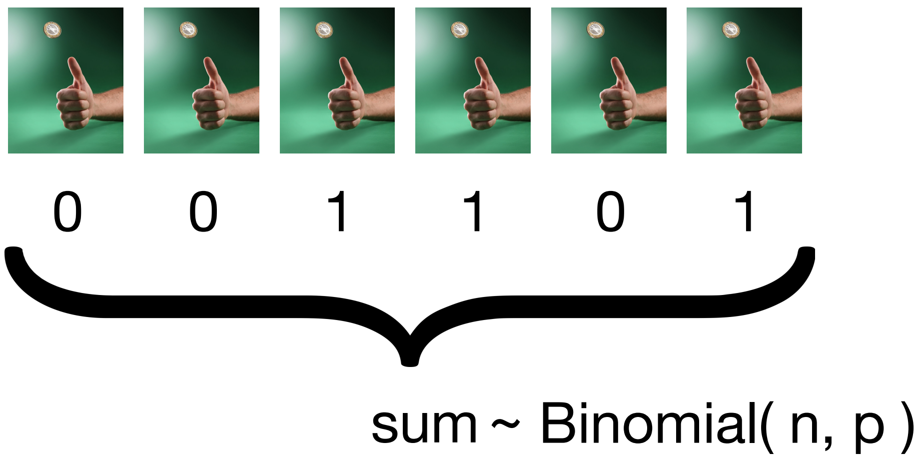
- What is the average number of heads I expect to get?
- What is the probability that I will get exactly 4 heads?
- What is the probability that I will get 4 or more heads?
- What is the probability that I will get 3 or less heads?

Exercises in R

- Bernoulli distribution: tossing a coin
- Binomial distribution: adding up coin tosses

The Binomial distribution

- Two parameters: n and p
- $P[X = k] = \binom{n}{k} p^k (1-p)^{n-k}$



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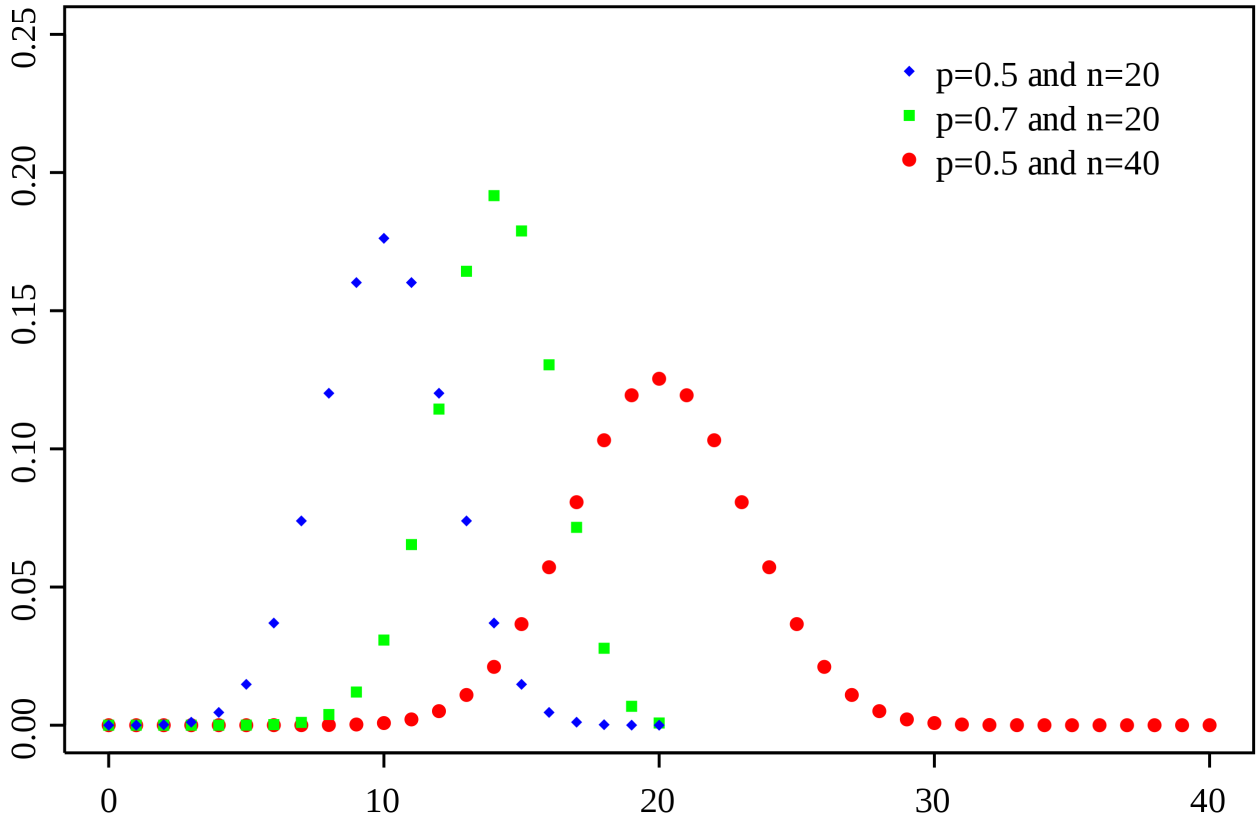

Two broad families of random variables

Discrete

- Examples:
  - Single coin flip (Bernoulli)
  - Number of heads in a series of flips (Binomial)
  - Number of times I roll a dice and get 4 or more
  - Number of trials until I roll a six
  - Number of rain drops in a bucket
  - Number of trees in a forest

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- Examples:
  - Height in a population
  - Waiting time till the next bus arrives
  - Time until a particle decays
  - Temperature in a room
  - Lifespan of a species
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We can model all of these (and much more) using common probability distributions

The Expectation of a Random Variable

- $E[X] = \sum_i x_i P[X = x_i]$

Properties of the expectation

- If a is a constant: $E[aX] = aE[X]$
- If a is a constant: $E[X + a] = E[X] + a$
- For two random variables X and Y: $E[X + Y] = E[X] + E[Y]$
- More generally: $E[\sum X_i] = \sum E[X_i]$

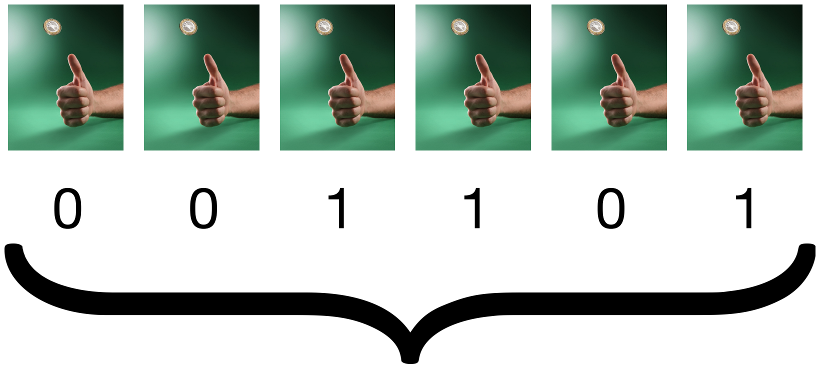
The Average or Sample Mean

- $E[X] = \sum_i x_i P[X = x_i]$
- The average or sample mean ($\bar{x}$) is an approximation to E[X] when one has observed N samples
    - $\bar{x} = \sum_{i=1}^{N} \frac{x_i}{N}$

The Binomial distribution

- $P[X = k] = \binom{n}{k} p^k (1-p)^{n-k}$

- $E[X] = np$



sum ~ Binomial(n, p)

The Expectation - exercises

- In the case of the Bernoulli distribution: $E[X] = p$ …LET'S PROVE THIS!

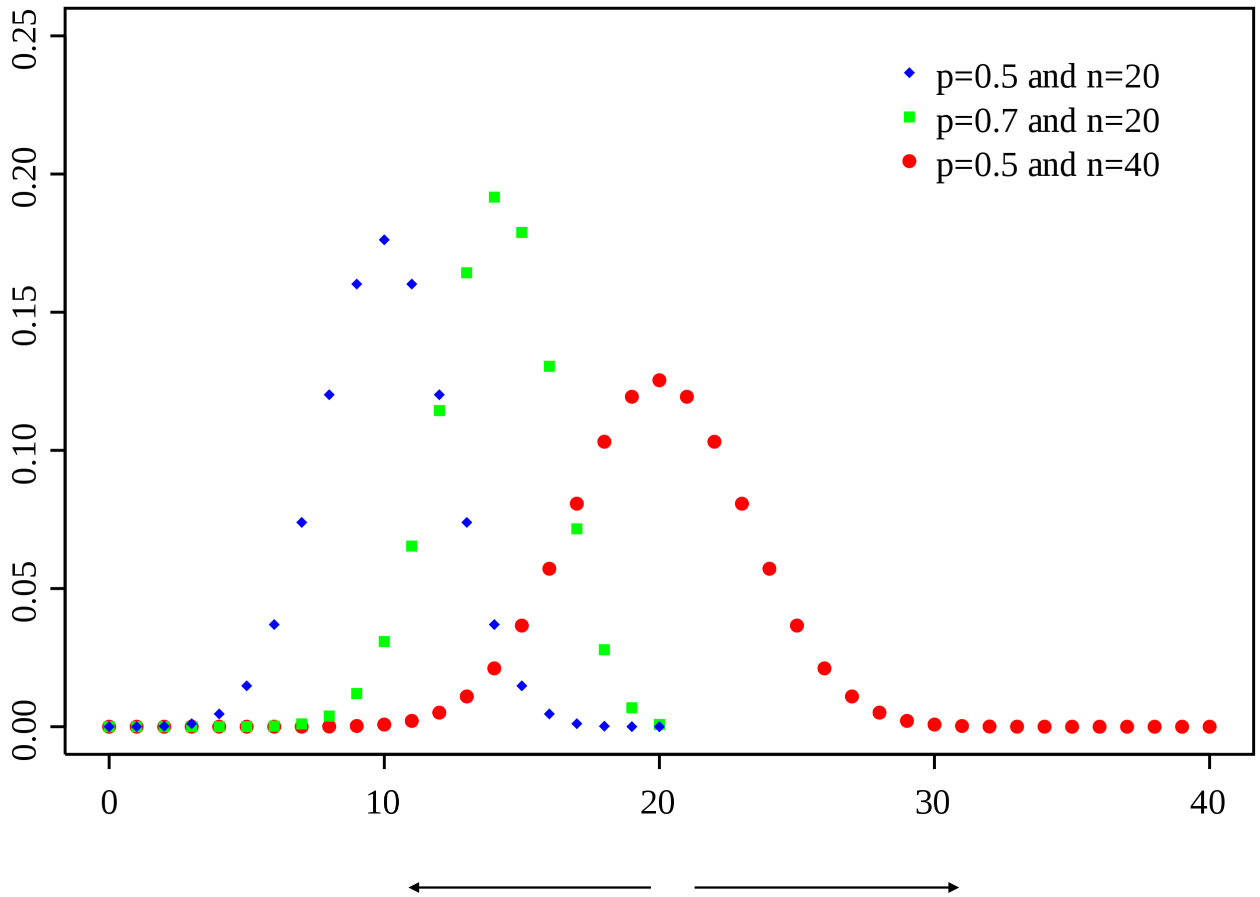
- In the case of the Binomial distribution: $E[X] = np$ …LET'S PROVE THIS!

The Variance

- $Var[X] = E[(X - E[X])^2]$

- This can also be written as: $Var[X] = E[X^2] - E[X]^2$

The Variance



The Sample Variance

- $Var[X] = E[(X - E[X])^2]$

- The sample variance (s^2) is an unbiased approximation to $Var[X]$
- $s^2 = \dfrac{\sum_i (x_i - \bar{x})^2}{n - 1}$

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Why "n-1", and not "n"? See: https://web.ma.utexas.edu/users/mks/M358KInstr/SampleSDPf.pdf

The Binomial distribution

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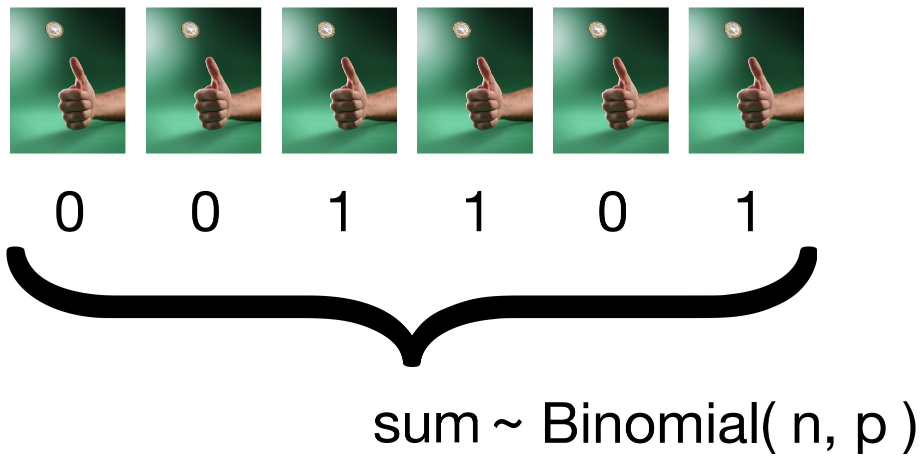
- $E[X] = np$

- $Var[X] = np(1-p)$



Exercises in R

- The Expectation
- Our first probability mass function
- The Variance

The Geometric distribution

The Geometric distribution

0

The Geometric distribution

0 0

The Geometric distribution

0 0 0

The Geometric distribution

0 0 0 0

The Geometric distribution

0 0 0 0 0

The Geometric distribution

0 0 0 0 0 1

The Geometric distribution

$$\underbrace{0 \quad 0 \quad 0 \quad 0 \quad 0}_{\text{no. tails until 1 heads}} \quad 1$$

The Geometric distribution

$$\underbrace{0 \qquad 0 \qquad 0 \qquad 0 \qquad 0}_{\text{no. tails until 1 heads} \sim \text{Geom}(p\ )} \qquad 1$$

The Geometric distribution

- PMF: $P[T = t] = \overbrace{(1-p)^{t-1}}^{\text{t-1 failures}}\overbrace{p}^{\text{1 success}}$

- Expectation: $E[T] = \dfrac{1-p}{p}$

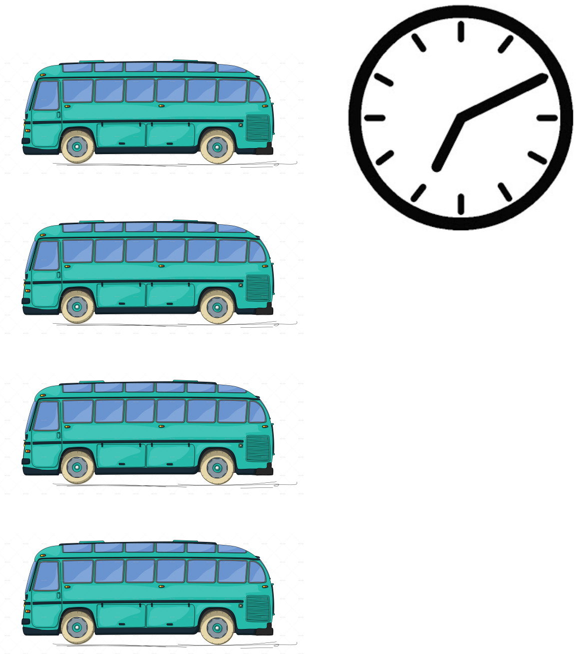
- Variance: $Var[T] = \dfrac{(1-p)}{p^2}$

$$\underbrace{0 \quad 0 \quad 0 \quad 0 \quad 0}_{\text{no. tails until 1 heads}} \quad 1$$

no. tails until 1 heads ~ Geom(p)

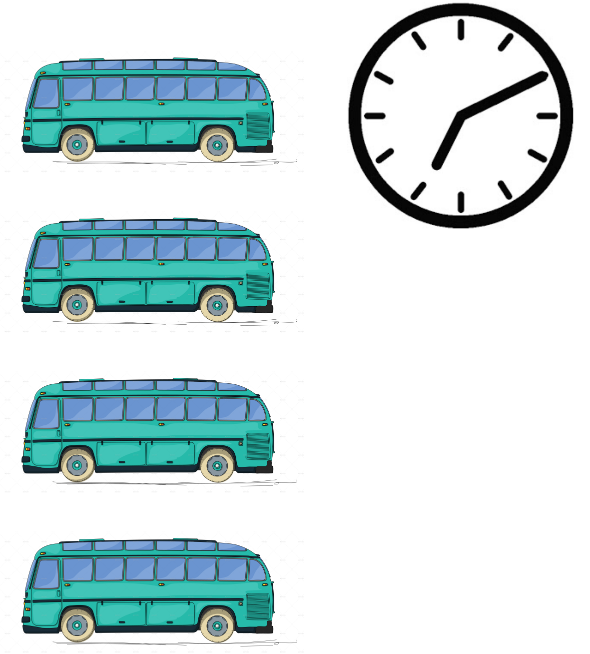
The Poisson distribution

- How many buses arrive at this station every hour?

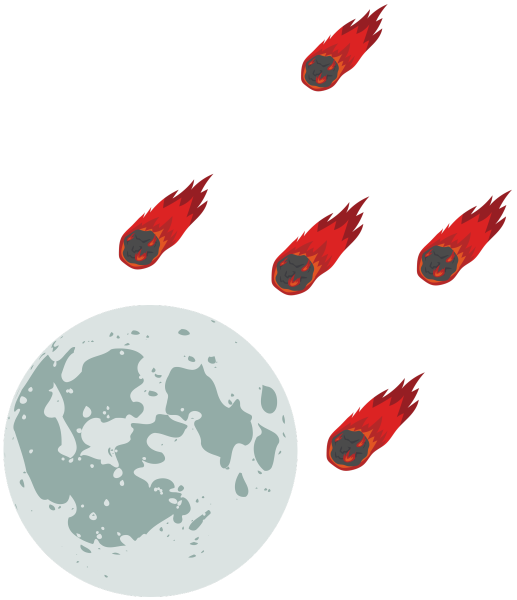


The Poisson distribution

- How many buses arrive at this station every hour?



- How many meteors hit the moon every year?



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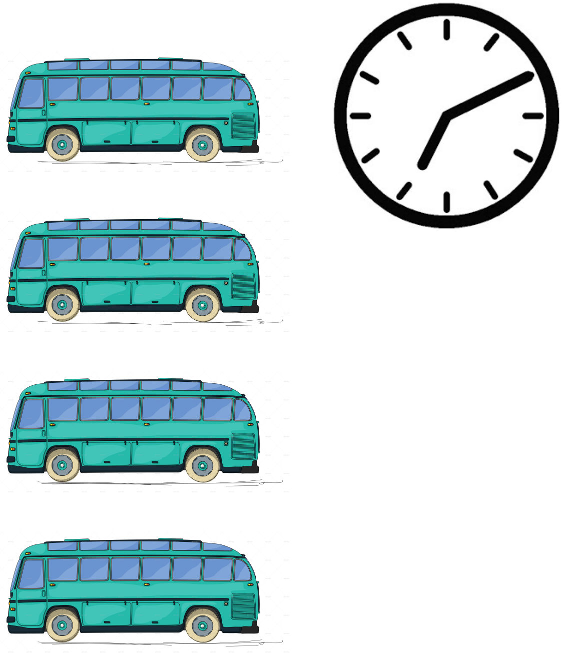

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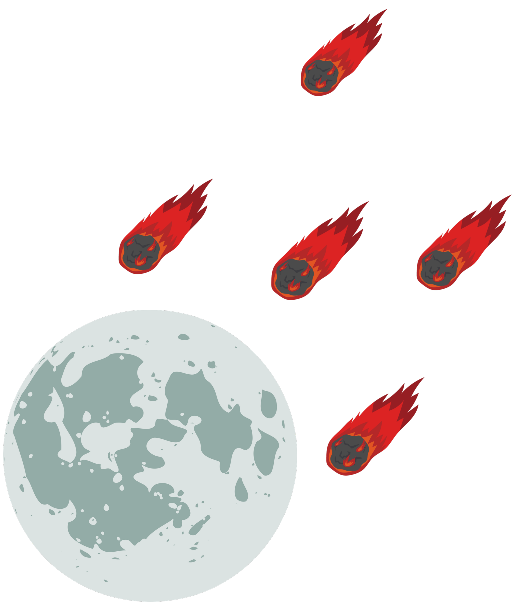
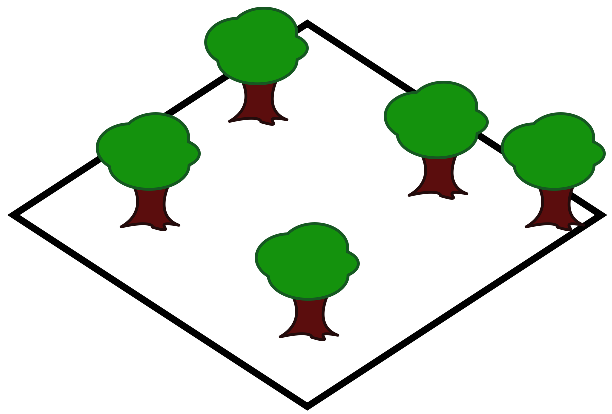


- How many trees are there in a hectare of land?

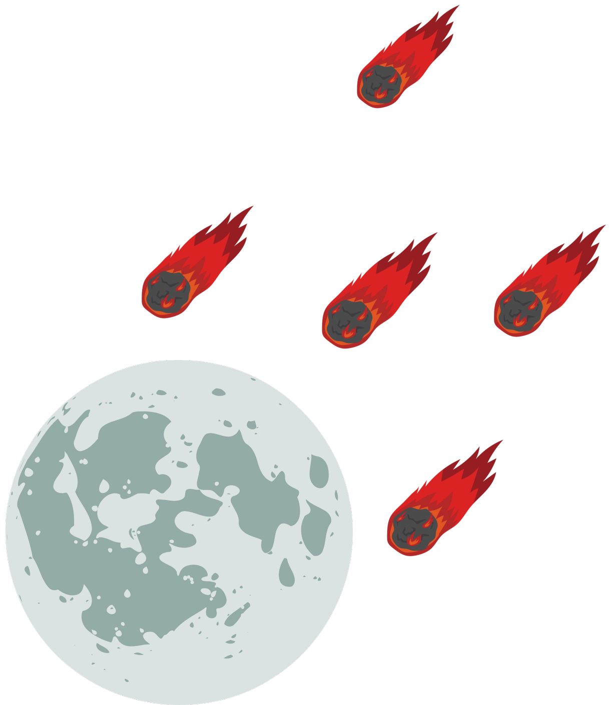


The Poisson distribution

- Events or objects that occur **over a given stretch** of time and/or space
- The **rate or intensity** of occurrence (per unit of time and/or space) is the **same for all** events or objects
- Each event or object occurs **independently** of the other events or objects

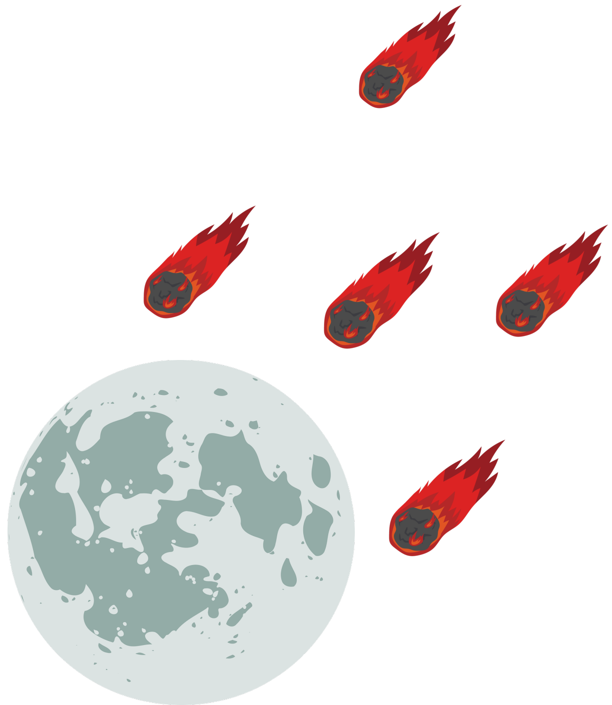
The Poisson distribution

- **Average rate** at which meteorites fall on a planet is 5 per year
- Some years, there might be 4 meteorites. Other years, there might be 6, or 3, or 5, or 7, etc…
- It is **technically possible** that 1,000,000 meteorites fall on a given year, but **extremely unlikely**

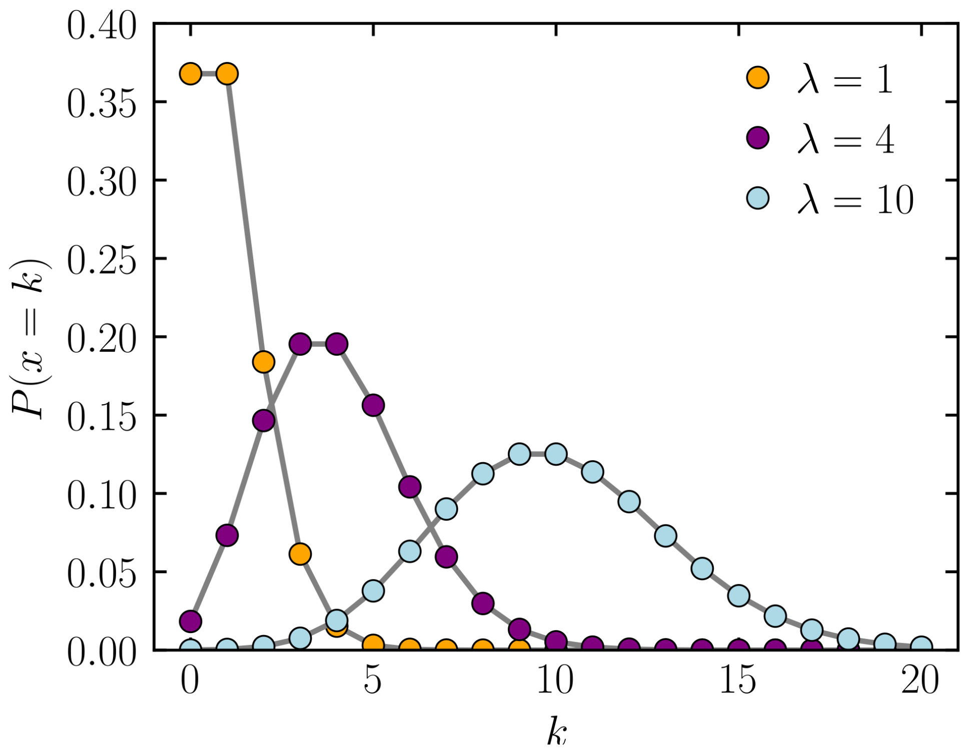


The Poisson distribution

- One parameter: λ ("intensity" or "rate" of an event or object)
- $E[X] = \lambda$
- $Var[X] = \lambda$
- $P[X = k] = \dfrac{\lambda^k e^{-\lambda}}{k!}$



The Poisson distribution: PMF



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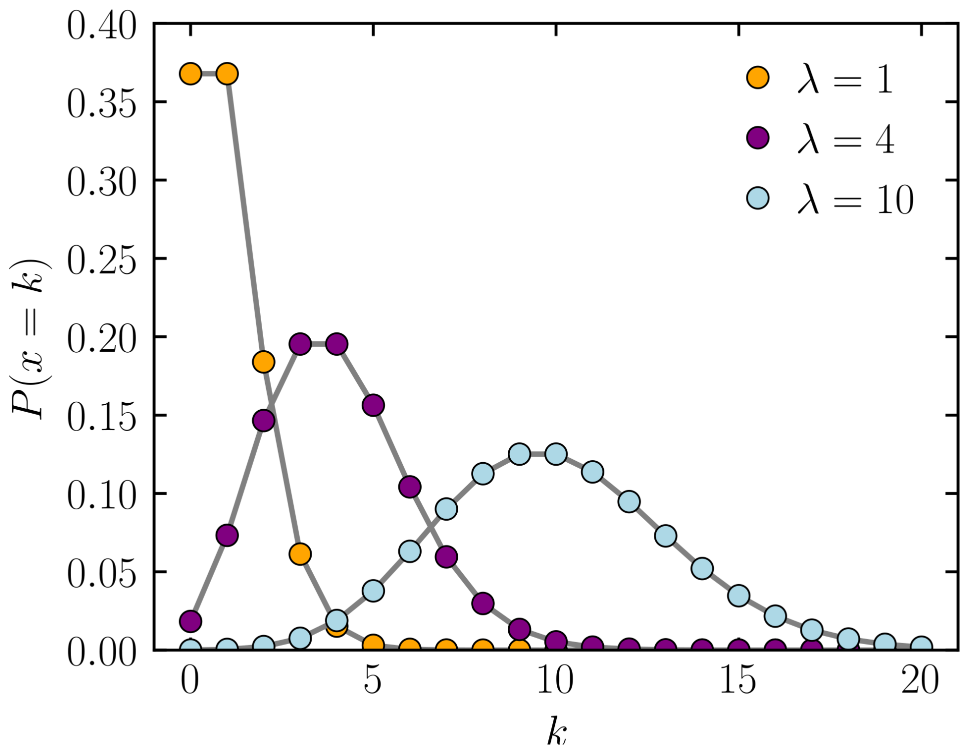


NOTE: unlike the Binomial distribution, there is no "maximum" number of possible events

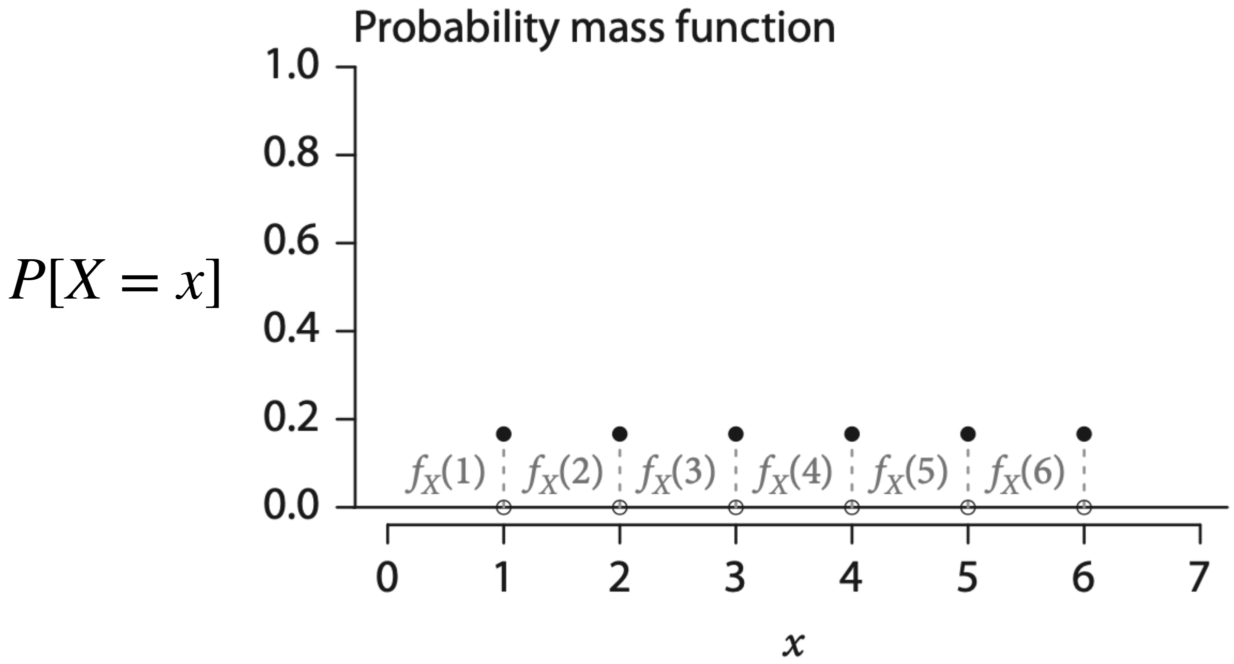
The Uniform distribution (discrete)

- Every integer in a given range is **equally likely**
- Example: throwing a fair dice
- Two parameters: a and b, with $b \geq a$
- For example, in the case of a dice, a=1 and b=6, so the sample set is 1, 2, 3, 4, 5 and 6.

The Uniform distribution (discrete)

- Two parameters: a and b, with $b \geq a$
- $E[X] = \dfrac{a+b}{2}$
- $P[X = k] = \dfrac{1}{n}$, where $n = b - a + 1$

The Uniform distribution (discrete)



Discrete distributions re-cap

- Bernoulli (p)
- Binomial (n, p)
- Geometric (p)
- Poisson (λ)
- Uniform [a,b]

Discrete distributions re-cap

- Bernoulli (p) $\longrightarrow$ "What is the probability I get heads?"
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Discrete distributions re-cap

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- Binomial (n, p) → "What is the probability I get 5 heads in 20 tosses?"
- Geometric (p) → "What is the probability I get 8 tails before 1 head?"
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- Uniform [a,b] → “What is the probability I get a “3” in a dice toss?”

Exercises in R

- Playing with discrete distributions