# EE16B — Midterm 2 Review

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# Disclaimer

This is an unofficial review session and HKN is not affiliated with this course. All of the topics we're reviewing will reflect the material you have covered, our experiences in EE16B, and past exams. We make no promise that what we cover will necessarily reflect the content of this midterm. While some course staff members may be among the presenters, this review session is still not official.

# Eigenvalue Placement

# Why?

- Recall that we are always interested in determining if a given system is BIBO (bounded input bounded output) stable.
- More precisely, if we have a system described by $\vec{x}(t+1) = A\vec{x}(t) + Bu(t) + \vec{\omega}(t)$ we would like the eigenvalues of $A \in \mathbb{R}^{n \times n}$, to satisfy the following property : $|\lambda_i| < 1$.
- So what if we have a $\lambda$ that does not satisfy this property?
- This is where eigenvalue placement comes into play!
- Assuming the system is controllable, we will use closed loop controls to change the eigenvalues such that they satisfy this property.

# How?

- Assume e.g. a DT system. Input: $u[t]$ If the system is controllable then we can use feedback, which means that we can let the input depend on the output, $\vec{x}[t]$.
- We would like to change the matrix multiplying $\vec{x}[t]$ such that $|\lambda_i| < 1$, so let's see what happens when we let $u[t] = K\vec{x}[t]$, where $K \in \mathbb{R}^{1\times n}$.
- Using this input we have:

$$\begin{aligned}\vec{x}[t+1] &= A\vec{x}[t] + Bu[t] + \vec{\omega}[t] \\ &= A\vec{x}[t] + BK\vec{x}[t] + \vec{\omega}[t] \\ &= (A + BK)\vec{x}[t] + \vec{\omega}[t]\end{aligned}$$

- Strategically choosing K allows us to have specific $\lambda$'s for A + BK (Good!).
- This process is called coefficient matching.

# Example

- Suppose we are given a controllable system defined by:

$$\vec{x}[t+1] = \begin{bmatrix} 2 & -1 \\ 0 & 1 \end{bmatrix} \vec{x}[t] + \begin{bmatrix} 2 \\ -1 \end{bmatrix} u[t]$$

- Is the system stable? No! $\lambda = 2, 1$
- What if we let

$$u[t] = \begin{bmatrix} f_1 & f_2 \end{bmatrix} \vec{x}[t]$$

Then we have:

$$\vec{x}(t+1) = \begin{bmatrix} -2 & -1 \\ 0 & 1 \end{bmatrix} \vec{x}[t] + \begin{bmatrix} 2 \\ -1 \end{bmatrix} \begin{bmatrix} f_1 & f_2 \end{bmatrix} \vec{x}[t]$$

- Solve for the values of $f_1$ and $f_2$ such that $\lambda_1 = -0.25$ and $\lambda_2 = 0$
- The answer is $f_1 = -1.50$ and $f_2 = 0.25$
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## Controllable Canonical Form

- Controllable Canonical Form (CCF) for any controllable system is a special form that allows us to simplify the process of eigenvalue placement. It takes on the following form:

$$A^* = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \dots & \dots & \dots & 0 \\ 0 & \dots & \dots & \dots & 1 \\ \alpha_0 & \alpha_1 & \dots & \dots & \alpha_{n-1} \end{bmatrix} \qquad B^* = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$

- The characteristic polynomial of $A^*$ is $\lambda_n - \sum_{i=0}^{n-1} \alpha_i \lambda^i = 0$.
- So how does it help with eigenvalue placement? The last row of this matrix determines the eigenvalues of $A^*$ so modifying the last row will allow us to (easily) modify the eigenvalues.

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# How to convert to CCF

- Let $A, B$ be the matrices in standard form and let $A^*, B^*$ be the matrices in CCF.
- Recall the matrix we used to check controllability?

$$C = \begin{bmatrix} B & AB & \dots & A^{n-1}B \end{bmatrix}$$

$$C^* = \begin{bmatrix} B^* & A^*B^* & \dots & A^{*n-1}B^* \end{bmatrix}$$

- We then have $T = C^*C^{-1}$, which means $A^* = TAT^{-1}$ and $B^* = TB$.
- Remember, all controllable matrices with single input can be transformed into CCF!

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# Example

Consider the following discrete time system:

$$\vec{x}[t+1] = \begin{bmatrix} -2 & 2 & 0 \\ 2 & -4 & 2 \\ 0 & 1 & -1 \end{bmatrix} \vec{x}[t] + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} u[t]$$

(a) Is the system stable? Is it controllable?

(b) Using an appropriate transformation ($\vec{z}[t] = T\vec{x}[t]$), bring the system to controllable canonical form.

(c) Using the state feedback $u[t] =$

$$\begin{bmatrix} f_1 & f_2 & f_3 \end{bmatrix}$$

$\vec{z}[t]$ bring the eigenvalues of the system to $0, 0.75, -0.25$.

# Solutions to Example

(a) The characteristic polynomial is: $\lambda^3 + 7\lambda^2 + 8\lambda = \lambda(\lambda^2 + 7\lambda + 8) = 0$, therefore the eigenvalues of A are $\{0, -5.56, -1.44\}$. As we can see there are $|\lambda_i| > 1$ therefore the system is not stable.
The controllability matrix $C =$

$$\begin{bmatrix} 1 & -2 & 8 \\ 0 & 2 & -12 \\ 0 & 0 & 2 \end{bmatrix}$$

$C$ has full rank so the system is controllable

(b) As we previously mentioned the coefficients of the characteristic polynomial are closely related to the last row of the $A^*$ matrix. Therefore, the CCF of the system is:

$$\vec{z}[t+1] = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -8 & -7 \end{bmatrix} \vec{x}[t] + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u[t]$$

## Example Solutions Continued

(c) Our system then becomes:

$$\vec{z}[t+1] = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ f_1 & f_2 - 8 & f_3 - 7 \end{bmatrix} \vec{x}[t]$$

Which means its characteristic polynomial is :
$\lambda^3 - (f_3 - 7)\lambda^2 - (f_2 - 8)\lambda - f_1 = 0$.
Now, we know the characteristic polynomial should be
$\lambda(\lambda - \frac{3}{4})(\lambda + \frac{1}{4})$, so we can equate the two and solve for the feedback vector $\vec{f}^{\,T} = \begin{bmatrix} 0 & \frac{1}{2} & \frac{3}{16} \end{bmatrix}$.

## Linearization

- Recall that if we have $\frac{dx}{dt} = \lambda x(t) + bu(t)$ we know a solution to this is:

$$x(t) = x(0)e^{\lambda t} + \int_0^t e^{\lambda(t-\tau)}u(\tau)\ d\tau$$

- What if we had $\frac{dx}{dt} = f(x(t)) + bu(t)$, where $f$ is nonlinear (e.g $sin$)?
- Big Picture: linearize $f$ around an operating point and then treat it as a linear function in a small neighborhood of that point.
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# Linearizing a Single-Variable Function

- Suppose we have $f(x)$ that is a non linear function. We can use a first order Taylor polynomial to linearize the function, this is equivalent to finding the slope of the tangent line of $f(x)$ at a particular point.
- From calculus: $f(x) \approx f(x^*) + f'(x^*)(x - x^*)$.
- As long as we are within some (very small) $\delta$ neighborhood of $x^*$ the linearization is valid.
- Example: Linearize $f(x) = 3e^{x^2+2}$ around $x^*$
- Solution:
  $f(x^*) = 3e^{x^{*2}+2}$
  $f'(x) = 3e^{x^2+2}(2x) = 6xe^{x^2+2}$
  $f'(x^*) = 6x^*e^{x^{*2}+2}$
  Therefore : $f(x) \approx 3e^{x^{*2}+2} + 6x^*e^{x^{*2}+2}(x - x^*)$

## Linearizing Steps for $\frac{dx(t)}{dt} = f(x(t)) + bu(t)$

(i) Choose, or you may be given, a DC input point. That is, a point $u^* \equiv u(t)$ that is constant with time.

(ii) Find a DC operating point, $x^* \equiv x(t)$. That is, solve $\frac{dx^*}{dt} = f(x^*) + bu^*$. Notice that this boils down to finding an $x^*$ such that $f(x^*) + bu^* = 0$.

(iii) Define $x_l(t) = x(t) - x^*$ and $u_l(t) = u(t) - u^*$, and re-write the ODE in terms of $x_l(t)$ and $u_l(t)$. By plugging in you get: $\frac{dx_l(t)}{dt} = f(x_l(t) + x^*) + b(u_l(t) + u^*)$

(iv) It is ok to assume at this point that $u_l(t)$ is small, that means that the $u(t)$ in step 1 does not deviate too much from $u^*$.

(v) Linearize the ODE: $f(x_l(t) + x^*) \approx f(x^*) + f'(x^*)x_l(t)$. Here we assume that $x_l(t)$ is also small. This is something that we will need to verify in the next step!

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# Linearizing Steps for $\frac{dx(t)}{dt} = f(x(t)) + bu(t)$ (Continued)

(vi) Plug (vi) back into (iii) and we obtain :
$\frac{dx_l(t)}{dt} \approx f'(x^*)f(x_l(t)) + \cancel{f(x^*)} + bu_l(t) + \cancel{bu^*} =$
$f'(x^*)f(x_l(t)) + bu_l(t)$

(vii) Verify the linearization!
How do we know if the linearization is valid? Well, if we have $\frac{dx_l(t)}{dt} = \lambda x_l(t) + bu(t)$ we know the solution doesn't blow up if $\lambda < 0$ as we will have a term $e^{\lambda t}$.
This means that we want $m = f'(x^*) < 0$.

So what do we do if $m > 0$?
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# Practice Problem

Linearize $\frac{dx(t)}{dt} = \cos(x(t)) + bu(t)$ about $u^* = 0$.
*Hint:* $\cos(x^*) = 0$ has multiple solutions, which means that we can find numerous DC operating points, can you guess which one would result in a stable system ?

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# Practice Problem Solution

(i) We were given the DC input, $u^* = 0$

(ii) $\cos(x^*) = 0$, which means that $x^* = k\frac{\pi}{2}$ for $k \in \{... - 2, -1, 1, 2, ...\}$. We will choose $x^* = \frac{\pi}{2}$

(iii) Let $x_l(t) = x(t) - \frac{\pi}{2}$ and $u_l(t) = u(t) - 0$. By plugging in we get: $\frac{dx_l(t)}{dt} = \cos(x_l(t) + \frac{\pi}{2}) + u_l(t)$

(iv) We assume that $u_l(t)$ is small.

(v) Linearize the ODE: $\cos(x_l(t) + \frac{\pi}{2}) \approx \cos(\frac{\pi}{2}) - \sin(\frac{\pi}{2})x_l(t)$.

(vi) Plug (v) back into ODE: $\frac{dx_l(t)}{dt} = -x_l(t) + u_l(t)$

We see that our assumption that $x_l(t)$ is small is indeed satisfied as we will have a $e^{-t}$ term in the solution which means that $x_l(t)$ will decay.

What if we had chosen a different DC Operating point, say $-\frac{\pi}{2}$? When we linearize the system we see that the solution will "explode" around that particular DC operating point.

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What if we had $\frac{d\vec{x}}{dt} = \vec{f}(\vec{x}, \vec{u})$ ? We will see that the process is very similar to the scalar case we just analyzed!

First, let's see how to linearize $\vec{f}(\vec{x})$ around a DC operating point $\vec{x}^*$. Where $\vec{f} \in \mathbb{R}^{n \times 1}$ is a vector of scalar functions.

The idea is to linearize individually each one of the $f_i$ around the DC operating point.

For example:

$f_1 \approx f_1(\vec{x}^*) + \frac{\partial f_1}{\partial x_1}(\vec{x}^*)(x_1 - x_1^*) + ... + \frac{\partial f_n}{\partial x_1}(\vec{x}^*)(x_n - x_n^*)$

Repeating this for all $n$ functions in $\vec{f}$ we see we get a system of scalar, linearized, multivariate functions which makes you think, wouldn't it be nice to express it in a shorthand matrix notation?

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# Jacobian Matrix

We can use the Jacobian to express everything nicely and neatly. The Jacobian is the name given to the matrix of partial derivatives of $\vec{f}$, and it is denoted by $J_{\vec{x}}$ or $\nabla_{\vec{x}}\vec{f}$.

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# Linearization with Jacobians Example

Linearize $\vec{f}(\vec{x}(t)) = \begin{bmatrix} \sin(x_1(t) * x_2(t)) + 2x_1(t)x_3^2(t) \\ x_3(t)\cos(x_2(t)) + \frac{x_1(t)}{x_3(t)} \\ x_1(t) + 2x_3(t)x_2^3(t) \end{bmatrix}$ about $\vec{x}^* = \begin{bmatrix} 0 \\ 2\pi \\ \frac{2\pi}{3} \end{bmatrix}$

## Solutions

Find the Jacobian:

$$\begin{bmatrix} x_2(t)\cos(x_1(t)*x_2(t)) + 2x_3^2(t) & x_1(t)\cos(x_1(t)*x_2(t)) & 4x_1(t)x_3(t) \\ \frac{1}{x_3(t)} & -x_3(t)\sin(x_2(t)) & \cos(x_2(t)) - \frac{x_1(t)}{x_3^2(t)} \\ 1 & 6x_3(t)x_2^2(t) & 2x_2^3(t) \end{bmatrix}$$

Evaluate the Jacobian about $\vec{x}^*$:

$$\begin{bmatrix} 5\pi & 0 & 0 \\ \frac{2\pi}{3} & 0 & 1 \\ 1 & 36\pi^3 & 16\pi^3 \end{bmatrix}$$

Linearize:

$$\vec{f}(\vec{x}(t)) \approx \begin{bmatrix} 0 \\ \frac{3\pi}{4} \\ 24\pi^4 \end{bmatrix} + \begin{bmatrix} 5\pi & 0 & 0 \\ \frac{2\pi}{3} & 0 & 1 \\ 1 & 36\pi^3 & 16\pi^3 \end{bmatrix} \begin{bmatrix} x_1(t) - 0 \\ x_2(t) - \frac{3\pi}{4} \\ x_3(t) - 24\pi^4 \end{bmatrix}$$

# Steps to Linearize Vector ODE Systems

To linearize $\frac{d\vec{x}}{dt} = \vec{f}(\vec{x}(t), \vec{u}(t))$ we use a similar procedure as we did for the scalar case.

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# Linearizing Vector ODE Systems Example

Given a DC input $u^* = 1$, linearize:

$$\frac{d\vec{x}(t)}{dt} = \begin{bmatrix} x_1^2(t) - x_2(t)u(t) \\ x_2^2(t)(1 + x_1(t)) + \sin(\pi x_1(t)u(t)) \end{bmatrix}$$

# Solutions

Again, we will do this in steps:

(i) We are given $u^* = 1$

(ii) We need to find a DC operating point, this means solving the following system of equations:

$$x_1^{*2} - x_2^* u^* = 0 \tag{1}$$

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The solution is $x_1^* = -1$ and $x_2^* = 1$.

(iii) Let $\vec{x}_l(t) = \vec{x}(t) - \vec{x}^*$ and $\vec{u}_l(t) = \vec{u}(t) - \vec{u}^*$

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# Solutions Continued

(v) Substitute linear approximation back into the system,

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