Mixed-dimensional models of linear elasticity and poroelasticity

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1 Introduction

In this report we describe a dimension reduction for the Biot system of poroelasticity in a domain containing a fracture, which is assumed to be a thin manifold around a geometrical entity of codimension 1. The obtained equations (eq. (35), (45)) are expressed in terms of mean pressure and mean displacement in the fracture. The resulting problem consists of equations of flow and mechanics in the fracture and in the surrounding domain, accompanied by appropriate interface conditions (eq. (36), (46)). Finally we analyse the well-posedness of the steady-state mechanical part of the problem.

The original idea of dimension reduction comes from [1], where it was applied to the darcian flow model. Here we deal with the linear elasticity system, for which the approach has to be generalized using tangential and normal calculus.

The structure of the paper is as follows. In section 2 we describe the original mathematical model, the full and reduced geometry. The tangential and normal calculus is presented in section 3. In sections 4 and 5 we derive the reduced problem for mechanics and flow, respectively. Finally, in section 6 we prove the existence and uniqueness of weak solution to the mechanical part of the resulting problem.

2 Biot poroelasticity in domain with fracture

We consider a bounded, simply connected domain $\Omega \subset \mathbf{R}^d$, $d \in \{2,3\}$ with Lipschitz boundary, which contains a thin layer

$$\Omega_f := \Omega \cap \left((-\tfrac{\delta}{2}, \tfrac{\delta}{2}) \times \mathbf{R}^{d-1}\right) \tag{1}$$

called fracture, with aperture $\delta > 0$ (see Figure 2).

The surrounding domain $\Omega_m := \Omega \setminus \overline{\Omega}_f$, so-called matrix, is divided into two parts, which are interacting with Ω_f via two interfaces

$$\gamma^+ := \Omega \cap \left(\{\tfrac{\delta}{2}\} \times \mathbf{R}^{d-1}\right), \quad \gamma^- := \Omega \cap \left(\{-\tfrac{\delta}{2}\} \times \mathbf{R}^{d-1}\right). \tag{2}$$

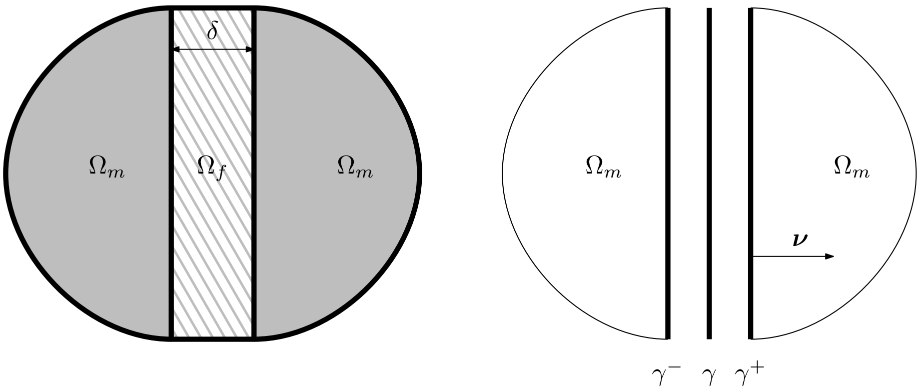

Figure 1: The domain of the full model (left) and the reduced geometry (right).

Further, we introduce the reduced fracture

$$\gamma := \Omega \cap \left(\{0\} \times \mathbf{R}^{d-1}\right) \tag{3}$$

lying in the center of Ω_f. By $\boldsymbol{\nu}$ we denote the unit normal vector to γ in the direction from γ^- to γ^+. The symbol $\partial\Omega$ stands for the boundary of Ω, while $\partial\gamma$ shall denote the relative boundary of γ.

We shall derive a model of poroelasticity on the reduced geometry consisting of Ω_m and γ. The starting point is the Biot system:

$$-\operatorname{div} \sigma + \nabla(\alpha p) = \boldsymbol{f} \qquad \text{in } \Omega_m \cup \Omega_f, \tag{4a}$$

$$\partial_t \left(Sp + \operatorname{div}(\alpha \boldsymbol{u})\right) + \operatorname{div} \boldsymbol{q} = g \qquad \text{in } \Omega_m \cup \Omega_f, \tag{4b}$$

where σ is the stress tensor given by the Hooke law

$$\sigma = \mathbb{C}\nabla \boldsymbol{u}, \tag{5}$$

$\mathbb{C}$ is the 4^{th}-order elasticity tensor, $\boldsymbol{u}$ the displacement, α the Biot effective stress, p the pressure, $\boldsymbol{f}$ the body force, S the storativity, g the fluid source and $\boldsymbol{q}$ the flux given by the Darcy law:

$$\boldsymbol{q} = -\mathbb{K}\nabla p$$

via the hydraulic conductivity tensor $\mathbb{K}$. On the interface between Ω_m and Ω_f we require that

$$p, \boldsymbol{u}, \boldsymbol{q}\cdot\boldsymbol{\nu}, \sigma\boldsymbol{\nu} \text{ are continuous on } \gamma^{\pm}. \tag{6}$$

In what follows, we shall assume that the physical parameters $\alpha, S, \mathbb{C}, \mathbb{K}$ are constant in Ω_m, Ω_f, respectively. To distinguish values on the interfaces $\gamma^{\pm}$ we shall use the subscripts "m" and "f", i.e. $\alpha_m := \alpha_{|\Omega_m}$, $\alpha_f := \alpha_{|\Omega_f}$ etc. In addition, it is assumed that $\mathbb{C}_*$ and $\mathbb{K}_*$, $* \in \{m, f\}$, have the usual symmetries:

$$\forall i,j,k,l = 1,\dots,d: \ [\mathbb{C}_*]_{ijkl} = [\mathbb{C}_*]_{jikl} = [\mathbb{C}_*]_{ijlk} = [\mathbb{C}_*]_{klij}, \tag{7}$$

$$\mathbb{K}_* = \mathbb{K}_*^\top, \tag{8}$$

and are positive definite:

$$\forall \mathbb{A} \in \mathbf{R}^{d\times d}_{sym}: \; \mathbb{C}_*[\mathbb{A}] : \mathbb{A} \geq C_1 |\mathbb{A}|^2, \tag{9}$$

$$\forall \boldsymbol{v} \in \mathbf{R}^d: \; \mathbb{K}_* \boldsymbol{v} \cdot \boldsymbol{v} \geq C_2 |\boldsymbol{v}|^2, \quad * \in \{m, f\}. \tag{10}$$

Here $|\mathbb{A}|$ denotes the Frobenius norm, i.e. $|\mathbb{A}|^2 = \mathbb{A} : \mathbb{A}$. A further natural requirement is that $\boldsymbol{\nu}$ is an eigenvector of $\mathbb{K}_f$, i.e.

$$\mathbb{K}_f \boldsymbol{\nu} = k_f \boldsymbol{\nu}, \tag{11}$$

where $k_f > 0$ is the hydraulic conductivity in orthogonal direction to γ.

3 Tangential and normal calculus in fracture

Let $\mathbb{P} := \boldsymbol{\nu} \otimes \boldsymbol{\nu}$ be the orthogonal projector to γ. For any $\boldsymbol{v} \in \mathbf{R}^d$ and $\mathbb{A} \in \mathbf{R}^{d\times d}$ we shall define the orthogonal decomposition into normal and tangential direction to γ:

$$\boldsymbol{v} = \mathbb{P}\boldsymbol{v} + (\mathbb{I} - \mathbb{P})\boldsymbol{v} =: \boldsymbol{v}_\nu + \boldsymbol{v}_\tau, \tag{12}$$

$$\mathbb{A} = \mathbb{A}\mathbb{P} + \mathbb{A}(\mathbb{I} - \mathbb{P}) =: \mathbb{A}_\nu + \mathbb{A}_\tau. \tag{13}$$

Likewise we decompose differential operators acting on vector- and tensor-valued functions:

$$\nabla f = (\nabla f)_\nu + (\nabla f)_\tau =: \nabla_\nu f + \nabla_\tau f, \tag{14}$$

$$\nabla \boldsymbol{v} = (\nabla \boldsymbol{v})_\nu + (\nabla \boldsymbol{v})_\tau =: \nabla_\nu \boldsymbol{v} + \nabla_\tau \boldsymbol{v}, \tag{15}$$

$$\operatorname{div} \boldsymbol{v} = \operatorname{div} \boldsymbol{v}_\nu + \operatorname{div} \boldsymbol{v}_\tau =: \operatorname{div}_\nu \boldsymbol{v} + \operatorname{div}_\tau \boldsymbol{v}, \tag{16}$$

$$\operatorname{div} \mathbb{A} = \operatorname{div}(\mathbb{P}\mathbb{A}) + \operatorname{div}((\mathbb{I} - \mathbb{P})\mathbb{A}) =: \operatorname{div}_\nu \mathbb{A} + \operatorname{div}_\tau \mathbb{A}. \tag{17}$$

Let w be a function defined in Ω_f. We denote its trace on $\gamma^\pm$ as follows:

$$w^{\oplus}(\boldsymbol{x}) := w(\boldsymbol{x} \pm \tfrac{\delta}{2}\boldsymbol{\nu}), \; \boldsymbol{x} \in \gamma, \tag{18}$$

and introduce the average and the jump between the traces:

$$\{\{w\}\} := \frac{w^\oplus + w^\ominus}{2}, \quad [\![w]\!] := w^\oplus - w^\ominus. \tag{19}$$

When integrating across the fracture, we shall write

$$\int w \, \mathrm{d}\boldsymbol{\nu} := \int w(\cdot + s\boldsymbol{\nu}) \, \mathrm{d}s.$$

The symbol $\overline{w}$ will denote the integral mean of w across the fracture aperture, i.e.

$$\overline{w} := \frac{1}{\delta} \int_{-\frac{\delta}{2}}^{\frac{\delta}{2}} w \, \mathrm{d}\boldsymbol{\nu}. \tag{20}$$

Then it holds:

$$\int_{-\frac{\delta}{2}}^{\frac{\delta}{2}} \nabla_\nu f \, \mathrm{d}\boldsymbol{\nu} = \int_{-\frac{\delta}{2}}^{\frac{\delta}{2}} \boldsymbol{\nu} \frac{\mathrm{d}}{\mathrm{d}s} \left(f(\cdot + s\boldsymbol{\nu}) \right) \, \mathrm{d}s = [\![f]\!]\boldsymbol{\nu}, \tag{21}$$

$$\int_{-\frac{\delta}{2}}^{\frac{\delta}{2}} \mathrm{div}_\nu \, \boldsymbol{v} \, \mathrm{d}\boldsymbol{\nu} = \int_{-\frac{\delta}{2}}^{\frac{\delta}{2}} \boldsymbol{\nu} \cdot \frac{\mathrm{d}}{\mathrm{d}s} \left(\boldsymbol{v}(\cdot + s\boldsymbol{\nu})\right) \, \mathrm{d}s = [\![\boldsymbol{v}]\!] \cdot \boldsymbol{\nu}, \tag{22}$$

$$\int_{-\frac{\delta}{2}}^{\frac{\delta}{2}} \nabla_\nu \boldsymbol{v} \, \mathrm{d}\boldsymbol{\nu} = [\![\boldsymbol{v}]\!] \otimes \boldsymbol{\nu}, \tag{23}$$

$$\int_{-\frac{\delta}{2}}^{\frac{\delta}{2}} \mathrm{div}_\nu \, \mathbb{A} \, \mathrm{d}\boldsymbol{\nu} = [\![\mathbb{A}^\top]\!] \boldsymbol{\nu}. \tag{24}$$

We also have the Green formula:

$$\int_\gamma (\mathrm{div}_\tau \, \boldsymbol{v}) w = \int_{\partial\gamma} (\boldsymbol{v} \cdot \boldsymbol{n}) w - \int_\gamma \boldsymbol{v} \cdot \nabla_\tau w, \tag{25}$$

where $\boldsymbol{n}$ denotes the unit outward normal to $\partial\gamma$.

4 Continuum-fracture model for elasticity

Our aim is to obtain an equation for the averages $(\overline{\boldsymbol{u}}, \overline{p})$ in γ. To do so, we integrate (4a) over the fracture aperture:

$$-\int_{-\frac{\delta}{2}}^{\frac{\delta}{2}} \mathrm{div} \, \mathbb{\sigma} \, \mathrm{d}\boldsymbol{\nu} + \int_{-\frac{\delta}{2}}^{\frac{\delta}{2}} \nabla(\alpha p) \, \mathrm{d}\boldsymbol{\nu} = \delta \overline{\boldsymbol{f}}. \tag{26}$$

Next, we decompose the integrands on the left hand side of (26) into tangential and normal parts and use the formulae (21), (24). We get:

$$\int_{-\frac{\delta}{2}}^{\frac{\delta}{2}} \mathrm{div} \, \mathbb{\sigma} \, \mathrm{d}\boldsymbol{\nu} = \int_{-\frac{\delta}{2}}^{\frac{\delta}{2}} \left(\mathrm{div}_\tau \, \mathbb{\sigma} + \mathrm{div}_\nu \, \mathbb{\sigma}\right) \, \mathrm{d}\boldsymbol{\nu} = \delta \, \mathrm{div}_\tau \, \mathbb{C}_f \overline{\nabla \boldsymbol{u}} + [\![\mathbb{\sigma}]\!] \boldsymbol{\nu}, \tag{27}$$

$$\int_{-\frac{\delta}{2}}^{\frac{\delta}{2}} \nabla(\alpha p) \, \mathrm{d}\boldsymbol{\nu} = \int_{-\frac{\delta}{2}}^{\frac{\delta}{2}} \left(\nabla_\tau(\alpha p) + \nabla_\nu(\alpha p)\right) \, \mathrm{d}\boldsymbol{\nu} = \delta \nabla_\tau(\alpha_f \overline{p}) + \alpha_f [\![p]\!] \boldsymbol{\nu}. \tag{28}$$

Using (23) we can express the averaged gradient:

$$\overline{\nabla \boldsymbol{u}} = \nabla_\tau \overline{\boldsymbol{u}} + [\![\boldsymbol{u}]\!] \otimes \tfrac{\boldsymbol{\nu}}{\delta}. \tag{29}$$

Now we turn to approximation of the normal stress $\mathbb{\sigma}\boldsymbol{\nu}$ on $\gamma^\pm$. We define the approximate normal gradient in the positive/negative half of the fracture:

$$\mathbb{G}_\nu^\oplus = \mathbb{G}_\nu^\oplus[\boldsymbol{u}^\oplus, \overline{\boldsymbol{u}}] := \frac{\pm(\boldsymbol{u}^\oplus - \overline{\boldsymbol{u}})}{\frac{\delta}{2}} \otimes \boldsymbol{\nu} = \pm \frac{2}{\delta} (\boldsymbol{u}^\oplus - \overline{\boldsymbol{u}}) \otimes \boldsymbol{\nu}. \tag{30}$$

Assuming that $\nabla^2 \boldsymbol{u}$ is bounded in Ω_f, we have that

$$(\nabla_\nu \boldsymbol{u})^\oplus = \mathbb{G}_\nu^\oplus + O(\delta). \tag{31}$$

For the tangential gradient we shall use the approximation

$$(\nabla_\tau \boldsymbol{u})^\oplus = \nabla_\tau \overline{\boldsymbol{u}} + O(\delta). \tag{32}$$

We can observe that the approximate normal and tangential gradient satisfy

$$\overline{\nabla \boldsymbol{u}} = \nabla_\tau \overline{\boldsymbol{u}} + \{\!\{\mathbb{G}_\nu\}\!\}. \tag{33}$$

Then the approximate normal stress is given by

$$\boldsymbol{T}^\oplus = \boldsymbol{T}^\oplus[\boldsymbol{u}^\oplus, \overline{\boldsymbol{u}}] := \left(\mathbb{C}_f \left(\nabla_\tau \overline{\boldsymbol{u}} + \mathbb{G}_\nu^\oplus\right)\right) \boldsymbol{\nu}, \tag{34}$$

so that $\sigma^\oplus \boldsymbol{\nu} = \boldsymbol{T}^\oplus + O(\delta)$. Collecting (26), (27), (28), (29) and (34) and neglecting $O(\delta)$ we obtain the averaged counterpart of (4a):

$$\delta \left(-\operatorname{div}_\tau \mathbb{C}_f \left(\nabla_\tau \overline{\boldsymbol{u}} + \{\!\{\mathbb{G}_\nu\}\!\}\right) + \nabla_\tau(\alpha_f \overline{p})\right) - [\![\boldsymbol{T}]\!] + \alpha_f [\![p]\!] \boldsymbol{\nu} = \delta \overline{\boldsymbol{f}}. \tag{35}$$

The continuity of $\sigma\boldsymbol{\nu}$ on $\gamma^\pm$ is approximated by the condition

$$(\mathbb{C}_m \nabla \boldsymbol{u}_{|\Omega_m})\boldsymbol{\nu} = \boldsymbol{T}^\oplus \text{ on } \gamma^\pm. \tag{36}$$

5 Continuum-fracture model for flow

Let us integrate (4b) over the fracture aperture. We get:

$$\partial_t \left(\delta S \overline{p} + \int_{-\frac{\delta}{2}}^{\frac{\delta}{2}} \operatorname{div}(\alpha \boldsymbol{u}) \,\mathrm{d}\boldsymbol{\nu}\right) + \int_{-\frac{\delta}{2}}^{\frac{\delta}{2}} \operatorname{div} \boldsymbol{q} \,\mathrm{d}\boldsymbol{\nu} = \delta \overline{g}. \tag{37}$$

The first integral in (37) will be rewritten using (24) and (22):

$$\begin{aligned}\int_{-\frac{\delta}{2}}^{\frac{\delta}{2}} \operatorname{div}(\alpha \boldsymbol{u}) \,\mathrm{d}\boldsymbol{\nu} &= \int_{-\frac{\delta}{2}}^{\frac{\delta}{2}} \left(\operatorname{div}_\tau(\alpha_f \boldsymbol{u}_f) + \operatorname{div}_\nu(\alpha \boldsymbol{u})\right) \,\mathrm{d}\boldsymbol{\nu} \\ &= \delta \operatorname{div}_\tau(\alpha_f \overline{\boldsymbol{u}}) + \alpha_f [\![\boldsymbol{u}]\!] \cdot \boldsymbol{\nu}. \end{aligned} \tag{38}$$

Similarly we get

$$\int_{-\frac{\delta}{2}}^{\frac{\delta}{2}} \operatorname{div} \boldsymbol{q} = \int_{-\frac{\delta}{2}}^{\frac{\delta}{2}} (\operatorname{div}_\tau \boldsymbol{q} + \operatorname{div}_\nu \boldsymbol{q}) = \delta \operatorname{div}_\tau(\mathbb{K}_f \overline{\nabla p}) + [\![\boldsymbol{q}]\!] \cdot \boldsymbol{\nu}, \tag{39}$$

where

$$\overline{\nabla p} = \nabla_\tau \overline{p} + \tfrac{1}{\delta} [\![p]\!] \boldsymbol{\nu}. \tag{40}$$

From the definition of div_τ and (11) it then follows that

$$\operatorname{div}_\tau(\mathbb{K}_f \overline{\nabla p}) = \operatorname{div}_\tau(\mathbb{K}_f \nabla_\tau \overline{p}). \tag{41}$$

For the approximation of the normal flux $\boldsymbol{q} \cdot \boldsymbol{\nu}$ on $\gamma^\pm$ we use the approximation

$$(\nabla_\nu p)^\oplus \approx \frac{\pm(p^\oplus - \overline{p})}{\frac{\delta}{2}} \boldsymbol{\nu} + O(\delta), \tag{42}$$

provided that $\nabla^2 p$ is bounded in Ω_f. Then the approximate normal flux is given by

$$F^\oplus = F^\oplus[p^\oplus, \overline{p}] := \pm k_f \frac{2}{\delta}(p^\oplus - \overline{p}), \tag{43}$$

so that

$$-\boldsymbol{q}^{\oplus}\cdot\boldsymbol{\nu} = \mathbb{K}_f\nabla p^{\oplus}\cdot\boldsymbol{\nu} = k_f(\nabla_\nu p)^{\oplus}\cdot\boldsymbol{\nu} = F^{\oplus} + O(\delta). \tag{44}$$

Now from (37), (38), (39), (41), (43) and (44) we obtain:

$$\delta\partial_t\left(S_f\overline{p} + \operatorname{div}_\tau(\alpha_f\overline{\boldsymbol{u}})\right) - \delta\operatorname{div}_\tau(\mathbb{K}_f\nabla_\tau\overline{p}) + \alpha_f[\![\partial_t\boldsymbol{u}]\!]\cdot\boldsymbol{\nu} - [\![F]\!] = \delta\overline{g}. \tag{45}$$

Continuity of flux through $\gamma^{\pm}$ is approximated by the condition

$$\mathbb{K}_m\nabla p_{|\Omega_m}\cdot\boldsymbol{\nu} = F^{\oplus} \text{ on } \gamma^{\pm}. \tag{46}$$

6 Well-posedness of mixed-dimensional linear elasticity

In this section we consider only the mechanical part of the problem. We want to find the displacement fields $\boldsymbol{u}:\Omega_m\to\mathbf{R}^d$ and $\boldsymbol{U}:\gamma\to\mathbf{R}^d$ satisfying the following boundary-value problem:

$$-\operatorname{div}(\mathbb{C}_m\nabla\boldsymbol{u}) = \boldsymbol{f} \qquad \text{in } \Omega_m, \tag{47a}$$

$$-\delta\operatorname{div}_\tau\left(\mathbb{C}_f\left(\nabla_\tau\boldsymbol{U} + \{\{\mathbb{G}_\nu\}\}\right)\right) - [\![\boldsymbol{T}]\!] = \delta\overline{\boldsymbol{f}} \qquad \text{in } \gamma, \tag{47b}$$

$$(\mathbb{C}_m\nabla\boldsymbol{u})\boldsymbol{\nu} = \boldsymbol{T}^{\oplus} \qquad \text{on } \gamma^{\pm}, \tag{47c}$$

$$\boldsymbol{u} = \boldsymbol{0} \qquad \text{on } \partial\Omega_m, \tag{47d}$$

$$\boldsymbol{U} = \boldsymbol{0} \qquad \text{on } \partial\gamma. \tag{47e}$$

Here and in what follows, $\mathbb{G}_\nu := \mathbb{G}_\nu[\boldsymbol{u}^{\oplus},\boldsymbol{U}]$ and $\boldsymbol{T}^{\oplus} = \boldsymbol{T}^{\oplus}[\boldsymbol{u}^{\oplus},\boldsymbol{U}]$. Let us define the space

$$V := H^1_0(\Omega_m;\mathbf{R}^d)\times H^1_0(\gamma;\mathbf{R}^d) \tag{48}$$

equipped by the norm $\|(\boldsymbol{v},\boldsymbol{V})\|_V := (\|\nabla\boldsymbol{v}\|^2_{L^2(\Omega)} + \|\nabla_\tau\boldsymbol{V}\|^2_{L^2(\gamma)})^{1/2}$ and the forms

$$a((\boldsymbol{u},\boldsymbol{U}),(\boldsymbol{v},\boldsymbol{V})) := \int_{\Omega_m}\mathbb{C}_m\nabla\boldsymbol{u}:\nabla\boldsymbol{v} + \delta\int_\gamma\mathbb{C}_f(\nabla_\tau\boldsymbol{U} + \{\{\mathbb{G}_\nu\}\}):\nabla_\tau\boldsymbol{V} + \int_\gamma\left([\![\boldsymbol{T}\cdot\boldsymbol{v}]\!] - [\![\boldsymbol{T}]\!]\cdot\boldsymbol{V}\right), \tag{49}$$

$$l((\boldsymbol{v},\boldsymbol{V})) := \int_{\Omega_m}\boldsymbol{f}\cdot\boldsymbol{v} + \delta\int_\gamma\overline{\boldsymbol{f}}\cdot\boldsymbol{V}. \tag{50}$$

Then the weak formulation of (47) reads:

$$\left.\begin{array}{l}\text{Find } (\boldsymbol{u},\boldsymbol{U})\in V \text{ such that}\\ \forall(\boldsymbol{v},\boldsymbol{V})\in V:\ a((\boldsymbol{u},\boldsymbol{U}),(\boldsymbol{v},\boldsymbol{V})) = l((\boldsymbol{v},\boldsymbol{V})).\end{array}\right\} \tag{51}$$

Before establishing the well-posedness of (51) we prove a suitable Korn type inequality. For any function $\boldsymbol{V}$ defined in γ we denote its symmetrized tangential gradient by

$$\boldsymbol{\varepsilon}_\tau\boldsymbol{V} := \frac{1}{2}(\nabla_\tau\boldsymbol{V} + (\nabla_\tau\boldsymbol{V})^\top). \tag{52}$$

Then we have the following Korn inequality in $H^1_0(\gamma)$.

Lemma 1. *There exists a constant $C_3 > 0$ such that for all $\boldsymbol{V} \in H_0^1(\gamma; \mathbf{R}^d)$ it holds:*

$$\|\boldsymbol{\varepsilon}_\tau \boldsymbol{V}\|^2_{L^2(\gamma)} \geq C_4 \|\nabla_\tau \boldsymbol{V}\|^2_{L^2(\gamma)}. \tag{53}$$

Proof. Let $\boldsymbol{V} \in H_0^1(\gamma; \mathbf{R}^d)$. Then, by mapping $\boldsymbol{V}$ isometrically from γ to a subset of $\mathbf{R}^{d-1}$, we apply the Korn inequality to the tangential part $\boldsymbol{V}_\tau$, which implies

$$\|\boldsymbol{\varepsilon}_\tau \boldsymbol{V}_\tau\|^2_{L^2(\gamma)} \geq C_K \|\nabla_\tau \boldsymbol{V}_\tau\|^2_{L^2(\gamma)}. \tag{54}$$

From the definition of tangential and normal operators in follows that

$$|\boldsymbol{\varepsilon}_\tau \boldsymbol{V}|^2 = |\boldsymbol{\varepsilon}_\tau \boldsymbol{V}_\tau|^2 + \tfrac{1}{2}|\nabla_\tau \boldsymbol{V}_\nu|^2, \tag{55}$$

so that

$$\begin{aligned}\|\boldsymbol{\varepsilon}_\tau \boldsymbol{V}\|^2_{L^2(\gamma)} &= \|\boldsymbol{\varepsilon}_\tau \boldsymbol{V}_\tau\|^2_{L^2(\gamma)} + \tfrac{1}{2}\|\nabla_\tau \boldsymbol{V}_\nu\|^2_{L^2(\gamma)} \\ &\geq C_K \|\nabla_\tau \boldsymbol{V}_\tau\|^2_{L^2(\gamma)} + \tfrac{1}{2}\|\nabla_\tau \boldsymbol{V}_\nu\|^2_{L^2(\gamma)} \geq \min\{C_K, \tfrac{1}{2}\}\|\nabla_\tau \boldsymbol{V}\|^2_{L^2(\gamma)}.\end{aligned} \tag{56}$$

□

Now we can prove a variant of the Korn inequality which takes into account the approximate normal gradient in γ.

Lemma 2. *There exists a constant $C_4 > 0$ such that for all $(\boldsymbol{v}, \boldsymbol{V}) \in V$ it holds:*

$$\|\boldsymbol{\varepsilon v}\|^2_{L^2(\Omega_m)} + \{\{\|\boldsymbol{\varepsilon}_\tau \boldsymbol{V} + \tfrac{1}{2}(\mathbb{G}_\nu[\boldsymbol{v}, \boldsymbol{V}] + \mathbb{G}_\nu^\top[\boldsymbol{v}, \boldsymbol{V}])\|^2_{L^2(\gamma)}\}\} \geq C_4 \|(\boldsymbol{v}, \boldsymbol{V})\|^2_V. \tag{57}$$

Proof. Let us assume for contradiction that there is a sequence $\{(\boldsymbol{v}_n, \boldsymbol{V}_n)\}_{n=1}^\infty$ in V such that

$$\|\boldsymbol{\varepsilon v}_n\|^2_{L^2(\Omega_m)} + \{\{\|\boldsymbol{\varepsilon}_\tau \boldsymbol{V}_n + \tfrac{1}{2}(\mathbb{G}_\nu[\boldsymbol{v}_n, \boldsymbol{V}_n] + \mathbb{G}_\nu^\top[\boldsymbol{v}_n, \boldsymbol{V}_n])\|^2_{L^2(\gamma)}\}\} < \frac{1}{n}\|(\boldsymbol{v}_n, \boldsymbol{V}_n)\|^2_V. \tag{58}$$

Without loss of generality we may assume that $\|(\boldsymbol{v}_n, \boldsymbol{V}_n)\|_V = 1$, so that

$$(\boldsymbol{v}_n, \boldsymbol{V}_n) \rightharpoonup (\boldsymbol{v}, \boldsymbol{V}) \text{ weakly in } V,\ n \to \infty, \tag{59}$$

where $(\boldsymbol{v}, \boldsymbol{V}) \in V$, passing eventually to a subsequence. From (58) and the Korn inequality in $H_0^1(\Omega_m; \mathbf{R}^d)$ it follows that

$$\boldsymbol{v}_n \to \mathbf{0} \text{ strongly in } H_0^1(\Omega_m; \mathbf{R}^d),\ n \to \infty. \tag{60}$$

From (58) it also follows that

$$\boldsymbol{\varepsilon}_\tau \boldsymbol{V}_n + \tfrac{1}{2}(\mathbb{G}_\nu^\oplus[\boldsymbol{v}_n^\oplus, \boldsymbol{V}_n] + \mathbb{G}_\nu^{\oplus\top}[\boldsymbol{v}_n^\oplus, \boldsymbol{V}_n]) \to 0 \text{ strongly in } L^2(\gamma; \mathbf{R}^{d\times d}),\ n \to \infty. \tag{61}$$

Using this together with (59), (60) and the definition of $\mathbb{G}_\nu^\oplus$ we obtain:

$$\boldsymbol{\varepsilon}_\tau \boldsymbol{V} = \pm\frac{1}{\delta}(\boldsymbol{V} \otimes \boldsymbol{\nu} + \boldsymbol{\nu} \otimes \boldsymbol{V}), \tag{62}$$

from which it follows that $\boldsymbol{V} = \mathbf{0}$. With help of (61) and the compact embeddings $H^1(\Omega_m) \hookrightarrow L^2(\gamma)$ and $H^1(\gamma) \hookrightarrow L^2(\gamma)$ we find that

$$\boldsymbol{\varepsilon}_\tau \boldsymbol{V}_n \to 0 \text{ strongly in } L^2(\gamma; \mathbf{R}^{d\times d}),\ n \to \infty. \tag{63}$$

Lemma 1 then implies:

$$\boldsymbol{V}_n \to \mathbf{0} \text{ strongly in } H_0^1(\gamma;\mathbf{R}^d),\ n \to \infty. \tag{64}$$

Finally, having shown the strong convergence of $\{(\boldsymbol{v}_n, \boldsymbol{V}_n)\}_{n=1}^{\infty}$, we observe that

$$1 = \lim_{n\to\infty} \|(\boldsymbol{v}_n, \boldsymbol{V}_n)\|_V = \|(\boldsymbol{v}, \boldsymbol{V})\|_V = 0, \tag{65}$$

which is a contradiction. Therefore the proof is finished. □

In the following lemma we prove the ellipticity of a.

Lemma 3. *The bilinear form a is elliptic in the following sense: There exists a constant $C_5 > 0$ such that*

$$a((\boldsymbol{u}, \boldsymbol{U}), (\boldsymbol{u}, \boldsymbol{U})) \geq C_5 \|(\boldsymbol{u}, \boldsymbol{U})\|_V^2 \tag{66}$$

for all $(\boldsymbol{u}, \boldsymbol{U}) \in V$.

Proof. For any $(\boldsymbol{u}, \boldsymbol{U}) \in V$ we have:

$$\begin{aligned} a((\boldsymbol{u}, \boldsymbol{U}), (\boldsymbol{u}, \boldsymbol{U})) = \int_{\Omega_m} \mathbb{C}\nabla\boldsymbol{u} : \nabla\boldsymbol{u} + \delta \int_\gamma \{\{\mathbb{C}_f(\nabla_\tau \boldsymbol{U} + \mathbb{G}_\nu) : \nabla_\tau \boldsymbol{U}\}\} \\ + \int_\gamma [\![\boldsymbol{T} \cdot (\boldsymbol{u} - \boldsymbol{U})]\!]. \end{aligned} \tag{67}$$

From the definition of $\boldsymbol{T}^\oplus$ and $\mathbb{G}_\nu^\oplus$ we get:

$$\boldsymbol{T}^\oplus \cdot (\boldsymbol{u}^\oplus - \boldsymbol{U}) = \mathbb{C}_f(\nabla_\tau \boldsymbol{U} + \mathbb{G}_\nu^\oplus) : (\boldsymbol{u}^\oplus - \boldsymbol{U}) \otimes \boldsymbol{\nu} = \pm\frac{\delta}{2}\mathbb{C}_f(\nabla_\tau \boldsymbol{U} + \mathbb{G}_\nu^\oplus) : \mathbb{G}_\nu^\oplus, \tag{68}$$

which together with (67) and (9) yields:

$$\begin{aligned} a((\boldsymbol{u}, \boldsymbol{U}), (\boldsymbol{u}, \boldsymbol{U})) &= \int_{\Omega_m} \mathbb{C}_m\nabla\boldsymbol{u} : \nabla\boldsymbol{u} + \delta \int_\gamma \{\{\mathbb{C}_f(\nabla_\tau \boldsymbol{U} + \mathbb{G}_\nu) : (\nabla_\tau \boldsymbol{U} + \mathbb{G}_\nu)\}\} \\ &\geq C_1 \|\boldsymbol{\varepsilon}\boldsymbol{u}\|_{L^2(\Omega_m)}^2 + \frac{\delta}{2} C_1 \{\{\|\boldsymbol{\varepsilon}_\tau \boldsymbol{U} + \tfrac{1}{2}(\mathbb{G}_\nu + \mathbb{G}_\nu^\top)\|_{L^2(\gamma)}^2\}\}. \end{aligned} \tag{69}$$

Applying (57) we arrive at (66). □

It is not difficult to show that a and l are bounded in V provided that $\boldsymbol{f} \in L^2(\Omega;\mathbf{R}^d)$. Hence we have the following result.

Theorem 4. *Let $\boldsymbol{f} \in L^2(\Omega;\mathbf{R}^d)$. Then problem (51) has a unique solution.*

References

[1] V. Martin, J. Jaffré, and J. E. Roberts. Modeling fractures and barriers as interfaces for flow in porous media. SIAM Journal on Scientific Computing, 26(5):1667, 2005. ISSN 10648275. doi: 10.1137/S1064827503429363.