On the solution of static contact problems with Coulomb friction via the semismooth* Newton method

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Ad (i): Introduction

We want to solve an inclusion

$$0 \in F(x),$$

where $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is a closed-graph multifunction.

Outline

(i) Introduction
(ii) Variational geometry and generalized differentiation
(iii) Semismoothness* of sets and multifunctions
(iv) Semismooth* Newton method for inclusions
(v) The used model
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Introduction: Josephy-Newton method

Consider the *generalized equation (GE)*

$$0 \in F(x) = f(x) + \Psi(x),$$

where $f : \mathbb{R}^n \to \mathbb{R}^n$ is single-valued and smooth and $\Psi : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is a set-valued mapping (multifunction).

Josephy-Newton method: Given $x^{(k)}$, the next iterate $x^{(k+1)}$ solves the auxiliary problem

$$0 \in f(x^{(k)}) + \nabla f(x^{(k)})(x - x^{(k)}) + \Psi(x) \tag{1}$$

(only f is linearized).

Drawback: Problems (1) may be difficult to solve.

Remark

There are various modern developments of this method in which one admits Lipschitzian f and Ψ is approximated via its graphical derivative.

Ad (ii) Variational geometry

Definition

Given a closed set $A \subset \mathbb{R}^n$ and $\bar{x} \in A$, we define

(i) the *tangent (Bouligand, contingent) cone* to A at $\bar{x}$ by

$$T_A(\bar{x}) := \{u \in \mathbb{R}^n | \exists u_k \to u, t_k \searrow 0 : \bar{x} + t_k u_k \in A \forall k\};$$

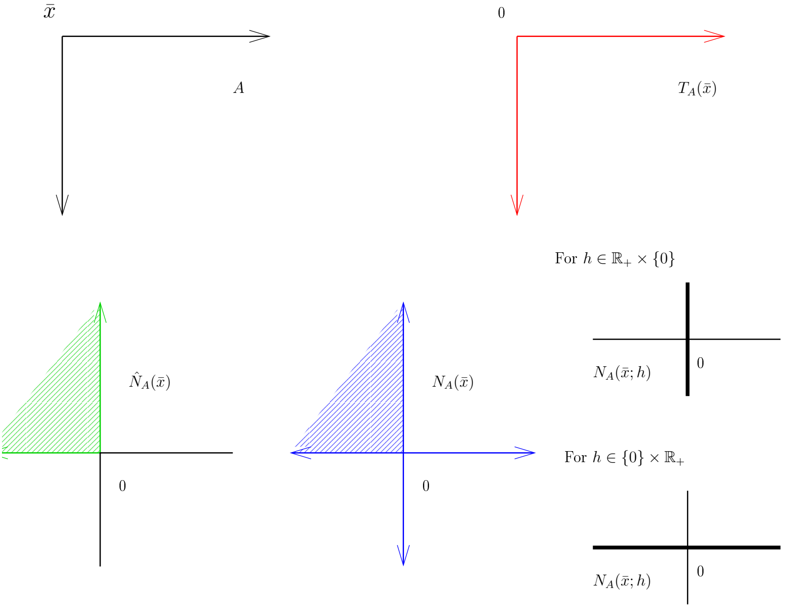
(ii) the *regular (Fréchet) normal cone* to A at $\bar{x}$ by

$$\widehat{N}_A(\bar{x}) := (T_A(\bar{x}))^\circ;$$

(iii) the *limiting (Mordukhovich) normal cone* to A at $\bar{x}$ by

$$N_A(\bar{x}) := \{u^* \in \mathbb{R}^n | \exists x_k \xrightarrow{A} \bar{x}, u_k^* \to u^* : u_k^* \in \widehat{N}_A(x_k) \forall k\}.$$

Example



$\bar{x}$
A
0
$T_A(\bar{x})$
For $h \in \mathbb{R}_+ \times \{0\}$
$\hat{N}_A(\bar{x})$
0
$N_A(\bar{x})$
0
$N_A(\bar{x}; h)$
0
For $h \in \{0\} \times \mathbb{R}_+$
$N_A(\bar{x}; h)$
0

Generalized derivatives

Definition

Consider a multifunction $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ and a point $(\bar{x}, \bar{y}) \in \operatorname{gph} F := \{(x, y) \mid y \in F(x)\}$. Then

(i) the multifunction $DF(\bar{x}, \bar{y}) : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$, defined by

$$DF(\bar{x}, \bar{y})(u) := \{v \in \mathbb{R}^m | (u, v) \in T_{\operatorname{gph} F}(\bar{u}, \bar{v})\}, h \in \mathbb{R}^n,$$

is called the *graphical derivative* of F at $(\bar{x}, \bar{y})$;

(ii) the multifunction $D^* F(\bar{x}, \bar{y}) : \mathbb{R}^m \rightrightarrows \mathbb{R}^n$, defined by

$$D^* F(\bar{x}, \bar{y})(v^*) := \{u^* \in \mathbb{R}^n | (u^*, -v^*) \in N_{\operatorname{gph} F}(\bar{x}, \bar{y})\}, v^* \in \mathbb{R}^m,$$

is called the *limiting (Mordukhovich) coderivative* of F at $(\bar{x}, \bar{y})$.

Definition

(i) A set-valued mapping $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ is called *metrically regular* around a point $(\bar{x}, \bar{y}) \in \operatorname{gph} F$ with modulus κ if there is a constant $\kappa \geq 0$ along with neighborhoods U of $\bar{x}$ and V of $\bar{y}$ such that

$$d(x, F^{-1}(y) \leq \kappa\, d(y, F(x)) \quad \forall (x, y) \in U \times V.$$

(ii) A set-valued mapping $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ is called *strongly metrically regular* around a point $(\bar{x}, \bar{y}) \in \operatorname{gph} F$ with modulus $\kappa \geq 0$ if it is metrically regular around $(\bar{x}, \bar{y})$ and its inverse F^{-1} has a localization around $(\bar{y}, \bar{x})$ that is nowhere multi-valued.

Theorem (Mordukhovich criterion)

F is metrically regular around a point $(\bar{x}, \bar{y})$ iff

$$\operatorname{Ker} D^* F(\bar{x}, \bar{y}) := \{y^* | 0 \in D^* F(\bar{x}, \bar{y})(y^*)\} = \{0\}.$$

Definition

A subset $\Gamma \subset \mathbb{R}^n$ is *subanalytic* if each $a \in \Gamma$ has a neighborhood V such that $\Gamma \cap V$ is a projection of a relatively compact *semianalytic* set.

Recall that every closed semianalytic subset X of $\mathbb{R}^n$ is a finite union of sets having the form

$$\{x \in \mathbb{R}^n | \, f_i(x) \geq 0, i = 1, 2, ..., k\},$$

where the functions f_i are analytic on X.

Ad (iii): Semismoothness* of sets and multifunctions

Definition ([2])

Let $\tilde{x} \in A \subset \mathbb{R}^n$. We say that A is *semismooth** at $\tilde{x}$ provided that for every $\epsilon > 0$ there is some $\delta > 0$ such that the inequality

$$|\langle x^*, x - \tilde{x}\rangle| \leq \epsilon \|x - \tilde{x}\| \, \|x^*\| \tag{2}$$

is valid for all $x \in \mathcal{B}_\delta(\tilde{x})$ and for all $x^* \in N_A(x)$.

Clearly, (2) amounts to the equality $\quad \langle x^*, x - \tilde{x}\rangle = o(\|x - \tilde{x}\| \, \|x^*\|)$.

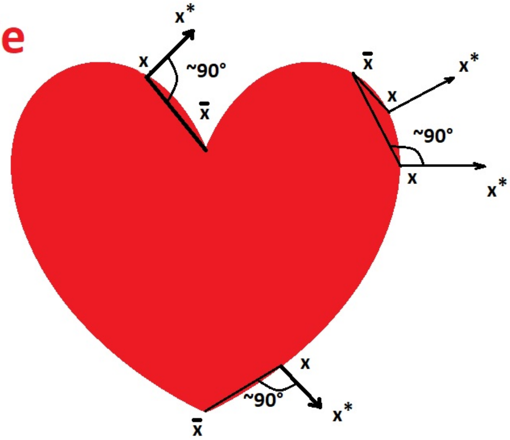
Definition ([2])

Let $(\tilde{x}, \tilde{y}) \in \operatorname{gph} F$. We say that F is *semismooth** at $(\tilde{x}, \tilde{y})$ provided that $\operatorname{gph} F$ is semismooth* at $(\tilde{x}, \tilde{y})$, i.e., for every $\epsilon > 0$ there is some $\delta > 0$ such that the inequality

$$|\langle x^*, x - \tilde{x}\rangle + \langle y^*, y - \tilde{y}\rangle| \leq \epsilon \|(x, y) - (\tilde{x}, \tilde{y})\| \, \|(x^*, y^*)\| \tag{3}$$

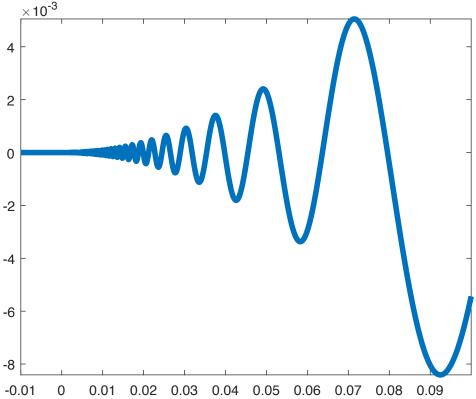
is valid for all $(x, y) \in \mathcal{B}_\delta(\tilde{x}, \tilde{y})$ and for all $(x^*, y^*) \in N_{\operatorname{gph} F}(x, y)$.

We



semismooth* sets

Example



The set $A = \mathrm{gph}\,(x^2 \sin \frac{1}{x})$ is not semismooth* at $(0,0)$ because, e.g., for the sequence $x_k = 1/(2\pi k)$ we have $(1,-1) \in N_A(x_k, 0)$ and therefore $(1,-1) \in N_A((0,0);(1,0))$. But $\langle (1,-1),(1,0)\rangle = 1 \neq 0$ and therefore A is not semismooth* at $(0,0)$.

Criteria for semismoothness*

On the basis of the definition and the above statements we may conclude that:

1) A closed convex set $A \subset \mathbb{R}^s$ is semismooth* at each $\bar{x} \in A$;
2) Given closed sets $A_i \in \mathbb{R}^s$, $i = 1, 2, \ldots, p$, and $\bar{x} \in A := \bigcup_{i=1}^{p} A_i$, then one has the implication

$$\text{the sets } A_i, i \in \{j | \bar{x} \in A_j\} \text{ are semismooth}^* \text{ at } \bar{x} \Rightarrow A \text{ is semismooth}^* \text{ at } \bar{x};$$

Theorem (Jourani 2007)

Let A be a closed subanalytic set and $\bar{x} \in A$. Then A is semismooth at $\bar{x}$.*

Relationship to semismoothness

Definition (Qi,Sun 1993)

A mapping $F:\mathbb{R}^n\to\mathbb{R}^m$ is called *semismooth* at $\bar{x}$, provided it is Lipschitz near $\bar{x}$ and

$$\lim_{\substack{A\in\operatorname{conv}\bar{\nabla}F(\bar{x}+tu')\\ t\searrow 0,\,u'\to u}} Au'$$

exists for all $u\in\mathbb{R}^n$.

Theorem

Let $F:\mathbb{R}^n\to\mathbb{R}^m$ be Lipschitzian near $\bar{x}$.

1. *If F is semismooth at $\bar{x}$, it is semismooth* at $(\bar{x},F(\bar{x}))$ as well;*
2. *Conversely, if F is directionally differentiable at $\bar{x}$ and semismooth* at $(\bar{x},F(\bar{x}))$, it is semismooth.*

- There are Lipschitz continuous functions F which are semismooth* at $(\bar{x},F(\bar{x}))$, but not directionally differentiable
- Semismoothness* is also defined for non-Lipschitzian mappings.

Ad (iv): Semismooth* Newton method for inclusions - basic idea

From now on let $\bar{x}$ be a solution of the inclusion

$$0 \in F(x),$$

where $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ has closed graph and assume that F is semismooth* at $(\bar{x}, 0)$. Consider an iterate $x^{(k)}$ close to $\bar{x}$.

- Since we are dealing with general set-valued mappings F, we can expect neither $F(x^{(k)}) \neq \emptyset$ nor that 0 is close to $F(x^{(k)})$. Hence, in the first step we have to compute an element $(\hat{x}^{(k)}, \hat{y}^{(k)}) \in \operatorname{gph} F$ "close" to $(x^{(k)}, 0)$ (approximation step (AS)).
- By the definition of semismoothness*, for every $(y^*, x^*) \in \operatorname{gph} D^*F(\hat{x}^{(k)}, \hat{y}^{(k)})$ there holds

  $$\langle x^*, \hat{x}^{(k)} - \bar{x} \rangle = \langle y^*, \hat{y}^{(k)} \rangle + o(\|(\hat{x}^{(k)}, \hat{y}^{(k)}) - (\bar{x}, 0)\| \|(x^*, y^*)\|).$$

  We thus choose n pairs $(y_i^*, x_i^*) \in \operatorname{gph} D^*F(\hat{x}^{(k)}, \hat{y}^{(k)})$, $i = 1, \ldots, n$ and determine $x^{(k+1)}$ as solution of the n linear equations

  $$\langle x_i^*, \hat{x}^{(k)} - x \rangle = \langle y_i^*, \hat{y}^{(k)} \rangle, \; i = 1, \ldots, n,$$

  in variable x (Newton step (NS)).

Formalism of the basic idea

- Given $(x, y) \in \mathrm{gph}\, F$, we denote by $\mathcal{A}F(x, y)$ the collection of all pairs of $[n \times n]$ matrices (A, B), such that there are n elements $(y_i^*, x_i^*) \in \mathrm{gph}\, D^* F(x, y)$, $i = 1, \dots, n$, and the i-th row of A and B is ${x_i^*}^T$ and ${y_i^*}^T$, respectively.
- Further we introduce the set

$$\mathcal{A}_{\mathrm{reg}} F(x, y) := \{(A, B) \in \mathcal{A}F(x, y) \mid A \text{ nonsingular } \}.$$

- Then, with some $(A, B) \in \mathcal{A}_{\mathrm{reg}} F(\hat{x}^{(k)}, \hat{y}^{(k)})$, the Newton step can be written as

$$x^{(k+1)} = \hat{x}^{(k)} - A^{-1} B \hat{y}^{(k)}.$$

- When is $\mathcal{A}_{\mathrm{reg}} F(\hat{x}, \hat{y}) \neq \emptyset$?

Theorem

Assume that F is strongly metrically regular around $(\hat{x}, \hat{y}) \in \mathrm{gph}\, F$ with modulus $\kappa > 0$. Then there is an $n \times n$ matrix C with $\|C\| \leq \kappa$ such that $(I, C) \in \mathcal{A}_{\mathrm{reg}} F(\hat{x}, \hat{y}) \neq \emptyset$, where I denotes the identity matrix.

In this way we arrive at the following conceptual algorithm.

Algorithm (1)

1. *Choose a starting point $x^{(0)}$, set the iteration counter $k := 0$.*
2. *If $0 \in F(x^{(k)})$, stop the algorithm.*
3. *Approximation step: Compute $(\hat{x}^{(k)}, \hat{y}^{(k)}) \in \mathrm{gph}\, F$ close to $(x^{(k)}, 0)$ such that $\mathcal{A}_{\mathrm{reg}} F(\hat{x}^{(k)}, \hat{y}^{(k)}) \neq \emptyset$.*
4. *Newton step: Select $(A, B) \in \mathcal{A}_{\mathrm{reg}} F(\hat{x}^{(k)}, \hat{y}^{(k)})$ and compute the new iterate $x^{(k+1)} = \hat{x}^{(k)} - A^{-1} B \hat{y}^{(k)}$.*
5. *Set $k := k + 1$ and go to 2.*

Convergence

Given two reals $L, \kappa > 0$, we assign to each x the set

$$\mathcal{G}^{L,\kappa}_{F,\bar{x}}(x) := \{(\hat{x}, \hat{y}, A, B) \mid \|(\hat{x} - \bar{x}, \hat{y})\| \leq L\|x - \bar{x}\|, (\hat{x}, \hat{y}) \in \operatorname{gph} F,$$

$$(A, B) \in \mathcal{A}_{\mathrm{reg}} F(\hat{x}, \hat{y}), \|A^{-1}\| \|(A \,\vdots\, B)\|_F \leq \kappa\}.$$

Theorem

Assume that F is semismooth at $(\bar{x}, 0) \in \operatorname{gph} F$ and assume that there are $L, \kappa > 0$ such that for every $x \notin F^{-1}(0)$ sufficiently close to $\bar{x}$ we have $\mathcal{G}^{L,\kappa}_{F,\bar{x}}(x) \neq \emptyset$. Then there exists some $\delta > 0$ such that for every starting point $x^{(0)} \in \mathcal{B}_\delta(\bar{x})$ Algorithm 1 either stops after finitely many iterations at $\bar{x}$ or produces a sequence $x^{(k)}$ which converges superlinearly to $\bar{x}$, provided we choose in every iteration $(\hat{x}^{(k)}, \hat{y}^{(k)}, A, B) \in \mathcal{G}^{L,\kappa}_{F,\bar{x}}(x^{(k)})$.*

Ad approximation step

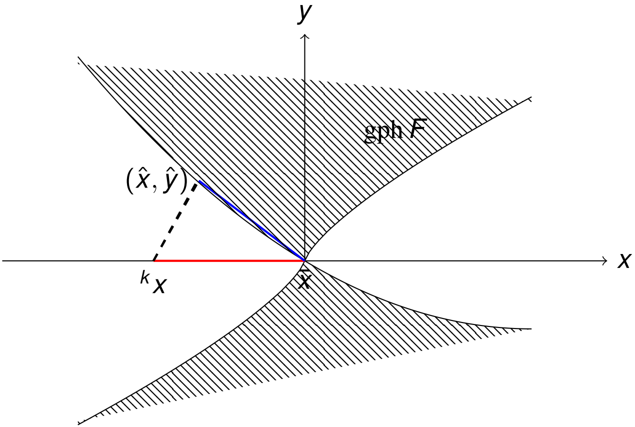


Figure: Approximation step.

Convergence

In applications we may often compute matrices A, B such that

$$\mathcal{R}((A \vdots B)^T) \subset \operatorname{gph} D^*F(\hat{x}, \hat{y}).$$

Define $$\mathcal{A}^{\text{lin}}F(\hat{x}, \hat{y}) := \{(A, B) | \mathcal{R}((A \vdots B)^T) \subset \operatorname{gph} D^*F(\hat{x}, \hat{y})\} \subset \mathcal{A}F(\hat{x}, \hat{y}),$$
$$\mathcal{A}^{\text{lin}}_{\text{reg}}F(\hat{x}, \hat{y}) := \{(A, B) \in \mathcal{A}^{\text{lin}}F(\hat{x}, \hat{y}) | A \text{ nonsingular}\}.$$

Then it may be shown that the main convergence statement remain valid with a modified $\mathcal{G}^{L,\kappa}_{F,\bar{x}}(x)$, where the last two conditions are replaced by

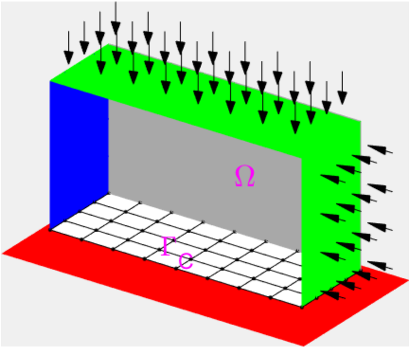
$$(A, B) \in \mathcal{A}^{\text{lin}}_{\text{reg}}F(\hat{x}, \hat{y}), \qquad \text{and } \|A^{-1}B\| \leq \kappa.$$

Theorem

Assume that the mapping F is both semismooth* at $(\bar{x}, 0)$ and strongly metrically regular around $(\bar{x}, 0)$. Then all assumptions of the previous theorem are fulfilled.

Of course, even in case of strong metric regularity one has to check whether our concrete suggested AS and NS fulfill the above posed requirements.

Ad (v): The used model



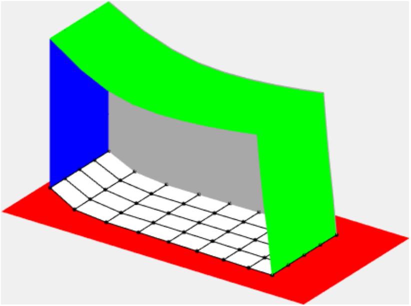


Figure: The left figure depicts an undeformed elastic prism occupying domain Ω with the left (blue) face attached (Dirichlet condition) and some boundary tractions applied to the right and top faces (depicted in green). They press the contact face Γ_C against the (red) rigid plane foundation. Example of the resulting deformed body is depicted in the right figure. Front faces are not visualized.

The fundamental results concerning mechanical problems of this type have been established in [8]. We perform a FEM discretization in such a way that all nodes, not lying in Γ_C, are eliminated (by the Schur complement technique) and with i-th node from Γ_C one associates the pair

$$(u_t^i, u_n^i) \in \mathbb{R}^2 \times \mathbb{R},$$

denoting its *tangential* and *normal displacements*, respectively.
Let p be the number of nodes on Γ_C,

$$u = (u^1, u^2, \dots, u^p) \qquad \text{with } u^i = (u_t^i, u_n^i) \in \mathbb{R}^3$$

and let us simplify the notation via

$$u_{12}^i = (u_1^i, u_2^i) = u_t^i, \qquad u_3^i = u_n^i$$

for all i.

In this way we arrive at the GE

$$0 \in Pu - f + \widetilde{Q}(u), \tag{4}$$

where the $[3p \times 3p]$ matrix P is computed from the original *stiffness* matrix K via elimination of the nodes, not belonging to Γ_C, $f = (f^1, r^2, \ldots, f^p) \in (\mathbb{R}^3)^p$ reflects the action of boundary tractions and the multifunction

$$\widetilde{Q} : \mathbb{R}^{3p} \rightrightarrows \mathbb{R}^{3p}$$

is given by

$$\widetilde{Q}(u) = \bigtimes_{i=1}^{p} Q(u^i) \quad \text{with } Q(u^i) = \left\{ \left[\begin{array}{c} -\phi\lambda\partial\|u^i_{12}\| \\ \lambda \end{array} \right] | \lambda \in N_{\mathbb{R}_+}(u^i_3) \right\}. \tag{5}$$

In (5), $\phi > 0$ is the *friction coefficient*. Note that $Q : \mathbb{R}^3 \rightrightarrows \mathbb{R}^3$ is a composite multifunction, having the structure $Q = S_2 \circ S_1$ where, for some $x \in \mathbb{R}^3$,

$$S_1 : (x_{12}, x_3) \mapsto \begin{bmatrix} x_{12} \\ \lambda \end{bmatrix} \in \begin{bmatrix} x_{12} \\ N_{\mathbb{R}_+}(x_3) \end{bmatrix} \text{ and } S_2 : (x_{12}, \lambda) \mapsto \begin{bmatrix} h_{12} \\ h_3 \end{bmatrix} \in \begin{bmatrix} -\phi\lambda\partial\|x_{12}\| \\ \lambda \end{bmatrix}.$$

Thanks to the separable structure of $\widetilde{Q}$ we observe that for $g = (g^1, g^2, \ldots, g^p) \in \widetilde{Q}(u)$ and $a = (a^1, a^2, \ldots, a^p) \in (\mathbb{R}^3)^p$ one has

$$D^*\widetilde{Q}(u,g)(a) = \sum_{i=1}^{p} D^*Q(u^i, g^i)(a^i),$$

and $D^*Q(u^i, g^i)(\cdot)$ can be computed via the standard coderivative chain and product rule. Further note that the "intermediate" variable λ, taken with the opposite sign, amounts to the i-th component of the *Lagrange multiplier*, associated with the *non-penetrability* constraint $u_3^i \geq 0$ for all i. So, model (4) is just a light modification of a corresponding model in [1], used there in a shape optimization context.

That is why one can describe $\operatorname{gph} Q$ as follows:

$$\operatorname{gph} Q = \{(u, g) \in \mathbb{R}^3 \times \mathbb{R}^3 | g \in Q(u)\} = L \cup M_1 \cup M_3^+ \cup M_2 \cup M_3^- \cup M_4, \quad (6)$$

where all sets on the right-hand side of (6) are disjoint and defined by the following table:

	no contact $u_3 > 0, g_3 = 0$	weak contact $u_3 = 0, g_3 = 0$	strong contact $u_3 = 0, g_3 < 0$
sliding $u_{12} \neq 0$	L	M_2	M_1
weak sticking $u_{12} = 0, \|g_{12}\| = -\phi\, g_3$		M_4	M_3^-
strong sticking $u_{12} = 0, \|g_{12}\| < -\phi\, g_3$	–	–	M_3^+

Table: 1

Note that in Table 1 the impossible combinations of variables are crossed out.

Important facts

(1) The GE

$$y \in Pu - f + \widetilde{Q}(u)$$

Is strongly metrically regular at any pair (u, y) whenever ϕ is sufficiently small (cf. [1, Theorem 3.13]).

(2) The sets L, M_1 and M_3^+ exhibit a stable behavior in the sense that, for a sufficiently small $\varrho > 0$,

$$\left.\begin{array}{l}(\bar{u}, \bar{g}) \in L(\text{or } M_1 \text{ or } M_3^+) \\ (u, g) \in \mathcal{B}_\varrho(\bar{u}, \bar{g}) \cap \operatorname{gph} Q\end{array}\right\} \Rightarrow (u, g) \in L(\text{or } M_1 \text{ or } M_3^+).$$

(3) Let $(\bar{u}, \bar{g}) \in \operatorname{gph} Q$. Then $\operatorname{gph} Q \cap \mathcal{B}_1(\bar{v}, \bar{h})$ is subanalytic, because it is a canonical projection of a semianalytic set (intersection of finitely many polynomial equalities and inequalities).

On the basis of property (5) and the appropriate rules of generalized differential calculus we compute that

(1) for $(\bar{v}, \bar{h}) \in L$

$$D^* Q(\bar{v}, \bar{h})(h^*) = \{0\} \text{ for any } h^* \in \mathbb{R}^3, \tag{7}$$

(2) for $(\bar{v}, \bar{h}) \in M_1$ and $h^* \in \mathbb{R}^*$ one has

$$D^* Q(\bar{v}, \bar{h})(h^*) = \{v^* | v^*_{12} = -\phi \bar{h}_3 H(\bar{v}_{12})^T h^*_{12}, v^*_3 = \langle v^*_{12}, \frac{\bar{v}_{12}}{\|\bar{v}_{12}\|} \rangle\}, \tag{8}$$

where $H(v_{12}) = \nabla(\frac{v_{12}}{\|v_{12}\|})$;

(3) for $(\bar{v}, \bar{h}) \in M_3^-$

$$D^* Q(\bar{v}, \bar{h})(h^*) = \begin{cases} \mathbb{R}^3 & \text{provided } h^* = 0 \\ \emptyset & \text{otherwise .} \end{cases} \tag{9}$$

Thanks to property (2) and the definition of limiting coderivative formulas (7), (8) and (9) yield nonempty subsets of the coderivatives also in case when

$$(\bar{v}, \bar{h}) \in M_2 \text{ or } M_4, \quad (\bar{v}, \bar{h}) \in M_2 \text{ or } M_3^- \text{ or } M_4 \quad \text{and} \quad (\bar{v}, \bar{h}) \in M_3^- \text{ or } M_4,$$

respectively.

Ad (vi) Implementation

In order to facilitate the AS we will solve, instead of GE (4) the enhanced system

$$0 \in \mathcal{F}(u,d) := \left[\begin{array}{l} Au - f + \tilde{Q}(d) \\ u - d \end{array} \right] \tag{10}$$

in variables $(u,d) \in \mathbb{R}^{3p} \times \mathbb{R}^{3p}$. Clearly, $\bar{u}$ is the solution of (4) iff $(\bar{u},\bar{u})$ solves (10).

The separable structure of $\widetilde{Q}$, fact (3) and the theorem by Jourani, shown in (iii), imply the next statement.

Proposition

The mapping $\mathcal{F} : \mathbb{R}^{3p} \times \mathbb{R}^{3p} \rightrightarrows \mathbb{R}^{6p}$ is semismooth* at any point (u,d,y) from $\operatorname{gph} \mathcal{F}$.

Approximation step

1. Given the k-th iterate ${}^k u = ({}^k u^1, {}^k u^2, \dots, {}^k u^p)$, we compute for all i the (unique) solutions $\hat{z}_3^i, \hat{z}_{12}^i$ of the strictly convex optimization problems

$$\underset{z_3^i \in \mathbb{R}}{\text{minimize}} \quad \frac{1}{2}(z_3^i)^2 + c_3^i({}^k u) z_3^i + \delta_{\mathbb{R}_+}({}^k u_3^i + z_3^i), \tag{11}$$

$$\underset{z_{12}^i \in \mathbb{R}^2}{\text{minimize}} \quad \frac{1}{2}\|z_{12}^i\|^2 + \langle c_{12}^i({}^k u), z_{12}^i \rangle + \phi(\hat{z}_3^i + c_3^i({}^k u))\|{}^k u_{12}^i + z_{12}^i\|, \tag{12}$$

respectively. In these auxiliary problems $c({}^k u) = A({}^k u) - f$ stands for the single-valued part of GE (4) for $u = {}^k u$. Note that the solution z_3^i of (11) arises in (12) as a parameter.

2. Compute the outcome of the AS via

$$\hat{u} = {}^k u, \qquad \hat{d} = {}^k u + \hat{z}, \qquad \hat{y} = (-\hat{z}, -\hat{z}),$$

where $\hat{z} = (\hat{z}^1, \hat{z}^2, \dots, \hat{z}^p) \in (\mathbb{R}^3)^p$ with $\hat{z}^i = (z_{12}^i, z_3^i), i = 1, 2, ..., p,$. One can verify that $\hat{y} \in \mathcal{F}(\hat{u}, \hat{d})$ and there exists $L \geq 0$ such that

$$\|((\hat{u} - \bar{u}, \hat{d} - \bar{u}), \hat{y})\| \leq L\|({}^k u - \bar{u}, {}^k d - \bar{u})\|$$

holds for all $({}^k u, {}^k d)$ close to $(\bar{u}, \bar{u})$.

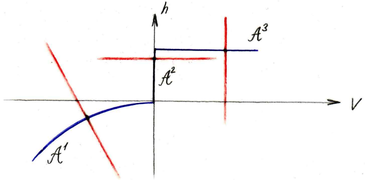
Newton step

To compute matrices A, B needed in the NS one has to find suitable linear subspaces of the limiting normal cone to $\operatorname{gph} Q$ at the respective points. To grasp this problem in a more general setting, assume that a multifunction $\Lambda : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ has closed graph and $\operatorname{gph} \Lambda = \bigcup_{i=1}^{s} \operatorname{cl} \mathcal{A}^i$, where each $\mathcal{A}^i \subset \mathbb{R}^n \times \mathbb{R}^n$ belongs to one of the following two groups:

(1) $\dim(\operatorname{aff} \mathcal{A}^i) \leq n$;

(2) there exist a $[2n \times 2n]$ matrix C, an open set $O \subset \mathbb{R}^n$ and a C^1 mapping $\Psi : O \to \mathbb{R}^n$ such that

$$\mathcal{A}^i = \left\{ (v, h) \in \mathbb{R}^n \times \mathbb{R}^n \,\middle|\, C \begin{bmatrix} v \\ h \end{bmatrix} \in \operatorname{gph} \Psi \right\}.$$



Newton step

Proposition

(i) *Let $\mathcal{A}^i$ belong to the group (1), $(\bar{v}, \bar{h}) \in \mathcal{A}^i$, and let the $[n \times n]$ matrices G^i, H^i fulfill the condition*

$$\mathcal{R}\begin{bmatrix} G^i \\ H^i \end{bmatrix} = \operatorname{aff} \mathcal{A}^i.$$

Then

$$\mathcal{N}[(G^i)^T \vdots (H^i)^T] \subset N_{\operatorname{gph}\Lambda}(\bar{v}, \bar{h}). \tag{13}$$

(ii) *Let $\mathcal{A}^i$ belong to the group (2), $(\bar{v}, \bar{h}) \in \mathcal{A}^i$, let C be non-singular and $\begin{bmatrix} \bar{w} \\ \bar{z} \end{bmatrix} = C \begin{bmatrix} \bar{v} \\ \bar{h} \end{bmatrix}$. Then incl. (13) holds true with*

$$G^i = (C^{-1})^T, \qquad H^i = \nabla\Psi(\bar{w})^T (C^{-1})^T.$$

Remark

Alternatively, a suitable linear subspace of $N_{\operatorname{gph}\Lambda}(\bar{v}, \bar{h})$ can be constructed as the range space of a linear mapping.

Newton step

Given the output $\hat{x}, \hat{d}, \hat{y} = (\hat{y}_1, \hat{y}_2)$ of the AS, we compute

$$\hat{b} = (\hat{b}^1, \hat{b}^2, \dots, \hat{b}^p) = \hat{y} - c(\hat{x}) \in \widetilde{Q}(\hat{d}),$$

implying that $\hat{b}^i \in Q(\hat{d}^i)$ for all i. By virtue of the above results we can now proceed as follows:

- For $(\hat{b}^i, \hat{d}^i) \in L \cup M_2 \cup M_4$ set $(G^i)^T = I, (H^i)^T = 0$;
- For $(\hat{b}^i, \hat{d}^i) \in M_3^+ \cup M_3^-$ set $(G^i)^T = 0, (H^i)^T = I$;
- For $(\hat{b}^i, \hat{d}^i) \in M_1$ set

$$(G^i)^T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \qquad (H^i)^T = \left[\begin{array}{cc|c} \multicolumn{2}{c|}{W} & 0 \\ & & 0 \\ \hline -\phi\omega_1 & -\phi\omega_2 & 1 \end{array}\right],$$

where the $[2 \times 2]$ matrix W is given by

$$W = -\frac{\phi \hat{b}_3^i}{\|\hat{d}_{12}^i\|} \begin{bmatrix} \omega_2^2 & -\omega_1\omega_2 \\ -\omega_1\omega_2 & \omega_1^2 \end{bmatrix}, \qquad \omega_j = -\frac{\hat{d}_j^i}{\|\hat{d}_{12}^i\|}, j = 1, 2.$$

NS attains now the form stated in Algorithm 1 with $A = I$ and

$$B = \begin{bmatrix} I & 0 \\ 0 & G \end{bmatrix} D^{-1},$$

where

$$D = \begin{bmatrix} A & -H \\ I & G \end{bmatrix}, \quad G = \begin{bmatrix} G^1 & & \\ & \ddots & \\ & & G^p \end{bmatrix}, \quad H = \begin{bmatrix} H^1 & & \\ & \ddots & \\ & & H^p \end{bmatrix}.$$

The strong metric regularity of GE (4) around $(\bar{u}, 0)$ implies the (strong) metric regularity of $\mathcal{F}$ around $(\bar{u}, \bar{u}, 0)$. From this it follows by the Mordukhovich criterion that D is non-singular whenever $(\hat{u}, \hat{d}, \hat{y})$ is sufficiently close to $(\bar{u}, \bar{u}, 0)$.
NS amounts thus to the solution of a (dense) linear system with $6p$ equations. Thanks to its structure it can be reduced to merely $3p$ equations. This reduction can be, however, achieved also by a different choice of matrices G, H.

Stopping rule

Due to the strong metric regularity of (10) around $(\bar{u}, \bar{u}, 0)$ there is a Lipschitz constant $l > 0$ such that,

$$\|(\hat{u}, \hat{d}) - (\bar{u}, \bar{u})\| \leq l\|\hat{y}\|,$$

whenever the output of the AS lies in a sufficiently small neighborhood of $(\bar{u}, \bar{u}, 0)$. It follows that the condition

$$\|\hat{z}\| \leq \varepsilon,$$

with $\hat{z}$ composed from the solutions of the optimization problems in the approximation step, may serve as a simple yet efficient stopping rule.

Ad (vii) Numerical tests

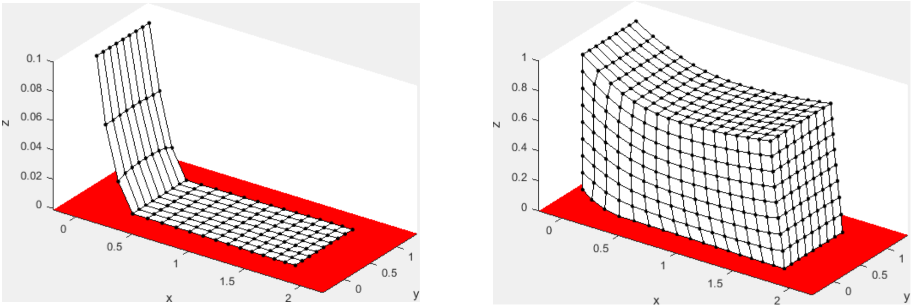


Figure: The left figure depicts a deformed contact boundary and the right figure shows the corresponding deformed elastic prism, both pictures together with the (red) rigid plane foundation. The right picture is obtained by (a linear elasticity) post-processing of non-contact boundary nodes.

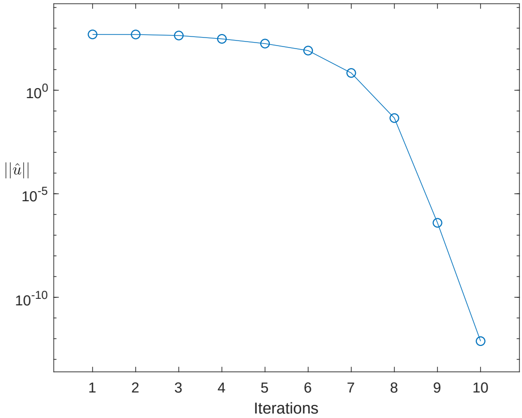
Running times

mesh level	nodes of Ω	assembly of K	Cholesky + Schur	nodes of Γ_C	semismooth* solv. time	iters
3	637	0.13 sec	0.02 sec	84	0.03 sec	6
4	1377	0.19 sec	0.09 sec	144	0.07 sec	6
5	2541	0.28 sec	0.24 sec	220	0.27 sec	9
6	4225	0.61 sec	0.69 sec	312	0.51 sec	9
7	6525	1.11 sec	1.76 sec	420	0.93 sec	9
8	9537	1.61 sec	3.73 sec	544	1.60 sec	9
9	13357	2.48 sec	7.20 sec	684	3.05 sec	10
10	18081	3.30 sec	13.44 sec	840	4.69 sec	9
11	23805	4.38 sec	23.07 sec	1012	8.17 sec	10
12	30625	5.73 sec	40.71 sec	1200	12.48 sec	10
13	38637	7.33 sec	107.63 sec	1404	17.55 sec	9
14	47937	9.16 sec	293.50 sec	1624	30.24 sec	11
15	58621	11.92 sec	684.17 sec	1860	43.00 sec	11
16	70785	13.78 sec	1217.38 sec	2112	57.80 sec	11

Table: 2

Practical convergence

Computational data:
Mesh level 6 -
312 nodes, 936 unknowns
Stopping rule: ε=1e-12
Number of iterations: 9



Ad (viii) Conclusion

The respective code is in fact slightly more advanced when compared with its description in the previous section. It contains, among other things, an adaptive scaling of the AS accelerating the convergence and a heuristic globalization strategy for the case when we do not dispose with a suitable starting point.

When ϕ is not "sufficiently" small, then, as shown in [4, 7], GE (4) may loose its metric regularity at the solution. Consequently, in the vicinity of the solution the linear system in the NS becomes unsolvable. The GE (4) may, however, also have multiple isolated solutions, around which it is strongly metrically regular. In such a case, expectantly, the semismooth* Newton method works well.

As already mentioned, strong metric regularity around the solution is not necessary for the convergence. The method has potential to converge also in case of non-isolated solutions, which opens further application areas.

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