Luni, 20 Februarie 2017

Calcul Numeric

Cursul 1

ancai@info.uaic.ro → întrebări / Cabinet : C307

$\boxed{\text{ancai\_fii@yahoo.ro}}$ → temele (sf. semestrului)

Laborator : 8 teme — 4 teme - S. 6-7
4 teme - S. 13-14

Examen : S15-16 / Criterii trecere : Examen ≥ 3
Final ≥ 400
(Final = Lab + Examen × 12)

Sist. liniare:

$$\begin{cases} x_1 + 0,5x_2 + 3x_3 = 4,5 \\ 2x_2 - 0,7x_3 = 13 \\ x_1 + \frac{1}{7}x_3 = \frac{8}{7} \end{cases}$$

$$\boxed{x_i = \frac{\det A_i}{\det A}}$$

Cap. 1 & 3 → alg. liniară

Ec. neliniare : $\sin x - xe^{-x} = 0$

Interpolare numerică : ~~[illegible]~~ ex. recensământul
aprox. unei funcții pe baza ~~unor val. discrete~~

Calcul numeric — ~~Retelele sociale~~

* Centralitate → nodurile care au cei mai mulți vecini
* Drum geodesic → drumul de lung. min. dintre 2 vârfuri (noduri)
* ~~Centralitate de intermediere → proporția nr. de drumuri între~~ două noduri care trec printr-un anumit nod
* Vector propriu = "eigen vector" (engl. / german)
* Nodurile importante infl. importanța nodurilor vecine → page-rank (google)

! $\boxed{A * x = y}$ → vector / $y_i = \sum_{j=1}^{n} a_{ij} \cdot x_j \ ; \ i = \overline{1,n}$

matrice → matrici rare !

→ Compresia imaginilor
- Reducerea dim. problemei
- compresie ⟨ cu pierdere de informație (JPEG)
fără — // — (TIFF)
- Idom faus → alg. de identif. a fețelor

vector → vector coloană (=) $u \in \mathbb{R}^n = \mathbb{R}^{n\times 1}$

$u = \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_m \end{pmatrix}$; $v^T = (v_1, v_2, \dots, v_n) \to \mathbb{R}^{1\times n}$

$u \to \mathbb{R}^{m\times 1}$

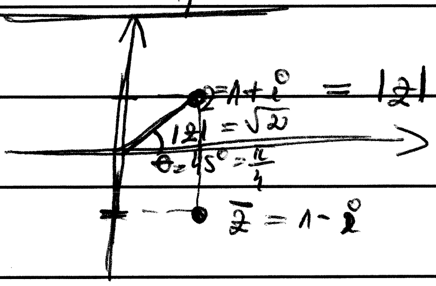
$\Rightarrow \boxed{u \cdot v^T \in \mathbb{R}^{m\times n}}$

$\begin{pmatrix} u_1 v_1 & \dots & u_1 v_n \\ & \dots u_i v_j \dots & \\ u_m v_1 & \dots & u_m v_n \end{pmatrix}$

170, 744 - jpg
3 249, 546 - tiff

→ Lucru cu vectori și matrici
→ sol. aproximative → ~~norme~~

* Nr. complexe



$z = |z| e^{i\theta} = \sqrt{2}\left(\cos\frac{\pi}{4} + i \sin\frac{\pi}{4}\right)$ = 44 - jpg
6 - tiff

$e^{i\theta} \to \cos\theta + i\sin\theta$

$\boxed{z\bar{z} = |z|^2}$

$\mathbb{R}^{m\times n}$
$\mathbb{C}^{m\times n}$
$\mathbb{Z}^{m\times n}$

* operația realizată de calculator nu este asociativă!
(asociativ., comutativ.)

↓ la laborator

$\mathbb{R}^3 : \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$; $\begin{pmatrix} 5 \\ 7 \\ 9 \end{pmatrix} = 5\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + 7\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + 9\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$; $e_j = \begin{pmatrix} 0 \\ 0 \\ 1 \\ \vdots \end{pmatrix}$ → poz j

$c_1\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + c_2\begin{pmatrix} 2 \\ 2 \\ 5 \end{pmatrix} + c_3\begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix} + c_4\begin{pmatrix} 1 \\ 2 \\ 7 \end{pmatrix} = 0 \Rightarrow c_1 = c_2 = c_3 = c_4 = 0$

vect. liniar independenți
↓
nu pot să-l scriu pe unul în funcție de ceilalți

* ~~Matricea adjunctă~~ !

$$A = \begin{pmatrix} 1+i & 2-i & 3 \\ 3+5i & 0 & 1+2i \end{pmatrix}$$

$$\Rightarrow A^H = \begin{pmatrix} 1-i & 3-5i \\ 2+i & 0 \\ 3 & 1-2i \end{pmatrix} = \overline{A^T} = (\overline{a_{ji}})_{j=\overline{1,n},\ i=\overline{1,m}}$$

$A = A^H \Rightarrow$ $\boxed{\text{matr. autoadjunctă}}$

simetrica : pe diag. principală $\begin{pmatrix} 1 & 1 & 5 \\ 1 & 2 & 7 \\ 5 & 7 & 3 \end{pmatrix}$

avantaj : economie de memorie

$$\begin{pmatrix} 1 & 2 & 3 \\ 5 & 7 & 9 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ 7 \end{pmatrix} = \begin{pmatrix} a_{21} \\ a_{22} \end{pmatrix}$$

$\boxed{e_2}$

$$(1 \quad 0) \begin{pmatrix} 1 & 2 & 3 \\ 5 & 7 & 9 \end{pmatrix} = (1, 2, 3)$$

27 Martie 2016

Symfony

symfony.com/doc/current/book

cd /var/www

symfony.com/downloads

192.168.33.10:8000

sudo chmod 777

cd nume_proiect

php bin/console server:run 192.168.33.10:8000

192.168.33.10 www.blog.dev

→ www.blog.dev:8000

symfony.com

app.php → front controller

Bundles → module

! acesa

php bin/console generate:bundle --namespace=Acme/BlogBundle --format=yml

cat app/AppKernel.php

{{ }} → afișare

{% %} → executare

Twig

Miercuri, 22 Februarie 2017

Calcul Numeric

Laboratorul 1

$\boxed{\text{calculnumeric2017@gmail.com}}$

Bonus predare teme în avans

Bonus interfață grafică

Tema 1:

1. Precizia mașină → pragul de la care mașina produce erori de calcul

$\approx 10^{-15} \sim 10^{-16}$

→ o buclă

3. Optimizare calcule
   - grupare termeni
   - nr. minim de operații

$$P_1(x) = x - c_1x^3 + c_2x^5 = x(1 - c_1x^2 + c_2x^4) = \boxed{x(1 - x^2(c_1 - c_2x^2))}$$

$$P_2(x) = x - c_1x^3 + c_2x^5 - c_3x^7 = x(1 - c_1x^2 + c_2x^4 - c_3x^6) =$$

$$= x(1 + x^2(-c_1 + c_2x^2 + c_3x^4)) =$$

$$= \boxed{x(1 + x^2(-c_1 + x^2(c_2 - c_3x^2)))}$$

$$P_3(x) = x - c_1x^3 + c_2x^5 - c_3x^7 + c_4x^9 = x(1 - c_1x^2 + c_2x^4 - c_3x^6 + c_4x^8) =$$

$$= x(1 + x^2(-c_1 + c_2x^2 - c_3x^4 + c_4x^6)) =$$

$$= x(1 + x^2(-c_1 + x^2(+c_2 - c_3x^2 + c_4x^4))) =$$

$$= \boxed{x(1 + x^2(-c_1 + x^2(+c_2 + x^2(-c_3 + c_4x^2))))}$$

$$P_4(x) = x - 0{,}166x^3 + 0{,}00833x^5 - c_3x^7 + c_4x^9 =$$

$$= x(1 - 0{,}166x^2 + 0{,}00833x^4 - c_3x^6 + c_4x^8) =$$

$$= x(1 + x^2(-0{,}166 + 0{,}00833x^2 - c_3x^4 + c_4x^6)) =$$

$$= x(1 + x^2(-0{,}166 + x^2(0{,}00833 - c_3x^2 + c_4x^4))) =$$

$$= \boxed{x(1 + x^2(-0{,}166 + x^2(0{,}00833 + x^2(-c_3 + c_4x^2))))}$$

$$P_5(x) = x - c_1x^3 + c_2x^5 - c_3x^7 + c_4x^9 - c_5x^{11} =$$

$$= x(1 - c_1x^2 + c_2x^4 - c_3x^6 + c_4x^8 - c_5x^{10}) =$$

$$= x(1 + x^2(-c_1 + c_2x^2 - c_3x^4 + c_4x^6 - c_5x^8)) =$$

$$= x(1 + x^2(-c_1 + x^2(c_2 - c_3x^2 + c_4x^4 - c_5x^6))) =$$

$$= x(1 + x^2(-c_1 + x^2(c_2 + x^2(-c_3 + c_4x^2 - c_5x^4)))) =$$

$$= \boxed{x(1 + x^2(-c_1 + x^2(c_2 + x^2(-c_3 + x^2(c_4 - c_5x^2)))))}$$

$$P_6(x) = x - c_1x^3 + c_2x^5 - c_3x^7 + c_4x^9 - c_5x^{11} + c_6x^{13} =$$

$$= x(1 - c_1x^2 + c_2x^4 - c_3x^6 + c_4x^8 - c_5x^{10} + c_6x^{12}) =$$

$$= \boxed{x(1 + x^2(-c_1 + x^2(c_2 + x^2(-c_3 + x^2(c_4 - x^2(-c_5 + c_6x^2))))))}$$

Luni, 27 Februarie 2017

CN

Cursul 2

Vectori → vect. coloană

$$Ax = b \Leftrightarrow \begin{cases} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n = b_2 \\ \dots \\ a_{n1}x_1 + a_{n2}x_2 + \dots + a_{nn}x_n = b_n \end{cases}$$

x → vector coloană

$$\underbrace{\underset{n\times n}{A} * \underset{n\times 1}{x}}_{n\times 1} = y = \begin{pmatrix} a_{11} & \dots & a_{1n} \\ a_{21} & \dots & a_{2n} \\ \dots & \dots & \dots \\ a_{n1} & \dots & a_{nn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} =$$

$$(x, y)_{\mathbb{R}^3} = x_1y_1 + x_2y_2 + x_3y_3$$

x → x_1, x_2, x_3

y → y_1, y_2, y_3

$$(x, y)_{\mathbb{C}^3} = x_1\overline{y_1} + x_2\overline{y_2} + x_3\overline{y_3} = y^H x$$

$$(x, y)_{\mathbb{R}^3} = y^T x = \underset{\mathbb{R}^{1\times 3}}{(y_1, y_2, y_3)} \underset{\mathbb{R}^{3\times 1}}{\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}} = \underbrace{y_1x_1 + y_2x_2 + y_3x_3}_{\mathbb{R}^{1\times 1}}$$

$A = (a_{ij})\ i, j = \overline{1, n}$

$A : \mathbb{R}^n \to \mathbb{R}^n$

$A(x) = A * x$

Op. liniar $\boxed{A(\alpha x + \beta y) = \alpha A x + \beta A y}$

$A : C[a, b] \to (x, y)$

$(Ax, y) = (x, By)$; $B = ?$ $\qquad$ $\boxed{(AB)^T = B^T A^T}$ *

$$(Ax, y)_{\mathbb{R}^n} = y^T (Ax) = y^T A \cdot x = y^T (A^T)^T x =$$
$$= (A^T y)^T x = (x, A^T y)$$

$A^T A = A \cdot A^T = I$ $\qquad$ | $\quad Ax = b$ sol. $x = A^{-1} b$

$\Rightarrow \exists A^{-1}$; $A^{-1} = A^T$ $\qquad$ $\underline{x = A^T \cdot b}$

$\boxed{\text{matr. ortogonală}}$

Matrici "bune"

* Matr. triunghiulare inferior / superior

$$\begin{pmatrix} 2 & 0 & 0 \\ 3 & 1 & 0 \\ 5 & 1 & 1,5 \end{pmatrix} \to$$ inferior triunghiulară (deasupra diag. principale am numai 0)

transpusa $\to$ superior triunghiulară (dedesubt —"—)

$$\begin{pmatrix} 2 & 3 & 5 \\ 0 & 1 & 1 \\ 0 & 0 & 1,5 \end{pmatrix}$$

$\Rightarrow$ $\boxed{\text{sisteme triunghiulare}}$

$$\begin{cases} 2x_1 = 4 \\ 3x_1 + x_2 = 5 \\ -x_1 + x_2 + x_3 = -1 \end{cases}$$

$\Rightarrow \boxed{x_1 = \frac{4}{2} = 2}$

$3 \cdot 2 + x_2 = 5 \Rightarrow \boxed{x_2 = 5 - 6 = -1}$

$-2 - 1 + x_3 = -1 \Rightarrow \boxed{x_3 = -1 + 3 = 2}$

$\boxed{\text{sist. inf. triunghiular}}$

$$\begin{cases} 3x_1 + x_2 + x_3 = 2 \\ 2x_2 - x_3 = 1 \\ x_3 = 1 \end{cases}$$

$\Rightarrow$ Metoda substituției inverse (de jos în sus)

sist. sup. triunghiular

$\boxed{D \cdot A}$

matr. diagonală

$$\begin{pmatrix} d_1 & 0 & 0 \\ 0 & d_2 & 0 \\ 0 & 0 & d_3 \end{pmatrix} \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} = \begin{pmatrix} d_1 a_{11} & d_1 a_{12} & d_1 a_{13} \\ d_2 a_{21} & d_2 a_{22} & d_2 a_{23} \\ d_3 a_{31} & d_3 a_{32} & d_3 a_{33} \end{pmatrix}$$

$$\boxed{A \cdot D} \cdot e_j = A(D e_j) = A \begin{pmatrix} 0 \\ 0 \\ \vdots \\ d_j \\ \vdots \\ 0 \end{pmatrix} \begin{matrix} \\ \\ \\ \to \text{poz. } j \\ \\ \\ \end{matrix} =$$

$$= A(d_j e_j) = d_j (A \cdot e_j)$$

$(AD)e_j =$ col. j a matr. $\underline{A \cdot D}$ $= d_j (A \cdot e_j) = \underline{d_j \cdot \text{col. } j \text{ a matr. } A}$

$$\underline{A \cdot D} = \begin{pmatrix} d_1 a_{11} & d_2 a_{12} & d_3 a_{13} \\ d_1 a_{21} & d_2 a_{22} & d_3 a_{23} \\ d_1 a_{31} & d_2 a_{32} & d_3 a_{33} \end{pmatrix}$$

$$Dx = b \Rightarrow x_i = \frac{b_i}{d_i}$$

= Norme =

$\Rightarrow$ au valori în $\mathbb{R}^*$

$$x = \begin{pmatrix} 1 \\ -3 \\ 2 \end{pmatrix} \Rightarrow \|x\|_1 = 1+3+2 = 6$$

$$\|x\|_2 = |x| = \sqrt{1+9+4} = \sqrt{14}$$

Pt nr. complexe ($\mathbb{C}$) se ia cu $\sqrt{\ }$ modul / ($i^2 = -1$)

$$\|x\|_c = 3$$

Normele se fol. la evaluarea erorilor!

$Ax = b$; $A \in \mathbb{R}^{100 \times 100}$

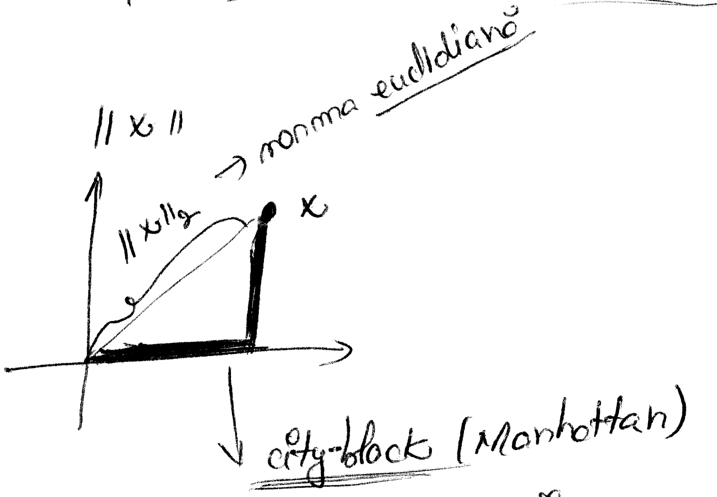
calc. x_{apx} $\to$ $\simeq 10^{-\text{ceva}}$

pp. cum. sol. exactă $x_{exactă} = x^*$

$\dfrac{\| x_{apx} - x^* \|}{\|x^*\|}$ $\to$ se evaluează norma

[illegible] confid. o normă

Adaptez alegerea normei în funcție de situație (ex. în funcție de nr. de componente)



Cebâșev → tabla de șah

$\|x\|_1$

P = matr. nesingulară $\iff \det(P) \neq 0$

$\iff \exists P^{-1}$

$\iff$ sist. $Px = b$ are soluție ; $\forall b$

$\iff$ sist. $Px = 0$ are doar sol. $x = 0$

$\|x\|_{1,P} = \|Px\|_1$

P = matr. diagonală $\begin{pmatrix} p_1 & 0 & 0 \\ 0 & p_2 & 0 \\ 0 & 0 & p_3 \end{pmatrix}$

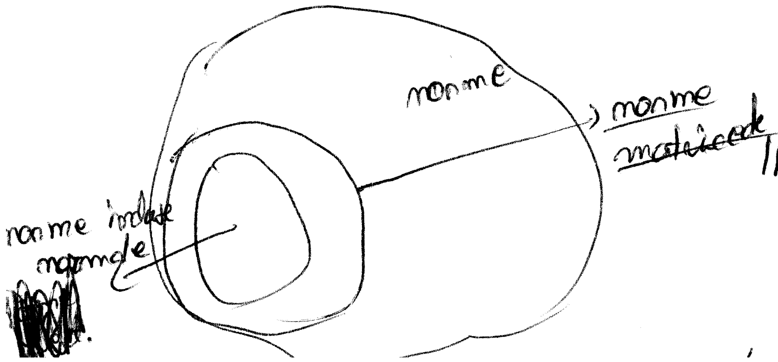
$Px = \begin{pmatrix} p_1 x_1 \\ p_2 x_2 \\ p_3 x_3 \\ \vdots \\ p_n x_n \end{pmatrix}$ $\quad p_1 > 0$

$\|Px\|_1 = p_1|x_1| + p_2|x_2| + \dots + p_n|x_n| < 10^{-6}$

$\|x - y\|_{1,P} = p_1|x_1 - y_1| + \dots + p_n|x_n - y_n| \leq 10^{-6}$

↓

Componentele sunt tratate inegal



A, B două matrici

$\|A \cdot B\| \leq \|A\| \cdot \|B\|$

→ normă matriceală

→ Norma ~~Cebâșev~~ nu este normă matriceală

$$\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$ → matr. de rotație
↓
ortogonală

Norma Frobenius:

$$\| I \|_{FRO} = \sqrt{n}$$

$$\| I \|_i = 1$$

↓ Normă indusă

$$\|Ax\|_v \le \|A\|_i \cdot \|x\|_v$$

$$\|A\|_1 = \max\left\{ \sum_{i=1}^{n} |a_{ij}|,\ j=\overline{1,n} \right\}$$

↓ suma elem. de pe coloana j

$$\|A\|_\infty = \max\left\{ \sum_{j=1}^{n} |a_{ij}|,\ i=\overline{1,n} \right\}$$

$$\begin{array}{r} 1+|-3|+5=9 \\ 2+7+9=18 \\ |-1|+2+3=6 \end{array} \begin{bmatrix} 1 & -3 & 5 \\ 2 & 7 & 9 \\ -1 & 2 & 3 \end{bmatrix} \Rightarrow \|A\|_1 = \max\{4, 12, 17\} = 17$$

$$\|A\|_\infty = \max\{9, 18, 6\} = 18$$

$1+2+|-1| = 4$ $|-3|+7+2 = 12$ $5+9+3 = 17$

$$\begin{bmatrix} 1 & 2 & 3 \\ -1 & 2 & 4 \end{bmatrix} \begin{array}{l} 1+2+3=6 \\ |-1|+2+4=7 \end{array} \Rightarrow \|A\|_1 = \max\{2, 4, 7\} = 7$$

$$\|A\|_\infty = \max\{6, 7\} = 7$$

$1+|-1| = 2$ $2+2=4$ $3+4=7$

$$\|x\|_v \longrightarrow \|A\|_i$$

P ↓   ↓ ?

$$\|x\|_{v,P} \longrightarrow \|A\|_{P,i} = \|PAP^{-1}\|_i$$

$$\underline{PAP^{-1}} = \begin{pmatrix} \frac{p_1}{p_1} a_{11} & \frac{p_1}{p_2} a_{12} & \frac{p_1}{p_3} a_{13} & \dots & \frac{p_1}{p_n} a_{1n} \\ \dots & & \frac{p_i}{p_j} a_{ij} & & \dots \\ \frac{p_n}{p_1} a_{n1} & \dots & & & \frac{p_n}{p_n} a_{nn} \end{pmatrix}$$

$P = \text{diag}\{p_1, \dots, p_n\}$

$P^{-1} = \text{diag}\{\frac{1}{p_1}, \dots, \frac{1}{p_n}\}$

$\|x\|_2 \longrightarrow \|A\|_{2,i}$

Val. și vectorii proprii → doar pt matr. pătratice

Val. → eigen-value

vectorii → eigen vector

$\lambda \in \mathbb{R}$ a.î. $Ax = \lambda x$

$$\underbrace{(\lambda I - A)}_{B} x = 0, \quad x \neq 0 \searrow B \cdot x = 0 \ ; \ x \neq 0$$

$$\Downarrow$$

$$\underline{\det B = 0}$$

$$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

$$\lambda I - A = \begin{bmatrix} \lambda & +1 \\ -1 & \lambda \end{bmatrix} \Rightarrow \det(\lambda I - A) = \lambda^2 + 1 = 0$$

$$\Rightarrow \lambda^2 = -1 \Rightarrow \lambda \in \{-i, i\}$$

↓ nr. complex ($\mathbb{C}$)

~~Val. proprii~~

Vect. proprii: multiplicitate algebrică

ex: $(\lambda - 5)^3 (\lambda - 2)^6$

$A \in \mathbb{R}^{m \times n}$

~~[illegible]~~

$A^T \in \mathbb{R}^{n \times m}$

$\Rightarrow \boxed{A^T * A \in \mathbb{R}^{n \times n}}$

$\varepsilon > 0$;

$\begin{bmatrix} 1 \\ \lambda \end{bmatrix}$

$\rho(A) \le \|A\|_\varepsilon \le \rho(A) + \varepsilon$

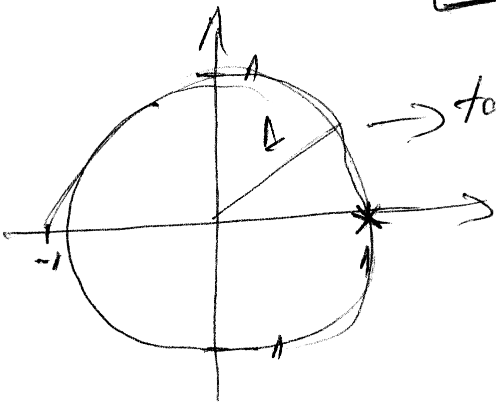
$A^k \longrightarrow 0$

$2^k \longrightarrow \infty$

$\left(\frac{1}{2}\right)^k \longrightarrow 0$

$a^k \longrightarrow \infty \Leftrightarrow |a| < 1$

$\boxed{\text{rază spectrală}}$



toate val. proprii sunt în cercul de rază $r = 1$

$$I + A + A^2 + \dots + A^k + \dots = S = (I - A)^{-1}$$

$\rho(A) \leq 1 \rightarrow$ val. propr. ale lui A $|\lambda| < 1$

$$(I - A)^{-1}$$

$$\det(\lambda I - A) \neq 0$$

$|\lambda| < 1 \Rightarrow \lambda = \pm 1$ nu este val. propr.

$$\det(\pm 1 I - A) \neq 0$$

$$A = \begin{pmatrix} 10^{-8} & 0 & 10^{-6} \\ 0 & 10^{-3} & 0 \\ 0 & 10^{-4} & 10^{-1} \end{pmatrix}$$

$* \rightarrow I \pm A \;;\; \|A\| < 1$

I

Luni, 6 Martie 2017

CN

Cursul 3

$$\mathring{I} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad \mathring{I}+A = \begin{pmatrix} 1 & 10^{-8} & 10^{-6} \\ 0 & 1 & 10^{-10} \\ -10^{-8} & 0 & 1-10^{-7} \end{pmatrix}$$

$\det \mathring{I} = 1 \neq 0$ $\qquad$ $\det(\mathring{I}+A) \neq 0?$

$$A = \begin{pmatrix} 0 & 10^{-8} & 10^{-6} \\ 0 & 0 & 10^{-10} \\ -10^{-8} & 0 & -10^{-7} \end{pmatrix}$$

$Ax = b$

$$\begin{cases} \frac{1}{3}x_1 + 2x_2 + x_3 = 4 \\ -\frac{1}{7}x_1 + 2x_2 + 5x_3 = 3 \\ x_2 + \frac{2}{3}x_3 = 6 \end{cases}$$

$$\begin{cases} 0,3333\ldots x_1 + 2x_2 + x_3 = 4 \\ -0,142\ldots x_1 + 2x_2 + 5x_3 = 3 \\ x_2 + 0,66\ldots x_3 = 6 \end{cases} \rightarrow \text{ce calculează calculatorul}$$

$(A+E)x = b+e$

$\det A \neq 0 \qquad \det(A+E) \neq 0\,?$

$A \to \det A \neq 0 \Rightarrow \exists A^{-1}$

$A+E = A\,\underline{(\mathring{I} + A^{-1}E)}$

$\|A^{-1}E\| < 1$

$\Rightarrow \|E\| < \frac{1}{\|A^{-1}\|}$

Aproximările trebuiesc făcute f bine ar să nu perturbe matricea

Erorile apar de la datele de intrare până la cele de ieșire.

pot fi induse și de ignorarea param. mici (0,00...)

$P(A) = \frac{3}{5}$, $P(B) = \frac{2}{3}$; $P(A \cap B) = 0,4$

$P(A \cap B) = P(A) \cdot P(B)$

$0,4 = \frac{\cancel{3}}{5} \cdot \frac{2}{\cancel{3}}$

$0,4 = \frac{2}{5}$ (A)

$0,4 \overset{?}{=} \frac{3}{5} \cdot 0,66...(\approx \frac{2}{3}) \to 0,399...$

eroare

Erori la reprez. numerelor : val. exacte și val. aprox.

- absolută = diferența dintre val. exactă și val. aprox.
- relativă = eroarea absolută / val. exactă (%)

- Raportez eroarea la val. exactă.

Pb exactă: A, b, rez. sist. $Ax = b \to x^*$

Pb. rez. num. $\tilde{A}, \tilde{b}$, rez. aprox. $\tilde{A}x = \tilde{b} \to \tilde{x}$

date aproximative de intrare

$\tilde{x} \approx x^*$?

P

x $\quad P(x) \to$ sol. exactă

$\tilde{x}$ $\quad P(\tilde{x}) \to$ sol. aproximativă

$x \approx \tilde{x}$

Schimbări mici în coeficienții polinoamelor pot duce la schimbări mari ale soluțiilor.

$x \to P(x)$

$x \to \tilde{P}(x)$

$\| P(x) - \tilde{P}(x) \|$

	1	2	3
stabilă	0.999...9	2	3
instabilă	0.9999853	2.000001	3

Se dorește proiectarea de alg. numerici stabili (aprox. f. apropiată de sol. reală)

Metode de rezolvare

→ directe

→ Iterative – sist. de dimensiuni mari

– matrici rare

Eliminarea Gauss (⇒) desc. LU

$$\begin{cases} 5x + 6y = 5 \\ 2x + 3y = 2 \\ x + 3y = 3 \end{cases}$$

→ sist. cu 3 ecuații și 2 necunoscute

Forme biliniare:

(Ax, x)

ex: $5x_1^2 + 2x_1x_2 + 4x_2^2 + 7$

$\Delta A, \Delta b, \Delta x$ → erorile absolute

$\|A^{-1}\| \cdot \|A\|$ → mare ⇒ eroare mare la ieșire

$1 - \|A^{-1}\| \cdot \|\Delta A\|$ → mic

$\|\Delta A\| < \frac{1}{\|A^{-1}\|}$

↓

aprox. lui A

$k(A)$ = nr. de condiționare

≥ 1

$Ax = \lambda x$ → val. proprii

$A^2 x = \lambda A x = \lambda^2 x$

$A^{-1} \quad x = \lambda A^{-1} x$

$A^{-1} x = \frac{1}{\lambda} x$

$\rightarrow \nexists A^{-1}$; $\det(0 \cdot I - A) = 0 \Rightarrow \det(A) = 0$

→ Matrici ortogonale

$A \cdot A^T = A^T \cdot A = I \cdot A$

$A^T = A^{-1}$

→ matrici f. bine condiționate

$k(A) = 1$ (spectral)

$k(A)$ mare $\cancel{\Leftrightarrow}$ $\det(A) \approx 0$

$$A = \begin{pmatrix} 1 & & & 0 \\ & 0.1 & & \\ & & 0.1 \ \ddots & \\ 0 & & & 0.1 \end{pmatrix}_{100 \times 100}$$ → Norma spectrală: 1

$$A^{-1} = \begin{pmatrix} 1 & & & 0 \\ & 10 & & \\ & & 10 \ \ddots & \\ 0 & & & 10 \end{pmatrix}$$ → Norma spectrală: 10

elem. maxim

$\|A\|_\infty = 3$

$\|A\|_\infty = \frac{2^{n}-1}{2-1}$

Sist. prost condiționate ($k(A)$ mare) → rezultate complet neașteptate (eronate) cu metodele de calcul numeric

Matricea lui Hilbert → prost condiționate

Luni, 13 Martie 2014

CN

Cursul 4

Metode de rezolvare a sist. liniare

Regula lui Cramer:

$$\boxed{x_i = \frac{\det A_i(b)}{\det A}} ; \quad i = \overline{1,n} ; \quad \det A \neq 0$$

Erori mari $\Rightarrow$ alg. instabil

Regula lui Cramer este un alg. instabil!

Q ortogonală: $Q^T = Q^{-1}$; $Qx = b \Rightarrow \boxed{x = Q^T * b}$

Se încearcă aducerea la un sist. echivalent (cu aceeași soluție), a cărui matrice este superior triunghiulară

$\boxed{\text{Alg. de eliminare Gauss}}$

$$\begin{pmatrix} 1 & 3 & 5 \\ 0 & 2 & 1 \\ 0 & 0 & 1 \end{pmatrix} \Rightarrow \det = 1 \cdot 2 \cdot 1 = 2$$ (elem. de pe diag. principală)

$$\begin{cases} 2x_1 = 2 \\ x_1 + 3x_2 = 2 \\ -x_1 + 6x_2 + x_3 = 1 \end{cases} \qquad \begin{cases} x_1 = 1 \\ x_2 = \frac{2-1}{3} = \frac{1}{3} \\ x_3 = 1 + 1 - 6 \cdot \frac{1}{3} = 2 - 2 = 0 \end{cases}$$

inf. triunghiular $\to$ metoda substituției directe (de sus în jos)

sist. sup. triunghiular $\to$ metoda subst. inverse (de jos în sus)

M = complex. algoritmului (nr. de operații - înmulțiri / împărțiri)

$$\begin{cases} 3x_1 + 2x_2 - x_3 = 4 \\ \frac{1}{3}x_2 + \frac{2}{3}x_3 = 1 \\ \frac{1}{7}x_3 = \frac{1}{7} \end{cases}$$

$x_1 = \frac{4 - 2 \cdot 1 + 1}{3} = 1$

$x_2 = \frac{1 - \frac{2}{3}}{\frac{1}{3}} = \frac{1}{3} \cdot \frac{3}{1} = 1$

$\Rightarrow x_3 = 1$

<u>Op. ale alg. de eliminare Gauss</u>

- Primii n pași - transf. coloanelor în col. sup. triunghiulare (în coloane)
- Alg. se blochează la <u>det = 0</u>

$$\begin{cases} x_1 + x_2 + x_3 = 3 & / \cdot (-2) + \text{ec. 2} \quad / \cdot \frac{1}{3} + \text{ec. 3} \\ 2x_1 + \quad + x_3 = 3 \\ -\frac{1}{3}x_1 - \frac{2}{3}x_2 + 2x_3 = 1 \end{cases}$$

<u>Pasul 1</u> : transf. col. 1 în $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$

$$\begin{cases} x_1 + x_2 + x_3 = 3 \\ -2x_2 - x_3 = -3 & / \cdot \left(-\frac{1}{6}\right) + \text{ec. 3} \\ -\frac{1}{3}x_2 + \frac{7}{3}x_3 = 2 \end{cases}$$

$+\frac{1}{6}x_3 + \frac{7}{3}x_3 = \frac{+1+14}{6}x_3 = \frac{15}{6}x_3$

<u>Pasul 2</u> : transf. coloanei 2

$$\begin{cases} x_1 + x_2 + x_3 = 3 \\ -2x_2 - x_3 = -3 \\ -\frac{15}{6}x_3 = \frac{5}{2} \end{cases}$$

$\Rightarrow$ <u>rezolv sist.</u>

$\searrow -\frac{5}{2}$

<u>Pivotarea</u>

$\rightarrow$ <u>Fără pivotare</u> : caut primul coef $\neq 0$ și interschimb ecuațiile

$\Rightarrow$ aduce instabilitate

Pivotare → parțială: caut coef. cu modulul cel mai mare și interschimb liniile

↓ cu scalare: $|s_i| = \max\{|a_{ij}|;\ j=\overline{1,n}\};\ i=\overline{1,n}$

$\dfrac{|a_{i_0 r}|}{s_{i_0}} = \max\left\{\dfrac{|a_{jr}|}{s_j};\ j=\overline{r,n}\right\}$

↓ pivotarea propriu-zisă

↓ totală

↓ coef. cu modulul cel mai mare, schimbarea liniei + coloane

$$\begin{cases} x_2 + 2x_3 = 3 \\ x_1 + 3x_2 + x_3 = 5 \\ 2x_1 - x_2 + 2x_3 = 3 \end{cases}$$

1) Pivotare parțială:

Pasul 1: coloana 1 în formă sup. triunghiulară

$\max\{|a_{11}|, |a_{21}|, |a_{31}|\} = \max\{0, 1, 2\} = 2 = |a_{31}|$

interschimb ec. 3 cu ec. 1

Pasul 2
$$\begin{cases} 2x_1 - x_2 + 2x_3 = 3 \quad /\left(-\frac{1}{2}\right) + \text{ec. } 2 \\ x_1 + 3x_2 + x_3 = 5 \\ x_2 + 2x_3 = 3 \end{cases} \Rightarrow \begin{cases} 2x_1 - x_2 + 2x_3 = 3 \\ \frac{7}{2}x_2 = \frac{7}{2} \quad /\left(-\frac{2}{7}\right) \\ x_2 + 2x_3 = 3 \end{cases}$$

Pasul 3: Pivotare $\max\{|a_{22}|, |a_{32}|\} = \max\left\{\frac{7}{2}, 1\right\} = \frac{7}{2} = |a_{22}|$

$$\Rightarrow \begin{cases} 2x_1 - x_2 + 2x_3 = 3 \\ \frac{7}{2}x_2 = \frac{7}{2} \\ 2x_3 = 2 \end{cases} \Rightarrow \begin{cases} x_1 = \frac{3-2+1}{2} = 1 \\ x_2 = 1 \\ x_3 = 1 \end{cases}$$

2) Pivotare totală:

Pasul 1: transf. col. 1 în forma sup. Δ

$$\max\{|a_{ij}|,\ \forall i,j\} = 3 = |a_{22}|$$

ec. 2 $\leftrightarrow$ ec. 1

col. 2 $\leftrightarrow$ col. 1

Schimb. linie:

$$\begin{cases} x_1 + 3x_2 + x_3 = 5 \\ x_2 + 2x_3 = 3 \\ 2x_1 - x_2 + 2x_3 = 3 \end{cases}$$

Schimb. coloană:

$$\begin{cases} 3x_2 + x_1 + x_3 = 5 \\ x_2 + \quad + 2x_3 = 3 \\ -x_2 + 2x_1 + 2x_3 = 3 \end{cases} \quad \Big/ \left(-\frac{1}{3}\right) + ec.2 \Big/ \cdot \frac{1}{3} + ec.3$$

$$\Rightarrow \begin{cases} 3x_2 + x_1 + x_3 = 5 \\ -\frac{1}{3}x_1 + \frac{5}{3}x_3 = \frac{4}{3} \\ \frac{7}{3}x_1 + \frac{7}{3}x_3 = \frac{14}{3} \end{cases} \qquad \cancel{3}/3 + \frac{5}{3} = \frac{9+5}{3}$$

Pasul 2: transf. col. 2 în f. sup. Δ

$$\max\{|a_{22}|, |a_{23}|, |a_{32}|, |a_{33}|\} = \left\{\frac{1}{3}, \frac{5}{3}, \frac{7}{3}, \frac{7}{3}\right\} = \frac{7}{3} = a_{33}$$

linia 2 $\leftrightarrow$ linia 3

col. 2 $\leftrightarrow$ col. 3

$$\begin{cases} 3x_2 + x_3 + x_1 = 5 \\ \frac{7}{3}x_3 + \frac{7}{3}x_1 = \frac{14}{3} \quad \Big/ \left(-\frac{5}{7}\right) + ec.3 \\ \frac{5}{3}x_3 - \frac{1}{3}x_1 = \frac{4}{3} \end{cases}$$

$$\Rightarrow \begin{cases} 3x_2 + x_3 + x_1 = 5 \\ \frac{7}{3}x_3 + \frac{7}{3}x_1 = \frac{14}{3} \\ -2x_1 = -2 \end{cases} \Leftrightarrow \begin{cases} x_2 = \dfrac{5-1-1}{3} = \dfrac{3}{3} = 1 \\ x_3 = \dfrac{\frac{14}{3} - \frac{7}{3}}{\frac{7}{3}} = \frac{7}{3} \cdot \frac{3}{7} = 1 \\ x_1 = 1 \end{cases}$$

$$\begin{cases} x_1 + x_2 + x_3 = 2 \\ 2x_1 + 2x_2 + 2x_3 = 4 \\ -x_2 + x_3 = -1 \end{cases} \Rightarrow \begin{cases} 2x_1 + 2x_2 + 2x_3 = 4 & / \cdot \left(-\frac{1}{2}\right) + ec_2 \\ x_1 + x_2 + x_3 = 2 \\ -x_2 + x_3 = -1 \end{cases}$$

$$\Rightarrow \begin{cases} 2x_1 + 2x_2 + 2x_3 = 4 \\ 0x_2 + 0x_3 = 0 \\ -x_2 + x_3 = -1 \end{cases} \Rightarrow \begin{cases} 2x_1 + 2x_2 + 2x_3 = 4 \\ -x_2 + x_3 = -1 \\ \boxed{0 \cdot x_3 = 0} \end{cases}$$

<u>Metoda chinezească</u> → la fel ca Gauss, linii/e pe coloane (linia 1 → col. 3, linia 2 → col. 2, linia 3 → col. 1)

<u>Descompuneri LU</u> → scrierea unei matrici ca produs dintre o mat. inf. Δ și o mat. sup. Δ

$$\begin{pmatrix} 1 & 2 & 3 \\ 5 & 1 & 2 \\ 2 & 1 & 7 \end{pmatrix} = \begin{pmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{pmatrix} \times \begin{pmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{pmatrix}$$

$$6 + 6 = \underline{\underline{12 \text{ nec.}}}$$

<u>Irelații</u>

$Ly = b$; $y = ?$ / l_{ii}

$Ux = y \Rightarrow x = ?$ / u_{ii}

LDL^T?

$$\left.\begin{array}{l} L \cdot U \text{ cu } l_{ii} = 1 \quad \forall i = \overline{1,n} \\ ? \text{ cu } u_{ii} = 1 \quad \forall i = \overline{1,n} \end{array}\right\} \text{fac } n \text{ împărțiri mai puține}$$

$\boxed{\text{det. tuturor minorilor} \neq 0}$, altfel nu pot face desc. LU

$l_{ij} = 1$ → 4 alg. de determinare

Luni, 20 Martie 2017

CN

Cursul 5

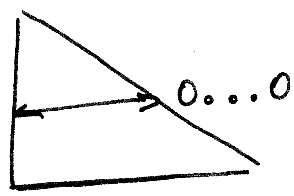
$A = L \cdot U$ → L : $l_{ii} = 1$

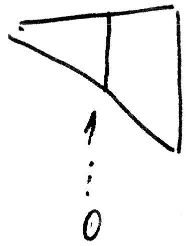
sau

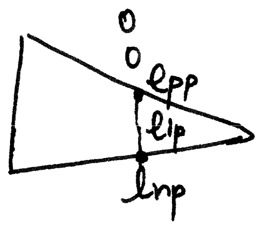
→ U : $u_{ii} = 1$

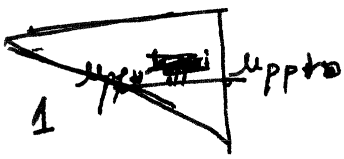
$\det A \neq 0 \not\Rightarrow \exists L * U = A$

A_p $p = 1$; $a_{11} = 0$ $\nexists LU$

L: 

U: 

L: 



l_{ip} • $a_{ip} = (LU)_{ip}$

$u_{pi} = a_{pi} = (LU)_{pi}$

$\det A_p \neq 0$; $p = \overline{1,n}$

$p = 1$; $\det A_1 = a_{11}$

.. .. — — —

$p = n$; $\det A_n = \det A$ $(A_n = A)$

$\det(A) \neq 0$

$A = L \cdot U$

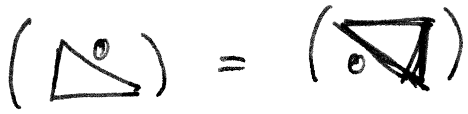
$U - u_{ii} = 1$ $\det U = 1$

$\det A \neq 0 \Rightarrow \det L \neq 0$

T_1, T_2 – matr. triunghiulare sup/inferior

$\Rightarrow \boxed{T_1 * T_2 \ \text{– matr. triungh. sup./inf.}}$

T sup/inf. Δ ; $\det T \neq 0 \Rightarrow \boxed{\exists\, T^{-1} \ \text{matr. sup./inf. } \Delta}$

 $\Rightarrow$ matr. diagonală

T matr. Δ; $\det T \neq 0$; $T^{-1} = w_{ij}$; $w_{ii} = \dfrac{1}{t_{ii}}$

$\|$

t_{ij}

$$A = \begin{pmatrix} 2 & 0 & 2 \\ 1 & 2 & 5 \\ 1 & 1 & 7 \end{pmatrix} = \begin{pmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{pmatrix} \cdot \begin{pmatrix} 1 & u_{12} & u_{13} \\ 0 & 1 & u_{23} \\ 0 & 0 & 1 \end{pmatrix} =$$

$$= \begin{pmatrix} l_{11} & l_{11}u_{12} & l_{11}u_{13} \\ l_{21} & l_{21}u_{12} + l_{22} & l_{21}u_{13} + l_{22}u_{23} \\ l_{31} & l_{31}u_{12} + l_{32} & l_{31}u_{13} + l_{32}u_{23} + l_{33} \end{pmatrix}$$

Pas 1: col. 1L: l_{11}, l_{21}, l_{31}

lin. 1U: u_{12}, u_{13}

Pas 2: col. 2L: l_{22}, l_{32}

col. 2U: u_{23}

Pas 3: col. 3L: l_{33}

lin 3U: —

l_{11} $\quad a_{11} = (LU)_{11}$; $\boxed{2 = l_{11}}$

l_{21} $\quad a_{21} = (LU)_{21}$; $\boxed{1 = l_{21}}$

Pasul 1: l_{31} $\quad a_{31} = (LU)_{31}$; $\boxed{1 = l_{31}}$

u_{12} $\quad a_{12} = (LU)_{12}$; $0 = l_{11}u_{12}$; $\boxed{u_{12} = 0}$

u_{13} $\quad a_{13} = (LU)_{13}$; $2 = l_{11}u_{13}$; $\boxed{u_{13} = 1}$

l_{22} : $\quad 2 = l_{21}u_{12} + l_{22} \Rightarrow \boxed{l_{22} = 2}$

Pasul 2: l_{32} : $\quad 1 = l_{31}u_{12} + l_{32} \Rightarrow \boxed{l_{32} = 1}$

$u_{23}: l_{21} u_{13} + l_{22} u_{23} = 5 \Rightarrow \boxed{u_{23} = 2}$

Pasul 3: $l_{33}: 7 = l_{31} u_{13} + l_{32} u_{23} + l_{33} \Rightarrow \boxed{l_{33} = 4}$

$L \to$ linii
$U \to$ linii

Pasul 1: lin. 1 L, U
l_{11}, u_{12}, u_{13}

Pasul 2: l_{21}, l_{22}, u_{23}

Pasul 3: l_{31}, l_{32}, l_{33}.

Pasul 1: $l_{11} = 1$
$u_{12} = 0$
$u_{13} = 2$

Pasul 2: $l_{21} = 1$
$l_{22} = 2 - 1 \cdot 0 = 2$
~~$u_{23} = 5 - \frac{0 \cdot 3}{2}$~~ $\quad u_{23} = 5 - 1 \cdot \cdot$

$$\begin{pmatrix} 2 & 0 & 2 \\ 1 & 2 & 5 \\ 1 & 1 & 7 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{pmatrix} \begin{pmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{pmatrix}$$

$$= \begin{pmatrix} u_{11} & u_{12} & u_{13} \\ l_{21} u_{11} & l_{21} u_{12} + u_{22} & l_{21} u_{13} + l_{21} u_{23} \\ l_{31} u_{11} & l_{31} u_{12} + l_{32} u_{22} & l_{31} u_{13} + l_{32} \cdot u_{23} + u_{33} \end{pmatrix}$$

col. U :
lin. L

Pas 1: lin. 1L: —
col. 1U: u_{11} ; $\boxed{u_{11} = 2}$

$l_{21} \cdot u_{11} = 1 \Rightarrow \boxed{l_{21} = \frac{1}{2}}$

Pas 2: lin. 2L: l_{21} ;
col. 2U: u_{12}, u_{22}

$\boxed{u_{12} = 0}$

$l_{21} u_{12} + u_{22} = 2 \Rightarrow$

$\Rightarrow \frac{1}{2} \cdot 0 + u_{22} = 2 \Rightarrow$

$\Rightarrow \boxed{u_{22} = 2}$

Pas 3: lin. 3L: l_{31}, l_{32}
col. 3U: u_{13}, u_{23}, u_{33}

Pasul 3: $\cancel{l_{31}} \cdot u_{11} = 1 \Rightarrow \boxed{l_{31} = \frac{1}{2}}$

$l_{31} \cdot u_{12} + l_{32} u_{22} = 1 \Rightarrow$

$\Rightarrow \frac{1}{2} \cdot 0 + l_{32} \cdot 2 = 1 \Rightarrow \boxed{l_{32} = \frac{1}{2}}$

$l_{31} u_{13} + l_{32} u_{23} + u_{33} = 7 \Rightarrow$

$= \frac{1}{2}.$

Desc. Choleski

$A = A^T$

$a_{ij} = a_{ji} ; \forall i, j$

$A > 0$ (pozitiv definită) $\Leftrightarrow$ $\underline{\det A_p > 0}$; $\forall p = \overline{1,n}$

sau

$\Leftrightarrow$ toate val. proprii sunt $\lambda_i > 0$

$P_A(\lambda) = \det(\lambda I - A) = 0$

semi-pozitiv definită $\Leftrightarrow$ $\underline{\det A_p \geq 0}$; $p = \overline{1,n}$

$$A = \begin{bmatrix} 1 & -1 & 0 \\ 2 & 1 & 0 \\ -4 & 0 & 4 \end{bmatrix}$$

$$x = e_j = \begin{pmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{pmatrix} \rightarrow \text{poz } j$$

$$A \cdot e_j = \text{col. } j = \begin{pmatrix} a_{1j} \\ a_{2j} \\ \vdots \\ a_{jj} \\ \vdots \\ a_{nj} \end{pmatrix}, \begin{pmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{pmatrix} \Rightarrow a_{jj} > 0$$

$$A = \begin{bmatrix} 4 & -1 & 1 \\ -1 & 1 & 0 \\ 1 & 0 & 2 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

$$\Rightarrow A x = \begin{pmatrix} 4x_1 - x_2 + x_3 \\ -x_1 + x_2 \\ x_1 - 2x_3 \end{pmatrix} \begin{matrix} \cdot x_1 \\ \cdot x_2 \\ \cdot x_3 \end{matrix}$$

$\rightarrow$ Produs scalar (Ax, x)

$(Ax, x) = 4x_1^2 - x_1x_2 + x_1x_3 - x_1x_2 + x_2^2 + x_1x_3 - 2x_3^2$

$A > 0 \Leftrightarrow \det A_p > 0 \text{ , } \forall p)$

$$(x_1-x_2)^2 + (x_1+x_3)^2 + 2x_1^2 + x_3^2 \geq 0$$

$$= 0 : \begin{cases} x_1 - x_2 = 0 \\ x_1 + x_3 = 0 \\ x_1 = 0 \\ x_3 = 0 \end{cases} \Rightarrow X = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$A = L * U$

$A^T = U^T * L^T$

$A = A^T$

Pt matr. care <u>nu sunt pozitiv definite</u> se poate calcula o desc. în <u>numere complexe</u>!

$$\begin{cases} 2x_1 + x_2 + x_3 = 4 & / \cdot (-3) + ec. 2 \; / \cdot (-1) + ec. 3 \\ 6x_1 + 7x_2 + 4x_3 = 17 \\ 2x_1 + 9x_2 + 5x_3 = 16 \end{cases}$$

$$\Rightarrow \begin{cases} 2x_1 + x_2 + x_3 = 4 \\ 4x_2 + x_3 = 5 & / \cdot (-2) + ec. 3 \\ 8x_2 + 4x_3 = 12 \end{cases} \Rightarrow \begin{cases} 2x_1 + x_2 + x_3 = 4 \\ 4x_2 + x_3 = 5 \\ 2x_3 = 2 \end{cases}$$

$\underline{\underline{U}}$

$\underline{\underline{L = ?}}$

$$\overset{\text{inf. }\Delta}{T_2} \cdot \underbrace{\overset{\text{inf. }\Delta}{T_1} \cdot A}_{} = U$$

m cu col. 1 inf/sup Δ

$$T_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{pmatrix}$$

$$T_1 = \begin{pmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 2 & 1 & 1 \\ 6 & 7 & 4 \\ 2 & 9 & 5 \end{pmatrix} = \begin{pmatrix} 2 & 1 & 1 \\ 0 & 4 & 1 \\ 0 & 8 & 4 \end{pmatrix}$$

$$* = \begin{pmatrix} 2 & 1 & 1 \\ 0 & 4 & 1 \\ 0 & 0 & 2 \end{pmatrix}$$

$$T_1^{-1} T_2^{-1} / \underbrace{T_2 T_1 A} = U$$

$$A = \underbrace{(T_2^{-1} T_1^{-1})}_{L} \cdot U$$

$$A = L \cdot U \qquad \boxed{L = T_2^{-1} \cdot T_1^{-1}}$$

$$T_1^{-1} \cdot T_2^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 2 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 1 & 2 & 1 \end{pmatrix}$$

$$\begin{cases} 2x_1 + \quad + 2x_3 = 4 \quad / \cdot (-\frac{1}{2}) + ec.2 \;/\; (-\frac{1}{2}) + ec.3 \\ x_1 + 2x_2 + 5x_3 = 8 \\ x_1 + x_2 + 7x_3 = 9 \end{cases}$$

$$\begin{cases} 2x_1 + \quad + 2x_3 = 4 \\ 2x_2 + 4x_3 = 6 \quad / \cdot (-\frac{1}{2}) + ec_3 \\ x_2 + 6x_3 = 7 \end{cases}$$

$$\Rightarrow L = \begin{pmatrix} 1 & 0 & 0 \\ \frac{1}{2} & 1 & 0 \\ 0{,}5 & 0{,}5 & 1 \end{pmatrix}$$

$$\Rightarrow \begin{cases} 2x_1 + \ldots + 2x_3 = 4 \\ 2x_2 + 4x_3 = 6 \\ 4x_3 = 4 \end{cases} \Rightarrow U = \begin{pmatrix} 2 & 0 & 2 \\ 0 & 2 & 4 \\ 0 & 0 & 4 \end{pmatrix}$$

$\dot{I}_{ij}$ = matr. $\dot{I}$ în care lin. $i \leftrightarrow$ lin. j
col. $i \leftrightarrow$ col. j $\rightarrow$ $\boxed{\text{matr. de transpoziție}}$

$$\dot{I}_{2,3} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

$\dot{I}_{i,j} * A$ = interschimb lin. i din A cu lin. j

$A * \dot{I}_{i,j}$ = interschimb col. i cu col. j

$\cancel{I_{3i_3} T_2 \cdot T_1 \cdot \dot{I}_{1i_1} A}$

$$T_3 \cdot \dot{I}_{3i_3} \cdot T_2 \cdot \dot{I}_{2i_2} \cdot T_1 \cdot \dot{I}_{1i_1} \cdot A = U \qquad (A_{4\times4})$$

$$(\dot{I}_{2i_2} \cdot T_1 \cdot \dot{I}_{2i_2}) \cdot \dot{I}_{2i_2} \cdot \dot{I}_{1i_1} A$$

$i_2 = 3 \Rightarrow \underbrace{\underbrace{I_{23} T_1} I_{23}}_{T_1'} = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 3 & 1 & 0 \end{pmatrix} \cdot I_{23} = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 3 & 0 & 1 \end{pmatrix}$

$$T_3 \dot{I}_{3 i_3} \overbrace{T_2 \cdot T_1}^{T} \dot{I}_{2 i_2} \dot{I}_{1 i_1} A = U$$

$$T_3 \underbrace{\dot{I}_{3 i_3} T \dot{I}_{3 i_3} I_{3 i_3}}_{\tilde{T}} T_{2 i_2} \dot{I}_{1 i_1}$$

$$\tilde{T} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ a & 1 & 0 & 0 \\ b & d & 1 & 0 \\ c & e & 0 & 1 \end{pmatrix}$$

$$\underbrace{T_3 \tilde{T}}_{=L^{-1}} \underbrace{\dot{I}_{3 i_3} \dot{I}_{2 i_2} \dot{I}_{1 i_1} A}_{\text{permutare a lui } A} = U$$

$$\begin{cases} 2x_1 + x_2 + x_3 = 4 \\ 6x_1 + 7x_2 + 4x_3 = 17 \\ 2x_1 + 9x_2 + 5x_3 = 16 \end{cases}$$

$$\begin{cases} 6x_1 + 7x_2 + 4x_3 = 17 \quad \Big| \left(-\frac{1}{3}\right) + ec \cdot 2 \Big| ec_3 \\ 2x_1 + x_2 + x_3 = 4 \\ 2x_1 + 9x_2 + 5x_3 = 16 \end{cases}$$

$$\Rightarrow \begin{cases} 6x_1 + 7x_2 + 4x_3 = 17 \\ -\frac{4}{3}x_2 - \frac{1}{3}x_3 = -\frac{5}{3} \\ \frac{20}{3}x_2 + \frac{11}{3}x_3 = \frac{31}{3} \end{cases} \Rightarrow \begin{cases} 6x_1 + 7x_2 + 4x_3 = 17 \\ \frac{20}{3}x_2 + \frac{11}{3}x_3 = \frac{31}{3} \\ -\frac{4}{3}x_2 - \frac{1}{2}x_3 = -\frac{5}{3} \end{cases}$$

pt mat. aste

$$\Rightarrow \begin{cases} 6x_1 + 7x_2 + 4x_3 = 17 \\ \frac{20}{3}x_2 + \frac{11}{3}x_3 = \frac{31}{3} \\ \frac{2}{5}x_3 = \frac{2}{5} \end{cases}$$

$$\frac{31}{15} - \frac{5}{3} =$$

$$L = \begin{pmatrix} 1 & 0 & 0 \\ 1/3 & 1 & 0 \\ 1/3 & -4/20 & 1 \end{pmatrix}$$

Luni, 27 Martie 2017

CN
Cursul 6

= Descompuneri QR =

A bine condiționată $\xrightarrow{\text{desc. LU}}$ L/U prost condiționată

(Erori mici intrare $\rightsquigarrow$ erori mari ieșire)

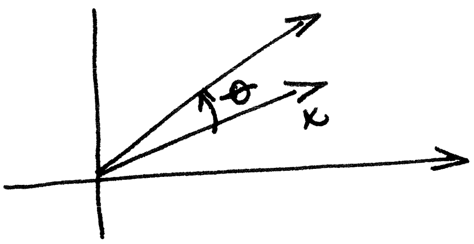
$A = Q * R$ → sup. triunghiulară

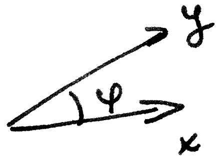
↓ matr. ortogonală $(Q^{-1} = Q^T)$

$\|Qx\|_2 = \|x\|_2$; $x \to$ vector

Rx ; $R = \begin{bmatrix} \cos\theta & \sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$

↓ matricea de rotație





$$\cos\varphi = \frac{(x, y)}{\|x\|_2 \|y\|_2}$$

↓ măsură de similaritate între vectori
$(\varphi \approx 0 \Rightarrow \cos\varphi \approx 1)$

$(x, y) \rightsquigarrow \varphi$

$(Qx, Qy) = (x, \underbrace{Q^T Q}_{I} y) = (x, y)$

$(Ax, y) = (x, A^T y)$

$$\cos\varphi(x, y) = \frac{(x, y)}{\|x\|_2 \|y\|_2}$$

$$\cos\varphi(Qx, Qy) = \frac{(Qx, Qy)}{\|Qx\|_2 \|Qy\|_2} = \frac{(x, y)}{\|x\|_2 \|y\|_2} = \cos\varphi(x,y$$

$Q \cdot Q^T = Q^T \cdot Q = I$

$Q = [q^1, q^2 \dots q^n]$

$Q^T \cdot Q = I \quad (q^i, q^j) = \begin{cases} 1; & i=j \\ 0, & i \neq j \end{cases}$

$(q^i, q^i) = \|q^i\|_2^2 = 1$

$(q^i, q^j) = 0 \; ; \; i \neq j$

Produs scalar $= 0 \Rightarrow$ ortogonalitate $\rightarrow$ bază ortonormată ($\mathbb{R}^n$)

$e_1 = \begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{matrix}$

$Ax = b$

$A = [a^1, a^2 \dots a^n]$

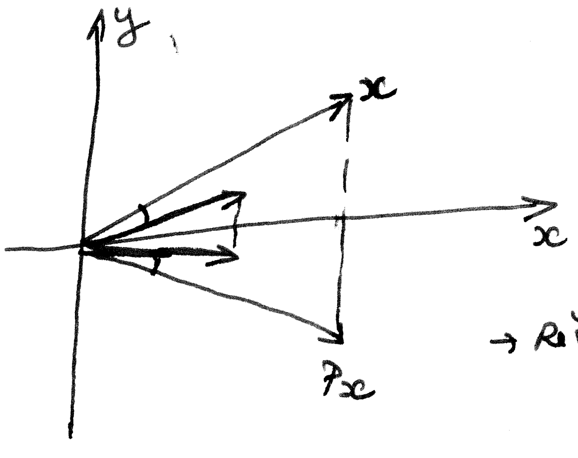
$Ax = b \Leftrightarrow a^1 x_1 + a^2 x_2 + \dots + a^n x_n = b$

$R(A) = \underset{\downarrow}{\text{spam}} [a^1, a^2, \dots, a^n] = \{ c_1 a^1 + c_2 a^2 + \dots + c_n a^n ; \; c_i \in \mathbb{R} \}$

spațiu generat

$Ax = b \Rightarrow b \overset{?}{\in} R(A$

$A = QR \Rightarrow Q \; q^1 \dots q^n \rightarrow$ bază ortonormată pt $R(A)$



$\underbrace{v}_{n \times 1} \cdot \underbrace{v^T}_{1 \times n}$

$\rightarrow$ Reflectatul vectorului

$v \begin{pmatrix} \frac{3}{5} \\ \frac{4}{5} \end{pmatrix} \qquad e = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = v \quad ; \; P = I_3 - 2v \cdot v^T$

$$v \cdot v^T = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\Rightarrow P = I_3 - 2vv^T = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} - \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 2 \end{pmatrix} =$$

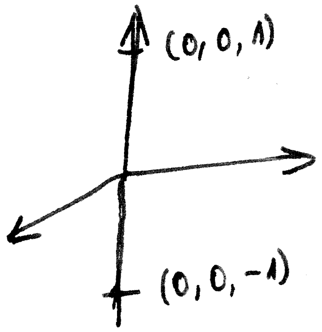
$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

$$v \cdot v^T = \begin{pmatrix} \frac{3}{5} \\ \frac{4}{5} \end{pmatrix} \cdot \begin{pmatrix} \frac{3}{5} & \frac{4}{5} \end{pmatrix} = \frac{1}{25} \begin{pmatrix} 9 & 12 \\ 12 & 16 \end{pmatrix}$$

$$I_2 - 2vv^T = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} \frac{18}{25} & \frac{24}{25} \\ \frac{24}{25} & \frac{32}{25} \end{pmatrix} = \begin{pmatrix} \frac{7}{25} & -\frac{24}{25} \\ \frac{24}{25} & \frac{-7}{25} \end{pmatrix}$$

<u>Matricea de reflexie</u> = matr. <u>simetrică</u> + <u>ortogonală</u>

$$x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \qquad P(x) = \begin{pmatrix} x_1 \\ x_2 \\ -x_3 \end{pmatrix}$$



$$P_{n-1} \dots P_2 \, P_1 \, A = R$$

$$\underbrace{P_{n-1} \dots P_2 \, P_1}_{Q} \, I = Q^T$$

$$P = I - 2vv^T$$

$$P = \begin{pmatrix} 1 & 0 \\ 0 & \boxed{\neq 0} \end{pmatrix}$$

$u = (2\alpha)\, v$

$v = \frac{1}{2\alpha}\, u$

$\beta = 0 \Rightarrow$ matrice singulară
nu oprește calculele

$\frac{1}{\beta} \rightarrow$ β cât mai mare

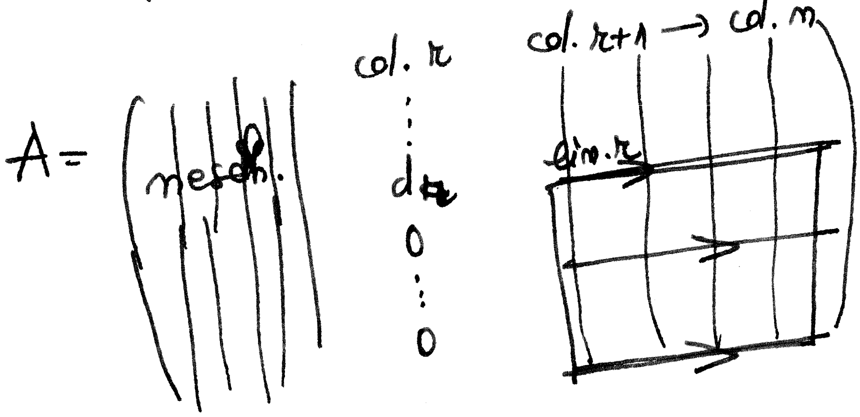
$A = R_r A$ → col. r nu este în forma sup. Δ

↓

col. r în f. sup. Δ

$$u = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ a_{rr} - r \\ a_{r+1\,r} \\ \vdots \\ a_{nr} \end{pmatrix} \qquad A e_1 = \begin{pmatrix} a_{11} \\ 0 \\ \vdots \\ a_{kk} \\ 0 \end{pmatrix} \qquad \nu_1 = (A e_i, u), \quad i = \overline{1, k-1}$$

$A e_1 - c \cdot u$



Pasul 1:

A → k →
0 →
⋮ →
0 →

Pasul 2:

Mat. de rotație:

$$R = \begin{bmatrix} c & -s \\ s & c \end{bmatrix} \to \mathbb{R}^2$$

$\mathbb{R}^n$: În I_n care înlocuiește elem. de pe lin. și col. p și q cu $\begin{pmatrix} c & \cdots & -s \\ \cdots & & \cdots \\ (-s) & \cdots & (c) \end{pmatrix}$

$$R_{12}(\theta) = \begin{bmatrix} c & s & 0 \\ -s & c & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$c^2+s^2=1$ ($\exists\ \theta$ a.î. $c = \cos\theta$, $s = \sin\theta$)

$$R_{13}(\theta) = \begin{pmatrix} c & 0 & s \\ 0 & 1 & 0 \\ -s & 0 & c \end{pmatrix}$$

$$R_{23}(\theta) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & c & s \\ 0 & -s & c \end{pmatrix}$$

$R_{pq}(\theta) \cdot A \to$ nu se schimbă f. mult.

→ Alg. lui Givens cu mat. de rotație

$$\begin{pmatrix} 3 & 1 & 3 \\ 2 & 1 & 0 \\ 1 & 1 & 5 \end{pmatrix}$$

$a_{ir} = 0$

$R_{ki}(\theta)$

$b_{ir} = -s a_{kr} + c a_{ir}$

$c = a_{kr}$; $s = a_{ir}$

$c^2 + s^2 = 1$?

} → normalizare:

$c = a_{kr} \cdot f$

$s = a_{ir} \cdot f$

$$a_{kr}^2 f^2 + a_{ir}^2 f^2 = 1$$

Miercuri, 5 Aprilie 2017

CN

Cursul 7

$Ax = b$

$A = Q \times R$

$\downarrow$

- proiecții pe spații / subspații
- sisteme nepătratice

Householder $\rightarrow$ reflex

$P_{n-1} \dots P_2 \underbrace{P_1 \cdot A}$

transf. col. 1 în formă sup. Δ

transf. col. 2 —"—

Givens : mai costisitor decât Householder

$\rightarrow$ util pt matrici rare

Matrici de rotație

$$R = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

$c = \cos\theta$

$s = \sin\theta$ $\Leftrightarrow$ $c^2 + s^2 = 1$

$\downarrow$

le pot considera 2 const. c.î. cu această proprietate

$p \neq q$

$n = 3$:

(p, q) : $(1, 2)$, $(1, 3)$, $(2, 3)$

$$R_{12}(\theta) = \begin{bmatrix} c & s & 0 \\ -s & c & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$R_{23}(\theta) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & c & s \\ 0 & -s & c \end{bmatrix}$$

$$R_{13}(\theta) = \begin{bmatrix} c & 0 & s \\ 0 & 1 & 0 \\ -s & 0 & c \end{bmatrix}$$

$a_{i,k} \rightsquigarrow 0; \quad k = \overline{1, n-1}$
$\qquad\qquad\qquad\qquad i = \overline{k+1, n}$

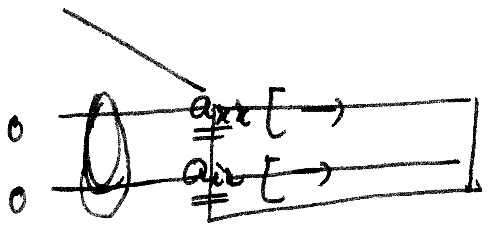
$b_{ik} = -s\, a_{kk} + c\, a_{ik} = 0$

$s = a_{ik} * f$
$c = a_{kk} * f$
$c^2 + s^2 = 1$?

$f = ? \qquad c^2 + s^2 = 1$

$$f = \frac{1}{\sqrt{\underbrace{a_{ik}^2 + a_{kk}^2}_{\text{nu are sens să facem calculele (1/0!)}}}}$$

$a_{ik} = 0$, $a_{kk} = 0$



$$A = \begin{bmatrix} 3 & 1 & 0 \\ 4 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

Pasul 1: $\underbrace{R_{23} R_{13} R_{12}}_{Q^T} \cdot A = R$

Pasul 2:

$$\begin{pmatrix} c & s & 0 \\ -s & c & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 3 & 1 & 0 \\ 4 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 3c+4s & c & s \\ -3s+4c = 0 & -s & c \\ 0 & 1 & 1 \end{pmatrix}$$

$c, s = ?$ a.î. $-3s + 4c = 0$
$c^2 + s^2 = 1$

$s = 4f$
$c = 3f$
$c^2 + s^2 = 1 \Rightarrow f = \frac{1}{5}$

$16f^2 + 9f^2 = 1$
$\Rightarrow 25 f^2 = 1$
$\Rightarrow f = \pm\frac{1}{5}$

$$\Rightarrow \boxed{\begin{aligned} s &= \tfrac{4}{5} \\ c &= \tfrac{3}{5} \end{aligned}}$$

$$R_{12} \cdot A = \begin{pmatrix} 5 & \frac{3}{5} & \frac{4}{5} \\ 0 & -\frac{4}{5} & \frac{3}{5} \\ 0 & 1 & 1 \end{pmatrix}$$

$\downarrow \frac{a_{32}}{0}$ (pasul 2)

$$R_{23} \cdot A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & c & s \\ 0 & -s & c \end{pmatrix} \begin{pmatrix} 5 & \frac{3}{5} & \frac{4}{5} \\ 0 & -\frac{4}{5} & \frac{3}{5} \\ 0 & 1 & 1 \end{pmatrix} =$$

$\frac{4}{5} c + s = 0$

$$= \begin{pmatrix} 5 & 3/5 & 4/5 \\ 0 & -4/5\,c + s & 3/5\,c + s \\ 0 & \underset{\substack{\| \\ 0}}{4/5\,c + s} & -3/5\,c + s \end{pmatrix} = R$$

$s = -1$
$c = \frac{4}{5}$ sau $\boxed{\begin{array}{l} s = 1 \\ c = -\frac{4}{5} \end{array}}$

$s = f$
$c = -\frac{4f}{5}$ $\Rightarrow f = \frac{1}{\sqrt{1 + \frac{16}{25}}} = \frac{1}{\sqrt{\frac{41}{25}}} = \sqrt{\frac{25}{41}} = \frac{5}{\sqrt{41}}$

Householder:

$P_1 = I - \frac{1}{\beta} u u^T$

$\beta = \sigma - k * a_{11}$

$u = \begin{pmatrix} a_{11} \\ a_{21} \\ a_{31} \end{pmatrix}$

$\sigma = a_{11}^2 + a_{21}^2 + a_{31}^2$

$k = \pm\sqrt{k} = -\text{semn}(a_{11})\sqrt{\sigma}$

$\sigma = 3^2 + 4^2 + 0^2 = 25$

$k = -5$

$\beta = 25 - (-5) * 3 = 40$

$u = \begin{pmatrix} 3-(-5) = 8 \\ 4 \\ 0 \end{pmatrix}$; $u u^T = \begin{pmatrix} 8 \\ 4 \\ 0 \end{pmatrix} (8 \;\; 4 \;\; 0) =$

$= \begin{pmatrix} 64 & 32 & 0 \\ 32 & 16 & 0 \\ \dots \end{pmatrix}$

$$P_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} - \frac{1}{40} u u^T = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} - \begin{pmatrix} \frac{64}{40} & \frac{32}{40} & 0 \\ \frac{32}{40} & \frac{16}{40} & 0 \\ 0 & 0 & 0 \end{pmatrix} =$$

$$= \begin{pmatrix} -\frac{3}{5} & -\frac{4}{5} & 0 \\ -\frac{4}{5} & \frac{3}{5} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$P_1 \cdot A = \begin{pmatrix} -5 & -\frac{3}{5} & -4/5 \\ 0 & -4/5 & 3/5 \\ 0 & 1 & 1 \end{pmatrix}$$

$P_2 \cdot A$:

$$P_2 = I - \frac{1}{\beta} u u^T$$

$$\sigma = a_{22}^2 + a_{32}^2 = \left(-\frac{4}{5}\right)^2 + 1^2 = \frac{16}{25} + 1 = \frac{41}{25}$$

$$k_2 = -\text{semn}(a_{22}) \sqrt{\sigma} = \sqrt{\sigma} = \frac{\sqrt{41}}{5}$$

$$\beta = \sigma - k a_{22} = \frac{41}{25} - \frac{\sqrt{41}}{5} \cdot \left(-\frac{5}{4}\right) = \frac{41}{25} - \frac{\sqrt{41}}{4}$$

$$u = \begin{pmatrix} 0 \\ a_{22} - k = -\frac{4}{5} - \sqrt{\sigma} \\ 1 \end{pmatrix}$$

= Gram Schmidt =

$a^1, a^2, \ldots, a^n$ vect. lin. dep. din $\mathbb{R}^n$

= bază în $\mathbb{R}^n$

o bază ortonormată $q^1, q^2, \ldots, q^n$

$(q^i, q^j) = 0 \ \forall\, i \neq j$

$\|q^i\|_2 = 1$

$$e_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \quad \begin{matrix} 0 \\ 1 \\ 0 \end{matrix} \quad \begin{matrix} 0 \\ 0 \\ 1 \end{matrix}$$

$a^1, a^2, \dots, a^n$ lin. indep $\Leftrightarrow$ $\det A \neq 0$

$A = [a^1 \; a^2 \; \dots \; a^n]$

$$\begin{cases} \|q^i\|_2 = 1 \\ (q^i, q^j) = 0; \; \forall i \neq j \end{cases}$$

$$\begin{cases} a_1 = r_{11} q^1 \\ a^2 = r_{12} q^1 + r_{22} q^2 \\ a^3 = r_{13} q^1 + r_{23} q^2 + r_{33} q^3 \end{cases}$$

Pasul 1: $q^1 = \frac{1}{r_{11}} a^1$

$(\det A \neq 0 \Rightarrow r_{11} \neq 0)$

$$\|q^1\|_2^2 = 1 = \frac{1}{r_{11}^2} \|a^1\|_2^2$$

$\Rightarrow r_{11} = \pm \|a^1\|_2$

$\Rightarrow r_{11} = \sqrt{3^2 + 4^2 + 0^2} = \sqrt{25} = 5$

$q^1 = \frac{1}{r_{11}} a_1 = (3/5 \quad 4/5 \quad 0)$

Pasul 2: $(a^2, q^1) = r_{12} \underbrace{(q^1, q^1)}_{\|q^1\|_2 = 1} + r_{22} \underbrace{(q^1, q^2)}_{=0}$

$$r_{12} = (a^2, q^1) = \left(\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 3/5 \\ 4/5 \\ 0 \end{pmatrix} \right) = \frac{3}{5}$$

$a^2 - r_{12} q^1 = b = r_{22} q^2$

$r_{22} = \|b\|_2$

$q_2 = \frac{1}{r_{22}} b$

$$b = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} - \frac{3}{5} \begin{pmatrix} 3/5 \\ 4/5 \\ 0 \end{pmatrix} = \begin{pmatrix} 16/25 \\ -12/25 \\ 1 \end{pmatrix}$$

$$r_{22} = \sqrt{\frac{156 + 144 + 625}{625}} = \sqrt{\frac{1025}{625}} = \sqrt{\frac{41}{25}} = \frac{\sqrt{41}}{5}$$

$= \sqrt{5}$

implementat sub forma unor variante modificate
(alg. original poate fi numeric instabil)

= Metode iterative =

B ușor inversabilă:
→ știu să calculez B^{-1} repede
→ $Bx = f$

$$Bx^* = Cx^* + b$$

$x^{(1)}$, $x^{(2)}$; $x^{(0)}$, $x^{(1)}$

$$x^{(k+1)} = B^{-1}(Cx^{(k)} + b)$$
$$Bx = Cx^{(k)} + b; \quad k = 0, 1, \ldots$$

(Sist. 4 din tema 4 este rezolvabil cu o altă metodă)

$$\|M\| < 1 \Rightarrow x^{(k)} \longrightarrow x^*$$

$$1 - \|M\| \simeq 0$$

$$\|M\| \simeq 1$$

Luni, 24 Aprilie 2017

CN
Cursul 8

$Ax = b$ det A

$A \to$ matr. rară

$x^* = A^{-1}b \Rightarrow$ sol. exactă

$x^{(k)} \to x^*$

$\{x^{(k)}\}$

$A = B - C$; B "ușor" inversabilă

det $B \neq 0$

$x^{(k+1)} = Mx^{(k)} + d$; $x^{(0)}$ det.

$M = B^{-1}C$ / $d = B^{-1}b$

$x^{(k)} \to x^* \Leftrightarrow \rho(M) < 1$

$\forall x^{(0)}$

$\exists \|M\| < 1 \Rightarrow x^{(k)} \to x^{(*)}$

$A * B \to A$ în ord. liniilor

B în ord. coloanelor

$A * x$: $y_i = a_{i1}x_1 + a_{i2}x_2 + \ldots + a_{in}x_n$

Memorare în raport cu diagonala:

diag$(5,1) = 3 = a_{ij} = a_{5\,5+(-2)} =$ (lin, col)

$= a_{53}$

$x^{(k+1)}$ este sol. sistemului $Bx = Cx^{(k)} + b$

Jacobi: $x_i^{(k+1)} = \left(b_i - \sum_{j=1}^{i-1} a_{ij} x_j^{(k)} - \sum_{j=i+1}^{n} a_{ij} \cdot x_j^{(k)}\right) / a_{ii}$

$i = \overline{1,n}$

cunosc: $x_1^{(k+1)}, x_2^{(k+1)}, \ldots, x_{i-1}^{(k+1)}$

informația actualizată

$x_1^{(k)}, x_2^{(k)}, \ldots$

ecu. sunt parcurse în ordine

$$\begin{cases} 3x_1 = 5 \\ 2x_1 + 2x_2 = 4 \\ x_1 + 3x_2 + x_3 = 7 \end{cases}$$

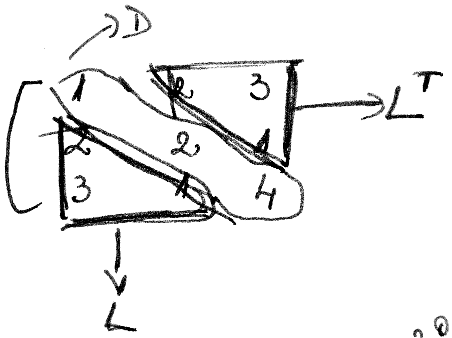
$x^{(k)} \to x^*$ → **Jacobi**

$\{x^{(k)}\}$ $\quad \|x^{J} - x^*\| \leq 10^{-7}$

$\|x^{GS} - x^*\| \leq 10^{-7}$ → mai puțini pași

↓

Gauss - Seidel



A poz. def. $\Leftrightarrow \det \begin{bmatrix} a_{11} & \dots & a_{1p} \\ \vdots & & \\ a_{p1} & \dots & a_{pp} \end{bmatrix} > 0$

$\forall p = \overline{1,n}$

$2^\circ \to$ toate val. p ale lui A $\quad a_{11} > 0$

$x_i > 0$ $\quad \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} > 0$

$$M = B^{-1}C = B^{-1}(B - A) = I - B^{-1}A$$

A simetrică ptr. care $P = B + B^T - A$ este poz. def.

metodele iterative sunt convergente $\Leftrightarrow$ matr. A este poz. definită

Metodele relaxării → problemă de minimizare

$x^* - y = e$ → eroarea

$f(y) = (Ae, e) > 0$; $e \neq 0$

$= 0$; $e = 0$

$\min f(y) = 0 = f(x^*)$

$y^{(k)} \to y^{(k+1)}$

$\begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix} \quad \begin{pmatrix} z_1 \\ z_2 \\ \vdots \\ z_n \end{pmatrix}$

$z_i = y_i$; $\forall i \neq l$

$z_l = y_l + c_k$

comp. pe care o schimb

Alternativa: metoda relaxării succesive

$$a_{ll}\underbrace{c_k^2}_{x^2} - 2\, r_l^{(k)} c_k < 0$$

$$x = 0$$
$$x = \frac{2 r_l^{(k)}}{a_{ll}}$$

$$ax^2 + bx + c < cy^{(k)} \rightsquigarrow y^{(k+1)}$$

$$a > 0 \qquad x \in [x_1, x_2]$$

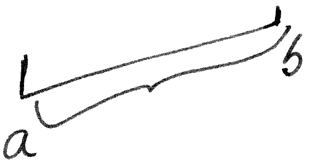
$$e_l = \begin{pmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{pmatrix} \to \text{poziția } l$$

$$x_i^{(k+1)} = (1-w)\, x_i^{(k)} + w \underbrace{\left(b_i - \sum_{j=1}^{i-1} a_{ij} x_j^{(k+1)} - \sum_{j=i+1}^{n} a_{ij} x_j^{(k)}\right) \Big/ a_{ii}}_{x_i^{(k+1)}\ GS} \qquad i = \overline{1, n}$$

$$\downarrow$$

$$\underline{(1-w)a + wb}$$

$$bA = B - C$$



$$Bx^{(k+1)} = Cx^{(k)} + b$$

$$\frac{a_{ii}}{w} x_i^{(k+1)} + \sum_{j=1}^{i-1} a_{ij} x_j^{(k+1)} = \frac{1-w}{w} a_{ii} x_i^{(k)} - \sum_{j=i+1}^{n} a_{ij} x_j^{(k)}$$

$$\frac{1}{w} D x^{(k+1)} L x^{(k+1)} = \frac{1-w}{w} D x^{k} - L^T x^{(k)} + b$$

$$B = \frac{1}{w} D + L \ ; \quad C = \frac{1-w}{w} D - L^T$$

$$\begin{cases} 2x_1 + \quad + x_3 = 3 \\ x_1 + x_2 + \ \ldots = 2 \\ \quad x_2 + 2x_3 = 3 \end{cases} \Rightarrow A = \begin{pmatrix} 2 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 2 \end{pmatrix}$$

Jacobi: $B = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$; $B^{-1} = \begin{pmatrix} 1/2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1/2 \end{pmatrix}$; $C = \begin{pmatrix} 0 & 0 & -1 \\ -1 & 0 & 0 \\ 0 & -1 & 0 \end{pmatrix}$

$$M = B^{-1}C = \begin{pmatrix} 0 & 0 & -1/2 \\ -1 & 0 & 0 \\ 0 & -1/2 & 0 \end{pmatrix}$$

$$\exists \|M\| < 1 \Rightarrow x^{(k)} \to x^*$$

$$\det(\lambda \mathring{I} - M) = 0 \qquad \begin{vmatrix} \lambda & 0 & 1/2 \\ 1 & \lambda & 0 \\ 0 & 1/2 & \lambda \end{vmatrix} = \lambda^3 + \frac{1}{4} = 0$$

$$|\lambda| = \sqrt[3]{\frac{1}{4}} < 1 \Rightarrow \text{metoda Jacobi converge}$$

$$\lambda_1 = -\sqrt[3]{\frac{1}{4}} \quad + 2 \text{ răd. complexe}$$

$$B = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}; \quad C = B - A$$

- Construiți M, d
- Convergența ?
- $x^{(0)}$ det. $\to$ construiți $x^{(1)}$

<u>Gauss-Seidel</u>:

$$B = \begin{pmatrix} 2 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 2 \end{pmatrix} \qquad C = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$B^{-1} = \begin{bmatrix} 1/2 & 0 & 0 \\ x & 1 & 0 \\ y & z & 1/2 \end{bmatrix}$$

$$B \cdot B^{-1} = \mathring{I}$$

$$\begin{pmatrix} 2 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 2 \end{pmatrix} \begin{pmatrix} 1/2 & 0 & 0 \\ x & 1 & 0 \\ y & z & 1/2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$1/2 + x = 0 \Rightarrow \boxed{x = -\frac{1}{2}}$$

$$x + 2y = 0 \Rightarrow -\frac{1}{2} + 2y = 0 \Rightarrow \boxed{y = \frac{1}{4}}$$

$$1 + 2z = 0 \Rightarrow \boxed{z = -\frac{1}{2}}$$

$B^{-1} \to$ col. lui B^{-1} $[x_1, x_2 \quad x_3]$

$$Bx^1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} ; \quad Bx^2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} ; \quad Bx^3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$$BX = I$$

$$\begin{cases} 2x_1 = 1 \\ x_1 + x_2 = 0 \\ x_2 + 2x_3 = 0 \end{cases} \Rightarrow \begin{array}{l} x_1 = \frac{1}{2} \\ x_2 = -\frac{1}{2} \\ x_3 = \frac{1}{4} \end{array}$$

$$\begin{cases} 2x_1 = 0 \\ x_1 + x_2 = 1 \\ x_2 + 2x_3 = 0 \end{cases} \Rightarrow \begin{array}{l} x_1 = 0 \\ x_2 = 1 \\ x_3 = -\frac{1}{2} \end{array}$$

$$\Rightarrow B^{-1} = \begin{pmatrix} 1/2 & 0 & 0 \\ -1/2 & 1 & 0 \\ 1/4 & -1/2 & 1/2 \end{pmatrix}$$

$$\begin{cases} 2x_1 = 0 \\ x_1 + x_2 = 0 \\ x_2 + 2x_3 = 1 \end{cases} \Rightarrow \begin{array}{l} x_1 = 0 \\ x_2 = 0 \\ x_3 = \frac{1}{2} \end{array}$$

$$M = B^{-1}C = \begin{pmatrix} 1/2 & 0 & 0 \\ -1/2 & 1 & 0 \\ 1/4 & -1/2 & 1/2 \end{pmatrix} \cdot \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & -1/2 \\ 0 & 0 & 1/2 \\ 0 & 0 & -1/4 \end{pmatrix}$$

$$\|M\| = \sqrt{\frac{1}{4} + \frac{1}{4} + \frac{1}{16}} < 1$$

$$\|M\|_1 = \|M\|_\infty = \frac{1}{2} < 1$$

$$\Downarrow$$

GS converge

Miercuri, 3 Mai 2017

CN

Cursul 9

Valori și vectori proprii

→ Matrici pătratice!

$$\underbrace{(\lambda I - A)}_{B} u = 0$$

$B u = 0$

→ o sol. $u = 0$
→ încă o sol. $u \neq 0$ } ⟺ $\det B = 0$

$\det(B) \neq 0$
⇓
sol. unică $u = B^{-1} \cdot 0 = 0$

$$A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 2 & -4 & 3 \end{pmatrix} \qquad \det(\lambda I - A) = \det\begin{pmatrix} \lambda-1 & -1 & 0 \\ 0 & \lambda & -1 \\ -2 & 4 & \lambda-3 \end{pmatrix} =$$

$= \lambda(\lambda-1)(\lambda-3) - 2 + 4(\lambda-1) =$
$= \lambda^3 - 4\lambda^2 + 3\lambda - 2 + 4\lambda - 4 =$
$= \lambda^3 - 4\lambda^2 + 7\lambda - 6$

$\lambda^3 \to a_0$, $-4\lambda^2 \to a_1$, $7\lambda \to a_2$, $-6 \to a_3$

$\Rightarrow \lambda_1 = 2$ rădăcină

$$\begin{array}{l|l} \lambda^3 - 4\lambda^2 + 7\lambda - 6 & \lambda - 2 \\ \hline -\lambda^3 + 2\lambda^2 & \lambda^2 - 2\lambda + 3 \\ \quad -2\lambda^2 + 7\lambda & \\ \quad\ 2\lambda^2 - 4\lambda & \\ \qquad\quad 3\lambda - 6 & \end{array}$$

$$\lambda_{2,3} = \frac{2 \pm \sqrt{4-12}}{2} = \frac{2 \pm \sqrt{-8}}{2} = 1 \pm i\sqrt{2}$$

$$p_A(\lambda) = (\lambda - 2)(\lambda^2 - 2\lambda + 3)$$

$P(\lambda_0)$

$p_A(\lambda) = (((\lambda - 4)\cdot\lambda + 7)\cdot\lambda - 6)$

$b_0 = a_0 = 1$
$b_1 = b_0\lambda - 4$
$b_2 = b_1\lambda + 7$
$b_3 = b_2\lambda - 6 = P(\lambda)$

$\lambda = 2$
$\Rightarrow b_0 = 1$
$b_1 = 1 \cdot 2 - 4 = -2$
$b_2 = -2 \cdot 2 + 7 = 3$
$b_3 = 3 \cdot 2 - 6 = 0$

$p_A(\lambda) = (\lambda - 2)\, q(\lambda) + r$

$$\begin{cases} q(\lambda) = b_0\lambda^2 + b_1\lambda + b_2 p_A(\lambda) \\ r = b_3 \end{cases}$$

$$p_A(\lambda) = (\lambda-1)^5(\lambda-2)(\lambda^2+1) \Rightarrow n = 8$$

1 este de multiplicitate algebrică 5

$$A u = \lambda u$$

$$v = c \cdot u \; ; \quad c \in \mathbb{R} ; \; c \neq 0$$

$$A v = c A u = c \lambda u = \lambda v \quad \Rightarrow v = \text{vector propriu asociat } \lambda$$

Nr. de vect. proprii lin. indep. care se pot asocia unei valori prop

= multiplicitatea geometrică

$$\lambda = 2 ; \quad A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 2 & -4 & 3 \end{pmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}$$

$$(2I - A) u = 0$$

$$A u = 2u$$

$$\begin{cases} u_1 + u_2 = 2u_1 \\ u_3 = 2u_2 \\ 2u_1 - 4u_2 + 3u_3 = 2u_3 \end{cases} \Rightarrow \begin{cases} u_1 = u_2 \\ u_3 = 2u_2 \\ 2u_2 - 4u_2 + 6u_2 = 4u_2 \end{cases} \Rightarrow$$

$$\Rightarrow \begin{cases} u_1 = u_2 \\ u_3 = 2u_2 \\ 0 = 0 \end{cases}$$

$$u_2 = a \neq 0$$
$$= 1$$

$$u = \begin{pmatrix} a \\ a \\ 2a \end{pmatrix} \quad \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$$

vectori proprii asoc. val. proprii

$$\lambda = 2$$

$$\boxed{\|u\|_2 = 1}$$

$$\sqrt{a^2 + a^2 + 4a^2} = 1 \Rightarrow 6a^2 = 1$$
$$\Rightarrow a = \frac{1}{\sqrt{6}}$$

$$c_1 u^1 + c_2 u^2 + c_3 u^3 = 0 \Rightarrow$$

$$\Rightarrow c_1 = c_2 = \dots = c_n = 0$$

$$c_i u^i + c_j u^j = 0$$

$A x = b$

$\underbrace{T \cdot A}\, x = T \cdot b$

↓ o transformare pt care sist. este ușor de rezolvat

A diagonală: $[d_1, d_2, \dots, d_n] \to$ val. proprii $\lambda_i = d_i$; $i = \overline{1,n}$

A sup. Δ sau inf. $\Delta \Rightarrow$ val. p $\lambda_i = a_{ii}$; $\forall i = \overline{1,n}$

A = f. Schur reală

A transf. $\to B$ - polinomul caracteristic este ușor de calculat

Matrici companion:

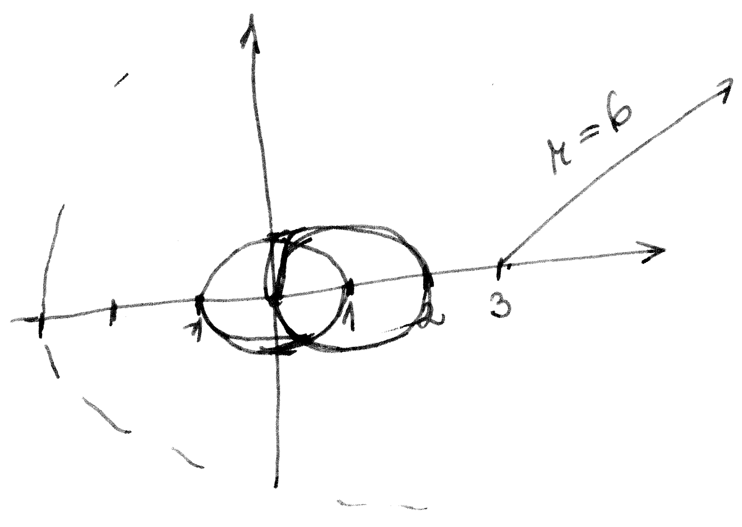
$$p(\lambda) = a_0 + a_1\lambda_1 + \dots + a_{n-1}\lambda^{n-1} + \lambda^n$$

$$C = \begin{bmatrix} 0 & 0 & \dots & 0 & -a_0 \\ 1 & 0 & \dots & 0 & -a_1 \\ 0 & 1 & \dots & 0 & -a_2 \\ \vdots & & & & \\ 0 & 0 & \dots & 1 & -a_{n-1} \end{bmatrix}$$

$$\Rightarrow C^T \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & & & & \\ 0 & 0 & 0 & \dots & 1 \\ -a_0 & -a_1 & a_2 & \dots & a_{n-1} \end{bmatrix}$$

$A = T^{-1} B T$

1	1	0
0	0	1
2	-1	3



$A - \lambda_1, \lambda_2, \ldots, \lambda_n$ - val. pr. distincte

$\exists\ u^1, u^2, \ldots, u^n$ - vect. p. lin. indep.

$U = [u^1, u^2 \ldots u^n]$

$A u^i = \lambda_i u^i$

$A \cdot U = \Lambda \cdot U$

$\Lambda = \text{diag.}[\lambda_1, \ldots, \lambda_n]$

$\exists\ U^{-1}$; $\quad U^{-1} A U = \Lambda$

$r(u)$

$u \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$ $\qquad A \cdot u = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 2 & -4 & 3 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \\ 4 \end{pmatrix}$

$(Au, u) = 2 \cdot 1 + 2 \cdot 1 + 2 \cdot 4 = 12$

$\|u\|_2^2 = 1^2 + 1^2 + 2^2 = 6$

$r(u) = \dfrac{(Au, u)}{\|u\|_2^2} = \dfrac{12}{6} = 2 \Rightarrow \underline{\underline{\lambda = 2}}$

$P_A(\lambda) = (\lambda - \lambda_1)^1 g(\lambda)$

$v_0 \in \mathbb{R}^n$; $v_0 \neq 0$

$v_1 = A v^0$

$v_2 = A v^1$

$\ldots$

$v^p = A v^{p-1}$

$\underline{v^p \to u_{max}}$ - vect. propriu asociat val. proprii λ_1

$v^p = A^p v_0 \rightarrow \boxed{\text{metoda puterii}}$

$u^0 : \|u^0\|_2 = 1$

$w = A u^0$

$u^1 = \frac{1}{\|w\|_2} w$

$w = A u^1$; $u^2 = \frac{1}{\|w\|_2} \cdot w$

$r(u_{max}) \approx \lambda_{max}$

$A = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$ $\Rightarrow \det A = 0 \Leftrightarrow \lambda = 0$ val. prop.

$\det(\lambda I - A) = 0$; $\det(-A) = -\det A = 0$

$P_A(\lambda) = \lambda(\lambda+1)(\lambda-2)$

$\lambda_{max} = 2$; $u_{max} = \begin{pmatrix} a \\ a \\ 2a \end{pmatrix}$

u^0 a.î. $(u^0, u_{max}) = 0$

$u^0 (1, 1, -1)^T$

$1 a + 1 \cdot a - 1 \cdot 2a = 0$

Metoda puterii este folosită de Google pt calc. page-rank-ul paginilor

Metoda iterației inverse → matrici simetrice
→ det. unei val. propr. apropiate de o val. dată

$\det(\mu I - A) = 0 \Rightarrow \mu$ val. proprie

~~[illegible]~~

$\lambda_1, \lambda_2, \dots, \lambda_n$

$\lambda_1 - \mu, \lambda_2 - \mu, \dots, \lambda_n - \mu$

$\left|\frac{1}{\lambda_1 - \mu}\right| \geq \left|\frac{1}{\lambda_2 - \mu}\right| \geq \dots \geq \left|\frac{1}{\lambda_n - \mu}\right|$ → o matr. ale cărei val. proprii sunt $\frac{1}{\lambda_i - \mu}$; λ_i val. p. pt A

$A - cI$

$A \to$ val. propr. $\lambda_1, \dots, \lambda_n$

$A - c\dot{I}$ are val. propr. $\lambda_1 - c, \lambda_2 - c, \dots, \lambda_n - c$

$$\det((\lambda + c)\dot{I} - A) = p_A(\lambda - c)$$

A: $\det A \neq 0 \Rightarrow \lambda = 0$ nu este val. proprie

$\exists A^{-1}$

val. propr. ale lui A^{-1} $\frac{1}{\lambda_1}, \frac{1}{\lambda_2}, \dots, \frac{1}{\lambda_n}$

$(A - \mu\dot{I})^{-1}$ - cea mai mare val. propr. a lui $(A - \mu\dot{I})^{-1}$ este val. propr. cea mai apropiată μ pt A

metoda puterii: $w = Au$

$(\mu\dot{I} - A)^{-1}$: $w = (\mu\dot{I} - A)^{-1}u \Leftrightarrow$

$\Leftrightarrow w$ este sol. sist. liniar $(\mu\dot{I} - A)x = u$

- fac o singură dată alg. de elim. Gauss pt $\mu\dot{I} - A$

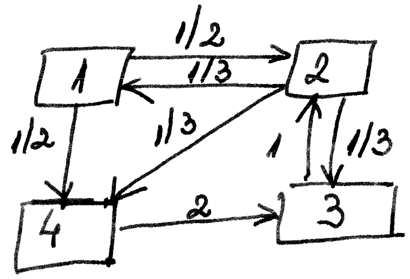
$\mathcal{Z}$

Luni, 8 Mai 2017

CN

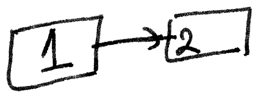
Cursul 10

Problema **Page-Rankului** (utilizată de Google)



$$A_{4\times 4} = \begin{pmatrix} 0 & 1/3 & 0 & 0 \\ 1/2 & 0 & 1 & 0 \\ 0 & 1/3 & 0 & 1 \\ 1/2 & 1/3 & 0 & 0 \end{pmatrix}$$

$$\begin{cases} x_1 = 1/3\, x_2 \\ x_2 = 1/2\, x_1 + x_3 \\ x_3 = 1/3\, x_2 + x_4 \\ x_4 = 1/2\, x_1 + 1/3\, x_2 \end{cases} \qquad x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}$$



$\boxed{x = Ax}$

$\lambda = 1$ val. propr.

→ probl. de val. propr.

$\frac{1}{n}$ $\quad M = p\cdot A + (1-p)B$

$$B = \frac{1}{n}\begin{pmatrix} 1 & 1 & \dots & 1 \\ \vdots & & & \\ 1 & 1 & \dots & 1 \end{pmatrix}; \quad p = 0,85$$

≈ 0 (n f. mare → mld.)

$$A \rightsquigarrow S$$

$$A = A^{(0)} = A^{(1)}\ A^{(2)}\ \dots\ A^{(k)}, \dots$$

aceleași valori proprii

$$A^{(0)} \sim A^{(1)} \sim A^{(2)} \sim \dots$$

def. matr. asemenea:

$A \sim B$ dacă $\exists T$ cu $\det T$

$$A = TBT^{-1}$$

sau

$$T^{-1}BT$$

Matr. în formă Hessemberg:

$$H = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1.5 & 2 & 4 & 7 & 9.5 \\ 0 & 2 & 3 & 1 & 2 \\ 0 & 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 3 & -1 \end{pmatrix}$$

$TA = H$

$TAT^{-1} = H$

Householder: $P = I - 2vv^T$; $v \in \mathbb{R}^n$; $\|v\|_2 = 1$

$P = I - \frac{1}{\beta} u u^T$

ptr. păstr. val. proprii

$$P_{n-2} \dots \underbrace{\Big(P_2 \underbrace{(P_1 A P_1)}_{\text{transf. col. 1 în f. Hessemberg sup.}} P_2\Big)}_{\text{transf. col. 2 în f. Hess. sup.}} \dots P_{n-2} = H = TAT^{-1}$$

$$u = \begin{pmatrix} 0 \\ 0 \\ * \\ * \\ * \end{pmatrix}$$

Pasul $k = 1$: $u = \begin{pmatrix} 0 \\ * \\ * \\ * \\ * \end{pmatrix}$; Pasul $k = 2$: $\begin{pmatrix} 0 \\ 0 \\ * \\ * \\ * \end{pmatrix}$ ....

$A^{(0)} = A^{(1)}, A^{(2)}, \dots\dots$

$A^{(p)} \longrightarrow R$ sup. Δ

val. prop. A $\quad \lambda_i = r_{ii}$; $i = \overline{1, n}$

- forma Schur $\rightarrow$ sup. Δ pe blocuri

$$\begin{array}{cccccc} S_{11} & S_{12} & & S_{13} & & S_{14} \\ \boxed{1} & 2 & 3 & 4 & 5 & \boxed{6} \\ 0 & 0 & 1 & 2 & 1 & 3 \\ 0 & -1 & 0 & 2 & 1 & 2 \\ 0 & 0 & 0 & 2 & 1 & 3 \\ 0 & 0 & 0 & 3 & 0 & 2 \end{array} \quad S_{22},\ S_{33}$$

$\lambda_{1,2} = \frac{2 \pm \sqrt{4-12}}{2} = 1 \pm i\sqrt{2}$

$\begin{vmatrix} \lambda - 2 & 1 \\ -3 & \lambda \end{vmatrix} = \lambda^2 - 2\lambda + 3$

$h_{i\,i-1} \neq 0 \Rightarrow h_{i+1\,i} = 0$ + verif. că blocul corespunzător $h_{i\,i-1}$ are val. propr. $\mathbb{C}$

$$\begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 1 & 3 & 3 & 3 \\ 0 & 2 & 1 & 3 & 2 \\ 0 & 0 & 1.5 & 2 & 4 \\ 0 & 0 & 0 & 3 & 2 \end{bmatrix}$$

$R_{p,q}(\theta) = \begin{pmatrix} 1 & & & & 0 \\ & \underset{(p,p)}{c} & & \underset{(p,q)}{s} & \\ & & \ddots & & \\ & -\underset{(q,p)}{s} & & \underset{(q,q)}{c} & \\ 0 & & & & 1 \end{pmatrix}$

$R_{p,q}$ → matr. de rotație

$c = \cos\theta$

$s = \sin\theta$

$$1 + 2 + \dots + n-2 = \frac{(n-2)(n-1)}{2}$$

Alg. QR

$$\overbrace{\underbrace{\left(R_{n\,n-1}(\theta) \dots R_{23}(\theta) R_{12}(\theta) A\right)}_{R \text{ sup. } \Delta}}^{Q^T} R_{12}^T(\theta) R_{23}^T(\theta) \dots R_{n\,n-1}^T(\theta) = A^{\text{următor}}$$

$A \cdot R_{12}^T$ = se schimbă col. 1, 2

c col. 1 + s col. 2 = noua col. 1

− s col. 1 + c col. 2 = noua col. 2

$$\begin{pmatrix} * & * & * & * & * \\ 0 & * & * & * & * \\ 0 & 0 & * & * & * \\ 0 & 0 & 0 & * & * \\ 0 & 0 & 0 & 0 & * \end{pmatrix} * R_{12}^T * R_{23}^T \dots \Rightarrow \text{matr. Hessenberg sup.}$$

se poate aplica în mai multe variante!

→ disc. QR se aplică unei matrici obținute prin deplas. lui A

→ QR cu deplasare (shiftare) simplă

deplasare dublă → pt. valori proprii $\in \mathbb{C}$

→ ajung mai rapid la f. Schur reală

$a + ib$ $\quad$ $a - ib$

$a_1 = a + ib$, $\quad$ $a_2 = a - ib$

$a_1 + a_2$, $\quad$ $a_1 \cdot a_2 \in \mathbb{R}$

$A \longrightarrow H \longrightarrow S$; $\quad S = Q^T A Q$

col. lui Q sunt vect. proprii

$$A = U \Sigma V^T$$

$$A = \sum_{i=1}^{r} \sigma_i \underbrace{u^i}_{m \times 1} \underbrace{(v^i)^T}_{1 \times n} = \sigma_1 u^1 (v^1)^T + \sigma_2 u^2 (v^2)^T + \dots +$$

$$+ \sigma_r (u^r)(v^r)^T$$

$$\sigma_1 \geq \sigma_2 \geq \dots \geq \sigma_r > 0$$

$\| A - A_s \| \to$ mare

$$\begin{pmatrix} 1 & 2 & 3 \\ 5 & 1 & 2 \end{pmatrix}$$

$$A A^{-1} = A^{-1} . A = I$$

$(C \cdot D \cdot E)^{-1} = E^{-1} \cdot D^{-1} \cdot C^{-1}$ (Proprietatea 1)

$$\Sigma = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \end{pmatrix} \qquad \Sigma^{I} = \begin{bmatrix} [redacted] & 0 \\ 0 & 1/2 \\ 0 & 0 \end{bmatrix}$$

$\downarrow$

pseudo-inversa Moore-Penrose

sol. $Ax = b$ $\quad x = A^{I} b$ $\to$ nu se păstrează prop. 1

Luni, 15 Mai 2017

CN

Cursul 11

Probl. în care nr. de nec. > nr. de ecuații

$$\begin{cases} x+y+z=3 \longrightarrow \text{ec. unui plan din } \mathbb{R}_3 \\ x+2y-z=2 \longrightarrow \text{ec. altui plan} \end{cases}$$

planuri paralele $\Rightarrow$ 0 soluții

Vectorul reziduu: $r(x) = b - Ax$

Scopul: minimizarea normei euclidiene $(=0)$

sol. în sensul celor mai mici pătrate

$$\begin{cases} x+y=2 \\ x+2y \neq 3 \\ x-3.5y=7 \end{cases}$$

nu are sol. clasică

$\min\{f(x)\}$ $\quad f'(x)=0$

f' $\quad x \in \mathbb{R}^n$; $n>1$

$f' \sim \nabla f = \begin{matrix} \frac{\partial f}{\partial x_1} \\ \vdots \\ \frac{\partial f}{\partial x_n} \end{matrix}$

$\boxed{\text{gradientul lui } f}$

Ex din curs: $\begin{cases} -10+6x_2+12x_1=0 & \text{(deriv. în rap. cu } x_1) \\ -8+6x_1+12x_2=0 \end{cases}$

sol. $\frac{2}{3}$, $\frac{16}{3}$

Norma Manhattan nu este derivabilă!

$A : 3 \times 2$

$A^T : 2 \times 3$

$\underbrace{A^T \cdot A}$: 2×2

- simetrică
- pozitiv semidefinită

$Ax = b$

$x_1 A^1 + x_2 A^2 + \dots + x_n A^n = b$

$A = [A_1 A_2 \dots A_n]$

$(Ax, x) \geq 0, \forall x$ poz. semidefinite

$(Ax, x) > 0, \forall x$ poz. def.

matr. poz. def.

$\det A \neq 0$

$$\begin{cases} 12x_1 + 6x_2 = 10 \\ 6x_1 + 12x_2 = 8 \end{cases} \Rightarrow \begin{pmatrix} 12 & 6 \\ 6 & 12 \end{pmatrix}; \begin{pmatrix} 10 \\ 8 \end{pmatrix}$$

$$(A^T A)x = A^T b$$

$$x = \underbrace{(A^T A)^{-1} A^T}_{\text{pseudoinversă}} b$$

Cel mai recomandat:

alg. QR!

$$A = U \Sigma V^T$$

$$A^{\dot{I}} = U \Sigma^{\dot{I}} U^T$$

$$\Sigma = \begin{bmatrix} D & 0 \\ 0 & 0 \end{bmatrix}$$

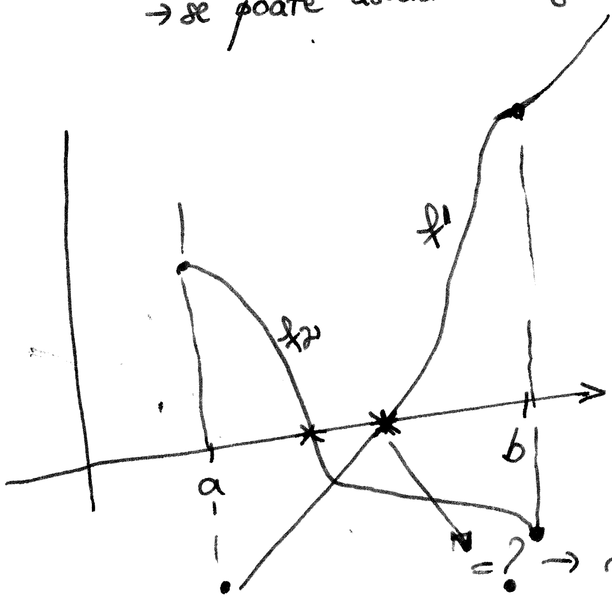
$$D = \begin{bmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \end{bmatrix}$$

$$\Sigma^{\dot{I}} = \begin{bmatrix} D^{-1} & 0 \\ 0 & 0 \end{bmatrix}$$

(examen: pseudoinversa Moore-Penrose)

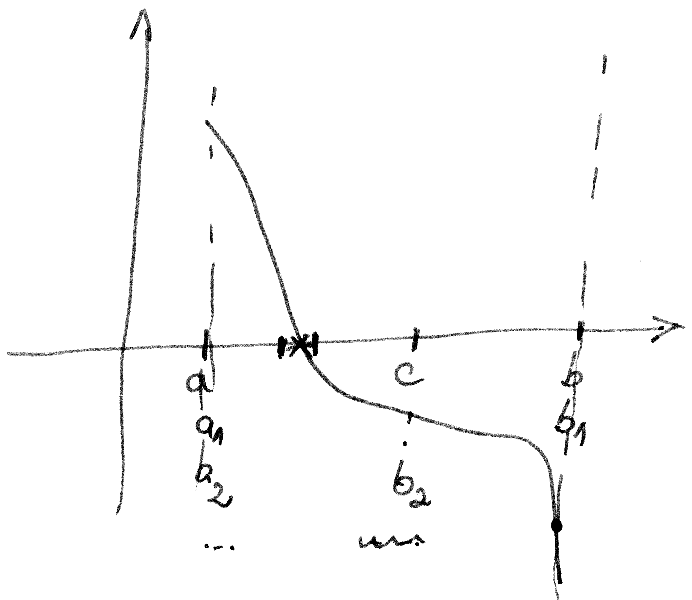
Rez. ecuațiilor neliniare

→ se poate asocia cu rez. polinoamelor de grad mare (> 4)



= ? → metode de aprox. neliniare

Metoda înjumătățirii intervalului



Înjumătățesc până ajung la un int. mic împrejurul sol. (precizia)

De regulă: interval de lung. 1 + metode de aflare a rădăcinii

↓

iterative (construiesc un șir care să converge la sol.)

↓

alegerea primelor elem. din șir influențează convergența!

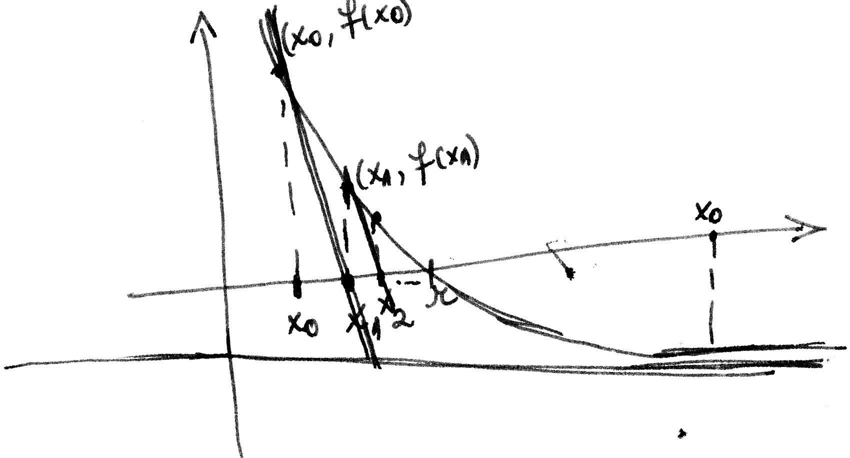
(x_0 în apropierea soluției)

↓

pt. asta se recurge la met. înjumătățirii

(Tema 8)

1) Metoda tangentei :



$$f(x) = \overbrace{f(x_k) + (x - x_k) f'(x_k)}^{g(x)} + \frac{(x-x_k)^2}{2!} f''(x_k)$$

$f(x) \simeq g(x)$ pt $x \simeq x_k$

$f(x) = 0$ nu știu, înlocuiesc cu $g(x) = 0$

$g(x) = 0$ are sol. $\overline{x} = x_k - \frac{f(x_k)}{f'(x_k)} = x_{k+1}$

Rezultat de convergență locală: știu unde e sol. ca să mă plasez în apropierea ei

Teorema de convergență:

$$[a,b] \quad \begin{cases} f' \neq 0 \rightarrow f' > 0 \text{ sau } f' < 0 \\ f'' \neq 0 \rightarrow f'' > 0 \text{ sau } f'' < 0 \end{cases}$$

→ conv. globală

→ fortează un interval relativ mic!

$$\underbrace{|x_{k+1} - l|}_{10^{-7}} \leq \mu \, |x_k - l|^{2}$$

$\mu = 10$; $q = 2$

$|x_k - l| = 10^{-4}$

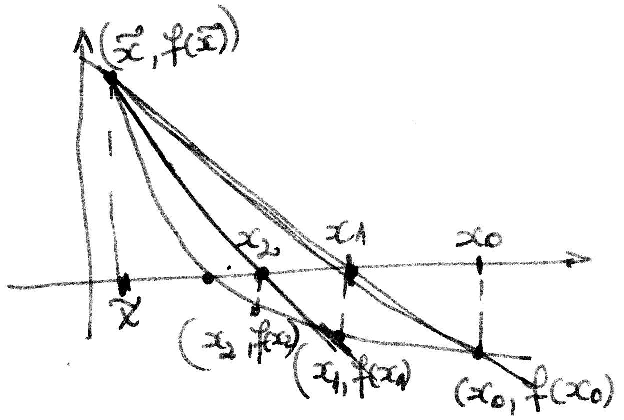
$q = 2 \rightarrow$ conv. pătratică

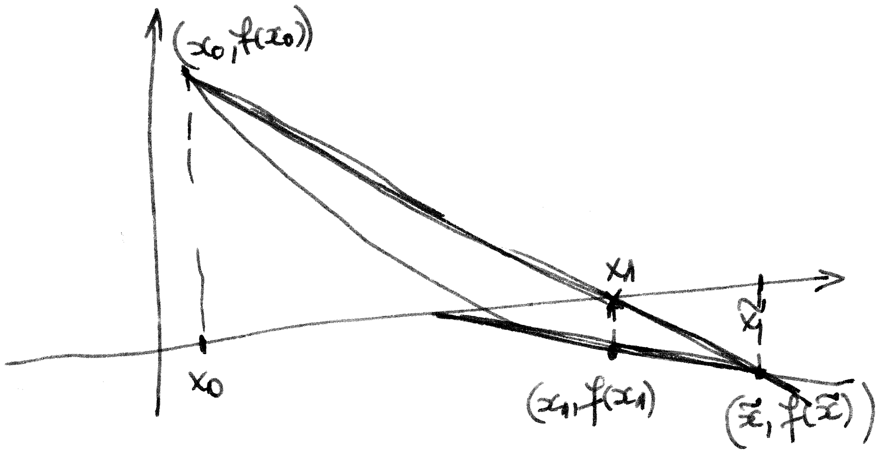
$q = 3 \rightarrow$ conv. cubică → f. rapidă

$q = 1 \rightarrow$ conv. liniară

$$|x_{k+1} - l| \leq \underset{0.1}{\mu} \underbrace{|x_k - l|}_{10^{-4}} \Rightarrow |x_k - l| = 10^{-5}$$

2) Metoda falsei poziții (metoda coardei)

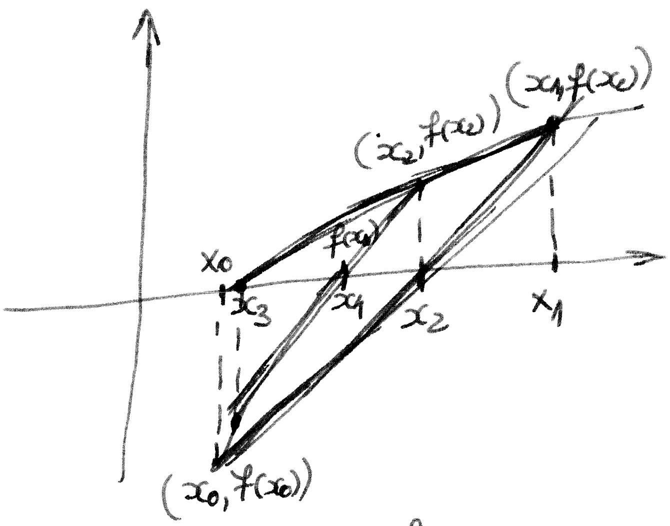




3) Metoda secantei : fixez: x_0, x_1

↓

metodă cu 2 pași



$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)}$$

$$f'(x) = \lim_{y \to x} \frac{f(y) - f(x)}{y - x}$$

$$f'(x) \simeq \frac{f(y) - f(x)}{y - x}$$ ptr y suf. de aproape de x

$$f'(x_k) \simeq \frac{f(x_{k-1}) - f(x_k)}{x_{k-1} - x_k} = \frac{f(x_k) - f(x_{k-1})}{x_k - x_{k-1}}$$

$$x_{k+1} = x_k - \frac{f(x_k)(x_k - x_{k-1})}{f(x_k) - f(x_{k-1})}$$

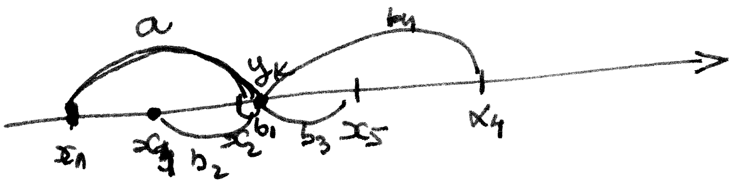
4) Metoda Laguere → specif. polinoamelor

$$p(x) = a_0 x^n + a_1 x^{n-1} + \dots + a_{n-1} x + a_n \quad ; \quad a_0 \neq 0$$

$$\sum \frac{1}{\bar{x} - x_i} = \frac{p'(\bar{x})}{p(\bar{x})}$$; nu cun. x_i

$$\sum \frac{1}{(\bar{x} - x_i)^2}$$ cunoscând doar coef.

$$\underbrace{\frac{1}{y_k - x_1}}_{=a} + \underbrace{\frac{1}{y_k - x_2}}_{b} + \dots + \underbrace{\frac{1}{x_k - x_n}}_{b}$$



$y_k - x_1 = a$

$y_k - y_{k+1} = a$ calc.

Sist. de ec. neliniare :

$$\begin{cases} x_1^2 + 2x_1 x_2 + 5x_2^2 = 3 \\ -x_1^2 + 2x_2^3 - 5x_2 = 7 \end{cases}$$

$$\nabla F(x) = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & \vdots & & \vdots \\ \frac{\partial f_n}{\partial x_1} & \frac{\partial f_n}{\partial x_2} & \cdots & \frac{\partial f_n}{\partial x_n} \end{bmatrix}$$

$$\nabla F(x) = \begin{bmatrix} 2x_1 + 2x_2 & 2x_1 + 10x_2 \\ -2x_1 & 6x_2^2 - 5 \end{bmatrix}$$

$$\nabla F(X^{(k)}) = \begin{bmatrix} 4 & 12 \\ -2 & 1 \end{bmatrix}$$

$k = 0$

$$X^{(0)} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$\begin{cases} 4\delta_1 + 12\delta_2 = -5 \\ -2\delta_1 + \delta_2 = 11 \end{cases}$$

$$F(X) = F(X^{(k)}) + \nabla F(X^{(k)})(X - X^{(k)}) + O(\|X - X^{(k)}\|^2)$$

Luni, 29 Mai 2017

CN

Cursul 13

= Optimizare numerică =

Examen : 6 iunie → progr. săli

→ Prob. de minim / maxim — global, local

— cu restricții ($\min\{f(x);\ g(x) \le 0\}$)

— fără restricții

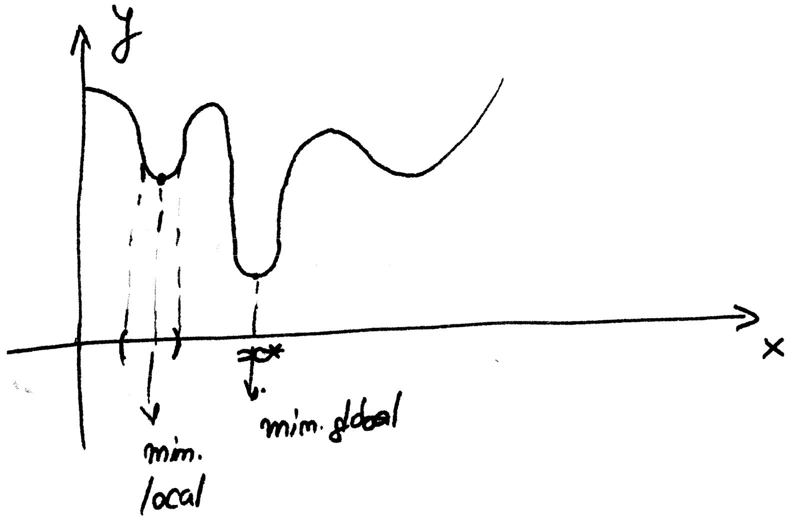
$$\max\{f(x)\} = -\min\{-f(x)\}$$

$f: \mathbb{R} \to \mathbb{R};\ \min\{f(x)\} \Rightarrow$

$\Rightarrow f'(x) = 0$

$f''(x) \geq 0$

(fcție convexă)



$$\left\{\begin{array}{l} f'(x) = 0 \\ + \\ f''(x) \geq 0 \end{array}\right. \overset{?}{\Rightarrow} x \text{ pct. de minim ?}$$

↓ pct. staționar

x - mim. local

$f(x^*) < f(x)$; $\forall x \in$ Vec. , $x^* \neq x$

Leg. cu rez. sist. $\boxed{Ax = b}$

$A = A^T$; $A > 0$ poz. def.

$x^{(0)}, x^{(1)}, \dots, x^{(k)}, \dots$; $f(x^{(0)}) \geq f(x^{(1)}) \geq \dots$

→ Direcție de descreștere

$f(x+\alpha d)$ $\qquad f(x, y) = x^2+y^2$

~~min { f(x), g(x) = 0 }~~

$$f(x+\alpha d) = \underbrace{(x+\alpha d_1)^2 + (y+\alpha d_2)^2}_{g(\alpha) : \mathbb{R} \to \mathbb{R}}$$

$d = \begin{pmatrix} d_1 \\ d_2 \end{pmatrix}$ $\qquad$ → Ajustarea pasului (line search)

$x = \begin{pmatrix} x \\ y \end{pmatrix}$

steepest descent (cea mai mare descreștere)

$$A = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \quad b = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$$

$$f(x, x_2) = \frac{1}{2}(Ax, x) - (b, x)$$

$$= \frac{1}{2}\left(\begin{pmatrix} 2x_1 + x_2 \\ x_1 + x_2 \end{pmatrix}, \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}\right) - (3x_1 + 2x_2) =$$

$$= \frac{1}{2}(2x_1^2 + x_1x_2 + x_1x_2 + x_2^2) - 3x_1 - 2x_2$$

$$\frac{\partial f}{\partial x_1} = 2x_1 + x_2 - 3 = 0 \qquad \Rightarrow \begin{cases} 2x_1 + x_2 = 3 \\ x_1 + x_2 = 2 \end{cases}$$

$$\frac{\partial f}{\partial x_2} = x_1 + x_2 - 2 = 0$$

$$\begin{bmatrix} \frac{\partial^2 f}{\partial x_1^2} = 2 & \frac{\partial^2 f}{\partial x_1 x_2} = 1 \\ \frac{\partial^2 f}{\partial x_2 x_1} = 1 & \frac{\partial^2 f}{\partial x_2^2} = 1 \end{bmatrix} = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$$

→ Desc. QR: 3 alg.

Atenție la calcule!!

1) Givens cu matr. de rotație → 1 matr. de rotație + cel. matr. unitate (I)

2) Householder - matr. de reflexie (pt. fiecare col.)

3) Gram-Schmidt

→ Metode iterative

- constr. matr. iterației M pt Jacobi / Gauss-Seidel (relaxare) + converg. + + $x^{(1)}$ din $x^{(0)}$

Jacobi: B = diag. A

GS: B = partea inf Δ a lui A

→ Metodele relaxării

→ vect. + val. proprii (intră la rez. sist. liniare)

→ QR

→ Metoda iterației inverse = metoda puterii pt matr. $(\mu I - A)^{-1}$

$$\boxed{(\mu I - A) w = u^{(0)}} \qquad u^{(0)} = \begin{pmatrix} 0.5 \\ 0 \\ 0 \end{pmatrix} ; \quad u^{(1)} \to \text{trb. rez. sist. liniar}$$

$$Ax = b \iff (A^T A) x = A^T b$$

$$\begin{cases} x + y = 2 \\ 2x + 3y = 5 \\ x - 3y = 1 \end{cases} \to \text{sol. în sensul celor mai mici pătrate}$$

$$A_{3\times 2} ; \; A^T_{2\times 3} \Rightarrow (A^T \cdot A)_{2\times 2}$$

→ Interpolare numerică → tabel (x, y) : aproximare f pt un pct $\nexists$ în tabel prin:

- polin. Lagrange
- Newton + Aitken (dif. divizate)
- Neville + Aitken
- Funcția spline liniară continuă

Newton → mai multe forme!

→ Precondiționarea sist. → înmulț. cu o matr. rară de precondiționare (la stg. / la dreapta)

Metoda Newton
$$f(x) = f(x_k) + f'(x_k)(x - x_k) + f''(x_k)\frac{(x-x_k)^2}{2}$$
x_k → pct. curent de aprox.

= Recapitulare + sub. examen =

3-4 exerciții – sist. liniare, aprox. răd. funcții, interpolare
optimizare (min. rez. unui sistem)

1) Tehnici de rez. a sist. liniare – directe / iterative

↓

valori proprii

– metodele substituției (sist. Δ)
– elim. Gauss (tip de pivotare: parțială / totală)

$\boxed{\text{aducere f. sup. } \Delta + \text{aplic. met. subst.}}$

$A = LU$
$Ly = b$ → metoda subst. directe
$Ux = y$ → met. subst. inv.

– desc. LU : col. L, lin. U

Pasul 1 : det. col. 1 L
lin. 1 U
(l_{11}, l_{21}, l_{31} + u_{11}, u_{12}, u_{13})

Pasul 2 : col. 2 L (l_{22}, l_{32})
lin. 2 U (u_{23})

Pasul 3 : col. 3 L (l_{33})
lin. 3 U (–)

A = sim., poz. def.
$A = L \cdot L^T$; L inf Δ
↓
pe col: (elem. diag., restul)

$l_{ip} \Rightarrow a_{ip} = (LU)_{ip}$

u_{pj}

$0 = l_{11} u_{13} + 0 \, u_{23} + 0 \cdot 1$