

Open Data for Open Cities: Re-use and discovery level
applied to the spatial point analysis process on linear
networks
point pattern session

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Outline

Linear networks

- Traffic accidents, Medellin, Colombia
- Mathematical structure

Linear network point processes

Intensity estimator

- Voronoi tessellations
- Kernel smoothing

Reference

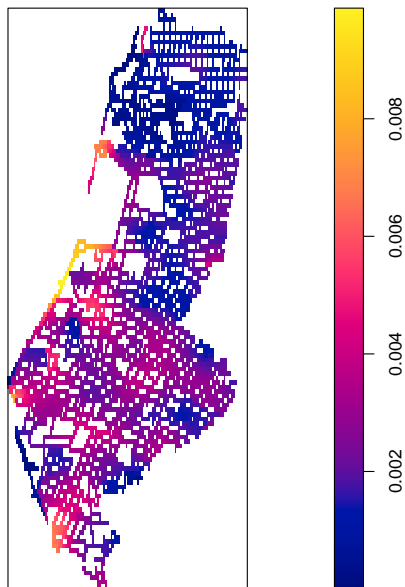
Traffic accidents in an area of Medellin, Colombia



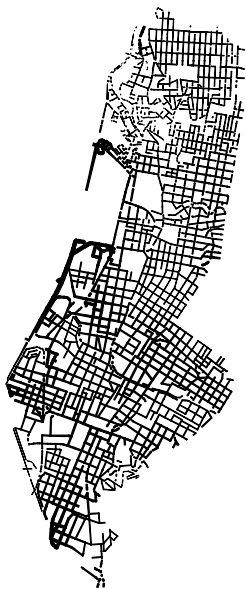
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Mathematical structure of linear networks

A line segment with endpoints $u \in \mathbb{R}^2$ and $v \in \mathbb{R}^2 \setminus \{u\}$ is given by $l = [u, v] = \{tu + (1 - t)v : 0 \leq t \leq 1\} \subseteq \mathbb{R}^2$, $u \neq v$ and a *linear network* L is defined as the union

$$L = \bigcup_{i=1}^k l_i$$

of $k \geq 1$ line segments $l_i = [u_i, v_i]$, the *components* of L , which we assume satisfy $l_i \not\subseteq l_j$, $i \neq j$, and boundedness, i.e. $|l_i| = |u_i - v_i| < \infty$.

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The *total network length* is given by the sum of the component lengths, i.e.

$$|L| = \sum_{i=1}^k |l_i|,$$

whereby $|L| < \infty$ if $k < \infty$. On the other hand, if $k = \infty$, since each component has positive length, we have that $|L| = \infty$.

Mathematical structure of linear networks

A *path* from a starting point $u \in L$ to some end point $v \in L$ is a sequence of points $u_0, \dots, u_n \in L$, $n > 1$, such that $u_0 = u$, $u_n = v$ and $[u_i, u_{i+1}] \subseteq L$ for all $i = 0, \dots, n-1$.

Denoting by $p_{u,v}$ the collection of all paths between $u \in L$ and $v \in L$, the shortest-path (geodesic) distance between u and v is given by

$$\begin{aligned} d_L(u, v) &= \min_{(u_0, \dots, u_n) \in p_{u,v}} \sum_{i=0}^{n-1} |u_i - u_{i+1}| \\ &= \min_{(u_0, \dots, u_n) \in p_{u,v}} |u - u_1| + \sum_{i=1}^{n-2} |u_{i+1} - u_i| + |v - u_{n-1}|. \end{aligned}$$

Network example

Example

The first very simple example is any $L = \bigcup_{i=1}^{\infty} l_i$, $|L| = \infty$, where each node has degree 2. Since the 1-dimensional Euclidean space \mathbb{R} may be expressed as a countable union of compacts, we note that L is isometric to \mathbb{R} and may be viewed as a bending of the real line at $k = \infty$ places. For the particular case where $L = \mathbb{R}$ it additionally follows that $d_L(\cdot, \cdot)$ coincides with the Euclidean metric.

Example

The second example is the graph $L = \mathbb{Z}^2$, which has its nodes at all integer pairs $(i, j) \in \mathbb{R}^2$, $i, j \in \mathbb{Z}$. The edges \mathcal{E}_L are formed by the line segments joining the nodes horizontally and vertically. Note that we may scale everything by an arbitrary positive constant $\alpha > 0$ so that all edges have length α .

Linear network point processes

Assume X is a point process on linear network L with intensity measure μ ,

$$\mu(A) = E(N(A)) = \int_A \rho(u) d_1 u, \quad A \subseteq L,$$

where $N(A)$ is the number of points belong to subnetwork A , we then say X has *intensity function* $\rho(u)$, $u \in L$.

Example

A Poisson process X on L with intensity function $\rho(\cdot)$ is a LNPP such that $X(A_1), \dots, X(A_n)$ are mutually independent for any disjoint line segments $A_1, \dots, A_n \subseteq L$, $n \geq 1$, and $X(A)$ is Poisson distributed with mean $\mu(A) = \int_A \rho(u) d_1 u$. Conditioning on $X(L) = n$, we obtain a classical random sample with density $f(u) = \rho(u)/\mu(L) = \rho(u)/n$, a so-called Binomial point process (Møller and Waagepetersen; 2004).

In the context of a linear network L , the *Voronoi cell/subnetwork* of $x \in X$ is given by (Okabe and Sugihara; 2012)

$$\mathcal{V}_x = \mathcal{V}_x(X) = \{u \in L : d_L(x, u) \leq d_L(y, u) \text{ for all } y \in X \setminus \{x\}\}.$$

Definition

Given a LNPP X on a linear (sub)network L with intensity function $\rho(u)$, $u \in L$, the *Voronoi intensity estimator* is given by

$$\hat{\rho}^V(u) = \hat{\rho}^V(u; X, L) = \sum_{x \in X \cap L} \frac{\mathbf{1}\{u \in \mathcal{V}_x(X) \cap L\}}{|\mathcal{V}_x(X) \cap L|}, \quad u \in L,$$

if $X \cap L \neq \emptyset$; set $\hat{\rho}^V(u) = 0$ otherwise.

The Kernel-based intensity estimator (Diggle corrected) of a point pattern on a linear network L is as

$$\hat{\lambda}_{\epsilon}^{(D)}(\mathbf{u}) = \sum_{i=1}^n \frac{\kappa_{\epsilon}(d_L(\mathbf{u}, \mathbf{v}_i))}{C_{L,\epsilon}(\mathbf{v}_i)}, \quad \mathbf{u} \in L, \quad (1)$$

where κ_{ϵ} is a one-dimensional kernel function with bandwidth ϵ , and

$$C_{L,\epsilon}(\mathbf{v}) = \int_L \kappa_{\epsilon}(d_L(\mathbf{v}, \mathbf{u})) d_1 u, \quad \mathbf{v} \in X \cap L, \quad (2)$$

is an edge-correction factor (Moradi et al.; 2016).

The diffusion estimator of intensity function of a point pattern on a linear network L with bandwidth $\epsilon = \sqrt{t}$ is as

$$\hat{\lambda}_{\epsilon}(\mathbf{u}) = \sum_{i=1}^n \kappa_{\epsilon^2}(\mathbf{u}|x_i), \quad \mathbf{u} \in L \quad (3)$$

where κ_t is the heat kernel (McSwiggan et al.; 2016).

How to choose bandwidth parameter ϵ

- use the smoothing bandwidth parameter ϵ which minimizes (Cronie and van Lieshout (2016))

$$\left| \sum_{i=1}^n \frac{1}{\hat{\lambda}_{\epsilon}^{(D)}(u_i)} - |L| \right|.$$

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- ▶ use the smoothing bandwidth parameter ϵ which miximaize (Loader (1999))

$$\sum_{i=1}^n \log(\hat{\lambda}_{\epsilon}^{(D)}(u_i)) - \int_L \hat{\lambda}_{\epsilon}^{(D)}(u) d_1 u \quad (4)$$

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