Homework 7

Math 123

Due March 16, 2023 by midnight

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Topics covered: Graph coloring, chromatic polynomial, Turan graphs

Instructions:

- This assignment must be submitted on Gradescope by the due date.
- If you collaborate with other students (which is encouraged!), please mention this somewhere on the assignment.
- If you are stuck, please ask for help (from me, a TA, a classmate). Use Campuswire!
- You may freely use any fact proved in class. In general, you should provide proof for facts used that were not proved in class.
- Please restrict your solution to each problem to a single page. Usually solutions can be even shorter than that. If your solution is very long, you should think more about how to express it concisely.

Problem 1. *Give an example or explain why no example exists: A graph G that is neither complete nor an odd cycle, but for which the greedy coloring uses $\Delta(G) + 1$ colors.*

Solution. Counterexample:

$$1 \text{ ——— } 4 \text{ ——— } 3 \text{ ——— } 2$$

Let a, b, c be the colorings in alphabetic order $a \leq b \leq c$ then

1 is colored a since a is the minimum coloring and it has no colored neighbors

$$a \text{ ——— } 4 \text{ ——— } 3 \text{ ——— } 2$$

2 is colored a since a is the minimum coloring and it has no colored neighbors

$$a \text{ ——— } 4 \text{ ——— } 3 \text{ ——— } a$$

3 is colored b since a is used by its neighbor 2

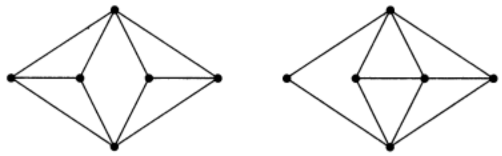
$$a \text{ ——— } 4 \text{ ——— } b \text{ ——— } a$$

4 is colored c since a is used by its neighbor 1 and b is used by its neighbor 3

$$a \text{ ——— } c \text{ ——— } b \text{ ——— } a$$

□

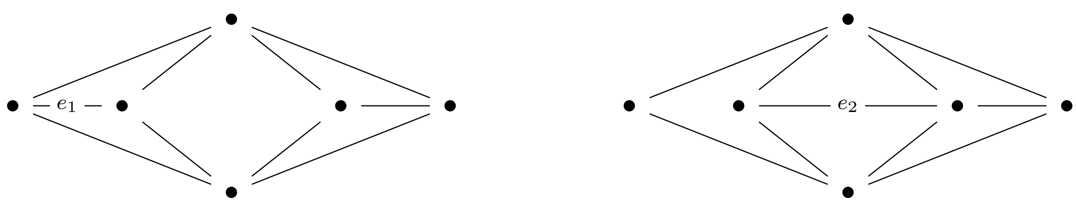
Problem 2. *Give a very short proof that the following two graphs have the same chromatic number.*[1]



Solution. Let the left graph be called G_1 and the right graph be called G_2. We know that $\chi(G,t) = \chi(G \setminus e, t) - \chi(G \cdot e, t)$. Therefore, we can show the two graphs are have the same chromatic polynomial if they both have edges e_1, e_2 s.t.

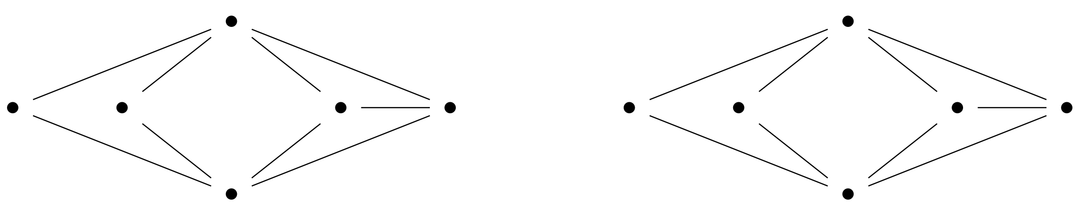
$$\chi(G_1,t) = \chi(G_1 \setminus e_1, t) - \chi(G_1 \cdot e_1, t) = \chi(G_2 \setminus e_2, t) - \chi(G_2 \cdot e_2, t) = \chi(G_2, t)$$

where $G_1 \setminus e_1 = G_2 \setminus e_2$ and $G_1 \cdot e_1 = G_2 \cdot e_2$ by selecting e_1 and e_2 as shown below:

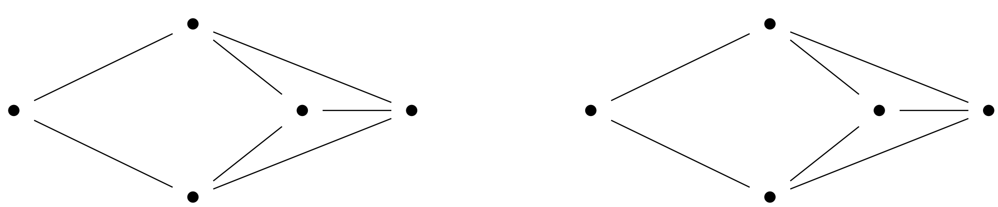


Here is some work of showing the equivalence of the statements.

$G_1 \setminus e_1 = G_2 \setminus e_2$:



$G_1 \cdot e_1 = G_2 \cdot e_2$:



□

[1]Note: solutions that construct optimal colorings of these graphs will not receive credit.

Problem 3. *Let $G = M_{n_1,\ldots,n_k}$ be a complete k-partite graph with $n = n_1 + \cdots + n_r$ vertices. Show that if $n_i - n_j \geq 2$ for some i, j, then there exists a k-partite graph with n vertices and more edges than G.*

Solution. □

Problem 4. *Given a set of lines in the plane with no three meeting at a point, form a graph G whose vertices are the intersections of the lines, with two vertices adjacent if they appear consecutively on one of the lines. Prove that $\chi(G) \leq 3$.* [2] [3]

Solution. □

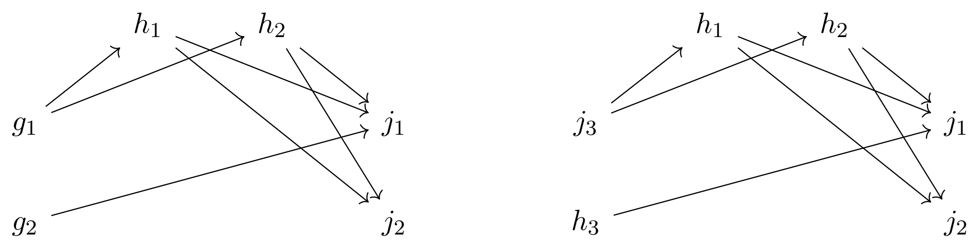
[2] Suggestion: start by looking at some explicit examples.

[3] Hint: use a greedy coloring with an appropriate vertex ordering.

Problem 5. *Let G be a graph with chromatic number k. Show that for every k-coloring of G and for each color i, there is a vertex of color i that is adjacent to vertices of the other $k-1$ colors.* [4]

Solution. Suppose for some k coloring of G there does not exist a vertex of color i s.t. it is adjacent to vertices of the all the other $k-1$ colors. Since no vertex is connected all to the other $k-1$ colors then there exists at least one color group that each vertex is not connected to. Therefore, we can take the vertices in the i colored group and disperse them amongst the other colors, resulting in a coloring of G with $k-1$ colors - a contradiction because G has chromatic number k.

Example: Here the g colored group does not contain a vertex adjacent to the other $k-1$ colors so the vertices in the g colored group can be assigned to the other colors $g_1 \to j_3, g_2 \to h_3$:



$\square$

[4]Hint: think back to the proof that a graph with chromatic number k has at least $\binom{k}{2}$ edges.

Problem 6. *Prove that $\chi(G) = \omega(G)$ when the complement $\bar{G}$ is bipartite.* [5] [6] [7]

Solution. □

[5] Here $\omega(G)$ is the clique number: the largest m so that G contains K_m.

[6] Hint: look to apply König's theorem. (!)

[7] This is a pretty challenging problem. If you want more hints, please ask.

Problem 7 (Bonus)**.** *Let $G = (V, E)$ be the unit distance graph in the plane: $V = \mathbb{R}^2$, and two points are adjacent if their Euclidean distance is 1. (a) Use the hexagonal tiling to prove $\chi(G) \leq 7$. (b) Prove that $\chi(G) \geq 3$.*

Solution. □