Homework 9

Math 123

Due April 14, 2023 by midnight

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Topics covered: Ramsey theory

Instructions:

- This assignment must be submitted on Gradescope by the due date.
- If you collaborate with other students (which is encouraged!), please mention this somewhere on the assignment.
- If you are stuck, please ask for help (from me, a TA, a classmate). Use Campus wire!
- You may freely use any fact proved in class. In general, you should provide proof for facts used that were not proved in class.
- Please restrict your solution to each problem to a single page. Usually solutions can be even shorter than that. If your solution is very long, you should think more about how to express it concisely.

Problem 1. *Prove that $R(k,\ell) \leq \binom{k+\ell-2}{k-1}$ and deduce that $R(k,k) \leq 4^k$.* [1]

Solution. First consider $R(n,2) = n$. Suppose K_n does not have all edges of the same color (blue), then the must be an edge of a different color (red) and so there exists K_2 with red edges. However, in K_{n-1} we could make all the edges the same color (blue) and it wouldn't meet either criteria.

We prove $R(k,\ell) \leq \binom{k+\ell-2}{k-1}$ by induction (without loss of generality let $k \geq \ell$):

Base Case: ($\ell = 2$)

Earlier we showed $R(k,2) = k$ then $R(k,2) = k \leq \binom{k+2-2}{k-1} = \binom{k}{k-1} = \binom{k}{1} = k$

Inductive Step: By Ramsey's theorem we know:

$$R(k,l) \leq R(k-1,l) + R(k,l-1)$$

Therefore, using the inductive hypothesis:

$$R(k,l) \leq R(k-1,l) + R(k,l-1) \leq \binom{k+\ell-3}{k-2} + \binom{k+\ell-3}{k-1} = \binom{k+\ell-2}{k-1}$$

To find the 4^k upper bound we can evaluate $(x+1)^{2k-2}$ at $x = 1$ since one of the coeffecents of $(x+1)^{2k-2}$ must be $\binom{k+k-2}{k-1} = \binom{2k-2}{k-1}$. Therefore, the upper bound must be $2^{2k-2} = 2^{2(k-1)} = 4^{k-1} \leq 4^k$

$\square$

[1] Use induction for the first part. You will probably want to use some well-known facts about binomial coefficients, which you can google.

Problem 2. *Use the pigeonhole principle*[2] *to prove that every set of n integers $\{a_1, \ldots, a_n\}$ contains a nonempty subset whose sum is divisible by n.* [3][4] *Give a collection of $n-1$ integers with no such subset.*

Solution. Let the partial sums be $S_i = a_1 + \cdots + a_i$.

If any S_i is divisible by n then we are done, but assume this is not the case. Therefore, by modding out by n, S_i are in $n-1$ equivalence classes. Since there are n partial sums one of the classes must contain two sums S_i, S_j, by pigeonhole principle. Therefore, we can take the difference of the sums ($a_i + \cdots + a_j$ where $k > i$) and see that it mod n is zero.

$\{2, 2, 2, 2\}$ is a set where no subset is divisible by $4 + 1 = 5$ $\square$

[2] The pigeonhole principle says that if you put m pigeons into n holes, then there is a hole with at least $\lceil m/n \rceil$ pigeons. We used this implicitly multiple times in the lecture 4/4 (it may be helpful to look back over the notes to see where it was used).

[3] Hint: consider the partial sums $S_i = a_1 + \cdots + a_i$.

[4] Remark: The set $\{1, n+1, 2n+1, \ldots, (n-1)n+1\}$ shows you cannot improve this result to a set of $n-1$ integers.

Problem 3. *You're given two concentric discs, each with* 20 *radial sectors of equal size. For each disc,* 10 *sectors are painted red and* 10 *blue, in some (arbitrary) arrangement. Prove that the two discs can be aligned so that at least* 10 *sectors on the inner disc match colors with the corresponding sector on the outer disc.* [5] [6]

Solution. Each section can have a corresponding match with 10 other colors and there are 20 sections, so there are 200 possible corresponding matches. Since there are 20 ways to match the inner wheel with the outer wheel, one position of the inner wheel must have $200/20 = 10$ corresponding matches, by pigeonhole (suppose there is one position which has less than 10 corresponding matches, then, since the total is 200, there must be one with more than 10 corresponding matches). □

[5] Hint: There is a very short solution using the pigeonhole principle.
[6] Hint: count the total number of inner/outer sector pairs with matching colors.

Problem 4. *Fix $r \geq 2$ and let $R(k, \ldots, k)$ (with r copies of k) denote the minimal n so that any r-coloring of the edges of K_n contains at a monochromatic K_k (i.e. we are generalizing the Ramsey numbers to more colors). Prove that every r coloring of the numbers $1, \ldots, R(3, \ldots, 3)$ (with r copies of 3) contains a monochromatic x, y, z so that $x + y = z$.* [7] [8]

Solution. Let $n = R(3, \ldots, 3)$. Take K_n, we will color the edges of the vertices $v_1 \cdots v_n$ based on a function that considers the distance $|i - j|$ between two vertices v_i and v_j.

Because K_n has $R(3, \ldots, 3)$ vertices there will be a monochromatic K_3, $\{v_a, v_b, v_c\}$, in K_n. Therefore, there exists $x = b - a, y = c - b, z = c - a$, so $x + y = z$ $\square$

[7] Hint: set $n = R(3, \ldots, 3)$. Given a coloring of $1, \ldots, n$, construct an edge 2-coloring of K_n so that a monochromatic triangle in K_n corresponds to monochromatic x, y, z with $x + y = z$.

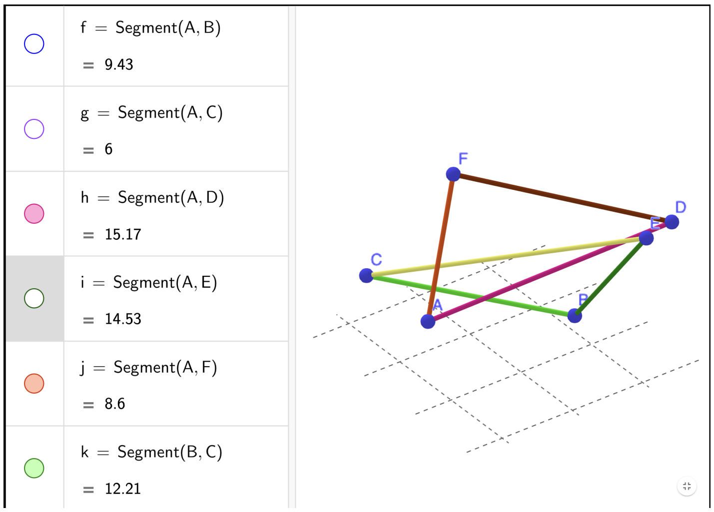
[8] Hint: labelling the vertices of K_n by $v_1, \ldots, v_n$, choose the coloring of the edge $v_i v_j$ using the color of $|j - i|$ as a guide.

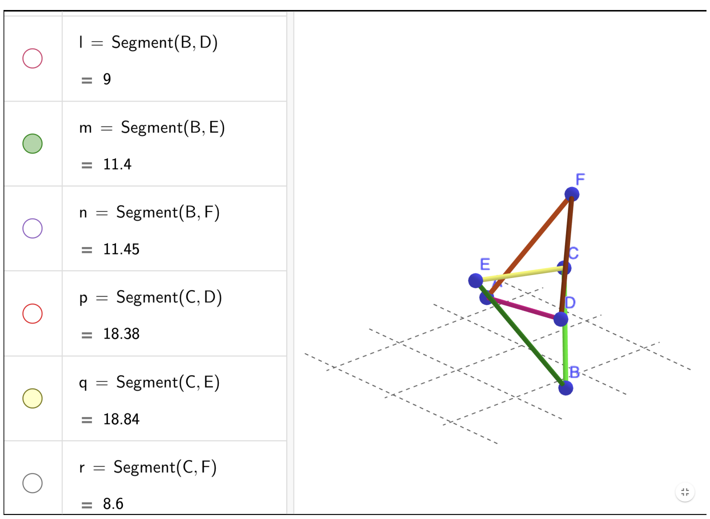
Problem 5. *A theorem similar to Nepali's theorem says that for any embedding of K_6 in $\mathbb{R}^3$, there exists a pair of triangles that form a Hope link.*[9] *Verify this result for the following set of 6 points in $\mathbb{R}^3$.*

$$A = (0, 3, 2), \quad B = (-2, -6, 0), \quad C = (6, 3, 2), \quad D = (-6, -10, 7), \quad E = (-9, -6, 9), \quad F = (3, -1, 9)$$

To help solve this, use this Geo-gebra notebook: `https://www.geogebra.org/m/ng6pspkh` [10] *Include a screenshot in your solution.*

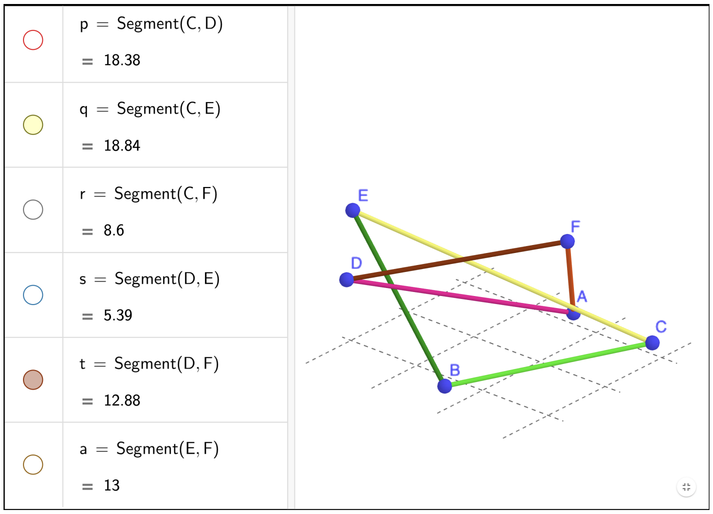
Solution. Here we can see a Hopf link with A, D, F and B, C, D





[9] Google this.

[10] Click the circles on the left to make line segments appear/disappear.


p = Segment(C, D)
= 18.38
q = Segment(C, E)
= 18.84
r = Segment(C, F)
= 8.6
s = Segment(D, E)
= 5.39
t = Segment(D, F)
= 12.88
a = Segment(E, F)
= 13
E
F
D
A
C
B

$\square$

Submit a final project outline. See other document for instructions.