

# Graph Theory

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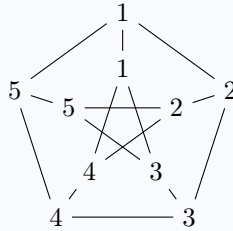
## Lecture 1

### 0 Fundamental Concepts

1/26/2023

**Definition 1 (graph).** A graph is a pair  $(V, E)$  where  $V$  is the vertex space and  $E$  is the edge space.

**Example (petersen graph).** An element subset of  $\{1, 2, 3, 4, 5\}$  connected by disjointness



**Note.** In this course we are excluding multiple edges and self loops

**Definition 2 (vertex degrees).** Let,  $G = (V, E)$ ,  $v \in V$ ,  $e \in E$   $e$  are incident if  $v \in e$  i.e.  $v$  is an endpoint of  $e$

**Lemma 1.**

$$\sum_{v \in V} \deg(v) = \sum_{v \in V} \sum_{e \in E} 1 = \sum_{v \in E} \sum_{v \in V} 1 = 2|E|$$

**Proof.** Every edge has two vertices ■

**Definition 3 (complete graph).** Represented  $K_n$ , the graph has  $V = 1 \dots n$  and all possible edges

$$|E| = \frac{n(n-1)}{2} = \binom{n}{2}$$

**Definition 4 (isomorphic).** Graphs  $G_1 = (V_1, E_1)$  and  $G_2 = (V_2, E_2)$  are isomorphic if there exists a bijection,  $f : v_1 \rightarrow v_2$ , s.t.  $\{u, v\} \in E_1 \Leftrightarrow \{f(u), f(v)\} \in E_2$

#### 0.1 Connectivity

**Definition 5 (connected).**  $u$  and  $v$  are connected if there exists a path from  $u$  to  $v$ . A graph is connected when all vertices are connected

## Lecture 2

### 0.2 Degree Sequences

2023-01-31

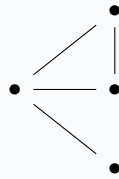
**Definition 6** (degree sequence). List of vertex degrees in decreasing order

**Note.** Isomorphic graphs  $\Rightarrow$  same degree sequence; however, if they have the same degree sequence they are not necessarily isomorphic

**Definition 7** (graphic). A sequence is graphic if it's the degree sequence of some graph

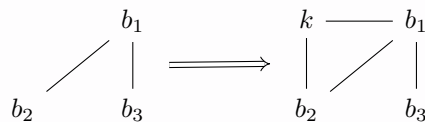
**Example.** Here are some degree sequences which may or may not exist:

- $(3, 2, 1, 1)$  - not possible since they don't sum to an even number
- $(3, 3, 1, 1)$  - not graphic since there are not enough vertices
- $(3, 2, 2, 1)$  - graphic



**Theorem 1** (Havel-Hakimi).  $(a_1 \dots a_n)$  is graphic iff  $(a_2 - 1, a_3 - 1, \dots, a_{a_1+1} - 1, a_{a_1+2} \dots a_n)$  is graphic

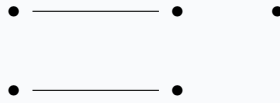
**Proof.** We can apply the theorem in reverse to understand the intuition. If we have a graphic sequence,  $(b_1 \dots b_m)$ , we can add a vertex with degree  $k$  such that  $(k, b_1 + 1, \dots, b_k + 1, b_{k+1} \dots b_m)$ .



In the reverse direction, we can subtract a vertex from a graph, and we make a transformation (2-switch) which preserves degree sequences but makes the graph maximal, to get the reverse result. ■

**Example.**  $(3, 3, 3, 3, 3, 2, 2, 1)$  eventually becomes  $(1, 1, 1, 1, 0)$  which we can show is

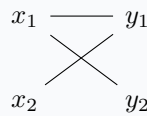
graphic:



### 0.3 Bipartite graphs

**Definition 8 (bipartite graph).** A graph is bipartite if it's possible to color vertices using only 2 colors. A simple check for it being bipartite is to check if there are no odd cycles

**Example.**



Here  $G = (X \sqcup Y, E)$  where  $x_1, x_2 \in X$  and  $y_1, y_2 \in Y$

**Theorem 2 (bipartite iff no odd cycle).** A graph is bipartite  $\Leftrightarrow$  if it contains no odd cycle

**Proof.** It is clear that if a graph contains an odd cycle that it is not bipartite, so we have to prove the reverse direction: if a graph does not contain an odd cycle then it is bipartite. It suffices to prove the statement for connected graphs - apply the argument to each component.

By induction:

*Base Case:* When  $|E| = 0$  there are no connected vertices so there is no odd cycle and the graph must be bipartite since no vertices are connected to vertices of the same color.

*Inductive Step:* Assume a graph s.t.  $|E| = n - 1$  and no odd cycles is bipartite. Any connected vertices of the same color must be connected by a path of even vertices. Let us add an edge which maintains the restriction that there is no odd cycle in the graph. Suppose an edge is added between these vertices, then the graph is no longer bipartite since two vertices of the same color are connected, the addition of an edge will create an odd cycle — a contradiction. Therefore, when adding an edge to keep the graph bipartite, there must be no odd cycle. ■

### 0.4 Walks and Paths

**Definition 9 (walk).** A walk is a sequence of vertices that are connected by edges

**Definition 10 (length).** The number of edges contained in the walk

**Definition 11 (path).** A path is a walk that has unique vertices



**Lemma 2.** A walk from  $v_0$  to  $v_n$  implies a path from  $v_0$  to  $v_n$

## 0.5 Closed Walks and Cycles

**Definition 12** (closed walk). A closed walk is a walk which starts and ends at the same vertex

**Definition 13** (cycle). A cycle is a closed walk that has unique vertices

**Lemma 3** (closed walk). A closed walk of odd length contains an odd cycle

**Proof.** By induction:

*Base Case:*  $k = 1$  A closed walk of length 3 ( $2k + 1$ ) must be a cycle

*Inductive Step:* If all vertices in the walk are distinct, we are done since it is a cycle of odd length. In the other case where there are repeated, we can split the walk on repeated vertices to get smaller walks which are proved in the previous cases ■

## Lecture 3

**Remark.** Algorithmic Bipartite Testing

2023-02-02

- Brute force:  $2^{|V|} \cdot |E|$
- Proof Algorithm: where we color the vertices and check the edges  $|V| + |E|$

**Remark** (local to global). Global properties always lead to local results (ex. bipartite implies no odd cycles). In West it is called "TONCAS"

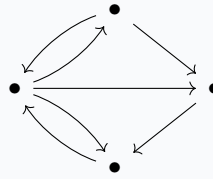
## 0.6 Eulerian Graphs

**Definition 14** (euler tour). A closed walk that visits each edge of a graph exactly once

**Definition 15** (eulerian). A graph with an euler tour is Eulerian.

**Proposition 1** (eulerian local property). A graph is Eulerian iff every vertex has even degree.

**Example** (seven bridges of Königsberg).



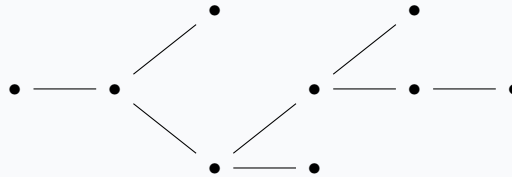
Using a graph to model the seven bridges of Königsberg shows that an Euler tour is not possible.

**Note.** This is another example of local and global properties where the parity of the degree is the local property and Eulerian is the global property

## 1 Trees

**Definition 16** (tree). A tree is a **connected** and **acyclic** graph

**Example.**



**Definition 17** (leaf). A node of degree 1

**Lemma 4.** Any connected subgraph of a tree is a tree

**Proof.** By contradiction:

Assume that the connected subgraph is not a tree, then the subgraph has a cycle, therefore the graph has a cycle, but trees are acyclic. Therefore, contradiction. ■

**Lemma 5.** A tree with  $n$  vertices has  $n - 1$  edges

**Proof.** By induction:

*Base Case:* There are 0 edges in a 1 vertex tree

*Inductive Step:* Suppose this holds for  $n$ . Let  $T$  be a tree with  $n + 1$  vertices, then by removing one vertex, which must be a leaf — removing a non-leaf would make the graph unconnected, you remove one edge and the graph becomes the  $n$  case. Therefore, the  $n + 1$  tree has 1 more edge than an  $n$  tree. ■

## 1.1 Spanning Trees

**Definition 18** (spanning tree). A spanning tree (ST) is a subgraph of a **connected** that is a **tree** that contains all vertices of the original graph

**Lemma 6.** There is a spanning tree in every connected graph

**Proof.** By contradiction:

Assume  $G$  is a connected graph with no ST. Let  $T$  be a connected subgraph of  $G$  that has the same vertices as  $G$  with the smallest number of edges. Since  $T$  is not a tree, it does not have a cycle. However,  $T$  containing a cycle would imply that  $T$  is not a subgraph that has the smallest number of edges. Therefore, contradiction test ■

**Proposition 2.** With a graph  $G$  where  $T, T' \subset G$  s.t.  $e \in T, e \notin T'$ . Then there is edge  $e' \in T'$  s.t.  $T - e + e'$  is a spanning tree.

**Proof.**  $T$  is a tree  $\Rightarrow T - e$  is disconnected.  $T, T'$  have the same vertex set  $\Rightarrow \exists e \in T'$  that connects two components of  $T - e$ . Let's claim that  $T - e + e'$  is a spanning tree. It has the same number of edges and is connected, so it's a tree, and is spanning since it connects all edges. ■

## Lecture 4

### 1.2 Prüfer Codes

2023-02-09

**Definition 19** (prüfer code). A code derived from a tree by iteratively removing the smallest degree vertex and adjoining its neighbor until there are only two vertices left.

**Example** (prüfer code to tree). Let's take the example  $(1, 4, 1, 1)$

We know that vertices of deg 1 do not appear, so we start with an edge from 2 to 1 then we remove 1 from the list. Next we start with 4 and add the next deg 1 vertex. Now we have a connection from 4 to 1 since it's left in the Prüfer code. Then we finally create a tree with vertices 1, 5, 6 and adjoin it.

**Theorem 3.** For each sequence  $(a_1 \dots a_{n-2})$  there exists a unique tree  $T$  with  $P(T) = (a_1 \dots a_{n-2})$

**Proof.** By induction:

*Base Case:*  $n = 2$

Sequences have length  $n - 2 = 0$  and there is only one tree with 2 vertices

*Inductive Step:*

For any tree with code  $(a_1 \dots a_{n-2})$  Let  $x$  be the smallest index not in the sequence then we can construct an edge between it and  $a_1$  and remove  $a_1$  from the code. Therefore, we have a code with one less element. And, by induction,  $\exists!$  a tree  $T'$  with vertices

$\{1 \dots n\} \setminus x$  and code  $(a_2 \dots a_{n-2})$ .

The uniqueness comes from the fact that  $T'$  is unique and there is only one way to add an edge ■

**Theorem 4 (Cayley).** There is a bijection between trees and these sequences. Therefore, we can count trees easily by counting sequences.

Therefore, we find there are  $n^{n-2}$  trees given  $n$  vertices

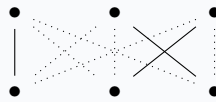
## Lecture 5

### Matchings

2023-02-14

**Definition 20 (matching).** A matching is a subset of  $M \subset E$  such that no two edges in  $M$  share a vertex

**Example.**



**Definition 21 (saturates).** For  $U \subset V$  say  $M$  saturates  $U$  if each  $u \in U$  is incident to some  $e \in M$ . A matching that saturates  $V$  is called perfect

**Remark.** Perfect matching relate to bijections.

### 1.3 Matchings in Bipartite Graphs

For  $G = (X \sqcup Y, E)$ , is there a matching that saturates  $X$ ?

**Proposition 3 (saturated matching for bipartite graph).** Local to Global: Let  $N(S)$  be set of vertices connected to vertices in  $S$ . The local property is that for  $S \subseteq X$ ,  $|S| \leq |N(S)|$

**Theorem 5 (Hall's Matching).**  $G = (X \sqcup Y, E)$  bipartite has matching saturating  $X$  iff  $(\forall S \subseteq X)(|S| \leq |N(S)|)$

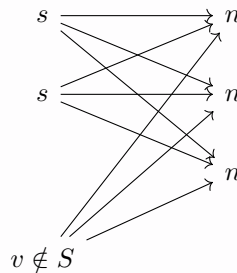
**Proof.** The proof requires the maximum matching theorem. Suppose  $G$  does not have a matching that saturates  $X$ . Let  $M$  max matching that does not saturate  $X$ ,  $u \in X$  unsaturated.  $S = \{x \in X \text{ connected to } u \text{ by a } M\text{-alt path}\}$   $T = \{y \in Y \text{ connected to } u \text{ by a } M\text{-alt path}\}$  Any  $M$ -alt path from  $u$  to  $y \in T$  can be extended uniquely to  $M$ -alt path ending in  $S$ . Therefore, there is a bijection from  $T$  to  $S \setminus u$

$N(S) = T$  since  $T \subset N(S)$  by definition and  $N(S) \subset T$  because we can fix a  $y \in N(S)$  which is connected to an  $x$  which is connected to  $u$ . ■

**Corollary.** Any  $k$ -regular bipartite graph has perfect matching

**Proof.** Fix  $G = (X \cup Y, E)$  bipartite and  $k$ -regular

Show  $|X| = |Y|$ :  $k|X| = \sum_{x \in X} \deg(x) = |E| = \sum_{y \in Y} \deg(y) = k|Y|$  Fix  $S \in X$ . Each edge leaving  $S$  lands in  $N(S)$ . There are  $k|S|$  edges leaving  $S$ . Therefore,  $k|S| \leq k|N(S)|$ .

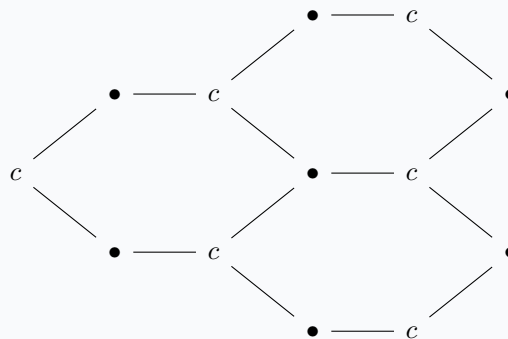


## 1.4 Matching Dual Problem

**Definition 22 (maximum).** A matching  $M \in E$  is maximum if it has the most edges of any matching

**Definition 23 (vertex cover).** A vertex cover of  $G$  is a subset  $Q \subset V$  s.t. every edge is incident to some vertex in  $Q$

**Example.**



Where vertices labeled  $c$  are a vertex cover

**Remark.** Matchings and vertex covers are dual problems

## Lecture 6

**Lemma 7.** Let  $Q$  be a vertex cover,  $M$  a matching Then  $|M| \leq |Q|$

2023-02-16

**Proof.** For each  $e \in M$  there is at least one vertex of  $Q$  incident to  $e$ . No two  $e, e' \in M$  share a vertex of  $Q$  ■

**Note.** We can use this to show that if we have a vertex cover that covers all vertices with the same size as a matching we must have a maximum matching/minimum vertex cover.

**Theorem 6 (König).** Let  $G$  bipartite, then the max size of the matching is equal to the min size of the vertex cover

**Remark.** Therefore, matching and vertex covers are dual problems

**Definition 24 (m-augmenting-path).** A path whose edges alternate between  $M$  and  $E \setminus M$  whose end points are unsaturated.

**Theorem 7 (maximum matching).** Local to Global property: If there's no M-augmenting path then  $M$  is maximum

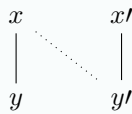
**Proof.** Suppose  $M$  is not a maximum there exists an  $M'$  s.t.  $|m| < |M'|$  consider the symmetric difference  $M \triangle M'$ . Here vertices have either degree 1 or 2 and the graph is a union of even cycles and paths.

Let  $M$  be a non-maximal matching,  $M'$  maximum matching. Consider  $M \triangle M'$  By above this is a union of paths and even cycles. Since  $M$  and  $M'$  share the same size in cycles then some component in  $M \triangle M'$  has a path ■

## Lecture 7

**Definition 25 (stable matching).** A matching is stable if

2023-02-23



then either  $x$  prefers  $y$  to  $y'$  or  $y'$  prefers  $x'$  to  $x$

**Theorem 8 (Gale-Shapely).** Stable matchings always exist in bipartite graphs

**Proof.** We perform an algorithm to construct a stable matching:

**Round 1:**

Each  $x \in X$  proposes to top choice

Each  $x \in X$  proposes to top choice

**Subsequent Rounds:** Each unmatched  $x$  proposes to top choice not yet proposed to. Each  $y$  accepts the best proposal possible breaking an engagement

**Repeat**

*Features*

1. Algorithm stops after  $< |E|$  rounds
2. End matching is stable
3. Proposers get the best match among all stable matchings. Proposees get worse.

■

## Lecture 8

### 2 Connectivity

2023-02-28

**Definition 26** (vertex cut). A vertex cut on a graph  $G$  is a subset  $S \subset V$  s.t.  $G \setminus S$  is disconnected

**Definition 27** (vertex connectivity). Represented  $\kappa(G)$ , the vertex connectivity is the minimum size of a vertex cut of  $G$ . For  $K_n$ ,  $\kappa$  is not defined.

**Lemma 8.** If  $G$  is not complete then  $G$  has a vertex cut

**Proof.** Let  $G$  not be complete, then there exist  $u, v \in V$  s.t.  $\{u, v\} \notin E$ . Then  $S = V \setminus \{u, v\}$  is a vertex cut ■

#### 2.1 Connectivity Duel Problem

**Definition 28** (vertex connectivity between two vertices). Let  $G = (V, E)$ , for  $x, y \in V$  define  $\kappa(x, y)$  min size of vertex cut that disconnects  $x$  and  $y$

**Note.** We see that the points we have to remove are related to paths from  $x$  to  $y$  which are disjoint.

**Definition 29 (disjoint path).** A path is disjoint with another path if there are no interior vertices they share

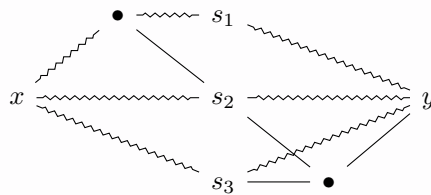
**Definition 30 (max pairwise disjoint).** Let  $\lambda(x, y)$  be the max number of pairwise disjoint  $x, y$ -paths

**Theorem 9 (Menger).** Let  $G = (V, E)$ ,  $x, y \in V$ ,  $\{x, y\} \notin E$  then  $\kappa(x, y) = \lambda(x, y)$

**Proof.**  $\kappa(x, y) \geq \lambda(x, y)$  is easy  
 $\kappa(x, y) \leq \lambda(x, y)$ :

Let  $S$  be a minimum vertex cut. We want to show that there are  $|S|$  disjoint  $(x, y)$  paths. Consider two subgroups  $G_x$  and  $G_y$  where  $G_x$  is the union of all paths from  $x$  to  $S$  (doesn't contain vertices from  $S$ ) and  $G_y$  is the union of all paths from  $y$  to  $S$  (doesn't contain vertices from  $S$ ). We have broken up the problem into just working with finding paths to  $S$  from  $x$  and  $y$ .

To form a picture like:



■

## 2.2 Max Flow, Min Cut

**Definition 31 (network).** Is a tuple  $G, s, t, c$  where  $G = (V, E)$  is a directed graph,  $s, t \in V$  "source" and "terminus",  $c : E \rightarrow \mathbb{N}$  capacity.

**Definition 32 (flow).** A flow on a network is  $f : E \rightarrow \mathbb{N}$  s.t.  $f(e) \leq c(e)$  and conservation law  $f^+(v) = \sum_{e \rightarrow v} f(e) = \sum_{v \rightarrow e} f(e) = f^-(v)$

**Definition 33 (value).** Value of  $f$  is  $f^-(s)$

The general problem is that given a network what is the max value of a flow

**Definition 34 (cut).** A cut is a partition  $V = S \sqcup T$  with  $s \in S$ ,  $t \in T$

**Definition 35 (capacity).** The capacity of a cut  $S, T$  is  $\sum_e c(e)$  where  $e$  is an edge from  $S$  to  $T$



## 2.3 Ford-Fulkerson

**Theorem 10 (Ford-Fulkerson).** Max value of the flow is equal to the min capacity of a cut

## Lecture 9

**Note.** Capacity of any cut gives upper bound on value of any flow. From this we can see the reasoning of why Ford-Fulkerson finds this as a dual problem

2023-03-02

## Lecture 10

## 3 Graph Coloring

2023-03-07

**Example (exam scheduling).** Define  $G = (V, E)$  where  $V$  are courses and  $E$  are conflicts if a student takes both courses. Here a graph coloring can help us answer how to schedule classes to limit conflicts.

**Definition 36 (vertex coloring).** Is a function of  $V \rightarrow \mathbb{N}$  s.t. if  $\{u, v\} \in E$  then  $c(u) \neq c(v)$

**Definition 37 (chromatic number).** Represented  $\chi(G)$ , the chromatic number is the minimum number of colors needed to color a graph.

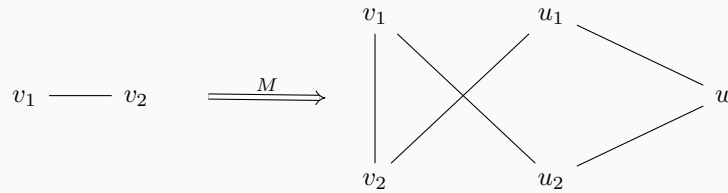
**Lemma 9.**  $\chi(G) \leq \Delta G + 1$  where  $\Delta G$  is the max degree of a vertex in  $G$

**Theorem 11 (Brooks).**  $\chi(G) = \Delta(G) + 1 \Rightarrow G$  is an odd cycle or complete

**Definition 38 (clique number).** Represented  $\omega(G)$ , the clique number is the largest number s.t.  $K_n$  is subgraph of  $G$  (this implies  $\chi(G) \geq n$ )

### 3.1 Mycielski Construction

**Example** (mycielski construction).



With construction that if  $G = (V, E)$  where  $V = \{v_1 \dots v_n\}$  then  $M(G)$  has vertices  $\{v_1 \dots v_n, u_1 \dots u_n, w\}$  and edges  $\{v_i, v_j\}, \{v_i, u_j\}, \{u_i, w\}$ , for  $\{v_i, v_j\} \in E$

**Theorem 12** (Mycielski).

1.  $\chi(G) = k \Rightarrow \chi(M(G)) = k + 1$
2.  $G$  doesn't contain  $K_3 \Rightarrow M(G)$  doesn't contain  $K_3$

**Proof.** *First statement:* Given  $k$ -coloring of  $G$ , we can  $(k + 1)$  color  $M(G)$ , where you color  $u_i$  the same as  $v_i$  and  $w$  the  $k + 1$  color. We also want to show that the graph has no smaller  $k$  coloring. Suppose  $M(G)$  has a  $k$ -coloring then  $U$  uses a  $k - 1$  coloring and since the  $U$  coloring can be sent to the  $V$  coloring. Therefore,  $G$  has a  $k - 1$  coloring - a contradiction. ■

**Remark.** Therefore, the clique number is not a very strong property

### 3.2 Coloring Extremal Problem

**Note.** Coloring graphs is an NP problem, so it might be better to try to solve extremal problems

**Question** (coloring extremal problems). Among graphs with a  $\chi(G) = k$  what is the maximal/minimal number of edges?

## Lecture 11

**Definition 39** (turan graphs). Graphs of the form

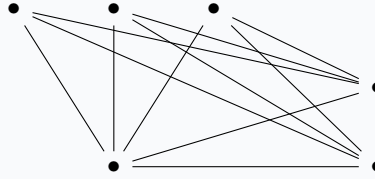
2023-03-09

$$T_{n,k} = M_{\underbrace{q \dots q}_{k-r}, \underbrace{q+1 \dots q+1}_r}$$

where  $n = qk + r$  s.t.  $r < k$  are the solutions to the maximal coloring problem.

**Proof.** Pretty obvious with calculus based approach ■

**Example.**  $T_{6,3} = M_{3,2,1}$



### 3.3 Chromatic Polynomial

**Definition 40 (chromatic polynomial).**  $\chi(G, t)$  = number of colorings of  $G$  using at most  $t$  colors

1. Degree =  $|V|$
2.  $a_{n-1} = |E|$
3.  $\chi(G)$  = smallest  $t$  s.t.  $\chi(G, t) \neq 0$  ( $0, \dots, \chi(G) - 1$  are roots of  $\chi(G, t)$ )
4.  $t^d$  divides  $\chi(G, t) \Rightarrow G$  has  $\geq d$  components
5. coefficients are log concave

**Example (chromatic polynomial of a complete graph).**  $\chi(K_n, t) = t(t-1)(t-2)\dots(t-(n+1)) = \binom{t}{n}$

**Example (chromatic polynomial of a tree).** Given a tree  $T$ , and a subtree  $S$  with one less vertex. Inductively,  $\chi(T, t) = (t-1)\chi(S, t)$ . And the coloring does not depend on what tree it is, only the number of vertices.  $\chi(T, t) = t \cdot (t-1)^{n-1}$

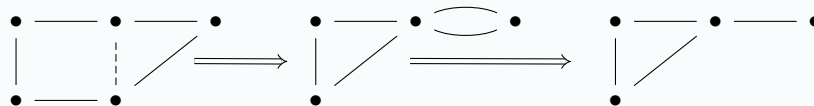
**Theorem 13.** The chromatic polynomial is a polynomial

**Proof.**  $\chi(G, t) = \chi(G \setminus e, t) - \chi(G \cdot e, t)$

Where  $G \cdot e$  is the contraction formula

Write  $V = I_1 \sqcup \dots \sqcup I_r$  then  $\chi(G, t) = \sum_{r=1}^{|V|} a_r(G) \cdot \chi(K_r, t)$  where  $a_r(G)$  are the number of ways to write  $V = I_1 \sqcup \dots \sqcup I_r$  ■

**Definition 41 (contraction formula).**



**Proof.**

$$\chi(G, t) = \chi(G \setminus e, t) - \chi(G \cdot e, t)$$

Any  $t$  coloring gives a  $t$  coloring of  $G \setminus e$ . A coloring of  $G \setminus e$  gives a coloring of  $G$  when the endpoints  $e$  are independent. Colorings of  $G \setminus e$  correspond to colorings of  $G \cdot e$  ■

**Example (sudoku).** We form a graph where vertices are squares on the board and vertices are connected if they are in the same row column or square on the grid. Now this is made into a coloring problem.

How many  $9 \times 9$  Sudoku puzzles?

Each graph has 81 vertices with degree 20, so there are 810 edges.

## Lecture 12

**Definition 42 (independent).** A subset  $I \subseteq V$  is independent if there are no edges  $\{u, v\}$  where  $u, v \in I$

2023-03-14

## 4 Planar Graphs

**Definition 43 (planar).** A graph is planar if it can be drawn on the plane with no edges crossing.

**Example.**  $K_4$



### 4.1 Theorems on the Real Plane

**Theorem 14 (Jordan Curve).** Any circle in  $\mathbb{R}^2$  splits  $\mathbb{R}^2$  into two regions, one bounded and one unbounded

**Theorem 15 (Euler's Formula).** Let  $G = (V, E)$  be embedded in  $\mathbb{R}^2$  and connected,  $|F|$  be the number of regions of  $\mathbb{R}^2 \setminus G$ , then the  $|V| - |E| + |F| = 2$

**Proof.** By induction:

*Base Case:*  $(|E| - |V| = -1)$

Here  $G$  is a tree, so  $|V| - |E| = 1$  and  $|F| = 1$ ; therefore  $|V| - |E| + |F| = 2$

*Inductive Step:* If  $|E| - |V| \geq -1$  then  $G$  has a cycle,  $C$ . Then by fixing an edge  $e$  on  $C$  we can consider

$$\begin{aligned} & |V \setminus e| - |E \setminus e| + |F \setminus e| \\ & |V| - (|E| + 1) + (|F| + 1) \\ & |V| - |E| - 1 + |F| + 1 \\ & |V| - |E| + |F| \end{aligned}$$

and by induction we know  $|V \setminus e| - |E \setminus e| + |F \setminus e| = 2$ , so  $|V| - |E| + |F| = 2$  ■

**Corollary.**  $K_5$  is not planar

**Proof.** By contradiction:

Assume there exists an embedding  $K_5$  in  $\mathbb{R}^2$  then by Euler's Formula

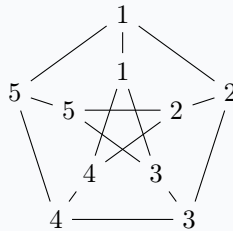
$$\begin{aligned} |F| &= 2 - |V| + |E| \\ |F| &= 2 - 5 + 10 \\ 7 &= 2 - 5 + 10 \end{aligned}$$

But each edge has at most 2 faces and each face has  $\geq 3$  sides therefore  $2|E| \geq 3|F|$ . However,  $2 \cdot 10 \geq 21$ ; therefore  $K_5$  is not planar ■

**Note.** Similar proof can be done for showing  $K_{3,3}$  is not planar

**Theorem 16 (Kuratowski).**  $G$  is planar  $\Leftrightarrow G$  does not contain a subdivision of  $K_5$  or  $K_{3,3}$

**Example.**



The Petersen graph is not planar since it contains a subdivision of  $K_5$

**Remark.** This is an example of another local to global property

## Lecture 13

### 4.2 Linear Planer Embeddings

2023-03-16

**Note.** Our definition of planer does not require straight lines

**Question.** If  $G$  is planar, then can it be drawn using straight lines?

**Definition 44 (linear embedding).** An embedding drawn with only straight lines

**Theorem 17 (Fary).** Every planar graph has a linear embedding

**Proof.** By induction:

*Base Case:* ( $|V| =$ )

*Inductive Step:*

■

**Proposition 4.** A planar graph has at least four vertices of degree  $\leq 5$ .

**Proof.** Suppose the degree of all the vertices are  $\geq 6$  then  $2|E| = \sum \deg(v) \geq 6|V|$  but this contradicts  $|E| \leq 3|V| - 6$  ■

### 4.3 Maximal Planar Graphs

**Observe.** For Fary's theorem adding edges makes it harder to linearly embedded

**Definition 45 (maximal planar).**  $G$  is maximal planar if adding any edge to  $G$  gives a nonplanar graph

**Proposition 5 (max planar properties).** Fix  $G = (V, E)$  The following are equivalent:

1.  $G$  is maximal planar
2.  $|E| = 3|V| - 6$
3. Complementary regions of  $G \hookrightarrow \mathbb{R}^2$  is a triangle

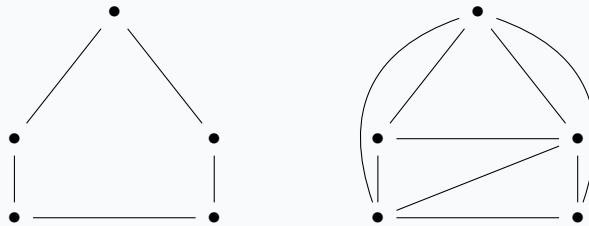
**Proof.**

$2 \Leftrightarrow 3$  : We saw that  $3|F| \leq |E|$  and with Euler we can see that  $|E| \leq 3|V| - 6$ . We have equality of the second statement when the first statement has equality. Therefore, all the faces are triangles if and only if  $|E| \leq 3|V| - 6$ .

$1 \Rightarrow 3$  :  $G \hookrightarrow \mathbb{R}^2$  where some region has  $\geq 4$  sides. Then since we can add an edge in this case (ex. diagonal across square).

$3 \Rightarrow 1$  :  $G$  is not max planar then we can add edge  $e$  s.t.  $G' = G \cup e$  is a;so planar. Then we have an embedding  $G' \hookrightarrow \mathbb{R}^2$  where we can remove  $e$  and see that we get an embedding of  $G$  with a region that's not a triangle. ■

**Example** (complementary region as triangles).



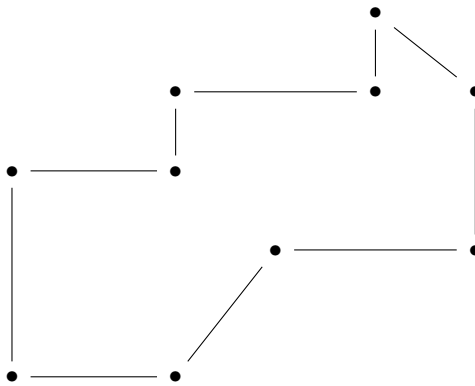
Here the left graph is not maximal planar and does not have all triangle complementary regions (basically faces) while the right one does, including the outer region

## Lecture 14

### 4.4 Art Gallery Problem

2023-03-21

**Question.** Given a room, how many light sources can fill the entire space?

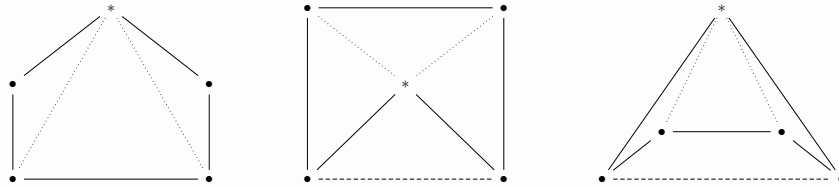


**Theorem 18 (Art Gallery Thm).** You only need  $\lfloor \frac{n}{3} \rfloor$  for a polygon with  $n$  sides

**Proof.** Given a 5 sided polygon, consider the convex hull of the vertices.

The cases for the convex hull can be either 5, 4, 3. For these cases, if you split the

polygon into triangles there exists a vertex where it is incident to every triangle. And the solution is that you put the source near this vertex.



■

## 4.5 Map Coloring

**Question.** Given a map how many colors are needed to color all the regions s.t. adjacent regions have different colors

**Observe.** The dual graph of them map is planar since we can move the vertices to the border, the borders do not intersect, and by Fary's.

**Theorem 19 (4 Color).** Every planar graph is 4 colorable

**Theorem 20 (6 Color).** Every planar graph is 6 colorable

**Proof.** A planar graph has a vertex of degree  $\leq 5$  by Euler's Formula. By induction, 6 color  $G \setminus v$ . If we remove  $v$  we get another planer graph that has a vertex of degree  $\leq 5$  By applying this repeatedly we get an ordering  $v_1 \cdots v_n$ . Now we can be greedy color by the order  $v_n \cdots v_i$  where  $v_i$  has degree  $\leq 5$  ■

## 4.6 Planar Regular Graph

**Definition 46 (regular planar).** A connected graph is called planar regular if it has an embedding where the vertices all have the same number of sides, which implies the dual is regular.

**Question.** What are all planer regular graphs?

$G$  is  $k$  regular  $2|E| = k|V|$  planar regular  $2|E| = l|F|$   
and using Euler we get  $\frac{1}{k} + \frac{1}{l} = \frac{1}{|E|} + \frac{1}{2} > \frac{1}{2}$