Homework 6

Math 123

Due March 10, 2023 by midnight

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Topics covered: vertex cuts, connectivity, Menger's theorem, network flows

Instructions:

- This assignment must be submitted on Gradescope by the due date.
- If you collaborate with other students (which is encouraged!), please mention this somewhere on the assignment.
- If you are stuck, please ask for help (from me, a TA, a classmate). Use Campuswire!
- You may freely use any fact proved in class. In general, you should provide proof for facts used that were not proved in class.
- Please restrict your solution to each problem to a single page. Usually solutions can be even shorter than that. If your solution is very long, you should think more about how to express it concisely.

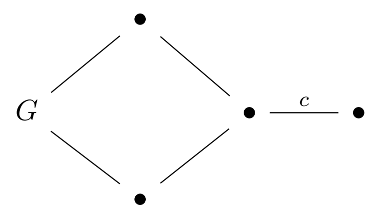
Problem 1. *Let G be a graph.*

(a) Give a counterexample to the following statement: If e is a cut-edge of G, then at least one vertex of e is a cut-vertex of G. [1]

(b) Add a hypothesis to correct the above statement.

Solution.

a Let G be a graph attached to two vertices s.t.



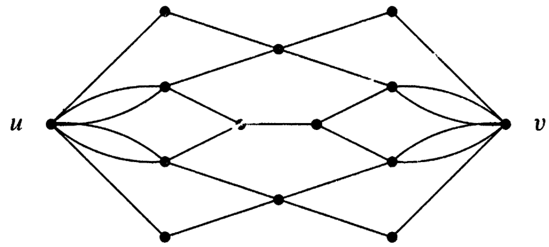
Where the edge label c is the cut edge, but neither vertex connected to it is a cut vertex.

b If e is a cut-edge of G connecting two vertices of degree greater than 1, then at least one vertex of e is a cut-vertex of G.

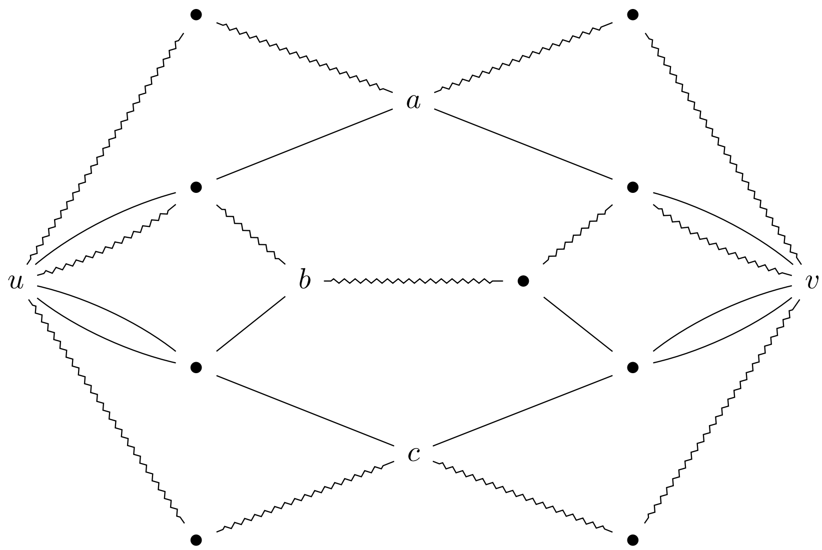
□

[1] We did not define cut edge in class, but it means what you most likely guess.

Problem 2. *Compute (with proof) $\kappa(u, v)$ for the graph below.*



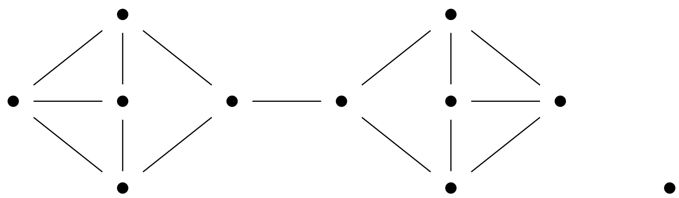
Solution.



By removing a, b, c we disconnect u and v, and we have 3 unique disjoint u, v-paths therefore, $\kappa(u, v) = 3$ by Menger's theorem. $\square$

Problem 3. *Find (with proof) the smallest* 3*-regular graph with* $\kappa(G) = 1$*.* [2]

Solution.



This graph has 10 vertices, so now we only need to show that there is no graph that satisfies the condition with fewer vertices/edges.

In G, a 3-regular graph with $\kappa(G) = 1$, let v be a cut vertex. The resulting components (of which there are at least 2) either contain 1 or 2 vertices that connected to v. This implies that there exists a vertex u which was not connected to v. Therefore, each component has ≥ 4 vertices. Since there are at least 2 components with at least 4 vertices, G has have had at least 9 vertices. And since a graph must have an even number of odd degree vertices the graph can only have $10, 12, 14 \ldots$ vertices with 10 being the minimum. $\square$

[2] Hint: consider a 1-element vertex cut S. What does $G \setminus S$ look like?

Problem 4. *Fix $k \geq 2$ and let Q_k be the hypercube graph. Prove that for any pair of vertices x, y there exist k pairwise disjoint (x, y)-paths.*

Solution. By induction:

Base Case: $k = 0$ Since there is one vertex there are 0 pairwise disjoint paths between any two vertices of the graph

Inductive Step: $k \to k-1$ Let's consider vertices of Q_k as $\overbrace{\mathbb{Z}_2 \times \ldots \times \mathbb{Z}_2}^{k} = (\mathbb{Z}_2)^k$.

If they share a value in the tuple then they are in a Q_{k-1} sub hypercube graph, and this is the $k-1$ case, so there are $k-1$ paths. We can get another one by going from x to its mirror vertex in the other Q_{k-1} sub hypercube graph and then traveling to y's mirror vertex. Therefore, there are k paths.

If they share no values (corners of the hypercube) then we can start each ith path by adding a 1 to the ith place of the tuple and then adding $(1, 0 \ldots 0)$, $(0, 1 \ldots 0) \ldots (0, 0 \ldots 1)$ to so all the bits are fliped to travel to the other corner. $\square$

Problem 5. *Use Menger's theorem to prove König's theorem: if $G = (X \sqcup Y, E)$ is bipartite the maximum size of a matching of G is equal to the minimum size of a vertex cover of G.* [3]

Solution. Consider the graph G' obtained by adding vertices a, b to G and connecting a to every vertex of X and b to every vertex of Y. Therefore, the matchings correspond the disjoint paths from a to b and the vertex cover corresponds to the connectivity of a and b.

By removing a and b from the disjoint paths from a to b we get matchings from vertices in X to vertices in Y Therefore, there are at least as many matchings as disjoint paths. (max # of matchings in $G \geq$ # of disjoint paths from a to b)

By removing S, an x, y-seperating set, the set must contain vertices that, in total, are connected to all edges in G; therefore, S is a vertex cover over G. (# of vertices in the separating set $\geq$ min # of vertices in the vertex cover of G)

By Menger's theorem, # of disjoint paths from a to b = # of vertices in the separating set, so max # of matchings in $G \geq$ # of disjoint paths from a to b = # of vertices in the separating set $\geq$ min # of vertices in the vertex cover of G, proving König's theorem □

[3] Hint: consider graph G' obtained by adding vertices a, b to G and connecting a to every vertex of X and b to every vertex of Y.

Problem 6. *Use the matrix-tree theorem[4] to prove Cayley's theorem.[5]*

Solution.

$$\begin{bmatrix} n-1 & ... & -1 \\ ... & n-1, ... & ... \\ -1 & ... & n-1 \end{bmatrix} = \begin{bmatrix} n & ... & 0 \\ ... & n, ... & ... \\ 0 & ... & n \end{bmatrix} - \begin{bmatrix} 1 & ... & 1 \\ ... & ... & ... \\ 1 & ... & 1 \end{bmatrix}$$

$$nI - \begin{bmatrix} 1 & ... & 1 \\ ... & ... & ... \\ 1 & ... & 1 \end{bmatrix}$$

using matrix diagonalization (which does not effect the determinant) we can find that

$$P^{-1} \begin{bmatrix} 1 & ... & 1 \\ ... & ... & ... \\ 1 & ... & 1 \end{bmatrix} P = \begin{bmatrix} n-1 & ... & 0 \\ ... & 0, ... & ... \\ 0 & ... & 0 \end{bmatrix}$$

by knowing that there are $n-1$ cofactors

$$nI - P^{-1} \begin{bmatrix} 1 & ... & 1 \\ ... & ... & ... \\ 1 & ... & 1 \end{bmatrix} P = \begin{bmatrix} 1 & ... & 0 \\ ... & n, ... & ... \\ 0 & ... & n \end{bmatrix}$$

Which we find that the determinant is $\overbrace{n \cdot \ldots \cdot n}^{n-2} = n^{n-2}$ by multiplying the elements in the diagonal.

□

[4] From the end of lecture 2/28

[5] Use a connection between the determinant and eigenvalues. It may help to first try to guess the form of the answer. For the love of algebra, do NOT compute any determinants!