Homework 5

Math 123

Due March 3, 2023 by *midnight*

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Topics covered: matchings, König's theorem, vertex covers, Gale–Shapely algorithm

Instructions:

- This assignment must be submitted on Gradescope by the due date.
- If you collaborate with other students (which is encouraged!), please mention this somewhere on the assignment.
- If you are stuck, please ask for help (from me, a TA, a classmate). Use Campuswire!
- You may freely use any fact proved in class. In general, you should provide proof for facts used that were not proved in class.
- **Please restrict your solution to each problem to a single page.** Usually solutions can be even shorter than that. If your solution is very long, you should think more about how to express it concisely.

Problem 1. *Let $G = (V, E)$ be a bipartite graph with maximum vertex degree Δ.*

(a) Use König's theorem to prove that G has a matching of size at least $|E|/\Delta$.

(b) Use (a) to conclude that every subgraph of $K_{n,n}$ with more than $(k-1)n$ edges has a matching of size at least k.

Solution.

(a) If the maximum vertex degree is Δ then each vertex covers at most Δ edges. Therefore, vertex cover must be at least of size $|E|/\Delta$, so by König's Theorem the matching of the bipartite graph G must be at least size $|E|/\Delta$

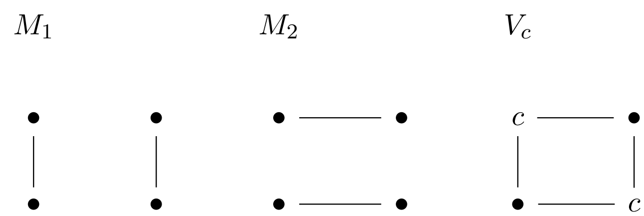
(b) The $K_{n,n}$ graph has $\Delta \leq n$ so $|E|/\Delta > (k-1)n/n = k-1$. Therefore, $|E|/\Delta \geq k$ and by König's Theorem the matching must be at least size k

$\square$

Problem 2. *Fix $k \geq 2$, and let Q_k denote hypercube graph (from HW1). Prove that Q_k has at least $2^{2^{k-2}}$ perfect matchings.*

Solution. By induction:

Base Case: When $k = 2$ we see that $2^{2^{k-2}} = 2^{2^0} = 2^1 = 2$, and we know that there are 2 perfect matchings:



The fact the vertex cover V_c has only two vertices proves that these matchings (M_1 and M_2) are perfect.

Inductive Step: When we have graph of size k we can recognize that there exist matchings of each $k-1$ subgraph where they are independent of the other ones, and there are $2^{2^{k-3}} \cdot 2^{2^{k-3}} = 2^{2^{k-3}+2^{k-3}} = 2^{2 \cdot 2^{k-3}} = 2^{2^{k-2}}$. Therefore, with only counting the matchings arising from taking the subgraphs independent of each other we find that there must be at least $2^{2^{k-2}}$ perfect matchings. $\square$

Problem 3. *Determine the stable matchings resulting from the proposal algorithm run with cats proposing and with giraffes proposing, given the preference lists below.*

Cats $\{u, v, w, x, y, z\}$	Giraffes $\{a, b, c, d, e, f\}$
$u: a > b > d > c > f > e$	$a: z > x > y > u > v > w$
$v: a > b > c > f > e > d$	$b: y > z > w > x > v > u$
$w: c > b > d > a > f > e$	$c: v > x > w > y > u > z$
$x: c > a > d > b > e > f$	$d: w > y > u > x > z > v$
$y: c > d > a > b > f > e$	$e: u > v > x > w > y > z$
$z: d > e > f > c > b > a$	$f: u > w > x > v > z > y$

To receive full credit, you should show your work.

Solution. Let $q \to r$ represent a proposal and $q \not\to r$ represent a rejection.

Cats $\to$ Giraffes

1. P $u \to a, v \to a, w \to c, x \to c, y \to c, z \to d$
   R $v \not\to a, w \not\to c, y \not\to c$
2. P $u \to a, v \to b, w \to b, x \to c, y \to d, z \to d$
   R $v \not\to b, z \not\to d$
3. P $u \to a, v \to c, w \to b, x \to c, y \to d, z \to e$
   R $x \not\to c$
4. P $u \to a, v \to c, w \to b, x \to a, y \to d, z \to e$
   R $u \not\to a$
5. P $u \to b, v \to c, w \to b, x \to a, y \to d, z \to e$
   R $u \not\to b$
6. P $u \to d, v \to c, w \to b, x \to a, y \to d, z \to e$
   R $u \not\to d$
7. P $u \to c, v \to c, w \to b, x \to a, y \to d, z \to e$
   R $u \not\to c$
8. P $u \to f, v \to c, w \to b, x \to a, y \to d, z \to e$
   R None, therefore the computation is complete

Giraffes $\to$ Cats

1. P $a \to z, b \to y, c \to v, d \to w, e \to u, f \to u$
   R $e \not\to u$

2. P $a \to z, b \to y, c \to v, d \to w, e \to v, f \to u$
   R $e \not\to v$
3. P $a \to z, b \to y, c \to v, d \to w, e \to x, f \to u$
   R None, therefore the computation is complete

$\square$

Problem 4. *Let* $G = (X \sqcup Y, E)$ *be a bipartite graph satisfying* $|N(S)| > |S|$ *for each nonempty* $S \subset X$*. Prove that every edge of* G *belongs to some matching that saturates* X*.*

Solution. By induction:

Base Case: $|E| = 1$ Therefore, there is only one matching between $\{x, y\}$ where $x \in X$ and $y \in Y$. This saturates x

Inductive Step: We know from Hall's Theorem that given $G = (X \sqcup Y, E)$, a bipartite graph satisfying $|N(S)| \geq |S|$ for each nonempty $S \subset X$ every edge of G belongs to some matching that saturates X. Therefore, by removing an edge we now have a graph with $n - 1$, so by induction we can see that the remaining graph has some matching that saturates X

□

Problem 5. *Complete the proof of König's theorem that we started in class.*

Solution. Let $G = (X \sqcup Y, E)$ be a bipartite graph, and let M be a maximum matching. To prove the theorem, it suffices to find a vertex cover Q with one vertex from each edge of M.

Define Q as follows. Given an edge $e = \{x, y\}$ of M (here $x \in X$ and $y \in Y$), if there is an M-alternating path from an unsaturated vertex $u \in X$ that passes through e, then we put $y \in Q$. Otherwise we put $x \in Q$.

We want to show that Q is a vertex cover. Let $f = \{a, b\}$ be an edge of E, and assume $a \in X$ and $b \in Y$. If f is in M we are done, so assume not. Since M is maximum, at least one of a or b is saturated by some edge $e = x, y$ in M. Now consider cases depending on whether $a = x$ or $b = y$ and whether or not there is an M-alternating path starting from an unsaturated vertex of $u \in X$ and passing through e.

1. $a = x, b = y$ and there exists an M-alternating path - By definition y must be in Q; therefore f is covered
2. $a = x, b = y$ and there doesn't exist an M-alternating path - By definition x must be in Q; therefore f is covered
3. $a = x$ and there exists an M-alternating path - Suppose b is not saturated then there would be an M-augmenting path connecting u and b, a contradiction since M is maximal. Since b is in the matching then let e' be the edge of M connected to b and c. Therefore, $c \in Q$
4. $a = x$ and there doesn't exist an M-alternating path - Let $e' = \{b, c\} \in M$. Suppose there exists an M-alternating path s.t. it connected b to an unsaturated vertex, then the path could contain e, a contradiction. Therefore, e' cannot be in an M-alternating path, so $a = x \in Q$
5. $b = y$ and there exists an M-alternating path - By definition y must be in Q; therefore f is covered
6. $b = y$ and there doesn't exist an M-alternating path - Let e' be the edge of M connected to b and c. Suppose, there exists an M-alternating path from u to e' then there would be an M-alternating path from u to e, a contradiction. Therefore, $c \in Q$

$\square$

Problem 6. *A deck with mn cards with m values and n suits consists of one card for each value in each suit. The cards are dealt into an $n \times m$ array. Prove that there is a set of m cards, one in each column, having distinct values.*

Solution. Let G be a bipartite graph with vertex set $X \sqcup Y$ where $x \in X$ are columns in the array and $y \in Y$ are the distinct values and E is the edge set where $e \in E$ correspond to value y appearing in a column x. Each column consists of n cards and each value appears in n times in total, so the graph is n-regular, meaning we can apply the Marriage Theorem (a corollary of Hall's Theorem), so see that there is a perfect matching which shows that there is a set of m cards, one in each column, having distinct values. $\square$

Problem 7 (Bonus). *Let T_1 be the tiling of the plane by unit squares whose vertices have integer coordinates. Let T_2 be the result of rotating T_1 about the origin by some angle θ. Prove that it is possible to find a bijection between squares of T_1 and squares of T_2 in such a way that the matched squares are within 10 units of each other. The matching will depend on θ.*

Solution. □