Homework 5

Math 123

Due March 3, 2023 by midnight

Name: George Chemmala

Topics covered: matchings, König's theorem, vertex covers, Gale–Shapely algorithm Instructions:

- This assignment must be submitted on Gradescope by the due date.
- If you collaborate with other students (which is encouraged!), please mention this somewhere on the assignment.
- If you are stuck, please ask for help (from me, a TA, a classmate). Use Campuswire!
- You may freely use any fact proved in class. In general, you should provide proof for facts used that were not proved in class.
- Please restrict your solution to each problem to a single page. Usually solutions can be even shorter than that. If your solution is very long, you should think more about how to express it concisely.

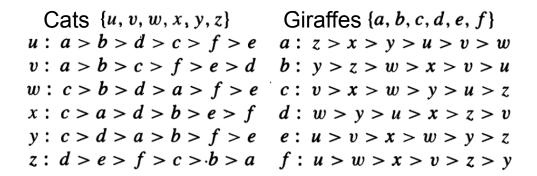
Problem 1. Let G = (V, E) be a bipartite graph with maximum vertex degree Δ .

- (a) Use König's theorem to prove that G has a matching of size at least $|E|/\Delta$.
- (b) Use (a) to conclude that every subgraph of $K_{n,n}$ with more than (k-1)n edges has a matching of size at least k.

 \Box

Problem					$let Q_k$	denote	hypercube	graph	(from	HW1).	Prove	that	Q_k	has a	t
least $2^{2^{k-2}}$	perfec	et mate	chir	ngs.											

Problem 3. Determine the stable matchings resulting from the proposal algorithm run with cats proposing and with giraffes proposing, given the preference lists below.



To receive full credit, you should show your work.

 \Box

Problem 4. Let $G = (X \sqcup Y, E)$ be a bipartite graph satisfying $ N(S) > S $	for each	nonempty
$S \subset X$. Prove that every edge of G belongs to some matching that saturates X		

Problem 5.	Complete the proof of König's theorem that we started in class.	
Solution.		

Problem 6.	A deck with mn cards with m values and n suits consists of one card for each value
in each suit.	The cards are dealt into an $n \times m$ array. Prove that there is a set of m cards, one in
$each\ column,$	having distinct values.

Problem 7 (Bonus). Let T_1 be the tiling of the plane by unit squares whose vertices have integer coordinates. Let T_2 be the result of rotating T_1 about the origin by some angle θ . Prove that it is possible to find a bijection between squares of T_1 and squares of T_2 in such a way that the matched squares are within 10 units of each other. The matching will depend on θ .