Homework 7

Math 123

Due March 16, 2023 by midnight

Name: George Chemmala

Topics covered: Graph coloring, chromatic polynomial, Turan graphs

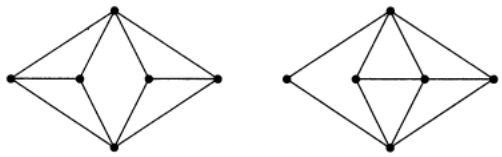
Instructions:

- This assignment must be submitted on Gradescope by the due date.
- If you collaborate with other students (which is encouraged!), please mention this somewhere on the assignment.
- If you are stuck, please ask for help (from me, a TA, a classmate). Use Campuswire!
- You may freely use any fact proved in class. In general, you should provide proof for facts used that were not proved in class.
- Please restrict your solution to each problem to a single page. Usually solutions can be even shorter than that. If your solution is very long, you should think more about how to express it concisely.

Problem 1. *Give an example or explain why no example exists: A graph G that is neither complete nor an odd cycle, but for which the greedy coloring uses $\Delta(G) + 1$ colors.*

Solution. $\square$

Problem 2. *Give a very short proof that the following two graphs have the same chromatic number.*[1]



Solution. □

[1] Note: solutions that construct optimal colorings of these graphs will not receive credit.

Problem 3. *Let $G = M_{n_1,\ldots,n_k}$ be a complete k-partite graph with $n = n_1 + \cdots + n_r$ vertices. Show that if $n_i - n_j \geq 2$ for some i, j, then there exists a k-partite graph with n vertices and more edges than G.*

Solution. $\square$

Problem 4. *Given a set of lines in the plane with no three meeting at a point, form a graph G whose vertices are the intersections of the lines, with two vertices adjacent if they appear consecutively on one of the lines. Prove that $\chi(G) \leq 3$.* [2] [3]

Solution. □

[2] Suggestion: start by looking at some explicit examples.

[3] Hint: use a greedy coloring with an appropriate vertex ordering.

Problem 5. *Let G be a graph with chromatic number k. Show that for every k-coloring of G and for each color i, there is a vertex of color i that is adjacent to vertices of the other $k-1$ colors.* [4]

Solution. $\square$

[4] Hint: think back to the proof that a graph with chromatic number k has at least $\binom{k}{2}$ edges.

Problem 6. *Prove that $\chi(G) = \omega(G)$ when the complement $\bar{G}$ is bipartite.* [5] [6] [7]

Solution. □

[5] Here $\omega(G)$ is the clique number: the largest m so that G contains K_m.
[6] Hint: look to apply König's theorem. (!)
[7] This is a pretty challenging problem. If you want more hints, please ask.

Problem 7 (Bonus). *Let $G = (V, E)$ be the unit distance graph in the plane: $V = \mathbb{R}^2$, and two points are adjacent if their Euclidean distance is 1. (a) Use the hexagonal tiling to prove $\chi(G) \leq 7$. (b) Prove that $\chi(G) \geq 3$.*

Solution. $\square$