Homework 1

Math 123

Due February 3, 2023 by 5pm

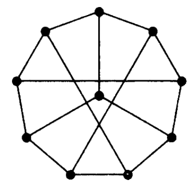
Name: George Chemmala

Topics covered: graph, subgraph, cycle, path, vertex degrees,

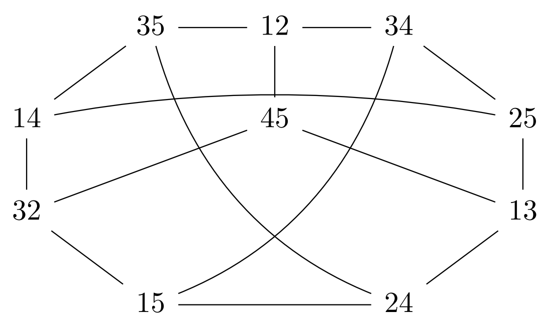
Instructions:

- This assignment must be submitted on Grade scope by the due date. Grade scope Entry Code: RZ277D.
- If you collaborate with other students (which is encouraged!), please mention this near the corresponding problems.
- If you are stuck, please ask for help (from me, a TA, a classmate).

Problem 1. *Prove that the graph below is isomorphic to the Petersen graph.*[1]



Solution. I tried to best replicate the graph using tikz:



This labeling of the graph fits the definition of the Petersen graph. □

Problem 2. *How many cycles of length n are there in the complete graph K_n?*

Solution. We can define a cycle as a list of vertices (like how we label walks). For example, $v_1 \ldots v_n$ There are $n!$ different ways to order this list, but we must divide this by n (permutations are isomorphic since a cycle can start from n starting points) and by 2 since the permutations are isomorphic backwards and forwards. Therefore, there are $\frac{(n-1)!}{2}$ cycles of length n □

Problem 3. *Define the hypercube graph Q_k as the graph with a vertex for each tuple $(a_1, \ldots, a_k)$ with coordinates $a_i \in \{0, 1\}$ and with an edge between $(a_1, \ldots, a_k)$ and $(b_1, \ldots, b_k)$ if they differ in exactly one coordinate.*[2]

(a) Prove that two 4-cycles in Q_k are either disjoint, intersect in a single vertex, or intersect in a single edge.

(b) Let $K_{2,3}$ be the complete bipartite graph with 2 red vertices, 3 blue vertices, and all possible edges between red and blue vertices. Prove that $K_{2,3}$ is not a subgraph of any hypercube Q_k.

Solution. (a) The following will show that cases where the 4-cycles are disjoint, intersect in a single vertex, or intersect in a single edge are possible and then show intersection beyond 2 points is not possible:

Example of existence of 2 disjoint 4-cycles:

$\{(0000), (0001), (0011), (0010)\}$ and $\{(0101), (0111), (0100), (0110)\}$

Example of existence of 2 4-cycles that intersect at a single vertex:

[1] Hint: label the graph.

[2] Suggestion: Draw Q_k for $k = 2$ and $k = 3$.

$\{(0000), (0001), (0011), (0010)\}$ and $\{(0000), (0100), (1000), (1100)\}$

Example of existence of 2 4-cycles that intersect in a single edge:

$\{(0000), (0001), (0011), (0010)\}$ and $\{(0000), (0010), (1000), (1010)\}$

Explanation of nonexistence 4-cycles that intersect at 3 vertices:

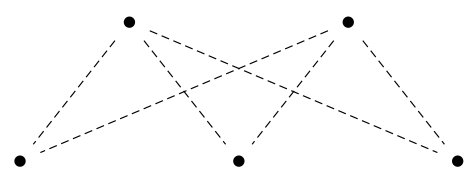
For all 4-cycles the addition of all points must be equal to $(0 \dots 0)$ in order to be closed cycles. For example, in the case of $\{(0000), (0001), (0011), (0010)\}$, $(0+0+0+0, 0+0+0+0, 0+0+1+1, 0+1+1+0) = (0, 2, 2, 0) = (0, 0, 0, 0)$ Essentially, we are working in $\mathbb{Z}_2 \times \dots \times \mathbb{Z}_2$. When we add 3 points in a cycle we get the 4th point since $p_1 + p_2 + p_3 + p_4 = (0, 0, 0, 0) \iff p_1 + p_2 + p_3 = -p_4 = p_4$ Therefore, if there were two 4-cycles that intersect at 3 vertices, they must be the same cycle - a contradiction.

Explanation of nonexistence 4-cycles that intersect at more than 4 vertices:

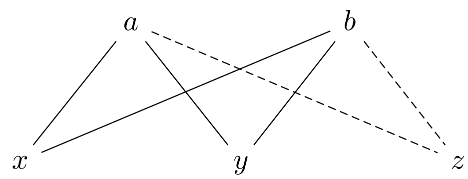
If the two 4-cycles intersect at 4 vertices they must be the same cycle - a contradiction. If they intersect at more than 4 vertices that is a contradiction with the condition that there are 4 vertices in 4-cycles

(b) In the last problem we showed that a 4-cycle in a hypercube graph cannot intersect another separate 4-cycle with 3 points. However, this is possible in $K_{2,3}$:

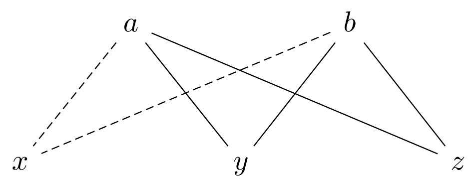
$K_{2,3}$



1st 4-cycle in $K_{2,3}$



2nd 4-cycle in $K_{2,3}$



Both 4-cycles share a, b, y, but are not the same 4-cycle

$\square$

Problem 4. *For a graph $G = (V, E)$, the complement of G is the graph $\bar{G} = (V, \bar{E})$, where $\{u, v\} \in \bar{E}$ if and only if $\{u, v\} \notin E$. Prove or disprove: If G and H are isomorphic, then the complements $\bar{G}$ and $\bar{H}$ are also isomorphic.*

Solution. For all edges in $\widetilde{G}$, $\widetilde{G}$ has an edge $\iff$ G doesn't have an edge $\iff$ H doesn't have an edge $\iff$ $\widetilde{H}$ has an edge. Therefore, for all edges in $\widetilde{G}$, $\widetilde{G}$ has an edge $\iff$ $\widetilde{H}$ has an edge. $\square$

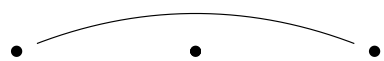
Problem 5. *(a) Determine the complement of the graphs P_3 and P_4. (Recall that P_n is the path with n vertices. It has $n-1$ edges.)*

(b) We say that G is self-complementary if G is isomorphic $\bar{G}$. Prove that if G is self-complementary with n vertices, then either n is divisible by 4 or $n-1$ is divisible by 4. [3]

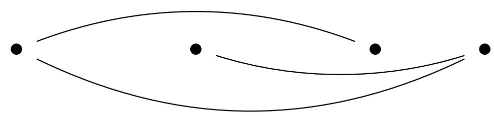
In fact, whenever n or $n-1$ is divisible by 4, there is a self-complementary graph with n vertices – see the bonus problem below.

Solution. (a) $\widetilde{P_3}$ & $\widetilde{P_4}$:

$\widetilde{P_3}$



$\widetilde{P_4}$

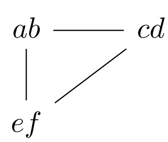


(b) The sum of the number of edges of a graph and it's complement has to be the number of edges in K_n i.e $\binom{n}{2} = n(n-1)/2$ and if G is self-complementary G, it's complement, $\widetilde{G}$, must have the same number of edges; therefore G and $\widetilde{G}$ have $n(n-1)/4$ edges. Since the number of edges is an integer, G can only be self-complementary when $n(n-1)/4$ is an integer, which is equivalent to either n or $(n-1)$ being divisible by 4 since only either n or $(n-1)$ could divide 4 (n and $n-1$ are consecutive). ☐

Problem 6. *Prove that the Petersen graph has no cycles of length 3 or 4.* [4]

Solution. We will suppose the cycles are possible and show how they lead to contradictions of the definition of the Petersen graph:

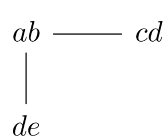
Suppose there exists a 3-cycle then this would require:



Where a, b, c, d, e, and f are all different, to satisfy one of the Petersen graph definition conditions, but there is a contradiction with the Petersen graph definition condition that are only 5 different choices (there can be no f).

Suppose there exists a 4-cycle:

There must be 2 vertices which are disjoint from 1 vertex:



[3] Hint: count edges

[4] Hint: use the definition of Petersen graph given in class.

There must be one repeated value between the two non-connected vertices since there are only 2 choices up to isomorphism from choosing between c d, and e (we are limited to 5 different choices, a, b, c, d, and e, by definition of the Petersen graph)

When we add another vertex, it also needs to be disjoint from the 2 vertices; therefore, it cannot be b, c, and e forcing it to choose the remaining a and b, but this is a contradiction.

$$\begin{tikzcd} ab \arrow[r, dash] \arrow[d, dash] & cd \arrow[d, dash] \\ de \arrow[r, dash] & ab \end{tikzcd}$$

$\square$

Problem 7 (Bonus)**.** *Let G, H be a self-complementary graphs, and assume G has with $4k$ vertices. Construct a self-complementary graph obtained by taking the union of G and H and adding some edges.*[5] *Deduce that if either n or $n-1$ is divisible by 4, then there is a self-complementary graph with n vertices.*

Solution. I believe an algorithm for doing this would be to connect all vertices of H to all vertices in G that have even degree. Unfortunately, I have not yet found a proof of this but playing around adjacency matrices seems to suggest this (e.g, $P_4 + P_4$ results in the same odd/even degree switch while $P_4 + C_5$ results is all the degrees having the same parity). We have shown in problem 5 that P_4 is self-complementary and C_5 is also self-complementary. Therefore, we can create any graph of n, where either n or $n-1$ is divisible by 4, using a combination of P_4 and C_5 with the aforementioned algorithm. $\square$

[5] Hint: How does the degree of even/odd vertices of G change after taking the complement?