Homework 8

Math 123

Due April 7, 2023 by midnight

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Topics covered: planar graphs, Kuratowski's theorem

Instructions:

- This assignment must be submitted on Gradescope by the due date.
- If you collaborate with other students (which is encouraged!), please mention this somewhere on the assignment.
- If you are stuck, please ask for help (from me, a TA, a classmate). Use Campuswire!
- You may freely use any fact proved in class. In general, you should provide proof for facts used that were not proved in class.
- Please restrict your solution to each problem to a single page. Usually solutions can be even shorter than that. If your solution is very long, you should think more about how to express it concisely.

Problem 1. *Prove that the Petersen graph is not planar using Euler's formula (do not use Kuratowski's theorem).* [1]

Solution. Let P represent the Petersen graph

By contradiction:

Assume there exists an embedding P in $\mathbb{R}^2$ then by Euler's Formula

$$\begin{aligned} |F| &= 2 - |V| + |E| \\ |F| &= 2 - 10 + 15 \\ |F| &= 2 - 5 + 10 \\ |F| &= 7 \end{aligned}$$

But each edge has at most 2 faces and each face has ≥ 5 sides (the minimum cycle size in P is 5) therefore $2|E| \geq 3|F|$. However, $2 \cdot 15 = 30 \geq 35 = 5 \cdot 7$ is a contradiction; therefore P is not planer. $\square$

[1] Hint: you will need to use a problem from HW1.

Problem 2. *Let G be a connected graph embedded in the plane. Prove that G is bipartite if and only if every region of $\mathbb{R}^2 \setminus G$ has an even number of sides.* [2] [3]

Solution. A bipartite graph contains no closed cycle of odd length, so it contains no faces with odd sides.

Reverse direction: Suppose every face has an even number of sides.

By induction:

Base Case: ($|F| = 1$)

If G has only one face then it is a tree, and so bipartite

Inductive Step: Fix edge e on a cycle in G, this edge borders faces F_1, F_2. By removing e, we remove one side from F_1, F_2 and combine them to have a bigger face F_3 that has the sum of sides of F_1 and F_2 minus 2, so it has even sides.

By inductive hypothesis, we know that $G \setminus e$ is bipartite. Since the vertices e is connected to have an odd walk between them, adding e maintains that G is bipartite.

$\square$

[2] Note: the boundary of a region does <u>not</u> necessarily correspond to a cycle in the graph. Make sure you understand why!

[3] Hint: for one direction, use induction on the number of regions of $\mathbb{R}^2 \setminus G$.

Problem 3. *Prove that the complement of a planar graph with at least 11 vertices is nonplanar.*[4] *Give an example of a planar graph with 8 vertices whose complement is also planar.*[5]

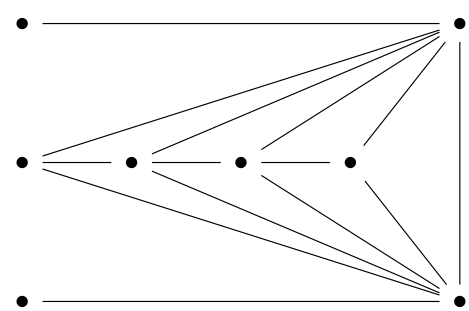
Solution. Let $T(n)$ represent the nth triangle number.

$$|F| = 2 - |V| + |E|$$
$$|F| = 2 - 11 + |E|$$

Since $2|E| = 3|F|$:

$$\begin{aligned} 2|E| &\geq 6 - 3(11) + 3|E| \\ 2|E| &\geq 6 - 33 + 3|E| \\ 2|E| &\geq -27 + 3|E| \\ -|E| &\geq -27 \\ |E| &\leq 27 \end{aligned}$$

G being planar implies that $|E| \leq 27$. Therefore, $\overline{G}$ has greater than $T(11-1) = 55 - 27 = 28$ edges (K_{11} has $T(11-1) = 55$ edges), which is more than 27, which means it's not planar.



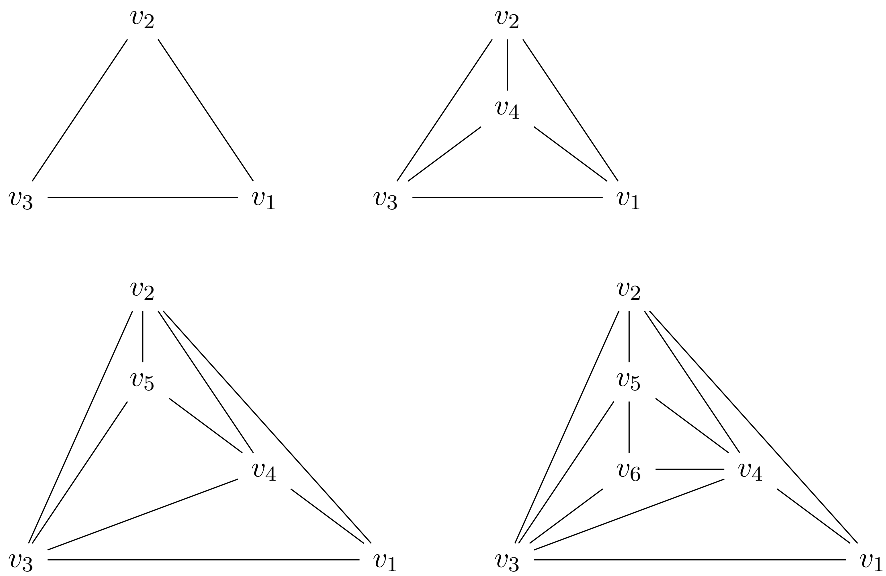
This graph was creating using the algorithm from the bonus problem in HW1 $\square$

[4]Hint: your solution should be very short.
[5]Hint: in fact you can find one that is self-complementary. This should simplify your search.

Problem 4. *Let G_n be the graph with vertices $v_1, \ldots, v_n$ and an edge between v_i and v_j whenever $|i-j| \leq 3$. Prove that G_n is a maximal planar graph.*

Solution. From class we know that G is maximal planar $\iff$ G is planer and $|E| = 3|V| - 6$

We can see that G is planer by constuction:



We can place the v_{n+1} vertex in the region bounded by v_n, v_{n-1}, v_{n-2}. As repersented in the picture above.

Since the vertices are connected to the 3 vertices to the left $(v_{i-3}, v_{i-2}, v_{i-1})$ and right $(v_{i+1}, v_{i+2}, v_{i+3})$. All vertices, except for the first 3 and last 3, have degree 6. However, $v_{i-3}, v_{i-2}, v_{i-1}$ do not exist for $i = 1$, v_{i-3}, v_{i-2} do not exist for $i = 2$, v_{i-3} does not exist for $i = 3$, and analogously for $i = n, n-1, n-2$; therefore, these 6 vertices have in total 12 $(2(1+2+3))$ less connections than if the aforementioned vertices existed. Therefore, the sum of the degrees of the vertices is $6n - 12$ and by the degree formula $2|E| = \sum_{v \in V} \deg(v)$, there must be $|E| = 3n - 6$, and so G is maximal planar. $\square$

Submit a final project proposal. See other document for instructions. This should be submitted separately from the HW assignment: one submission per group.