Homework 10

Math 123

Due April 21, 2023 by midnight

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Topics covered: Ramsey theory, random graphs

Instructions:

- This assignment must be submitted on Gradescope by the due date.
- If you collaborate with other students (which is encouraged!), please mention this near the corresponding problems.
- Some problems from this assignment come from West's book, as indicated next to the problem. In some cases, the statements on this assignment differ slightly from the book.
- If you are stuck please ask for help (from me or your classmates). Occasionally problems may require ingredients not discussed in the course.
- You may freely use any fact proved in class. In general, you should provide proof for facts used that were not proved in class.

Problem 1. *Prove $R(4,4) > 17$ using the 17-vertex graph described in class.*[1]

Solution. Let all the vertices be labeled from v_0 to v_16. Now any vertex v_i is connected to $v_{i\pm1}, v_{i\pm2}, v_{i\pm4}, v_{i\pm8}$ where the labels are $\mod 17$.

By rotational symmetry we can consider a vertex v_i. Any K_4 will correspond to v_i and 3 vertices from $\{v_{i\pm1}, v_{i\pm2}, v_{i\pm4}, v_{i\pm8}\}$. To simplify notation let $v_i = 0, v_{i\pm1} = 1, v_{i\pm2} = 2 \cdots$

Assuming we start with 0, we can notice if we choose 1 we are forced to choose at least one from $\{4, 8\}$ but we cannot connect 1 to $\{4, 8\}$ since they are either distance $3, 7 \not\in \{1, 2, 4, 8\}$ away.

Therefore, we are forced to choose 2 and connect to $4, 8$. However, 2 is a distance $6 \not\in \{1, 2, 4, 8\}$ from 8.

Therefore, there exist no combination of 3 vertices that connect, so there exist no K_4 in the graph.

Now we must check the complement, where vertices are a distance 3, 5, 6, 7 away. We will use the same notation.

Assuming we start with 0 we can if we choose 3 we are forced to choose at least one from $\{5, 7\}$ but we cannot connect 3 to $\{5, 7\}$ since they are either distance $2, 4 \not\in \{3, 5, 6, 7\}$ away.

Therefore, we are forced to choose 5 and connect to $6, 7$. However, 5 is a distance $1, 2 \not\in \{3, 5, 6, 7\}$ from $6, 7$.

Therefore, there exist no combination of 3 vertices that connect, so there exist no K_4 in the graph. $\square$

[1] 17 vertices around a circle; connected a given vertex to the vertices distance 1, 2, 4, 8 away.

Problem 2. *Fix a graph H with k vertices. Prove that almost every graph contains H as an induced subgraph.*[2] [3]

Solution. I'm a little confused what the problem is asking but:

The limit of

$$\binom{n}{k}(\frac{1}{2})^k$$

approaches infinity as n tends to infinity since $\binom{n}{k}$ grows faster than 2^k □

[2]Recall: given a collection of vertices in a graph G, the induced subgraph is the subgraph consisting of those vertices and all the edges between them that belong to G.

[3]Hint: Decompose the vertices into groups of size k. Consider the event that one these groups spans H.

Problem 3. *Recall that a graph G satisfies property $(\star)$ if for any collection $u_1, \ldots, u_p$ and $v_1, \ldots, v_q$ of distinct vertices of G there exists a vertex z of G so that z is adjacent to all of the u_i and to none of the v_j. Let G_1, G_2 be graphs whose vertex sets are countably infinite. Prove that if G_1 and G_2 satisfy $(\star)$, then G_1 and G_2 are isomorphic.*[4]

Solution. Suppose G_1 and G_2 satisfy $(\star)$ and let the vertices of G_1 and G_2 be enumerated by $x_1, x_2, \ldots$ and $y_1, y_2, \ldots$, respectively.

Let G_{1_i}, G_{2_i} be isomorphic subgraphs of G_1, G_2. We will construct them by the following.

By induction:

Base Case: $(i = 1)$

Here we have G_{1_1}, G_{2_1}, both consisting of one vertex each. It is clear that these two vertices are isomorphic since they are not connected to anything

Inductive Step: By the extension property, there exist $x_i \in G_1, y_i \in G_2$ that are connected to the vertex sets of G_{1_i}, G_{2_i} but not to the remaining vertices in G_1, G_2. Since these vertices x_i, y_i are connected to vertices which are isomorphic to each other, they themselves must be isomorphic, and we can add them to G_{1_i}, G_{2_i} to get $G_{1_{i+1}}, G_{2_{i+1}}$

□

[4] Hint: Enumerate the vertices of G_1 and G_2 by $x_1, x_2, \ldots$ and $y_1, y_2, \ldots$, respectively. Inductively define an isomorphism $f : V(G_1) \to V(G_2)$. On odd (resp. even) steps of the induction extend f so that the smallest unmatched vertex of $V(G_1)$ (resp. $V(G_2)$) is in the domain (resp. image) of f.

Problem 4. *Prove that the Radio graph has the following "pigeonhole" property: For any partition of the vertex set $V = U_1 \cup \cdots \cup U_m$, there exists j so that the subgraph spanned by U_j is isomorphic to R.*

Solution. Suppose all partitions $G_1, \cdots, G_m$ which correspond to vertex sets $U_1, \cdots, U_m$ are not Rado.

Since G_1 is not Rado, there exist subsets p, q of the vertex set U_1 where no vertex in G_1 is adjacent to p and not adjacent to q.

Since $G_2, \cdots G_m$ are not Rado, there exist subsets w, r of the vertex set $U_2 \cup \cdots \cup U_m$ where no vertex in the corresponding graph G' is adjacent to w and not adjacent to r.

Now we have sets $p \cup w$ and $q \cup r$ s.t. they are disjoint, so in G, which is Rado, there must be a vertex that connects to either $p \cup w$ or $q \cup r$. However, this vertex cannot be in either G_1 or the rest of the graphs by the above reasoning, so we have a contradiction. $\square$

Problem 5 (Bonus)**.** *Create a meme related to the course. Please submit to the campuswire page for everyone's enjoyment.*

Solution. https://campuswire.com/c/GCDD00E4D/feed/228 □

Submit a draft of your final project slides. See other document for instructions.