

Homework 7

Math 123

Due March 16, 2023 by midnight

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Topics covered: Graph coloring, chromatic polynomial, Turan graphs

Instructions:

- This assignment must be submitted on Gradescope by the due date.
- If you collaborate with other students (which is encouraged!), please mention this somewhere on the assignment.
- If you are stuck, please ask for help (from me, a TA, a classmate). Use Campuswire!
- You may freely use any fact proved in class. In general, you should provide proof for facts used that were not proved in class.
- Please restrict your solution to each problem to a single page. Usually solutions can be even shorter than that. If your solution is very long, you should think more about how to express it concisely.

Problem 1. Give an example or explain why no example exists: A graph G that is neither complete nor an odd cycle, but for which the greedy coloring uses $\Delta(G) + 1$ colors.

Solution. Counterexample:

$$1 \text{ --- } 4 \text{ --- } 3 \text{ --- } 2$$

Let a, b, c be the colorings in alphabetic order $a \leq b \leq c$ then

1 is colored a since a is the minimum coloring and it has no colored neighbors

$$a \text{ --- } 4 \text{ --- } 3 \text{ --- } 2$$

2 is colored a since a is the minimum coloring and it has no colored neighbors

$$a \text{ --- } 4 \text{ --- } 3 \text{ --- } a$$

3 is colored b since a is used by its neighbor 2

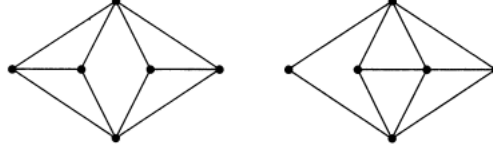
$$a \text{ --- } 4 \text{ --- } b \text{ --- } a$$

4 is colored c since a is used by its neighbor 1 and b is used by its neighbor 3

$$a \text{ --- } c \text{ --- } b \text{ --- } a$$

□

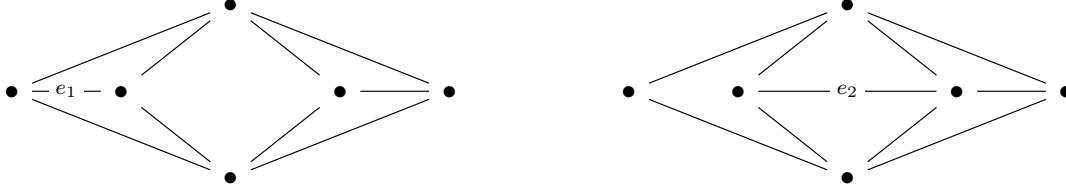
Problem 2. Give a very short proof that the following two graphs have the same chromatic number.¹



Solution. Let the left graph be called G_1 and the right graph be called G_2 . We know that $\chi(G, t) = \chi(G \setminus e, t) - \chi(G \cdot e, t)$. Therefore, we can show the two graphs have the same chromatic polynomial if they both have edges e_1, e_2 s.t.

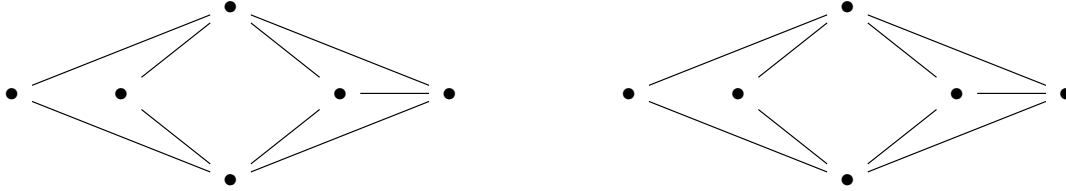
$$\chi(G_1, t) = \chi(G_1 \setminus e_1, t) - \chi(G_1 \cdot e_1, t) = \chi(G_2 \setminus e_2, t) - \chi(G_2 \cdot e_2, t) = \chi(G_2, t)$$

where $G_1 \setminus e_1 = G_2 \setminus e_2$ and $G_1 \cdot e_1 = G_2 \cdot e_2$ by selecting e_1 and e_2 as shown below:



Here is some work of showing the equivalence of the statements.

$$G_1 \setminus e_1 = G_2 \setminus e_2:$$



$$G_1 \cdot e_1 = G_2 \cdot e_2:$$



□

¹Note: solutions that construct optimal colorings of these graphs will not receive credit.

Problem 3. Let $G = M_{n_1, \dots, n_k}$ be a complete k -partite graph with $n = n_1 + \dots + n_k$ vertices. Show that if $n_i - n_j \geq 2$ for some i, j , then there exists a k -partite graph with n vertices and more edges than G .

Solution.

□

Problem 4. *Given a set of lines in the plane with no three meeting at a point, form a graph G whose vertices are the intersections of the lines, with two vertices adjacent if they appear consecutively on one of the lines. Prove that $\chi(G) \leq 3$.*^{2 3}

Solution.

□

²Suggestion: start by looking at some explicit examples.

³Hint: use a greedy coloring with an appropriate vertex ordering.

Problem 5. *Let G be a graph with chromatic number k . Show that for every k -coloring of G and for each color i , there is a vertex of color i that is adjacent to vertices of the other $k - 1$ colors.*⁴

Solution.

□

⁴Hint: think back to the proof that a graph with chromatic number k has at least $\binom{k}{2}$ edges.

Problem 6. *Prove that $\chi(G) = \omega(G)$ when the complement \bar{G} is bipartite.* ^{5 6 7}

Solution.

□

⁵Here $\omega(G)$ is the clique number: the largest m so that G contains K_m .

⁶Hint: look to apply König's theorem. (!)

⁷This is a pretty challenging problem. If you want more hints, please ask.

Problem 7 (Bonus). Let $G = (V, E)$ be the unit distance graph in the plane: $V = \mathbb{R}^2$, and two points are adjacent if their Euclidean distance is 1. (a) Use the hexagonal tiling to prove $\chi(G) \leq 7$. (b) Prove that $\chi(G) \geq 3$.

Solution.

□