Graph Theory

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Lecture 1

0 Intro

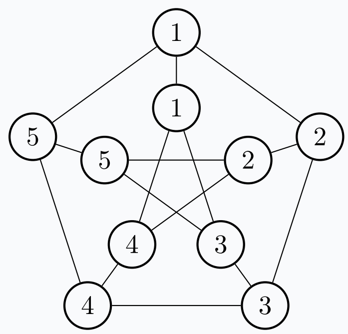
1/26/2023

Graph Theory is important

1 Fundamental Concepts

Definition 1 (graph). A graph is a pair (V, E) where V is the vertex space and E is the edge space.

Example (petersen graph). A element subset of $\{1, 2, 3, 4, 5\}$ connected by disjointedness



Note. In this course we are excluding multiple edges and self loops

Definition 2 (vertex degrees). Let, $G = (V, E)$, $v \in V$, $e \in E$ e are incident if $v \in e$ i.e. v is an endpoint of e

Lemma 1.

$$\sum_{v \in V} deg(v) = \sum_{v \in V} \sum_{v \in e} 1 = \sum_{v \in e} \sum_{v \in V} 1 = 2|E|$$

Proof. Every edge has two vertices ■

Definition 3 (complete graph). Represented K_n, the graph has $V = 1 \dots n$ and all possible edges

$$|E| = \frac{n(n-1)}{2} = nCr(n, 2)$$

Definition 4 (isomorphic). Graphs G_1 and G_2 are isomorphic if there exists a bijection, $f : v_1 \to v_2$, s.t. $\{u, v\} \in E_1 \Leftrightarrow \{f(u), f(v)\}$

1.1 Connectivity

Definition 5 (connected). u and v are connected if there exists a path from u to v A graph is connected when all vertices are connected

Lecture 2

1.2 Degree Sequences

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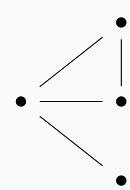
Definition 6 (degree sequence). List of vertex degrees in decreasing order

Note. Isomorphic graphs have the same degree sequence; however, if they have the same degree sequence they are not necessarily isomorphic

Definition 7 (graphic). A sequence is graphic if it's the degree sequence of some graph

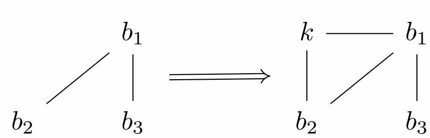
Example. Here are some degree sequences which may or may not exist:

- $(3, 2, 1, 1)$ - not possible since they don't sum to an even number
- $(3, 3, 1, 1)$ - not graphic since there are not enough vertices
- $(3, 2, 2, 1)$ - graphic



Theorem 1 (Havel-Hakimi). $(a_1 \ldots a_n)$ is graphic iff $(a_2 - 1, a_3 - 1, \ldots a_{a_1+1} - 1, a_{a_1+2} \ldots a_n)$ is graphic

Proof. We can apply the theorem in reverse to understand the intuition. If we have a graphic sequence, $(b_1 \ldots b_m)$, we can add a vertex with degree k such that $(k, b_1 + 1, \ldots b_k + 1, b_{k+1} \ldots b_m)$.



In the reverse direction, we can subtract a vertex from a graph, and we make a transformation (2-switch) which preserves degree sequences but makes the graph maximal, to get the reverse result. ■

Example. $(3, 3, 3, 3, 3, 2, 2, 1)$ eventually becomes $(1, 1, 1, 1, 0)$ which we can show is graphic:

1.3 Bipartite graphs

Definition 8 (bipartitie graph). A graph is bipartite if it's possible to color vertices using only 2 colors. A simple check for it being bipartite is to check if there are no odd cycles

1.4 Walks and Paths

Definition 9 (walk). A walk is a sequence of vertices that are connected by edges It has the property:

> **Definition 10** (length). The number of edges contained in the walk

Definition 11 (path). A path is a walk that has unique vertices

Lemma 2. A walk from v_0 to v_n implies a path from v_0 to v_n

1.5 Closed Walks and Cycles

Definition 12 (closed walk). A closed walk is a walk which starts and ends at the same vertex

Definition 13 (cycle). A cycle is a closed walk that has unique vertices

Lemma 3 (closed walk). A closed walk of odd length contains an odd cycle

Proof. By induction:

Base Case: $k = 1$ A closed walk of length 3 $(2k + 1\ )$ must be a cycle

Inductive Step: If all vertices in the walk are distinct we are done since it is a cycle of odd length. In the other case where there are repeated, we can split the walk on repeated vertices to get smaller walks which are proved in the previous cases ■

Theorem 2 (bipartie). A graph is bipartite iff there are no odd cycles

Lecture 3

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Remark. Algorithmic Bipartite Testing

- Brute force: $2^{\|V\|} \cdot \|E\|$
- Proof Algorithm: where we color the vertices and check the edges $\|V\| + \|E\|$

Remark. Local to Global Results
Global properties always lead to local results (ex. bipartite implies no odd cycles). In West it is called "TONCAS"

1.6 Eulerian Graphs

Definition 14 (euler tour)**.** A closed walk that visits each edge of a graph exactly once

Definition 15 (eulerian graph)**.** A graph with a euler tour is Eulerian.

Proposition 1. A graph is Eulerian iff every vertex has even degree.

Proof. By induction:
Base Case: Trivial: 0 edges
Inductive Step: Let F be an edge of C a cycle in G. Then consider $H = (V, E\backslash)$. H has fewer edges, so each component has an Euler tour by induction ■

Note. This is another example of local and global properties where the parity of the degree is the local property and Eulerian is the global property

2 Trees

Definition 16 (tree)**.** A tree is a connected and acyclic graph

> **Definition 17** (leaf)**.** A node of degree 1

Lemma 4. Any connected subgraph of a tree is a tree

Proof. By contradiction:
Assume that the connected subgraph is not a tree, then the subgraph has a cycle, therefore the graph has a cycle, but trees are acyclic. Therefore, contradiction. ■

Lemma 5. A tree with n vertices has $n - 1$ edges

Proof. By induction:

Base Case: There are 0 edges in a 1 vertex tree

Inductive Step: Suppose this holds for n. Let T be a tree with $n+1$ vertices, then by removing one vertex, which must be a leaf — removing a non-leaf would make the graph unconnected, you remove one edge and the graph becomes the n case. Therefore, the $n+1$ tree has 1 more edge than an n tree. ■

Definition 18 (spanning tree)**.** A spanning tree (ST) is a subgraph of a connected that is a tree that contains all vertices of the original graph

Lemma 6. There is a spanning tree in every connected graph

Proof. By contradiction:
Assume G is a connected graph with no ST. Let T be a connected subgraph of G that has the same vertices as G with the smallest number of edges. Since T is not a tree, it does not have a cycle. However, T containing a cycle would imply that T is not a subgraph that has the smallest number of edges. Therefore, contradiction test ■