Graph Theory

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Contents

List of Theorems

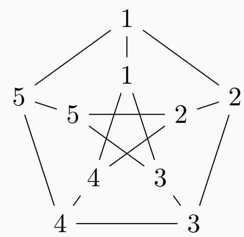
Lecture 1

0 Fundamental Concepts

1/26/2023

Definition 1 (graph). A graph is a pair (V, E) where V is the vertex space and E is the edge space.

Example (petersen graph). An element subset of $\{1, 2, 3, 4, 5\}$ connected by disjointedness



Note. In this course we are excluding multiple edges and self loops but we are considering that it may be considered

Definition 2 (vertex degrees). Let, $G = (V, E)$, $v \in V$, $e \in E$ e are incident if $v \in e$ i.e. v is an endpoint of e

Lemma 1.

$$\sum_{v \in V} deg(v) = \sum_{v \in V} \sum_{v \in e} 1 = \sum_{v \in e} \sum_{v \in V} 1 = 2|E|$$

Proof. Every edge has two vertices ■

Definition 3 (complete graph). Represented K_n, the graph has $V = 1 \dots n$ and all possible edges

$$|E| = \frac{n(n-1)}{2} = \binom{n}{2}$$

Definition 4 (isomorphic). Graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ are isomorphic if there exists a bijection, $f : v_1 \to v_2$, s.t. $\{u, v\} \in E_1 \Leftrightarrow \{f(u), f(v)\} \in E_2$

0.1 Connectivity

Definition 5 (connected). u and v are connected if there exists a path from u to v A graph is connected when all vertices are connected

Lecture 2

0.2 Degree Sequences

2023-01-31

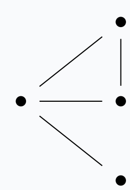
Definition 6 (degree sequence)**.** List of vertex degrees in decreasing order

Note. Isomorphic graphs $\Rightarrow$ same degree sequence; however, if they have the same degree sequence they are not necessarily isomorphic

Definition 7 (graphic)**.** A sequence is graphic if it's the degree sequence of some graph

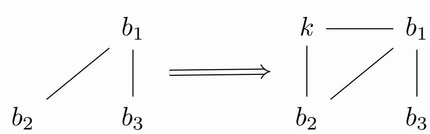
Example. Here are some degree sequences which may or may not exist:

- $(3, 2, 1, 1)$ - not possible since they don't sum to an even number
- $(3, 3, 1, 1)$ - not graphic since there are not enough vertices
- $(3, 2, 2, 1)$ - graphic



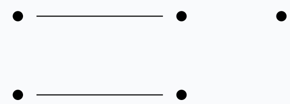
Theorem 1 (Havel-Hakimi)**.** $(a_1 \dots a_n)$ is graphic iff $(a_2 - 1, a_3 - 1, \dots a_{a_1+1} - 1, a_{a_1+2} \dots a_n)$ is graphic

Proof. We can apply the theorem in reverse to understand the intuition. If we have a graphic sequence, $(b_1 \dots b_m)$, we can add a vertex with degree k such that $(k, b_1 + 1, \dots b_k + 1, b_{k+1} \dots b_m)$.



In the reverse direction, we can subtract a vertex from a graph, and we make a transformation (2-switch) which preserves degree sequences but makes the graph maximal, to get the reverse result. ∎

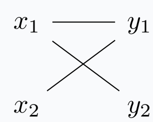
Example. $(3, 3, 3, 3, 3, 2, 2, 1)$ eventually becomes $(1, 1, 1, 1, 0)$ which we can show is graphic:



0.3 Bipartite graphs

Definition 8 (bipartitie graph). A graph is bipartite if it's possible to color vertices using only 2 colors. A simple check for it being bipartite is to check if there are no odd cycles

Example.



Here $G = (X \sqcup Y, E)$ where $x_1, x_2 \in X$ and $y_1, y_2 \in Y$

Theorem 2 (bipartite iff no odd cycle). A graph is bipartite $\Leftrightarrow$ if it contains no odd cycle

Proof. It is clear that if a graph contains an odd cycle that it is not bipartite, so we have to prove the reverse direction: if a graph does not contain an odd cycle then it is bipartite. It suffices to prove the statement for connected graphs - apply the argument to each component.

By induction:

Base Case: When $|E| = 0$ there are no connected vertices so there is no odd cycle and the graph must be bipartite since no vertices are connected to vertices of the same color.

Inductive Step: Assume a graph s.t. $|E| = n - 1$ and no odd cycles is bipartite. Any connected vertices of the same color must be connected by a path of even vertices. Let us add an edge which maintains the restriction that there is no odd cycle in the graph. Suppose an edge is added between these vertices, then the graph is no longer bipartite since two vertices of the same color are connected, the addition of an edge will create an odd cycle — a contradiction. Therefore, when adding an edge to keep the graph bipartite, there must be no odd cycle. ■

0.4 Walks and Paths

Definition 9 (walk). A walk is a sequence of vertices that are connected by edges

Definition 10 (length). The number of edges contained in the walk

Definition 11 (path). A path is a walk that has unique vertices

Lemma 2. A walk from v_0 to v_n implies a path from v_0 to v_n

0.5 Closed Walks and Cycles

Definition 12 (closed walk). A closed walk is a walk which starts and ends at the same vertex

Definition 13 (cycle). A cycle is a closed walk that has unique vertices

Lemma 3 (closed walk). A closed walk of odd length contains an odd cycle

Proof. By induction:
Base Case: $k = 1$ A closed walk of length 3 $(2k + 1)$ must be a cycle
Inductive Step: If all vertices in the walk are distinct, we are done since it is a cycle of odd length. In the other case where there are repeated, we can split the walk on repeated vertices to get smaller walks which are proved in the previous cases ■

Lecture 3

2023-02-02

Remark. Algorithmic Bipartite Testing

- Brute force: $2^{|V|} \cdot |E|$
- Proof Algorithm: where we color the vertices and check the edges $|V| + |E|$

Remark (local to global). Global properties always lead to local results (ex. bipartite implies no odd cycles). In West it is called "TONCAS"

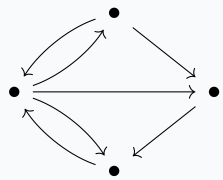
0.6 Eulerian Graphs

Definition 14 (euler tour). A closed walk that visits each edge of a graph exactly once

Definition 15 (eulerian). A graph with an euler tour is Eulerian.

Proposition 1 (eulerian local property). A graph is Eulerian iff every vertex has even degree.

Example (seven bridges of königsberg).



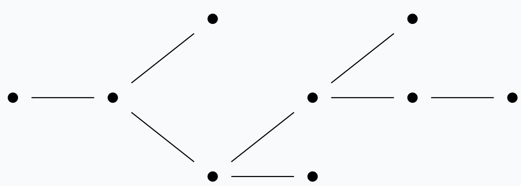
Using a graph to model the seven bridges of Königsberg shows that an euler tour is not possible.

Note. This is another example of local and global properties where the parity of the degree is the local property and Eulerian is the global property

1 Trees

Definition 16 (tree). A tree is a connected and acyclic graph

Example.



Definition 17 (leaf). A node of degree 1

Lemma 4. Any connected subgraph of a tree is a tree

Proof. By contradiction:

Assume that the connected subgraph is not a tree, then the subgraph has a cycle, therefore the graph has a cycle, but trees are acyclic. Therefore, contradiction. ■

Lemma 5. A tree with n vertices has $n-1$ edges

Proof. By induction:

Base Case: There are 0 edges in a 1 vertex tree

Inductive Step: Suppose this holds for n. Let T be a tree with $n+1$ vertices, then by removing one vertex, which must be a leaf — removing a non-leaf would make the graph unconnected, you remove one edge and the graph becomes the n case. Therefore, the $n+1$ tree has 1 more edge than an n tree. ■

1.1 Spanning Trees

Definition 18 (spanning tree). A spanning tree (ST) is a subgraph of a connected that is a tree that contains all vertices of the original graph

Lemma 6. There is a spanning tree in every connected graph

Proof. By contradiction:
Assume G is a connected graph with no ST. Let T be a connected subgraph of G that has the same vertices as G with the smallest number of edges. Since T is not a tree, it does not have a cycle. However, T containing a cycle would imply that T is not a subgraph that has the smallest number of edges. Therefore, contradiction test ■

Proposition 2. With a graph G where $T, T' \subset G$ s.t. $e \in T, e \notin T'$. Then there is edge $e' \in T'$ s.t. $T - e + e'$ is a spanning tree.

Proof. T is a tree $\Rightarrow T - e$ is disconnected. T, T' have the same vertex set $\Rightarrow \exists e \in T'$ that connects two components of $T - e$ Lets claim that $T - e + e'$ is a spanning tree. It has the same number of edges and is connected, so it's a tree, and is spanning since it connects all edges. ■

Lecture 4

1.2 Prüfer Codes

2023-02-09

Definition 19 (prüfer code). A code derived from a tree by iteratively removing the smallest degree vertex and adjoining its neighbor until there are only two vertices left.

Example (prüfer code to tree). Let's take the example $(1, 4, 1, 1)$

We know that vertices of deg 1 do not appear, so we start with an edge from 2 to 1 then we remove 1 from the list. Next we start with 4 and add the next deg 1 vertex. Now we have a connection from 4 to 1 since it's left in the Prüfer code. Then we finally create a tree with vertices $1, 5, 6$ and adjoin it.

Theorem 3. For each sequence $(a_1 \ldots a_{n-2})$ there exists a unique tree T with $P(T) = (a_1 \ldots a_{n-2})$

Proof. By induction:

Base Case: $n = 2$

Sequences have length $n = 2 = 0$ and there is only one tree with 2 vertices

Inductive Step:

For any tree with code $(\vec{a_1} \ldots a_{n-2})$ Let x be the smallest index not in the sequence then we can construct an edge between it and a_1 and remove a_1 from the code. Therefore, we have a code with one less element. And, by induction, $\exists!$ a tree T' with vertices

$\{1 \ldots n\} \setminus x$ and code $(a_2 \ldots a_{n-2})$.

The uniqueness come from the fact that T' is unique and there is only one way to add an edge ■

Theorem 4 (Cayley). There is a bijection between trees and these sequences. Therefore, we can count trees easily by counting sequences.

Therefore, we find there are n^{n-2} trees given n vertices

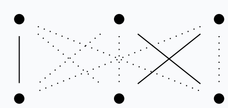
Lecture 5

2023-02-14

Matchings

Definition 20 (matching). A matching is a subset of $M \subset E$ such that no two edges in M share a vertex

Example.



Definition 21 (saturates). For $U \subset V$ say M saturates U if each $u \in U$ is incident to some $e \in M$ A matching that saturates V is called perfect

Remark. Perfect matching relate to bijections.

1.3 Matchings in Bipartite Graphs

For $G = (X \sqcup Y, E)$, is there a matching that saturates X?

Proposition 3 (saturated matching for bipartite graph). Local to Global: Let $N(S)$ be set of vertices connected to vertices in S. The local property is that for $S \in X$, $|S| \leq |N(S)|$

Theorem 5 (Hall's Matching). $G = (X \sqcup Y, E)$ bipartite has matching saturating X iff $(\forall S \subset X)(|S| \leq |N(S)|)$

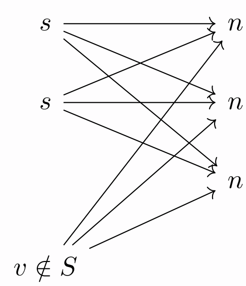
Proof. The proof requires the maximum matching theorem. Suppose G does not have a matching that saturates X. Let M max matching that does not saturate X, $u \in X$ unsaturated. $S = \{x \in X \text{connected to u by a M-alt path}\}$ $T = \{y \in Y \text{connected to u by a M-alt path}\}$ Any M-alt path from u to $y \in T$ can be extended uniquely to M-alt path ending in S Therefore, there is a bijection from T to $S \setminus u$

$N(S) = T$ since $T \subset N(S)$ by definition and $N(S) \subset T$ because we can fix a $y \in N(S)$ which is connected to an x which is connected to u. ■

Corollary. Any k-regular bipartite graph has perfect matching

Proof. Fix $G = (X \cup Y, E)$ bipartite and k=regular

Show $|X| = |Y|$: $k|X| = \sum_{x \in X} deg(x) = |E| = \sum_{y \in Y} deg(y) = k|Y|$ Fix $S \in X$. Each edge leaving S lands in $N(S)$. There are $k|N(S)|$ edges entering $N|S|$. Therefore, $k|S| \leq k|N(S)$.



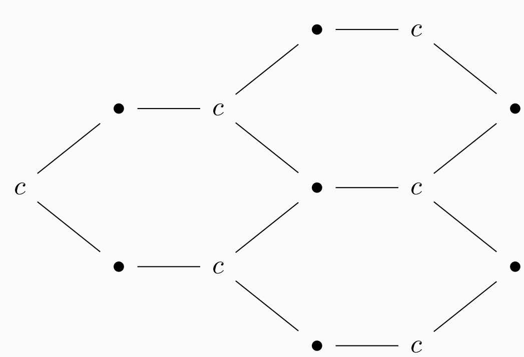
■

1.4 Matching Duel Problem

Definition 22 (maximium)**.** A matching $M \in E$ is maximum if it has the most edges of any matching

Definition 23 (vertex cover)**.** A vertex cover of G is a subset $Q \subset V$ s.t. every edge is incident to some vertex in Q

Example.



Where vertices labeled c are a vertex cover

Remark. Matchings and vertex covers are dual problems

Lecture 6

2023-02-16

Lemma 7. Let Q be a vertex cover, M a matching Then $|M| \leq |Q|$

Proof. For each $e \in M$ there is at least one vertex of Q incident to e. No two $e, e' \in M$ share a vertex of Q ■

Note. We can use this to show that if we have a vertex cover that covers all vertices with the same size as a matching we must have a maximum matching/minimum vertex cover.

Theorem 6 (König)**.** Let G bipartite, then the max size of the matching is equal to the min size of the vertex cover

Remark. Therefore, matching and vertex covers are duel problems

Definition 24 (m-augmenting-path)**.** A path whose edges alternate between M and $E \setminus M$ whose end points are unsaturated.

Theorem 7 (maximum matching)**.** Local to Global property: If there's no M-augmenting path then M is maximum

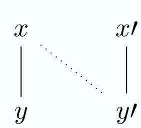
Proof. Suppose M is not a maximum there exists an M' s.t. $|m| < |M'|$ consider the symmetric difference $M \triangle M'$. Here vertices have either degree 1 or 2 and the graph is a union of even cycles and paths.

Let M be a non-maximal matching, M' maximum matching. Consider $M \triangle M'$ By above this is a union of paths and even cycles. Since M and M' share the same size in cycles then some component in $M \triangle M'$ has a path ■

Lecture 7

2023-02-23

Definition 25 (stable matching)**.** A matching is stable if



then either x prefers y to y' or y' prefers x' to x

Theorem 8 (Gale-Shapely)**.** Stable matchings always exist in bipartite graphs

Proof. We preform an algorithm to construct a stable matching:

Round 1:

Each $x \in X$ proposes to top choice

Each $x \in X$ proposes to top choice

Subsequent Rounds: Each unmatched x proposes to top choice not yet proposed to. Each y accepts the best proposal possible breaking an engagement

Repeat

Features

1. Algorithm stops after $< |E|$ rounds
2. End matching is stable
3. Proposers get the best match among all stable matchings. Proposed get worse.

■

Lecture 8

2023-02-28

2 Connectivity

Definition 26 (vertex cut). A vertex cut on a graph G is a subset $S \subset V$ s.t. $G \setminus S$ is disconnected

Definition 27 (vertex connectivity). Represented $\kappa(G)$, the vertex connectivity is the minimum size of a vertex cut of G. For K_n, κ is not defined.

Lemma 8. If G is not complete then G has a vertex cut

Proof. Let G not be complete, then there exist $u, v \in V$ s.t. $\{u, v\} \notin E$. Then $S = V \setminus u, v$ is a vertex cut ■

2.1 Connectivity Duel Problem

Definition 28 (vertex connectivity between two vertices). Let $G = (V, E)$, for $x, y \in V$ define $\kappa(x, y)$ min size of vertex cut that disconnects x and y

Note. We see that the points we have to remove are related to paths from x to y which are disjoint.

Definition 29 (disjoint path). A path is disjoint with another path if there are no interior vertices they share

Definition 30 (max pairwise dijoint). Let $\lambda(x, y)$ be the max number of pairwise disjoint x, y-paths

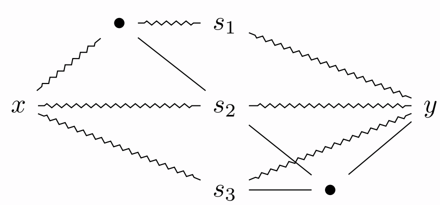
Theorem 9 (Menger). Let $G = (V, E)$, $x, y \in V$, $\{x, y\} \notin E$ then $\kappa(x, y) = \lambda(x, y)$

Proof. $\kappa(x, y) \geq \lambda(x, y)$ is easy

$\kappa(x, y) \leq \lambda(x, y)$:

Let S be a minimum vertex cut. We want to show that there are $|S|$ disjoint (x, y) paths Consider two subgroups G_x and G_y where G_x is the union of all paths from x to S (doesn't contain vertices form S) and G_y is the union of all paths from y to S (doesn't contain vertices form S). We have broken up the problem into just working with finding paths to S from x and y

To form a picture like:



■

2.2 Max Flow, Min Cut

Definition 31 (network). Is a tuple G, s, t, c where $G = (V, E)$ is a directed graph, $s, t \in V$ "source" and "terminus", $c : E \to N$ capacity.

Definition 32 (flow). A flow on a network is $f : E \to N$ s.t. $f(e) \leq c(e)$ and conservation law $f^+(v) = \sum_{e \to v} f(e) = \sum_{v \to e} f(e) = f^-(v)$

Definition 33 (value). Value of f is $f^-(s)$

The general problem is that given a network what is the max value of a flow

Definition 34 (cut). A cut is a partition $V = S \sqcap T$ with $s \in S$, $t \in T$

Definition 35 (capacity). The capacity of a cut S, T is $\sum_e c(e)$ where e is an edge from S to T

2.3 Ford-Fulkerson

Theorem 10 (Ford-Fulkerson). Max value of the flow is equal to the min capacity of a cut

Lecture 9

Note. Capacity of any cut gives upper bound on value of any flow. From this we can see the reasoning of why Ford-Fulkerson finds this as a duel problem 2023-03-02

Lecture 10

3 Graph Coloring

2023-03-07

Example (exam scheduling). Define $G = (V, E)$ where V are courses and E are conflicts if a student takes both courses. Here a graph coloring can help us answer how to schedule classes to limit conflicts.

Definition 36 (vertex coloring). Is a function of $V \to \mathbb{N}$ s.t. if $\{u, v\} \in E$ then $c(u) \neq c(v)$

Definition 37 (chromatic number). Represented $\chi(G)$, the chromatic number is the minimum number of colors needed to color a graph.

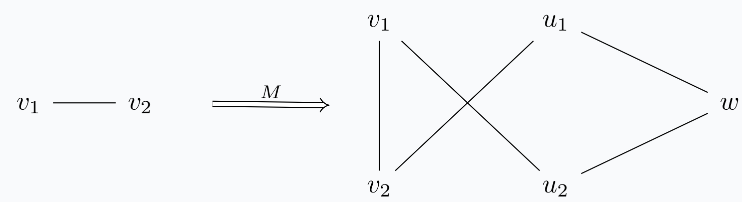
Lemma 9. $\chi(G) \leq \Delta G + 1$ where ΔG is the max degree of a vertex in G

Theorem 11 (Brooks). $\chi(G) = \Delta(G) + 1 \Rightarrow G$ is an odd cycle or complete

Definition 38 (clique number). Represented $\omega(G)$, the clique number is the largest number s.t. K_n is subgraph of G (this implies $\chi(G) \geq n$)

3.1 Mycielski Construction

Example (mycielski construction).



With construction that if $G = (V, E)$ where $V = \{v_1 \dots v_2\}$ then $M(G)$ has vertices $\{v_1 \dots v_n, u_1 \dots u_n, w\}$ and edges $\{v_i, v_j\}, \{v_i, u_j\}, \{u_i, w\}$, for $\{v_i, v_j\} \in E$

Theorem 12 (Mycielski).

1. $\chi(G) = k \Rightarrow \chi(M(G)) = k + 1$
2. G doesn't contain $K_3 \Rightarrow M(G)$ doesn't contain K_3

Proof. *First statement:* Given k-coloring of G, we can $(k+1)$ color $M(G)$, where you color u_i the same as v_1 and w the $k+1$ color. We also want to show that the graph has no smaller k coloring. Suppose $M(G)$ has a k-coloring then U uses a $k-1$ coloring and since the U coloring can be sent to the V coloring. Therefore, G has a $k-1$ coloring - a contradiction. ∎

Remark. Therefore, the clique number is not a very strong property

3.2 Coloring Extremal Problem

Note. Coloring graphs is an NP problem, so it might be better to try to solve extremal problems

Question (coloring extremal problems). Among graphs with a $\chi(G) = k$ what is the maximal/minimal number of edges?

Lecture 11

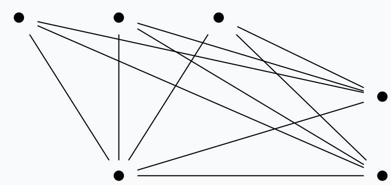
Definition 39 (turan graphs). Graphs of the form 2023-03-09

$$T_{n,k} = M_{\underbrace{q \dots q}_{k-r}, \underbrace{q+1 \dots q+1}_{r}}$$

where $n = qk + r$ s.t. $r < k$ are the solutions to the maximal coloring problem.

Proof. Pretty obvious with calculus based approach ∎

Example. $T_{6,3} = M_{3,2,1}$



3.3 Chromatic Polynomial

Definition 40 (chromatic polynomial). $\chi(G,t) =$ number of colorings of G using at most t colors

1. Degree $= |V|$
2. $a_{n-1} = |E|$
3. $\chi(G) =$ smallest t s.t. $\chi(G,t) \neq 0$ $(0, \dots \chi(G) - 1$ are roots of $\chi(G,t))$
4. t^d divides $\chi(G,t) \Rightarrow$ G has $\geq d$ components
5. coefficients are log concave

Example (chromatic polynomial of a complete graph). $\chi(K_n, t) = t(t-1)(t-2)\dots(t-(n+1)) = \binom{t}{n}$

Example (chromatic polynomial of a tree). Given a tree T, and a subtree S with one less vertex. Inductively, $\chi(T,t) = (t-1)\chi(S,t)$. And the coloring does not depend on what tree it is, only the number of vertices. $\chi(T,t) = t \cdot (t-1)^{n-1}$

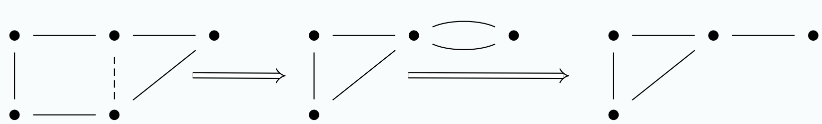
Theorem 13. The chromatic polynomial is a polynomial

Proof. $\chi(G,t) = \chi(G \setminus e, t) - \chi(G \cdot e, t)$

Where $G \cdot e$ is the contraction formula

Write $V = I_1, \sqcup \dots \sqcup I_r$ then $\chi(G,t) = \sum_{r=1}^{|V|} a_r(G) \cdot \chi(K_r, t)$ where $a_r(G)$ are the number of ways to write $V = I_1 \sqcup \dots \sqcup I_r$ ■

Definition 41 (contraction formula).



Proof.

$$\chi(G,t) = \chi(G \setminus e, t) - \chi(G \cdot e, t)$$

Any t coloring gives a t coloring of $G \setminus e$. A coloring os $G \setminus e$ gives a coloring of G when the edpoints e are independent. Colorings of $G \setminus e$ corapsond to colorings of $G \cdot e$ ∎

Example (sudoku). We form a graph where vertices are squares on the board and vertices are connected if they are in the same row column or square on the grid. Now this is made into a coloring problem.

How many 9×9 Sudoku puzzles?

Each graph has 81 vertices with degree 20, so there are 810 edges.

Lecture 12

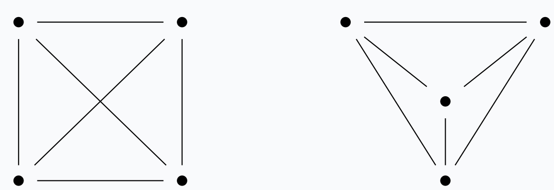
2023-03-14

Definition 42 (independent). A subset $I \subseteq V$ is independent if there are no edges $\{u, v\}$ where $u, v \in I$

4 Planar Graphs

Definition 43 (planer). A graph is planar if it can be drawn on the plane with no edges crossing.

Example. K_4



4.1 Theorems on the Real Plane

Theorem 14 (Jordan Curve). Any circle in $\mathbb{R}^2$ splits $\mathbb{R}^2$ into two regions, one unbounded and one unbounded

Theorem 15 (Euler's Formula). Let $G = (V, E)$ be embedded in $\mathbb{R}^2$ and connected, $|F|$ be the number of regions of $\mathbb{R}^2 \setminus G$, then the $|V| - |E| + |F| = 2$

Proof. By induction:

Base Case: $(|E| - |V| = -1)$

Here G is a tree, so $|V| - |E| = 1$ and $|F| = 1$; therefore $|V| - |E| + |F| = 2$

Inductive Step: If $|E| - |V| \geq -1$ then G has a cycle, C. Then by fixing an edge e on C we can consider

$$\begin{aligned} |V \setminus e| - |E \setminus e| + |F \setminus e| \\ |V| - (|E| + 1) + (|F| + 1) \\ |V| - |E| - 1 + |F| + 1 \\ |V| - |E| + |F| \end{aligned}$$

and by induction we know $|V \setminus e| - |E \setminus e| + |F \setminus e| = 2$, so $|V| - |E| + |F| = 2$

■

Corollary. K_5 is not planar

Proof. By contradiction:
Assume there exists an embedding K_5 in $\mathbb{R}^2$ then by Euler's Formula

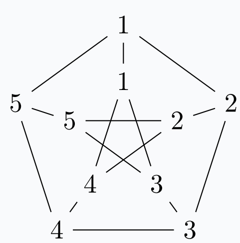
$$\begin{aligned} |F| &= 2 - |V| + |E| \\ |F| &= 2 - 5 + 10 \\ 7 &= 2 - 5 + 10 \end{aligned}$$

But each edge has at most 2 faces and each face has ≥ 3 sides therefore $2|E| \geq 3|F|$. However, $2 \cdot 10 \geq 21$; therefore K_5 is not planer ■

Note. Similar proof can be done for showing $K_{3,3}$ is not planar

Theorem 16 (Kuratowski). G is planar $\Leftrightarrow$ G does not contain a subdivision of K_5 or $K_{3,3}$

Example.



The Peterson graph is not planer since it contains a subdivision of K_3

Remark. This is an example of another local to global property

Lecture 13

4.2 Linear Planer Embeddings

2023-03-16

Note. Our definition of planer does not require straight lines

Question. If G is planar, then can it be drawn using straight lines?

Definition 44 (linear embedding)**.** An embedding drawn with only straight lines

Theorem 17 (Fary)**.** Every planar graph has a linear embedding

Proof. By induction:

Base Case: $(|V| =)$

Inductive Step:

■

Proposition 4. A planar graph has at least four vertices of degree ≤ 5.

Proof. Suppose the degree of all the vertices are ≥ 6 then $2|E| = \sum \deg(v) \geq 6|V|$ but this contradicts $|E| \leq 3|V| - 6$ ■

4.3 Maximal Planar Graphs

Observe. For Fary's theorem adding edges makes it harder to linearly embedded

Definition 45 (maximal planar)**.** G is maximal planar if adding any edge to G gives a nonplanar graph

Proposition 5 (max planar properties)**.** Fix $G = (V, E)$ The following are equivalent:

1. G is maximal planar
2. $|E| = 3|V| - 6$
3. Complementary regions of $G \hookrightarrow \mathbb{R}^2$ is a triangle

Proof.

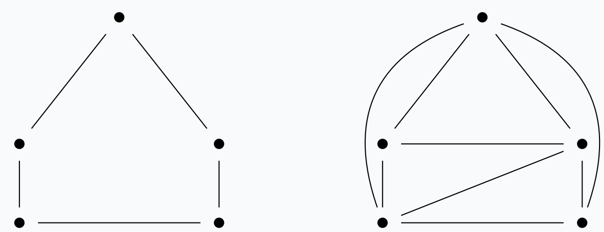
$2 \Leftrightarrow 3$: We saw that $3|F| \leq |E|$ and with Euler we can see that $|E| \leq 3|V| - 6$. We have equality of the second statement when the first statement has equality. Therefore, all the faces are triangles if and only if $|E| \leq 3|V| - 6$.

$1 \Rightarrow 3$: $G \hookrightarrow \mathbb{R}^2$ where some region has ≥ 4 sides. Then since we can add an edge in this case (ex. diagonal across square).

$3 \Rightarrow 1$: G is not max planer then we can add edge e s.t. $G' = G \cup e$ is a;so planar. Then we have an embedding $G' \hookrightarrow \mathbb{R}^2$ where we can remove e and see that we get an embedding of G with a region that's not a triangle.

■

Example (complementary region as traingles).



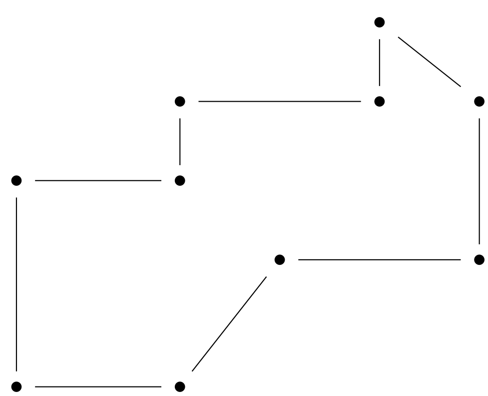
Here the left graph is not maximal planar and does not have all triangle complementary regions (basically faces) while the right one does, including the outer region

Lecture 14

4.4 Art Gallery Problem

2023-03-21

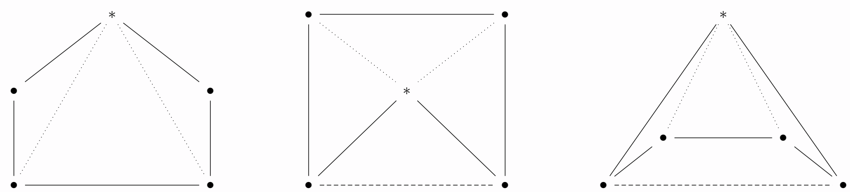
Question. Given a room, how many light sources can fill the entire space?



Theorem 18 (Art Gallery Thm). You only need $\left\lfloor \frac{n}{3} \right\rfloor$ for a polygon with n sides

Proof. Given a 5 sided polygon, consider the convex hull of the vertices.

The cases for the convex hull can be either $5, 4, 3$. For these cases, if you split the polygon into traingles there exists a vertex where it is incident to every triangle. And the solution is that you put the source near this vertex.



Here are each of the convex hull cases $(5, 4, 3)$ with the convex hull with long dashed lines and the triangle sections separated by the dotted lines. ■

4.5 Map Coloring

Question. Given a map how many colors are needed to color all the regions s.t. adjacent regions have different colors

Observe. The dual graph of them map is planar since we can move the vertices to the border, the borders do not intersect, and by Fary's.

Theorem 19 (4 Color)**.** Every planar graph is 4 colorable

Theorem 20 (6 Color)**.** Every planar graph is 6 colorable

Proof. A planar graph has a vertex of degree ≤ 5 by Euler's Formula. By induction, 6 color $G \setminus v$. If we remove v we get another planer graph that has a vertex of degree ≤ 5 By applying this repeatedly we get an ordering $v_1 \cdots v_n$. Now we can be greedy color by the order $v_n \cdots v_i$ where v_i has degree ≤ 5 ■

4.6 Planar Regular Graph

Definition 46 (regular planar)**.** A connected graph is called planar regular if it has an embedding where the vertices all have the same number of sides, which implies the duel is regular.

Question. What are all planer regular graphs?

G is k regular $2|E| = k|V|$ planar regular $2|E| = l|F|$
and using Euler we get $\frac{1}{k} + \frac{1}{l} = \frac{1}{|E|} + \frac{1}{2} > \frac{1}{2}$

Lecture 15

2023-04-04

5 Ramsey Theory

Question. Given a social network with n people where any two are whether friends or strangers, how large are there needed for there to exist either 3 mutual friends or 3 mutual strangers?

Proposition 6. You only need 6 vertices in order for the above to hold ($R(3,3)=6$)

Proof. Fix any edge 2-coloring of K_6. Fix a vertex v. Notice that it has either more red or more blue edges. Among the incident vertices there are at least 3 red edges which connect to x, y, z. If any edge between x, y, z is red there is a 3 cycle. Therefore, the edges must all be blue which shows that there is a 3 cycle among them. ■

Definition 47 (ramsey numbers)**.** For k, l define

$$R(k,l) = \min(\{n : \text{every edge 2-colring of } K_n \text{ has either a red } K_k \text{ or blue } K_l\})$$

Theorem 21 (ramsey)**.**

$$R(k,l) \leq R(k-1,l) + R(k,l-1)$$

Proof. Fix vertex v. Among edges incident to v, if there are $R(k-1,l)$ red edges or $R(k,l-1)$ blue edges we win.

By contradiction, assume this is not the case, then the number of edges incident to v is $\leq R(k-1,l)+R(k,l-1) = N-2$ but in K_N every vertex has degree $N-1$ ■

5.1 Ramsey Theory for Numbers

Theorem 22 (Van Der Waerden)**.** Given r, k there exist $N(r,k)$ s.t. for $n \geq N(r,k)$ every r-coloring of $\{1 \cdots n\}$ contains monochromatic arithmetic progression of length k

Lecture 16

Lower bounds 2023-04-11

Proposition 7. $R(m,m) > n$

1. We can find an edge 2-coloring of K_n that does not contain monochromatic K_m, or we can find H s.t. G and $\bar{G}$ doesn't contain K_m.

2. Probabilistic Method: There are $2^{\binom{n}{2}}$ coloring. And there are $\binom{n}{m}$ K_n in K_m. Let's label them $X_1, X_2, \cdots X_{\binom{n}{m}}$. And there are $2 \cdot 2^{\binom{n}{2}-\binom{m}{2}}$ coloring which have

X_i monochromatic. Therefore, at maximum the number of total coloring with some X_i monochromatic is $\leq \binom{n}{m} 2 \cdot 2^{\binom{n}{2} - \binom{m}{2}}$.

$$\binom{n}{m} 2 \cdot 2^{\binom{n}{2} - \binom{m}{2}} \leq 2^{\binom{n}{2}}$$
$$\binom{n}{m} 2 \cdot 2^{-\binom{m}{2}} \leq 1$$
$$n^m 2 \cdot 2^{-\binom{m}{2}} \leq 1$$
$$n < 2^{\frac{m}{2}}$$

5.2 Ramsey and Fermat

Theorem 23 (Fermat). For $n \geq 3$, $X^n + Y^n = Z^n$ has no nontrivial integer solutions

Proof. There is a truly marvelous proof of this which is too long to fit in these notes ■

Question. Does $X^n + Y^n = Z^n$ have solutions in $\mathbb{Z}/m\mathbb{Z}$?

Theorem 24 (Schur). $\exists N$ s.t. for a prime $p > N$ then

$$X^n + Y^n = Z^n$$

has solution in $\mathbb{Z}/p\mathbb{Z}$

Proposition 8. Any n-coloring of $1 \cdots m$ has monochromatic solution to $x + y = z$ if m is sufficiently large

$\mathbb{Z}/p\mathbb{Z}$ cyclic nth power has index n implies there is an n-coloring of $1 \cdots p-1$ Therefore, by Schur's $\exists X, Y, Z, \eta \in 1..p$ $\eta X^n + \eta Y^n = \eta Z^n$

6 Random Graphs

Lecture 18

6.1 Erdos-Renyi Random Graphs and Phase Transitions

2023-04-18

Definition 48 (labled n vertex graphs). Let $\Omega_{n,p} = \{$labeled n vertex graphs$\}$ with probability function given by flipping biased coin for each edge where an edge occurs with p probability.

Definition 49 (phase transition probability function). We can write $P(G) = p^{|E|}(1 - p)^{\binom{n}{2} - |E|}$

Example (threshold function).

1. $t(n) = \ln(n)/n$ is the threshold for connectedness
2. $t(n) = \sqrt{2\ln(n)/n}$ is the threshold for a diameter of 2
3. $p(n) = o(1/n)$ ($p < c/n$ for n sufficiently large) then G is a forest
4. $p(n)$ c/n where $0 < c < 1$ have cycle of any size
5. $p(n)$ c/n where $1 < c$ has large component

Remark. Properties that are monotone, i.e. adding edges to the graph doesn't change the property (ex. connectedness)

6.2 Rado's Graph

Definition 50 (rado's graph). Let $V = \mathbb{N}$ and $\exists i, j \in E \Leftrightarrow$ the ith digit of j is odd

Properties

1. Robustness - any finite edge/vertex cut does not destroy R
2. $R \cong \bar{R}$
3. Universal $=$ any finite/countable G is induced subgraph of R
4. Homogeneity - Every isomerism between subgraphs if R extension of R

Definition 51 (extension property). Given $u_1 \cdots u_p$ and $v_1 \cdots v_q$ distinct $\exists z$ adjacent to all us and none of the vs

Theorem 25. A $= \{G \in \Omega_\infty \text{isomophic to } R\} \in \Omega_\infty$

Proof. $B = \{G \textit{ has the extension property } \} \Rightarrow P(B) = 1$ Let us compute the probability that the extension property fails

Given $u_1 \cdots u_p$, $v_1 \cdots v_q$, $z_1 \cdots z_r$, then the probability is $\binom{p+q+r}{p+q}(1 - \frac{1}{2^{p+q}})^r$ ■

Lecture 19

2023-04-20

7 Graphs and Matrices

7.1 Adjacency matrix

Definition 52 (adjacency matrix). A matrix where

$$(A_G)_{i,j} = 1 \Leftrightarrow \{v_1, v_j\} \in E$$

Lemma 10. Given an adjacency matrix A, $(A^k)_{i,j}$ is the number of walks from v_i to v_j

Corollary. G is connected $\Leftrightarrow (A + A^2 + \cdots A^{n-1})_{i,j} \neq 0$ where $n = |V|$

Definition 53 (diameter).

$$d(v_i, v_j) = \min\{k : (A^k)_{i,j} \neq 0\}$$

7.2 Eigenvalues

Theorem 26 (Spectral). A n by n real symmetric matrix has n eigenvalues

Corollary. Non-directed graphs should have adjacency matrix with eigenvalues

Example. K_4 has eigenvalues $3, -1, -1, -1$

Example. P_4 has eigenvalues $\frac{\pm 1 \pm 5}{2}$

We can notice the max eigenvalue is bounded

$$\begin{aligned}(Ax)_j &= \sum_{k=1}^{n} A_{j,k} x_k \\ &= \sum_{\{v_j v_k\} \in E} x_k \\ &\leq \deg(v_j) \cdot \max\{x_k\} \\ &\leq \max \deg(v_j) \cdot \max\{x_k\} \\ &\leq \Delta(G) \cdot \max\{x_k\}\end{aligned}$$

For each j

$$\lambda \cdot x_j \leq \Delta(G) \cdot \max\{x_k\} \lambda \leq \Delta(G) \quad \text{Given the max is equal to the avg}$$

Theorem 27 (test). Max eigenvalue equaling $\Delta(G) \Leftrightarrow G$ is regular

Proof. We have shown this in the previous inequality. This is only true if x_k are all the same ∎

7.3 Multiplicities

Question. How many district eigenvalues for A_G?

Proposition 9. For any nxn A:

$$A^m + c_1 A^{m-1} \cdots c_m I = 0$$

for some $c_1 \cdots c_m \in \mathbb{R}$

Definition 54 (min polynomial)**.** The polynomial of min degree that A satisfies by the above construction

For A_G the min polynomial has roots of the eigenvalues with multiplicity one

Proposition 10. If $diam(G) = k$ then A does not satisfy polynomial of degree k

Proof. Suppose A satisfies a polynomial of degree k, the diameter. Therefore, there exists v_i, v_j s.t. $d(v_i, v_j) = k$ Therefore, $(A^l)_{i,j} = 0$ for all $l < k$. From here the proof is trivial ■

Corollary. The number of distinct eigenvalue is greater than the diameter of G

7.4 Matrix Tree Theorem

Definition 55 (laplacian)**.**

$$L_G = \text{degree matrix} - A_G$$

Note.

$$\det(L) = \frac{1}{n}(\text{product of the nonzero eigenvalues of } L_G)$$

Definition 56 (incidence matix of directed graph)**.**

$$B_{i,j} = \begin{cases} -1 & v_i \text{ is the target} \\ 1 & v_j \text{ is the source} \\ 0 & \text{else} \end{cases}$$

Lemma 11. $L_G = B_G \cdot B_G^t$

Proof. Just matrix multiplication ■

Theorem 28 (Cauchy-Binet)**.** Given X, Y, then $\det(XY) \sum_{s=nxn} \det(X_s) \det(Y_s)$

Theorem 29 (matrix tree). Let L be the $(n-1)x(n-1)$ matrix created by removing the first row/collums of L_G. Then $\det(L)$ is the number of spanning trees of G.

Proof. G_s tree iff $\det(B_s') = 0$: Suppose G_s is not a tree, then G_s has a cycle, C. By the definition of B_s', when we consider the column of B' with edges in C, we see that the columns are linearly dependent. Therefore, the determent is 0.

If G_s is a tree then G_s has a leaf v_i and $\det(B_s) = \pm\det(B'')$, where B'' = remove the row of v_i and column of B_s' where v_i has nonzero entry. B'' is minor corresponding to spanning tree of $G \setminus v_i$ where we can proceed inductively. ∎

Lecture 20

7.5 Interoperation of the Laplacian

2023-04-25

Definition 57 (harmonic). Given a function $f : \mathbb{R}^2 \to \mathbb{R}$ we can apply the Laplacian operator where

$$Lf - f_{xx} + f_{yy}$$

A function is harmonic if $Lf = 0$

Proposition 11. If a function is harmonic a point's value is equal to the avg value of the ball around it

If $Lf = 0$ then for each i, $f(v_i) = \frac{1}{\deg(v_i)} \sum_{v_i, v_j \in E} f(v_j)$, which is the avg of f given neighbors of v_i

The multiplicity of 0 as an eigenvalue of L is the number of components of G