Homework 5

Math 123

Due March 3, 2023 by *midnight*

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Topics covered: matchings, König's theorem, vertex covers, Gale–Shapely algorithm

Instructions:

- This assignment must be submitted on Gradescope by the due date.
- If you collaborate with other students (which is encouraged!), please mention this somewhere on the assignment.
- If you are stuck, please ask for help (from me, a TA, a classmate). Use Campuswire!
- You may freely use any fact proved in class. In general, you should provide proof for facts used that were not proved in class.
- **Please restrict your solution to each problem to a single page.** Usually solutions can be even shorter than that. If your solution is very long, you should think more about how to express it concisely.

Problem 1. *Let $G = (V, E)$ be a bipartite graph with maximum vertex degree Δ.*

(a) Use König's theorem to prove that G has a matching of size at least $|E|/\Delta$.

(b) Use (a) to conclude that every subgraph of $K_{n,n}$ with more than $(k-1)n$ edges has a matching of size at least k.

Solution. □

Problem 2. *Fix $k \geq 2$, and let Q_k denote hypercube graph (from HW1). Prove that Q_k has at least $2^{2^{k-2}}$ perfect matchings.*

Solution. □

Problem 3. *Determine the stable matchings resulting from the proposal algorithm run with cats proposing and with giraffes proposing, given the preference lists below.*

Cats $\{u, v, w, x, y, z\}$	Giraffes $\{a, b, c, d, e, f\}$
$u : a > b > d > c > f > e$	$a : z > x > y > u > v > w$
$v : a > b > c > f > e > d$	$b : y > z > w > x > v > u$
$w : c > b > d > a > f > e$	$c : v > x > w > y > u > z$
$x : c > a > d > b > e > f$	$d : w > y > u > x > z > v$
$y : c > d > a > b > f > e$	$e : u > v > x > w > y > z$
$z : d > e > f > c > b > a$	$f : u > w > x > v > z > y$

To receive full credit, you should show your work.

Solution. $\square$

Problem 4. *Let $G = (X \sqcup Y, E)$ be a bipartite graph satisfying $|N(S)| > |S|$ for each nonempty $S \subset X$. Prove that every edge of G belongs to some matching that saturates X.*

Solution. $\square$

Problem 5. *Complete the proof of König's theorem that we started in class.*

Solution. □

Problem 6. *A deck with mn cards with m values and n suits consists of one card for each value in each suit. The cards are dealt into an $n \times m$ array. Prove that there is a set of m cards, one in each column, having distinct values.*

Solution. □

Problem 7 (Bonus)**.** *Let T_1 be the tiling of the plane by unit squares whose vertices have integer coordinates. Let T_2 be the result of rotating T_1 about the origin by some angle θ. Prove that it is possible to find a bijection between squares of T_1 and squares of T_2 in such a way that the matched squares are within 10 units of each other. The matching will depend on θ.*

Solution. □