Math 1540 Spring 2023 - Homework 4

Instructions: This assignment is worth twenty points. Please complete the following problems assigned below. Submissions with insufficient explanation may lose points due to a lack of reasoning or clarity. If you are handwriting your work, please ensure it is readable and well-formatted for the grader.

Be sure when uploading your work to **assign problems to pages**. Problems with pages not assigned to them **may not be graded**.

Textbook Problems: 78, 79, 80

Additional Problems: We're largely working in fields, so you can typically assume E, F, B are fields unless otherwise stated. Also, typically if not explicitly stated E/F is the splitting field of the polynomial in question.

1. Consider a field extension E/F where $E = F(\alpha_1, \ldots, \alpha_n)$ and σ be a field automorphism of E. Assume σ fixes F pointwise, i.e. $\sigma(c) = c$ for all $c \in F$, and $\sigma(\alpha_i) = \alpha_i$ for each $i = 1, \ldots, n$. Prove σ is the identity map.

2. Let $f(x) \in \mathbb{Q}[x]$ be a degree 3 irreducible polynomial. Prove that $\mathrm{Gal}(E/\mathbb{Q})$ is either isomorphic to S_3 or A_3.

3. Let p be prime and $f(x)$ be an irreducible polynomial of degree p in $\mathbb{Q}[x]$ with all but two real roots. Prove $\mathrm{Gal}(E/\mathbb{Q})$ is isomorphic to S_p. (Hint: It suffices to find a p-cycle and a transposition)

4. Consider the field extension $\mathbb{Z}_3(t)/\mathbb{Z}_3$ where $\mathbb{Z}_3(t)$ is the field of rational functions in the variable t with coefficients in $\mathbb{Z}_3$. Let $a, b, c, d \in \mathbb{Z}_3$ such that $ad - bc \neq 0$. Prove that the map $\phi : \mathbb{Z}_3(t) \longrightarrow \mathbb{Z}_3(t)$ induced by

$$\phi(t) = \frac{at + b}{ct + d}$$

is an element of $\mathrm{Gal}(\mathbb{Z}_3(t)/\mathbb{Z}_3)$.

5. [Bonus] (3 points): Let α be a root of $f(x) = x^3 - 7x + 7$. Write out the other roots of $f(x)$ as polynomials with rational coefficients in the variable α. What does this mean about $\mathrm{Gal}(E/\mathbb{Q})$?