# Math 1540 Spring 2023 - Homework 3

**Instructions:** This assignment is worth twenty points. Please complete the following problems assigned below. Submissions with insufficient explanation may lose points due to a lack of reasoning or clarity. If you are handwriting your work, please ensure it is readable and well-formatted for the grader.

Be sure when uploading your work to **assign problems to pages**. Problems with pages not assigned to them **may not be graded**.

**Textbook Problems:** 72, 76, 77

**Additional Problems:** We're largely working in fields, so you can typically assume $E, F, B$ are fields unless otherwise stated.

1. Let $\alpha \in E/F$. Prove that $F(\alpha) = \{f(\alpha)/g(\alpha) \in E \mid f, g \in F[x] \text{ and } g(\alpha) \neq 0\}$.

2. Let $f(x) \in \mathbb{Z}_2[x]$ be defined by $f(x) = x^4 + x + 1$. Let $E$ be a splitting field of $f(x)$. Calculate, with proof, $[E : \mathbb{Z}_2]$.

3. Consider $\mathbb{Q}(\sqrt{2+\sqrt{3}}, \sqrt{2-\sqrt{3}})/\mathbb{Q}$. Is this extension simple? If so, find an element that generates the extension, along with its minimal polynomial. Calculate the degree of this extension.

4. Let $x^2 + x + 2 \in \mathbb{Z}_5[x]$ and consider a splitting field $E$. As observed in class, $E$ is a vector space over $\mathbb{Z}_5$. Using a root of $f(x)$, construct a faithful representation of $\mathbb{Z}_5^\times \longrightarrow \mathrm{GL}(2, \mathbb{Z}_5)$. Write out the image of your representation.