Math 1540 Spring 2023 - Homework 1

Instructions: This assignment is worth twenty points. Please complete the following problems assigned below. Submissions with insufficient explanation may lose points due to a lack of reasoning or clarity. If you are handwriting your work, please ensure it is readable and well-formatted for the grader.

Be sure when uploading your work to **assign problems to pages**. Problems with pages not assigned to them **may not be graded**.

Textbook Problems: 49, 50, 63, 69

Additional Problems:

1. Let u be a root solving the cubic $x^3 + qx + r = 0$. Choose y and z such that $y + z = u$ and $yz = -q/3$. Solve the equation $y^6 + ry^3 - q^3/27 = 0$ for y^3 as we did in class. Prove the choice the of the plus or minus in $y^3 = \frac{1}{2}\left(-r \pm \sqrt{r^2 + 4q^3/27}\right)$ ultimately yields the same set of solutions in $x^3 + qx + r = 0$ without defering to the fundamental theorem of algebra. (Hint: Take the two choices of y^3 and multiply them together)

2.. Define the characteristic of a field F the be the smallest positive integer n such that $n1_F = 0$ where 1_F is the unit of F. If no such n exists, say the field is characteristic zero. Prove if the characteristic is non-zero, then the characteristic is prime.

3. For a given field F define its prime subfield as the intersection of all subfields of F; denote this subfield P. If the characteristic of F is some prime p, prove that P is isomorphic to $\mathbb{Z}_p$. If the characteristic of F is zero, prove that P is isomorphic to $\mathbb{Q}$.

4. Consider the polynomial $p(x) = x^3 - 19x + 30$. Using the cubic formula, we know that a solution is given by

$$s_1 = \sqrt[3]{\frac{1}{2}\left(-30 + \sqrt{\frac{-3136}{27}}\right)} + \sqrt[3]{\frac{1}{2}\left(-30 - \sqrt{\frac{-3136}{27}}\right)}$$

We may also solve $p(x) = 0$ using the rational root theorem, Problem 63. Explain why this expression must be an integer which divides thirty. Denote the remaining solutions of $p(x) = 0$ by s_2 and s_3. Calculate $s_1 + s_2 + s_3$.