

### 2.1.1

Show the following properties:

- (a) If there exists a function  $g : B \rightarrow A$  such that  $f \circ g = I_A$ , then  $f : A \rightarrow B$  is injective.

By contradiction, assume  $f$  is not injective. Then it must collapse two distinct elements into one, i.e.  $\exists a_1, a_2 \in A$  such that  $f(a_1) = f(a_2) = b$  and  $a_1 \neq a_2$ . But if  $f \circ g = I_A$ , then  $g(f(a_1)) = g(b) = a_1$  **and**  $g(f(a_2)) = g(b) = a_2$ . Since  $a_1$  and  $a_2$  are distinct,  $g$  must be non-deterministic, so  $g$  cannot be a function as claimed.

If  $f : A \rightarrow B$  is injective and  $A \neq \emptyset$ , then there exists a function  $g : B \rightarrow A$  such that  $f \circ g = I_A$ .

Let  $f^T = \{ \langle b, a \rangle \mid \langle a, b \rangle \in f \}$ . Since  $f$  is injective,  $f^T$  is a partial function (deterministic), and since  $f$  is total and  $f^T$  is a partial function,  $f \circ f^T = I_A$ .

Now let  $g$  be any total function which extends  $f^T$  (this can be done because  $f^T$  is a partial function). Since  $g$  extends  $f^T$ ,  $I_A \subset f \circ g$ , and since  $g$  is functional/deterministic (in particular on  $\text{range}(f) = \text{dom}(f^T)$ ) and  $I_A$  is total,  $f \circ g = I_A$ .

- (b) A function  $f$  is surjective if and only if there exists a function  $g : B \rightarrow A$  such that  $g \circ f = I_B$ .

Forward direction: for each  $b \in B$ , let  $g(b) = a$  for **some**  $a \in f^{-1}(b)$ . Since  $f$  is surjective,  $f^{-1}(b)$  is non-empty, so we can always pick some  $a$ , and so  $g$  is a (total) function. And of course  $f(g(b)) = b = I_B(b)$  for all  $b \in B$ .

Backward direction: by contradiction, assume  $f$  is not surjective. Then there exists  $b \in B$  such that  $f^{-1}(b) = \emptyset$ . Nothing maps to  $b$ , so  $f(g(b)) \neq b$ , so  $g \circ f \neq I_B$ .

- (c) A function  $f : A \rightarrow B$  is bijective if and only if there is a function  $f^{-1}$  called its *inverse* such that  $f \circ f^{-1} = I_A$  and  $f^{-1} \circ f = I_B$ .

Forward direction: given  $f$ 's injectivity, we know from the problem on injectivity that any functional extension  $g$  of  $f^T$  satisfies  $f \circ g = I_A$ , and since  $f$  is also surjective the only functional extension is trivially  $f^T = f^{-1}$  itself. Furthermore, since  $f$  is total and surjective,  $f^{-1}$  is as well, and because  $(f^T)^T = f$ , by symmetry we have  $f^{-1} \circ f = I_B$ .

Backward direction: we know from the problems on injectivity and surjectivity that both  $f$  and  $f^{-1}$  are both injective and surjective, and are therefore bijective.

### 2.1.2

Prove that a function  $f : A \rightarrow B$  is injective if and only if, for all functions  $g, h : C \rightarrow A$ ,  $g \circ f = h \circ f$  implies that  $g = h$ .

Forward direction: by contradiction, let  $f$  be injective and  $g \circ f = h \circ f$ , but assume  $g \neq h$ . If  $g \neq h$  then there exists  $c \in C$  where  $g(c) \neq h(c)$ . But since  $f$  is injective then  $f(g(c)) \neq f(h(c))$ , so  $g \circ f$  cannot equal  $h \circ f$ .

Backward direction: by contradiction, assume  $f$  is not injective, i.e.  $\exists a_1, a_2 \in A$  such that  $f(a_1) = f(a_2)$  and  $a_1 \neq a_2$ . Let  $g$  and  $h$  be indistinguishable except at  $c$ , where  $g(c) = a_1$  and  $h(c) = a_2$ . Then the difference is washed out by  $f$ , and we have  $g \circ f = h \circ f$  but  $g \neq h$ .

A function  $f : A \rightarrow B$  is surjective if and only if, for all functions  $g, h : B \rightarrow C$ ,  $f \circ g = f \circ h$  implies that  $g = h$ .

Forward direction: by contradiction, let  $f$  be surjective and  $f \circ g = f \circ h$ , but assume  $g \neq h$ . Then  $g(b) \neq h(b)$  for some  $b \in B$ . Since  $f$  is surjective,  $f^{-1}(\{b\})$  is nonempty. Pick any element  $a \in f^{-1}(\{b\})$ . But then  $(f \circ g)(a) \neq (f \circ h)(a)$ , so  $f \circ g \neq f \circ h$ .

Backward direction: by contradiction, assume  $f$  is not surjective, i.e.  $f^{-1}(\{b\}) = \emptyset$  for some  $b \in B$ . Let  $g$  and  $h$  be indistinguishable except at  $b$ , where  $g(b) \neq h(b)$ . But then because nothing maps to  $b$  and  $g$  and  $h$  are otherwise identical,  $f \circ g = f \circ h$ .

### 2.1.3

Given a relation  $R$  on a set  $A$ , prove that  $R$  is transitive if and only if  $R \circ R$  is a subset of  $R$ .

By the definition of transitivity,  $R$  is transitive if and only if  $\forall x, y, z (xRy \wedge yRz \implies xRz)$ . By the definition of composition,  $\forall x, y, z (xRy \wedge yRz \iff x(R \circ R)z)$ . Therefore,  $R$  is transitive if and only if  $\forall x, y (x(R \circ R)y \implies xRy)$ , i.e. if and only if  $R \circ R$  is a subset of  $R$ .

### 2.1.4

Given two equivalence relations  $R$  and  $S$  on a set  $A$ , prove that if  $R \circ S = S \circ R$ , then  $R \circ S$  is the least equivalence relation containing  $R$  and  $S$ .

First, let's check that  $R \circ S$  is an equivalence relation.

- Reflexivity: because  $R$  and  $S$  are reflexive, for all  $x \in A$  we have  $xRx \wedge xSx$ , which implies  $x(R \circ S)x$ .
- Symmetry: consider some  $x, y \in A$  such that  $x(R \circ S)y$ . Since  $R$  and  $S$  commute we also know  $x(S \circ R)y$  (i.e.  $xSa \wedge aRy$  for some  $a \in A$ ). Finally, since  $R$  and  $S$  are symmetric, we know  $yRa \wedge aSx$ , and hence  $y(R \circ S)x$ .
- Transitivity: adopting a point-free notation, showing transitivity amounts to showing  $RSRS \subset RS$ . Since  $RS = SR$ ,  $RSRS = RRSS$ . Since  $R$  and  $S$  are transitive,  $RR \subset R$  and  $SS \subset S$ , and since  $M \subset N \implies (\forall x, y (xFMGy \implies xFNGy))$  for any relations  $F$  and  $G$  (monotonicity), we can conclude  $RSRS \subset RS$ .

Alternatively, with explicit arguments, consider some  $x, y, z \in A$  such that  $x(R \circ S)y(R \circ S)z$ , i.e.  $xRaSyRbSz$  for some  $a, b \in A$ . Since  $R \circ S = S \circ R$ ,  $xRaSyRbSz \iff xRaRwSbSz$  for some  $w \in A$ . Since  $R$  and  $S$  are transitive,  $xRaRwSbSz \implies xRwSz \implies x(R \circ S)z$ .

Now let's finish by checking that  $R \circ S$  is a subset of the least equivalence relation containing  $R$  and  $S$ . By definition of composition,  $\langle x, z \rangle \in (R \circ S) \iff \langle x, y \rangle \in R \wedge \langle y, z \rangle \in S$  for some  $y$ . The least equivalence relation containing  $R$  and  $S$  certainly contains  $R$  and  $S$ , so it must contain both  $\langle x, y \rangle$  and  $\langle y, z \rangle$ , and since any equivalence relation is transitive it must also contain  $\langle x, z \rangle$ . Therefore, it contains  $R \circ S$ .