2.1.1

Show the following properties:

(a) If there exists a function $g : B \to A$ such that $f \circ g = I_A$, then $f : A \to B$ is injective.

By contradiction, assume f is not injective. Then it must collapse two distinct elements into one, i.e. $\exists a_1, a_2 \in A$ such that $f(a_1) = f(a_2) = b$ and $a_1 \neq a_2$. But if $f \circ g = I_A$, then $g(f(a_1)) = g(b) = a_1$ **and** $g(f(a_2)) = g(b) = a_2$. Since a_1 and a_2 are distinct, g must be non-deterministic, so g cannot be a function as claimed.

If $f : A \to B$ is injective and $A \neq \emptyset$, then there exists a function $g : B \to A$ such that $f \circ g = I_A$.

Let $f^T = \{< b, a > \mid < a, b > \in f\}$. Since f is injective, f^T is a partial function (deterministic), and since f is total and f^T is a partial function, $f \circ f^T = I_A$.

Now let g be any total function which extends f^T (this can be done because f^T is a partial function). Since g extends f^T, $I_A \subset f \circ g$, and since g is functional/deterministic (in particular on $range(f) = dom(f^T)$) and I_A is total, $f \circ g = I_A$.

(b) A function f is surjective if and only if there exists a function $g : B \to A$ such that $g \circ f = I_B$.

Forward direction: for each $b \in B$, let $g(b) = a$ for **some** $a \in f^{-1}(b)$. Since f is surjective, $f^{-1}(b)$ is non-empty, so we can always pick some a, and so g is a (total) function. And of course $f(g(b)) = b = I_B(b)$ for all $b \in B$.

Backward direction: by contradiction, assume f is not surjective. Then there exists $b \in B$ such that $f^{-1}(b) = \emptyset$. Nothing maps to b, so $f(g(b)) \neq b$, so $g \circ f \neq I_B$.

(c) A function $f : A \to B$ is bijective if and only if there is a function f^{-1} called its *inverse* such that $f \circ f^{-1} = I_A$ and $f^{-1} \circ f = I_B$.

Forward direction: given f's injectivity, we know from the problem on injectivity that any functional extension g of f^T satisfies $f \circ g = I_A$, and since f is also surjective the only functional extension is trivially $f^T = f^{-1}$ itself. Furthermore, since f is total and surjective, f^{-1} is as well, and because $(f^T)^T = f$, by symmetry we have $f^{-1} \circ f = I_B$.

Backward direction: we know from the problems on injectivity and surjectivity that both f and f^{-1} are both injective and surjective, and are therefore bijective.

2.1.2

Prove that a function $f : A \to B$ is injective if and only if, for all functions $g, h : C \to A$, $g \circ f = h \circ f$ implies that $g = h$.

Forward direction: by contradiction, let f be injective and $g \circ f = h \circ f$, but assume $g \neq h$. If $g \neq h$ then there exists $c \in C$ where $g(c) \neq h(c)$. But since f is injective then $f(g(c)) \neq f(h(c))$, so $g \circ f$ cannot equal $h \circ f$.

Backward direction: by contradiction, assume f is not injective, i.e. $\exists a_1, a_2 \in A$ such that $f(a_1) = f(a_2)$ and $a_1 \neq a_2$. Let g and h be indistinguishable except at c, where $g(c) = a_1$ and $h(c) = a_2$. Then the difference is washed out by f, and we have $g \circ f = h \circ f$ but $g \neq h$.

A function $f : A \to B$ is surjective if and only if, for all functions $g, h : B \to C$, $f \circ g = f \circ h$ implies that $g = h$.

Forward direction: by contradiction, let f be surjective and $f \circ g = f \circ h$, but assume $g \neq h$. Then $g(b) \neq h(b)$ for some $b \in B$. Since f is surjective, $f^{-1}(\{b\})$ is nonempty. Pick any element $a \in f^{-1}(\{b\})$. But then $(f \circ g)(a) \neq (f \circ h)(a)$, so $f \circ g \neq f \circ h$.

Backward direction: by contradiction, assume f is not surjective, i.e. $f^{-1}(\{b\}) = \emptyset$ for some $b \in B$. Let g and h be indistinguishable except at b, where $g(b) \neq h(b)$. But then because nothing maps to b and g and h are otherwise identical, $f \circ g = f \circ h$.

2.1.3

Given a relation R on a set A, prove that R is transitive if and only if $R \circ R$ is a subset of R.

By the definition of transitivity, R is transitive if and only if $\forall x, y, z(xRy \wedge yRz \implies xRz)$. By the definition of composition, $\forall x, y, z(xRy \wedge yRz \iff x(R \circ R)z)$. Therefore, R is transitive if and only if $\forall x, y(x(R \circ R)y \implies xRy)$, i.e. if and only if $R \circ R$ is a subset of R.

2.1.4

Given two equivalence relations R and S on a set A, prove that if $R \circ S = S \circ R$, then $R \circ S$ is the least equivalence relation containing R and S.

First, let's check that $R \circ S$ is an equivalence relation.

- Reflexivity: because R and S are reflexive, for all $x \in A$ we have $xRx \wedge xSx$, which implies $x(R \circ S)x$.
- Symmetry: consider some $x, y \in A$ such that $x(R \circ S)y$. Since R and S commute we also know $x(S \circ R)y$ (i.e. $xSa \wedge aRy$ for some $a \in A$). Finally, since R and S are symmetric, we know $yRa \wedge aSx$, and hence $y(R \circ S)x$.
- Transitivity: adopting a point-free notation, showing transitivity amounts to showing $RSRS \subset RS$ (see 2.1.3). Since $RS = SR$, $RSRS = RRSS$. Since R and S are transitive, $RR \subset R$ and $SS \subset S$, and since $M \subset N \implies (\forall x, y(xFMGy \implies xFNGy))$ for any relations F and G (monotonicity), we can conclude $RRSS \subset RS$ and therefore $RSRS \subset RS$.

  Alternatively, with explicit arguments, consider some $x, y, z \in A$ such that $x(R \circ S)y(R \circ S)z$, i.e. $xRaSyRbSz$ for some $a, b \in A$. Since $R \circ S = S \circ R$, $xRaSyRbSz \iff xRaRwSbSz$ for some $w \in A$. Since R and S are transitive, $xRaRwSbSz \implies xRwSz \implies x(R \circ S)z$.

Now let's finish by checking that $R \circ S$ is a subset of the least equivalence relation containing R and S. By definition of composition, $< x, z > \in (R \circ S) \iff < x, y > \in R \wedge < y, z > \in S$ for some y. The least equivalence relation containing R and S certainly contains R and S, so it must contain both $< x, y >$ and $< y, z >$, and since any equivalence relation is transitive it must also contain $< x, z >$. Therefore, it contains $R \circ S$.

2.1.5

Prove that $R^+ = \bigcup_{n \geq 1} R^n$ is the smallest transitive relation on A containing R,

R^+ is transitive because $R^+R^+ = \bigcup_{n \geq 1} R^{2n}$ is a subset of $R^+ = \bigcup_{n \geq 1} R^n$ (see 2.1.3).

Any transitive extension T of R must extend R^+ because (by contradiction) if it does not then there is some smallest $n \geq 2$ (≥ 2 because T at least extends R) such that $R^n \not\subset T$, i.e. $< x, y > \in R^n$ and $< x, y > \notin T$ for some x, y. Since $R^n = RR^{n-1}$, $xR^ny \iff xRa \wedge aR^{n-1}y$ for some a. But $R \subset T \implies < x, a > \in T$ and $R^{n-1} \subset T \implies < a, y > \in T$, and since $< x, y > \notin T$, T cannot be transitive.

and $R^* = \bigcup_{n \geq 0} R^n$ is the smallest reflexive and transitive relation on A containing R.

TODO

Prove that for any relation R on a set A, $(R \cup R^{-1})^*$ is the least equivalence relation containing R.

TODO