2.1.1

Show the following properties:

(a) If there exists a function $g : B \to A$ such that $f \circ g = I_A$, then $f : A \to B$ is injective.

> By contradiction, assume f is not injective. Then it must collapse two distinct elements into one, i.e. $\exists a_1, a_2 \in A$ such that $f(a_1) = f(a_2) = b$ and $a_1 \neq a_2$. But if $f \circ g = I_A$, then $g(f(a_1)) = g(b) = a_1$ **and** $g(f(a_2)) = g(b) = a_2$. Since a_1 and a_2 are distinct, g must be non-deterministic, so g cannot be a function as claimed.

If $f : A \to B$ is injective and $A \neq \emptyset$, then there exists a function $g : B \to A$ such that $f \circ g = I_A$.

> Let $f^T = \{< b, a > \mid < a, b > \in f\}$. Since f is injective, f^T is a partial function (deterministic), and since f is total and f^T is a partial function, $f \circ f^T = I_A$.
>
> Now let g be any total function which extends f^T (this can be done because f^T is a partial function). Since g extends f^T, $I_A \subset f \circ g$, and since g is functional/deterministic (in particular on $range(f) = dom(f^T)$) and I_A is total, $f \circ g = I_A$.

(b) A function f is surjective if and only if there exists a function $g : B \to A$ such that $g \circ f = I_B$.

> Forward direction: for each $b \in B$, let $g(b) = a$ for **some** $a \in f^{-1}(b)$. Since f is surjective, $f^{-1}(b)$ is non-empty, so we can always pick some a, and so g is a (total) function. And of course $f(g(b)) = b = I_B(b)$ for all $b \in B$.
>
> Backward direction: by contradiction, assume f is not surjective. Then there exists $b \in B$ such that $f^{-1}(b) = \emptyset$. Nothing maps to b, so $f(g(b)) \neq b$, so $g \circ f \neq I_B$.

(c) A function $f : A \to B$ is bijective if and only if there is a function f^{-1} called its *inverse* such that $f \circ f^{-1} = I_A$ and $f^{-1} \circ f = I_B$.

> Forward direction: given f's injectivity, we know from the problem on injectivity that any functional extension g of f^T satisfies $f \circ g = I_A$, and since f is also surjective the only functional extension is trivially $f^T = f^{-1}$ itself. Furthermore, since f is total and surjective, f^{-1} is as well, and because $(f^T)^T = f$, by symmetry we have $f^{-1} \circ f = I_B$.
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> Backward direction: we know from the problems on injectivity and surjectivity that both f and f^{-1} are both injective and surjective, and are therefore bijective.

2.1.2

Prove that a function $f : A \to B$ is injective if and only if, for all functions $g, h : C \to A$, $g \circ f = h \circ f$ implies that $g = h$.

> Forward direction: by contradiction, let f be injective and $g \circ f = h \circ f$, but assume $g \neq h$. If $g \neq h$ then there exists $c \in C$ where $g(c) \neq h(c)$. But since f is injective then $f(g(c)) \neq f(h(c))$, so $g \circ f$ cannot equal $h \circ f$.
>
> Backward direction: by contradiction, assume f is not injective, i.e. $\exists a_1, a_2 \in A$ such that $f(a_1) = f(a_2)$ and $a_1 \neq a_2$. Let g and h be indistinguishable except at c, where $g(c) = a_1$ and $h(c) = a_2$. Then the difference is washed out by f, and we have $g \circ f = h \circ f$ but $g \neq h$.

A function $f : A \to B$ is surjective if and only if, for all functions $g, h : B \to C$, $f \circ g = f \circ h$ implies that $g = h$.

> Forward direction: by contradiction, let f be surjective and $f \circ g = f \circ h$, but assume $g \neq h$. Then $g(b) \neq h(b)$ for some $b \in B$. Since f is surjective, $f^{-1}(\{b\})$ is nonempty. Pick any element $a \in f^{-1}(\{b\})$. But then $(f \circ g)(a) \neq (f \circ h)(a)$, so $f \circ g \neq f \circ h$.
>
> Backward direction: by contradiction, assume f is not surjective, i.e. $f^{-1}(\{b\}) = \emptyset$ for some $b \in B$. Let g and h be indistinguishable except at b, where $g(b) \neq h(b)$. But then because nothing maps to b and g and h are otherwise identical, $f \circ g = f \circ h$.

2.1.3

Given a relation R on a set A, prove that R is transitive if and only if $R \circ R$ is a subset of R.

Symbolically, we have (with implicit universal quantification)

$$R \text{ is transitive iff } xRy \wedge yRz \implies xRz \qquad \text{(definition of transitivity)}$$

$$xRy \wedge yRz \Leftrightarrow x(R \circ R)z \qquad \text{(definition of composition)}$$

We can plug the definition of composition into the definition of transitivity (modus ponens) to eliminate $xRy \wedge yRz$ and derive

$$R \text{ is transitive iff } x(R \circ R)z \implies xRz$$

2.1.4

Given two equivalence relations R and S on a set A, prove that if $R \circ S = S \circ R$, then $R \circ S$ is the least equivalence relation containing R and S.

TODO